Random Numners

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Contents

1 Uniform Random Numbers

1.1 **Solution:** Download the following files and execute the C program.

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    coeffs.h
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    exrand.c
gcc exrand.c
./a.out
```

1.2 **Solution:** The following code plots Fig. 1.1

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    cdf_plot.py
python cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since U is a uniform distribution

$$F_U(x) = x \tag{1.1}$$

1.4 **Solution:** Download the following files and execute the C program.

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    mean.c
gcc mean.c
./a.out
```

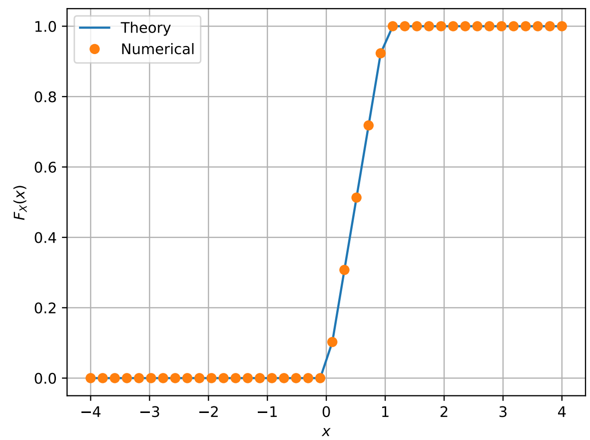

Fig. 1.1. The CDF of U

```
Mean: 0.500004
Variance: 0.083269
```

1.5

Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.2}$$

Solution:

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.3}$$

$$= \int_0^1 x dx \tag{1.4}$$

$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.5}$$

$$= \frac{1}{2} \tag{1.6}$$

$$
\begin{aligned}
E\left[U - E\left[U\right]\right]^2 &= E\left[U^2\right] - E\left[U\right]^2 && (1.7)\\
E\left[U^2\right] &= \int_{-\infty}^{\infty} x^2 dF_U(x) && (1.8)\\
&= \int_0^1 x^2 dx && (1.9)\\
&= \left[\frac{x^3}{3}\right]_0^1 && (1.10)\\
&= \frac{1}{3} && (1.11)\\
E\left[U - E\left[U\right]\right]^2 &= \frac{1}{3} - \left(\frac{1}{2}\right)^2 && (1.12)\\
&= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} && (1.13)
\end{aligned}
$$

2 Central Limit Theorem

2.1 **Solution:** Download the following files and execute the C program.

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    coeffs.h
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    exrand2.c
gcc exrand2.c
./a.out
```

2.2 What properties does a CDF have?
Solution: The following code plots Fig. 2.1

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    cdf_plot2.py
python cdf_plot2.py
```

- CDF reaches 0.5 as it reaches the mean(0)
- It reaches 1 as the graph reaches the end
- Differentiating the graph we obtain the graph of the PDF.

2.3 What properties does the PDF have?
Solution: The following code plots Fig. 2.2

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    pdf_plot.py
python pdf_plot.py
```

- PDF is symmetric about x = mean(0)

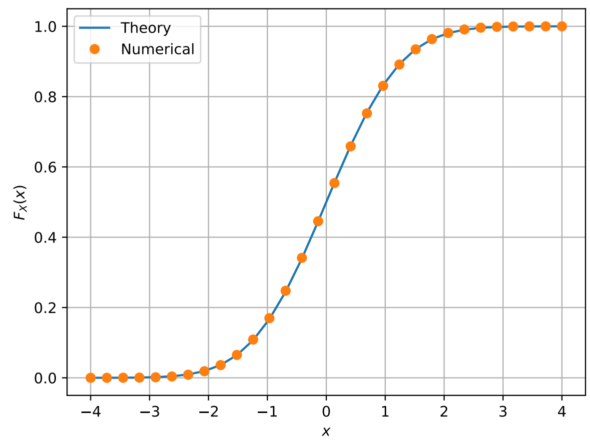

Fig. 2.1. The CDF of X

- Graph is shpaed like a bell

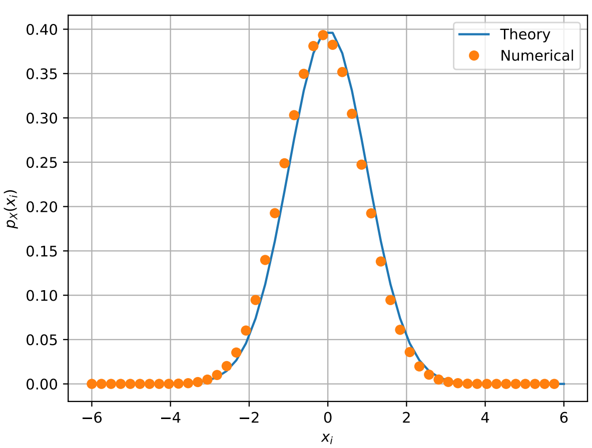

Fig. 2.2. The PDF of X

2.4 **Solution:** Download the following files and execute the C program.

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    mean2.c
gcc mean2.c
./a.out
```

```
Mean: 0.000326
Variance: 1.000467
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.1}$$

repeat the above exercise theoretically.
Solution:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.2}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \tag{2.3}$$

$$\text{Taking } \frac{x^2}{2} = t \implies xdx = dt \tag{2.4}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{-t} dt = 0 \tag{2.5}$$

$$E[X^2] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{x^2}{2}} dx \tag{2.6}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(xe^{-\frac{x^2}{2}}) dx \tag{2.7}$$

$$= \frac{1}{\sqrt{2\pi}} \left[-xe^{-\frac{-x^2}{2}} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right]_{-\infty}^{\infty} \tag{2.8}$$

$$= 1 \tag{2.9}$$

$$\text{Variance} = E[X^2] - E[X]^2 = 1 - 0 = 1 \tag{2.10}$$

3 From Uniform to Other

3.1 **Solution:** The following code plots Fig. 3.1

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/codes/
    cdf_plot3.py
python cdf_plot3.py
```

3.2 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{3.1}$$

repeat the above exercise theoretically.

$$F_V(x) = P(V \le x) \tag{3.2}$$

$$= P(-2\ln(1-U) \le x) \tag{3.3}$$

$$= P(U \le 1 - e^{\frac{-x}{2}}) \tag{3.4}$$

$$P(U < x) = \int_0^x dx = x \tag{3.5}$$

$$\therefore P(U \le 1 - e^{\frac{-x}{2}}) = 1 - e^{\frac{-x}{2}} \tag{3.6}$$

$$\implies F_V(x) = 1 - e^{\frac{-x}{2}} \tag{3.7}$$

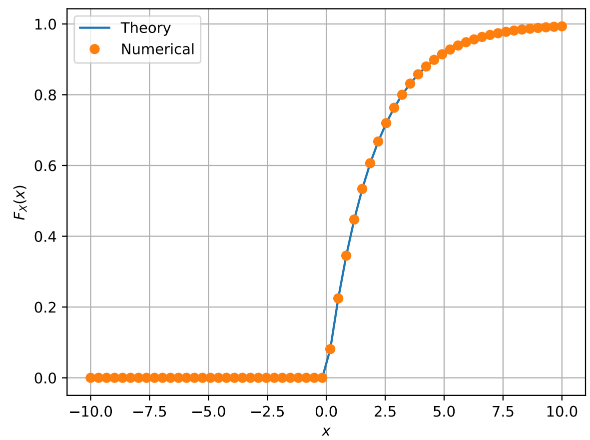

Fig. 3.1. The CDF of V

4 Triangular distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/
    codes/coeffs.h
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/
    codes/triang.c
gcc triang.c
./a.out
```

4.2 Find the CDF of T
Solution:

$$P_T(x) = P(U_1 + U_2 = x) \tag{4.2}$$

$$= \int_{-\infty}^{\infty} P_{U_1}(a) P_{U_2}(x-a) da \tag{4.3}$$

As $P_{U_1}(a) = 1$ for $0 < a < 1$ and 0 otherwise

$$\implies \int_0^1 P_{U_2}(x-a) da \tag{4.4}$$

$P_{U_2}(x-a) = 1$ for $0 < x-a < 1 \implies x-1 < a < x$
If $x < 1, 0 < a < x$

$$= \int_0^x 1 da = x \tag{4.5}$$

If $x > 1, x-1 < a < 1$

$$= \int_{x-1}^{1} 1da = 2 - x \tag{4.6}$$

$$\therefore P_T(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \tag{4.7}$$

Integrating $P_T(x)$ we obtain $F_T(x)$

$$\therefore F_T(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 < x < 1 \\ 1 - \frac{(2-x)^2}{2}, & 1 < x < 2 \\ 1, & x > 2 \end{cases} \tag{4.8}$$

4.3 Find the PDF of x

$$P_T(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \tag{4.9}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.

$$P_T(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \tag{4.10}$$

$$F_T(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 < x < 1 \\ 1 - \frac{(2-x)^2}{2}, & 1 < x < 2 \\ 1, & x > 2 \end{cases} \tag{4.11}$$

4.5 Verify your results through a plot.

5 Maximul Likelihood

5.1 solution Download the following files and execute the C program.

```
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/
    codes/coeffs.h
wget https://github.com/gunjitmittal/
    Random_Numbers/blob/main/
    codes/generate5.c
gcc generate5.c
./a.out
```

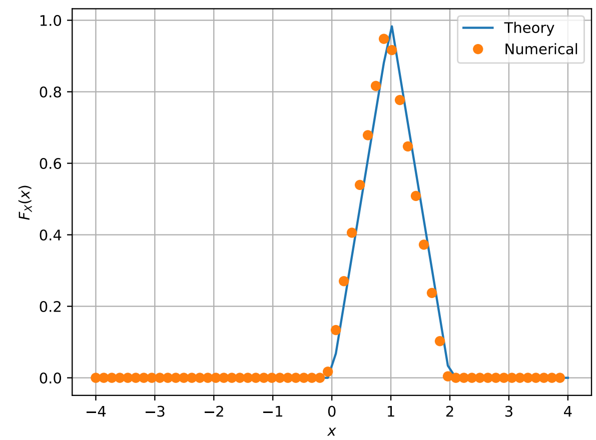

Fig. 4.1. The PDF of T

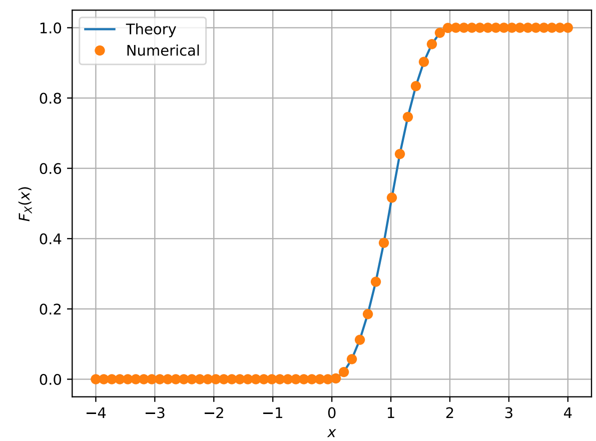

Fig. 4.2. The CDF of T

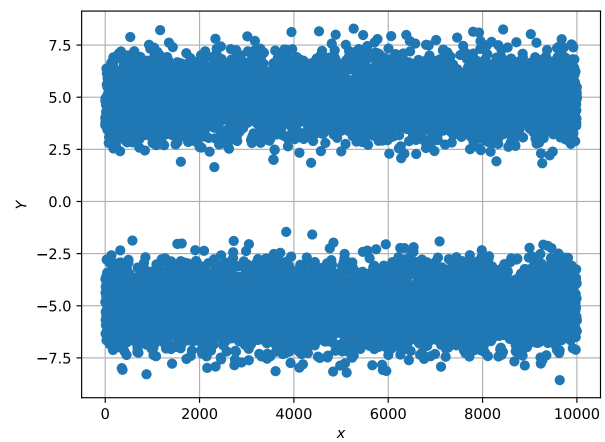

Fig. 5.1. The plot of Y

5.2 Plot Y.

5.3 Guess how to estimate X from Y. Since X lies in {-1,1} and is multiplied by 5, while the normal distribution is concentrated mostly in the [-1,1] region so if X is 1 then Y lies mostly in the region [4,6] and if X is -1 then Y lies mostly in the region [-6,-4] so we can take a threshold δ and say that if $Y > \delta$ then X is most probably 1 and if $Y < \delta$ then X is mostly -1. Since it is symmetric about X axis let $\delta = 0$ for now.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.2}$$

Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.3}$$
$$= \Pr\left(Y < 0|X = 1\right) \tag{5.4}$$
$$= \Pr\left(A + N < 0\right) \tag{5.5}$$
$$= \Pr\left(N < -A\right) \tag{5.6}$$
$$= F_N(-A) = 1 - Q_N(-A) \tag{5.7}$$

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.8}$$
$$= \Pr\left(Y > 0|X = -1\right) \tag{5.9}$$
$$= \Pr\left(-A + N > 0\right) \tag{5.10}$$
$$= \Pr\left(N > A\right) \tag{5.11}$$
$$= Q_N(A) \tag{5.12}$$

5.5 Find P_e assuming that X has equiprobable symbols.

Solution:

$$P_e = \frac{1}{2}P_{e|0} + \frac{1}{2}P_{e|1} \tag{5.13}$$
$$P_e = \frac{1}{2}\left(Q_N(A) + Q_N(A)\right) \tag{5.14}$$
$$= Q_N(A) \tag{5.15}$$