AST1100 Project

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1 Part 6 - The atmosphere

1.1 Theoretical Model

To model the atmosphere, we require two important equations, the ideal gas equation:

$$P = \frac{\rho k T}{\mu m_H} \tag{1}$$

And the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -G\rho(r)\frac{M}{r^2} \tag{2}$$

We will first model the atmosphere as adiabatic.

1.1.1 Adiabatic model

Adiabatic implies that the following equation holds:

$$P^{1-\gamma}T^{\gamma} = C \tag{3}$$

Where C is constant. Now I will differentiate the above equation, according to an idea found here **REFERENCE UNI TEXAS HERE**. This gives:

$$\frac{d}{dT}\left(P^{1-\gamma}T^{\gamma}\right) = 0$$

Which can be written out as:

$$(1-\gamma)P^{-\gamma}T^{\gamma}\frac{dP}{dT} + \gamma T^{\gamma-1}P^{1-\gamma} = 0$$

Which can be rearranged as:

$$\frac{P}{T} = \frac{\gamma}{\gamma - 1}\frac{dP}{dT}$$

Solving the ideal gas equation for the density gives:

$$\rho = \frac{\mu m_H}{k}\frac{P}{T}$$

Inserting for P/T:

$$\rho = \frac{\gamma - 1}{\gamma} \frac{\mu m_H}{k} \frac{dP}{dT}$$

I can now insert this relation into the hydrostatic equilibrium to "cancel out" dP (mathematicians, please look away), giving:

$$\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma} \frac{\mu m_H}{k} \frac{GM}{r^2}$$

This is a straightforward separable differential equation:

$$dT = -\frac{\gamma - 1}{\gamma} \frac{GM\mu m_H}{k} \frac{1}{r^2} dr$$

Integrating gives:

$$T = \frac{\gamma - 1}{\gamma} \frac{GM\mu m_H}{k} \frac{1}{r} + C$$

The constant can be fixed by realizing that the temperature must be T_0, the surface temperature of the planet, when $r = r_p$, the radius of the planet. Inserting gives:

$$T_0 = \frac{\gamma - 1}{\gamma} \frac{GM\mu m_H}{k} \frac{1}{r_p} + C$$

Which immediately implies that:

$$C = T_0 + \frac{1 - \gamma}{\gamma} \frac{GM\mu m_H}{k} \frac{1}{r_p}$$

Which finally gives the explicit dependence of temperature upon radius as:

$$T(r) = \frac{\gamma - 1}{\gamma} \frac{GM\mu m_H}{k} \left(\frac{1}{r} - \frac{1}{r_p} \right) + T_0$$

I will model the atmosphere as transitioning to an isothermal atmosphere when $T = T_0/2$. This happens for $r = r_{T_0/2}$, which can be found from:

$$-\frac{T_0}{2} = \frac{\gamma - 1}{\gamma} \frac{GM\mu m_H}{k} \left(\frac{1}{r_{T_0/2}} - \frac{1}{r_p} \right)$$

Which gives:

$$-\frac{k}{GM\mu m_H} \frac{T_0 \gamma}{2(\gamma - 1)} = \frac{1}{r_{T_0/2}} - \frac{1}{r_p}$$

I.e:

$$\frac{1}{r_{T_0/2}} = \frac{1}{r_p} - \frac{T_0}{2} \frac{\gamma}{\gamma - 1} \frac{k}{GM\mu m_H}$$

From this, it is relatively straightforward to get the dependence of pressure, P upon distance, seeing as:

$$P = CT^{\frac{\gamma}{\gamma - 1}}$$

Which gives:

$$\frac{P}{T} = CT^{1/(\gamma - 1)}$$

Thus, to find the density as a function of distance, I can simply calculate:

$$\rho = \frac{\mu m_H}{k}\frac{P}{T} = \frac{\mu m_H}{k} C T^{1/(\gamma-1)}$$

I can fix the constant by requiring that $\rho = \rho_0$ when $r = r_p$, which gives:

$$\rho_0 = \frac{\mu m_H}{k} C T_0$$

I.e:

$$C = \frac{\rho_0 k}{\mu m_H T_0}$$

Which gives:

$$\rho = \frac{\rho_0}{T_0} T^{1/(\gamma-1)}$$

1.1.2 The isothermal region

Once the radius is above $r_{T_0/2}$, we assume an isothermal atmosphere, i.e. constant temperature. This makes the equation much easier, as the hydrostatic equilibrium now reads:

$$\frac{dP}{dr} = -\frac{GM}{r^2}\frac{\mu m_H}{kT} P$$

Which is a separable differential equation, which can be solved as:

$$\frac{1}{P} dP = -\frac{GM}{r^2}\frac{\mu m_H}{kT} dr$$

Integrating gives:

$$\ln P = \frac{GM}{r}\frac{\mu m_H}{kT} + C$$

Or.

$$P = \tilde{C} e^{\frac{GM}{r}\frac{\mu m_H}{kT}}$$

Realizing that $T = T_0/2$ and inserting in the ideal gas equation gives:

$$\rho = \frac{2\mu m_H}{kT_0}\tilde{C} e^{2GM\mu m_H/rkT_0}$$

The constant can be easily fixed, by requiring that $\rho = \rho_{T_0/2}$ (from the adiabatic model) when $r = r_{T_0/2}$, which gives:

$$\rho_{T_0/2} = \frac{2\mu m_H}{kT_0}\tilde{C} e^{2GM\mu m_H/r_{T_0/2}kT_0}$$

Which gives:

$$\tilde{C} = \frac{kT_0 \rho_{T_0/2}}{2\mu m_H} e^{-2GM\mu m_H/r_{T_0/2}kT_0}$$

From which it follows that:

$$\rho = \rho_{T_0/2} \exp\left(\frac{2GM\mu m_H}{kT_0}\left(\frac{1}{r} - \frac{1}{r_{T_0/2}}\right)\right)$$