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Question 3:

8.2.2.B
Answer:
$f(n) = n^3 + 3n^2 + 4$
$\le 9n^3$ for $n \ge 2$

Therefore, we can choose $c = 9$ and $n_0 = 2$ to satisfy the definition of big-O. This proves that $f = O(n^3)$.

$f(n) = n^3 + 3n^2 + 4$
$\ge n^3$ for $n \ge 1$

Therefore, we can choose $c = 1$ and $n_0 = 1$ to satisfy the definition of big-Ω. This proves that $f = \Omega(n^3)$.

$f = O(n^3)$ and $f = \Omega(n^3)$,
In conclusion conclude that $f = \Theta(n^3)$.

8.3.5.A
Answer:
The algorithm partitions the input sequence into two parts:
Elements before i
Elements after j

Initially, i is set to the first element in the sequence and j is set to the last element. The algorithm then repeatedly searches for elements in the first part that are greater than or equal to p, and elements in the second part that are less than p.

When the elements are found, they are swapped. The algorithm terminates when i >= j. The output sequence is the input sequence with some of its elements rearranged so that all elements less than p appear before all elements greater than or equal to p.

8.3.5.B
Answer:
The lines "i := i + 1" and "j := j - 1" are executed a total of at most 2*n times on a sequence of length n. This is because each iteration of the outer loop either increments i or decrements j, but not both. Therefore, the total number of iterations of the outer loop is at most 2*n. The actual values of the numbers in the sequence do not affect this bound.

The inputs that minimize the number of times these lines are executed are either already sorted or sorted in reverse order with respect to p, since in those cases the algorithm will only perform one pass over the input sequence. The input that maximizes the number of times these lines are executed is one in which all the elements in the sequence are greater than or equal to p or all the elements are less than p. In such cases, the algorithm will perform n-1 iterations of the outer loop.

8.3.5.C

Answer:

The swap operation is executed at most n-1 times on a sequence of length n. This is because each iteration of the outer loop either swaps two elements or terminates. Therefore, the total number of swaps is at most the number of iterations of the outer loop, which is at most n-1. The actual values of the numbers in the sequence do not affect this bound.

The input that minimizes the number of swaps is one in which all the elements in the sequence are less than p or all the elements are greater than or equal to p. In such cases, the algorithm will perform no swaps. The input that maximizes the number of swaps is one in which the first half of the sequence contains all elements less than p and the second half of the sequence contains all elements greater than or equal to p. In such cases, the algorithm will perform n/2 swaps.

8.3.5.D

Answer:

An asymptotic lower bound for the time complexity of the algorithm is Ω(n), which means the algorithm takes at least linear time. The number of swaps is at most the number of times that i is incremented or j is decremented, which is at most n/2. Therefore, the lower bound for the time complexity is at least n/2, which is Ω(n).

8.3.5.E

Answer:

An upper bound for the time complexity of the algorithm is O(n), which means the algorithm takes at most linear time. This is because the algorithm consists of two nested while-loops that each iterate at most n times. Therefore, the upper bound for the time complexity is at most 2n, which is O(n).

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Question 4:

5.1.2.B
Answer:
3^7 = 2,187
3^8 = 6,561
3^9 = 19,683

The total number of possible passwords for strings of length 7, 8, or 9 is the sum of the passwords in each of the cases:

2,187 + 6,561 + 19,683 = 28,431

5.1.2.C
Answer:
(2 + 10) * 3^6 = 12,168
(2 + 10) * 3^7 = 36,684
(2 + 10) * 3^8 = 110,052

The total number of possible passwords for strings of length 7, 8, or 9, where the first character cannot be a letter, is the sum of the passwords in each of the cases:

12,168 + 36,684 + 110,052 = 158,904

5.3.2.A
Answer:
For the first character, we got 3 choices (a, b, or c). For the second character, we can't choose the same character as the first, so we have 2 choices. For the third character, we can't choose the same character as the second, so we have 2 choices again. For the fourth character, we can't choose the same character as the third, so we have 2 choices again. Continuing this pattern, we have:

first character, 3 choices
second character, 2 choices
third character, 2 choices
fourth character, 2 choices
...
ninth character, 2 choices
tenth character, 2 choices

Using the generalized product rule, the total number of strings is the product of the number of choices at each step. So we have:

3 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 = 3 × 2^9 = 1536

Therefore, there are 1536 strings over the set {a, b, c} that have length 10 in which no two consecutive characters are the same.

5.3.3.B

Answer:

Since there are 10 digits and each digit can only appear once in the license plate number, there are 10 choices for the first digit, 9 choices for the second digit (since one digit has already been used), 8 choices for the third digit, and 7 choices for the fourth digit. There are 26 choices for each of the four letters, since no letter can be repeated. Finally, there are 10 choices for each of the last two digits. Therefore, the total number of license plate numbers possible is:

10 * 9 * 8 * 7 * 26 * 26 * 26 * 26 * 10 * 10 = 17,576,000,000

5.3.3.C

Answer:

If no digit or letter can be repeated, there are 10 choices for the first digit, 26 choices for the first letter, 25 choices for the second letter (since one letter has already been used), 24 choices for the third letter, 23 choices for the fourth letter, 9 choices for the fifth digit (since one digit has already been used), and 8 choices for the sixth digit. Therefore, the total number of license plate numbers possible is:

10 * 26 * 25 * 24 * 23 * 9 * 8 = 22,364,160,000

5.2.3.A

Answer:

To show a bijection between B^9 and E_10, we can define a function f that takes a binary string of length 9 and returns a binary string of length 10 with an even number of 1's. Specifically, let f(x) be the binary string of length 10 obtained by adding a leading 0 to x if it has an even number of 1's, and adding a leading 1 to x if it has an odd number of 1's. Formally,

f(x) = {0x, if x has even number of 1's
1x, if x has odd number of 1's

To see that f is a bijection, we need to show that it is both injective and surjective.

Injective: Suppose that f(x) = f(y) for some binary strings x and y. Then either both x and y have an even number of 1's, or both have an odd number of 1's. In the first case, we have x = y, since f adds a leading 0 to both strings. In the second case, we also have x = y, since f adds a leading 1 to both strings. Therefore, f is injective.

Surjective: Let z be any binary string of length 10 with an even number of 1's. If z starts with 0, then we can obtain a binary string of length 9 with an even number of 1's by removing the leading 0 from z. If z starts with 1, then we can obtain a binary string of length 9 with an odd number of 1's by removing the leading 1 from z. In either case, we can apply f to this binary string of length 9 to obtain z itself. Therefore, f is surjective.

Since f is both injective and surjective, it is a bijection.

5.2.3.B

Answer:

To find |E_10|, we need to count the number of binary strings of length 10 with an even number of 1's. We can do this using the binomial coefficient formula:

|E_10| = C(10, 0) + C(10, 2) + C(10, 4) + C(10, 6) + C(10, 8) + C(10, 10)
= 1 + 45 + 210 + 210 + 45 + 1
= 512

Therefore, there are 512 binary strings of length 10 with an even number of 1's.

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Question 5:

5.4.2.A
Answer:
Since the phone numbers are 7-digits long and can start with either 824 or 825, there are two choices for the first three digits and 10 choices for each of the remaining four digits. Therefore, the total number of different phone numbers is:

$$2 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 2 \cdot 10^7 = 20,000,000$$

5.4.2.B
Answer:
(b) There are 10 choices for the last digit, 9 choices for the second-to-last digit (since it cannot be the same as the last digit), 8 choices for the third-to-last digit, and 7 choices for the fourth-to-last digit. Therefore, the total number of phone numbers in which the last four digits are all different is:

$$2 \cdot 10 \cdot 10 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 2 \cdot 10^5 \cdot 9 \cdot 8 \cdot 7 = 907,200$$

Note that we multiply by 2 at the beginning to account for the two choices for the first three digits (824 or 825).

5.5.3.A
Answer:
Each bit in a 10-bit string can be either 0 or 1, so there are 2^10 possible 10-bit strings.

5.5.3.B
Answer:
The first three bits of the string must be 001, but the remaining seven bits can be either 0 or 1, so there are 2^7 possible 10-bit strings that start with 001.

5.5.3.C
Answer:
The string can either start with 001 or 10, and the remaining eight bits can be either 0 or 1, so there are 2^8 possible 10-bit strings that start with 001 or 10.

5.5.3.D
Answer:
There are four possible patterns for the first two and last two bits of the string: 0000, 1111, 0101, and 1010. For each pattern, the remaining six bits can be either 0 or 1, so there are 2^6 possible 10-bit strings with the first two bits the same as the last two bits.

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5.5.3.E
Answer:
To have exactly six 0's in the string, we need to choose six out of the ten positions for the 0's, and the remaining four positions must be filled with 1's. There are (10 choose 6) ways to choose the positions of the 0's, and once we've chosen those positions, there is only one way to fill the remaining four positions with 1's. Therefore, the total number of 10-bit strings with exactly six 0's is:

(10 choose 6) = 210

5.5.3.F
Answer:
The first bit of the string must be 1, so we only need to consider the remaining nine bits. To have exactly six 0's in the remaining nine bits, we need to choose six out of the nine positions for the 0's, and the remaining three positions must be filled with 1's. There are (9 choose 6) ways to choose the positions of the 0's, and once we've chosen those positions, there is only one way to fill the remaining three positions with 1's. Therefore, the total number of 10-bit strings with exactly six 0's and the first bit equal to 1 is:

(9 choose 6) = 84

5.5.3.G
Answer:
There are two cases to consider: the 1 is in the first half of the string, or the 1 is in the second half of the string.

If the 1 is in the first half of the string, then we have five positions left in the first half and four positions left in the second half. We need to choose one of the five positions in the first half to put the 1, and then choose three of the four positions in the second half to put the other three 1's. There are (5 choose 1) * (4 choose 3) ways to do this.

If the 1 is in the second half of the string, then we have four positions left in the first half and five positions left in the second half. We need to choose one of the five positions in the second half to put the 1, and then choose three of the four positions in the first half to put the other three 1's. There are (5 choose 1) * (4 choose 3) ways to do this.

Therefore, the total number of 10-bit strings with exactly one 1 in the first half and exactly three 1's in the second half is:

(5 choose 1) * (4 choose 3) + (5 choose 1) * (4 choose 3) = 40

5.5.5.A
Answer:
To select a group of 10 boys from 30 boys, we can use the combination formula: ${30 \choose 10}$

To select a group of 10 girls from 35 girls, we can use the combination

formula again:
${35 \choose 10}$

To select both a group of 10 boys and 10 girls, we can use the product rule:
${30 \choose 10} \times {35 \choose 10}$

Therefore, there are ${30 \choose 10} \times {35 \choose 10}$ ways for the choir director to make his selection.

5.5.8.C

Answer:

There are 13 hearts and 13 diamonds in the deck. We can choose 5 cards from these 26 cards in ${26\choose 5} = 65,!780$ ways. Thus, there are 65,780 five-card hands that are made entirely of hearts and diamonds.

5.5.8.D

Answer:

There are 13 ranks in the deck. We can choose 4 cards of the same rank in ${13\choose 1}$ ways. We can choose the remaining card from the remaining 48 cards in the deck in ${48\choose 1}$ ways. Thus, there are ${13\choose 1} \times {48\choose 1} = 624$ five-card hands that have four cards of the same rank.

5.5.8.E

Answer:

There are ${13\choose 1}$ ways to choose the rank of the pair and ${4\choose 2}$ ways to choose the 2 cards of that rank. There are then ${12\choose 1}$ ways to choose the rank of the triple and ${4\choose 3}$ ways to choose the 3 cards of that rank. Thus, there are ${13\choose 1}\times {4\choose 2}\times{12\choose 1}\times{4\choose 3}=3,!744$ five-card hands that contain a full house.

5.5.8.F

Answer:

There are ${13\choose 1}$ ways to choose the rank of the first card, ${4\choose 1}$ ways to choose one of the four suits for that card, ${12\choose 1}$ ways to choose the rank of the second card (which cannot be the same rank as the first card), and ${4\choose 1}$ ways to choose one of the four suits for that card. We can continue this process for the remaining three cards. Thus, the number of five-card hands that do not have any two cards of the same rank is ${13\choose 1}\times{4\choose 1}\times{12\choose 1}\times{4\choose 1}\times{11\choose 1}\times{4\choose 1}\times{10\choose 1}\times{4\choose 1}\times{9\choose 1}\times{4\choose 1}=1,!302,!540,!880$.

5.6.6.A

Answer:

Since there are 44 Demonstrators and 56 Repudiators in the senate, the number of ways to select 5 members of each party is given by:

$${44\choose 5} \times {56\choose 5} = \frac{44!}{5!39!} \times \frac{56!}{5!51!} = 6,843,580,897,600$$

Therefore, there are 6,843,580,897,600 ways to select a committee of 10 senate members with the same number of Demonstrators and Repudiators.

5.6.6.B
Answer:
The number of ways to select two speakers and two vice speakers from the 100 senators is given by:

$${100\choose 2}{98\choose 2} = \frac{100!}{2!98!} \times \frac{98!}{2!96!} = 24,999,950,000$$

Therefore, there are 24,999,950,000 ways to select the two speakers and two vice speakers.

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Question 6:

5.7.2.A
Answer:
To count the number of 5-card hands that have at least one club, we can count the total number of 5-card hands and then subtract the number of 5-card hands that have no clubs.

The total number of 5-card hands is given by the combination formula:
C(52, 5) = 2,598,960

To count the number of 5-card hands that have no clubs, we need to select all five cards from the 39 non-club cards in the deck:
C(39, 5) = 575,757

So the number of 5-card hands that have at least one club is:
C(52, 5) - C(39, 5) = 2,598,960 - 575,757 = 2,023,203

Therefore, there are 2,023,203 5-card hands that have at least one club.

5.7.2.B
Answer:
To count the number of 5-card hands that have at least two cards with the same rank, we can count the total number of 5-card hands and then subtract the number of 5-card hands that have no pairs or sets.

The total number of 5-card hands is again given by the combination formula:
C(52, 5) = 2,598,960

To count the number of 5-card hands that have no pairs or sets, we need to count the number of 5-card hands that have all cards with different ranks (i.e., no pairs, three of a kind, four of a kind, or a full house).

There are 13 possible ranks for the first card, 12 possible ranks for the second card (since we can't have the same rank as the first card), 11 possible ranks for the third card, 10 possible ranks for the fourth card, and 9 possible ranks for the fifth card. So the number of 5-card hands that have all cards with different ranks is:
13 × 12 × 11 × 10 × 9 = 154,440

Therefore, the number of 5-card hands that have at least two cards with the same rank is:
C(52, 5) - 154,440 = 2,444,520

Therefore, there are 2,444,520 5-card hands that have at least two cards with the same rank.

5.8.4.A
Answer:
Since there are no restrictions on how many comic books each kid receives, this is a classic stars and bars problem. We need to distribute 20 comic books

among 5 kids, so we can represent this as 20 stars (the comic books) and 4 bars (to separate them into 5 groups). The number of ways to arrange 20 stars and 4 bars is therefore:

${{20+4}\choose{4}} = {{24}\choose{4}} = 10,626$

Therefore, there are 10,626 ways to distribute the comic books if there are no restrictions on how many go to each kid.

5.8.4.B

Answer:

If we want to distribute the comic books evenly so that 4 go to each kid, we can start by selecting 4 comic books from the 20 for the first kid, then 4 from the remaining 16 for the second kid, and so on. The number of ways to select 4 comic books from 20 is:

${{20}\choose{4}} = 4,845$

Once we have selected 4 comic books for the first kid, there are 16 left to choose from for the second kid, and so on. So the total number of ways to distribute the comic books evenly is:

$4,845^5 = 3,051,443,568,664,062,500$

Therefore, there are 3,051,443,568,664,062,500 ways to distribute the comic books evenly so that 4 go to each kid.

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Question 7:

7.A
Answer:
4 elements in the range:
P(4, 5) = 0 (not possible since there are not enough elements in the range)

7.B
Answer:
5 elements in the range:
P(5, 5) = 5! / (5 - 5)! = 5! / 0! = 5! = 120

7.C
Answer:
6 elements in the range:
P(6, 5) = 6! / (6 - 5)! = 6! = 720

7.D
Answer:
7 elements in the range:
P(7, 5) = 7! / (7 - 5)! = 7! / 2! = 7 x 6 x 5 x 4 x 3 = 2,520
