ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 1

Prof. Pillai

Problem 1

Box 1 contains 3 red balls, 5 green balls and 2 white balls. Box 2 contains 5 red balls, 3 green balls and 1 white ball. One ball of unknown color is transferred from Box 1 to Box 2.

a. What is the probability that a ball drawn at random from Box 2 is green?

b. What is the probability that a ball drawn from Box 1 is not white?

Solution

Let the events T_R, T_G and T_W represent rtansferring a red, green and white ball respectively. Note that these event form a partition for the transfer.

a. Probability that a green ball is drawn from box 2 $P(G)$ is,

$$\begin{aligned}
P(G) &= P(G|T_R)P(T_R) + P(G|T_G)P(T_G) + P(G|T_W)P(T_W), \\
&= \frac{3}{10}.\frac{3}{10} + \frac{4}{10}.\frac{5}{10} + \frac{3}{10}.\frac{2}{10}, \\
&= 0.35.
\end{aligned}$$

b. This refers to the scenario after the draw. the probability that a ball drawn from box 2 is not white will be,

$$\begin{aligned}
P(\overline{B}) &= P(\overline{B}|T_R)P(T_R) + P(\overline{B}|T_G)P(T_G) + P(\overline{B}|T_W)P(T_W), \\
&= \frac{7}{9}.\frac{3}{10} + \frac{7}{9}.\frac{5}{10} + \frac{8}{9}.\frac{2}{10}, \\
&= 0.8.
\end{aligned}$$

Problem 2

In a batch of microprocessors, the probability that a microprocessor is defective is 10^{-3}. In one draw an assembly machine picks 10 microprocessors from this batch and tests each. It rejects the entire lot of 10 microporcessors if 2 or more of them are defective, else all the 10 are retained.

a. Find the probability that a lot is rejected.

b. If the machine draw 6 times, what is the probability that at least 60 microprocessors are retained.

Solution:

a. Let the event that a lot is rejected be R.

   Let the probablity that a microporcessor is defective be $p = 10^{-3}$.

   $P(R) = 1 - P(\bar{R}) = 1 - C_{10}^{0}(1-p)^{10} - C_{10}^{1}p(1-p)^{9}$

b. Probability that all 6 lots (i.e. 60 microporcessors) are retained $= C_6^0 P(R)^0 (1-P(R))^6 = (1-P(R))^6$.

Problem 3

a. Toss a coin n times, Let 'p' represent the probability of obtaining a "Head" in any toss. Show that the most likely number of "Heads" k_0 in n trials is given by

$$(n+1)p - 1 \leq k_0 \leq (n+1)p$$

$$\text{and hence} \frac{k_0}{n} \to p$$

Solution:

Let $a_k = P(X = k)$, we have

$$a_k = \binom{n}{k} p^k q^{n-k} \qquad \text{and} \qquad a_{k+1} = \binom{n}{k+1} p^{k+1} q^{n-k-1},$$

where as usual $q = 1 - p$ in binomial distribution.

We calculate the *ratio* $\frac{a_{k+1}}{a_k}$. Note that $\frac{\binom{n}{k+1}}{\binom{n}{k}}$ simplifies to $\frac{n-k}{k+1}$, and therefore

$$\frac{a_{k+1}}{a_k} = \frac{n-k}{k+1} \cdot \frac{p}{q} = \frac{n-k}{k+1} \cdot \frac{p}{1-p}.$$

From this equation we can follow:

$$\begin{aligned} k > (n+1)p - 1 &\implies a_{k+1} < a_k \\ k = (n+1)p - 1 &\implies a_{k+1} = a_k \\ k < (n+1)p - 1 &\implies a_{k+1} > a_k \end{aligned}$$

The calculation says that we have equality of two consecutive probabilities precisely if $a_{k+1} = a_k$, that is , if $k = np + p - 1$ implies that $np + p - 1$ is an integer.

So if $k = np + p - 1$ is not an integer, there is a single mode; and if $k = np + p - 1$ is an integer, there are two modes, at $np + p - 1$ and at $np + p$.

b. In a book of 200 pages long, it is not unreasonable to expect 20 misprints. Find the probability that a given page will contain

i) two misprints
ii) two or less prints
iii) two or more misprints.
Solution:
Let X be the number of misprints in a 200 page book with an average number of misprints =20 therefore $X \sim P_0(20)$.
Let Y be the number of misprints in a given page with an average number of misprints = $\frac{20}{200}$=0.1 therefore $Y \sim P_0(0.1)$.
(i)

$$P(Y = 2) = \frac{e^{-0.1}(0.1)^2}{2!}$$

(ii)

$$P(Y \leq 2) = P(Y = 0) + P(Y = 1) + P(Y = 2)$$
$$P(Y \leq 2) = e^{-0.1} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!}$$

(iii)

$$P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1) = 1 - e^{-0.1} - \frac{e^{-0.1}(0.1)^1}{1!}$$

Problem 4

The pdf of a continous random variable X is given by

$$f_X(x) = \begin{cases} \frac{1}{7} & -2 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find
(i) $P(X^2 > 1)$
(ii) $P(sin(\pi X) \leq 0)$

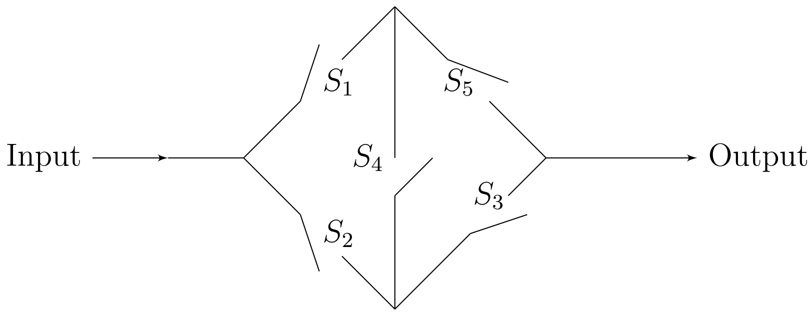
Solution:
(i)

$$\begin{aligned} P(X^2 > 1) &= P((X < -1)U(X > 1)) \\ &= P(X < -1) + P(X > 1) \\ &= \frac{1}{7}(1) + \frac{1}{7}(4) = \frac{5}{7} \end{aligned}$$

(ii)

$$\begin{aligned} P(sin(\pi X) < 0) &= P((-2 < X < 0)UP(2 < X < 4)) \\ &= P(-2 < X < 0) + P(2 < X < 4) = \frac{3}{7} \end{aligned}$$

Problem 5



The five switches in the figure operate independently. Each switch is closed with probability p and open with probability $(1-p)$.

a. Find the probability that the signal at the input will **not** be received at the output.

b. Find the conditional probability that the switch S_4 is open given that the signal is received at the output.

Solution:

a. Let A_i represent the event that "switch i is closed," for $i = 1, 2, 3, 4, 5$.
Then, $P(A_i) = p$ and $P(A_i^c) = 1 - p$ for $i = 1, 2, 3, 4, 5$.

From the diagram a signal is received at the output when, $\{S_1$ and $S_5\}$ are closed, or $\{S_2$ and $S_3\}$ are closed, or $\{S_1, S_4$ and $S_3\}$ are closed, or $\{S_2$, S_4 and $S_5\}$ are closed.

Let R = "input signal is received at the output".

Thus the probability of receiving a signal R equals,

$$P(R) = P((A_1 \cap A_5) \cup (A_2 \cap A_3) \cup (A_1 \cap A_4 \cap A_3) \cup (A_2 \cap A_4 \cap A_5)).$$

Rewriting $(A_1 \cap A_5) = B_1$,

$$
\begin{aligned}
&(A_2 \cap A_3) = B_2, \\
&(A_1 \cap A_3 \cap A_4) = B_3, \\
&(A_2 \cap A_4 \cap A_5) = B_4, \\
P(R) &= P(B_1 \cup B_2 \cup B_3 \cup B_4), \\
&= P(B_1) + P(B_2) + P(B_3) + P(B_4) - P(B_1 \cap B_2) - P(B_1 \cap B_3) - P(B_1 \cap B_4) \\
&\quad - P(B_2 \cap B_3) - P(B_2 \cap B_4) - P(B_3 \cap B_4) + P(B_1 \cap B_2 \cap B_3) + P(B_1 \cap B_2 \cap B_4) \\
&\quad + P(B_1 \cap B_3 \cap B_4) + P(B_2 \cap B_3 \cap B_4) - P(B_1 \cap B_2 \cap B_3 \cap B_4). \\
&= P(A_1 \cap A_5) + P(A_1 \cap A_4 \cap A_3) + P(A_2 \cap A_3) + P(A_2 \cap A_4 \cap A_5) \\
&\quad - P(A_1 \cap A_2 \cap A_3 \cap A_5) - P(A_1 \cap A_3 \cap A_4 \cap A_5) \\
&\quad - P(A_1 \cap A_2 \cap A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_3 \cap A_4) \\
&\quad - P(A_2 \cap A_3 \cap A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \\
&\quad + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \\
&\quad + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \\
&\quad - P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5. \\
&= P(A_1)P(A_5) + P(A_1)P(A_4)P(A_3) + P(A_2)P(A_3) + P(A_2)P(A_4)P(A_5) \\
&\quad - P(A_1)P(A_2)P(A_3)P(A_5) - P(A_1)P(A_3)P(A_4)P(A_5) \\
&\quad - P(A_1)P(A_2)P(A_4)P(A_5) - P(A_1)P(A_2)P(A_3)P(A_4) \\
&\quad - P(A_2)P(A_3)P(A_4)P(A_5) + 2P(A_1)P(A_2)P(A_3)P(A_4)P(A_5) \\
&= 2p^2 + 2p^3 - 5p^4 + 2p^5.
\end{aligned}
$$

$$P(R^c) = 1 - P(R) = 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$

Alternative solution Let us partition Ω as $A_4 \cup \overline{A_4}$, and $A_4 \cap \overline{A_4} = \phi$.

Thus,

$$P(R) = \underbrace{P(R|A_4)}_{\text{P(signal is received}|S_4\text{ is closed)}} \times P(A_4) + P(R|\overline{A_4})P(\overline{A_4}).$$

Note that,

$$
\begin{aligned}
P(R|A_4) &= P[(A_1 \cap A_5) \cup (A_2 \cap A_5) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3)], \\
&= 4p^2 - 4p^3 + p^4. \\
P(R|\overline{A_4}) &= P[(A_1 \cap A_5) \cup (A_2 \cap A_3)] \\
&= P(A_1 \cap A_5) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3 \cap A_5), \\
&= 2p^2 - p^4. \\
\implies P(R) &= (4p^2 - 4p^3 + p^4)p + (2p^2 - p^4)(1-p), \\
&= 2p^2 + 2p^3 - 5p^4 + 2p^5.
\end{aligned}
$$

$$P(R^c) = 1 - P(R) = 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$

b. The probability that S_3 is open given a signal is received at the output $= P(\overline{A_3}|R)$.
Using Bayes' Theorem we can write,

$$P(\overline{A_3}|R) = \frac{P(R|\overline{A_3})P(\overline{A_3})}{P(R)}.$$

$$P(R|\overline{A_3}) = P(A_1A_5 \cup A_2A_4A_5) = P(A_1A_5) + P(A_2A_4A_5) - P(A_1A_2A_4A_5) = p^2 + p^3 - p^4.$$

Substituting the results in part (a) we can write,

$$\begin{aligned} P(\overline{A_3}|R) &= \frac{(p^2 + p^3 - p^4)(1-p)}{p^2(2 + 2p - 5p^2 + 2p^3)} \\ &= \frac{(1 + p - p^2)(1-p)}{(2 + 2p - 5p^2 + 2p^3)}. \end{aligned}$$

Problem 6

Among a certain group of people 5 % are (professional) liars. A lie detector test on a liar is found to be positive with a probability of 0.94. If the test is positive for a non-liar, it is positive with a probability of 0.08. Given that the test is positive for a randomly picked person from that group, what is the probability that he is a liar.

Solution

Let the event L be that a person is a liar. By the problem,

$$P(L) = 0.05.$$

Let the event that the test is positive be T_p. Thus,

$$\begin{aligned} P(T_p|L) &= 0.94, \\ P(T_p|\overline{L}) &= 0.08. \end{aligned}$$

We need to compute $P(L|T_p)$, the probability that the person is a liar given that the test is positive.
Using total probability we can write,

$$\begin{aligned} P(T_p) &= P(T_p|L)P(L) + P(T_p|\overline{L})P(\overline{L}), \\ &= 0.94 * 0.05 + 0.08 * 0.95, \\ &= 0.123. \end{aligned}$$

Using Bayes Theorem we can write,

$$\begin{aligned} P(L|T_p) &= \frac{P(T_p|L)P(L)}{P(T_p)}, \\ &= \frac{0.94 \times 0.05}{0.123}, \\ &= 0.38211 \end{aligned}$$

Problem 7

We have two sealed boxes. In the first box we have 125 white and 75 black marbles. The second box contains 60 white and 90 black marbles. You pick a marble randomly from a box. For any given pick, the probability of picking from Box i, $P(B_i) = 0.5$.

a. What is probability that the marble drawn is black?

b. The marble picked turned to be black. What is the probability that it is picked out of Box 2?

Solution

a. Let the probability that a white marble is drawn be $P(M_w)$. Using the law of total probability we can write,

$$\begin{aligned} P(M_b) &= P(M_b|B_1).P(B_1) + P(M_b|B_2).P(B_2), \\ &= \frac{75}{200}.\frac{1}{2} + \frac{90}{150}.\frac{1}{2} \\ &= \frac{39}{80}. \end{aligned}$$

b. We need to find $P(B_2|M_b)$, where M_b is the event that a marble drawn is black. Using Bayes theorem,

$$\begin{aligned} P(B_2|M_b) &= \frac{P(M_b|B_2).P(B_2)}{P(M_b)}, \\ &= \frac{3/5 * 0.5}{39/80}, \\ &= 0.61538461 \end{aligned}$$