ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 4

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Problem 1

The random variable X is $\mathcal{N}(5, 2)$ and $Y = 2X + 4$. Find the mean, variance of Y and $f_Y(y)$.

Solution:
The r.v. Y is still Gaussian distributed.

$$E[Y] = E[2X + 4] = 2E[X] + 4 = 14,$$
$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = E[(2X + 4)^2] - (2E[X] + 4)^2 = 4\mathrm{Var}(X) = 8,$$
$$f_Y(y) = \frac{1}{4\sqrt{\pi}} e^{-\frac{(y-14)^2}{16}}, \quad -\infty < y < \infty.$$

Problem 2

The random variable X is $P(5)$ and $Y = 2X + 4$. Find the mean, variance of Y and $f_Y(y)$.

Solution:
Since X is Poisson distributed with parameter 5, $\mathbb{E}[X] = 5$, $\mathrm{Var}(X) = 5$.

$$E[Y] = E[2X + 4] = 2E[X] + 4 = 14,$$
$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = E[(2X + 4)^2] - (2E[X] + 4)^2 = 4\mathrm{Var}(X) = 20.$$

If $k = 2k' + 4$, $k' = 0, 1, \ldots$

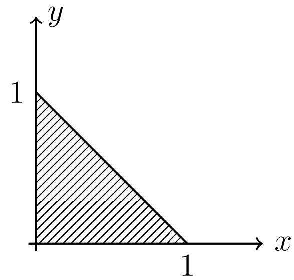
$$P(Y = k) = P(2X + 4 = 2k' + 4) = P(X = k') = e^{-5}\frac{5^{k'}}{k'!}.$$

Otherwise

$$P(Y = k) = 0.$$

Problem 3

Given the joint probability density function $f_{XY}(x,y)$ as,



$$f_{XY}(x,y) = \begin{cases} kxy, & (x,y) \in \text{shaded area} \\ 0 & \text{otherwise} \end{cases}$$

a. Find k, $f_X(x)$ and $f_Y(y)$.

b. Are X and Y independent?

Solution:

a.

$$1 = \int_0^1 \int_0^{1-x} kxydydx = \frac{k}{2}\int_0^1 (1-x)^2 xdx = \frac{k}{24} \quad \Rightarrow \quad k = 24.$$

$$f_X(x) = \int_0^{1-x} kxydy = 12(1-x)^2 x, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^{1-y} kxydx = 12(1-y)^2 y, \quad 0 \le y \le 1$$

b. No, as $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.

Problem 4

X and Y are jointly distributed random variables with joint p.d.f

$$f_{XY}(x,y) = \begin{cases} e^{-x} & \infty > x > y > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Find $f_X(x)$ and $f_Y(y)$.

b. Are X and Y independent?

Solution:

a.

$$f_X(x) = \int_0^x e^{-x}dy = xe^{-x}, \quad x > 0,$$

$$f_Y(y) = \int_y^\infty e^{-x}dx = e^{-y}, \quad y > 0.$$

b. No, as $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.

Problem 5

X and Y are jointly distributed random variables with joint p.d.f

$$f_{XY}(x,y) = \begin{cases} k & 0 < x < y < a \\ 0 & \text{otherwise} \end{cases}$$

a. Find k, $f_X(x)$ and $f_Y(y)$.

b. Are X and Y independent?

Solution:

a.

$$1 = \int_0^a \int_x^a k dy dx = \int_0^a k(a-x)dx = \frac{ka^2}{2} \quad \Rightarrow \quad k = \frac{2}{a^2}.$$

$$f_X(x) = \int_x^a k dy = \frac{2}{a^2}(a-x), \quad 0 < x < a$$

$$f_Y(y) = \int_0^y k dx = \frac{2}{a^2}y, \quad 0 < y < a$$

b. No, as $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.

Problem 6

a. X and Y are jointly distributed random variables with joint p.d.f

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

b. Given the joint probability dendity function $f_{XY}(x,y)$ as

$$f_{XY}(x,y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}.$$

Show that X and Y are independent random variables

Solution:

a.

$$f_X(x) = \int_0^\infty e^{-(x+y)}dy = e^{-x}, \quad x \geq 0,$$

$$f_Y(y) = \int_0^\infty e^{-(x+y)}dx = e^{-y}, \quad y \geq 0.$$

As $f_{XY}(x,y) = f_X(x)f_Y(y)$, they are independent.

b.

$$f_X(x) = \frac{1}{2\pi}\int_0^\infty e^{-(x^2+y^2)/2}dy = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

$$f_Y(y) = \frac{1}{2\pi}\int_0^\infty e^{-(x^2+y^2)/2}dx = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}.$$

As $f_{XY}(x,y) = f_X(x)f_Y(y)$, they are independent.