Midterm Exam

1. *(15 points)* Given $X \sim \text{Uniform}(-1, 1)$, and $Y = \dfrac{1-X}{1+X}$. Find the probability density function (p.d.f.) of Y.

2. *(20 points)* Given the joint probability density function

$$f_{XY}(x, y) = \begin{cases} \dfrac{2}{x^2y^2}, & x > y > 1 \\ 0 & \text{otherwise,} \end{cases}$$

Find the p.d.f. of $Z = \dfrac{X}{Y}$.

3. *(20 points)* Given the joint probability density function

$$f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)}, & x > y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

Find the p.d.f. of $Z = \dfrac{1}{\max(X, Y)}$.

4. *(35 points)* Joint p.d.f. of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)}, & x > y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

Define $Z = X + Y, W = X - Y$

a.) Find the joint p.d.f of Z and W

b.) Are Z and W independent?

c.) Are Z and W uncorrelated?

d.) Find $\text{Cov}(Z, W)$

5. *(10 points)* Given

$$f_{XY}(x, y) = \begin{cases} e^{-x}, & x > y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

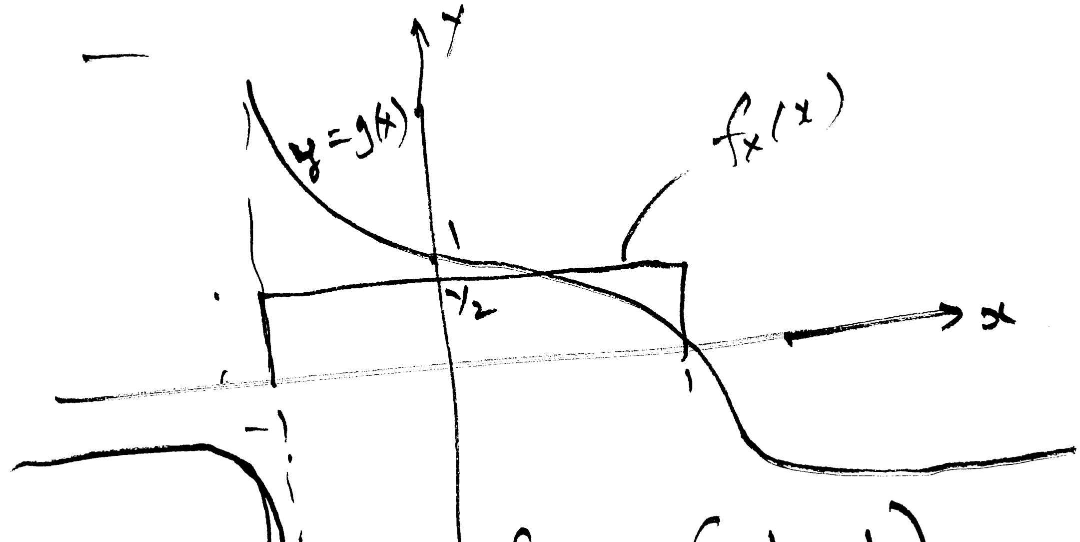
Find the conditional p.d.f of X given Y.

EL6303 Solutions to Midterm Exam

1) a) Given $X \sim U(-1,1)$ and

$$Y = \frac{1-X}{1+X}$$

Find the p.d.f. of Z



Notice that as x goes from $(-1,1)$ y goes from $0 \to \infty$, and there is one solution x_1 for every y. Hence

$$(1+x_1)y = 1 - x_1 \Rightarrow x_1 = \frac{y-1}{1+y}$$

$$\frac{dx_1}{dy} = \frac{(1+y)-(y-1)}{(1+y)^2} = \frac{2}{(y+1)^2}$$

$$f_y(y) = \sum_i \left|\frac{dx_1}{dy}\right| f_x(x_1) = \frac{2}{(y+1)^2}\cdot\frac{1}{2} = \frac{1}{(y+1)^2}, \quad y>0$$

check

$$\int_0^\infty f_y(y)\,dy = \int_0^\infty \frac{1}{(y+1)^2}\,dy = \int_1^\infty \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_1^\infty = \underline{1}.$$

2 a)

$$f_{XY}(x,y) = \begin{cases} \dfrac{2}{x^2y^2}, & x>y>1 \\ 0 & \text{otherwise} \end{cases}$$

find the p.d.f. of

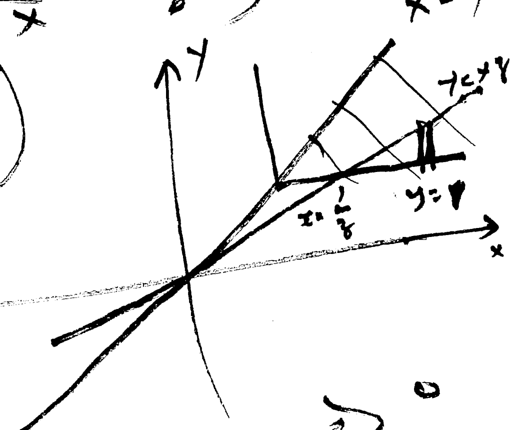
$$Z = \frac{Y}{X}$$

Solution

$$F_Z(z) = P(Z \le z) = P\left(\frac{Y}{X} \le z\right), \quad 0<z<1$$

$$= P(Y \le Xz,\ X>Y>1)$$

$$= \int_{x=\frac{1}{z}}^{\infty} \int_{y=1}^{xz} f_{XY}(x,y)\,dy\,dx$$



$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{1}{z^2}\int_{y=1}^{1} f_{XY}\left(\frac{1}{z}, y\right)dy \rightarrow 0$$

$$+ \int_{x=\frac{1}{z}}^{\infty} 2x\, f_{XY}(x, xz)\,dx$$

$$= \int_{1/z}^{\infty} 2x \cdot \frac{1}{x^2(xz)^2}\,dx = \frac{1}{z^2}\int_{1/z}^{\infty} \frac{2}{x^3}\,dx = \frac{-2}{z^2}\cdot\frac{1}{(2x^2)}\Bigg|_{1/z}^{\infty}$$

$$= \frac{z^2}{\cancel{2}z^2} = \cancel{\tfrac{1}{2}}\ 1. \quad 0<z<1$$

$$\Rightarrow Z \sim U(0,1)$$

3/2 a) $f_{xy}(x,y) = \begin{cases} \dfrac{2}{x^2 y^2}, & x > y > 1 \\ 0 & \text{otherwise} \end{cases}$

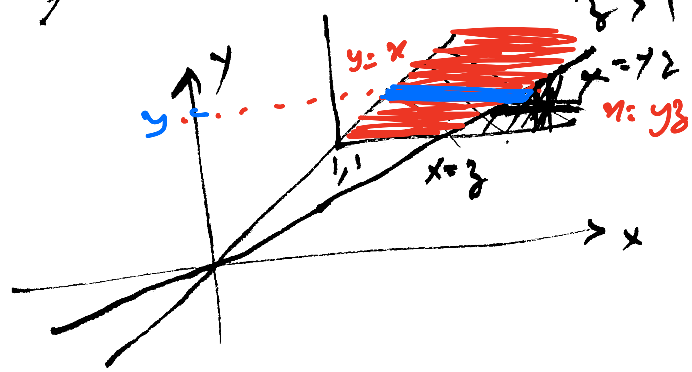
Find the p.d.f of $Z = \dfrac{X}{Y}$.

Solution

$F_Z(z) = P(Z \le z) = P\left(\dfrac{X}{Y} \le z\right) = P(X \le Yz,\ X > Y > 1)$

$z > 1$

$$= \int_{y=1}^{\infty} \int_{x=y}^{yz} f_{xy}(x,y)\, dx\, dy$$



$$f_Z(z) = \frac{d F_Z(z)}{dz} = \int_{y=1}^{\infty} (+y) f_{xy}(yz, y)\, dy$$

$$= \int_1^{\infty} (+y) \frac{2}{(yz)^2 y^2}\, dy = \frac{2}{z^2} \int_1^{\infty} \frac{1}{y^3}\, dy$$

$$= \frac{2}{z^2} \left(-\frac{1}{2y^2}\right)\Big|_1^{\infty} = \frac{1}{z^2}, \quad z > 1$$

Check

$$\int_1^{\infty} f_Z(z)\, dz = \int_1^{\infty} \frac{1}{z^2}\, dz = -\frac{1}{z}\Big|_1^{\infty} = 1$$

4(b) 2(b)

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)}, & x>y>0 \\ 0 \end{cases}$$

Find the p.d.f of $Z = \dfrac{1}{\max(X,Y)}$

solution

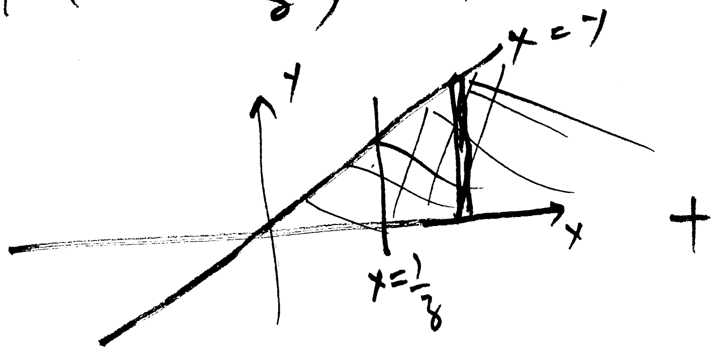
$$Z = \frac{1}{\max(X,Y)} = \begin{cases} \frac{1}{X}, & X \geq Y \\ \frac{1}{Y} & X < Y \end{cases}$$

$$F_Z(z) = P(Z \leq z) = P(\underbrace{Z \leq z}_{A}, \underbrace{(X \geq Y)}_{B} \cup \underbrace{(X<Y)}_{\bar{B}})$$

$$= P(A \cap B \cup \bar{B}) = P(AB) + P(A\bar{B})$$

$$= P\left(\frac{1}{X} \leq z, X \geq Y\right) + P\left(\frac{1}{Y} \leq z, X < Y\right)$$

$$= P\left(X \geq \frac{1}{z}, X \geq Y\right) + \underbrace{P\left(Y \geq \frac{1}{z}, X < Y\right)}_{0}$$



Hence

$$F_Z(z) = \int_{x=\frac{1}{z}}^{\infty} \int_{y=0}^{x} f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{z^2} \left(\int_0^{1/z} f_{XY}\left(\frac{1}{z}, y\right) dy \right.$$

5

$$f_Z(z) = \frac{1}{z^2} \int_0^{1/z} 2 e^{-\left(\frac{1}{z}+y\right)} \, dy$$

$$= \frac{2}{z^2} e^{-\frac{1}{z}} \int_0^{1/z} e^{-y} \, dy = \frac{2}{z^2} e^{-\frac{1}{z}} \left(1 - e^{-\frac{1}{z}}\right), \quad z \geq 0$$

check

$$\int_0^{\infty} f_Z(z) = \int_0^{\infty} \frac{2}{z^2} e^{-\frac{1}{z}} \left(1 - e^{-\frac{1}{z}}\right) dz, \qquad u = \frac{1}{z}, \quad du = -\frac{1}{z^2} dz$$

$$= 2 \int_0^{\infty} \left(e^{-u} - e^{-2u}\right) du = 2 \left(\frac{e^{-u}}{-1} - \frac{e^{-2u}}{(-2)} \right) \Bigg|_0^{\infty} = \frac{2}{2} = 1.$$

3) b/

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & , x>y>0 \\ 0 & \text{otherwise} \end{cases}$$

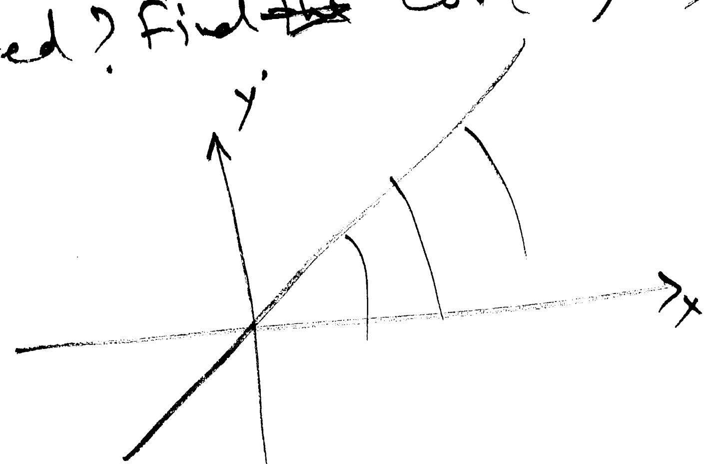
Define

$Z = X+Y \qquad W = X-Y$

a) Find the joint pdf of Z and W

b) Are Z and W independent?

c) Are they uncorrelated? Find Cov(Z, W).



$z = x+y$

$w = x-y$

$x_1 = \frac{z+w}{2}, \quad y_1 = \frac{z-w}{2}$

$0 < z < \infty, \quad 0 < w < \infty, \quad 0 < w < z < \infty.$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow |J| = 2$$

$$f_{ZW}(z,w) = \frac{1}{J} e^{-(x_1+y_1)} = \frac{2}{2} e^{-z} = e^{-z}, \quad 0<w<z<\infty$$

$$f_W(w) = \int_w^\infty \frac{2}{2} e^{-z}\, dz = \frac{e^{-z}}{-1}\Big|_w^\infty = e^{-w}, \quad w>0$$

$$f_Z(z) = \int_0^z \frac{2}{2} e^{-z}\, dw = z e^{-z}, \quad z>0$$

Z, W are not independent since $f_{ZW}(z,w) \neq f_Z(z) f_W(w)$

$$E(Z) = \int_0^\infty z f_Z(z) = \int_0^\infty z^2 e^{-z}\, dz = 2$$

$$E(W) = \int_0^\infty w f_W(w) = \int_0^\infty w e^{-w}\, dw = 1$$

$$E(ZW) = \int_{w=0}^{\infty}\int_{w}^{\infty} z w\, f_{ZW}(z,w)\, dz\, dw$$

$$= \int_{w=0}^{\infty} w \int_{w}^{\infty} z e^{-z}\, dz = \int_0^\infty w \left(-z e^{-z} - e^{-z}\right)\Big|_w^\infty dw$$

$$= \int_0^\infty w \left(w e^{-w} + e^{-w}\right) dw = \int_0^\infty w^2 e^{-w} + \int w e^{-w}\, dw$$

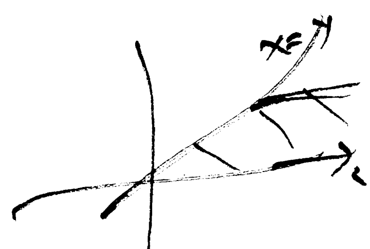
$$= 2 + 1 = \underline{\underline{3}}$$

$$\mathrm{Cov}(Z, W) = E(ZW) - E(Z)E(W)$$

$$= 3 - 2 \cdot 1 = 1 \neq 0$$

$\Rightarrow$ Z, W are correlated.

3) $f_{XY}(x, y) = e^{-x}, \quad x > y > 0$



$$f_Y(y) = \int_y^\infty e^{-x}\, dx = \frac{e^{-x}}{-1}\Big|_y^\infty = e^{-y}, \quad y > 0$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} e^{-(x-y)}, & x > y > 0 \\ 0 & \text{otherwise.} \end{cases}$$