Midterm Exam

1. *(15 points)* Given $X \sim \text{Uniform}(0, \pi)$, find the probability density function (p.d.f.) of $Y = \cot X$ and plot it.

2. a.) *(20 points)* Given the joint probability density function

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z = \frac{\max(X,Y)}{\min(X,Y)}$$

Find the p.d.f. of Z.

b.) *(20 points)* Given

$$f_{XY}(x,y) = \begin{cases} e^{-y}, & y > x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

i) Find the joint probability density function of $Z = X + Y$, $W = Y - X$, and marginal distributions of Z and W.

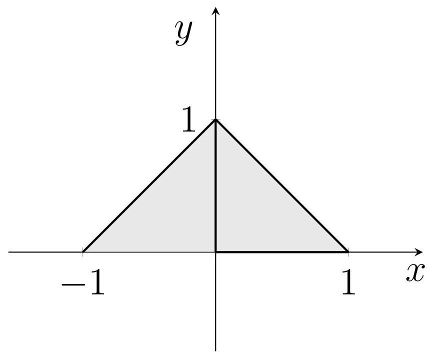
ii) Are Z and W independent? Prove your answer.

3. a.) *(15 points)*

The joint probability density function of random variables X and Y is given by,

$$f_{XY}(x,y) = \begin{cases} k(x^2+y^2) & (x,y), \in \text{shaded region,} \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional mean of X^2 given $Y = y$.



b.) *(15 points)* X and Y are independent Geometric random variables with

$$P(X=k) = P(Y=k) = pq^k, \quad k = 0, 1, 2, ...$$

Find the probability mass function of $Z = \max(X,Y)$. Verify that it is a valid probability mass function.

4. *(15 points)* X and Y are independent Poisson random variables with parameters λ and μ respectively. Thus, $P(X = k) = e^{-\lambda}\,\frac{\lambda^k}{k!}$, $k = 0, 1, 2, ..., \infty$.
Find the conditional probability mass function of X given $X + Y$. Can you identify this distribution?
(*Hint: Determine* $P(X = m | X + Y = n)$)