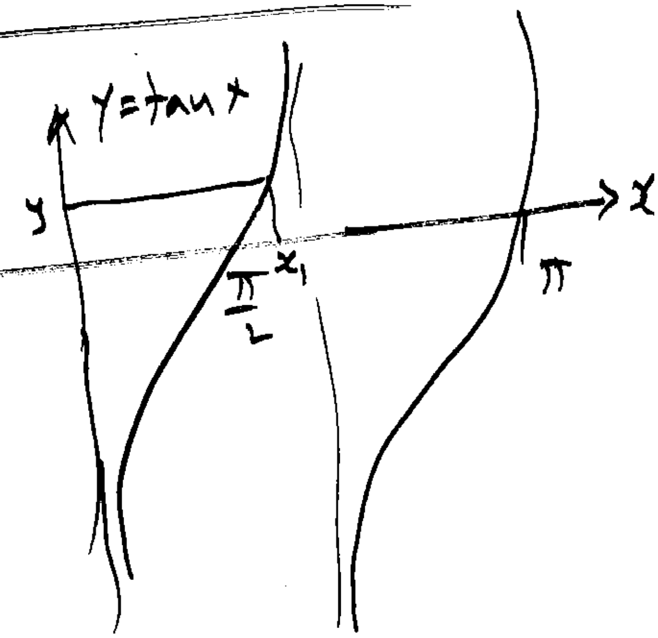
1) EL 6303 INET

Solutions to Midterm Exam



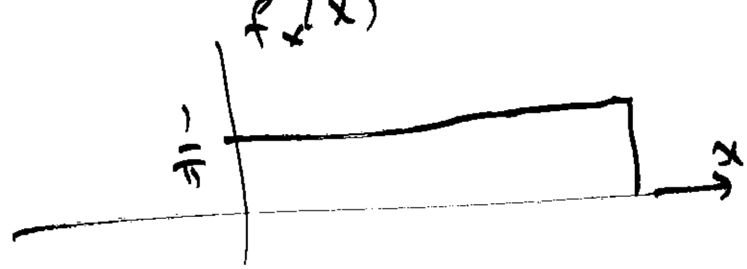
$y = \tan x$ has a unique solution in x given by

$$x_1 = \tan^{-1} y$$

for every $-\infty < y < +\infty$.

Also as $0 < x < \pi$,

$-\infty < y < +\infty$

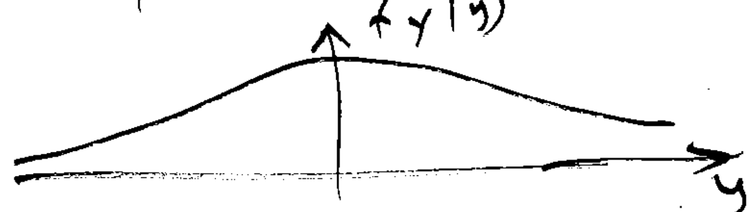


$$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

Hence $$f_Y(y) = \sum_i \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_i) = \frac{1/\pi}{1+y^2}$$

$-\infty < y < +\infty$

$\Rightarrow$ Y is a Cauchy random variable.



1a Note $Z = \frac{1}{Y} = \cot X$ is also Cauchy, since

$$f_Z(z) = \frac{1}{\left|\frac{dz}{dy}\right|} f_Y(1/z) = z^2 f_Y(1/z)$$

$$= \frac{1/\pi}{1+z^2}$$

2/2) a)

$$z = \frac{\max(x,y)}{\min(x,y)} = \begin{cases} \dfrac{x}{y}, & x \geq y \\ \dfrac{y}{x} & x < y \end{cases}$$

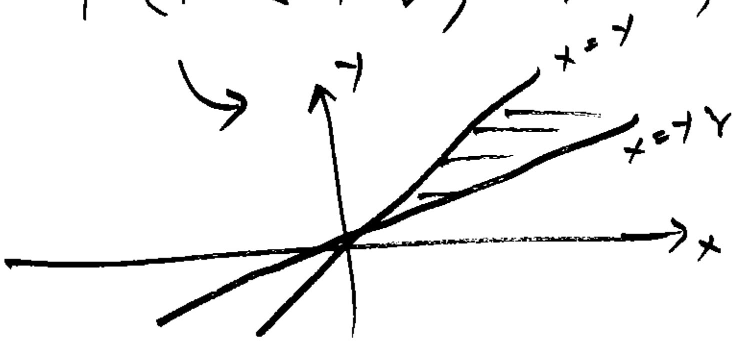
$\rightarrow$ $z > 1$

$F_Z(z) = P(Z \leq z) = P\left[(Z \leq z) \cap \{(x \geq y) \cup x < y\}\right]$

$= P(Z \leq z, x \geq y) + P(Z \leq z, x < y)$

$= P\left(\frac{x}{y} \leq z, x \geq y\right) + P\left(\frac{y}{x} \leq z, x < y\right)$

$= P(x \leq yz, x \geq y) + P(y < zx, x < y)$



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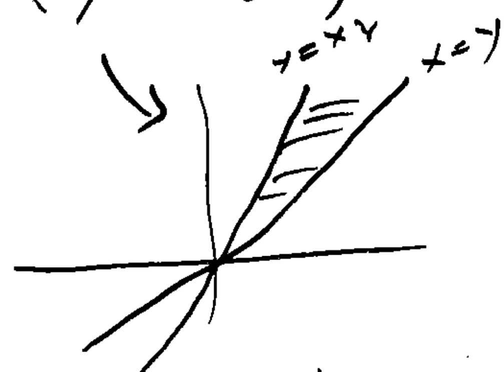
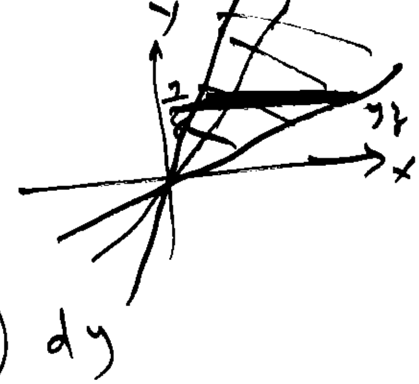


Hence

$$F_Z(z) = \int_0^\infty \int_{x=y/z}^{yz} f_{XY}(x,y)\,dx\,dy =$$



$$f_Z(z) = \int_0^\infty y\, f_{XY}(yz, y) + \frac{y}{z^2} f_{XY}\left(\frac{y}{z}, y\right) dy$$

$\underline{3}$

$$f_Z(z) = \int_0^\infty \left[y\, e^{-\underbrace{(1+z)y}_{u=(1+z)y}} + \frac{y}{z^2}\, e^{-\left(1+\frac{1}{z}\right)y} \right] dy$$

$$= \frac{1}{(1+z)^2} \underbrace{\int_0^\infty u e^{-u}\, du}_{1} + \frac{1}{z^2} \left(\frac{1}{1+\frac{1}{z}}\right)^2 \underbrace{\int_0^\infty u e^{-u}\, du}_{1}$$

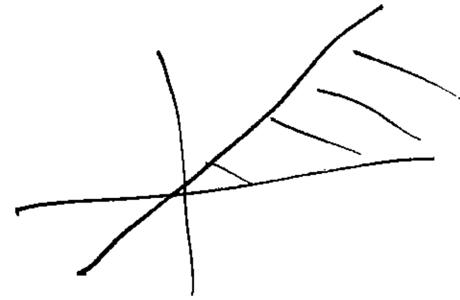
$$= \frac{2}{(1+z)^2}, \quad z \geq 1$$

check

$$\int_1^\infty f_Z(z) = 2 \int_1^\infty \frac{1}{(1+z)^2}\, dz = 2 \int_2^\infty \frac{1}{y^2}\, dy$$

$$= 2 \left(-\frac{1}{y}\right)\Big|_1^\infty = \underline{\underline{1}}.$$

4/2b) $f_{XY}(x,y) = \begin{cases} e^{-x}, & 0<y<x<\infty \\ 0 & \text{otherwise} \end{cases}$



$Z = X+Y \geq 0; \quad W = X - Y \geq 0$

$\Rightarrow x_1 = \dfrac{z+w}{2}, \quad y_1 = \dfrac{z-w}{2}$

is the unique solution $\quad z > 0, \quad z > w$

$x > y \Rightarrow w > 0,$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Rightarrow |J| = 2$$

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1,y_1)$$

$$= \begin{cases} \frac{1}{2} e^{-\left(\frac{z+w}{2}\right)}, & \infty > z > w > 0. \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$f_Z(z) = \int_0^z f_{ZW}(z,w)\,dw = \frac{1}{2} e^{-z/2} \int_0^z e^{-w/2}\,dw$$

$$= \frac{1}{2} e^{-z/2} \int_0^{z/2} e^{-u} \cdot (2\,du) = e^{-z/2}\left(1 - e^{-z/2}\right)$$

$$= \left(e^{-z/2} - e^{-z}\right), \quad z \geq 0 \qquad (2)$$

check $\int_0^\infty f_Z(z) = \int_0^\infty e^{-z/2}\,dz - \int_0^\infty e^{-z}\,dz = 2 - 1 = \underline{\underline{1}}.$

5 $f_W(w) = \int_w^\infty f_{ZW}(z,w)\,dz$

$$= \int_w^\infty \frac{1}{2} e^{-(z+w)/2}\,dz = \frac{1}{2} e^{-w/2}\int_w^\infty e^{-\overbrace{z/2}^{u}}\,dz$$

$$= e^{-w/2}\int_{w/2}^\infty e^{-u}\,du = e^{-\frac{w}{2}}\, \frac{e^{-u}}{-1}\Big|_{w/2}^\infty = e^{-w},\ w \geq 0 \qquad (3)$$

From (1) - (3) we have

$$f_{ZW}(z,w) \neq f_Z(z)\, f_W(w)$$

$\Rightarrow$ Z, W are not independent.

Note: Interchanging (x, y) gives the same result.

3d) $$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{— (1)}$$

$$f_{XY}(x,y) = \begin{cases} k(x^2+y), & \text{shaded area} \\ 0 \end{cases}$$

$$f_Y(y) = \int_{x=-(1-y)}^{1-y} f_{XY}(x,y)\,dx = k\int_{-(1-y)}^{1-y} (x^2+y)\,dx$$

$$= k\left(\frac{x^3}{3} + xy\right)\Big|_{-(1-y)}^{1-y} = 2k\left[\frac{(1-y)^3}{3} + (1-y)y\right],\ 0<y<1$$

Using (1)

$$f_{X|Y}(x|y) = \frac{k(x^2+y)}{2k(1-y)\left(\frac{(1-y)^2}{3}+y\right)} = \frac{3(x^2+y)}{2(1-y)(y^2+y+1)},$$

$0 < \cancel{1}-y < x < 1+y,$

$0 < y < 1$

b/ Hence

$$E(X^2 \mid Y=y) = \int_{-(1-y)}^{1-y} x^2 f_{X|Y}(x|y)\,dx$$

$$= \int_{-(1-y)}^{1-y} \frac{3x^2(x^2+y)}{2(1-y)(y^2+y+1)}\,dx = \frac{3}{2(1-y)(y^2+y+1)} \int_{-(1-y)}^{1-y} (x^4 + x^2 y)\,dx$$

$$= \frac{3}{2(1-y)(y^2+y+1)} \left(\frac{x^5}{5} + \frac{x^3 y}{3} \right)_{-(1-y)}^{(1-y)} = \frac{3(1-y)^2}{(y^2+y+1)} \left(\frac{(1-y)^2}{5} + \frac{y}{3} \right)$$

$$= \frac{(1-y)^2(3y^2-y+3)}{5(y^2+y+1)}.$$

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b) $P(X=k) = P(Y=k) = pq^k, \quad k = 0, 1, 2 \ldots \infty$

$$Z = \max(X, Y) = \begin{cases} X, & X \geq Y \\ Y, & X < Y \end{cases}$$

$$P(Z=h) = P\Big[\underbrace{(Z=k)}_{A} \cap \big\{\underbrace{(X \geq Y)}_{B} \cup \underbrace{(X<Y)}_{\bar{B}}\big\}\Big]$$

B — ME

$$= P(AB) + P(A\bar{B}) = P(Z=k, X \geq Y) + P(Z=k, X<Y)$$

$$= P(X=k, X \geq Y) + P(Y=k, X<Y)$$

$$= P(X=k, k \geq Y) + P(Y=k, \cancel{X} < \cancel{k})$$

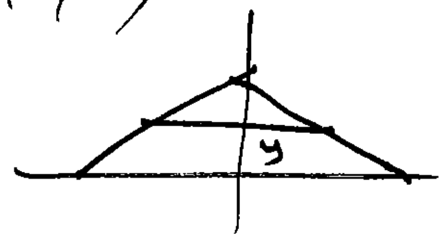
$$\qquad + P(Y=k) \cdot P(X<k)$$

$$= P(X=k, Y \geq k) + P(Y=k) \cdot P(X<k)$$

$$= P(X=k, Y \geq k) + P[(X<k)(Y=k)]$$

3a) Version-2

$f_{XY}(x,y) = k(x^2+y^2)$



$f_{X|Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

$f_Y(y) = \displaystyle\int_{-(1-y)}^{1-y} k(x^2+y^2)\,dx = \left(\frac{x^3}{3}+y^2x\right)\Bigg|_{-(1-y)}^{(1-y)}$

$= 2(1-y)\left[\cancel{2}\dfrac{(1-y)^2}{3}+y^2\right]$

$= \dfrac{2}{3}(1-y)(4y^2-2y+1)$

$f_{X|Y}(x|y) = \dfrac{3}{2}\,\dfrac{x^2+y^2}{(1-y)(4y^2-2y+1)}$

$E(X^2|Y) = \displaystyle\int x^2 f_{X|Y}(x|y)\,dx$

$= \dfrac{3}{2(1-y)(4y^2-2y+1)}\underbrace{\displaystyle\int_{-(1-y)}^{1-y}(x^4+x^2y^2)\,dx}_{\left(\frac{x^5}{5}+\frac{x^3}{3}y^2\right)\Big|_{-(1-y)}^{1-y}}$

$= \dfrac{3(1-y)^2}{2(4y^2-2y+1)}\,2\left(\dfrac{(1-y)^2}{5}+\dfrac{y^2}{3}\right) = \dfrac{(1-y)^2(8y^2-6y+3)}{5(4y^2-2y+1)}$ //

7 or

$$P(Z=k) = P(X=k)\,P(Y \le k) + P(Y=k)\,P(X<k)$$

$$= pq^k \sum_{i=0}^{k} P(Y=i) + pq^k \sum_{i=0}^{k-1} P(X=i)$$

$$= pq^k \left(\sum_{i=0}^{k} p q^i + \sum_{k=0}^{k-1} p q^i \right) = p^2 q^k \left(2 \sum_{k=0}^{k-1} q^i + q^k \right)$$

$$= p^2 q^k \left(2 \frac{1-q^k}{1-q} + q^k \right) = 2pq^k(1-q^k) + p^2 q^{2k}$$

$$= 2p(1-q^k)q^k + p^2 q^{2k}, \quad k = 0, 1, 2 .. \infty$$

check

$$\sum_{k=0}^{\infty} P(Z=k) = 2p \underbrace{\sum_{k=0}^{\infty} q^k}_{\frac{1}{1-q}} - 2p \underbrace{\sum_{k=0}^{\infty} (q^2)^k}_{\frac{1}{1-q^2}} + p^2 \underbrace{\sum_{k=0}^{\infty} (q^2)^k}_{\frac{1}{1-q^2}}$$

$$= \frac{2p}{p} - \frac{2}{1+q} + \frac{p}{1+q} = 2 - \frac{(1+p+q-p)}{1+q} = 2 - 1 = 1.$$

8) See Text Ch. 6, p. 227, Eq (6-222).

$$P(X=m \mid X+m=n) = \frac{P(X=m,\, X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=m,\, Y=n-m)}{P(X+Y=n)} = \frac{P(X=m)\,P(Y=n-m)}{P(X+Y=n)}$$

$$= \frac{e^{-\lambda} \frac{\lambda^m}{m!}\, e^{-\mu} \frac{\mu^{n-m}}{(n-m)!}}{\quad} = \binom{n}{m} \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\lambda+\mu}\right)^{n-m}, \quad m = 0, 1, \cdots n$$

~ Binomial.

Midterm Exam

1. *(15 points)* Given $X \sim \text{Uniform}(0, \pi)$, find the probability density function (p.d.f.) of $Y = \cot X$ and plot it.

2. a.) *(20 points)* Given the joint probability density function

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z = \frac{\max(X,Y)}{\min(X,Y)}$$

Find the p.d.f. of Z.

b.) *(20 points)* Given

$$f_{XY}(x,y) = \begin{cases} e^{-y}, & y > x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

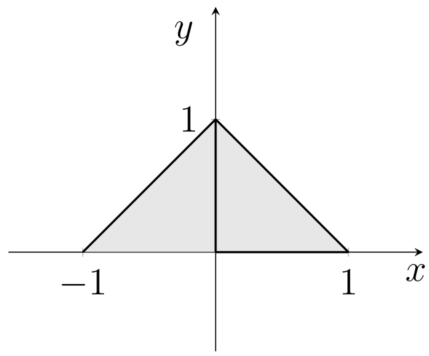
i) Find the joint probability density function of $Z = X + Y$, $W = Y - X$, and marginal distributions of Z and W.

ii) Are Z and W independent? Prove your answer.

3. a.) *(15 points)*

The joint probability density function of random variables X and Y is given by,

$$f_{XY}(x,y) = \begin{cases} k(x^2+y^2) & (x,y), \in \text{shaded region,} \\ 0 & \text{otherwise.} \end{cases}$$



Find the conditional mean of X^2 given $Y = y$.

b.) *(15 points)* X and Y are independent Geometric random variables with

$$P(X=k) = P(Y=k) = pq^k, \quad k = 0, 1, 2, ...$$

Find the probability mass function of $Z = \max(X,Y)$. Verify that it is a valid probability mass function.

4. *(15 points)* X and Y are independent Poisson random variables with parameters λ and μ respectively. Thus, $P(X=k) = e^{-\lambda}\,\frac{\lambda^k}{k!}$, $k = 0, 1, 2, ..., \infty$.
Find the conditional probability mass function of X given $X+Y$. Can you identify this distribution?
(*Hint: Determine* $P(X = m | X + Y = n)$)

Midterm Exam

1. *(15 points)* Given $X \sim \text{Uniform}(0, \pi)$, find the probability density function (p.d.f.) of $Y = \tan X$ and plot it.

2. a.) *(20 points)* Given the joint density function

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z = \frac{\max(X,Y)}{\min(X,Y)}$$

Find the p.d.f. of Z

b.) *(20 points)* Given

$$f_{XY}(x,y) = \begin{cases} e^{-x}, & 0 < y < x \\ 0 & \text{otherwise,} \end{cases}$$

i) Find the joint probability density function of $Z = X + Y$, $W = X - Y$, and marginal distributions of Z and W.

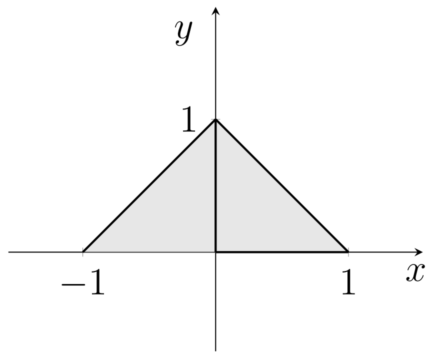
ii) Are Z and W independent? Prove your answer.

3. a.) *(15 points)*

The joint probability density function of random variables X and Y is given by,

$$f_{XY}(x,y) = \begin{cases} k(x^2 + y) & (x,y), \in \text{shaded region,} \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional mean for X^2 given $Y = y$.



b.) *(15 points)* X and Y are independent Geometric random variables with

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Find the conditional probability mass function of X given $X + Y$. Can you identify this distribution?
(*Hint: Determine* $P(X = m | X + Y = n)$)

ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 1

Prof. Pillai

Problem 1

Box 1 contains 3 red balls, 5 green balls and 2 white balls. Box 2 contains 5 red balls, 3 green balls and 1 white ball. One ball of unknown color is transferred from Box 1 to Box 2.

a. What is the probability that a ball drawn at random from Box 2 is green?

b. What is the probability that a ball drawn from Box 1 is not white?

Solution

Let the events T_R, T_G and T_W represent rtansferring a red, green and white ball respectively. Note that these event form a partition for the transfer.

a. Probability that a green ball is drawn from box 2 $P(G)$ is,

$$\begin{aligned}
P(G) &= P(G|T_R)P(T_R) + P(G|T_G)P(T_G) + P(G|T_W)P(T_W), \\
&= \frac{3}{10}.\frac{3}{10} + \frac{4}{10}.\frac{5}{10} + \frac{3}{10}.\frac{2}{10}, \\
&= 0.35.
\end{aligned}$$

b. This refers to the scenario after the draw. the probability that a ball drawn from box 2 is not white will be,

$$\begin{aligned}
P(\overline{B}) &= P(\overline{B}|T_R)P(T_R) + P(\overline{B}|T_G)P(T_G) + P(\overline{B}|T_W)P(T_W), \\
&= \frac{7}{9}.\frac{3}{10} + \frac{7}{9}.\frac{5}{10} + \frac{8}{9}.\frac{2}{10}, \\
&= 0.8.
\end{aligned}$$

Problem 2

In a batch of microprocessors, the probability that a microprocessor is defective is 10^{-3}. In one draw an assembly machine picks 10 microprocessors from this batch and tests each. It rejects the entire lot of 10 microporcessors if 2 or more of them are defective, else all the 10 are retained.

a. Find the probability that a lot is rejected.

b. If the machine draw 6 times, what is the probability that at least 60 microprocessors are retained.

Solution:

a. Let the event that a lot is rejected be R.

Let the probablity that a microporcessor is defective be $p = 10^{-3}$.

$P(R) = 1 - P(\bar{R}) = 1 - C_{10}^{0}(1-p)^{10} - C_{10}^{1}p(1-p)^{9}$

b. Probability that all 6 lots (i.e. 60 microporcessors) are retained $= C_6^0 P(R)^0 (1-P(R))^6 = (1-P(R))^6$.

Problem 3

a. Toss a coin n times, Let 'p' represent the probability of obtaining a "Head" in any toss. Show that the most likely number of "Heads" k_0 in n trials is given by

$$(n+1)p - 1 \leq k_0 \leq (n+1)p$$

$$\text{and hence} \frac{k_0}{n} \to p$$

Solution:

Let $a_k = P(X = k)$, we have

$$a_k = \binom{n}{k} p^k q^{n-k} \qquad \text{and} \qquad a_{k+1} = \binom{n}{k+1} p^{k+1} q^{n-k-1},$$

where as usual $q = 1 - p$ in binomial distribution.

We calculate the *ratio* $\dfrac{a_{k+1}}{a_k}$. Note that $\dfrac{\binom{n}{k+1}}{\binom{n}{k}}$ simplifies to $\frac{n-k}{k+1}$, and therefore

$$\frac{a_{k+1}}{a_k} = \frac{n-k}{k+1} \cdot \frac{p}{q} = \frac{n-k}{k+1} \cdot \frac{p}{1-p}.$$

From this equation we can follow:

$$\begin{aligned} k > (n+1)p - 1 &\implies a_{k+1} < a_k \\ k = (n+1)p - 1 &\implies a_{k+1} = a_k \\ k < (n+1)p - 1 &\implies a_{k+1} > a_k \end{aligned}$$

The calculation says that we have equality of two consecutive probabilities precisely if $a_{k+1} = a_k$, that is , if $k = np + p - 1$ implies that $np + p - 1$ is an integer.

So if $k = np + p - 1$ is not an integer, there is a single mode; and if $k = np + p - 1$ is an integer, there are two modes, at $np + p - 1$ and at $np + p$.

b. In a book of 200 pages long, it is not unreasonable to expect 20 misprints. Find the probability that a given page will contain

i) two misprints
ii) two or less prints
iii) two or more misprints.
Solution:
Let X be the number of misprints in a 200 page book with an average number of misprints =20 therefore $X \sim P_0(20)$.
Let Y be the number of misprints in a given page with an average number of misprints = $\frac{20}{200}$=0.1 therefore $Y \sim P_0(0.1)$.
(i)

$$P(Y=2)=\frac{e^{-0.1}(0.1)^2}{2!}$$

(ii)

$$P(Y \leq 2)=P(Y=0)+P(Y=1)+P(Y=2)$$
$$P(Y \leq 2)=e^{-0.1}+\frac{e^{-0.1}(0.1)^1}{1!}+\frac{e^{-0.1}(0.1)^2}{2!}$$

(iii)

$$P(Y \geq 2)=1-P(Y=0)-P(Y=1)=1-e^{-0.1}-\frac{e^{-0.1}(0.1)^1}{1!}$$

Problem 4

The pdf of a continous random variable X is given by

$$f_X(x)=\begin{cases} \frac{1}{7} & -2 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find
(i) $P(X^2>1)$
(ii) $P(sin(\pi X) \leq 0)$

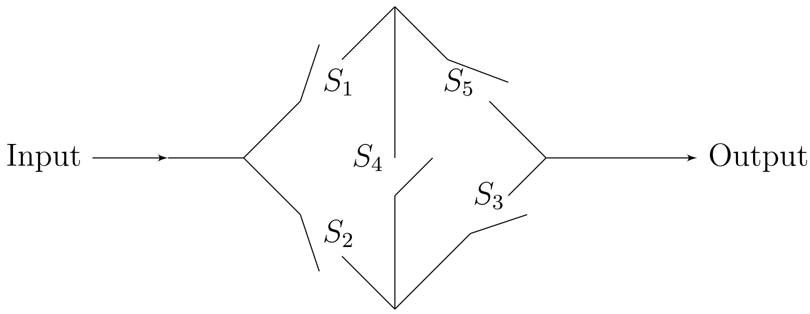
Solution:
(i)

$$\begin{aligned} P(X^2>1) &= P((X<-1)U(X>1)) \\ &= P(X<-1)+P(X>1) \\ &= \frac{1}{7}(1)+\frac{1}{7}(4)=\frac{5}{7} \end{aligned}$$

(ii)

$$\begin{aligned} P(sin(\pi X)<0) &= P((-2<X<0)UP(2<X<4)) \\ &= P(-2<X<0)+P(2<X<4)=\frac{3}{7} \end{aligned}$$

Problem 5



The five switches in the figure operate independently. Each switch is closed with probability p and open with probability $(1-p)$.

a. Find the probability that the signal at the input will **not** be received at the output.

b. Find the conditional probability that the switch S_4 is open given that the signal is received at the output.

Solution:

a. Let A_i represent the event that "switch i is closed," for $i = 1, 2, 3, 4, 5$.
Then, $P(A_i) = p$ and $P(A_i^c) = 1 - p$ for $i = 1, 2, 3, 4, 5$.

From the diagram a signal is received at the output when, $\{S_1$ and $S_5\}$ are closed, or $\{S_2$ and $S_3\}$ are closed, or $\{S_1, S_4$ and $S_3\}$ are closed, or $\{S_2$, S_4 and $S_5\}$ are closed.

Let R = "input signal is received at the output".

Thus the probability of receiving a signal R equals,

$$P(R) = P((A_1 \cap A_5) \cup (A_2 \cap A_3) \cup (A_1 \cap A_4 \cap A_3) \cup (A_2 \cap A_4 \cap A_5)).$$

Rewriting $(A_1 \cap A_5) = B_1$,

$$
\begin{aligned}
&(A_2 \cap A_3) = B_2,\\
&(A_1 \cap A_3 \cap A_4) = B_3,\\
&(A_2 \cap A_4 \cap A_5) = B_4,\\
P(R) &= P(B_1 \cup B_2 \cup B_3 \cup B_4),\\
&= P(B_1) + P(B_2) + P(B_3) + P(B_4) - P(B_1 \cap B_2) - P(B_1 \cap B_3) - P(B_1 \cap B_4)\\
&\quad - P(B_2 \cap B_3) - P(B_2 \cap B_4) - P(B_3 \cap B_4) + P(B_1 \cap B_2 \cap B_3) + P(B_1 \cap B_2 \cap B_4)\\
&\quad + P(B_1 \cap B_3 \cap B_4) + P(B_2 \cap B_3 \cap B_4) - P(B_1 \cap B_2 \cap B_3 \cap B_4).\\
&= P(A_1 \cap A_5) + P(A_1 \cap A_4 \cap A_3) + P(A_2 \cap A_3) + P(A_2 \cap A_4 \cap A_5)\\
&\quad - P(A_1 \cap A_2 \cap A_3 \cap A_5) - P(A_1 \cap A_3 \cap A_4 \cap A_5)\\
&\quad - P(A_1 \cap A_2 \cap A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_3 \cap A_4)\\
&\quad - P(A_2 \cap A_3 \cap A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)\\
&\quad + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)\\
&\quad + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)\\
&\quad - P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5.\\
&= P(A_1)P(A_5) + P(A_1)P(A_4)P(A_3) + P(A_2)P(A_3) + P(A_2)P(A_4)P(A_5)\\
&\quad - P(A_1)P(A_2)P(A_3)P(A_5) - P(A_1)P(A_3)P(A_4)P(A_5)\\
&\quad - P(A_1)P(A_2)P(A_4)P(A_5) - P(A_1)P(A_2)P(A_3)P(A_4)\\
&\quad - P(A_2)P(A_3)P(A_4)P(A_5) + 2P(A_1)P(A_2)P(A_3)P(A_4)P(A_5)\\
&= 2p^2 + 2p^3 - 5p^4 + 2p^5.
\end{aligned}
$$

$$P(R^c) = 1 - P(R) = 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$

Alternative solution Let us partition Ω as $A_4 \cup \overline{A_4}$, and $A_4 \cap \overline{A_4} = \phi$.

Thus,

$$P(R) = \underbrace{P(R|A_4)}_{\text{P(signal is received}|S_4\text{ is closed)}} \times P(A_4) + P(R|\overline{A_4})P(\overline{A_4}).$$

Note that,

$$
\begin{aligned}
P(R|A_4) &= P[(A_1 \cap A_5) \cup (A_2 \cap A_5) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3)],\\
&= 4p^2 - 4p^3 + p^4.\\
P(R|\overline{A_4}) &= P[(A_1 \cap A_5) \cup (A_2 \cap A_3)]\\
&= P(A_1 \cap A_5) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3 \cap A_5),\\
&= 2p^2 - p^4.\\
\implies P(R) &= (4p^2 - 4p^3 + p^4)p + (2p^2 - p^4)(1 - p),\\
&= 2p^2 + 2p^3 - 5p^4 + 2p^5.
\end{aligned}
$$

$$P(R^c) = 1 - P(R) = 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$

b. The probability that S_3 is open given a signal is received at the output $= P(\overline{A_3}|R)$.
Using Bayes' Theorem we can write,

$$P(\overline{A_3}|R) = \frac{P(R|\overline{A_3})P(\overline{A_3})}{P(R)}.$$

$$P(R|\overline{A_3}) = P(A_1A_5 \cup A_2A_4A_5) = P(A_1A_5) + P(A_2A_4A_5) - P(A_1A_2A_4A_5) = p^2 + p^3 - p^4.$$

Substituting the results in part (a) we can write,

$$\begin{aligned} P(\overline{A_3}|R) &= \frac{(p^2 + p^3 - p^4)(1-p)}{p^2(2 + 2p - 5p^2 + 2p^3)} \\ &= \frac{(1 + p - p^2)(1-p)}{(2 + 2p - 5p^2 + 2p^3)}. \end{aligned}$$

Problem 6

Among a certain group of people 5 % are (professional) liars. A lie detector test on a liar is found to be positive with a probability of 0.94. If the test is positive for a non-liar, it is positive with a probability of 0.08. Given that the test is positive for a randomly picked person from that group, what is the probability that he is a liar.

Solution

Let the event L be that a person is a liar. By the problem,

$$P(L) = 0.05.$$

Let the event that the test is positive be T_p. Thus,

$$\begin{aligned} P(T_p|L) &= 0.94, \\ P(T_p|\overline{L}) &= 0.08. \end{aligned}$$

We need to compute $P(L|T_p)$, the probability that the person is a liar given that the test is positive.
Using total probability we can write,

$$\begin{aligned} P(T_p) &= P(T_p|L)P(L) + P(T_p|\overline{L})P(\overline{L}), \\ &= 0.94 * 0.05 + 0.08 * 0.95, \\ &= 0.123. \end{aligned}$$

Using Bayes Theorem we can write,

$$\begin{aligned} P(L|T_p) &= \frac{P(T_p|L)P(L)}{P(T_p)}, \\ &= \frac{0.94 \times 0.05}{0.123}, \\ &= 0.38211 \end{aligned}$$

Problem 7

We have two sealed boxes. In the first box we have 125 white and 75 black marbles. The second box contains 60 white and 90 black marbles. You pick a marble randomly from a box. For any given pick, the probability of picking from Box i, $P(B_i) = 0.5$.

a. What is probability that the marble drawn is black?

b. The marble picked turned to be black. What is the probability that it is picked out of Box 2?

Solution

a. Let the probability that a white marble is drawn be $P(M_w)$. Using the law of total probability we can write,

$$\begin{aligned} P(M_b) &= P(M_b|B_1).P(B_1) + P(M_b|B_2).P(B_2), \\ &= \frac{75}{200}.\frac{1}{2} + \frac{90}{150}.\frac{1}{2} \\ &= \frac{39}{80}. \end{aligned}$$

b. We need to find $P(B_2|M_b)$, where M_b is the event that a marble drawn is black. Using Bayes theorem,

$$\begin{aligned} P(B_2|M_b) &= \frac{P(M_b|B_2).P(B_2)}{P(M_b)}, \\ &= \frac{3/5 * 0.5}{39/80}, \\ &= 0.61538461 \end{aligned}$$

ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 2

Prof. Pillai

Problem 1

Let the number of accidents on a highway on any given day be modelled as a Poission random variable with parameter $\lambda > 1.147$. Determine the following.

i) The probability that there is at most one accident on any given day.

ii) The probability that there is at least one accident on any given day.

iii) Which one of the above has higher probability?

Solution:

i) The probability that there is at most one accident on any given day:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-\lambda} + \lambda e^{-\lambda} = e^{-\lambda}(1 + \lambda)$$

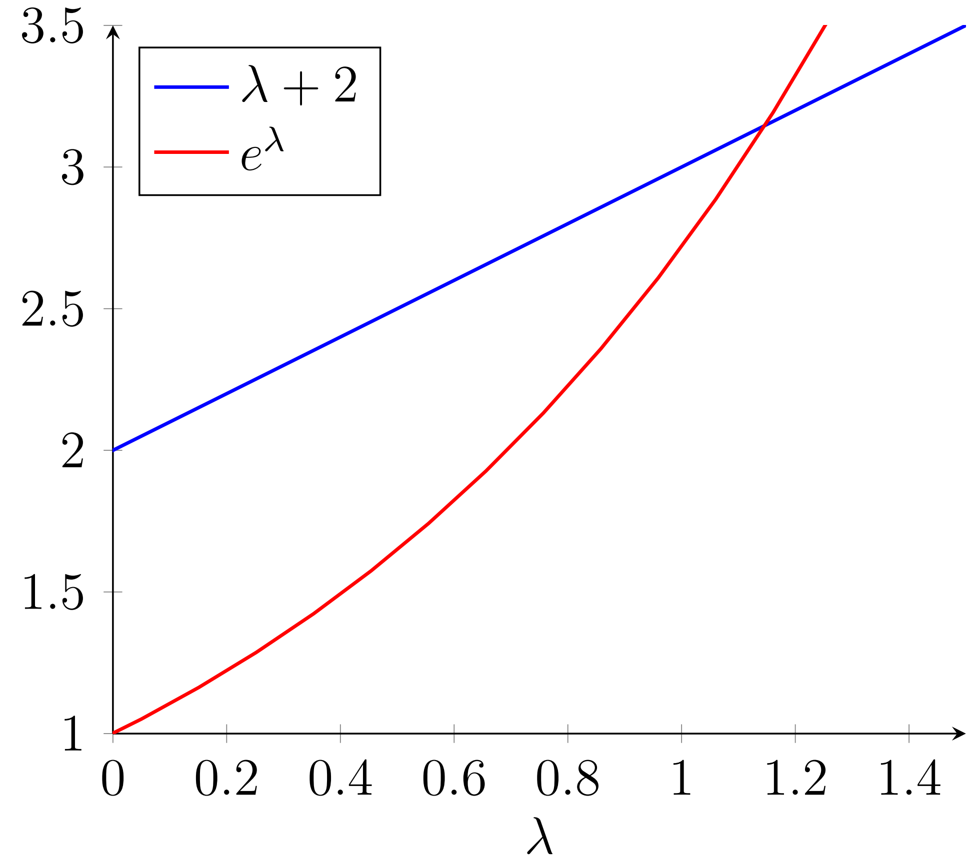
ii) The probability that there is at least one accident on any given day:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda}$$

iii)

$$P(X \leq 1) \ ? \ P(X \geq 1) \quad \Leftrightarrow \quad e^{-\lambda}(1 + \lambda) \ ? \ 1 - e^{-\lambda} \quad \Leftrightarrow \quad (2 + \lambda) \ ? \ e^{\lambda}$$

As $\lambda > 1.147$, (ii) is bigger than (i) as shown in the figure.

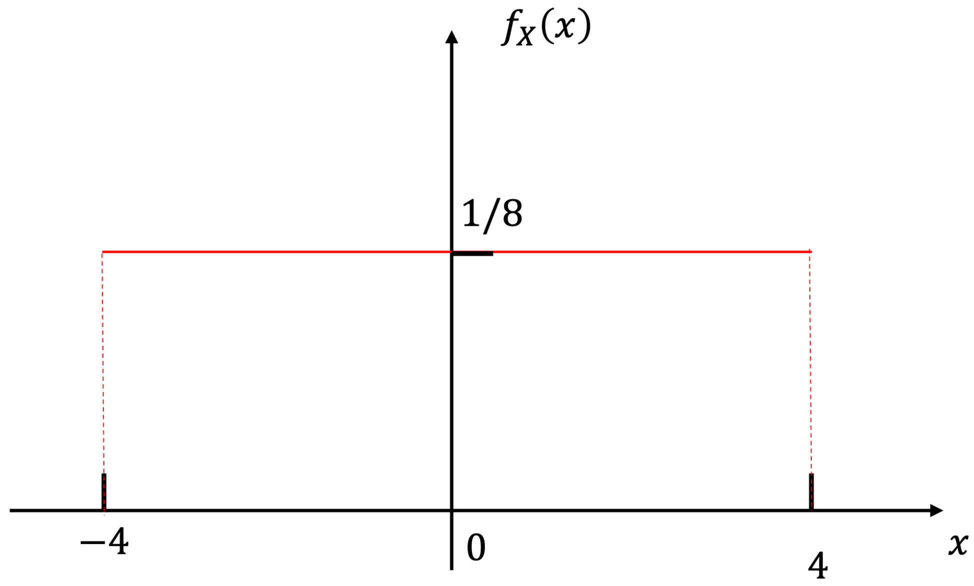


Problem 2

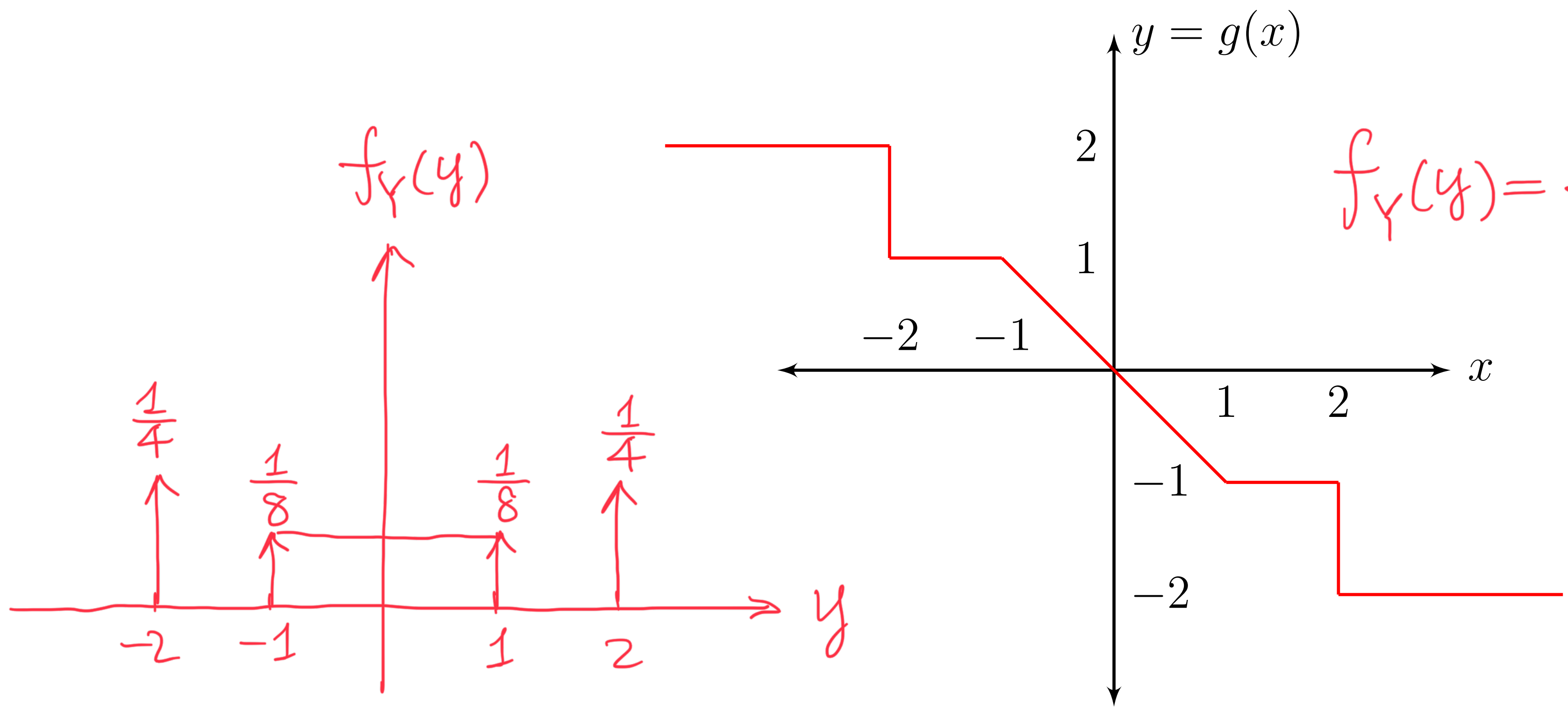
The pdf of the random variable X is is given by the following

$$f_X(x) = \begin{cases} 1/8 & x \in [-4, 4], \\ 0 & \text{otherwise,} \end{cases}$$

as shown in the figure.



Find the pdf of the random variable $Y = g(X)$ shown in the figure and sketch it.



$$f_Y(y) = \begin{cases} \frac{1}{4}\delta(y+2) & y=-2 \\ \frac{1}{8}\delta(y+1) & y=-1 \\ \frac{1}{8} & -1<y<1 \\ \frac{1}{8}\delta(y-1) & y=1 \\ \frac{1}{4}\delta(y-2) & y=2 \\ 0 & \text{OTW} \end{cases}$$

Solution:

$$P(Y = 2) = P(X \le -2) = F_X(-2) = \frac{1}{4},$$
$$P(Y = 1) = P(-2 \le X \le -1) = F_X(-1) - F_X(-2) = \frac{1}{8},$$
$$P(Y = -1) = P(1 \le X \le 2) = F_X(2) - F_X(1) = \frac{1}{8},$$
$$P(Y = -2) = P(X > 2) = 1 - F_X(2) = \frac{1}{4}.$$

For $-1 \le Y \le 1$, $Y = -X$, so $F_Y(y) = P(Y \le y) = P(X \ge -y) = 1 - F_X(-y)$. Therefore $f_Y(y) = \frac{d}{dy}F_Y(y) = f_X(-y)$. The sketch should include four impulses and a constant value of $1/8$ for $-1 \le Y \le 1$.

Problem 3

Given that 10% of entering college students do not complete their Degree programs, What is the probability that out of ~~5~~ 3 randomly selected students, more than half will get their degrees?.
(Simplify and find the exact answer)

Solution:

Let $p = P(\text{A randomly selected student completes the degree program}) = 1 - 0.1 = 0.9(\text{given data})$. Let $X =$ "number of students that complete the degree pragram out of 5".

$$X \sim \text{Binomial}(n = \overset{3}{\cancel{5}}, p = 0.9).$$

We need

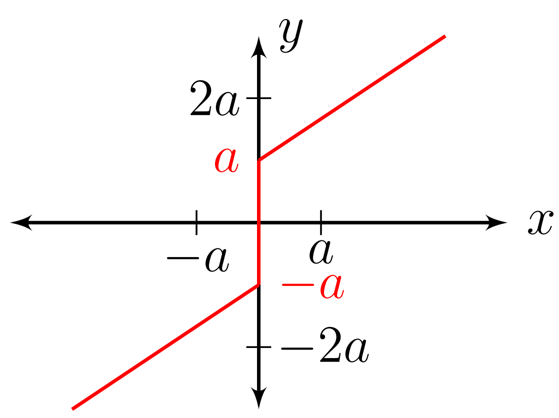
$$\begin{aligned} P(\text{3 or more students complete the degree program}) &= P(X \geq 3) \\ &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p)^1 + \binom{5}{5} p^5 (1-p)^0 \\ &= 0.99144. \end{aligned}$$

$$\begin{aligned} P(X \geq 2) = P(X = 2) + P(X = 3) &= \binom{3}{2} p^2 q^1 + \binom{3}{3} p^3 q^0 \\ &= 0.243 + 0.729 \\ &= 0.972 \end{aligned}$$

Problem 4

Given $f_X(x)$, and $y = g(x)$, find $f_Y(y)$ for the following.

a



Solution

$$Y = \begin{cases} X + a, & X > 0 \\ X - a, & X \le 0 \end{cases}$$

For $y > a$,

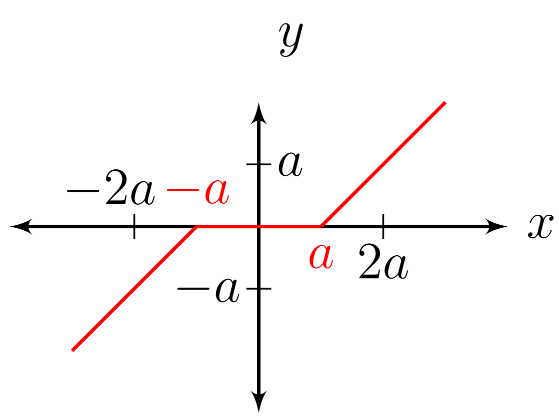
$$F_Y(y) = P(Y \le y) = P(X + a \le y) = P(X \le y - a) = F_X(y - a).$$

For $y < -a$,

$$F_Y(y) = P(Y \le y) = P(X - a \le y) = P(X \le y + a) = F_X(y + a)$$

$$f_Y(y) = \begin{cases} f_X(y - a) & y > a, \\ f_X(y + a) & y < -a, \\ 0 & \text{otherwise.} \end{cases}$$

b



Solution

$$Y = \begin{cases} X - a, & X > a \\ 0, & -a < X < a \\ X + a, & X \le -a \end{cases}$$

For $y < 0$,

$$F_Y(y) = P(Y \le y) = P(X + a \le y) = P(X \le y - a) = F_X(y - a).$$

For $y = 0$,

$$P(Y = 0) = P(-a \le X < a) = F_X(a) - F_X(-a).$$

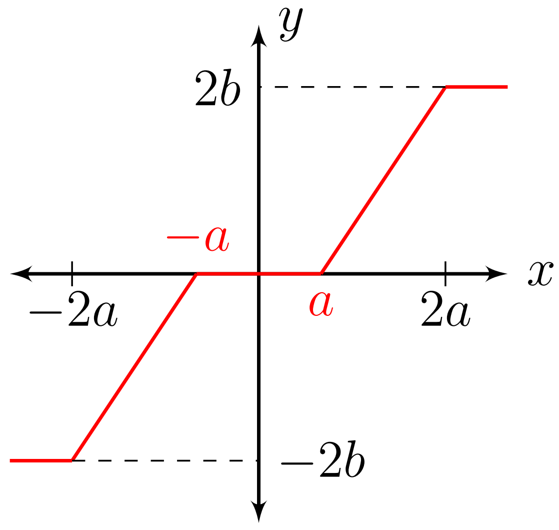
For $y > 0$,

$$F_Y(y) = P(Y \le y) = P(X - a \le y) = P(X \le y + a) = F_X(y + a).$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} f_X(y+a) & y > 0 \\ \cancel{F_X(a) - F_X(-a)} & y = 0 \\ f_X(y-a) & y < 0 \end{cases}$$

$$[F_X(a) - F_X(-a)]\,\delta(y)$$

c



Solution For $y = 2b$,

$$P(Y = 2b) = P(X > 2a) = 1 - F_X(2a)$$

For $0 < y < 2b$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(\frac{2b}{a}X - 2b \le y) = P(X \le \frac{ay + 2ab}{2b}) \\ &= F_X(\frac{ay + 2ab}{2b}). \end{aligned}$$

For $y = 0$,

$$P(Y = 0) = P(-a \le X \le a) = F_X(a) - F_X(-a)$$

For $-2b < y < 0$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(\frac{2b}{a}X + 2b \le y) = P(X \le \frac{ay - 2ab}{2b}) \\ &= F_X(\frac{ay - 2ab}{2b}). \end{aligned}$$

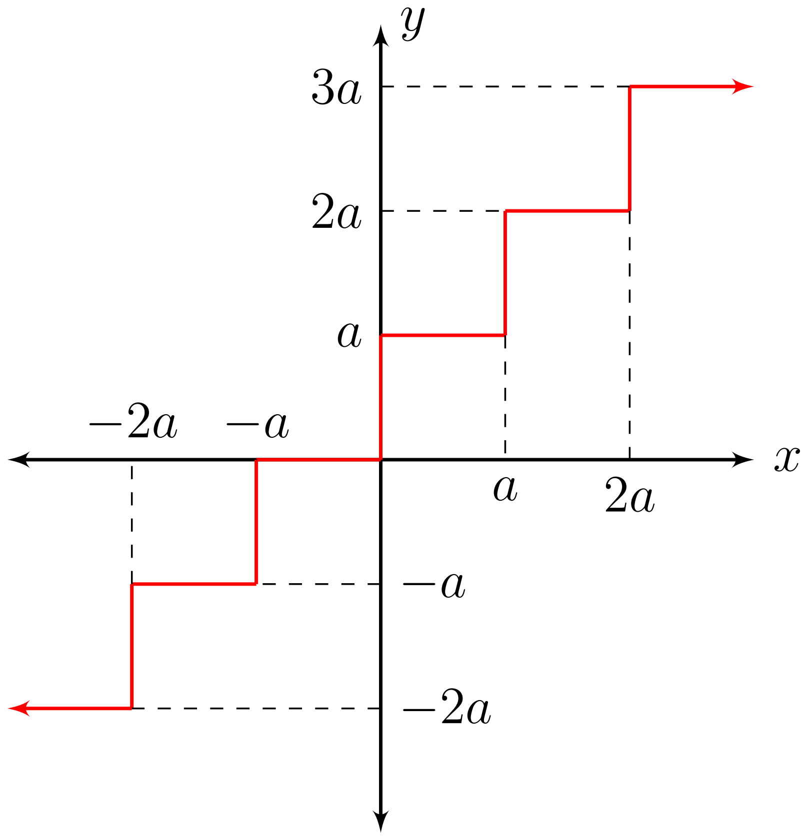
For $y = -2b$,

$$P(Y = -2b) = P(X \le -2a) = F_X(-2a).$$

Therefore,

$$f_Y(y) = \left\{ \begin{array}{lll} \cancel{F_X(-2a)} & y = -2b, & F_X(-2a)\,\delta(y+2b) \\ \frac{a}{2b} f_X(\frac{ay-2ab}{2b}) & -2b < y < 0 & \\ \cancel{F_X(a) - F_X(-a)} & y = 0 & [F_X(a) - F_X(-a)]\,\delta(y) \\ \frac{a}{2b} f_X(\frac{ay+2ab}{2b}) & 0 < y < 2b & \\ \cancel{1 - F_X(2a)} & y = 2b & [1 - F_X(2a)]\,\delta(y-2b) \end{array} \right.$$

d



Solution

$$
\begin{aligned}
&P(Y=-2a) = P(X<-2a) = F_X(-2a),\\
&P(Y=-a) = P(-a<X<-2a) = F_X(-a) - F_X(-2a),\\
&P(Y=0) = P(-a<X<0) = F_X(0) - F_X(-a),\\
&P(Y=a) = P(0<X<a) = F_X(a) - F_X(0),\\
&P(Y=2a) = P(a<X<2a) = F_X(2a) - F_X(a),\\
&P(Y=3a) = P(X>2a) = 1 - F_X(2a).
\end{aligned}
$$

$$
f_Y(y) = \begin{cases}
F_X(-2a) & y=-2a,\\
F_X(-a) - F_X(-2a) & y=-a,\\
F_X(0) - F_X(-a) & y=0,\\
F_X(a) - F_X(0) & y=a,\\
F_X(2a) - F_X(a) & y=2a,\\
1 - F_X(2a) & y=3a.
\end{cases}
$$

$$
f_Y(y) = \begin{cases}
F_X(-2a)\,\delta(y+2a) & y=-2a\\
[F_X(-a) - F_X(-2a)]\,\delta(y+a) & y=-a\\
[F_X(0) - F_X(-a)]\,\delta(y) & y=0\\
[F_X(a) - F_X(0)]\,\delta(y-a) & y=a\\
[F_X(2a) - F_X(a)]\,\delta(y-2a) & y=2a\\
[1 - F_X(2a)]\,\delta(y-3a) & y=3a
\end{cases}
$$

Problem 5

Solution

a

Given $f_X(x)$, find $f_Y(y)$ when,

i. $y = ax + b$.

$$x_1 = \frac{y-b}{a}, \quad \frac{dy}{dx} = a, \quad f_Y(y) = \frac{f_X(x_1)}{|\frac{dy}{dx}|_{x=x_1}} = \frac{f_X(\frac{y-b}{a})}{|a|}$$

ii. $y = ax^2 + b$.

$$x_1 = \sqrt{\frac{y-b}{a}}, \quad x_2 = -\sqrt{\frac{y-b}{a}}, \quad \frac{dy}{dx} = 2ax,$$

$$f_Y(y) = \frac{f_X(x_1)}{|\frac{dy}{dx}|_{x=x_1}} + \frac{f_X(x_2)}{|\frac{dy}{dx}|_{x=x_2}} = \frac{f_X(\sqrt{\frac{y-b}{a}})}{|2a|\sqrt{\frac{y-b}{a}}} + \frac{f_X(-\sqrt{\frac{y-b}{a}})}{|2a|\sqrt{\frac{y-b}{a}}}$$

iii. $y = |x|$.

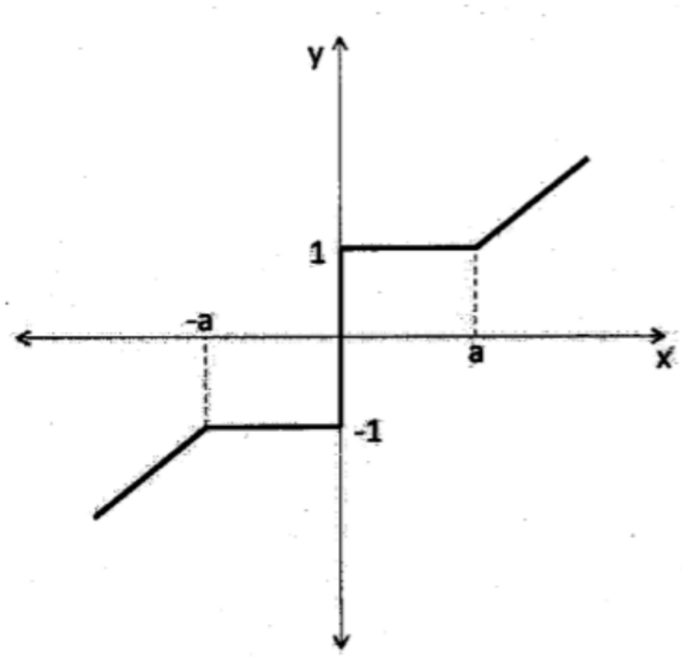
$$F_Y(y) = P(Y \le y) = P(-x \le X \le x) = F_x(x) - F_X(-x),$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \cancel{f_X(x) + f_X(-x)}$$

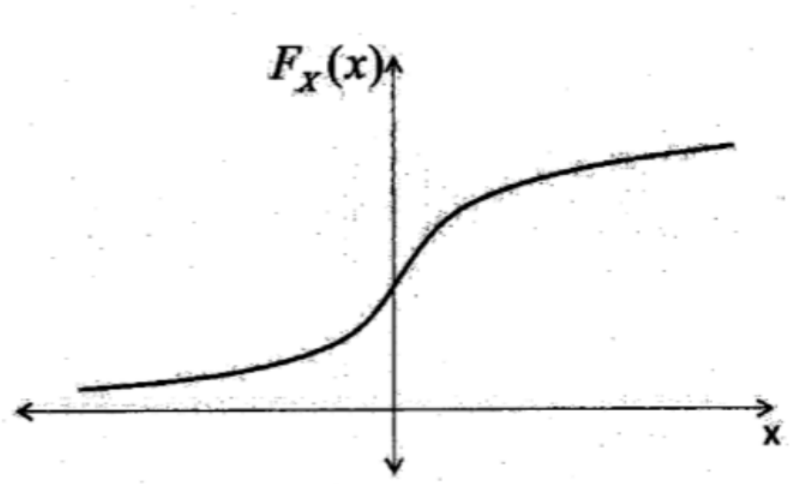
$f_X(y) + f_X(-y)$

b

Given $F_X(x)$, find and plot $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$.



(a)



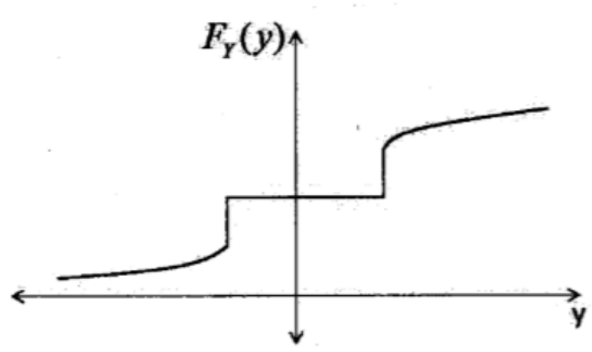
(b)

Solution: If $y = x$ for $x > a$ and $x < -a$, then $a = 1$:

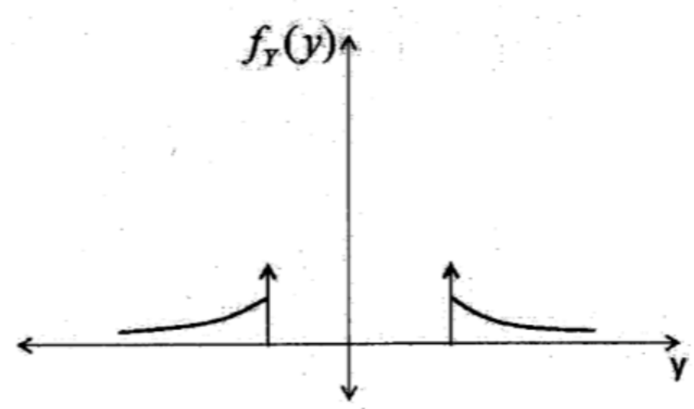
- If $y < -1$, $F_Y(y) = P(X < y) = F_X(y)$.
- If $y = -1$, $P(Y = -1) = P(-1 < x < 0) = F_X(0) - F_X(-1)$.
- If $-1 < y < 1$, $P(-1 \leq Y \leq 1) = P(X = 0) = 0$.
- If $y = 1$, $P(Y = 1) = P(0 < x < 1) = F_X(1) - F_X(0)$.
- If $y > 1$, $F_Y(y) = P(X < y) = F_X(y)$.

$$F_Y(y) = \begin{cases} F_X(y) & \text{if } y < -1 \\ F_X(0) & \text{if } -1 \leq y < 1 \\ F_X(1) & \text{if } y = 1 \\ F_X(y) & \text{if } y > 1 \end{cases} \tag{1}$$

$$f_Y(y) = \begin{cases} f_X(y) & \text{if } y < -1 \\ (F_X(0) - F_X(-1))\delta(y+1) & \text{if } y = -1 \\ 0 & \text{if } -1 < y < 1 \\ (F_X(1) - F_X(0))\delta(y-1) & \text{if } y = 1 \\ f_X(y) & \text{if } y > 1 \end{cases} \tag{2}$$



(c)



(d)

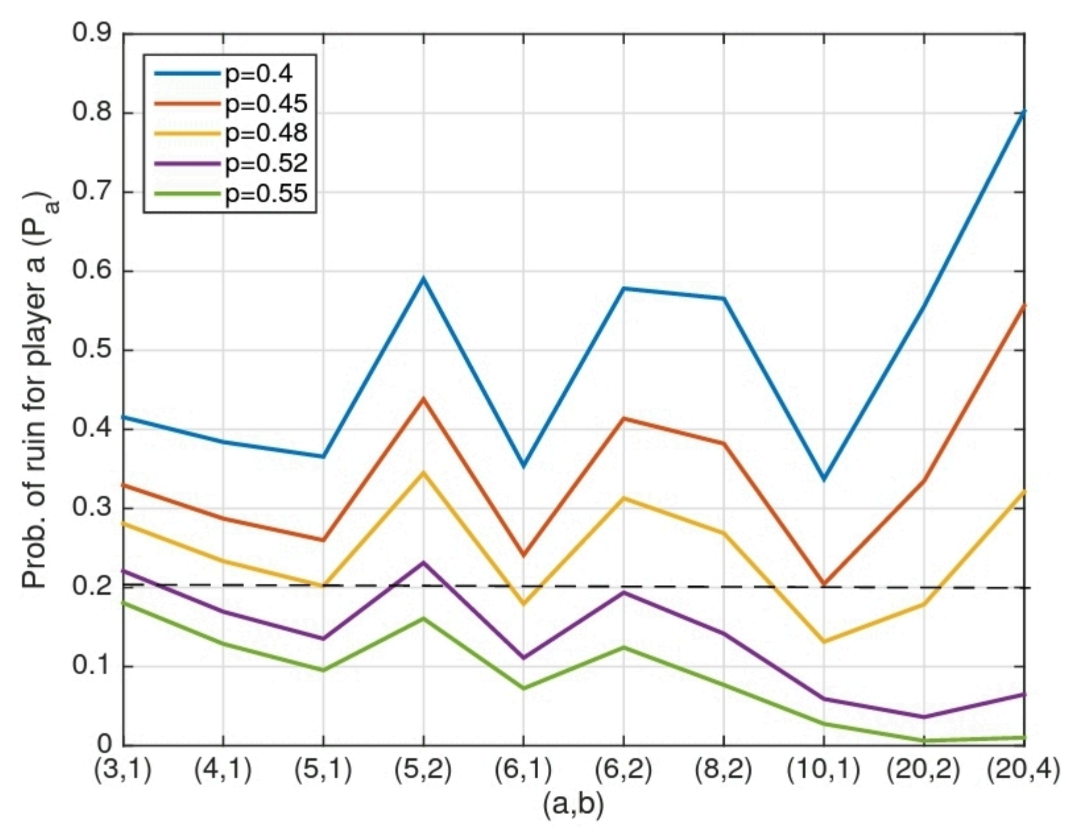
Note: If your solution is in terms of a, it is also correct.

Problem 6

Refer to Gambler's Ruin problem notations discussed in class. Plot risk (Probability of ruin) versus invesetment a for various returns b. Find the best combination (a, b) for 20%, 30% risk for p=0.45, 0.47, 0.52 and 0.55. You can present your results as a table or a figure.

Solution:

For example, as shown in the following figure, the combinations $(a, b) = (5, 1)$ and $(a, b) = (6, 1)$ satisfy 20% risk for $p = 0.48$. Similarly, the combination $(a, b) = (10, 1)$ also satisfies 20% risk for $p = 0.45$, etc.



(a) Prob. of ruin of A, i.e., B wins

Problem 7 (Problem 4 from HW#1)

The pdf of a continous random variable X is given by

$$f_X(x) = \begin{cases} \frac{1}{7} & -2 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find
(i) $P(X^2 > 1)$
(ii) $P(\sin(\pi X) \leq 0)$

Solution:
(i)

$$\begin{aligned} P(X^2 > 1) &= P((X < -1) \cup (X > 1)) = P(X < -1) + P(X > 1) \\ &= \frac{1}{7} + \frac{1}{7} \cdot 4 = \frac{5}{7}. \end{aligned}$$

(ii)

$$\begin{aligned} P(\sin(\pi X) \leq 0) &= P((-1 \leq X \leq 0) \cup P(1 \leq X \leq 2) \cup P(3 \leq X \leq 4)) \\ &= P(-1 \leq X \leq 0) + P(1 \leq X \leq 2) + P(3 \leq X \leq 4) = \frac{3}{7}. \end{aligned}$$

ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 3

Prof. Pillai

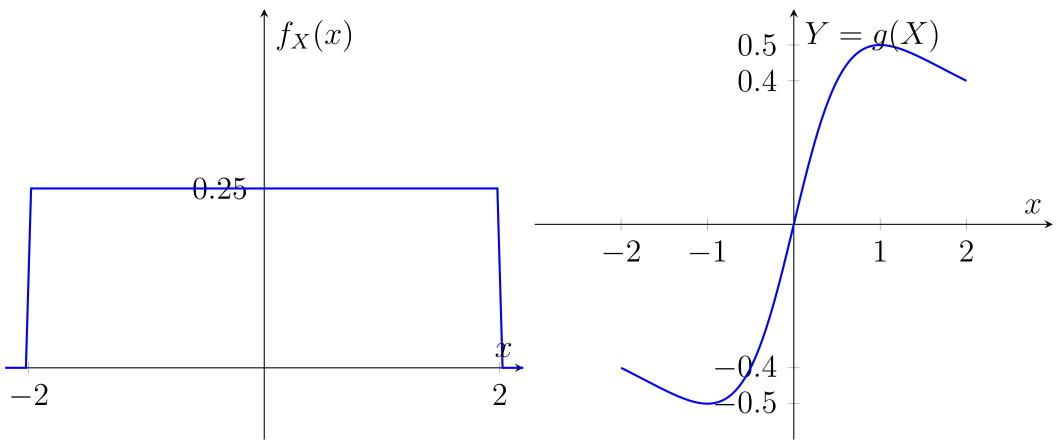
Problem 1

X is a uniform random variable in $(-2, 2)$. Consider the transformation,

$$Y = \frac{X}{1+X^2}.$$

Determine $f_Y(y)$.

Solution:



$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+x^2)\cdot 1 - x\cdot(2x)}{(1+x^2)^2} = 0 \quad \Rightarrow \quad x = \pm 1 \text{ are the stationary points.}$$

$$y = \frac{x}{1+x^2} \;\Rightarrow\; y + x^2 y - x = 0 \;\Rightarrow\; x_1 = \frac{1-\sqrt{1-4y^2}}{2y}, x_2 = \frac{1+\sqrt{1-4y^2}}{2y}$$

Case I: $|y| < 0.4$

$$f_Y(y) = f_X(x_1)\cdot\left|\frac{\mathrm{d}x_1}{\mathrm{d}y}\right| = \frac{1}{4}\cdot\frac{1-\sqrt{1-4y^2}}{2y^2\sqrt{1-4y^2}}$$
$$= \frac{1-\sqrt{1-4y^2}}{8y^2\sqrt{1-4y^2}}$$

Case II: $0.4 \leq |y| \leq 0.5$

$$\begin{aligned} f_Y(y) &= f_X(x_1) \cdot \left|\frac{\mathrm{d}x_1}{\mathrm{d}y}\right| + f_X(x_2) \cdot \left|\frac{\mathrm{d}x_2}{\mathrm{d}y}\right| \\ &= \frac{1}{4y^2\sqrt{1-4y^2}}. \end{aligned}$$

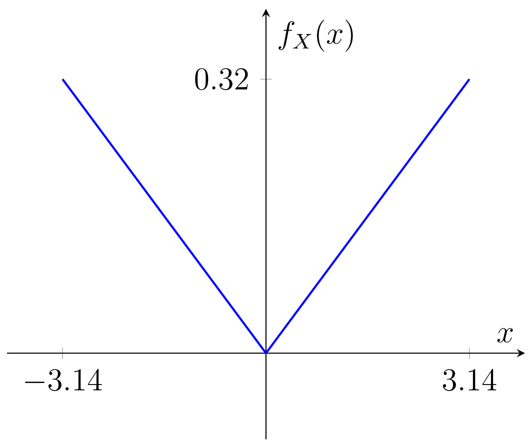
Problem 2

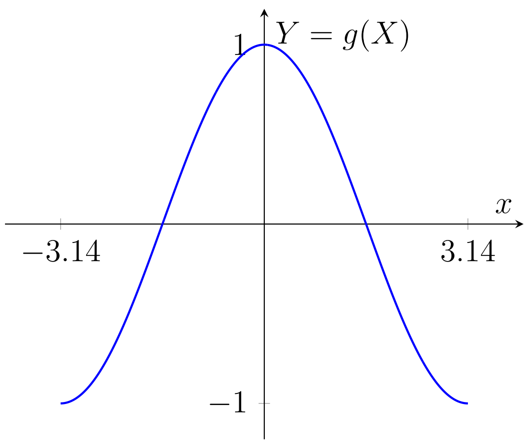
Let

$$f_X(x) = \frac{|x|}{\pi^2} : -\pi < x < \pi$$

, and define $Y = \cos X$. Determine $f_Y(y)$.

Solution:





Note that $x = \cos^{-1}(y)$.

$$f_Y(y) = f_X(x_1) \cdot \left|\frac{\mathrm{d}x_1}{\mathrm{d}y}\right| + f_X(x_2) \cdot \left|\frac{\mathrm{d}x_2}{\mathrm{d}y}\right| = \frac{\cos^{-1} y}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}} + \frac{\cos^{-1} y}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{2\cos^{-1} y}{\pi^2\sqrt{1-y^2}}.$$

Problem 3

A random variable X is Poisson with parameter λ.

i) Find its charecteristic function.

ii) Use the charectaristic function to find E[X] and Var(X).

Solution:

i)

$$\Phi_X(\omega) = E[e^{j\omega x}] = \sum_{k=0}^{\infty} e^{j\omega k} P(x = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{j\omega})^k}{k!} = e^{-\lambda(1-e^{j\omega})}.$$

ii)

$$E[X] = \frac{1}{j}\Phi_X^{(1)}(\omega)\Big|_{\omega=0} = e^{-\lambda} j\lambda e^{j\omega} e^{\lambda e^{j\omega}}\Big|_{\omega=0} = \lambda,$$
$$E[X^2] = \frac{1}{j^2}\Phi_X^{(2)}(\omega)\Big|_{\omega=0} = j\lambda e^{-\lambda}\left(je^{j\omega}e^{\lambda e^{j\omega}} + j\lambda e^{j\omega}e^{\lambda e^{j\omega}}\right)\Bigg|_{\omega=0} = \lambda + \lambda^2,$$
$$\text{Var} = E[X^2] - (E[X])^2 = \lambda.$$

Problem 4

A random variable X is geometric with parameter p.

i) Find its characteristic function.

ii) Use the characteristic function to find E[X] and Var(X)

Solution:

i) Let $q = 1 - p$.

$$\Phi_X(\omega) = E[e^{j\omega x}] = \sum_{k=0}^{\infty} e^{j\omega k} P(x = k) = \sum_{k=0}^{\infty} e^{j\omega k} pq^k = \frac{p}{1 - qe^{j\omega}}.$$

ii)

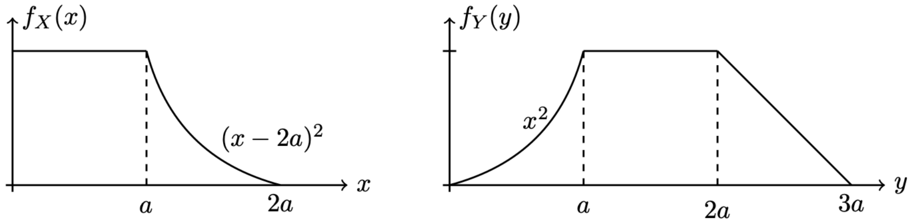
$$E[X] = \frac{1}{j}\Phi_X^{(1)}(\omega)\Big|_{\omega=0} = \frac{1}{j} \cdot \frac{jpqe^{j\omega}}{(1 - qe^{j\omega})^2}\Big|_{\omega=0} = \frac{q}{p}$$

$$E[X^2] = \frac{1}{j^2}\Phi_X^{(2)}(\omega)\Big|_{\omega=0} = \frac{1}{j^2} \cdot \frac{j^2 pqe^{j\omega}\left((1 - qe^{j\omega})^2 + 2qe^{j\omega}(1 - qe^{j\omega})\right)}{(1 - qe^{j\omega})^4}\Bigg|_{\omega=0} = \frac{q + q^2}{p^2}$$

$$\mathrm{Var} = E[X^2] - (E[X])^2 = \frac{q}{p^2}.$$

Problem 5

Find the mean and variance for the random variables X and Y with the probability density functions shown below.



Solution: For X,

$$1 = \int_0^{2a} f_X(x)dx = a^3 + \frac{1}{3}a^3 = \frac{4}{3}a^3 \quad \Rightarrow \quad a = \left(\frac{3}{4}\right)^{1/3}.$$

Mean Value:

$$\mathbb{E}[X] = \int_0^a xa^2 dx + \int_a^{2a} x(x-2a)^2 dx = \frac{1}{2}a^4 + \frac{5}{12}a^4 = \frac{11}{12}a^4.$$

Variance:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{13}{15}a^5 - \frac{121}{144}a^8$$

For Y,

$$1 = \int_0^{3a} f_Y(y)dy = \frac{1}{3}a^3 + a^3 + \frac{1}{2}a^3 = \frac{11}{6}a^3 \quad \Rightarrow \quad a = \left(\frac{6}{11}\right)^{1/3}.$$

Mean Value:

$$\mathbb{E}[Y] = \int_0^a y^3 dy + \int_a^{2a} a^2 y dy + \int_{2a}^{3a} y(3a^2 - ay)dy = \frac{1}{4}a^4 + \frac{3}{2}a^4 + \frac{7}{6}a^4 = \frac{35}{12}a^4.$$

Variance:

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{317}{60}a^5 - \frac{1225}{144}a^8.$$

Problem 6

Let X be a random variable with the following probability density function,

$$f_X(x) = \begin{cases} \frac{1}{2}\sin(x) & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Solution:

$$\mathbb{E}[X] = \frac{1}{2}\int_0^{\pi} x\sin(x)dx = \frac{1}{2}(-x\cos(x) + \sin(x)\big|_0^{\pi}) = \frac{\pi}{2}.$$

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{2}(-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\big|_0^{\pi}) - \frac{\pi^2}{4} \\ &= \frac{\pi^2}{4} - 2.\end{aligned}$$

Problem 7

Find the mean and variance of the random variable X^2 with the following distributions:
(i) $X \sim N(\mu, \sigma^2)$
(ii) $X \sim P(\lambda)$
(iii) $X \sim \exp(\lambda)$

Solution:

(i) The moment generating function is

$$M_x(t) = e^{\mu t} e^{\sigma^2 t^2/2}.$$

Hence,

$$\begin{aligned}\mathbb{E}[X^2] &= \frac{d^2}{dt^2} M_x(t)\big|_{t=0} = \mu^2 + \sigma^2. \\ \mathbb{E}[X^4] &= \frac{d^4}{dt^4} M_x(t)\big|_{t=0} = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \\ \mathrm{Var}(X^2) &= 4\mu^2\sigma^2 + 2\sigma^4.\end{aligned}$$

(ii)

$$\begin{aligned}\mathbb{E}[X^2] &= \mathrm{Var}(X) + (\mathbb{E}[X])^2 = \lambda + \lambda^2. \\ \mathbb{E}[X^4] &= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4 \\ \mathrm{Var}(X^2) &= \lambda + 6\lambda^2 + 4\lambda^3.\end{aligned}$$

(iii) Note that

$$\mathbb{E}[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx = \frac{n!}{\lambda^n}$$

Hence,

$$\begin{aligned}\mathbb{E}[X^2] &= \frac{2}{\lambda^2}. \\ \mathbb{E}[X^4] &= \frac{24}{\lambda^4} \\ \mathrm{Var}(X^2) &= \frac{20}{\lambda^4}.\end{aligned}$$

Problem 8

Find the mean and variance of the following random variables.
(i) $X \sim Gamma(\alpha, \beta)$
(ii) $E(X) = \mu, Var(X) = \sigma^2$. Find mean and variance of $Y = aX + b$

Solution:

(i)

$$f_X(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}, \quad \text{where} \quad \Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

Then

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{\beta^\alpha \int_0^\infty y^{\alpha-1}e^{-y}dy} \int_0^\infty x x^{\alpha-1} e^{-x/\beta} dx \\ &= \beta \frac{1}{\int_0^\infty y^{\alpha-1}e^{-y}dy} \int_0^\infty z^\alpha e^{-z} dz \\ &= \beta\alpha \\ \mathrm{Var}(X) &= \alpha\beta^2.\end{aligned}$$

(ii)

$$\mathbb{E}[Y] = \mathbb{E}[aX + b] = \int (ax + b) f_X(x) dx = a\mathbb{E}[X] + b.$$

$$\text{Var}(Y) = \mathbb{E}[(aX + b - (a\mathbb{E}[X] + b))^2] = a^2\text{Var}(X).$$

ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 4

Prof. Pillai

Problem 1

The random variable X is $\mathcal{N}(5,2)$ and $Y = 2X + 4$. Find the mean, variance of Y and $f_Y(y)$.

Solution:
The r.v. Y is still Gaussian distributed.

$$\begin{aligned} E[Y] &= E[2X+4] = 2E[X]+4 = 14, \\ \mathrm{Var}(Y) &= E[Y^2] - (E[Y])^2 = E[(2X+4)^2] - (2E[X]+4)^2 = 4\mathrm{Var}(X) = 8, \\ f_Y(y) &= \frac{1}{4\sqrt{\pi}} e^{-\frac{(y-14)^2}{16}}, \quad -\infty < y < \infty. \end{aligned}$$

Problem 2

The random variable X is $P(5)$ and $Y = 2X + 4$. Find the mean, variance of Y and $f_Y(y)$.

Solution:
Since X is Poisson distributed with parameter 5, $\mathbb{E}[X] = 5$, $\mathrm{Var}(X) = 5$.

$$\begin{aligned} E[Y] &= E[2X+4] = 2E[X]+4 = 14, \\ \mathrm{Var}(Y) &= E[Y^2] - (E[Y])^2 = E[(2X+4)^2] - (2E[X]+4)^2 = 4\mathrm{Var}(X) = 20. \end{aligned}$$

If $k = 2k' + 4$, $k' = 0, 1, ...$

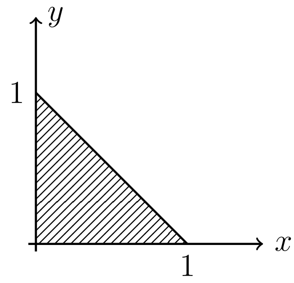
$$P(Y = k) = P(2X + 4 = 2k' + 4) = P(X = k') = e^{-5}\frac{5^{k'}}{k'!}.$$

Otherwise

$$P(Y = k) = 0.$$

Problem 3

Given the joint probability density function $f_{XY}(x, y)$ as,



$$f_{XY}(x, y) = \begin{cases} kxy, & (x, y) \in \text{shaded area} \\ 0 & \text{otherwise} \end{cases}$$

a. Find k, $f_X(x)$ and $f_Y(y)$.

b. Are X and Y independent?

Solution:

a.

$$1 = \int_0^1 \int_0^{1-x} kxydydx = \frac{k}{2} \int_0^1 (1-x)^2 xdx = \frac{k}{24} \quad \Rightarrow \quad k = 24.$$

$$f_X(x) = \int_0^{1-x} kxydy = 12(1-x)^2 x, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^{1-y} kxydx = 12(1-y)^2 y, \quad 0 \le y \le 1$$

b. No, as $f_{XY}(x, y) \neq f_X(x) f_Y(y)$.

Problem 4

X and Y are jointly distributed random variables with joint p.d.f

$$f_{XY}(x, y) = \begin{cases} e^{-x} & \infty > x > y > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Find $f_X(x)$ and $f_Y(y)$.

b. Are X and Y independent?

Solution:

a.

$$f_X(x) = \int_0^x e^{-x} dy = xe^{-x}, \quad x > 0,$$

$$f_Y(y) = \int_y^\infty e^{-x} dx = e^{-y}, \quad y > 0.$$

b. No, as $f_{XY}(x, y) \neq f_X(x) f_Y(y)$.

Problem 5

X and Y are jointly distributed random variables with joint p.d.f

$$f_{XY}(x,y) = \begin{cases} k & 0 < x < y < a \\ 0 & \text{otherwise} \end{cases}$$

a. Find k, $f_X(x)$ and $f_Y(y)$.

b. Are X and Y independent?

Solution:

a.

$$1 = \int_0^a \int_x^a k dy dx = \int_0^a k(a-x)dx = \frac{ka^2}{2} \quad \Rightarrow \quad k = \frac{2}{a^2}.$$

$$f_X(x) = \int_x^a k dy = \frac{2}{a^2}(a-x), \quad 0 < x < a$$

$$f_Y(y) = \int_0^y k dx = \frac{2}{a^2}y, \quad 0 < y < a$$

b. No, as $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.

Problem 6

a. X and Y are jointly distributed random variables with joint p.d.f

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

b. Given the joint probability dendity function $f_{XY}(x,y)$ as

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

Show that X and Y are independent random variables

Solution:

a.

$$f_X(x) = \int_0^\infty e^{-(x+y)} dy = e^{-x}, \quad x \geq 0,$$

$$f_Y(y) = \int_0^\infty e^{-(x+y)} dx = e^{-y}, \quad y \geq 0.$$

As $f_{XY}(x,y) = f_X(x)f_Y(y)$, they are independent.

b.

$$f_X(x) = \frac{1}{2\pi} \int_0^\infty e^{-(x^2+y^2)/2} dy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$f_Y(y) = \frac{1}{2\pi} \int_0^\infty e^{-(x^2+y^2)/2} dx = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

As $f_{XY}(x,y) = f_X(x)f_Y(y)$, they are independent.

ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 5

Prof. Pillai

Problem 1

The joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f of

a.) $Z = X + Y$.

b.) $Z = X - Y$.

c.) $Z = X/Y$.

Solution:

a.) $Z = X + Y$:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X + Y \le z) = P(Y \le z - X) \\ &= \int_0^{\frac{z}{2}} \int_x^{z-x} e^{-y} dy dx \\ f_Z(z) &= \int_0^{\frac{z}{2}} \frac{\mathrm{d}}{\mathrm{d}z} \int_x^{z-x} e^{-y} dy dx = \int_0^{\frac{z}{2}} 1 \cdot e^{-(z-x)} dx \\ &= e^{-z} \int_0^{\frac{z}{2}} e^x dx = e^{-z}(e^{\frac{z}{2}} - 1). \end{aligned}$$

Therefore

$$f_Z(z) = e^{-\frac{z}{2}} - e^{-z}, \qquad z \ge 0.$$

b.) $Z = X - Y$:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X - Y \le z) = 1 - P(Y \le X - z) \\ &= 1 - \int_0^{\infty} \int_x^{x-z} e^{-y} dy dx, \\ f_Z(z) &= -\frac{\mathrm{d}}{\mathrm{d}z} \int_z^{\infty} \int_x^{x-z} e^{-y} dy dx \\ &= -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^{\infty} \left(e^{-x} - e^{-(x-z)}\right) dx \\ &= e^z. \end{aligned}$$

Therefore

$$f_Z(z) = e^z, \qquad z \leq 0.$$

c.) $Z = X/Y$.

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) = P(X/Y \leq z) = 1 - P(Y \leq X/z) \\
&= 1 - \int_0^\infty \int_x^{x/z} e^{-y} dy dx, \\
f_Z(z) &= -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^\infty \int_x^{x/z} e^{-y} dy dx \\
&= -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^\infty \left(e^{-x} - e^{-x/z}\right) dx \\
&= -\frac{d}{dz}(-z) \\
&= 1.
\end{aligned}$$

Therefore

$$f_Z(z) = 1, \qquad 0 < z < 1.$$

Problem 2

X and Y are independent and uniform in the interval $(0, a)$. Find the p.d.f. of $Z = X - Y$.

Solution:

$$F_Z(z) = P(Z \leq z) = P(X - Y \leq z) = P(Y \geq X - z).$$

Case I: $z < 0$

$$\begin{aligned}
F_Z(z) &= P(Y \geq X - z) = \int_0^{a+z} \int_{x-z}^{a} \frac{1}{a^2} dydx, \\
f_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \int_0^{a+z} \int_{x-z}^{a} \frac{1}{a^2} dydx \\
&= 1 \cdot \int_{a+z-z}^{a} \frac{1}{a^2} dy - 0 + \int_0^{a+z} \frac{\mathrm{d}}{\mathrm{d}z} \int_{x-z}^{a} \frac{1}{a^2} dydx \\
&= \int_0^{a+z} (0 - (-1) \cdot \frac{1}{a^2}) dx \\
&= \frac{a+z}{a^2}.
\end{aligned}$$

Case II: $z \geq 0$

$$\begin{aligned}
F_Z(z) &= 1 - \int_0^{a-z} \int_{y+z}^{a} \frac{1}{a^2} dxdy \\
f_Z(z) &= -\int_0^{a-z} \frac{\mathrm{d}}{\mathrm{d}z} \int_{y+z}^{a} \frac{1}{a^2} dxdy = -\int_0^{a-z} -\frac{1}{a^2} dy \\
&= \frac{1}{a^2}(a - z)
\end{aligned}$$

Therefore,

$$f_Z(z) = \begin{cases} \frac{a+z}{a^2}, & z < 0, \\ \frac{a-z}{a^2}, & z \geq 0. \end{cases}$$

Problem 3

X and Y are independent exponential random variables with parameters α and β respectively, i.e.,

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \begin{cases} \alpha\beta e^{-(\alpha x+\beta y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define $Z = \min(X, 3Y)$. Show that Z is also an exponential random variable, and find the value of corresponding exponential parameter.

Solution: By definition,

$$Z = \min(X, 3Y) = \begin{cases} 3Y, & X \geq 3Y, \\ X, & X < 3Y. \end{cases}$$

With $z \geq 0$,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P((Z \leq z) \cap ((X \geq 3Y) \cup (X < 3Y))) \\ &= P((Z \leq z) \cap (X \geq 3Y)) + P((Z \leq z) \cap (X < 3Y)) \\ &= P((3Y \leq z) \cap (X \geq 3Y)) + P((X \leq z) \cap (X < 3Y)) \\ &= \int_0^{z/3} \int_{3y}^{\infty} f_{XY}(x,y)dxdy + \int_0^{z} \int_{x/3}^{\infty} f_{XY}(x,y)dydx \\ f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \frac{1}{3}\int_z^{\infty} \alpha\beta e^{-(\alpha x+\beta z/3)}dy + \int_{z/3}^{\infty} \alpha\beta e^{-(\alpha z+\beta y)}dx \\ &= \left(\alpha + \frac{\beta}{3}\right) e^{-\left(\alpha+\frac{\beta}{3}\right)z}, \quad z \geq 0. \end{aligned}$$

Therefore,

$$Z \sim \text{Exponential}\left(\alpha + \frac{\beta}{3}\right).$$

Problem 4

Given the joint density function

$$f_{XY}(x,y) = \begin{cases} xye^{-(x+y)} & x > 0, y > 0, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$Z = \frac{\min(X,Y)}{\max(X,Y)}.$$

Determine the p.d.f of Z.

Solution: By definition,

$$Z = \frac{\min(X,Y)}{\max(X,Y)} = \begin{cases} Y/X, & X \geq Y, \\ X/Y, & X < Y. \end{cases}$$

With $0 < z \leq 1$,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) = P((Z \leq z) \cap ((X \geq Y) \cup (X < Y))) \\
&= P((Z \leq z) \cap (X \geq Y)) + P((Z \leq z) \cap (X < Y)) \\
&= P((Y/z \leq X) \cap (X \geq Y)) + P((X \leq Yz) \cap (X < Y)) \\
&= P(Y/z \leq X)) + P(X \leq Yz)) \\
&= \int_0^\infty \int_0^{xz} f_{XY}(x,y)dydx + \int_0^\infty \int_0^{yz} f_{XY}(x,y)dxdy. \\
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \int_0^\infty x f_{XY}(x,xz)dx + \int_0^\infty y f_{XY}(yz,y)dy \\
&= \int_0^\infty x[f_{XY}(xz,x) + f_{XY}(x,xz)]dx \\
&= \int_0^\infty x[xzxe^{-(xz+x)} + xxze^{-(x+xz)}]dx \\
&= \frac{12z}{(1+z)^4}, \qquad 0 < z < 1.
\end{aligned}$$

Problem 5

X and Y are independent random variables with geometric p.m.f

$$P(X = k) = pq^k, k = 0, 1, 2, ...,$$
$$P(Y = m) = pq^m, m = 0, 1, 2.....$$

Find the p.m.f. of $Z = X + Y$.

Solution:

$$\begin{aligned} P(Z = k) &= \sum_{m=0}^{k} P(X = m)P(Y = k - m) \\ &= \sum_{m=0}^{k} pq^m \cdot pq^{k-m} = \sum_{m=0}^{k} p^2 q^k \\ &= p^2 q^k (k + 1). \end{aligned}$$

Therefore,

$$P(Z = k) = p^2 q^k (k + 1), \qquad k = 0, 1, 2, \ldots$$

.

Problem 6

X and Y are random variables with joint p.d.f.

$$f_{XY}(x, y) = \begin{cases} ke^{-(x+y)} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the p.d.f. of $Z = X - Y$.

Solution:

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X - Y \leq z) = P(X \leq Y + z) \\ &= \int_0^\infty \int_y^{y+z} ke^{-(x+y)} dxdy, \\ &= k\frac{1 - e^{-z}}{2}, \\ f_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = \frac{1}{2} ke^{-z}. \end{aligned}$$

Since $\lim_{z\to\infty} F(z) = 1$, $k = 2$. Therefore,

$$f_Z(z) = e^{-z}, \qquad z \geq 0.$$

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Solution to Homework # 6

Prof. Pillai

Problem 1

Let

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \qquad W = Y/X$$

a.) Find $f_{ZW}(z,w)$.

b.) Are Z and W independent random variables? Prove your answer.

Solution:

a.)

$$\begin{cases} z = x + y \\ w = y/x \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = wz/(w+1) \\ x_1 = z/(w+1) \end{cases}$$

Then,

$$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{w+1} & -\frac{z}{(w+1)^2} \\ \frac{w}{w+1} & \frac{z}{(w+1)^2} \end{vmatrix} = \frac{z}{(w+1)^2}$$

$$f_{ZW}(z,w) = \frac{z}{(w+1)^2} f_{XY}(x_1,y_1) = \frac{2e^{-z}z}{(w+1)^2}, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$x < y < \infty \;\Rightarrow\; \frac{z}{w+1} < \frac{wz}{w+1} < \infty \;\Rightarrow\; 1 < w < \infty,$$
$$0 < x < y \;\Rightarrow\; 0 < \frac{z}{w+1} < \frac{wz}{w+1} \;\Rightarrow\; 0 < z < wz \;\Rightarrow\; 0 < z < \infty.$$

b.) Marginal distributions:

$$f_Z(z) = \int_1^\infty f_{ZW}(z,w)dw = ze^{-z}, \quad z \geq 0,$$
$$f_W(w) = \int_0^\infty f_{ZW}(z,w)dz = \frac{2}{(w+1)^2}, \quad w \geq 1.$$

They are independent.

Problem 2

Given the joint density function

$$f_{XY}(x,y) = \begin{cases} 2e^{-(2x-y)} & 0 < y < x < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

and the two functions

$$Z = 2X - Y, \qquad W = Y/X.$$

a.) Find $f_{ZW}(z,w)$.

b.) Are Z and W independent random variables? Prove your answer.

Solution:

a.)

$$\begin{cases} 2x - y = z \\ y/x = w \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = zw/(2-w) \\ x_1 = z/(2-w) \end{cases}$$

Then,

$$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial 2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2-w} & \frac{z}{(2-w)^2} \\ \frac{w}{2-w} & \frac{2z}{(2-w)^2} \end{vmatrix} = \frac{z}{(2-w)^2},$$

and

$$f_{ZW}(z,w) = \frac{z}{(2-w)^2} f_{XY}(x_1,y_1) = \frac{2e^{-z}z}{(2-w)^2}, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$\begin{aligned} &x < y < \infty \ \Rightarrow z > 0, \\ &0 < x < y \ \Rightarrow \ 0 < \frac{wz}{2-w} < \frac{z}{2-w} \ \Rightarrow \ 0 < w < 1. \end{aligned}$$

b.) Marginal distributions:

$$\begin{aligned} f_Z(z) &= \int_0^1 \frac{2e^{-z}z}{(2-w)^2} dw = ze^{-z}, \quad z \geq 0, \\ f_W(w) &= \int_0^\infty \frac{2e^{-z}z}{(2-w)^2} dz = \frac{2}{(2-w)^2}, \quad 0 < w < 1. \end{aligned}$$

They are independent.

Problem 3

The joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 2xye^{-(x+y)} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \quad W = X/Y$$

a.) Find $f_{ZW}(z,w)$.

b.) Are Z and W independent random variables?

c.) Are Z and W uncorrelated random variables?

Solution:

a.)

$$\begin{cases} x + y = z \\ x/y = w \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = z/(w+1) \\ x_1 = wz/(w+1) \end{cases}$$

Then,

$$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{w}{w+1} & \frac{z}{(w+1)^2} \\ \frac{1}{w+1} & -\frac{z}{(w+1)^2} \end{vmatrix} = -\frac{z}{(w+1)^2},$$

and

$$f_{ZW}(z,w) = \frac{z}{(w+1)^2} f_{XY}(x_1,y_1) = \frac{z}{(w+1)^2} \frac{2wz^2}{(w+1)^2} e^{-z} = \frac{2wz^3e^{-z}}{(w+1)^4}, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$x < y < \infty \;\Rightarrow\; \frac{z}{w+1} < \frac{wz}{w+1} < \infty \;\Rightarrow\; 1 < w < \infty,$$
$$0 < x < y \;\Rightarrow\; 0 < \frac{z}{w+1} < \frac{wz}{w+1} \;\Rightarrow\; 0 < z < wz \;\Rightarrow\; 0 < z < \infty.$$

b.) Marginal distributions:

$$f_Z(z) = \int_1^\infty f_{ZW}(z,w)dw = \frac{1}{6} z^3 e^{-z}, \quad z \geq 0,$$
$$f_W(w) = \int_0^\infty f_{ZW}(z,w)dz = \frac{12w}{(w+1)^4}, \quad w \geq 1.$$

They are independent.

Problem 4

The joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{3}{4}(x+y)^2 & 0 < x < 1, -1 < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \qquad W = X - Y$$

a.) Find $f_{ZW}(z,w)$, $f_Z(z)$ and $f_W(w)$.

b.) Are Z and W independent random variables?

c.) Are Z and W uncorrelated random variables?

d.) Are Z and W orthogonal random variables ($E[ZW] = 0$)? Prove your answers.

Solution:

a.)

$$\begin{cases} x + y = z \\ x - y = w \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = (z-w)/2 \\ x_1 = (z+w)/2 \end{cases}$$

Then,

$$f_{ZW}(z,w) = f_{XY}(x_1,y_1) \cdot \begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \frac{3}{4}z^3 \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{3}{8}z^2, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$\mathcal{A} = \{(z,w) : \ 0 < z + w < 2, -2 < z - w < 2\}.$$

b.) Marginal distributions:

$$f_Z(z) = \begin{cases} \frac{3}{4}z^2(z+1) & -1 < z < 0, \\ \frac{3}{4}z^2 & 0 \le z < 1, \\ \frac{3}{4}z^2\cancel{(z-2)} & 1 \le z < 2, \end{cases} \qquad f_W(w) = \begin{cases} \frac{w^3+3w^2+6w+4}{4} & -1 < w < 0, \\ \frac{3w^2-6w+4}{4} & 0 \le w < 1, \\ \frac{1}{4}(2-w)^3 & 1 \le w < 2. \end{cases}$$

$(2-z)$

They are not independent.

c.)

$$E[ZW] = \int_{-1}^{0}\int_{-w}^{w+2} \frac{3}{8}z^3 w\,dz\,dw + \int_{0}^{1}\int_{-w}^{2-w} \frac{3}{8}z^3 w\,dz\,dw + \int_{1}^{2}\int_{w-2}^{2-w} \frac{3}{8}z^3 w\,dz\,dw$$

Upon solving we get $E[ZW] = 0$.

$$E[Z] = \int_{-1}^{0} \frac{3}{4}z^3(z+1)dz + \int_{0}^{1} \frac{3}{4}z^2 3dz + \int_{1}^{2} \frac{3}{4}z^3(z-2)dz = \frac{9}{8}.$$

Similarly,

$$E[W] = \frac{1}{8}$$

Since as $E[ZW] - E[Z]E[W] \neq 0$, they are correlated.

d.) Yes.

Problem 5

X, Y are independent, idential geometric random variables with common parameter p, i.e., with $q = 1-p$,

$$P(X = k) = P(Y = k) = pq^k, \quad k = 0, 1, 2, ...$$

a.) $Z = X + Y$, $W = \min\{X, Y\}$, find $f_{ZW}(z, w)$, $f_Z(z)$ and $f_W(w)$.

b.) $Z = \min\{X, Y\}$, $W = X - Y$, find $f_{ZW}(z, w)$, $f_Z(z)$ and $f_W(w)$.

Solution:

a.) Look at 16:30 on $Z = X + Y$, $W = \max(X, Y)$:

https://youtu.be/TU3Y9bagw9w

b.) $Z = \min(X, Y)$, $W = X - Y$:

https://youtu.be/V1EyqL1cqTE

(a) $$P(Z=m, W=n) = \begin{cases} p^2 q^m & m = 2n \\ 2p^2 q^m & m > 2n \end{cases}$$

$$P(Z=m) = (m+1) p^2 q^m \qquad m = 0, 1, 2, \ldots$$

$$P(W=n) = (q+1) p q^{2n} \qquad n = 0, 1, 2, \ldots$$

(b) $$P(Z=m, W=n) = p^2 q^{2m+|n|} \qquad m = 0, 1, 2, \ldots \quad n = 0, \pm 1, \pm 2, \ldots$$

$$P(Z=m) = (\overset{q}{\cancel{m}}+1) p q^{2m} \qquad m = 0, 1, 2, \ldots$$

$$P(W=n) = \frac{p}{1+q} \cdot q^{|n|} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Problem 6

X and Y are independent Geometric random variables with common parameter p, i.e., $P(X=k)=P(Y=k)=pq^k$ with $q=1-p$. Define

$$Z = X+Y, \qquad W = |X-Y|.$$

Find

a.) $P(Z=m, W=k)$

b.) $P(Z=m)$

c.) $P(W=k)$

Solution:

a.) When $X \geq Y$,

$$\begin{cases} X+Y=m, \\ X-Y=k, \end{cases} \Rightarrow \begin{cases} X=(m+k)/2, \\ Y=(m-k)/2. \end{cases}$$

$$\begin{aligned} X\geq Y \quad &\Rightarrow \quad (m+k)/2 \geq (m-k)/2 \quad \Rightarrow \quad k\geq 0,\\ X\geq 0 \quad &\Rightarrow \quad (m+k)/2\geq 0 \quad \Rightarrow \quad m\geq -k,\\ Y\geq 0 \quad &\Rightarrow \quad (m-k)/2\geq 0 \quad \Rightarrow \quad m\geq k. \end{aligned}$$

When $X < Y$,

$$\begin{cases} X+Y=m, \\ Y-X=k, \end{cases} \Rightarrow \begin{cases} X=(m-k)/2, \\ Y=(m+k)/2. \end{cases}$$

$$\begin{aligned} X< Y \quad &\Rightarrow \quad (m-k)/2 < (m+k)/2 \quad \Rightarrow \quad k> 0,\\ X\geq 0 \quad &\Rightarrow \quad (m-k)/2\geq 0 \quad \Rightarrow \quad m\geq k,\\ Y\geq 0 \quad &\Rightarrow \quad (m+k)/2\geq 0 \quad \Rightarrow \quad m\geq -k. \end{aligned}$$

$$\begin{aligned} P(Z=m,W=k) &= P(Z=m,W=k,X\geq Y)+P(Z=m,W=k,X<Y)\\ &= P(X+Y=m,X-Y=k,X\geq Y)+P(X+Y=m,Y-X=k,X<Y)\\ &= \begin{cases} P(X=\frac{m+k}{2})P(Y=\frac{m-k}{2}) & k=0, m=0,2,4,..\\ P(X=\frac{m+k}{2})P(Y=\frac{m-k}{2})+P(X=\frac{m-k}{2})P(Y=\frac{m+k}{2}) & k=0,1,2,...,m\geq k\\ 0 & \text{otherwise} \end{cases}\\ &= \begin{cases} p^2q^m & k=0, m=0,2,4,...\\ 2p^2q^m & \cancel{k=0,1,2,...,m=k+2,k+4,...} \quad k=1,2,\ldots \quad m=k,k+2,k+4,\ldots\\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

b.)

$$P(Z=m)=\sum_k P(Z=m,W=k) = \begin{cases} p^2 & m=0\\ p^2q^m+\sum_{k=0,2,4,...}^m 2p^2q^m & m=2,4,6,...\\ \sum_{k=1,3,5,...}^m 2p^2q^m & m=1,3,5,... \end{cases}$$

$$P(Z=m)=(m+1)p^2q^m$$

c.)

$$P(W=k)=\sum_m P(Z=m,W=k) = \begin{cases} \cancel{p^2} & k=0\\ \cancel{\sum_{m=k,k+2,k+4,...}^\infty} 2p^2q^m & k>0 \end{cases}$$

$$P(W=n)=\begin{cases} \dfrac{p}{1+q} & n=0\\ \dfrac{2pq^n}{1+q} & n=1,2,\ldots \end{cases}$$

Midterm Exam

1. *(15 points)* Given $X \sim \text{Uniform}(-1, 1)$, and $Y = \dfrac{1-X}{1+X}$. Find the probability density function (p.d.f.) of Y.

2. *(20 points)* Given the joint probability density function

$$f_{XY}(x,y) = \begin{cases} \dfrac{2}{x^2y^2}, & x > y > 1 \\ 0 & \text{otherwise,} \end{cases}$$

Find the p.d.f. of $Z = \dfrac{X}{Y}$.

3. *(20 points)* Given the joint probability density function

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)}, & x > y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

Find the p.d.f. of $Z = \dfrac{1}{\max(X,Y)}$.

4. *(35 points)* Joint p.d.f. of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)}, & x > y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

Define $Z = X + Y, W = X - Y$

a.) Find the joint p.d.f of Z and W

b.) Are Z and W independent?

c.) Are Z and W uncorrelated?

d.) Find $\text{Cov}(Z, W)$

5. *(10 points)* Given

$$f_{XY}(x,y) = \begin{cases} e^{-x}, & x > y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

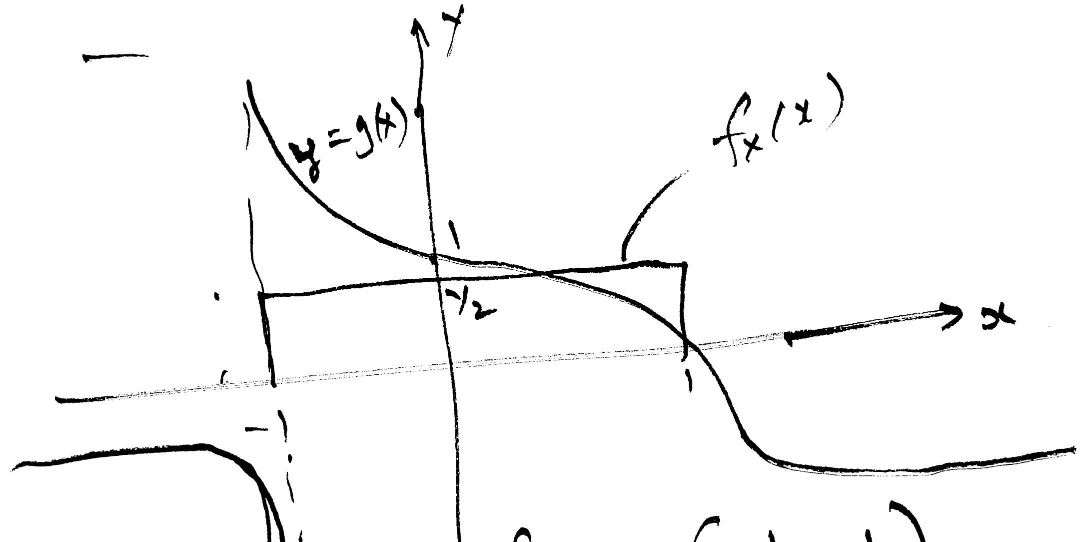
Find the conditional p.d.f of X given Y.

EL 6303 Solutions to Midterm Exam

1) a) Given $x \sim U(-1, 1)$ and

$$Y = \frac{1-x}{1+x}$$

Find the p.d.f. of Z



Notice that as x goes from (−1, 1) y goes from $0 \to \infty$, and there is one solution x_1 for every y. Hence

$$(1+x_1)y = 1 - x_1 \Rightarrow x_1 = \frac{y-1}{1+y}$$

$$\frac{dx_1}{dy} = \frac{(1+y)-(y-1)}{(1+y)^2} = \frac{2}{(y+1)^2}$$

$$f_y(y) = \sum_i \left|\frac{dx_i}{dy}\right| f_x(x_i) = \frac{2}{(y+1)^2} \cdot \frac{1}{2} = \frac{1}{(y+1)^2}, \quad y>0$$

check

$$\int_0^\infty f_y(y)\,dy = \int_0^\infty \frac{1}{(y+1)^2}\,dy = \int_1^\infty \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_1^\infty = \underline{\underline{1}}.$$

2 a)

$$f_{XY}(x,y) = \begin{cases} \dfrac{2}{x^2 y^2}, & x > y > 1 \\ 0 & \text{otherwise} \end{cases}$$

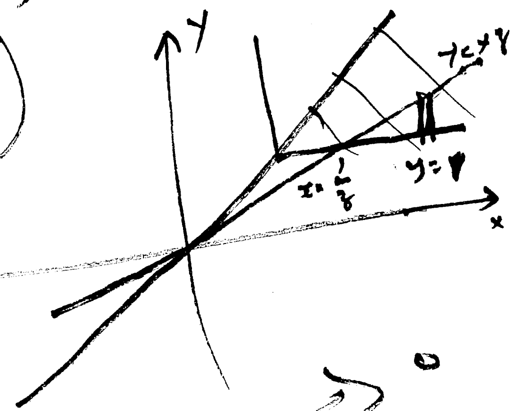
find the p.d.f. of

$$Z = \frac{Y}{X}$$

Solution

$$F_Z(z) = P(Z \le z) = P\left(\frac{Y}{X} \le z\right), \quad 0 < z < 1$$

$$= P(Y \le Xz,\ X > Y > 1)$$



$$= \int_{x=\frac{1}{z}}^{\infty} \int_{y=1}^{xz} f_{XY}(x,y)\, dy\, dx$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{z^2} \int_{y=1}^{1} f_{XY}\left(\frac{1}{z}, y\right) dy \rightarrow 0$$

$$+ \int_{x=\frac{1}{z}}^{\infty} 2x\, f_{XY}(x, xz)\, dx$$

$$= \int_{1/z}^{\infty} 2x \cdot \frac{1}{x^2 (xz)^2}\, dx = \frac{1}{z^2} \int_{1/z}^{\infty} \frac{2}{x^3}\, dx = -\frac{2}{z^2 (2x^2)} \Bigg|_{1/z}^{\infty}$$

$$= \frac{z^2}{z^2} = 1, \quad 0 < z < 1$$

$$\Rightarrow Z \sim U(0,1)$$

3/2 a) $f_{XY}(x,y) = \begin{cases} \dfrac{2}{x^2 y^2}, & x > y > 1 \\ 0 & \text{otherwise} \end{cases}$

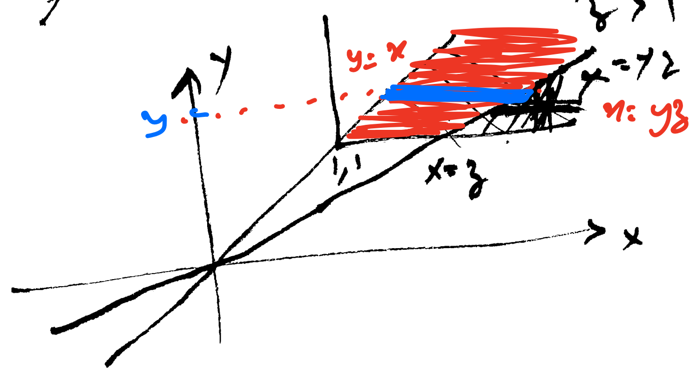
Find the p.d.f of $Z = \dfrac{X}{Y}$.

Solution

$F_Z(z) = P(Z \le z) = P\left(\frac{X}{Y} \le z\right) = P(X \le Yz, X > Y > 1)$

$z > 1$

$$= \int_{y=1}^{\infty} \int_{x=y}^{yz} f_{XY}(x,y)\, dx\, dy$$



$$f_Z(z) = \frac{d F_Z(z)}{dz} = \int_{y=1}^{\infty} (+y)\, f_{XY}(yz, y)\, dy$$

$$= \int_1^{\infty} (+y) \frac{2}{(yz)^2 y^2}\, dy = \frac{2}{z^2} \int_1^{\infty} \frac{1}{y^3}\, dy$$

$$= \frac{2}{z^2}\left(-\frac{1}{2y^2}\right)\Bigg|_1^{\infty} = \frac{1}{z^2}, \quad z \ge 1$$

Check

$$\int_1^{\infty} f_Z(z)\, dz = \int_1^{\infty} \frac{1}{z^2}\, dz = -\frac{1}{z}\Bigg|_1^{\infty} = 1$$

4/30

2b)

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)}, & x > y > 0 \\ 0 & \end{cases}$$

Find the p.d.f of $Z = \dfrac{1}{\max(X,Y)}$

solution

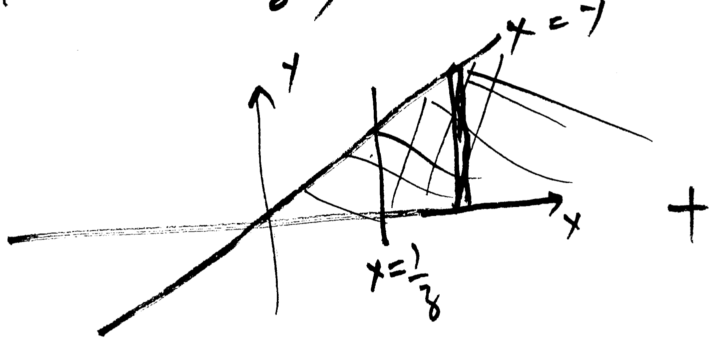
$$Z = \frac{1}{\max(X,Y)} = \begin{cases} \frac{1}{X}, & X \geq Y \\ \frac{1}{Y} & X < Y \end{cases}$$

$$F_Z(z) = P(Z \leq z) = P(\underbrace{Z \leq z}_{A}, \underbrace{(X \geq Y)}_{B} \cup \underbrace{(X<Y)}_{\bar{B}})$$

$$= P(A \cap B \cup \bar{B}) = P(AB) + P(A\bar{B})$$

$$= P\left(\frac{1}{X} \leq z, X \geq Y\right) + P\left(\frac{1}{Y} \leq z, X < Y\right)$$

$$= P\left(X \geq \frac{1}{z}, X \geq Y\right) + \underbrace{P\left(Y \geq \frac{1}{z}, X < Y\right)}_{0}$$



Hence

$$F_Z(z) = \int_{x=\frac{1}{z}}^{\infty} \int_{y=0}^{x} f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{z^2}\left(\int_0^{1/z} f_{XY}\left(\frac{1}{z}, y\right) dy\right)$$

5

$$f_Z(z) = \frac{1}{z^2} \int_0^{1/z} 2e^{-\left(\frac{1}{z}+y\right)} dy$$

$$= \frac{2}{z^2} e^{-\frac{1}{z}} \int_0^{1/z} e^{-y} dy = \frac{2}{z^2} e^{-\frac{1}{z}} \left(1 - e^{-\frac{1}{z}}\right), \quad z \geq 0$$

check

$$\int_0^\infty f_Z(z) = \int_0^\infty \frac{2}{z^2} e^{-\frac{1}{z}} \left(1 - e^{-\frac{1}{z}}\right) dz, \quad u = \frac{1}{z}, \quad du = -\frac{1}{z^2} dz$$

$$= 2\int_0^\infty \left(e^{-u} - e^{-2u}\right) du = 2\left(\frac{e^{-u}}{-1} - \frac{e^{-2u}}{(-2)}\right)\Bigg|_0^\infty = \frac{2}{2} = 1.$$

3) b/

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & , x>y>0 \\ 0 & \text{otherwise} \end{cases}$$

Define

$Z = X+Y \qquad W = X-Y$

a) Find the joint pdf of Z and W

b) Are Z and W independent?

c) Are they uncorrelated? Find Cov(Z, W).

$Z = X+Y$

$W = X-Y$

$x_1 = \frac{z+w}{2}, \quad y_1 = \frac{z-w}{2}$

$0 < z < \infty, \quad 0 < w < \infty, \quad 0 < w < z < \infty.$

$J = \begin{Vmatrix} 1 & 1 \\ 1 & -1 \end{Vmatrix} \Rightarrow |J| = 2$

$f_{ZW}(z,w) = \frac{1}{J} e^{-(x_1+y_1)} = \frac{2}{2} e^{-z} = e^{-z}, \ 0<w<z<\infty$

$f_W(w) = \int_w^\infty \frac{2}{2} e^{-z}\, dz = \frac{e^{-z}}{-1}\Big|_w^\infty = e^{-w}, \ w>0$

$f_Z(z) = \int_0^z \frac{2}{2} e^{-z}\, dw = z e^{-z}, \ z>0$

Z, W are not independent since $f_{ZW}(z,w) \neq f_Z(z) f_W(w)$

$$E(Z) = \int_0^\infty z f_Z(z) = \int_0^\infty z^2 e^{-z} dz = 2$$

$$E(W) = \int_0^\infty w f_W(w) = \int_0^\infty w e^{-w} dw = 1$$

$$E(ZW) = \int_{w=0}^{\infty}\int_{w}^{\infty} zw\, f_{ZW}(z,w)\, dz\, dw$$

$$= \int_{w=0}^{\infty} w \int_w^\infty z e^{-z} dz = \int_0^\infty w \left(-z e^{-z} - e^{-z}\right)\Big|_w^\infty dw$$

$$= \int_0^\infty w \left(w e^{-w} + e^{-w}\right) dw = \int_0^\infty w^2 e^{-w} + \int w e^{-w} dw$$

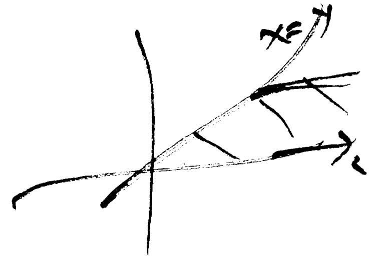
$$= 2 + 1 = \underline{\underline{3}}$$

$$\text{Cov}(Z,W) = E(ZW) - E(Z)E(W)$$

$$= 3 - 2 \cdot 1 = 1 \neq 0$$

$\Rightarrow$ Z, W are correlated.

3) $f_{XY}(x,y) = e^{-x}, \quad x > y > 0$



$$f_Y(y) = \int_y^\infty e^{-x} dx = \frac{e^{-x}}{-1}\Big|_y^\infty = e^{-y}, \quad y > 0$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} e^{-(x-y)}, & x > y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Quiz #1a

Name: ______________________ NetID: ______________________

Problem 1

On holidays the probability that there are no accidents on the highway is twice the probability that there is at least one accident. If we model the accidents occurring on the highway as a Poisson random variable, what is the condition on the Poisson parameter λ?

$$X \sim P(\lambda) \Rightarrow P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,\ldots$$

Prob.{"no accidents"} $= P(X=0) = e^{-\lambda}$ (1)

"At least one accident" $= (X \geq 1)$

$$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$$

Given $P(X=0) = 2P(X \geq 1)$

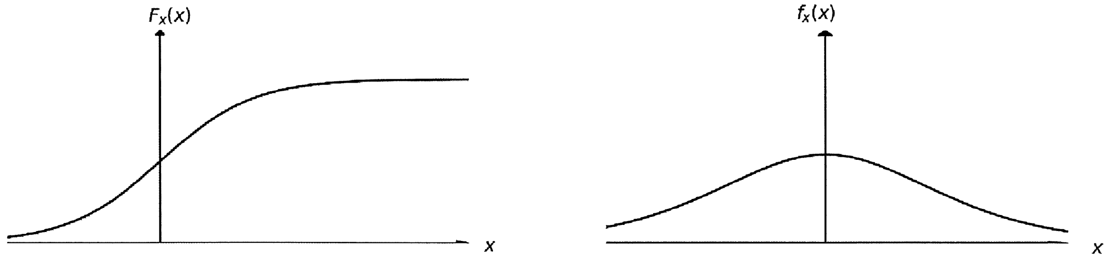
$$= 2(1 - P(X=0))$$

$$\Rightarrow 3P(X=0) = 2 \Rightarrow 3e^{-\lambda} = 2$$

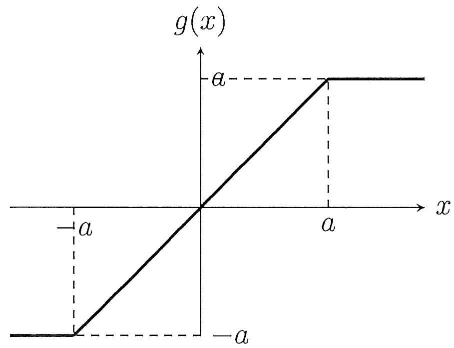
$$e^{\lambda} = \frac{3}{2} \Rightarrow \lambda = \ln\frac{3}{2}$$

Problem 2

Given $F_x(x)$, $f_x(x)$ as shown



Sketch $F_y(y)$ and $f_y(y)$ for $y = g(x)$ when



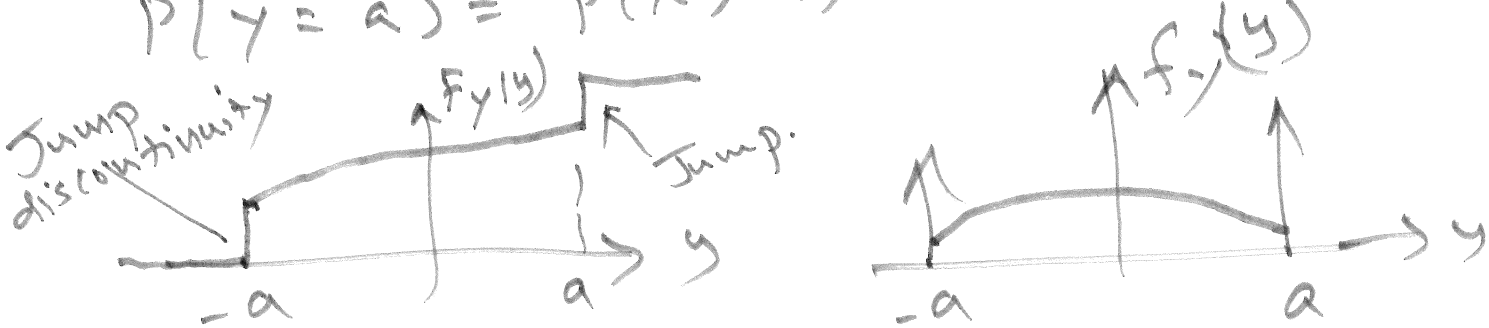
$$Y = \begin{cases} a, & x > a \\ x & -a < x \leq a \quad (-a < y \leq a) \\ -a & x \leq -a \end{cases}$$

$$P(Y = -a) = P(X \leq -a) = F_x(-a)$$

$$-a < y \leq a \Rightarrow Y = X$$

$$F_y(y) = P(Y \leq y) = P(X \leq y) = F_X(y)$$

$$P(Y = a) = P(X > a) = 1 - P(X \leq a) = 1 - F_X(a)$$



Quiz #1b

Name: ____________________ NetID: ____________________

Problem 1

On holidays the probability that there are no accidents on the highway is twice the probability that there is at most one accident. If we model the accidents occurring on the highway as a Poisson random variable, what is the condition on the Poisson parameter λ?

$$X \sim P(\lambda) \Rightarrow P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2\ldots$$

"No accident" $= (X=0)$

"At most one accident" $= \{X=0) \cup (X=1)$

ME

Given

$$P(X=0) = 2P(X\le 1)$$

$$= 2(P(X=0)+P(X=1))$$

$$\Rightarrow e^{-\lambda} = 2(e^{-\lambda}+\lambda e^{-\lambda})$$

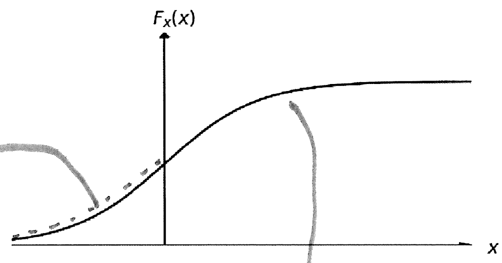
$$\Rightarrow 1 = 2(1+\lambda) \Rightarrow \lambda+1 = \frac{1}{2}$$

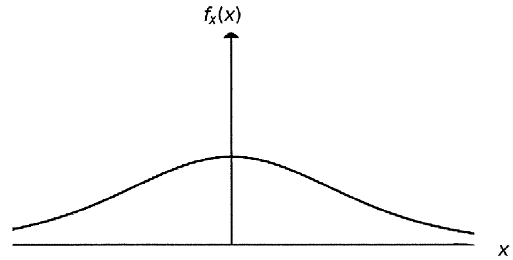
$$\lambda = -\frac{1}{2}$$

For $\lambda = -\frac{1}{2}$, this is unrealistic, since $\lambda > 0$ always.
There is no positive process for which the condition is $-\frac{1}{2}$.

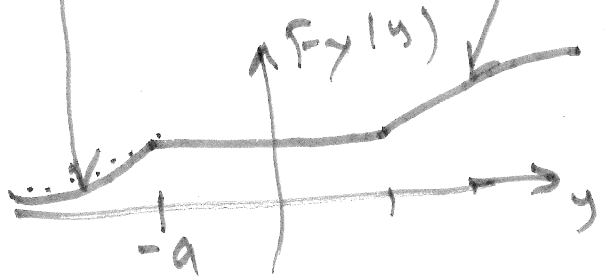
Problem 2

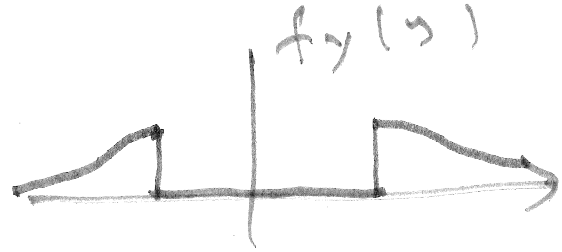
Given $F_x(x)$, $f_x(x)$ as shown

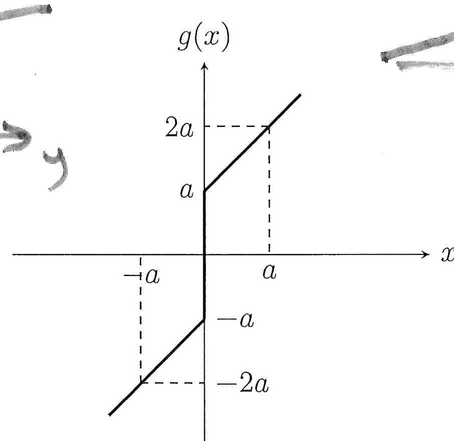




Sketch $F_y(y)$ and $f_y(y)$ for $y = g(x)$ when







$$Y = \begin{cases} x + a, & x > 0 \quad (y > a) \\ x - a, & x \le 0 \quad (y \le -a) \end{cases}$$

Notice Y doesn't take any values in $-a < y < a$.

$y > a$

$$F_y(y) = P(Y \le y) = P(x + a \le y) = P(x \le y - a)$$
$$= F_x(y - a), \quad y > a$$

$y < -a$

$$F_y(y) = P(Y \le y) = P(x - a \le y) = P(x \le y + a)$$
$$= F_x(y + a), \quad y < -a$$

See graph above.

Quiz #1c

Name: ____________________ NetID: ____________________

Problem 1

On holidays the probability that there are no accidents on the highway is twice the probability that there is exactly either one or two accidents. If we model the accidents occurring on the highway as a Poisson random variable, what is the condition on the Poisson parameter λ?

$$X \sim P(\lambda) \Rightarrow P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2$$

"No accidents" $= (X=0)$

"Exactly either one or two accidents"

$$= \underbrace{(X=1) \cup (X=2)}_{ME}$$

Given

$$P(X=0) = 2\left[P(X=1) + P(X=2)\right]$$

$$\Rightarrow e^{-\lambda} = 2\left[\lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda}\right]$$

$$\Rightarrow 1 = 2\lambda + \lambda^2$$

$$= \lambda^2 + 2\lambda - 1 = 0$$

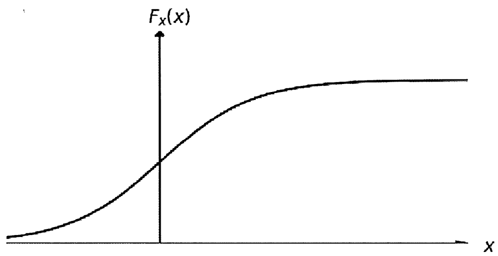
$$\lambda = \frac{-2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

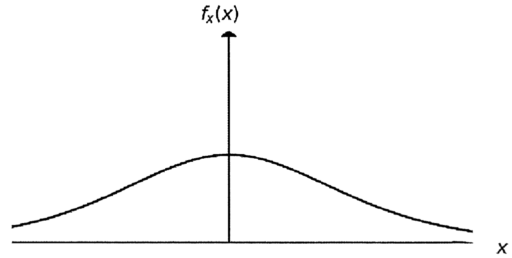
$$\lambda_1 = 1+\sqrt{2} \quad \text{or} \quad \lambda_2 = 1-\sqrt{2}.$$

λ_2 is unrealistic since $\lambda_2 < 0$.
$\lambda_1 > 0$, so λ_1 is the only solution.

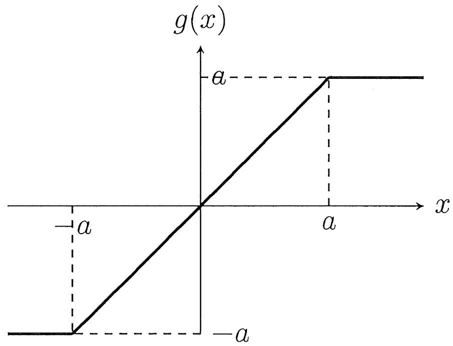
Problem 2

Given $F_x(x)$, $f_x(x)$ as shown





Sketch $F_y(y)$ and $f_y(y)$ for $y = g(x)$ when



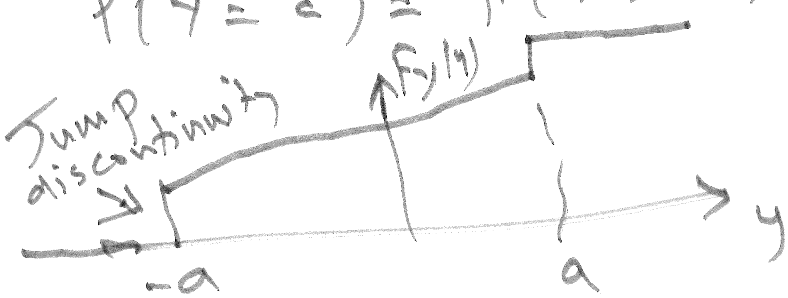
$$Y = \begin{cases} a, & x > a \\ x & -a < x \le a \\ -a, & x \le -a \end{cases}$$

$P(Y = -a) = P(X \le -a) = F_X(-a)$

$-a < y \le a \Rightarrow Y = X$

$F_Y(y) = P(Y \le y) = P(X \le y) = F_X(y)$

$P(Y = a) = P(X > a) = 1 - P(X \le a) = 1 - F_X(a)$

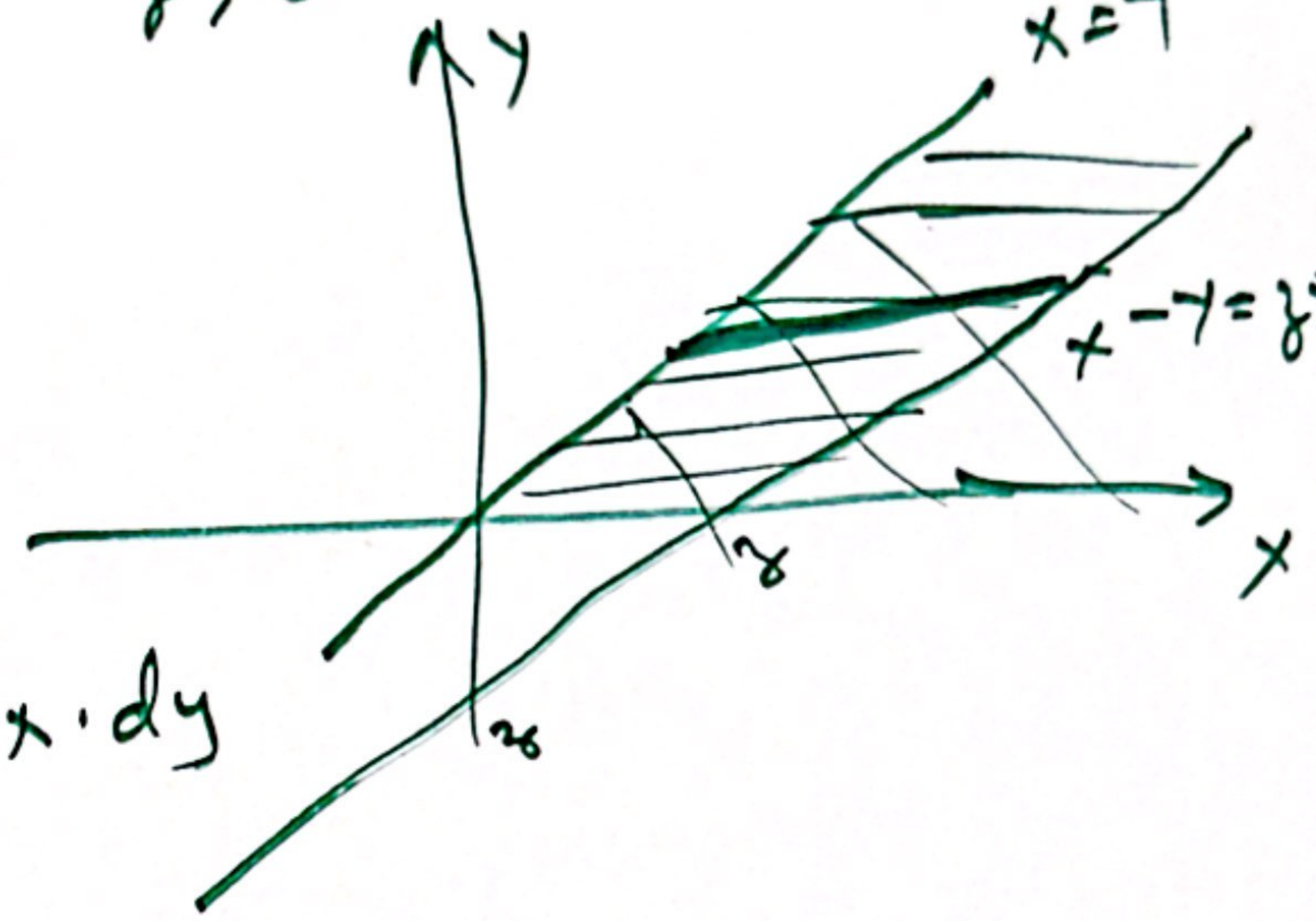


Solutions of Quiz #2

1) a)

$$f_{XY}(x,y) = \begin{cases} e^{-x} & x>y>0 \\ 0 & \text{otherwise} \end{cases}$$

$$Z = X - Y, \Rightarrow z>0$$



$z>0$

$$F_Z(z) = P(Z \le z)$$

$$= P(X-Y \le z)$$

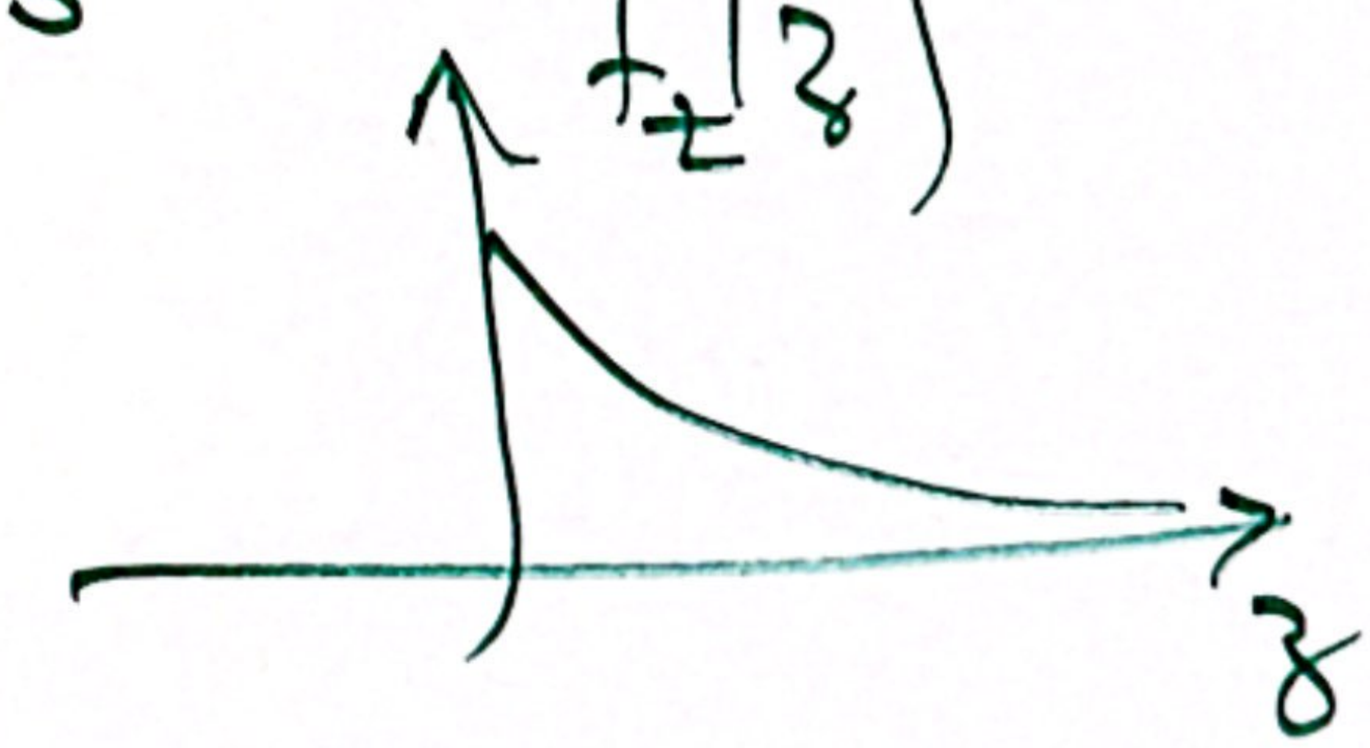
$$= \int_0^\infty \int_{x=y}^{y+z} f_{XY}(x,y)\,dx\cdot dy$$

$$f_Z(z) = \frac{d\,F_Z(z)}{dz} = \int_0^\infty 1\cdot f_{XY}(y+z,y)\,dy$$

$$= \int_0^\infty e^{-(y+z)}\,dy = e^{-z}\underbrace{\int_0^\infty e^{-y}\,dy}_{1}$$

$$= e^{-z}, \quad z>0$$

$\Rightarrow Z$ is exponential

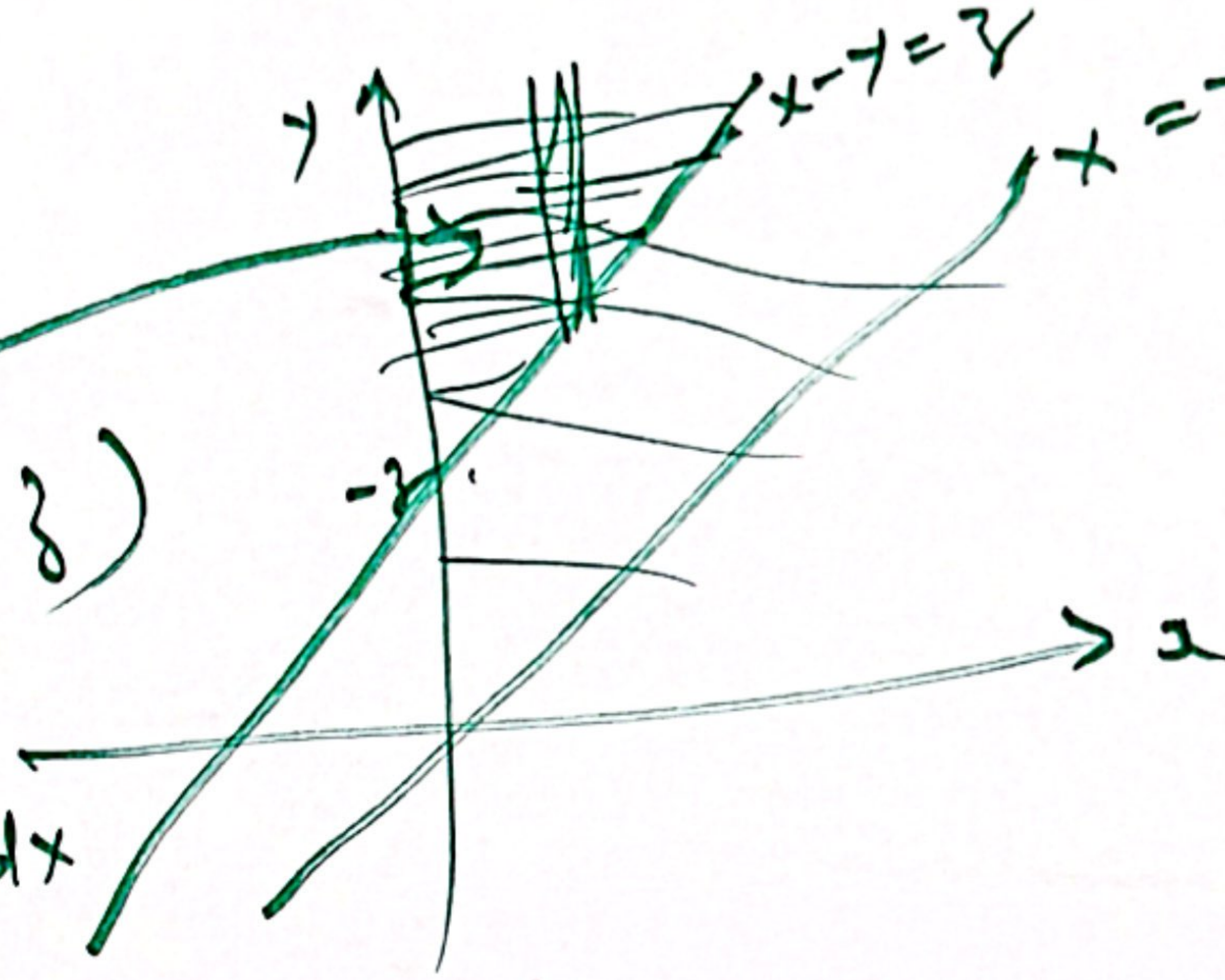


Solution to Quiz #2

2b) $f_{XY}(x,y) = \begin{cases} e^{-y}, & y > x > 0 \\ 0 & \text{otherwise} \end{cases}$

$y > x \Rightarrow x - y \le z < 0$

For $z < 0$



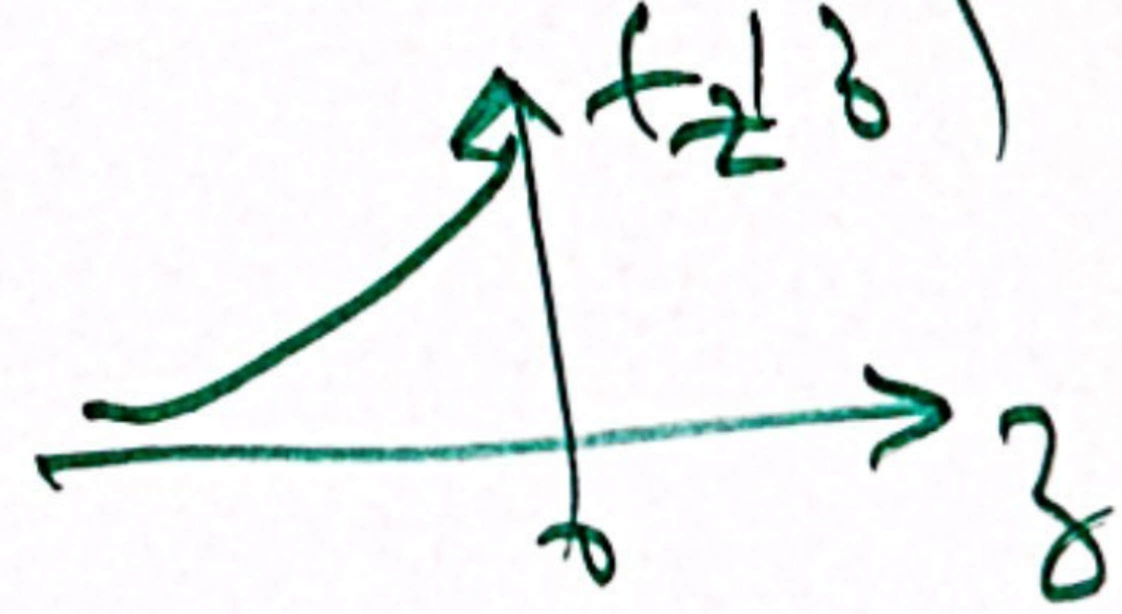
$F_Z(z) = P(X - Y \le z)$

$= \int_0^\infty \int_{x-z}^\infty f_{XY}(x,y)\,dy\,dx$

$f_Z(z) = \int_0^\infty (-1)(-1)\, f_{XY}(x, x-z)\,dx$

$= \int_0^\infty f_{XY}(x, x-z)\,dx = \int_0^\infty e^{-(x-z)}\,dx$

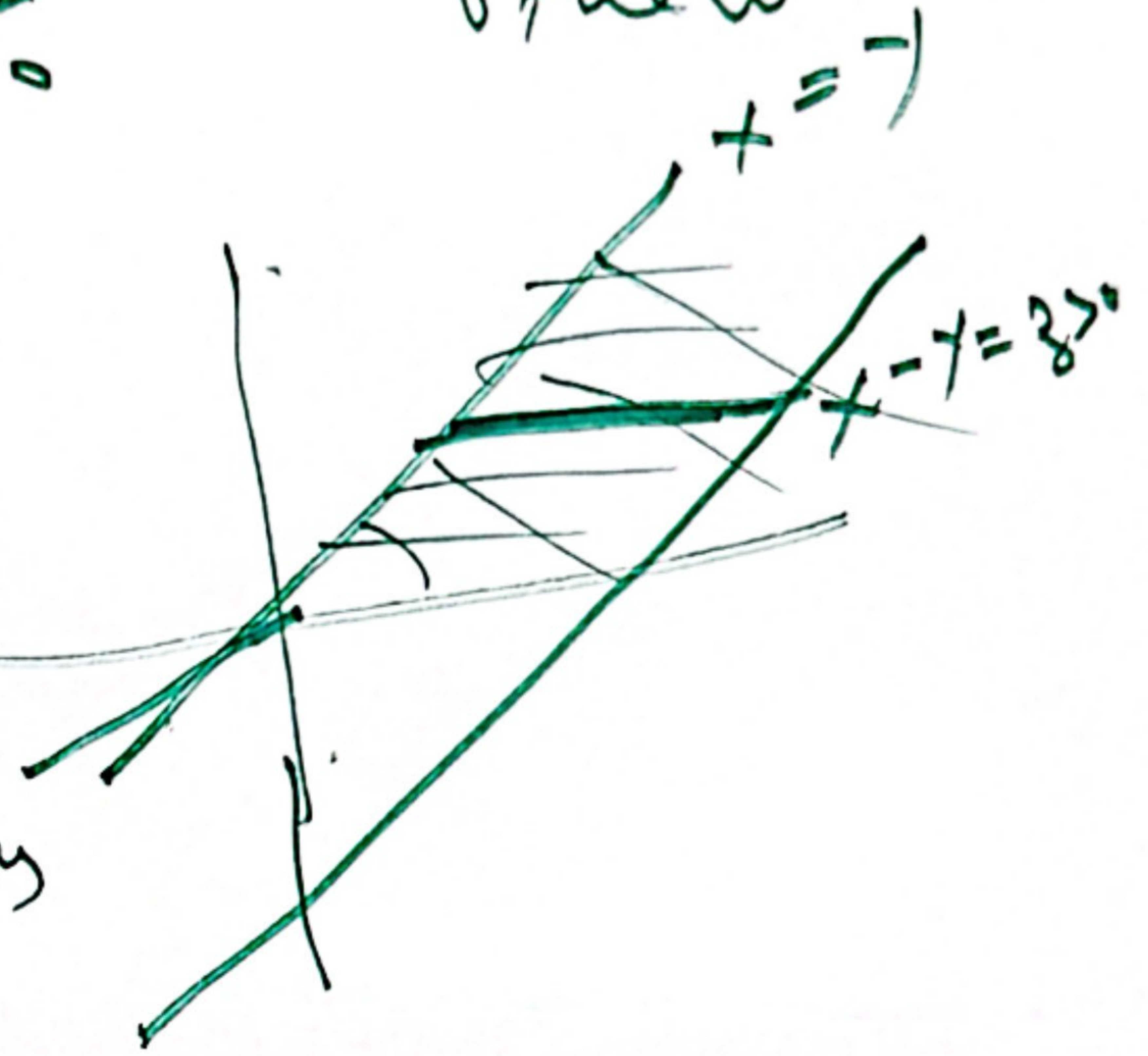
$= e^{z} \underbrace{\int_0^\infty e^{-x}\,dx}_{1} = e^{+z}, \quad z < 0$



Z is $-$ve exponential.

Solution to Quiz 2

1c) $f_{XY}(x,y) = \begin{cases} \frac{x}{2} e^{-x}, & x>y>0 \\ 0 & \text{otherw} \end{cases}$



$x > y \Rightarrow x - y = z > 0$

For $z > 0$

$F_Z(z) = P(X - Y \le z)$

$= \int_0^\infty \int_{x=y}^{y+z} f_{XY}(x,y)\,dx\,dy$

$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_0^\infty 1 \cdot f_{XY}(y+z, y)\,dy$

$= \frac{1}{2} \int_0^\infty (y+z) e^{-(y+z)}\,dy$

$= \frac{1}{2} e^{-z} \left[\underbrace{\int_0^\infty y e^{-y}}_{1} + z \underbrace{\int_0^\infty e^{-y}\,dy}_{1} \right]$

$= \frac{(1+z)}{2} e^{-z}, \quad z > 0$