ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 3

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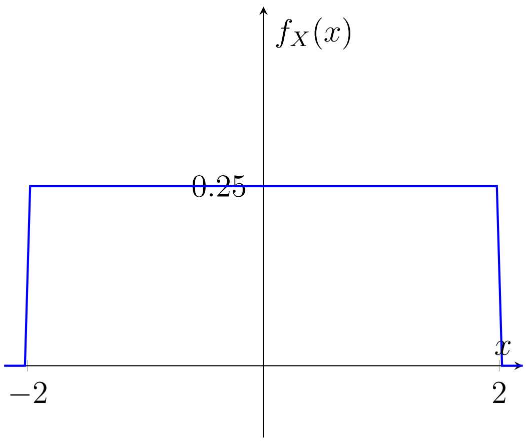
Problem 1

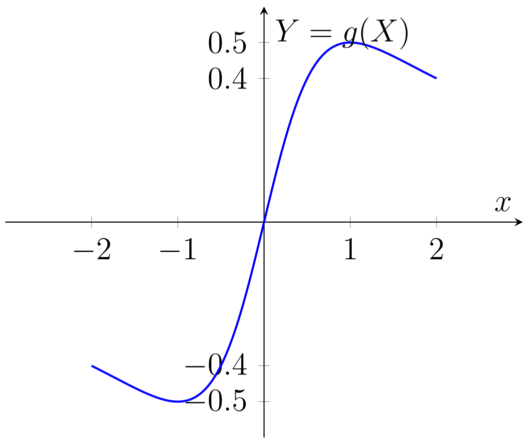
X is a uniform random variable in $(-2, 2)$. Consider the transformation,

$$Y = \frac{X}{1+X^2}.$$

Determine $f_Y(y)$.

Solution:





$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+x^2)\cdot 1 - x\cdot(2x)}{(1+x^2)^2} = 0 \quad \Rightarrow \quad x = \pm 1 \text{ are the stationary points.}$$

$$y = \frac{x}{1+x^2} \;\Rightarrow\; y + x^2y - x = 0 \;\Rightarrow\; x_1 = \frac{1-\sqrt{1-4y^2}}{2y}, x_2 = \frac{1+\sqrt{1-4y^2}}{2y}$$

Case I: $|y| < 0.4$

$$\begin{aligned} f_Y(y) &= f_X(x_1)\cdot\left|\frac{\mathrm{d}x_1}{\mathrm{d}y}\right| = \frac{1}{4}\cdot\frac{1-\sqrt{1-4y^2}}{2y^2\sqrt{1-4y^2}} \\ &= \frac{1-\sqrt{1-4y^2}}{8y^2\sqrt{1-4y^2}} \end{aligned}$$

Case II: $0.4 \le |y| \le 0.5$

$$\begin{aligned} f_Y(y) &= f_X(x_1) \cdot \left|\frac{\mathrm{d}x_1}{\mathrm{d}y}\right| + f_X(x_2) \cdot \left|\frac{\mathrm{d}x_2}{\mathrm{d}y}\right| \\ &= \frac{1}{4y^2\sqrt{1-4y^2}}. \end{aligned}$$

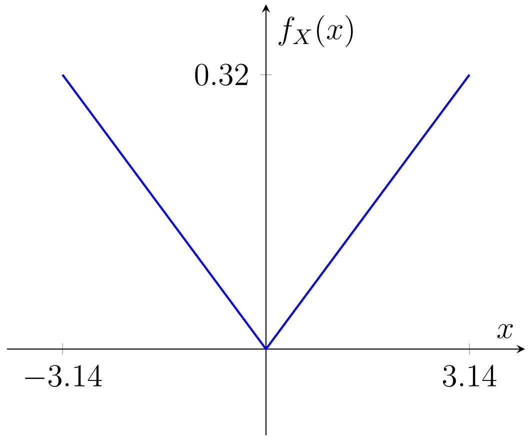
Problem 2

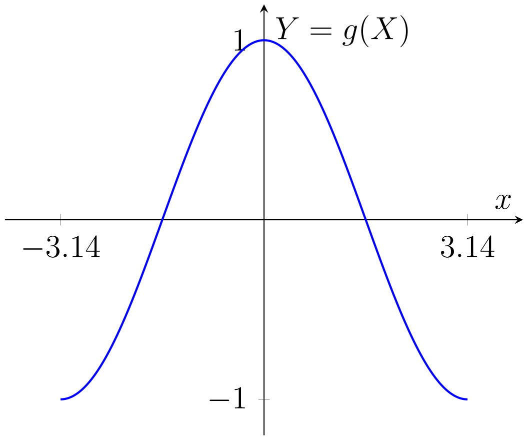
Let

$$f_X(x) = \frac{|x|}{\pi^2} : -\pi < x < \pi$$

, and define $Y = \cos X$. Determine $f_Y(y)$.

Solution:





Note that $x = \cos^{-1}(y)$.

$$f_Y(y) = f_X(x_1) \cdot \left|\frac{\mathrm{d}x_1}{\mathrm{d}y}\right| + f_X(x_2) \cdot \left|\frac{\mathrm{d}x_2}{\mathrm{d}y}\right| = \frac{\cos^{-1} y}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}} + \frac{\cos^{-1} y}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}}$$
$$= \frac{2\cos^{-1} y}{\pi^2 \sqrt{1-y^2}}.$$

Problem 3

A random variable X is Poisson with parameter λ.

i) Find its charecteristic function.

ii) Use the charectaristic function to find E[X] and Var(X).

Solution:

i)

$$\Phi_X(\omega) = E[e^{j\omega x}] = \sum_{k=0}^{\infty} e^{j\omega k} P(x = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{j\omega})^k}{k!} = e^{-\lambda(1-e^{j\omega})}.$$

ii)

$$\begin{aligned}
E[X] &= \frac{1}{j}\Phi_X^{(1)}(\omega)\Big|_{\omega=0} = e^{-\lambda} j\lambda e^{j\omega} e^{\lambda e^{j\omega}}\Big|_{\omega=0} = \lambda, \\
E[X^2] &= \frac{1}{j^2}\Phi_X^{(2)}(\omega)\Big|_{\omega=0} = j\lambda e^{-\lambda}\left(je^{j\omega}e^{\lambda e^{j\omega}} + j\lambda e^{j\omega}e^{\lambda e^{j\omega}}\right)\Bigg|_{\omega=0} = \lambda + \lambda^2, \\
\text{Var} &= E[X^2] - (E[X])^2 = \lambda.
\end{aligned}$$

Problem 4

A random variable X is geometric with parameter p.

i) Find its characteristic function.

ii) Use the characteristic function to find E[X] and Var(X)

Solution:

i) Let $q = 1 - p$.

$$\Phi_X(\omega) = E[e^{j\omega x}] = \sum_{k=0}^{\infty} e^{j\omega k} P(x = k) = \sum_{k=0}^{\infty} e^{j\omega k} pq^k = \frac{p}{1 - qe^{j\omega}}.$$

ii)

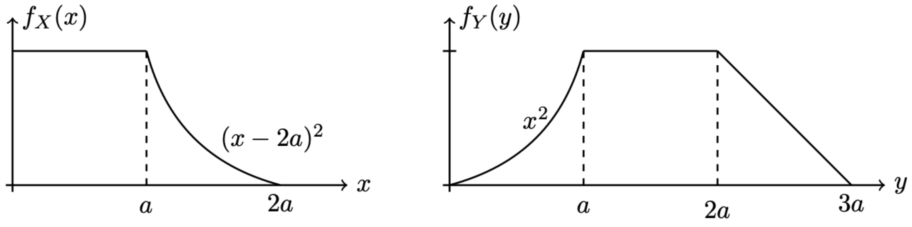
$$E[X] = \frac{1}{j}\Phi_X^{(1)}(\omega)\Big|_{\omega=0} = \frac{1}{j} \cdot \frac{jpqe^{j\omega}}{(1 - qe^{j\omega})^2}\Big|_{\omega=0} = \frac{q}{p}$$

$$E[X^2] = \frac{1}{j^2}\Phi_X^{(2)}(\omega)\Big|_{\omega=0} = \frac{1}{j^2} \cdot \frac{j^2 pqe^{j\omega}\left((1 - qe^{j\omega})^2 + 2qe^{j\omega}(1 - qe^{j\omega})\right)}{(1 - qe^{j\omega})^4}\Bigg|_{\omega=0} = \frac{q + q^2}{p^2}$$

$$\text{Var} = E[X^2] - (E[X])^2 = \frac{q}{p^2}.$$

Problem 5

Find the mean and variance for the random variables X and Y with the probability density functions shown below.



Solution: For X,

$$1 = \int_0^{2a} f_X(x)dx = a^3 + \frac{1}{3}a^3 = \frac{4}{3}a^3 \quad \Rightarrow \quad a = \left(\frac{3}{4}\right)^{1/3}.$$

Mean Value:

$$\mathbb{E}[X] = \int_0^a xa^2dx + \int_a^{2a} x(x-2a)^2dx = \frac{1}{2}a^4 + \frac{5}{12}a^4 = \frac{11}{12}a^4.$$

Variance:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{13}{15}a^5 - \frac{121}{144}a^8$$

For Y,

$$1 = \int_0^{3a} f_Y(y)dy = \frac{1}{3}a^3 + a^3 + \frac{1}{2}a^3 = \frac{11}{6}a^3 \quad \Rightarrow \quad a = \left(\frac{6}{11}\right)^{1/3}.$$

Mean Value:

$$\mathbb{E}[Y] = \int_0^a y^3dy + \int_a^{2a} a^2ydy + \int_{2a}^{3a} y(3a^2 - ay)dy = \frac{1}{4}a^4 + \frac{3}{2}a^4 + \frac{7}{6}a^4 = \frac{35}{12}a^4.$$

Variance:

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{317}{60}a^5 - \frac{1225}{144}a^8.$$

Problem 6

Let X be a random variable with the following probability density function,

$$f_X(x) = \begin{cases} \frac{1}{2}\sin(x) & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Solution:

$$\mathbb{E}[X] = \frac{1}{2}\int_0^{\pi} x\sin(x)dx = \frac{1}{2}(-x\cos(x) + \sin(x)\big|_0^{\pi}) = \frac{\pi}{2}.$$

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{2}(-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\big|_0^{\pi}) - \frac{\pi^2}{4} \\ &= \frac{\pi^2}{4} - 2.\end{aligned}$$

Problem 7

Find the mean and variance of the random variable X^2 with the following distributions:
(i) $X \sim N(\mu, \sigma^2)$
(ii) $X \sim P(\lambda)$
(iii) $X \sim \exp(\lambda)$

Solution:

(i) The moment generating function is

$$M_x(t) = e^{\mu t} e^{\sigma^2 t^2/2}.$$

Hence,

$$\begin{aligned}
\mathbb{E}[X^2] &= \frac{d^2}{dt^2} M_x(t)\big|_{t=0} = \mu^2 + \sigma^2. \\
\mathbb{E}[X^4] &= \frac{d^4}{dt^4} M_x(t)\big|_{t=0} = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \\
\mathrm{Var}(X^2) &= 4\mu^2\sigma^2 + 2\sigma^4.
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathbb{E}[X^2] &= \mathrm{Var}(X) + (\mathbb{E}[X])^2 = \lambda + \lambda^2. \\
\mathbb{E}[X^4] &= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4 \\
\mathrm{Var}(X^2) &= \lambda + 6\lambda^2 + 4\lambda^3.
\end{aligned}$$

(iii) Note that

$$\mathbb{E}[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx = \frac{n!}{\lambda^n}$$

Hence,

$$\begin{aligned}
\mathbb{E}[X^2] &= \frac{2}{\lambda^2}. \\
\mathbb{E}[X^4] &= \frac{24}{\lambda^4} \\
\mathrm{Var}(X^2) &= \frac{20}{\lambda^4}.
\end{aligned}$$

Problem 8

Find the mean and variance of the following random variables.
(i) $X \sim Gamma(\alpha, \beta)$
(ii) $E(X) = \mu, Var(X) = \sigma^2$. Find mean and variance of $Y = aX + b$

Solution:

(i)

$$f_X(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}, \quad \text{where} \quad \Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$$

Then

$$\begin{aligned}
\mathbb{E}[X] &= \frac{1}{\beta^\alpha \int_0^\infty y^{\alpha-1}e^{-y}dy} \int_0^\infty x x^{\alpha-1} e^{-x/\beta} dx \\
&= \beta \frac{1}{\int_0^\infty y^{\alpha-1}e^{-y}dy} \int_0^\infty z^\alpha e^{-z} dz \\
&= \beta\alpha \\
\mathrm{Var}(X) &= \alpha\beta^2.
\end{aligned}$$

(ii)

$$\mathbb{E}[Y] = \mathbb{E}[aX + b] = \int (ax + b) f_X(x) dx = a\mathbb{E}[X] + b.$$

$$\mathrm{Var}(Y) = \mathbb{E}[(aX + b - (a\mathbb{E}[X] + b))^2] = a^2 \mathrm{Var}(X).$$