ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 6

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Problem 1

Let

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \qquad W = Y/X$$

a.) Find $f_{ZW}(z,w)$.

b.) Are Z and W independent random variables? Prove your answer.

Solution:

a.)

$$\begin{cases} z = x + y \\ w = y/x \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = wz/(w+1) \\ x_1 = z/(w+1) \end{cases}$$

Then,

$$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{w+1} & -\frac{z}{(w+1)^2} \\ \frac{w}{w+1} & \frac{z}{(w+1)^2} \end{vmatrix} = \frac{z}{(w+1)^2}$$

$$f_{ZW}(z,w) = \frac{z}{(w+1)^2} f_{XY}(x_1,y_1) = \frac{2e^{-z}z}{(w+1)^2}, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$x < y < \infty \;\Rightarrow\; \frac{z}{w+1} < \frac{wz}{w+1} < \infty \;\Rightarrow\; 1 < w < \infty,$$
$$0 < x < y \;\Rightarrow\; 0 < \frac{z}{w+1} < \frac{wz}{w+1} \;\Rightarrow\; 0 < z < wz \;\Rightarrow\; 0 < z < \infty.$$

b.) Marginal distributions:

$$f_Z(z) = \int_1^\infty f_{ZW}(z,w)dw = ze^{-z}, \quad z \geq 0,$$
$$f_W(w) = \int_0^\infty f_{ZW}(z,w)dz = \frac{2}{(w+1)^2}, \quad w \geq 1.$$

They are independent.

Problem 2

Given the joint density function

$$f_{XY}(x,y) = \begin{cases} 2e^{-(2x-y)} & 0 < y < x < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

and the two functions

$$Z = 2X - Y, \qquad W = Y/X.$$

a.) Find $f_{ZW}(z,w)$.

b.) Are Z and W independent random variables? Prove your answer.

Solution:

a.)

$$\begin{cases} 2x - y = z \\ y/x = w \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = zw/(2-w) \\ x_1 = z/(2-w) \end{cases}$$

Then,

$$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial 2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2-w} & \frac{z}{(2-w)^2} \\ \frac{w}{2-w} & \frac{2z}{(2-w)^2} \end{vmatrix} = \frac{z}{(2-w)^2},$$

and

$$f_{ZW}(z,w) = \frac{z}{(2-w)^2} f_{XY}(x_1,y_1) = \frac{2e^{-z}z}{(2-w)^2}, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$x < y < \infty \ \Rightarrow z > 0,$$
$$0 < x < y \ \Rightarrow \ 0 < \frac{wz}{2-w} < \frac{z}{2-w} \ \Rightarrow \ 0 < w < 1.$$

b.) Marginal distributions:

$$f_Z(z) = \int_0^1 \frac{2e^{-z}z}{(2-w)^2} dw = ze^{-z}, \quad z \geq 0,$$
$$f_W(w) = \int_0^\infty \frac{2e^{-z}z}{(2-w)^2} dz = \frac{2}{(2-w)^2}, \quad 0 < w < 1.$$

They are independent.

Problem 3

The joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 2xye^{-(x+y)} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \quad W = X/Y$$

a.) Find $f_{ZW}(z,w)$.

b.) Are Z and W independent random variables?

c.) Are Z and W uncorrelated random variables?

Solution:

a.)

$$\begin{cases} x + y = z \\ x/y = w \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = z/(w+1) \\ x_1 = wz/(w+1) \end{cases}$$

Then,

$$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{w}{w+1} & \frac{z}{(w+1)^2} \\ \frac{1}{w+1} & -\frac{z}{(w+1)^2} \end{vmatrix} = -\frac{z}{(w+1)^2},$$

and

$$f_{ZW}(z,w) = \frac{z}{(w+1)^2} f_{XY}(x_1,y_1) = \frac{z}{(w+1)^2} \frac{2wz^2}{(w+1)^2} e^{-z} = \frac{2wz^3e^{-z}}{(w+1)^4}, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$x < y < \infty \;\Rightarrow\; \frac{z}{w+1} < \frac{wz}{w+1} < \infty \;\Rightarrow\; 1 < w < \infty,$$
$$0 < x < y \;\Rightarrow\; 0 < \frac{z}{w+1} < \frac{wz}{w+1} \;\Rightarrow\; 0 < z < wz \;\Rightarrow\; 0 < z < \infty.$$

b.) Marginal distributions:

$$f_Z(z) = \int_1^\infty f_{ZW}(z,w)dw = \frac{1}{6}z^3e^{-z}, \quad z \geq 0,$$
$$f_W(w) = \int_0^\infty f_{ZW}(z,w)dz = \frac{12w}{(w+1)^4}, \quad w \geq 1.$$

They are independent.

Problem 4

The joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{3}{4}(x+y)^2 & 0 < x < 1, -1 < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \qquad W = X - Y$$

a.) Find $f_{ZW}(z,w)$, $f_Z(z)$ and $f_W(w)$.

b.) Are Z and W independent random variables?

c.) Are Z and W uncorrelated random variables?

d.) Are Z and W orthogonal random variables ($E[ZW] = 0$)? Prove your answers.

Solution:

a.)

$$\begin{cases} x + y = z \\ x - y = w \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = (z-w)/2 \\ x_1 = (z+w)/2 \end{cases}$$

Then,

$$f_{ZW}(z,w) = f_{XY}(x_1,y_1) \cdot \begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \frac{3}{4}z^3 \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{3}{8}z^2, \quad (z,w) \in \mathcal{A}.$$

The region $\mathcal{A}$ is described as follows:

$$\mathcal{A} = \{(z,w) : \; 0 < z + w < 2, -2 < z - w < 2\}.$$

b.) Marginal distributions:

$$f_Z(z) = \begin{cases} \frac{3}{4}z^2(z+1) & -1 < z < 0, \\ \frac{3}{4}z^2 & 0 \le z < 1, \\ \frac{3}{4}z^2\cancel{(z-2)} \; (2-z) & 1 \le z < 2, \end{cases} \qquad f_W(w) = \begin{cases} \frac{w^3+3w^2+6w+4}{4} & -1 < w < 0, \\ \frac{3w^2-6w+4}{4} & 0 \le w < 1, \\ \frac{1}{4}(2-w)^3 & 1 \le w < 2. \end{cases}$$

They are not independent.

c.)

$$E[ZW] = \int_{-1}^{0}\int_{-w}^{w+2} \frac{3}{8}z^3 w\,dz\,dw + \int_{0}^{1}\int_{-w}^{2-w} \frac{3}{8}z^3 w\,dz\,dw + \int_{1}^{2}\int_{w-2}^{2-w} \frac{3}{8}z^3 w\,dz\,dw$$

Upon solving we get $E[ZW] = 0$.

$$E[Z] = \int_{-1}^{0} \frac{3}{4}z^3(z+1)dz + \int_{0}^{1} \frac{3}{4}z^2 3dz + \int_{1}^{2} \frac{3}{4}z^3(z-2)dz = \frac{9}{8}.$$

Similarly,

$$E[W] = \frac{1}{8}$$

Since as $E[ZW] - E[Z]E[W] \neq 0$, they are correlated.

d.) Yes.

Problem 5

X, Y are independent, idential geometric random variables with common parameter p, i.e., with $q = 1 - p$,

$$P(X = k) = P(Y = k) = pq^k, \quad k = 0, 1, 2, \ldots$$

a.) $Z = X + Y$, $W = \min\{X, Y\}$, find $f_{ZW}(z, w)$, $f_Z(z)$ and $f_W(w)$.

b.) $Z = \min\{X, Y\}$, $W = X - Y$, find $f_{ZW}(z, w)$, $f_Z(z)$ and $f_W(w)$.

Solution:

a.) Look at 16:30 on $Z = X + Y$, $W = \max(X, Y)$:

https://youtu.be/TU3Y9bagw9w

b.) $Z = \min(X, Y)$, $W = X - Y$:

https://youtu.be/V1EyqL1cqTE

(a) $$P(Z=m, W=n) = \begin{cases} p^2 q^m & m = 2n \\ 2p^2 q^m & m > 2n \end{cases}$$

$$P(Z=m) = (m+1) p^2 q^m \qquad m = 0, 1, 2, \ldots$$

$$P(W=n) = (q+1) p q^{2n} \qquad n = 0, 1, 2, \ldots$$

(b) $$P(Z=m, W=n) = p^2 q^{2m+|n|} \qquad \begin{matrix} m = 0, 1, 2, \ldots \\ n = 0, \pm 1, \pm 2, \ldots \end{matrix}$$

$$P(Z=m) = (\overset{q}{\cancel{m}}+1) p q^{2m} \qquad m = 0, 1, 2, \ldots$$

$$P(W=n) = \frac{p}{1+q} \cdot q^{|n|} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Problem 6

X and Y are independent Geometric random variables with common parameter p, i.e., $P(X = k) = P(Y = k) = pq^k$ with $q = 1 - p$. Define

$$Z = X + Y, \qquad W = |X - Y|.$$

Find

a.) $P(Z = m, W = k)$

b.) $P(Z = m)$

c.) $P(W = k)$

Solution:

a.) When $X \geq Y$,

$$\begin{cases} X + Y = m, \\ X - Y = k, \end{cases} \Rightarrow \begin{cases} X = (m+k)/2, \\ Y = (m-k)/2. \end{cases}$$

$$\begin{aligned} X \geq Y &\Rightarrow (m+k)/2 \geq (m-k)/2 \Rightarrow k \geq 0, \\ X \geq 0 &\Rightarrow (m+k)/2 \geq 0 \Rightarrow m \geq -k, \\ Y \geq 0 &\Rightarrow (m-k)/2 \geq 0 \Rightarrow m \geq k. \end{aligned}$$

When $X < Y$,

$$\begin{cases} X + Y = m, \\ Y - X = k, \end{cases} \Rightarrow \begin{cases} X = (m-k)/2, \\ Y = (m+k)/2. \end{cases}$$

$$\begin{aligned} X < Y &\Rightarrow (m-k)/2 < (m+k)/2 \Rightarrow k > 0, \\ X \geq 0 &\Rightarrow (m-k)/2 \geq 0 \Rightarrow m \geq k, \\ Y \geq 0 &\Rightarrow (m+k)/2 \geq 0 \Rightarrow m \geq -k. \end{aligned}$$

$$\begin{aligned} P(Z = m, W = k) &= P(Z = m, W = k, X \geq Y) + P(Z = m, W = k, X < Y) \\ &= P(X + Y = m, X - Y = k, X \geq Y) + P(X + Y = m, Y - X = k, X < Y) \\ &= \begin{cases} P(X = \frac{m+k}{2})P(Y = \frac{m-k}{2}) & k = 0, m = 0, 2, 4, .. \\ P(X = \frac{m+k}{2})P(Y = \frac{m-k}{2}) + P(X = \frac{m-k}{2})P(Y = \frac{m+k}{2}) & k = 0, 1, 2, ..., m \geq k \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} p^2 q^m & k = 0, m = 0, 2, 4, ..., \\ 2p^2 q^m & \cancel{k = 0, 1, 2, ..., m = k + 2, k + 4, ...,} \quad k = 1, 2, \ldots \quad m = k, k+2, k+4, \ldots \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

b.)

$$P(Z = m) = \sum_k P(Z = m, W = k) = \begin{cases} p^2 & m = 0 \\ p^2 q^m + \sum_{k=0,2,4,...}^{m} 2p^2 q^m & m = 2, 4, 6, ... \\ \sum_{k=1,3,5,...}^{m} 2p^2 q^m & m = 1, 3, 5, ... \end{cases}$$

$P(Z=m) = (m+1)p^2q^m$

c.)

$$P(W = k) = \sum_m P(Z = m, W = k) = \begin{cases} \cancel{p^2} & k = 0 \\ \cancel{\sum_{m=k,k+2,k+4,...}^{\infty}} 2p^2 q^m & k > 0 \end{cases}$$

$$P(W=n) = \begin{cases} \dfrac{p}{1+q} & n = 0 \\ \dfrac{2pq^n}{1+q} & n = 1, 2, \ldots \end{cases}$$