ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 5

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Problem 1

The joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f of

a.) $Z = X + Y$.

b.) $Z = X - Y$.

c.) $Z = X/Y$.

Solution:

a.) $Z = X + Y$:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X+Y \le z) = P(Y \le z - X) \\ &= \int_0^{\frac{z}{2}} \int_x^{z-x} e^{-y} dy dx \\ f_Z(z) &= \int_0^{\frac{z}{2}} \frac{\mathrm{d}}{\mathrm{d}z} \int_x^{z-x} e^{-y} dy dx = \int_0^{\frac{z}{2}} 1 \cdot e^{-(z-x)} dx \\ &= e^{-z} \int_0^{\frac{z}{2}} e^x dx = e^{-z}(e^{\frac{z}{2}} - 1). \end{aligned}$$

Therefore

$$f_Z(z) = e^{-\frac{z}{2}} - e^{-z}, \qquad z \ge 0.$$

b.) $Z = X - Y$:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X - Y \le z) = 1 - P(Y \le X - z) \\ &= 1 - \int_0^{\infty} \int_x^{x-z} e^{-y} dy dx, \\ f_Z(z) &= -\frac{\mathrm{d}}{\mathrm{d}z} \int_z^{\infty} \int_x^{x-z} e^{-y} dy dx \\ &= -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^{\infty} \left(e^{-x} - e^{-(x-z)}\right) dx \\ &= e^z. \end{aligned}$$

Therefore

$$f_Z(z) = e^z, \qquad z \le 0.$$

c.) $Z = X/Y$.

$$\begin{aligned}
F_Z(z) &= P(Z \le z) = P(X/Y \le z) = 1 - P(Y \le X/z) \\
&= 1 - \int_0^\infty \int_x^{x/z} e^{-y} dy dx, \\
f_Z(z) &= -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^\infty \int_x^{x/z} e^{-y} dy dx \\
&= -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^\infty \left(e^{-x} - e^{-x/z} \right) dx \\
&= -\frac{d}{dz}(-z) \\
&= 1.
\end{aligned}$$

Therefore

$$f_Z(z) = 1, \qquad 0 < z < 1.$$

Problem 2

X and Y are independent and uniform in the interval $(0, a)$. Find the p.d.f. of $Z = X - Y$.

Solution:

$$F_Z(z) = P(Z \le z) = P(X - Y \le z) = P(Y \ge X - z).$$

Case I: $z < 0$

$$\begin{aligned}
F_Z(z) &= P(Y \ge X - z) = \int_0^{a+z} \int_{x-z}^{a} \frac{1}{a^2} dy dx, \\
f_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \int_0^{a+z} \int_{x-z}^{a} \frac{1}{a^2} dy dx \\
&= 1 \cdot \int_{a+z-z}^{a} \frac{1}{a^2} dy - 0 + \int_0^{a+z} \frac{\mathrm{d}}{\mathrm{d}z} \int_{x-z}^{a} \frac{1}{a^2} dy dx \\
&= \int_0^{a+z} (0 - (-1) \cdot \frac{1}{a^2}) dx \\
&= \frac{a+z}{a^2}.
\end{aligned}$$

Case II: $z \ge 0$

$$\begin{aligned}
F_Z(z) &= 1 - \int_0^{a-z} \int_{y+z}^{a} \frac{1}{a^2} dx dy \\
f_Z(z) &= - \int_0^{a-z} \frac{\mathrm{d}}{\mathrm{d}z} \int_{y+z}^{a} \frac{1}{a^2} dx dy = - \int_0^{a-z} -\frac{1}{a^2} dy \\
&= \frac{1}{a^2}(a - z)
\end{aligned}$$

Therefore,

$$f_Z(z) = \begin{cases} \frac{a+z}{a^2}, & z < 0, \\ \frac{a-z}{a^2}, & z \ge 0. \end{cases}$$

Problem 3

X and Y are independent exponential random variables with parameters α and β respectively, i.e.,

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \begin{cases} \alpha\beta e^{-(\alpha x+\beta y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define $Z = \min(X, 3Y)$. Show that Z is also an exponential random variable, and find the value of corresponding exponential parameter.

Solution: By definition,

$$Z = \min(X, 3Y) = \begin{cases} 3Y, & X \geq 3Y, \\ X, & X < 3Y. \end{cases}$$

With $z \geq 0$,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P((Z \leq z) \cap ((X \geq 3Y) \cup (X < 3Y))) \\ &= P((Z \leq z) \cap (X \geq 3Y)) + P((Z \leq z) \cap (X < 3Y)) \\ &= P((3Y \leq z) \cap (X \geq 3Y)) + P((X \leq z) \cap (X < 3Y)) \\ &= \int_0^{z/3} \int_{3y}^{\infty} f_{XY}(x,y)dxdy + \int_0^{z} \int_{x/3}^{\infty} f_{XY}(x,y)dydx \\ f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \frac{1}{3}\int_z^{\infty} \alpha\beta e^{-(\alpha x+\beta z/3)}dy + \int_{z/3}^{\infty} \alpha\beta e^{-(\alpha z+\beta y)}dx \\ &= \left(\alpha + \frac{\beta}{3}\right) e^{-\left(\alpha+\frac{\beta}{3}\right)z}, \quad z \geq 0. \end{aligned}$$

Therefore,

$$Z \sim \text{Exponential}\left(\alpha + \frac{\beta}{3}\right).$$

Problem 4

Given the joint density function

$$f_{XY}(x,y) = \begin{cases} xye^{-(x+y)} & x>0, y>0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z = \frac{\min(X,Y)}{\max(X,Y)}.$$

Determine the p.d.f of Z.

Solution: By definition,

$$Z = \frac{\min(X,Y)}{\max(X,Y)} = \begin{cases} Y/X, & X \geq Y, \\ X/Y, & X < Y. \end{cases}$$

With $0 < z \leq 1$,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P((Z \leq z) \cap ((X \geq Y) \cup (X < Y))) \\ &= P((Z \leq z) \cap (X \geq Y)) + P((Z \leq z) \cap (X < Y)) \\ &= P((Y/z \leq X) \cap (X \geq Y)) + P((X \leq Yz) \cap (X < Y)) \\ &= P(Y/z \leq X)) + P(X \leq Yz)) \\ &= \int_0^\infty \int_0^{xz} f_{XY}(x,y)dydx + \int_0^\infty \int_0^{yz} f_{XY}(x,y)dxdy. \\ f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \int_0^\infty x f_{XY}(x, xz)dx + \int_0^\infty y f_{XY}(yz, y)dy \\ &= \int_0^\infty x[f_{XY}(xz, x) + f_{XY}(x, xz)]dx \\ &= \int_0^\infty x[xzxe^{-(xz+x)} + xxze^{-(x+xz)}]dx \\ &= \frac{12z}{(1+z)^4}, \qquad 0 < z < 1. \end{aligned}$$

Problem 5

X and Y are independent random variables with geometric p.m.f

$$P(X = k) = pq^k, k = 0, 1, 2, ...,$$
$$P(Y = m) = pq^m, m = 0, 1, 2.....$$

Find the p.m.f. of $Z = X + Y$.

Solution:

$$\begin{aligned} P(Z = k) &= \sum_{m=0}^{k} P(X = m)P(Y = k - m) \\ &= \sum_{m=0}^{k} pq^m \cdot pq^{k-m} = \sum_{m=0}^{k} p^2 q^k \\ &= p^2 q^k (k + 1). \end{aligned}$$

Therefore,

$$P(Z = k) = p^2 q^k (k + 1), \qquad k = 0, 1, 2, \ldots$$

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Problem 6

X and Y are random variables with joint p.d.f.

$$f_{XY}(x, y) = \begin{cases} ke^{-(x+y)} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the p.d.f. of $Z = X - Y$.

Solution:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X - Y \le z) = P(X \le Y + z) \\ &= \int_0^{\infty} \int_y^{y+z} ke^{-(x+y)} dx dy, \\ &= k \frac{1 - e^{-z}}{2}, \\ f_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = \frac{1}{2} ke^{-z}. \end{aligned}$$

Since $\lim_{z \to \infty} F(z) = 1$, $k = 2$. Therefore,

$$f_Z(z) = e^{-z}, \qquad z \ge 0.$$