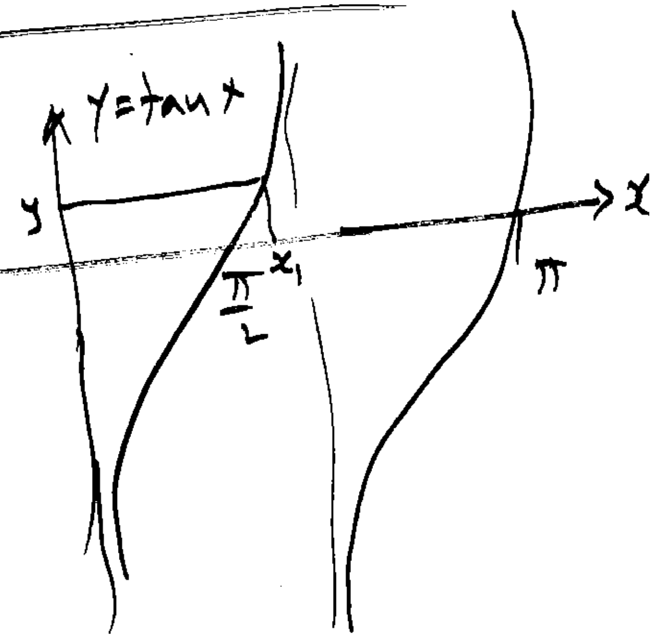
1)

EL 6303 INET

Solutions to ~~Mid~~ Midterm Exam



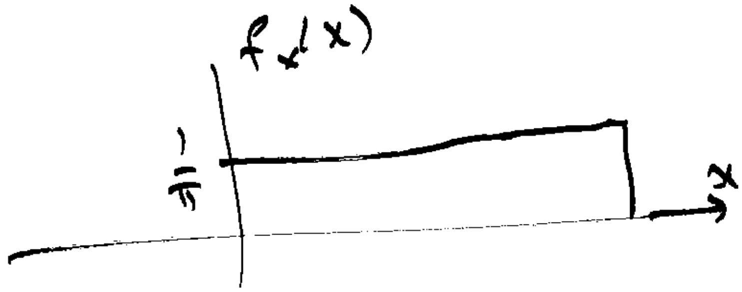
$y = \tan x$ has a unique solution in x given by

$x_1 = \tan^{-1} y$

for every $-\infty < y < +\infty$.

Also as $0 < x < \pi$,

$-\infty < y < +\infty$

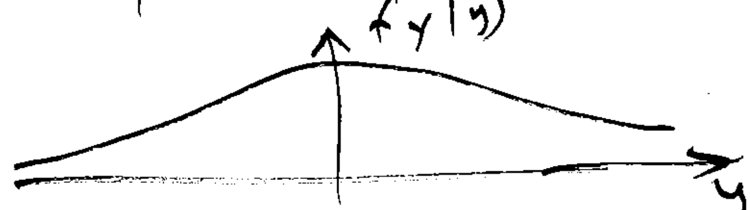


$$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

Hence $$f_Y(y) = \sum_i \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_i) = \frac{1/\pi}{1+y^2}$$

$-\infty < y < +\infty$

$\Rightarrow$ Y is a Cauchy random variable.



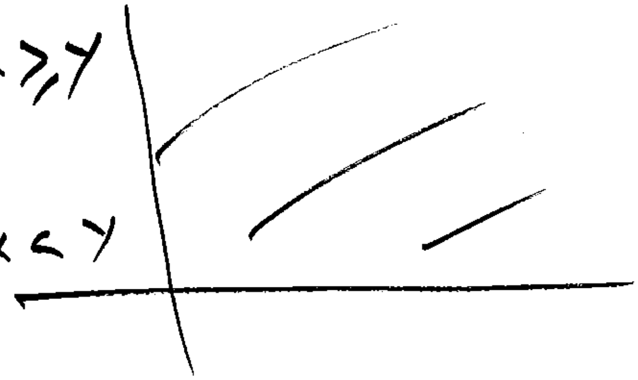
1a Note $Z = \frac{1}{Y} = \cot X$ is also Cauchy, since

$$f_Z(z) = \frac{1}{\left|\frac{dz}{dy}\right|} f_Y\left(\frac{1}{z}\right) = z^2 f_Y(1/z)$$

$$= \frac{1/\pi}{1+z^2}$$

2/2) a)

$$Z = \frac{\max(X,Y)}{\min(X,Y)} = \begin{cases} \frac{X}{Y}, & X \geq Y \\ \frac{Y}{X} & X < Y \end{cases}$$



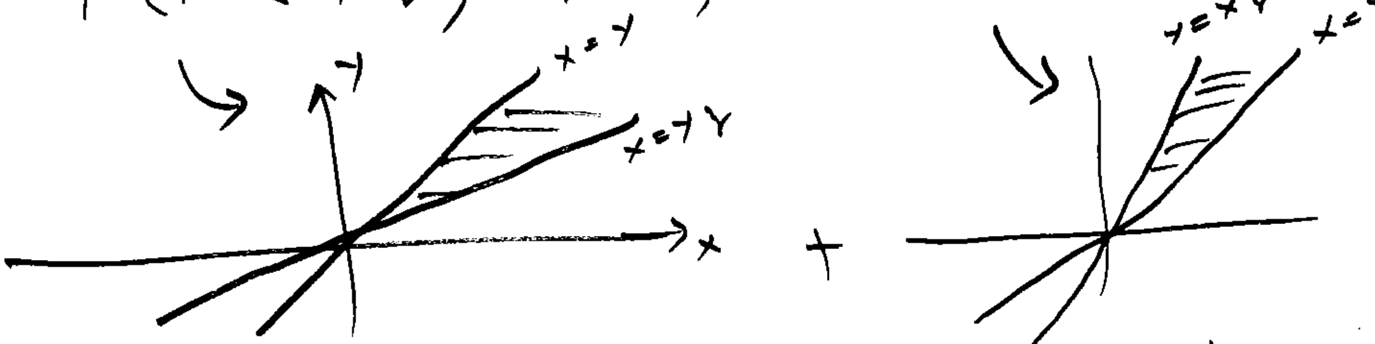
$\to \quad z > 1$

$$F_Z(z) = P(Z \leq z) = P\left[(Z \leq z) \cap \{(X \geq Y) \cup X < Y\} \right]$$

$$= P(Z \leq z, X \geq Y) + P(Z \leq z, X < Y)$$

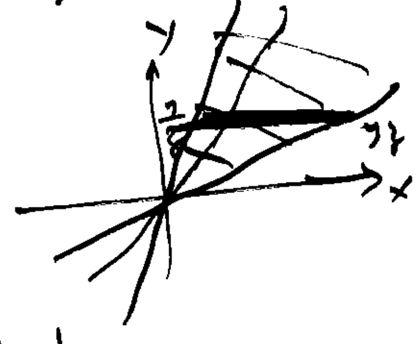
$$= P\left(\frac{X}{Y} \leq z, X \geq Y\right) + P\left(\frac{Y}{X} \leq z, X < Y\right)$$

$$= P(X \leq Yz, X \geq Y) + P(Y < zX, X < Y)$$



Hence

$$F_Z(z) = \int_0^\infty \int_{x=y/z}^{yz} f_{XY}(x,y)\,dx\,dy =$$



$$f_Z(z) = \int_0^\infty y f_{XY}(yz, y) + \frac{y}{z^2} f_{XY}\left(\frac{y}{z}, y\right) dy$$

3

$$f_Z(z) = \int_0^\infty \Big[y\, e^{-\underbrace{(1+z)y}_{u=(1+z)y}} + \frac{y}{z^2} e^{-\left(1+\frac{1}{z}\right)y} \Big] dy$$

$$= \frac{1}{(1+z)^2} \underbrace{\int_0^\infty u e^{-u}\, du}_{1} + \frac{1}{z^2} \left(\frac{1}{1+\frac{1}{z}}\right)^2 \underbrace{\int_0^\infty u e^{-u}\, du}_{1}$$

$$= \frac{2}{(1+z)^2}, \quad z \geq 1$$

check

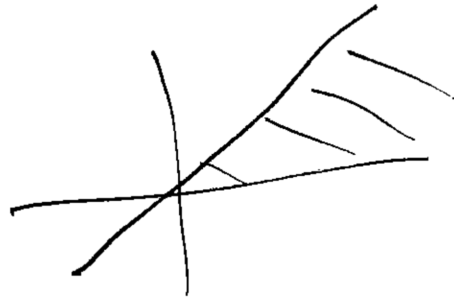
$$\int_1^\infty f_Z(z) = 2 \int_1^\infty \frac{1}{(1+z)^2}\, dz = 2 \int_2^\infty \frac{1}{y^2}\, dy$$

$$= 2 \left(-\frac{1}{y}\right)\Bigg|_1^\infty = \underline{\underline{1}}.$$

4/2b)

$$f_{XY}(x,y) = \begin{cases} e^{-x}, & 0<y<x<\infty \\ 0 & \text{otherwise} \end{cases}$$

$Z = X+Y \geq 0; \quad W = X - Y \geq 0$



$\Rightarrow x_1 = \dfrac{z+w}{2}, \quad y_1 = \dfrac{z-w}{2}$

is the unique solution

$x > y \Rightarrow w > 0, \quad z > 0, \quad z > w$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Rightarrow |J| = 2$$

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1, y_1)$$

$$= \begin{cases} \frac{1}{2} e^{-\left(\frac{z+w}{2}\right)}, & \infty > z > w > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

$$f_Z(z) = \int_0^z f_{ZW}(z,w)\,dw = \frac{1}{2} e^{-z/2} \int_0^z e^{-w/2}\,dw$$

$$= \frac{1}{2} e^{-z/2} \int_0^{z/2} e^{-u} \cdot (2\,du) = e^{-z/2}\left(1 - e^{-z/2}\right)$$

$$= \left(e^{-z/2} - e^{-z}\right), \quad z \geq 0 \qquad (2)$$

<u>check</u> $\int^\infty f_Z(z) = \int_0^\infty e^{-z/2}\,dz - \int_0^\infty e^{-z}\,dz = 2 - 1 = \underline{\underline{1}}.$

5 $f_W(w) = \int_w^\infty f_{ZW}(z, w)\, dz$

$$= \int_w^\infty \frac{1}{2} e^{-(z+w)/2}\, dz = \frac{1}{2} e^{-w/2} \int_w^\infty e^{-\overbrace{z/2}^{u}} \cdot dz$$

$$= e^{-w/2} \int_{w/2}^\infty e^{-u}\, du = e^{-\frac{w}{2}} \left.\frac{e^{-u}}{-1}\right|_{w/2}^{\infty} = e^{-w}, \quad w \geq 0 \qquad (3)$$

From (1) - (3) we have

$$f_{ZW}(z, w) \neq f_Z(z) f_W(w)$$

$\Rightarrow$ Z, W are not independent.

Note: Interchanging (x, y) gives the same result.

3a) $$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \qquad (1)$$

$$f_{XY}(x, y) = \begin{cases} k(x^2+y), & \text{shaded area} \\ 0 \end{cases}$$

$$f_Y(y) = \int_{x=-(1-y)}^{1-y} f_{XY}(x, y)\, dx = k \int_{-(1-y)}^{1-y} (x^2+y)\, dx$$

$$= k\left(\frac{x^3}{3} + xy\right)\Bigg|_{-(1-y)}^{1-y} = 2k\left[\frac{(1-y)^3}{3} + (1-y)y\right], \quad 0<y<1$$

Using (1)

$$f_{X|Y}(x|y) = \frac{k(x^2+y)}{2k(1-y)\left(\frac{(1-y)^2}{3}+y\right)} = \frac{3(x^2+y)}{2(1-y)(y^2+y+1)},$$

$0 < \cancel{1-}y < x < 1+y,$

$0 < y < 1$

b/ Hence

$$E(X^2|Y=y) = \int_{-(1-y)}^{1-y} x^2 f_{X|Y}(x|y)\, dx$$

$$= \int_{-(1-y)}^{1-y} \frac{3x^2(x^2+y)}{2(1-y)(y^2+y+1)}\, dx = \frac{3}{2(1-y)(y^2+y+1)} \int_{-(1-y)}^{1-y} (x^4+x^2y)\, dx$$

$$= \frac{3}{2(1-y)(y^2+y+1)} \left(\frac{x^5}{5}+\frac{x^3y}{3}\right)_{-(1-y)}^{(1-y)} = \frac{3(1-y)^2}{(y^2+y+1)}\left(\frac{(1-y)^2}{5}+\frac{y}{3}\right)$$

$$= \frac{(1-y)^2(3y^2-y+3)}{5(y^2+y+1)}.$$

b) $P(X=k) = P(Y=k) = pq^k, \quad k = 0, 1, 2 \ldots \infty$

$$Z = \max(X,Y) = \begin{cases} X, & X \geq Y \\ Y, & X < Y \end{cases}$$

$$P(Z=k) = P\Big[\underbrace{(Z=k)}_{A} \cap \big\{\underbrace{(X \geq Y)}_{B} \cup \underbrace{(X<Y)}_{\bar{B}}\big\}\Big]$$

B — ME — $\bar{B}$

$$= P(AB) + P(A\bar{B}) = P(Z=k, X \geq Y) + P(Z=k, X<Y)$$

$$= P(X=k, X \geq Y) + P(Y=k, X<Y)$$

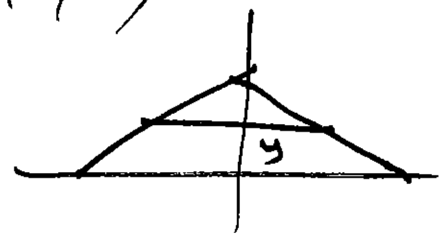
$$= P(X=k, k \geq Y) + P(Y=k, \cancel{X} < \cancel{k})$$

$$= P(X=k, Y \geq k) + P(Y=k) \cdot P(X<k)$$

$$= P(X=k, Y \geq k) + P\big[(X<k)(Y=k)\big]$$

3a) Version-2

$$f_{XY}(x,y) = k(x^2+y^2)$$



$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-(1-y)}^{1-y} k\,(x^2+y^2)\,dx = \left(\frac{x^3}{3}+y^2x\right)\Bigg|_{-(1-y)}^{(1-y)}$$

$$=2(1-y)\left[\frac{\cancel{2}(1-y)^2}{3}+y^2\right]$$

$$=\frac{2}{3}(1-y)(4y^2-2y+1)$$

$$f_{X|Y}(x|y) = \frac{3}{2}\,\frac{x^2+y^2}{(1-y)(4y^2-2y+1)}$$

$$E(X^2|Y) = \int x^2 f_{X|Y}(x|y)\,dx$$

$$= \frac{3}{2(1-y)(4y^2-2y+1)}\underbrace{\int_{-(1-y)}^{1-y}(x^4+x^2y^2)\,dx}_{\left(\frac{x^5}{5}+\frac{x^3}{3}y^2\right)\Big|_{-(1-y)}^{1-y}}$$

$$=\frac{3(1-y)^2}{2(4y^2-2y+1)}\,2\left(\frac{(1-y)^2}{5}+\frac{y^2}{3}\right) = \frac{(1-y)^2(8y^2-6y+3)}{5(4y^2-2y+1)} \quad //$$

7 or

$$P(Z=k) = P(X=k)P(Y \le k) + P(Y=k)P(X<k)$$

$$= pq^k \sum_{i=0}^{k} P(Y=i) + pq^k \sum_{i=0}^{k-1} P(X=i)$$

$$= pq^k \left(\sum_{i=0}^{k} pq^i + \sum_{k=0}^{k-1} pq^i \right) = p^2 q^k \left(2 \sum_{k=0}^{k-1} q^i + q^k \right)$$

$$= p^2 q^k \left(2 \frac{1-q^k}{1-q} + q^k \right) = 2pq^k(1-q^k) + p^2 q^{2k}$$

$$= 2p(1-q^k)q^k + p^2 q^{2k}, \quad k = 0, 1, 2 \ldots \infty$$

check

$$\sum_{k=0}^{\infty} P(Z=k) = 2p \underbrace{\sum_{k=0}^{\infty} q^k}_{\frac{1}{1-q}} - 2p \underbrace{\sum_{k=0}^{\infty} (q^2)^k}_{\frac{1}{1-q^2}} + p^2 \underbrace{\sum_{k=0}^{\infty} (q^2)^k}_{\frac{1}{1-q^2}}$$

$$= \frac{2p}{p} - \frac{2}{1+q} + \frac{p}{1+q} = 2 - \frac{(1+p+q-p)}{1+q} = 2 - 1 = 1.$$

A) See Text Ch. 6, p. 227, Eq (6-222).

$$P(X=m \mid X+Y=n) = \frac{P(X=m, X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=m, Y=n-m)}{P(X+Y=n)} = \frac{P(X=m)P(Y=n-m)}{P(X+Y=n)}$$

$$= \frac{e^{-\lambda} \frac{\lambda^m}{m!} e^{-\mu} \frac{\mu^{n-m}}{(n-m)!}}{\ldots} = \binom{n}{m} \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\lambda+\mu}\right)^{n-m}, \quad m = 0, 1, \ldots n$$

$\ldots$ Binomial.