ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 2

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Problem 1

Let the number of accidents on a highway on any given day be modelled as a Poission random variable with parameter $\lambda > 1.147$. Determine the following.

i) The probability that there is at most one accident on any given day.

ii) The probability that there is at least one accident on any given day.

iii) Which one of the above has higher probability?

Solution:

i) The probability that there is at most one accident on any given day:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-\lambda} + \lambda e^{-\lambda} = e^{-\lambda}(1 + \lambda)$$

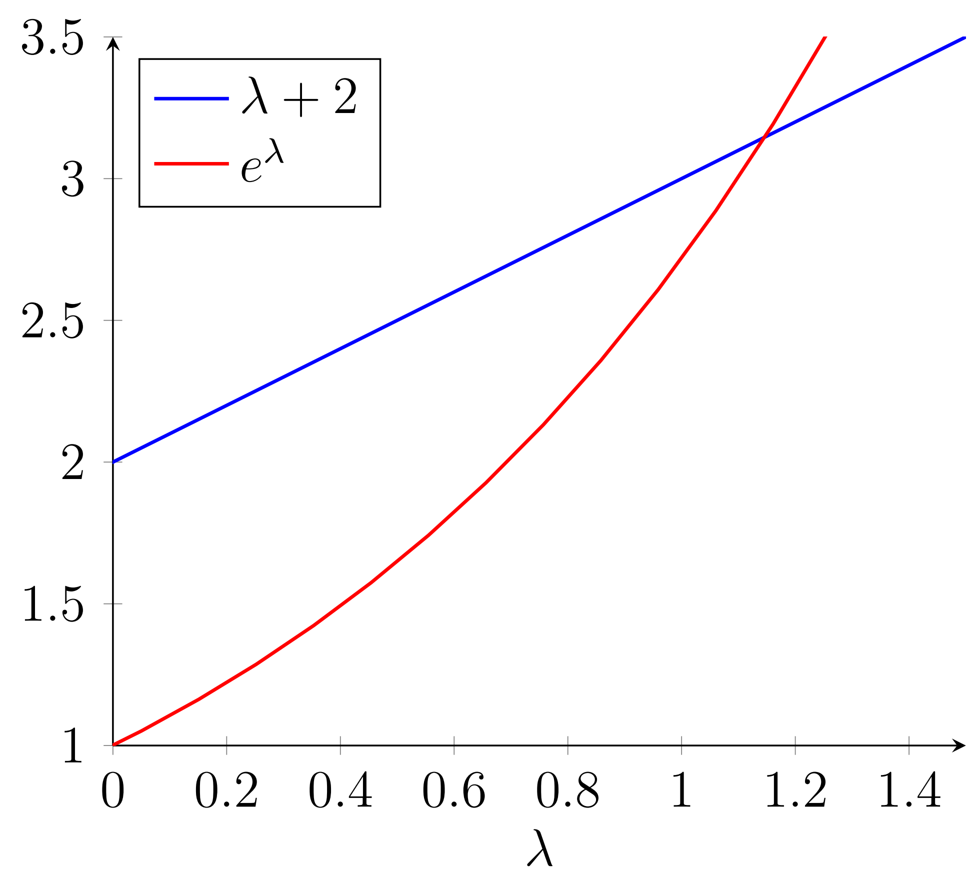
ii) The probability that there is at least one accident on any given day:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda}$$

iii)

$$P(X \leq 1) \ ? \ P(X \geq 1) \quad \Leftrightarrow \quad e^{-\lambda}(1 + \lambda) \ ? \ 1 - e^{-\lambda} \quad \Leftrightarrow \quad (2 + \lambda) \ ? \ e^{\lambda}$$

As $\lambda > 1.147$, (ii) is bigger than (i) as shown in the figure.

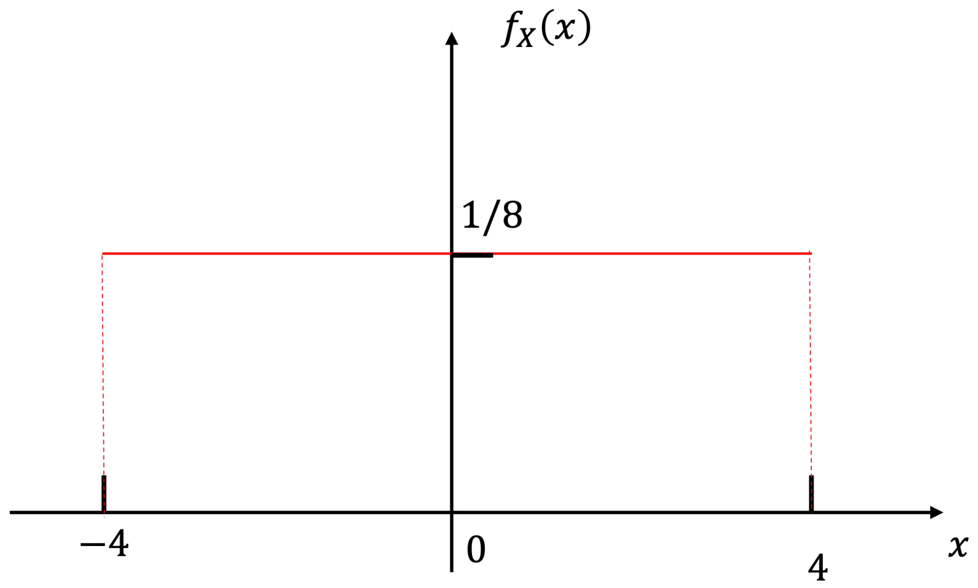


Problem 2

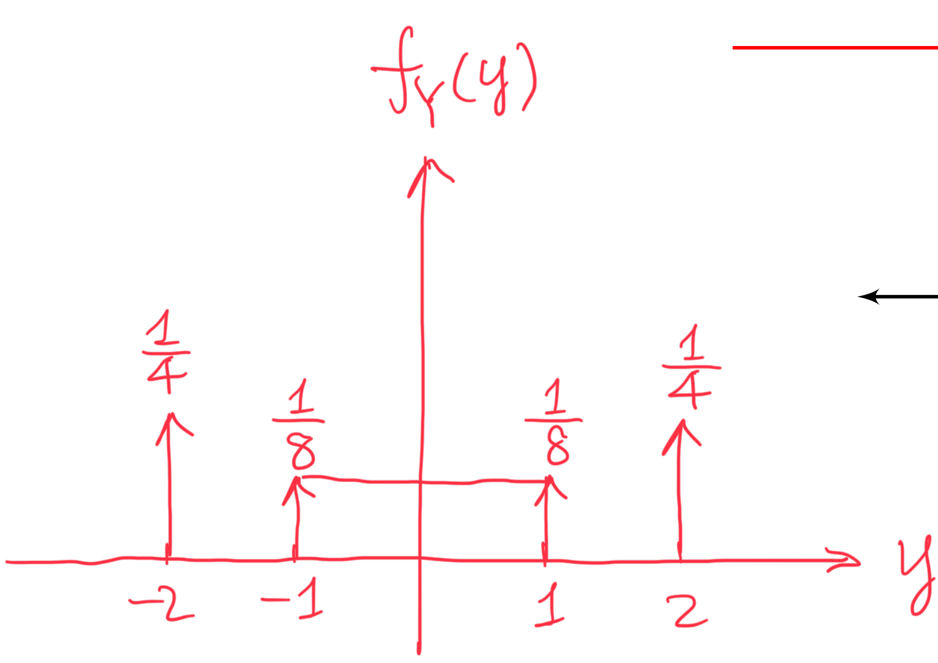
The pdf of the random variable X is is given by the following

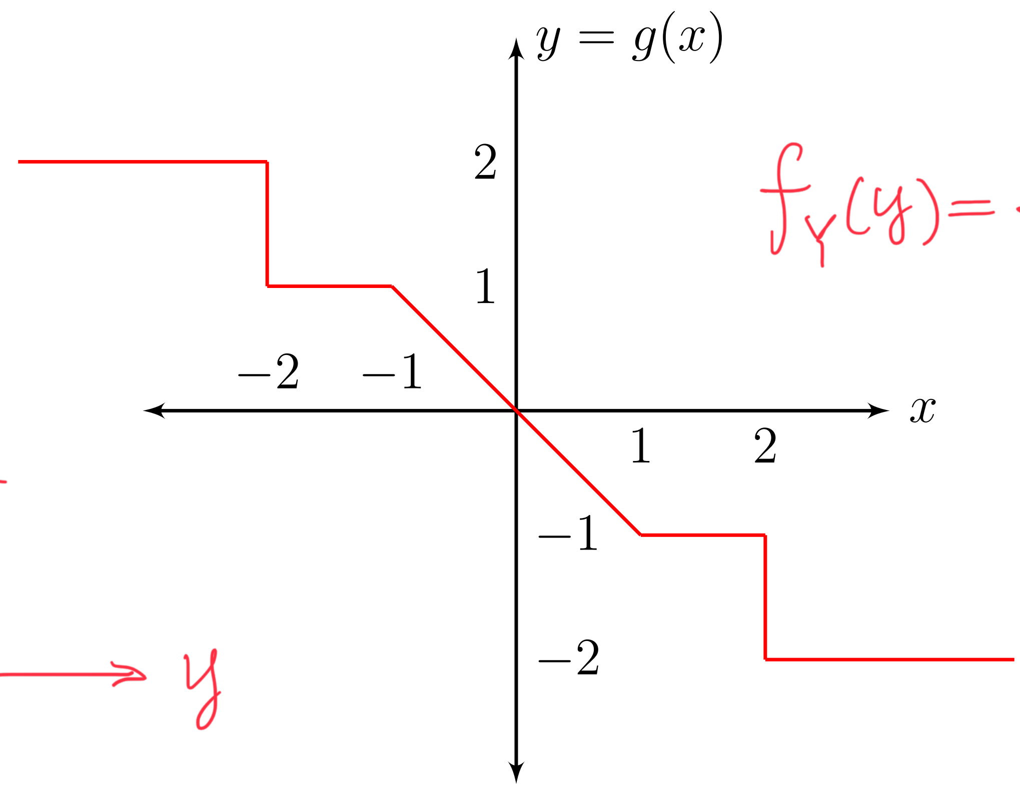
$$f_X(x) = \begin{cases} 1/8 & x \in [-4, 4], \\ 0 & \text{otherwise,} \end{cases}$$

as shown in the figure.



Find the pdf of the random variable $Y = g(X)$ shown in the figure and sketch it.





$$f_Y(y) = \begin{cases} \frac{1}{4}\delta(y+2) & y=-2 \\ \frac{1}{8}\delta(y+1) & y=-1 \\ \frac{1}{8} & -1<y<1 \\ \frac{1}{8}\delta(y-1) & y=1 \\ \frac{1}{4}\delta(y-2) & y=2 \\ 0 & \text{OTW} \end{cases}$$

Solution:

$$P(Y = 2) = P(X \le -2) = F_X(-2) = \frac{1}{4},$$
$$P(Y = 1) = P(-2 \le X \le -1) = F_X(-1) - F_X(-2) = \frac{1}{8},$$
$$P(Y = -1) = P(1 \le X \le 2) = F_X(2) - F_X(1) = \frac{1}{8},$$
$$P(Y = -2) = P(X > 2) = 1 - F_X(2) = \frac{1}{4}.$$

For $-1 \le Y \le 1$, $Y = -X$, so $F_Y(y) = P(Y \le y) = P(X \ge -y) = 1 - F_X(-y)$. Therefore $f_Y(y) = \frac{d}{dy}F_Y(y) = f_X(-y)$. The sketch should include four impulses and a constant value of $1/8$ for $-1 \le Y \le 1$.

Problem 3

Given that 10% of entering college students do not complete their Degree programs, What is the probability that out of ~~5~~ 3 randomly selected students, more than half will get their degrees?.
(Simplify and find the exact answer)

Solution:

Let $p = P(\text{A randomly selected student completes the degree program}) = 1 - 0.1 = 0.9(\text{given data})$. Let $X =$ "number of students that complete the degree pragram out of 5".

$$X \sim \text{Binomial}(n = \overset{3}{\cancel{5}}, p = 0.9).$$

We need

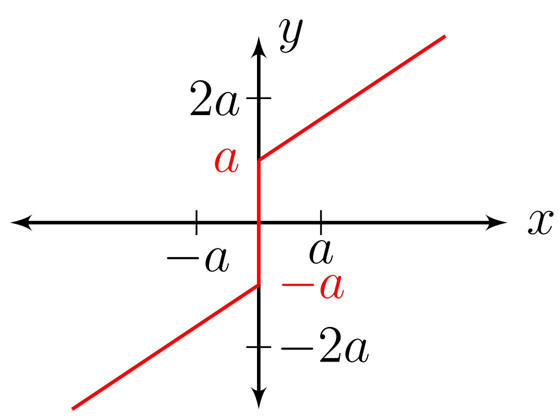
$$\begin{aligned} P(\text{3 or more students complete the degree program}) &= P(X \geq 3) \\ &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p)^1 + \binom{5}{5} p^5 (1-p)^0 \\ &= 0.99144. \end{aligned}$$

$$\begin{aligned} P(X \geq 2) = P(X=2) + P(X=3) &= \binom{3}{2} p^2 q^1 + \binom{3}{3} p^3 q^0 \\ &= 0.243 + 0.729 \\ &= 0.972 \end{aligned}$$

Problem 4

Given $f_X(x)$, and $y = g(x)$, find $f_Y(y)$ for the following.

a



Solution

$$Y = \begin{cases} X + a, & X > 0 \\ X - a, & X \leq 0 \end{cases}$$

For $y > a$,

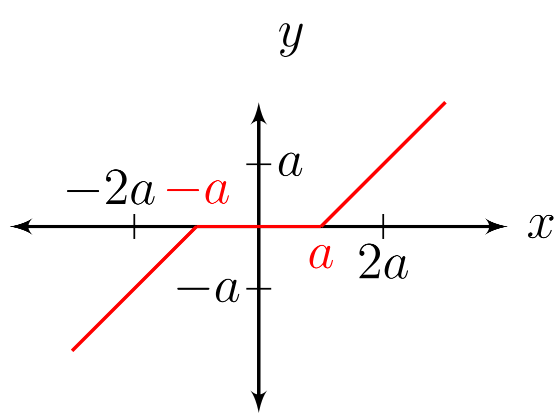
$$F_Y(y) = P(Y \leq y) = P(X + a \leq y) = P(X \leq y - a) = F_X(y - a).$$

For $y < -a$,

$$F_Y(y) = P(Y \leq y) = P(X - a \leq y) = P(X \leq y + a) = F_X(y + a)$$

$$f_Y(y) = \begin{cases} f_X(y - a) & y > a, \\ f_X(y + a) & y < -a, \\ 0 & \text{otherwise.} \end{cases}$$

b



Solution

$$Y = \begin{cases} X - a, & X > a \\ 0, & -a < X < a \\ X + a, & X \leq -a \end{cases}$$

For $y < 0$,

$$F_Y(y) = P(Y \leq y) = P(X + a \leq y) = P(X \leq y - a) = F_X(y - a).$$

For $y = 0$,

$$P(Y = 0) = P(-a \leq X < a) = F_X(a) - F_X(-a).$$

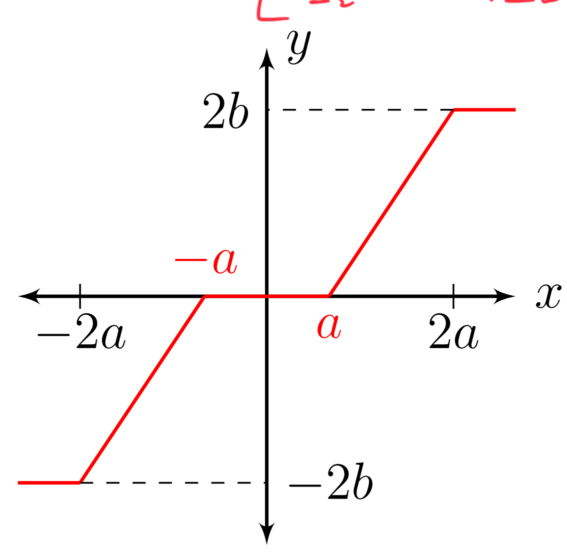
For $y > 0$,

$$F_Y(y) = P(Y \le y) = P(X - a \le y) = P(X \le y + a) = F_X(y + a).$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} f_X(y+a) & y > 0 \\ \cancel{F_X(a) - F_X(-a)} & y = 0 \\ f_X(y-a) & y < 0 \end{cases}$$

$$[F_X(a) - F_X(-a)]\,\delta(y)$$

c



Solution For $y = 2b$,

$$P(Y = 2b) = P(X > 2a) = 1 - F_X(2a)$$

For $0 < y < 2b$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(\frac{2b}{a}X - 2b \le y) = P(X \le \frac{ay + 2ab}{2b}) \\ &= F_X(\frac{ay + 2ab}{2b}). \end{aligned}$$

For $y = 0$,

$$P(Y = 0) = P(-a \le X \le a) = F_X(a) - F_X(-a)$$

For $-2b < y < 0$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(\frac{2b}{a}X + 2b \le y) = P(X \le \frac{ay - 2ab}{2b}) \\ &= F_X(\frac{ay - 2ab}{2b}). \end{aligned}$$

For $y = -2b$,

$$P(Y = -2b) = P(X \le -2a) = F_X(-2a).$$

Therefore,

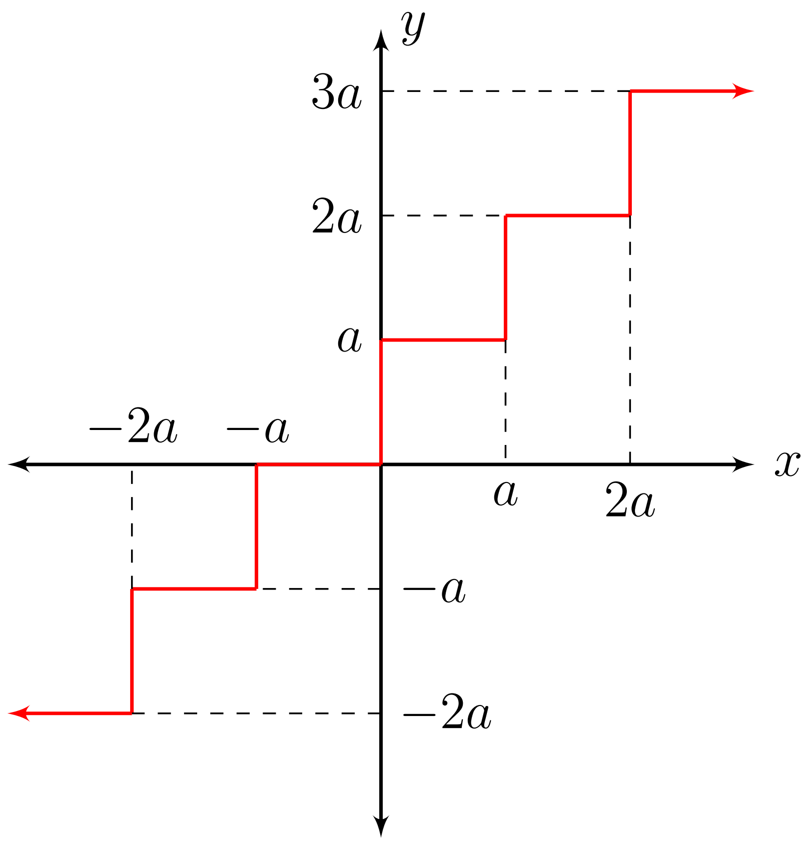
$$f_Y(y) = \begin{cases} \cancel{F_X(-2a)} & y = -2b, \\ \frac{a}{2b} f_X(\frac{ay-2ab}{2b}) & -2b < y < 0 \\ \cancel{F_X(a) - F_X(-a)} & y = 0 \\ \frac{a}{2b} f_X(\frac{ay+2ab}{2b}) & 0 < y < 2b \\ \cancel{1 - F_X(2a)} & y = 2b \end{cases}$$

$F_X(-2a)\,\delta(y+2b)$

$[F_X(a) - F_X(-a)]\,\delta(y)$

$[1 - F_X(2a)]\,\delta(y-2b)$

d



Solution

$$
\begin{aligned}
&P(Y=-2a) = P(X<-2a) = F_X(-2a),\\
&P(Y=-a) = P(-a<X<-2a) = F_X(-a) - F_X(-2a),\\
&P(Y=0) = P(-a<X<0) = F_X(0) - F_X(-a),\\
&P(Y=a) = P(0<X<a) = F_X(a) - F_X(0),\\
&P(Y=2a) = P(a<X<2a) = F_X(2a) - F_X(a),\\
&P(Y=3a) = P(X>2a) = 1 - F_X(2a).
\end{aligned}
$$

$$
\cancel{f_Y(y) = \begin{cases} F_X(-2a) & y=-2a,\\ F_X(-a) - F_X(-2a) & y=-a,\\ F_X(0) - F_X(-a) & y=0,\\ F_X(a) - F_X(0) & y=a,\\ F_X(2a) - F_X(a) & y=2a,\\ 1 - F_X(2a) & y=3a. \end{cases}}
$$

$$
f_Y(y) = \begin{cases} F_X(-2a)\,\delta(y+2a) & y=-2a\\ [F_X(-a) - F_X(-2a)]\,\delta(y+a) & y=-a\\ [F_X(0) - F_X(-a)]\,\delta(y) & y=0\\ [F_X(a) - F_X(0)]\,\delta(y-a) & y=a\\ [F_X(2a) - F_X(a)]\,\delta(y-2a) & y=2a\\ [1 - F_X(2a)]\,\delta(y-3a) & y=3a \end{cases}
$$

Problem 5

Solution

a

Given $f_X(x)$, find $f_Y(y)$ when,

i. $y = ax + b$.

$$x_1 = \frac{y-b}{a}, \quad \frac{dy}{dx} = a, \quad f_Y(y) = \frac{f_X(x_1)}{|\frac{dy}{dx}|_{x=x_1}} = \frac{f_X(\frac{y-b}{a})}{|a|}$$

ii. $y = ax^2 + b$.

$$x_1 = \sqrt{\frac{y-b}{a}}, \quad x_2 = -\sqrt{\frac{y-b}{a}}, \quad \frac{dy}{dx} = 2ax,$$

$$f_Y(y) = \frac{f_X(x_1)}{|\frac{dy}{dx}|_{x=x_1}} + \frac{f_X(x_2)}{|\frac{dy}{dx}|_{x=x_2}} = \frac{f_X(\sqrt{\frac{y-b}{a}})}{|2a|\sqrt{\frac{y-b}{a}}} + \frac{f_X(-\sqrt{\frac{y-b}{a}})}{|2a|\sqrt{\frac{y-b}{a}}}$$

iii. $y = |x|$.

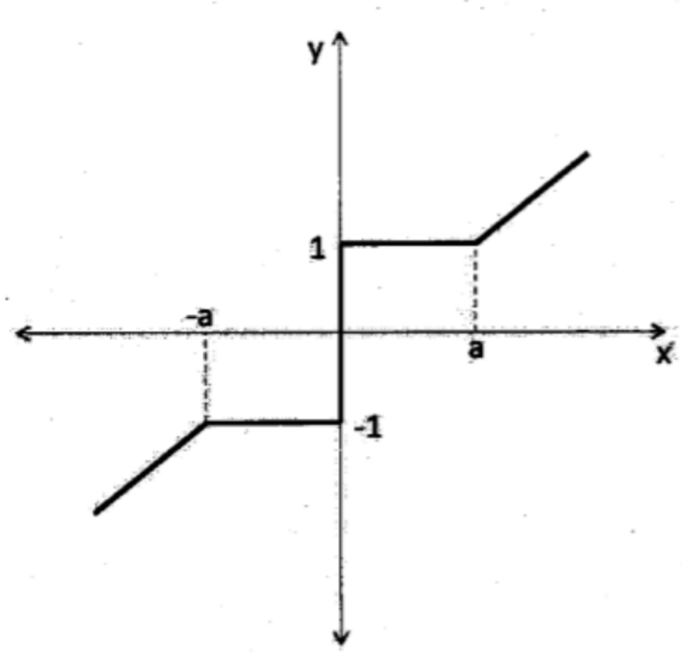
$$F_Y(y) = P(Y \leq y) = P(-x \leq X \leq x) = F_x(x) - F_X(-x),$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \cancel{f_X(x) + f_X(-x)}$$

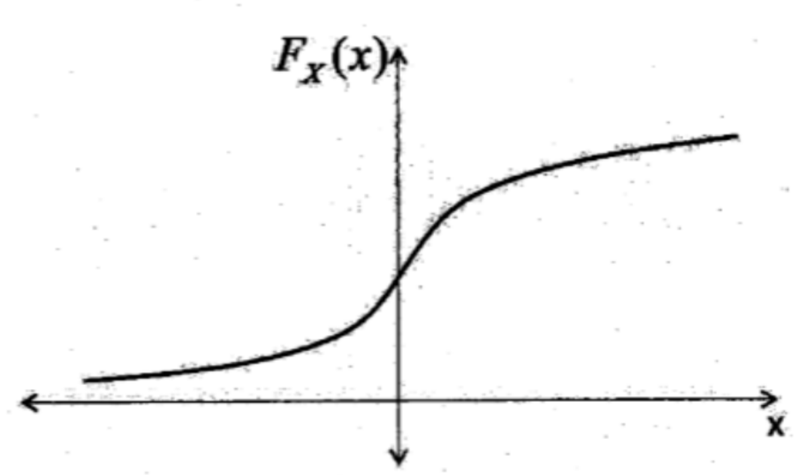
$$f_X(y) + f_X(-y)$$

b

Given $F_X(x)$, find and plot $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$.



(a)



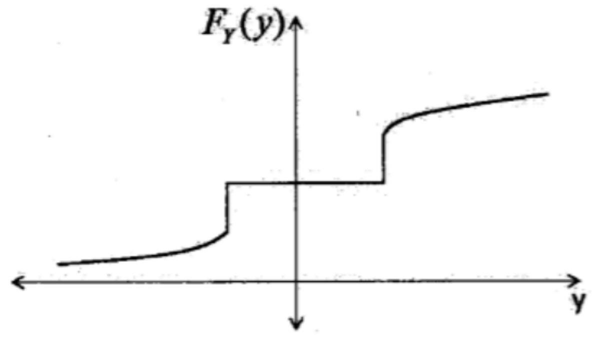
(b)

Solution: If $y = x$ for $x > a$ and $x < -a$, then $a = 1$:

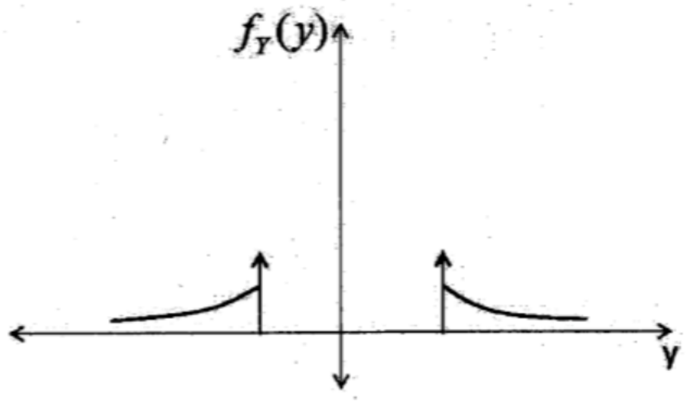
- If $y < -1$, $F_Y(y) = P(X < y) = F_X(y)$.
- If $y = -1$, $P(Y = -1) = P(-1 < x < 0) = F_X(0) - F_X(-1)$.
- If $-1 < y < 1$, $P(-1 \leq Y \leq 1) = P(X = 0) = 0$.
- If $y = 1$, $P(Y = 1) = P(0 < x < 1) = F_X(1) - F_X(0)$.
- If $y > 1$, $F_Y(y) = P(X < y) = F_X(y)$.

$$F_Y(y) = \begin{cases} F_X(y) & \text{if } y < -1 \\ F_X(0) & \text{if } -1 \leq y < 1 \\ F_X(1) & \text{if } y = 1 \\ F_X(y) & \text{if } y > 1 \end{cases} \tag{1}$$

$$f_Y(y) = \begin{cases} f_X(y) & \text{if } y < -1 \\ (F_X(0) - F_X(-1))\delta(y+1) & \text{if } y = -1 \\ 0 & \text{if } -1 < y < 1 \\ (F_X(1) - F_X(0))\delta(y-1) & \text{if } y = 1 \\ f_X(y) & \text{if } y > 1 \end{cases} \tag{2}$$



(c)



(d)

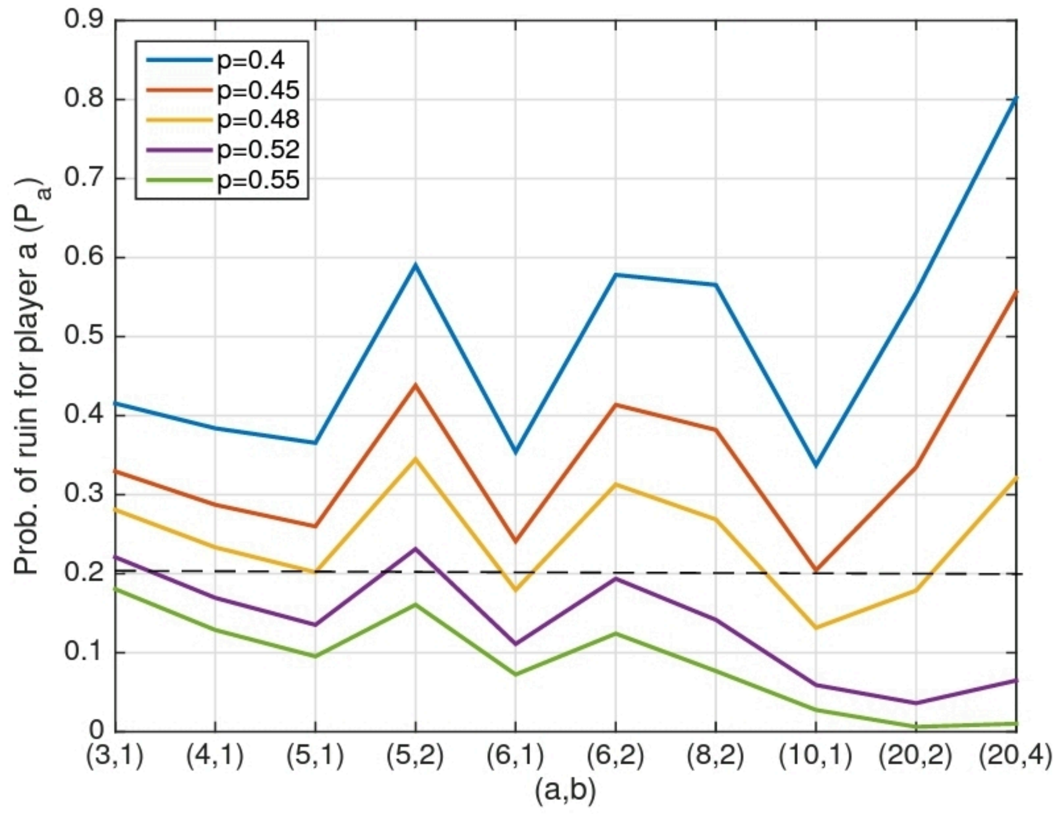
Note: If your solution is in terms of a, it is also correct.

Problem 6

Refer to Gambler's Ruin problem notations discussed in class. Plot risk (Probability of ruin) versus invesetment a for various returns b. Find the best combination (a, b) for 20%, 30% risk for p=0.45, 0.47, 0.52 and 0.55. You can present your results as a table or a figure.

Solution:

For example, as shown in the following figure, the combinations $(a, b) = (5, 1)$ and $(a, b) = (6, 1)$ satisfy 20% risk for $p = 0.48$. Similarly, the combination $(a, b) = (10, 1)$ also satisfies 20% risk for $p = 0.45$, etc.



(a) Prob. of ruin of A, i.e., B wins

Problem 7 (Problem 4 from HW#1)

The pdf of a continous random variable X is given by

$$f_X(x) = \begin{cases} \frac{1}{7} & -2 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find
(i) $P(X^2 > 1)$
(ii) $P(\sin(\pi X) \le 0)$

Solution:
(i)

$$\begin{aligned} P(X^2 > 1) &= P((X < -1) \cup (X > 1)) = P(X < -1) + P(X > 1) \\ &= \frac{1}{7} + \frac{1}{7} \cdot 4 = \frac{5}{7}. \end{aligned}$$

(ii)

$$\begin{aligned} P(\sin(\pi X) \le 0) &= P((-1 \le X \le 0) \cup P(1 \le X \le 2) \cup P(3 \le X \le 4)) \\ &= P(-1 \le X \le 0) + P(1 \le X \le 2) + P(3 \le X \le 4) = \frac{3}{7}. \end{aligned}$$