Math 797MF: Modular Forms

3 credits, Spring 2025, Time/Location TBA

Instructor Information. Paul Gunnells (LGRT 1115L, 545-6009, gunnells@umass.edu)
Office Hours: TBA

Appointment hours. TBA

Course Description. This course treats modular forms, their relation to arithmetic and geometry, and other applications. Modular forms are a fundamental tool in modern number theory and have applications to many other fields, such as algebraic geometry, representation theory, combinatorics, and mathematical physics. Topics include the following: geometry of the upper half plane and quotients by congruence subgroups, Eisenstein series/cuspforms/theta series, Fourier expansions, L-functions, Hecke theory, applications of modular forms to arithmetic and combinatorics. Prerequisites: Math 611–612 or the consent of the instructor. Other helpful courses: number theory, complex analysis.

Student Learning Objectives. The primary learning objective of this course is for students to learn the language of modular forms. The goal of the course is to prepare students to do research in mathematical disciplines which rely on modular forms as well as to prepare them for more advanced courses in number theory/arithmetic geometry. An additional objective involves working on mathematical writing. Specific learning objectives:

- Students will learn the language of modular forms.
- Students will become familiar with basic examples and theorems, such as the generation of the ring of modular forms of full level.
- Students will learn advanced applications of modular forms functions such as connections to elliptic curves and Galois representations.
- It is also important to practice mathematical exposition, since that forms a basic skill needed for writing a dissertation. To that end, in this class students will learn how to write an expository mathematics paper on an advanced topic.

Delivery Mode. The class will consist of 150 min of face-to-face lecture per week.

Course materials. There is no required textbook. Some materials for reading will be provided by the instructor. Some suggested books/papers that will occasionally be mentioned:

1. Bruinier, J. H., van der Geer, G., Harder, G., and Zagier, D. The 1-2-3 of modular forms: Lectures at a summer school in Nordfjordeid, Norway. Springer-Verlag, 2008.
2. Fred Diamond, Jerry Shurman, A First Course in Modular Forms, Springer-Verlag, New York, 2006, Graduate Texts in Mathematics, No. 228.
3. Anthony W. Knapp, Elliptic curves, Mathematical Notes, vol. 40, Princeton University Press, Princeton, NJ, 1992.
4. J.-P. Serre, A course in arithmetic, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7.
5. Don Zagier, Introduction to modular forms, From number theory to physics (Les Houches, 1989), Springer, Berlin, 1992, pp. 238291.

6. Goro Shimura, Introduction to the arithmetic theory of automorphic functions, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kano Memorial Lectures, 1.
7. Serge Lang, Introduction to modular forms, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 222, Springer-Verlag, Berlin, 1995, With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.

Course requirements and grade weights.

Course participation. Students will be expected to actively participate in the course. Participation is defined very broadly and includes, but is not limited to, the following:

1. Asking questions/making comments during lecture, by email or in office hours.
2. Attending class regularly.
3. Discussing course material with the instructor and their peers.
4. Discussing ideas and interests for the expository paper with the instructor.
5. Discussing the writing process and seeking additional help (as needed) with preparing the expository paper.

It is understood that students with a variety of obligations and interests will be taking the course. This component will count 50%.

Final paper. Together with the instructor, by the end of week 7 each student will pick a topic of interest to them that is related to the course material (A list of suggested topics will be made available, but students are free to pick others). The student will then prepare an expository paper of up to 10 pages on this topic and will submit a first draft by the end of week 10. The papers will be reviewed by the instructor and given back to students for revisions, which should be done by the end of the term. More details about timing are given below in the lecture schedule. The instructor will be available for additional writing support outside scheduled office hours. This assignment will count 50%.

Grading Scale. The final grade will be based on course participation (50%) and the final paper (50%). Letter grades will be assigned as follows:

A	90–100	B-	74–78
A-	86–90	C+	70–74
B+	82–86	C	66–70
B	78–82	F	–66

Course Schedule.

Week 1: Basics of modular forms of full level and higher level. The upper halfplane, fundamental domains, congruence subgroups.
Week 2: Geometry of quotients. Cusps and compactifications. Definition of modular forms. Weight and level. Ring structure.
Week 3: Basic examples (Eisenstein series, Δ, theta series). q-expansions. Condition at infinity. Cusp forms. Connection to cohomology.
Week 4: L-functions and functional equations.
Week 5: Convergence and special values.

Week 6: Hecke operators and Euler products of L-functions. Idea of Hecke operators. Lattice functions and Correspondences.
Week 7: Definition. Basic properties. Action on the space of modular forms. Petersson product. Basis of eigenforms. *Topic due for expository paper.*
Week 8: Implications for q-expansions. Euler products. Complements. The Bruhat-Tits tree and Hecke operators. *Meetings to discuss writing.*
Week 9: Applications I: Arithmetic geometry. Definition of elliptic curves and their L-functions. Modularity of elliptic curves. *Meetings to discuss writing.*
Week 10: Dihedral extensions. Weight 1 modular forms and dihedral Galois representations. *First draft of paper due.*
Week 11: Applications II: Ramanujan graphs. Definitions, examples. Deligne's theorem. *Drafts returned for revisions.*
Week 12: Construction of Ramanujan graphs.
Week 13: Applications III: sphere packings. *Revisions due. Papers re-read by instructor for final comments.*
Week 14: Review and summary. *Final version of paper due by last day of classes.*

Attendance Policy. Attendance at all classes is expected. However, formal attendance will not be taken.

Late or Make-up work policy. The only submitted assignment is the final paper, and it is not possible to submit it late or to have a make-up.

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