Perishable Inventory Theory: A Review

STEVEN NAHMIAS

The University of Santa Clara, Santa Clara, California

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This paper reviews the relevant literature on the problem of determining suitable ordering policies for both fixed life perishable inventory, and inventory subject to continuous exponential decay. We consider both deterministic and stochastic demand for single and multiple products. Both optimal and sub-optimal order policies are discussed. In addition, a brief review of the application of these models to blood bank management is included. The review concludes with a discussion of some of the interesting open research questions in the area.

THIS ARTICLE reviews the current literature on ordering policies for perishable inventories. Most inventory models assume that stock items can be stored indefinitely to meet future demands. However, certain types of inventories undergo change in storage so that in time they may become partially or entirely unfit for consumption. For example, lysis of red blood cells renders blood unacceptable for transfusion 21 days after it is drawn. Fresh produce, meats and other foodstuffs become unusable after a certain time has elapsed. Photographic film and drugs are further examples of items which have a limited useful lifetime.

We will consider two classifications of perishability: fixed lifetime and random lifetime. The former category includes those cases where the lifetime is known *a priori* to be a specified number of periods or a length of time independent of all other parameters of the system. As we will show, the latter category will include exponential decay as a special case and will also include those cases where the product lifetime is a random variable with a specified probability distribution. (Van Zyl [1964] has used the classification "age dependent" perishability and "age independent" perishability to distinguish between fixed life and exponential decay. Our scheme is broader; it includes other random lifetime models.)

The review is organized as follows:

- I. Fixed Life Perishability
  - 1. Deterministic Demand
  - 2. Stochastic Demand
    - a. Optimal Policies for a Single Product
    - b. Approximately Optimal Policies for a Single Product

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A problem which is closely related to finding ordering policies for perishable goods is that of finding suitable issuing policies. The issuing problem as originally formulated by Derman and Klein [1958] is to determine an optimal sequence to remove items from a stockpile of finitely many units of varying ages. It is assumed that an item which is issued at age s has a "field life" of $L(s)$, where L is some known function. An item is issued only when the previous item issued has expired so that the total field life of the stockpile will depend upon the sequence in which items are removed from the stockpile. The general approach has been to specify conditions on L for which issuing either the oldest units first (FIFO) or the newest units first (LIFO) is optimal.

In this review we will restrict attention only to ordering policies for perishables for two reasons. First, for most fixed life inventory problems, it is always optimal to issue the oldest items first (FIFO). Second, the classical issuing model is unrelated to the standard inventory management issues of regular balancing of stock levels to minimize system operating costs. (Issuing policies for perishables was treated by Pierskalla and Roach [1972]. The most recent work on the problem, which considers random field lives, is due to Albright [1976] and Nahmias [1974].)

Another related problem which we will not consider in the review is that of determining ordering policies for inventory which is subject to obsolescence. Obsolescence is modeled by assuming that the length of the planning horizon is random and is fundamentally different from perishability in that once items become obsolete they are not reordered. The problem has been considered by Barankin and Denny [1965], Brown et al. [1964], and Pierskalla [1969].

1. FIXED LIFE PERISHABILITY

The fixed life assumption is that units may be retained in stock to satisfy demand for some specified fixed time after which they must be discarded. Existing models assume that all units which have not expired are of equal utility.

1.1. Deterministic Demand

A general property of deterministic perishable inventory models is that under fairly general conditions an optimal policy will always order in such a way that items will never perish. Consider, for example, the simple EOQ model for a product with a lifetime of m. The optimal batch size which minimizes the holding and set-up cast rate is $Q^* = \sqrt{2K\lambda/h}$, where K is the set-up cost per order, λ the demand rate and h the cost of holding a unit for a unit time. Then it is easy to see that the optimal order size is $a = \min(Q^*, \lambda m)$ which ensures that no units expire.

Veinott [1960] treats periodic review and known demand. He examines three distinct problems: (1) determining an optimal ordering policy when the disposal and issuing policies are given, (2) determining optimal ordering and disposal policies when the issuing policy is given, and (3) determining optimal issuing and disposal policies when the ordering policy is given. (Disposal refers to the reduction of stock levels without satisfying demand.) For problem (1) he shows that when the field life function is nonincreasing in the item's age at issue and a FIFO issuing policy is used, an optimal policy will order an amount equal to demand over a sequence of periods. He also considers other types of issuing and disposal policies.

1.2. Stochastic Demand

Stochastic perishable inventory models are typically quite complex. However, one straightforward case occurs when inventory levels are reviewed periodically and units can be retained in stock no more than a single period. If the lifetime is exactly one period, the ordering decisions in successive periods are independent and the problem reduces to a sequence of simple newsboy problems (see Arrow et al. [1958]). The case where the delivery perishes immediately with probability p and after one period with probability $(1 - p)$ was treated by Bulinskaya [1964]. The analysis is a direct extension of that for the simple newsboy problem.

When demand is random and the product lifetime exceeds one period, determining optimal ordering policies is difficult. Since it is no longer possible to obtain an order policy so that there is no perishing, the state vector must include the stock level of each possible age category. Suppose stock levels are reviewed periodically and units have a useful lifetime of exactly m periods. Then the state vector $\mathbf{x} = (x_{m-1}, \cdots, x_1)$ gives starting stock levels of each age category where x_i = quantity of stock on hand which is scheduled to perish after i periods. Let $x = \sum_{i=1}^{m-1} x_i$ be the total starting stock of all age categories.

It is the one period transfer equation which expresses the process dynamics and indicates the complexity of the problem. Define the vector $\mathbf{s}(y, \mathbf{x}, t) = [s_{m-1}(y, \mathbf{x}, t), \cdots, s_1(y, \mathbf{x}, t)]$ as the levels of stock of each age category one period into the future when y is the quantity of fresh stock ordered, $\mathbf{x}$ is the vector of stocks currently on hand, and t is the realization of demand in the current period. Then, assuming FIFO issuing,

$$s_i(y, \mathbf{x}, t) = [x_{i+1} - (t - \textstyle\sum_{j=1}^{i} x_j)^+]^+ \quad \text{for} \quad 1 \le i \le m-2$$

and

$$s_{m-1}(y, \mathbf{x}, t) = \begin{cases} y - (t-x)^+ & \text{if excess demand is backlogged} \\ [y - (t-x)^+]^+ & \text{if sales are lost} \end{cases}$$

where $g^+ \equiv \max(g, 0)$.

1.2.a. Optimal Policies for a Single Product

(i) Two Period Lifetime

The first analysis of optimal policies for a fixed life perishable commodity was due to Van Zyl. He considered the case where the product lifetime is exactly two periods with proportional costs of ordering and shortage at c and p per unit respectively. Suppose that demands in successive periods, $D_1, D_2, \cdots$, are independent identically distributed random variables with cumulative distribution function F and density f. The dynamic programming functional equations describing an optimal policy are

$$C_n(x) = \min_{y \ge 0}\left\{cy + p\int_{x+y}^{\infty}(t-x-y)f(t)dt + \alpha\int_0^{\infty} C_{n-1}\{[y-(t-x)^+]^+\}f(t)dt\right\} \quad \text{for} \quad n \ge 1,$$

and $C_0(x) \equiv 0$. Interpret $0 < \alpha < 1$ as the one period discount factor. Van Zyl assumed lost sales and considered both finite and infinite horizons. He showed that the optimal order policy when n planning periods remain, say $y_n(x)$, is a continuously differentiable function of the state variable x and that $-1 < y_n'(x) \le 0$. That is, if the starting level of one period old stock increases by one unit, the optimal order quantity will decrease, but by less than one full unit.

Nahmias and Pierskalla [1973] take a slightly different approach to the $m = 2$ case. Suppose costs are levied only against outdating (perishing) and shortages. (The idea of developing a policy which balances the expected outdating and expected shortages arises in blood bank management. See Jennings [1973a].) Since the current order will not outdate until one period into the future, the optimal one period decision minimizes the expected shortage cost and thus orders an infinite quantity of stock.

To circumvent this problem, notice that the outdating of the current order y one period hence, say Z, is a function of both the current stock level x and demands over the next two periods. Clearly

$$Z = [y - (D_1 - x)^+ - D_2]^+,$$

from which one can show

$$E(Z) = \int_0^y F(v + x)F(y - v)dv.$$

Let θ be the per unit cost of outdating. Then a one period expected cost function which balances expected outdates and expected shortages is given by

$$L(x, y) = p \int_{x+y}^{\infty} (t - x - y)f(t)dt + \theta \int_0^y F(v + x)F(y - v)dv.$$

An optimal single period decision, say $y_1(x)$, will solve $L[x, y_1(x)] = \min_{y\geq 0} L(x, y)$. Optimal policies for the multiperiod version of this model and for a model with ordering and holding costs as well were also considered by Nahmias and Pierskalla [1973] by analyzing the appropriate functional equations. Both lost sales and full backordering are assumed.

When both holding and ordering costs are included, it is convenient to assume that stock remaining on hand at the end of the final period can be salvaged at a return equal to the purchase price c and that excess demand in the final period can be made up by an emergency shipment at a cost of c per unit. Mathematically, this is equivalent to assuming the initial condition $C_0(x) = -cx$. It is interesting to note that the salvage value assumption will result in the ordering region being stationary even though the optimal policy is not. That is, for each $n \geq 1$, $y_n(x) > 0$ if and only if $x < \bar{x}$ where

$$\bar{x} = F^{-1}[(p - (1 - \alpha)c)/(p + h)].$$

This particular form for $\bar{x}$ assumes backordering. A similar result holds for lost sales. Note that $\bar{x}$ is the optimal stationary critical number policy for associated nonperishable problems. Although the optimal ordering region is independent of θ the optimal order quantity is not.

Figure 1 compares the policy $y_\infty(x)$ (that is, the stationary optimal policy) when the lifetime is two periods with the corresponding optimal order policy for the same problem with an infinite lifetime for one particular case (see Nahmias [1975c] for the parameters used in this case). With perishability, an optimal policy will order less because of outdating. Furthermore, the difference between the two policies is greatest for small values of the starting stock. Perishability has the effect of discouraging large orders; when a great deal of stock of one age category enters the system at one time it is likely that some of this stock will perish.

Additional structural results which are evident in this case from Figure 1 and have been proved by Nahmias and Pierskalla [1973] are that $0 < x + y_n(x) < \bar{x}$ for all $0 < x < \bar{x}$ and $y_n(x) = |x| + y_n(0)$ if $x < 0$ (that is, if demand is backlogged, the optimal order quantity is the backlog plus what would be ordered from zero following an optimal policy).

(ii) The General m Period Lifetime Problem

Extensions of these two models to allow for lifetimes of $m \geq 2$ periods were developed simultaneously and independently by Fries [1975] and

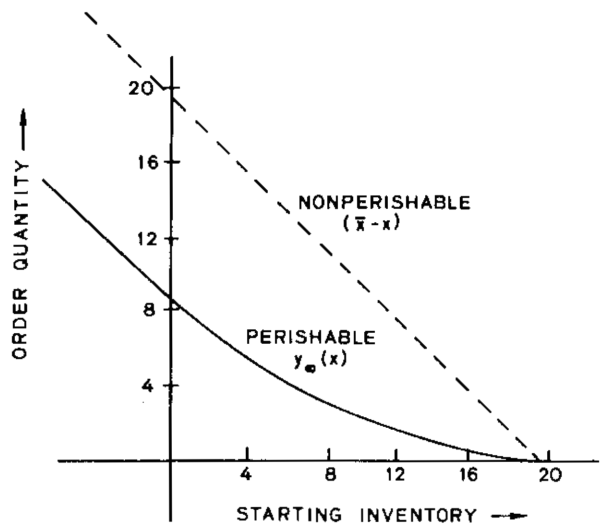

Figure 1. Optimal order quantities for perishable and nonperishable inventories.

Nahmias [1975a]. In principle, the analytical approach is similar to $m = 2$, but the analysis is far more complicated owing to the required multidimensional state variable and the involved form of the transfer function. Fries considered an expected one period cost function of the form

$$L(\mathbf{x}, y) = cy + R(x + y) + \theta \int_0^{x_1} (x_1 - t)f(t)dt$$

where R is the expected holding and shortage cost function, assumed to be convex. Note that the final term represents the expected outdating in the current period, which is independent of the current ordering decision, y. Fries allows for a restricted class of nonlinear outdating functions. Also, his decision variable is the total stock on hand after ordering rather than just the order quantity. Our notation follows Nahmias' work.

Nahmias (and Nahmias and Pierskalla [1975], which considered only shortage and outdate costs) included the expected future costs of the outdating of the current order into the expected one period cost function. He shows that the expected outdating of the current order y (which will occur m periods from now) is given by $\int_0^y G_m(u, \mathbf{x})du$ where G_m is obtained by recursively solving

$$G_n[\mathbf{x}(n)] = \int_0^{x_n} G_{n-1}[v + x_{n-1}, \mathbf{x}(n-2)]f(x_n - v)dv$$

with $G_1(x_1) = F(x_1)$ and $\mathbf{x}(n) = (x_n, \cdots, x_1)$. The one period expected cost function in Nahmias [1975a] is

$$L(\mathbf{x}, y) = cy + R(x + y) + \theta \int_0^y G_m(u, \mathbf{x})du.$$

Both authors analyzed functional equations of the form

$$C_n(x) = \min_{y \geq 0}[B_n(\mathbf{x}, y)],$$

where

$$B_n(\mathbf{x}, y) = L(\mathbf{x}, y) + \alpha \int_0^\infty C_{n-1}[\mathbf{s}(y, \mathbf{x}, t)]f(t)dt,$$

and proved in each case that $B_n(\mathbf{x}, y)$ is convex in y. Optimal ordering policies when n periods remain, say $y_n(\mathbf{x})$, differ structurally in the two approaches only in the final $m - 1$ periods of the horizon. In Fries' model $y_n(\mathbf{x})$ depends on $\mathbf{x}$ only through x and the vector $(x_n, \cdots, x_1)$ in periods $n < m$. When $n \geq m$, the optimal policies are structurally the same and are, in fact, identically the same when the future cost of outdating is discounted to the present in Nahmias' model by replacing θ with $\alpha^{m-1}\theta$. (See Nahmias [1977a].)

Let $y_n^{(i)}(\mathbf{x}) = (\partial/\partial x_{m-i})(y_n(\mathbf{x}))$. Then for $n \geq m$ in Fries' model and $n \geq 1$ in Nahmias' model, both authors show that

$$-1 \leq y_n^{(1)}(\mathbf{x}) \leq y_n^{(2)}(\mathbf{x}) \leq \cdots \leq y_n^{(m-1)}(\mathbf{x}) < 0$$

for all $\mathbf{x}$ in the positive ordering region (which is again $x = \sum_{i=1}^{m-1} x_i < \bar{x}$ when the appropriate salvage value is included). This says that if the initial stock of inventory at any age level is increased by one unit, the optimal order quantity decreases, but by less than a single unit. Furthermore, the optimal order quantity is more sensitive to changes in newer inventory.

Another significant result is that for $x < \bar{x}$, $y_n(\mathbf{0}) \leq x + y_n(\mathbf{x}) \leq \bar{x}$ which says that after ordering, total stock will always fall between the optimal order quantity when starting inventory is zero and $\bar{x}$. When $x < 0$, the lower bound holds with equality.

As with the $m = 2$ case discussed above, perishability decreases the size of the optimal order quantity, and the difference between the perishable and nonperishable solutions is largest for small values of starting stock.

In this writer's experience, Fries' approach is more efficient for actually computing an optimal policy. Both methods are roughly equivalent when the demand distribution is Erlang, since in this case one can use explicit formulas for G_m developed in Nahmias [1975c]. Owing to the multidimensional state variable, the computation time using either model is quite long for $m \geq 3$, making computation of an optimal policy impractical for a real problem.

(iii) Extension to Include Set-up Cost

When a set-up cost for ordering is included, optimal policies are extremely complex. Nahmias [1978] has extended the general model described above to include a positive set-up cost for ordering and derived the solution structure for the single period model. The optimal policy is specified by two nonlinear functions of the state vector $\mathbf{x}$ which define the optimal order quantity and the region of positive ordering respectively. Although no proof was presented, computations indicated that the optimal policy possessed a similar structure in the multiperiod case.

(iv) Continuous Review

Weiss [1980] treats optimal policies for a perishable inventory assuming inventory levels are reviewed continuously. Demands are assumed to follow a stationary Poisson process and the leadtime for placing an order is zero. The model generalizes Sivazlian's [1974] to the case of a fixed life commodity. The objective is to determine the ordering policy which minimizes the infinite horizon expected set-up, ordering, holding and outdating cost per unit time.

Weiss proves that an optimal policy orders only when the stock level is zero. This implies that there is no need to worry about the age distribution of stock, since a new order will not be placed until the previous one has been depleted by either demand or perishing. Under the Poisson assumption, successive cycles (that is, time between arrivals of orders) are independent. The usual method of regenerative processes is used to determine an expression for the expected average cost per unit time, say $C(S)$, which can then be minimized with choice of the order quantity, S. He shows that $-C(S)$ is unimodal and that the optimal S is a function of four variables: $\lambda K/h$, $\lambda(c - r)h$, λd and $\lambda(v + r)h$. (λ, K, h

and c are as defined above; r is unit revenue, v the disposal cost of perished goods and d the lifetime of each item.) His computations suggest that the optimal S is nondecreasing in all four variables.

Weiss' paper is interesting in that it is the only one published which considers a continuous review perishable inventory problem. However, its practical value seems limited because of the simultaneous assumptions of continuous review, single unit demand, and zero leadtime. These assumptions imply that an optimal policy orders only when stock level is zero. With positive leadtime or bulk demand this policy would clearly perform quite poorly.

1.2.b. Approximately Optimal Policies

(i) Types of Approximations

In addition to being difficult to compute, optimal policies are difficult to implement due to the accounting required to keep track of the age distribution of stock. Hence good approximations are of considerable interest.

The first issue is what is a suitable form for an approximation. This problem was considered by Nahmias [1975b] who compared the effectiveness of three simple policies. The three policies use only knowledge of total starting stock, x, so that the age distribution of the inventory is not required, and each policy depended on only a single parameter. One policy for consideration is the critical number, or base stock policy, which calls for ordering to a fixed level, say x^* (with x^* different from $\bar{x}$).

From Figure 1, it would appear that if we were able to estimate $y_\infty(0)$ then the order policy obtained by connecting that estimate to $\bar{x}$ with a straight line would also be a good policy (that is, the order quantity would be $\beta(\bar{x} - x)$ where $0 < \beta < 1$ and $\beta\bar{x}$ would estimate $y_\infty(0)$). This will be referred to as the linear policy.

The critical number, linear and a third policy suggested by Van Zyl were compared by means of simulation. For each value of the policy parameter, the inventory process was simulated for 10^6 periods. Optimal values of the parameters were then obtained by a Fibonacci search of the response surface corresponding to expected total discounted cost. The most salient result was that both the best critical number and best linear policies resulted in expected costs that were quite close to each other and close to the optimal policy.

(ii) Optimal Critical Number Policies

A critical number policy is easier to implement than a linear policy, so it is only natural that subsequent interest on approximations focused on it. If possible, one would like to determine the best critical number policy. This problem was considered by Cohen [1976] who applied the usual method of determining the stationary distribution of stock levels and subsequently finding the critical number S that minimized expected cost. Cohen assumed backordering of excess demand.

Define A_n as the total decrease in on hand stock in period n due to the combined effects of demand and outdating. Cohen shows that the vector valued stochastic process $\mathbf{Y}_n = (A_{n-m+1}, \cdots, A_{n-1})$ forms a positive recurrent discrete time continuous state Markov chain. Unfortunately, explicit results for the stationary distribution of $\mathbf{Y}_n$ could be obtained only for $m = 2$ (Van Zyl obtained similar results assuming lost sales). In that case, the stationary distribution, say $G(y)$, is given by

$$G(y) = F(y)\cdot[1 - F(S - y)]/[1 - F(y)F(S - y)], \qquad \text{for} \quad y \geq 0$$

where F is the cumulative distribution of one period's demand. An expression for the expected average cost per period in the steady state as a function of G and S is also derived and conditions obtained for its being convex in S. A problem with this method is that it appears to be very difficult to obtain the probability distribution of starting stock for $m \geq 3$, so that it is unlikely anyone with an actual problem would use this technique.

A similar approach was considered by Chazan and Gal [1977]. Daily demand is a discrete random variable so that starting stock levels form a discrete time, discrete state space Markov chain. They prove Cohen's conjecture that expected outdating in the steady state is a convex function of the order up to level S. In addition, they provide a rigorous proof that cumulative outdating is minimized under a FIFO issuing scheme.

An interesting observation made by Chazan and Gal is that if demand in a period, D, is less than or equal to S/m with probability 1, then the expected outdating per period will eventually be $S/m - E(D)$. (This occurs since starting stock before demand in every component will eventually be exactly S/m.) Although this case is of limited interest since it is quite restrictive, it leads to the general result that expected outdating per period is bounded below by $\mu_1(S) = S/m - \bar{D}_1$ and above by $\mu_2(S) = S/m - \bar{D}_2$ where $\bar{D}_1$ is $E(\min(D, S/m))$ and $\bar{D}_2$ is $E(\min(\bar{D}, S/m))$ where $\bar{D}$ is average demand over m days. These bounds are shown to converge to the true value of expected outdating as S grows large. The results obtained are extended to the case where demands vary in a cyclic manner. When demand is stationary and Poisson, an approximate model is constructed which leads to explicit expressions for the expected outdating per period and the average age of a unit used to meet demand.

(iii) Fixed Order Quantities

A property of the models discussed so far is that a complete description of the state of the system must include the age distribution of on hand stock. An exception was discovered by Brodheim et al. [1975]. If q, the number of fresh units entering stock each period, is fixed then the total stock on hand at the end of each period, Y_t, will form a stationary Markov chain. Then the $(m-1)$ dimensional vector of on hand stocks at the end of each period will be of the form: $(q, q, \cdots, q, s, 0, \cdots, 0)$ where $0 \leq s \leq q$. Note that stock will perish if and only if $(m-2)q \leq Y_t \leq (m-1)q$, since only in that case will the final component of the state vector be positive. The number of units which perish will be exactly $Y_t - (m-2)q$. Hence knowledge of both Y_t and the demand in period $t+1$ will be sufficient to determine Y_{t+1}.

The paper also considers the denumerable state chain Y_t' which corresponds to Y_t with $m = \infty$. A modified process, R_t, samples Y_t' at epochs where $Y_t' \leq (m-1)q$. The stationary distribution associated with R_t may be obtained directly from the stationary distribution of Y_t'. These results are used to obtain bounds on the stationary probabilities associated with Y_t. The results are used to derive operating characteristics of the system rather than for cost optimization.

(iv) Heuristic Critical Number Approximations

In general, a critical number order policy should result in significantly better performance than a fixed order quantity policy since it is responsive to fluctuations in demand. However, since the optimal S is extremely difficult to compute, heuristics are of considerable interest. Nahmias [1976] developed a very effective method of finding critical number approximations. The method involves replacing the expected outdating by a function which depends on $\mathbf{x}$ only through the sum x, and utilizing this expression in both the one period expected cost function and in the transfer function. Expected outdating is estimated by the upper bound $H(x+y) - H(x)$, where $H(t) = \int_0^t F^{m^*}(u)du$ and F^{m^*} is the m-fold convolution of one period's demand. The transfer function is approximated in the following manner: if in any period one has $\mathbf{x}$ on hand, an amount y of new stock is ordered and t is the realization of demand, then total starting stock one period hence is $x + y - t -$ the number of units which outdate. Letting $z = x + y$ we obtain the $\mathbf{s}(y, \mathbf{x}, t) \approx z - t - H(z) + H(z-t)$. This approximate transfer function and the expected one period cost approximation yields a new model for which critical number policies are both optimal and easy to compute. For the cases tested this heuristic gave expected costs generally less than 1% above the global optimal solution.

This same technique can be used with other bounds. In Nahmias [1977c] for the random lifetime problem (to be discussed below) and Nahmias [1978] for the set-up cost problem, the average of the upper and lower bounds derived by Chazan and Gal was used to approximate expected outdating as well. In each of these cases approximate order policies were developed using the approach outlined above. These approximations gave similar small increases in expected costs over the optimal policies.

One closing remark on approximate critical number policies: When a fixed fraction of stock is lost each period due to decay, and lead time is zero, optimal policies are of the critical number type. (This was proven originally by Van Zyl and can also be established from Veinott's [1965] more general work.) Hence, it might seem reasonable to utilize the optimal S from a model which assumes exponential decay in the fixed life situation. This was done in the unpublished technical report version of Nahmias [1976]. (See Technical Report No 15, Department of Industrial Engineering, University of Pittsburgh, November 1973.) It turns out, as one might expect, that the critical number obtained is very sensitive to the choice of the decay constant. Even when the decay constant was estimated exactly (that is, as the true fraction of stock which outdates each period), the approximation did not perform as well as that of Nahmias [1976].

Nahmias [1977b] developed a technique for finding approximations that are both more complex and more efficient than critical number policies. Noting that an optimal order policy is more sensitive to changes in newer than older stock, he considers policies that depend on the vector $(x_{m-1}, \cdots, x_{k+1}, \sum_{j=1}^{k} x_j)$ which has dimension $r - 1 < m - 1$ with $r = m - k + 1$. The approach is to approximate the expected outdating by a form like that for an r period lifetime and utilize the transfer function for an r period model. The resulting functional equations are then solved for an optimal policy. In computations for $m = 3$ and $r = 2$ the approximation yielded expected costs about halfway between the optimal cost and that obtained using the critical number approximation from Nahmias [1976].

1.2.c. LIFO Inventory Systems

The models which have been discussed thus far were developed under the implicit assumption that both the ordering and issuing policies were under the control of the decision maker. It is clear that for a fixed life commodity, issuing the oldest units first (FIFO) minimizes expected outdating. However, in many real systems the user determines the issuing policy and when the utility of new units is higher, LIFO issuing will be the result. An example is retail food distribution where consumers observe an expiration date on shelf items and choose the newest.

The problem of determining suitable ordering policies for a LIFO system under random demand was considered by Cohen and Pekelman [1978a]. They assume demands are independent and identically distributed, inventory levels are reviewed periodically, and excess demand is lost rather than backordered. Following the notation we have used above, the transfer function, $\mathbf{s}(y, \mathbf{x}, t)$, is now given by

$$s_j(y, \mathbf{x}, t) = \begin{cases} [x_{j+1} - (t - y - \sum_{i=j+2}^{m-1} x_i)^+]^+, & 1 \le j \le m-2 \\ (y-t)^+, & j = m-1 \end{cases}$$

The stochastic process, $B_j(n)$, defined in period n as

$$B_j(n) = D_n - y_n - \textstyle\sum_{i=j+2}^{m-1} x_i,$$

is shown to be determined by a related ladder height process whose stationary distribution satisfies a Weiner-Hopf integral equation when $m = \infty$. A solution to the integral equation is obtained when demand is exponential and a fixed order quantity policy is followed. Critical number order policies are also considered for $m = \infty$. (The integral equation does not apply when $m < \infty$.) These results and results for $m = 1$ are used to approximate the expected shortage and expected outdating.

Cohen and Pekelman [1978b] examined the effect of LIFO and FIFO accounting systems on inventory control policies. Although that paper does not deal directly with perishables, considering the age levels of the different stock is necessary for tax purposes. One very interesting result is that the optimal inventory policy does not vary greatly with the valuation scheme used (but does depend strongly on the explicit inclusion of taxation).

Cohen and Prastacos [1978] deal specifically with the effect of FIFO versus LIFO depletion policies on both the system performance and ordering decisions. The analysis is restricted to $m = 2$. Even for this case, a closed form expression for the stationary distribution of starting stock appears difficult to obtain under LIFO. However, they show that the expected outdating in the steady state when ordering to S each period and periodic demands have distribution function F can be approximated by

$$\int_0^S F^{2^*}(u)du/[1 - F(S)]$$

under LIFO. Using this approximation, approximately optimal values of S for the LIFO case are derived and compared with those obtained for FIFO systems (using results of Cohen [1976] and Van Zyl). The critical numbers obtained were relatively insensitive to the choice of the issuing policy even though the optimal expected cost was significantly higher for LIFO. This result is important in that it suggests that the relatively simple approximations derived in Nahmias [1976, 1977c] which assume FIFO could also be used effectively in the more complex LIFO case as well.

1.2.d. Multiproduct Models

Inventory models for multiple products when one or more of the products have fixed lifetimes are extremely complex. The only reasonably general dynamic multiproduct perishable inventory model for which the structure of an optimal policy has been derived is due to Nahmias and Pierskalla [1976]. There are two products: one with a finite lifetime of m periods, and one with an infinite lifetime. There is a single demand source which depletes the perishable stock first, according to FIFO, and then the nonperishable. This model is suggested by the operation of a central blood bank which is the facility to freeze blood. It can also be applied to certain food inventories such as whole and powdered milk.

Let product 1 be the perishable commodity and product 2 the nonperishable. Unit holding and procurement costs are (h_1, h_2) and (c_1, c_2) respectively. Shortages cost p per unit and occur only when the entire system is out of stock, and outdates are charged at a rate of θ per unit. The following relationships among the various cost parameters are assumed to hold:

(i) $0 \le h_2 \le h_1$

(ii) $0 < c_1 < c_2$

(iii) $p > (1 - \alpha)c_2$

(iv) $0 \le (1 - \alpha)(c_2 - c_1) + (h_2 - h_1) < \theta.$

See Nahmias and Pierskalla [1976] for a discussion of the economic interpretation of these relationships.

The optimal solution depends upon three constants:

$$u^* = F^{-1}\{[p - c_2(1 - \alpha)]/(p + h_2)\}$$

$$w^* = F^{-1}\{[p - c_1(1 - \alpha) + (h_2 - h_1)]/(p + h_2)\}$$

$$v^* = F^{-1}\{[p - c_1 + \alpha c_2]/(p + h_2)\},$$

where F is the cumulative distribution function of one period's demand.

The analysis proceeds by dynamic programming with state variable $(\mathbf{x}, x^2)$ ($\mathbf{x}$ is the vector of age differentiated stocks of product 1 and x^2 is total stock of product 2). The optimal policy in each period is characterized by three regions in m-space which respectively indicate whether or not it is optimal to order both products, product 1 only, or neither. (It is never optimal to order only product 2.) The policy is pictured in Figure 2 for $m = 2$. The arrows indicate stock levels before and after ordering.

There are a number of interesting properties of an optimal policy. If it is optimal to order both products when n periods remain, then the optimal policy orders in such a way that total stock on hand after ordering is a constant, u_n^*. Furthermore, $u_{n+1}^* \le u_n^*$ for all n. Also, when it is optimal to order both products (Region I in the figure) one orders into the interior of Region II while if it is optimal to order the perishable product one orders to a point short of the boundary of Regions II and III.

The methods developed in Nahmias [1976] may be applied to this problem to derive a simple approximation. Let the total starting perishable and nonperishable stock be (x, x^2) before ordering, and (w, z) after ordering. Let $\mu(w)$ be an approximation for the expected number of outdates per period. (See Nahmias [1977c] p. 87 for the mathematical

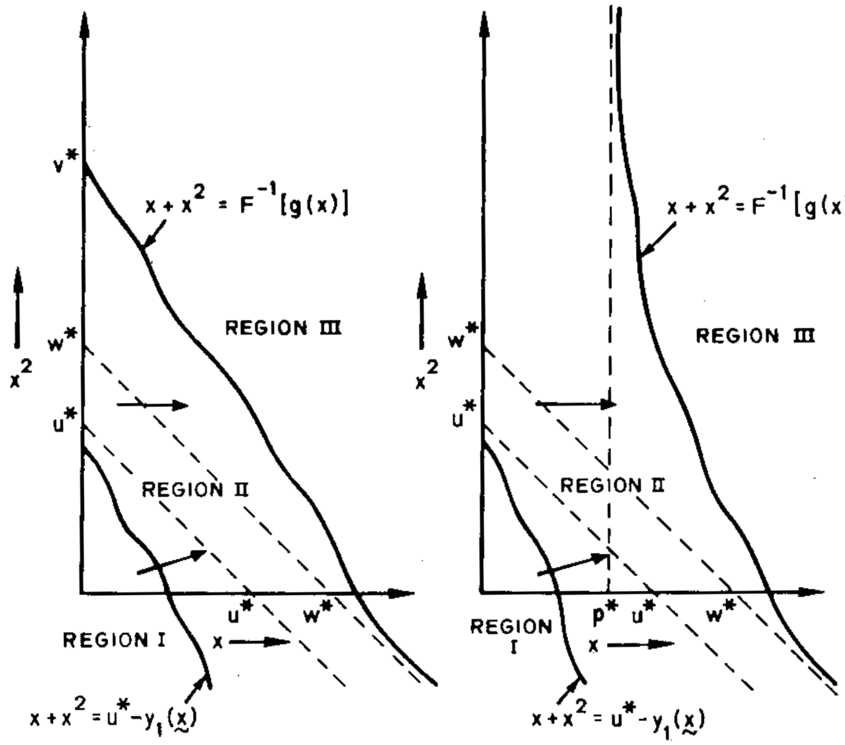

Figure 2. Optimal ordering regions for the Nahmias and Pierskalla [1976] two product model. (*Left*) $\alpha c_2 - h_2 < c_1$ and (*Right*) $\alpha c_2 - h_2 \geq c_1$.

form $\mu(w)$ based on the bounds derived in Chazan and Gal.) Then the approximate transfer equations take the form:

$$x_{n+1} = [w_n - D_n - \mu(w_n)]^+$$
$$x^2_{n+1} = z_n - (D_n - w_n)^+,$$

which results in a "myopic" expected cost function in the variables (w, z):

$$G(w, z) = c_1 w[1 - \alpha F(w)] + h_1 \int_0^w (w - t) f(t)dt$$
$$+ [\theta + \alpha c_1 F(w)]\mu(w) + c_2\{(1 - \alpha)z + \alpha[1 - F(w)]\}$$
$$+ h_2 \int_w^{w+z} (w + z - t) f(t)dt + p \int_{w+z}^{\infty} (t - w - z) f(t)dt.$$

Optimizing with respect to (w, z) jointly (by setting the first partial derivatives equal to zero) gives

$$w^* + z^* = F^{-1}[(p - c_2(1 - \alpha))/(p + h)]$$

and

$$(c_1 - c_2)[1 - \alpha F(w^*)] + (h_1 - h_2)F(w^*) + [\alpha c_1 F(w^*) + \theta]\mu^1(w^*) = 0.$$

Since w^* and z^* may be determined independently, computing the approximate policy requires essentially the same effort as for the single product case. Although this particular approximation has not been tested, similar approximations for a single product have performed extremely well (Nahmias [1976, 1977c]). Nakagami [1979] considers a more restrictive version in which the perishable inventory has a lifetime of exactly one period.

Two other multiproduct papers by Deuermeyer [1979, 1980] treat one period problems. The two product model of his latter paper is also suggested by blood banking. There are two production processes, say A and B. One unit of production from process A yields η_1 and η_2 units of products 1 and 2 respectively while a unit of production of process B results only in production of a unit of product 2. Both products are assumed to have finite lifetimes of at least two periods, and demands for each of the products are independent random variables.

Deuermeyer computes expected future outdating to levy the one period outdate cost as in 1.2.a(ii) above. Assuming that $\mathbf{x}$ and $\mathbf{w}$ are the vectors of products 1 and 2 on hand prior to ordering and y and z are the order quantities, the optimal order policy is characterized by four regions in $(\mathbf{x}, \mathbf{w})$ space which indicate the four combinations of using processes A or B or neither.

Deuermeyer [1980] treats a single period model of a general multiproduct perishable inventory problem and obtains first period results similar to those described above for one product. Although Deuermeyer was unable to develop a proof, it seems reasonable to conjecture that his results carry over to the multiperiod case as well.

1.2.e. Multiechelon Models

In both blood banking and food management, goods are produced at a central facility and subsequently shipped to regional centers for distribution. The first perishable inventory model which considers such multiechelon situations is Yen [1965]. He treats a two-echelon system in which a central facility, 0, supplies two other facilities, 1 and 2. External demand occurs only at facilities 1 and 2 and is satisfied according to a FIFO policy. The problem is to find both ordering and allocation policies at facility 0 that minimize expected costs of shortage and outdating.

Yen restricts attention to a stationary critical number order policy and assumes stock is allocated to facilities 1 and 2 in proportion to their previously observed demands. An important result obtained is that under certain conditions the total expected outdating of stock in period i is a convex function of the critical number in period i. Conditions are also established under which the proportional allocation policy described above is optimal.

Yen's results also appear in Cohen et al. [1979] who consider in addition a policy in which a fixed fraction of available stock is allocated to facilities 1 and 2. It is demonstrated that the optimal ordering policies associated with each of the two allocation policies are equivalent under certain circumstances.

A similar two-echelon model is considered by Prastacos [1978] who restricts attention to allocation policies. He defines a rotation policy as one in which stocks may be reallocated to the warehouses at the start of each planning period. Assuming demands are satisfied on a FIFO basis, the one period solution which minimizes expected shortage and outdate costs involves a two-stage allocation procedure: (1) first allocate those units which are scheduled to outdate in the current planning period so as to equalize the probability that a unit outdates among the warehouses, and (2) allocate all remaining stock so as to equalize the probability of shortage at any warehouse. (Yen obtains similar results.)

Prastacos [1981] treats the stationary version of the problem. The optimal allocation policy is shown to be independent of the unit costs of shortage and outdating. Furthermore, the simple one period solution described above is shown to be close to the optimal stationary solution.

An interesting difference between the allocation problem and the ordering problem is that an optimal allocation policy minimizes *both* shortage and outdating, while optimal ordering policy balances these two competing aspects. The reason is that when ordering we are choosing how much stock to bring into the system: more stock results in fewer shortages and increased outdating. In the allocation problem, the total stock in the system is fixed. By allocating older units to regions of high demand, both expected shortages *and* expected outdating can be minimized.

When stock is issued according to a LIFO policy, these results are no longer true. Prastacos [1979b] shows that in this case the optimal allocation policy will depend upon the specific unit costs of outdating and shortage. He indicates that an optimal allocation scheme can be obtained by dynamic programming but does not delineate any of its relevant properties. He suggests an allocation procedure which segregates older stock to some regions and newer stock to others based on the demand characteristics of the regions involved.

2. RANDOM LIFETIME

For many inventories the exact lifetime of stock items cannot be determined in advance. Items are discarded when they spoil and the time to spoilage may be uncertain. Fresh produce is a typical example.

There is one well known process which arises from a perishable inventory problem with random lifetimes. Suppose the lifetime of individual units is a random variable with a negative exponential distribution having parameter θ. Let $I(t)$ be the number of units surviving to time t exclusive of demand. Since each unit has probability of $e^{-\theta s}$ of surviving an additional s units of time, it follows that the number of units surviving to time $t + s$ is a binomial random variable with parameters $n = I(t)$ and $p = e^{-\theta s}$. It follows that the expected number of units surviving until time $t + s$ is np or $I(t)\exp\{-\theta s\}$. Hence we obtain the well known exponential decay process.

Exponential decay can also be derived by assuming that a fixed fraction of stock on hand is lost each period regardless of the age distribution of inventory. Suppose $I(t)$ is the inventory level at time t. Then, ignoring demand, the decay assumption is that $I(t + 1) = \gamma I(t)$ for some constant $0 < \gamma < 1$. This generalizes to $I(t + s) = \gamma^s I(t)$ which is precisely the same as above with $\theta = -\ln \gamma$.

2.1. Periodic Review

2.1.a. Deterministic Demand

Freidman and Hoch [1978] treat a model similar to that of Wagner and Whitin [1958] and Veinott [1960] in that inventory levels are reviewed periodically and demand is assumed known. In addition they assume that a fraction $0 \leq r_i \leq 1$ of the units on hand of age i survive into the next period. We may interpret r_i as the probability that an item of age i survives an additional period. One interesting feature of the solution is that the property that one only orders in periods in which starting stock is zero no longer holds when perishability is allowed. (This implies that an algorithm developed earlier by Smith [1975] for a similar problem is incorrect.)

2.1.b. Random Demand

The only analysis of ordering perishable goods subject to uncertainty in both the demand and the lifetime is due to Nahmias [1977c]. In order to obtain results similar to those described in 1.2a(ii) above, he required the assumption that successive orders outdate in the same sequence that they enter stock. That is, an order placed in period n_1 will outdate before an order placed in period n_2 whenever $n_1 < n_2$. This process is modeled in the following manner: Let $A_1, A_2, \cdots$ be a sequence of independent identically distributed random variables which assume values $\{1, 2, \cdots, m\}$ with respective probabilities $\{p_1, \cdots, p_m\}$. The event $\{A_n = k\}$ means that after demand is satisfied in period n all items remaining in stock that are k periods or older will outdate. (Note that A_n is *not* the lifetime of an order acquired in period n.) This approach is similar to Kaplan's [1969] for random leadtimes.

Since orders outdate in the same sequence that they enter stock, the expected future outdating of the current order is completely determined from knowledge of the current stock, $\mathbf{x}$, and realizations of future demands. Nahmias [1977c] shows that the optimal policy here is essentially the same as in the deterministic lifetime case.

In addition to the analysis of optimal policies, the heuristic described in 1.2.b(iv) is extended to the random lifetime case. Computations reported in Nahmias [1977c] indicate that the approximations are as effective for random lifetimes as they are for fixed lifetimes.

2.2. Exponential Decay

2.2.a. Deterministic Demand

Ghare and Schrader [1963] generalized the standard EOQ model to include exponential decay. The demand rate at time t is $D(t)$. In between orders, the on hand inventory level, $I(t)$, declines due to the simultaneous effects on both demand and decay according to the ordinary differential equation

$$dI(t)/dt + \theta I(t) = -D(t).$$

When $D(t) = \lambda$, independent of t, the solution obtained is

$$I(t+u) = \exp(-\theta u)\cdot[I(t) + \lambda/\theta] - \lambda/\theta.$$

This relationship may then be used to develop an expression for the cost incurred per unit time. The authors' procedure for determining the order quantity is based on approximating the exponential function by the first three terms of the Taylor series expansion.

A similar model was treated by Emmons [1968] who assumed that when on hand inventory reached a level $s \geq 0$, stock levels were immediately increased by Q units. He shows that a reasonable approximation for the cost incurred per unit time, is

$$C(Q) \cong \theta c Q/\ln[(Q + \lambda/\theta)/(s + \lambda/\theta)]$$

which depends only on the unit purchase cost, c. (The holding cost contribution was estimated to be negligible relative to the purchase cost in Emmons' application.)

Covert and Phillip [1973] generalized Ghare and Schrader's model in the case of deterioration governed by a Weibull rather than an exponential law. This is equivalent to saying that item lifetimes have a Weibull distribution. Their approach also involves solving an appropriate differential equation. It is

$$dI(t)/dt + \alpha\beta t^{\beta-1}I(t) = -\lambda,$$

where α and β are the shape parameters of the Weibull density. The optimal order quantity is given by an infinite series expansion which can be approximated by Newton's method.

Further extensions of Ghare and Schrader's model were considered by Shah [1977] and Tadikamalla [1978]. Tadikamalla assumed that the lifetime of individual units is governed by a gamma rather than Weibull distribution while Shah considered the case of an arbitrary distribution, say $\varphi(t)$. In the general case the differential equation governing changes in stock levels has the same form as above except that $\varphi(t)$ replaces $\alpha\beta t^{\beta-1}$.

Cohen [1977] considered an extension of Ghare and Schrader's model to the case where the demand rate, $\lambda(p)$, is a known function of the price of a unit, p. The goal is to obtain both order quantities and prices that minimize costs per unit time. Then the optimal time between placement of orders as a function of price is given by $T_p = \sqrt{2K/[\lambda(p)(c\theta + h)]}$ and the profit rate, assuming one orders in an optimal manner, is given by

$$\pi(p) = p\lambda(p) - c\lambda(p) - \sqrt{2K[c\theta + h]\lambda(p)}.$$

Conditions are established for which an optimal $p^* > c$ maximizing $\pi(p)$ exists. Similar results are obtained when shortages are allowed.

2.2.b. Stochastic Demand

The complexity of decay models having random demand depends strongly on the lead time assumptions. When lead times are zero, determining optimal order policies is relatively straightforward.

Consider the standard single product periodic review system when a fixed fraction, γ, of the stock on hand at the end of each period is lost due to decay. Assuming lost sales, the transfer function, $s(y, t)$, which gives starting stocks one period hence when ordering to y and realizing a demand of t, is given by

$$s(y, t) = \gamma(y - t)^+,$$

which will satisfy the requirements imposed by Veinott [1965] for the optimality of myopic policies. Hence it follows that a simple critical number policy will be optimal. The solution can be obtained with essentially the same effort required to solve the no decay case. When excess demand is backordered the transfer function is slightly more complex but these results still hold.

Another common approach used in inventory modeling is to assume a fixed operating policy and determine the stationary distribution of starting (or ending) stocks, from which expected costs per period in the steady state may be found and optimized with respect to the policy parameters. Emmons [1968] considered this method in trying to find an optimal stationary (s, S) policy for a decaying inventory. Suppose demand has density $f(t)$ in every period and $e^{-\theta}$ of each unit on hand at the start of a period is lost to decay. Then he shows that the stationary density $g(x)$ of ending stocks, x, satisfies the differential equations

$$g'(x) = g(x), \qquad Se^{-2\theta} < x < Se^{-\theta} \quad \text{or} \quad x < se^{-\theta}$$

$$g'(x) = g(x) - e^{\theta}g(xe^{\theta}), \qquad se^{-\theta} < x < Se^{-2\theta}.$$

Unfortunately, no explicit solution to this system of equations was obtained, so that anyone wanting to use this approach would have to resort to numerical techniques.

Shah and Jaiswal [1977a] treat an extension of Naddor's [1966] order level model to allow for continuous exponential decay. Suppose at the beginning of an infinite sequence of uniform scheduling periods of length T, stock levels are immediately raised to a level S. Total demand during the scheduling period is a random variable with continuous density $f(x)$. The authors derive an expression for the expected cost per period which is a fairly complex function of S. Unfortunately, their expression for the expected holding cost is not correct (see Aggarwal [1978]).

Determining an optimal S in these models requires integrating expressions of the form $\int_{t}^{\infty} f(x)/x^n dx$ where $f(x)$ is a probability density for demand. Nahmias and Wang [1978] developed an approximation for this integral when $f(x)$ is a normal density. Using this approximation they obtain a very accurate estimate for the expected shortage in a time period (such as a lead time or review period) whose demand density is $f(x)$.

Another probabilistic model was treated by Shah and Jaiswal [1977b]. It is of limited interest due to the unreasonable combination of assumptions that demand is random and shortages are not permitted.

In general, exponential decay problems with random demand are extremely difficult to handle when there is a positive lead time for ordering. The difficulty arises because decay applies only to the inventory on hand and not to the inventory on order. It follows that no optimal policy will simply be a function of the inventory position (sum of stocks on hand and on order). It is relatively straightforward to write down the functional equations defining an optimal policy, but for lead times of more than one or two periods, computing the optimal policy is impractical. It is interesting to note that no approximations have been suggested for the periodic review decay problem with positive lead time.

The only reasonably general model which allows for exponential decay, random demand and a positive lead time for ordering is due to Nahmias and Wang [1979]. Inventory levels are reviewed continuously and the operating doctrine is a stationary (Q, r) policy. That is, when the total stock on hand and on order reaches a level r, an order for Q units is placed which is scheduled to arrive after τ periods. Costs are holding at h per unit per unit time, shortage at p per unit short and ordering at $cQ + K$ per positive order of Q units. Total lead time demand is assumed to be a continuous variable with density $f(x)$ and mean demand per unit time is λ.

For optimization it is convenient to replace r by s, the safety stock given by $s = r - Q\tau/T$. Interpret T as the expected time between arrivals of orders which is given by $T = \theta^{-1}\ln[1 + \theta Q/s\theta + \lambda)]$. Average annual cost is approximately

$$A(Q, s) \cong T^{-1}\left\{K + cQ + hQ/\theta + p\int_{w}^{\infty}(x - w)f(x)dx\right\} - h\lambda/\theta - c\lambda$$

where $w = s(1 + \theta\tau/2) + \mu$ and μ is mean demand during lead time. The authors suggest that the minimizing values of (Q, s) be determined by a one-at-a-time interval bisection on the first partial derivatives of $A(Q, s)$.

Simulation was used to estimate the best (Q, r) policy and compare its performance with the heuristic. The heuristic policy gave average costs very close to those of the best (Q, r).

3. QUEUEING MODELS WITH IMPATIENCE

Results derived for queueing systems with customer impatience can be used to obtain control rules for certain types of perishable inventory systems. Consider a single server queue in which customers will wait a fixed time, say L, and will leave the system if service has not been completed by that time. The analogy with perishable inventory should be clear. The queue is identified with the inventory, the service process with the demand, the time to impatience with the lifetime of fresh stock, and the arrival of customers to the replenishment of inventory. The analogy has one serious shortcoming. In the inventory context, the timing and amounts of replenishment are controllable while in the queueing context arrivals occur at random.

Impatient queues have been studied by Barrier [1957], Gnedenko and Kovalenko [1968], Finch [1960], Gavish and Schweitzer [1977], and others. Gnedenko and Kovalenko demonstrated that the virtual waiting time process is Markovian even though the number of customers in the system is not. Recently Graves [1982] has shown that the process corresponding to the age of the oldest unit is also Markovian and can be used to derive various measures of system performance. Assuming that stock is replenished at a constant rate he obtains analytic expressions for the expected outdates per unit time, the expected shortages per unit time, the expected age of the oldest unit supplied to meet demand, and the expected inventory level.

Nahmias [1982] shows how these results can be incorporated into a model which determines the rate at which fresh units should enter the system in order to minimize the total outdating and shortage cost per unit time. A model of this type might be appropriate for a central blood bank which controls the rate at which donors enter the system in order to maintain the proper balance of blood supplies.

4. THE APPLICATION OF PERISHABLE INVENTORY THEORY

The interest among researchers in perishable inventory problems has been sparked primarily by problems of blood bank management. Since a complete review of the literature on inventory control of blood banks would constitute a paper in itself, we will merely highlight some of the more significant studies.

Millard [1960] seems to have been the first to recognize that inventory models could be applied to managing the stocking of whole blood. Elston and Pickrel [1963] used simulation to compare the effects of changing the size and age composition of the blood supply and compare the effects of FIFO and LIFO issuing policies on shortage and outdating.

Some of the issues concerning regionalization of blood banks were treated in a simulation study by Jennings [1968, 1973a]. Jennings appears to have been the first to realize the importance of distinguishing between the assigned and unassigned inventories. (The assigned inventory is blood that has been tested for compatibility with that of the recipient, i.e., cross-matched, and assigned for a specific requirement.)

Brodheim et al. [1976] used operational data to develop curves relating whole blood/red blood cell inventory levels and mean daily demand for various specified shortage rates. Cohen and Pierskalla [1975, 1979] develop an approach which involves setting optimal target levels based on a simple equation involving a variety of different factors involving demand rates, crossmatching and the return of unused assigned blood to the unassigned inventory. Rabinowitz [1973] constructed a detailed simulation model of a single hospital bank and included a statistical analysis of the demand patterns at the particular hospital considered. A Markov chain model of the blood inventory process was constructed by Pegels and Jelmert [1970] primarily for the purpose of comparing the effects of various issuing policies. However, their model was criticized on both theoretical (Kolesar [1973]) and practical grounds (Jennings [1973b]). Brodheim and Prastacos [1979] report on an operating system for blood management implemented in Long Island. This system utilizes results concerning optimal allocation policies described in 1.2.e above and in Prastacos [1978].

Although many of the existing theoretical perishable inventory models discussed in this review have been suggested by blood banking problems, the actual problem, even at the hospital level, seems too complex to be adequately described by any single mathematical model. Aspects of the problem that appear difficult to model include: (1) accounting for the separate assigned and unassigned inventories, (2) modeling the process by which unused assigned units return to unassigned stock after having aged some number of days, (3) dealing with practical constraints on the issuing policy (such as requiring very fresh blood for heart surgery), and (4) accounting for the substitutability relationships of the various blood types. A comprehensive review of the literature on blood bank inventory control prior to 1970 can be found in Elston [1970]. Prastacos [1979a] reviews work done since 1970.

An interesting point is that although blood banking problems appear to have dominated the interest of researchers and underlie most of the theoretical perishable inventory models developed, one would think that food management problems would have a far greater economic impact. Whether or not existing models could be useful in dealing with food management remains to be seen. Some of the possible reasons for the interest in blood banking might be that blood bank research has been supported by public funds (thus providing incentives for academics) and hospitals and blood banks are not profit-making centers (thus making operational data more accessible).

5. AREAS FOR FURTHER RESEARCH

Although considerable progress has been made in improving our understanding of perishable inventory models, a variety of important problems remain unsolved. One is determining good ordering policies for a fixed life commodity when there is a positive lead time for placing an order and inventory levels are reviewed continuously. The problem is extremely difficult because the state variable would have to be a possibly infinite vector of on hand and on order stocks since the age of any particular batch of items in stock could take on any one of a continuum of values.

The queueing models discussed in Section 3 suggest an approach to this problem. The virtual waiting time process in the queue with impatience is Markovian and analogous to the virtual lifetime of remaining stock in the inventory problem. This suggests that a control rule based on the expected remaining lifetime of on hand inventory might be more appropriate for perishable items than one based on the number of items in stock.

Another open issue is that of developing pricing policies for perishable items under random demand. In retailing there are obvious advantages to lowering prices on older items in order to turn them over more quickly and reduce outdating.

Existing models assume that fresh units have a lifetime that is fixed at some specified number of periods. In some contexts a more accurate description of the system might be obtained by assuming that the lifetime of an item first issued at age s follows some general field life function, $L(s)$, as is assumed in the analysis of issuing policies (Derman and Klein). Here one could be concerned about the simultaneous problems of both ordering and issuing.

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