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Note—Sensitivity to Distributions in Inventory Systems

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$\bar{Q}_j < \infty$; otherwise the cost of P' would be infinite. Choose $\bar{T} \geq \max\{\tau_2'; [h_j/h_{p(j)}] \cdot \bar{Q}_j/D_j\}$. Define $t_k = \min\{t_i \mid t_i \geq \bar{T}\}$, and let $\epsilon = \min_{i=2,\ldots,k}\{I_j(t_i)\} > 0$. Define $\tau_l' = \min_i\{\tau_i' \mid \tau_i' \geq t_k\}$ and $\tau_m' = \min_i\{\tau_i' \mid Q_j(\tau_i') > \epsilon\}$. Note that $l > m$ by definition.

Consider policy P which is identical to P' at $p(j)$ except at time $O(t_k)$, $p(j)$ produces ϵ less (more) than in P'. At j policy P produces at times τ_i where $\tau_i = \tau_i'$ for $i = 1, 2, \ldots, m$ and for $i > l$, while $\tau_i = \tau_i' - \epsilon/D_j$ for $i = m+1, \ldots, l$; accordingly the production quantities for policy P have $Q_j(\tau_m) = Q_j(\tau_m') - \epsilon$ and $Q_j(\tau_l) = Q_j(\tau_l') + \epsilon$, while $Q_j(\tau_i) = Q_j(\tau_i')$ otherwise. Policy P has a simultaneous production point by construction at one of the times $\tau_{m+1}, \ldots, \tau_{l-1}$. Furthermore, P has costs no larger than P'. To see this note that echelon holding costs are decreased on the interval $(0, t_k)$ by $\epsilon \cdot t_k \cdot h_{p(j)}$ and increased on the interval (t_k, ∞) by $\epsilon \cdot Q_j(\tau_l') \cdot h_j/D_j$. However, by definition $Q_j(\tau_l') \leq \bar{Q}_j$ and by the choice of t_k, we have $\epsilon \cdot t_k \cdot h_{p(j)} \geq \epsilon \cdot Q_j(\tau_l') \cdot h_j/D_j$. Finally, since P has the same number of order points as P', P is no more costly than P'. By repeated application of this construction plan, we can find a policy with no greater cost than P', for which the time interval between successive simultaneous production points is finite. Q.E.D.

Theorem 1 is correctly stated as follows: *Given zero initial inventories, or equivalent, if there exists an optimal policy which is stationary, that policy is a single cycle policy.* See [1, p. 540] for proof.

Reference

1. Graves, Stephen C. and Schwarz, Leroy B., "Single Cycle Continuous Review Policies for Arborescent Production/Inventory Systems," *Management Sci.*, Vol. 23, No. 5 (January 1977), pp. 529–540.

Notes* VI

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SENSITIVITY TO DISTRIBUTIONS IN INVENTORY SYSTEMS†

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The optimal decisions and costs of several inventory systems with the s, S policy are presented showing how they are affected by different distributions of demand, different shortage costs and different leadtimes.

The numerical results seem to imply that in many cases the optimal decisions depend on the means and standard deviations of demand but not on the specific forms of the distributions.

(INVENTORY/PRODUCTION—PARAMETRIC ANALYSIS)

Undocumented numerical solutions of many inventory systems seem to indicate that the precise form of the distribution of demand in a given system is not essential for the determination of the optimal decisions in the system. In many cases the optimal decisions only depend on the mean demand and its standard deviation.

* All Notes are refereed.

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The purpose of this note is to provide several sets of such numerical results. It is hoped that they will stimulate interest in a systematic investigation of the sensitivity of inventory systems (and possibly other systems) to probability distributions. An example of such an investigation is given in [1].

Inventory systems with the classical s, S policy are analyzed, where the carrying and shortage costs are linear, and the replenishing cost is independent of the amounts ordered. Two groups of discrete distributions are used. In the first group (distributions P, Q, R, and S) the mean is 3.0 and the standard deviation is 1.732. In the second group (distributions T, U, V, and W) the respective values are 1.6 and 3.2. The detailed properties of all the distributions are given in Table 1. The minimum demand in all cases is zero. The maximum demand, as well as the values of the probabilities of several demands, are given in the table. The Poisson and the Negative Binomial distributions used are truncated at the indicated maximum values. The Beta distribution is based on the density function:

$$f(x) = c(x - x_1)^{a-1}(x_2 - x)^{b-1}, \qquad x_1 \leqslant x \leqslant x_2, a > 0, b > 0.$$

The parameters a and b (and hence c) are chosen so that the corresponding discrete distribution in the range x_1 to x_2 has the desired mean, $\bar{x}$, and standard deviation, σ. This can be done whenever $(\bar{x} - x_1)(x_2 - \bar{x}) > \sigma^2$.

The 2-point distributions are included to indicate the effect of extreme conditions.

TABLE 1

The Probability Distributions P, Q, R, S and T, U, V, W

Distribution	Type	Maximum	Mean	Standard Deviation	P(0)	P(1)	P(2)	P(max − 1)	P(max)
P	Poisson	10	3.	1.732	0.050	0.149	0.224	0.001	0.001
Q	Beta	15	3.	1.732	0.027	0.176	0.242	0.000	0.000
R	Uniform	6	3.	1.732	0.072	0.171	0.171	0.171	0.072
S	2-Point	4	3.	1.732	0.250	0.000	0.000	0.000	0.750
T	Beta	20	1.6	3.2	0.624	0.116	0.061	0.001	0.001
U	Negative-Binomial	30	1.6	3.2	0.577	0.144	0.079	0.000	0.001
V	Beta	15	1.6	3.2	0.648	0.097	0.052	0.007	0.008
W	2-Point	8	1.6	3.2	0.800	0.000	0.000	0.000	0.200

In all systems the unit carrying cost is \$1 per period, shortages are backordered, and the cost of replenishing is \$20. It is assumed the unit cost of shortage, c_2, is given by [2, p. 1240]

$$c_2 = c_1 A/(1 - A)$$

where c_1 is the unit carrying cost and A is the fraction of time during which it is desired to have inventories in stock.

The availability fraction A is assumed to be either 0.90 or 0.99. Leadtime is either 0 or 3 periods.

Tables 2 and 3 give the optimal decisions, and the corresponding averages and total costs for all distributions. The computations are based on an algorithm in [2, pp. 1244–1246] for the s, S policy.

In Table 2, where the standard deviation is relatively small compared with the mean, the decisions are hardly affected by the form of the distribution. Because of the relative flatness of the total cost in the neighborhood of the optimum, even different

TABLE 2

Optimal Decisions and Costs for Distributions P, Q, R, S

Distribution	Maximum	Leadtime	Desired Availability	Decisions s	Decisions S	Averages Carry	Averages Short	Averages Replenish	Total Cost
P Poisson	10	0	90	0	11	5.27	0.11	0.24	11.02
Q Beta	15	0	90	0	11	5.26	0.11	0.24	11.02
R Uniform	6	0	90	0	11	5.28	0.11	0.24	11.04
S 2-Point	4	0	90	1	10	4.63	0.13	0.25	10.75
P Poisson	10	0	99	3	13	7.66	0.01	0.26	13.58
Q Beta	15	0	99	3	13	7.66	0.01	0.26	13.65
R Uniform	6	0	99	3	13	7.68	0.00	0.26	13.31
S 2-Point	4	0	99	3	12	6.50	0.00	0.25	11.50
P Poisson	10	3	90	10	22	6.88	0.21	0.22	13.26
Q Beta	15	3	90	10	22	6.88	0.22	0.22	13.31
R Uniform	6	3	90	10	22	6.88	0.20	0.22	13.13
S 2-Point	4	3	90	11	20	5.71	0.21	0.25	12.61
P Poisson	10	3	99	15	26	11.18	0.02	0.24	17.50
Q Beta	15	3	99	15	26	11.17	0.02	0.24	17.70
R Uniform	6	3	99	15	25	10.68	0.01	0.26	16.81
S 2-Point	4	3	99	15	24	9.50	0.00	0.25	14.50

TABLE 3

Optimal Decisions and Costs for Distributions T, U, V, W

Distribution	Maximum	Leadtime	Desired Availability	Decisions s	Decisions S	Averages Carry	Averages Short	Averages Replenish	Total Cost
T Beta	20	0	90	−1	6	3.46	0.26	0.15	8.80
U Negative-Binomial	30	0	90	−1	6	3.30	0.25	0.15	8.65
V Beta	15	0	90	−1	6	3.57	0.25	0.15	8.92
W 2-Point	8	0	90	−1	8	3.60	0.40	0.10	9.20
T Beta	20	0	99	5	13	9.82	0.03	0.14	15.36
U Neg.-Binomial	30	0	99	5	13	9.73	0.03	0.14	15.79
V Beta	15	0	99	6	12	9.71	0.02	0.17	14.78
W 2-Point	8	0	99	7	8	7.20	0.00	0.20	11.20
T Beta	20	3	90	8	18	9.58	0.51	0.12	16.55
U Neg.-Binomial	30	3	90	8	18	9.47	0.53	0.12	16.61
V Beta	15	3	90	9	17	9.44	0.45	0.14	16.34
W 2-Point	8	3	90	8	17	8.02	0.62	0.10	15.56
T Beta	20	3	99	20	28	19.97	0.04	0.14	26.54
U Neg.-Binomial	30	3	99	21	31	21.99	0.05	0.12	29.14
V Beta	15	3	99	19	27	19.02	0.04	0.14	25.63
W 2-Point	8	3	99	22	23	17.42	0.02	0.20	22.96

decisions have negligible effect on the total costs. For example, consider distributions Q and S when leadtime is 3 periods and the desired availability percent is 99. If the true distribution is Q, but one assumes that it is S and makes the decision $s = 15$ and $S = 24$, then the corresponding cost is \$17.93 per period, as compared with the optimal cost of \$17.70 per period.

When the standard deviation is relatively larger than the mean demand, as in Table 3, the decisions and costs are somewhat more sensitive to the form of the distribution. This is mostly notable in comparing distributions U and W, when the desired availability percent (and hence the corresponding unit cost of shortage) is relatively very high. In this extreme case, for zero leadtime, when the true distribution is W, but one assumes that it is U and makes the decisions $s = 5$ and $S = 13$, then the actual cost is \$16.20, as compared with the minimum cost of \$11.20. Conversely, if the true distribution is U and one assumes that it is W and makes the decision $s = 7$ and $S = 8$, the actual cost is then \$20.60, as compared with the minimum cost of \$15.79. But these seem to be very extreme cases that one would rarely encounter in practice.

A more realistic comparison would be to look at systems U and V, when leadtime is 3 periods and the availability percent 99. If the true distribution is V and one assumes it is U and uses the decisions $s = 21$, $S = 31$, then the actual cost is \$26.37, compared with the minimum cost of \$25.63 per period. Similarly, if U is the true distribution and one assumes that it is V and makes the decision $s = 19$ and $S = 27$, then the actual cost is \$29.99, as compared with the minimum cost of \$29.14 per period.

The insensitivity to the form of the demand distribution is even more dramatic when one also considers maximum demand in addition to the mean and standard deviation of demand. That is, if two different distributions have identical means, standard deviations, and maximum demands, then the optimal decisions in these systems are usually identical. And even when they are different, the effect on costs is negligible.

The author invites readers to send to him inventory systems for sensitivity analysis. In each case the parameters and the discrete distribution of demand should be given.

References

1. Archibald, Blyth C. and Silver, Eward A., "(s, S) Policies under Continuous Review and Discrete Compound Poisson Demand," *Management Sci.*, Vol. 24, No. 9 (May 1978), pp. 899–909.
2. Naddor, Eliezer, "Optimal and Heuristic Decisions in Single- and Multi-Item Inventory Systems," *Management Sci.*, Vol. 21, No. 11 (July 1975), pp. 1234–1249.

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Communication: Reply to Fox and Schruben

In the recent issue of *Management Science* Fox (1978) expressed doubts about the relevance of *steady-state* behavior. In a reply Schruben (1978) disagreed on this point. With Fox he was in favor of publishing *failures* of variance reduction techniques (VRT's). I would like to add the following remarks.

1. Steady-State Behavior

Interest in steady-state behavior is mainly academic, i.e., in academic research simulation is often used to develop and validate analytical models; for case studies I refer to Ignall et al. (1978). These analytic models concentrate on steady-state behavior. In my opinion this emphasis is based solely on mathematical convenience. Limiting asymptotic distribution theory can be used for steady-state analysis;