Mean-Variance Analysis of Basic Inventory Models

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Abstract

While financial planning models often involve a systematic trade-off analysis between an expected return criterion and a specific risk measure, this type of analysis is conspicuously absent in virtually all operational planning models under uncertainty. This paper focuses on several basic inventory models. The standard treatments of these models confine themselves to the optimization of the expected value of a given cost or profit measure, without any consideration of a risk measure. We revisit these models, exhibiting how a systematic mean-variance tradeoff analysis can be carried out efficiently, and how the resulting inventory strategies differ from those obtained in the standard analyses.

1 Introduction and Summary

Financial stochastic planning models invariably involve a systematic trade-off analysis between an expected return criterion and a specific risk measure. This paradigm was first introduced by Markowitz's (1959) mean-variance analysis for which contribution he was honored with the Nobel Prize in Economics: portfolios of financial instruments are chosen so as to minimize the variance of the return subject to a lower bound on its expected value or vice versa to maximize expected return subject to an upper bound on the variance of the return. By systematic variation of the (lower or upper) bound, one generates, in mean-variance space, the efficient frontier of all undominated portfolios, i.e., all portfolios whose expected return cannot be increased without a simultaneous increase of the variance of the return. Mean-variance analysis has become a standard analysis tool in portfolio management, see e.g., Fama (1976) or Copeland and Weston (1983).

More recently, other measures of risk are considered in financial studies, along with the expected return criterion, e.g., average downside risk (ADR), see Arterian (1996) and Harlow (1991) or Value At Risk (VAR), see Jorion (1997). The former assesses the expected downside deviation from the mean, or any other benchmark; the latter assesses a specific fractile of the return distribution.

While the above multi-criteria tradeoff analysis is standard in financial analyses, it is conspicuously absent in virtually all operational planning models under uncertainty. Indeed, we were motivated to write this paper when teaching a core MBA-course in Decision Models where the above multi-criteria tradeoff analysis is presented as a general approach for planning models under uncertainty, and subsequently applied to a variety of functional areas. It struck us that the resulting analysis of a couple of (admittedly elementary) production/inventory problems, went significantly beyond that described in our literature and standard textbooks, and led to recommendations that are qualitatively different.

It is sometimes argued, based on classical financial economics models such as that of

Modigliani-Miller, that the management of an individual company need not worry about the risks associated with its cash flows, provided the firm's shareholders can diversify the risks in a so-called perfect and complete market. This assertion is based on the observation that in a complete market a portfolio of assets can be found whose cash flows are perfectly correlated with those of the firm, allowing for a complete diversification of all risks. In practice, most firms devote major managerial resources to control the risks associated with the cash flows, all this for a variety of sound economic reasons, see, e.g., Froot et al. (1993, 1994). These reasons include the fact that capital markets may instead fail to be complete or spannable (see, e.g., Dixit and Pindyck (1994)), and that even in a complete market costless and complete diversification of assets may only be possible in the absence of transaction costs and under the assumption that firms and their shareholders can borrow unlimited amount of assets without facing the risk of bankruptcy. Similarly, in the classical view of finance, companies have unrestricted abilities to finance their ongoing investment needs from external sources (debt or new equity) when in practice firms' external funding sources are severely restricted. An additional rationale for corporate risk management, see Froot et al. (1993), is the prevalence of nonlinear (convex) tax functions. Finally, it is now understood that even if shareholders were able to diversify risks effectively, "the same cannot be said for managers who may hold a relatively large portion of their wealth in the firm's stock. The manager can be made strictly better off by reducing the variance of total firm value" (or profit). (The quote is from Froot et al. (1993), p. 1631, which is based on Stulz (1984).)

Having established that risks need to be managed *within* the corporation, the next question is how this objective can be achieved effectively. While in some cases, the firm's risks are primarily related to fluctuations in exchange rates or commodity prices for which simple hedging instruments exist. In many others, uncertainties about the firm's sales depend on factors for which such hedging instruments fail to be available, and it is inevitable to integrate the firm's risk management objectives within its core business planning processes, including all operational and marketing strategies.

As shown by von Neumann and Morgenstern (1944), see also Debreu (1959), the rational behavior of an economic agent can be modeled by the maximization of the expected value of an appropriate utility function of the performance measure under consideration. (This is usually referred to as the expected utility hypothesis.) If the utility function is quadratic, if it is approximated by a quadratic function or if the performance measure is normally distributed, maximization of the expected utility function reduces to a systematic tradeoff between the mean and variance of the performance measure under consideration.

This paper focuses on several basic inventory models. As mentioned, all standard treatments of these models confine themselves to the optimization of the expected value of a given cost or profit measure, without consideration of any of the above risk measures. The risk measures include the variance of profits/costs as well as that of the customer waiting times. We revisit these inventory models, exhibiting how a systematic mean-variance tradeoff analysis can be carried out efficiently, and how the resulting inventory strategies differ from those obtained in the standard analyses.

We start our analysis with the "newsboy" problem, the fundamental single-period model with stochastic demand, in which the risk of oversupply is traded off against that of undersupply. This most basic model appears to have arisen during the Second World War even though the first published material dates to the early fifties, in particular Arrow et al. (1951) and Dvoretzky et al.(1952). Since then, it has become part of every elementary textbook on Management Science or Operations Management. Most recently, Fisher et al. (1996) have propagated the model among a broad managerial audience. All of these treatments confine themselves to the expected value of costs or profits.

The newsboy problem has been analyzed in several basic versions focusing on different performance measures: (i) profits and (ii) one of several cost measures. In terms of the expected value, all of these performance measures are equivalent and the optimal order quantity or so-called "newsboy solution" is given by a specific fractile of the demand distribution where the fractile is a simple function of the cost/revenue rates. If variance is considered, we shall show that the profit and cost measures cease to be equivalent and often generate qualitatively different results. In the profit formulation, all order quantities larger than the newsboy solution are dominated, but the newsboy solution has the largest variance among all remaining order quantities. Therefore, if management is risk averse it should adopt an order quantity lower than the newsboy solution, and the amount by which the order quantity is reduced, increases monotonically with the degree of risk aversion. (This is diametrically opposed to the common "heuristic" practice to account for risk aversion by using an inflated lost sales penalty rate, while continuing to confine oneself to the expected profit criterion. The latter results, of course, in a solution *higher* than the newsboy solution.) For any of the cost measures, the situation is more complex. Here, the set of undominated or efficient order quantities, in general, and its position vis-a-vis the newsboy solution, in particular, depends heavily on the shape of the demand distribution. We show that for many such distributions, this set of efficient order quantities lies to the *right* of the newsboy solution, i.e., at the opposite side of its counterpart for the profit measure. The set of efficient order quantities and its position vis-a-vis the newsboy solution may even differ depending on which of the cost measures is being considered.

From a corporate point of view, the analysis should be based on profits. In fact, if the firm sells multiple items with correlated demand processes, the expected utility of aggregate profits across all such items should be maximized. In other words, in the presence of demand correlation between distinct items, the inventory planning process should not be developed on an item-by-item basis even in the absence of operational interdependence (e.g., joint setup costs, joint capacity constraints). Many organizations, however, evaluate operations managers on the basis of cost performance only. Our analysis indicates that this practice may result in distorted incentives even when, as in the simple newsboy problem, the revenue term in the profit equation is independent of any opertional decision variables.

We proceed with an analysis of the mean-variance tradeoff in two standard infinite-horizon models. We start with the base-stock model with Poisson arrivals and a constant (but positive) leadtime. The standard formulation of this model is to minimize the expected steady state costs possibly subject to a service-level constraint. The latter is often based on the expected waiting time an arbitrary customer experiences. In reality, customers usually care not only about the expected value, but also the variability of the waiting time. We therefore introduce a customer "disutility" measure, a general function which is increasing in the mean and variance of the customer waiting time. We provide a simple procedure to compute the efficient frontier between a general inventory measure (for instance, the expected value of steady state inventory costs, or a function of its expected value and variance as in the newsboy problem) and the above disutility measure.

Thereafter, we consider the standard periodic-review model with independent identically distributed demands, fixed-plus-linear order costs, and linear holding and backlogging costs. It is well known (see, e.g., Scarf (1960)) that an (s,S) policy minimizes long-run average costs. On the other hand, due to managerial reasons, other types of policies are frequently adopted in practice. One such policy is the (R,nQ) policy, where in each period a sufficiently large multiple of Q units is ordered to restore the inventory position to above R. (If the inventory position is already above R, no order is placed.) The managerial benefits of this policy are that orders always come in multiples of a constant batch size Q, facilitating packaging, stock counting, etc. (R,nQ) policies have been analyzed and shown to be quite close to being optimal (i.e., achieving an expected cost value close to that of the (s,S) policy), see Hadley and Whitin (1961) for the earlier literature and Zheng and Chen (1992) for more recent works. For the (R,nQ) model, the standard formulation is again to minimize the long-run average costs. We consider two measures of variability, the variance of on-hand inventory and the variance of costs incurred in an arbitrary period, and show how the long-run average costs can be traded off against each of these variability measures.

All of the above inventory models may be formulated as Markov Decision Problems (MDPs). Almost all of the voluminous literature on this class of problems concerns the expected value of some measure of effectiveness: the undiscounted or discounted sum of costs and the long-run average cost per period, see, e.g., Puterman (1994). Recently, Sobel (1994) and Filar et al. (1988) addressed mean-variance tradeoffs in general undiscounted and discounted MDPs, respectively. For example, Sobel characterizes the efficient frontier between the mean and variance of steady-state distributions among all stationary policies. See White (1988) for a survey of treatments of other risk-sensitive criteria in MDPs and White (1989), Bayal-Gursoy and Ross (1990), Huang and Killenberg (1990), Whittle (1990) and Chung (1994) for more recent contributions.

In the specific area of inventory models, Bouakiz and Sobel (1990) considered a special case of the periodic-review model considered here, with linear order costs. These authors address the objective of minimizing the expected utility of the present value of costs where the utility function is exponential. Lau (1980) considered for the newsboy model, and a specific class of demand distributions the problem of optimizing a linear function of the mean and standard deviation of profits, as well as that of maximizing the likelihood of reaching a given target profit level.

A different stream of papers (see Anvari (1987), Singhal (1988), Kim and Chung (1989), and Singhal et al. (1994)) address settings where the demand process of the item under consideration exhibits significant correlation with the stock market return process. Defining business risk by the covariance of the demand with the market return, these papers use the Capital Asset Pricing Model, as opposed to a fixed discount rate, to value the net present value of the cash flows in various standard inventory models, including the newsboy problem and the (R,Q) system. In doing so, the (approximate) evaluation method continues to assume that each period's or inventory cycle's cost is deterministically given by the long-run average cost per period or cycle.

2 The Newsboy Problem

In this, the most elementary of inventory models, a supply or purchase quantity Q needs to be chosen to cover a single period's demand D, which is random but with a known distribution. The decision maker is referred to as the newsboy as when Q denotes the number of copies of a newspaper to be purchased to cover a day's demand. D is assumed to be continuous and nonnegative with pdf $f(\cdot)$ and cdf $F(\cdot)$. Let μ and σ^2 be the mean and variance of D, both of which are assumed to be finite. Let s be the unit selling price, c the unit purchase cost, and v the net salvage value for each unit of unsold inventory (the salvage value, if any, minus the cost of holding/disposing the unit). To avoid trivial cases, we assume $v < c < s$.

The above newsboy problem is usually formulated in two different ways: one based on profits and the other based on one of several costs. If the objective is either maximizing expected profits or minimizing expected costs, then the two formulations, with properly chosen parameters, lead to the same optimal decision. However, as we shall show below, when the newsboy dislikes variability in either profits or costs, a systematic tradeoff of variability and expected value can lead to very different decisions, depending upon whether profits or costs are being considered.

To set the stage for later analysis, we now present a formulation based on profits and two others based on costs. Let $\pi(Q)$ be the total profit. Since the quantity sold is $\min\{Q, D\}$ and thus the unsold quantity is $Q - \min\{Q, D\}$, we have

$$\pi(Q) = s\min\{Q, D\} + v(Q - \min\{Q, D\}) - cQ$$

where the first two terms represent the total revenue and the last term total cost. Let $(x)^+ = \max\{0, x\}$. Since $\min\{Q, D\} = Q - (Q - D)^+$, the total profit can be rewritten as

$$\pi(Q) = (s - c)Q - (s - v)(Q - D)^+. \tag{1}$$

We assume that the newsboy's utility for profit x is

$$U(x) = ax - bx^2, \quad a, b > 0$$

where a and b are given parameters. (To ensure that the utility function is increasing for a sufficiently wide range of profits, we assume that $a/2b$ is large.) This type of utility function has been widely used in analyzing financial risks. Note that since $U(\cdot)$ is concave, the newsboy is risk averse. Finally, $U(\cdot)$ may be viewed as a two-term truncation of the

Taylor-series expansion of a more general utility function. The objective is to maximize the expected utility. Let $E_\pi(Q)$ and $V_\pi(Q)$ be the mean and variance of $\pi(Q)$. Then the objective function can be expressed as

$$E[U(\pi(Q))] = U(E_\pi(Q)) - bV_\pi(Q). \tag{2}$$

Therefore, maximizing expected utility reduces to a tradeoff between the mean and variance of profits. (An objection that is sometimes raised against the use of variance as a risk measure, is that it gives symmetric treatment to upside and downside deviations from the mean. This objection has given rise to alternative measures such as the above mentioned Average Downside Risk. The tradeoff procedure developed below can be adjusted to deal with such alternative risk measures.)

Even though the above formulation based on profits is rather straightforward, it has been conventional to address the newsboy problem based on costs. The reasoning seems impeccable. If the newsboy understocks (i.e., demand exceeds Q), he forgoes the opportunity to make $s-c$ per unit of excess demand. Call this the underage cost and denote it by c_u. On the other hand, if he overstocks (i.e., Q exceeds demand), he loses $c-v$ per unit of unsold inventory. Call this the overage cost and denote it by c_o. The total cost is thus

$$\phi(Q) = c_o(Q-D)^+ + c_u(Q-D)^- = (c-v)(Q-D)^+ + (s-c)(Q-D)^- \tag{3}$$

where $(x)^- = \max\{-x, 0\}$. Let $E_\phi(Q)$ and $V_\phi(Q)$ be the mean and variance of $\phi(Q)$.

Taking the above quadratic utility function and treating cost as negative profit, the newsboy's utility for incurring cost ϕ is

$$U(-\phi) = a(-\phi) - b\phi^2 \stackrel{def}{=} -u(\phi).$$

Call $u(\cdot)$ the 'disutility' function. Under the cost-based formulation, the newsboy minimizes the expected disutility, i.e.,

$$E[u(\phi(Q))] = aE_\phi(Q) + b(V_\phi(Q) + E_\phi^2(Q)) = u(E_\phi(Q)) + bV_\phi(Q). \tag{4}$$

Once again, the optimization problem reduces to a tradeoff between mean and variance. For convenience, we will refer to the above model based on underage and overage costs as the "Cost-I Model."

Clearly, there are other ways to define costs. One example is to write

$$\pi(Q) = sD - \psi(Q) \tag{5}$$

with sD the total (primary) revenue and $\psi(Q)$ the total cost. Interestingly, the previous cost model corresponds to a slightly different decomposition of the profit function:

$$\pi(Q) = (s - c)D - \phi(Q). \tag{6}$$

Clearly, from (3), (5) and (6)

$$\psi(Q) = \phi(Q) + cD = -v(Q - D)^+ + s(Q - D)^- + cQ. \tag{7}$$

Thus $\psi(Q)$ consists of the purchase cost, the cost of lost sales, and the cost associated with salvaging the unsold inventory. As in the above cost formulation, the objective is to minimize the expected disutilty, i.e., minimizing $u(E_\psi(Q)) + bV_\psi(Q)$ where $E_\psi(Q)$ and $V_\psi(Q)$ are the mean and variance of $\psi(Q)$. This model is referred to as the "Cost-II Model."

For the rest of this section, we will show that the above profit and cost models, although equivalent when the newsboy is risk neutral (i.e., U is linear), lead to very different solutions when the newsboy is risk averse. The intuition for this result is as follows: due to the strong dependence between the revenue term $(s - c)D$ (resp. sD) and the cost term $\phi(Q)$ (resp. $\psi(Q)$), var[profit] $\neq$ var[revenue] + var[cost]. In fact, var[profit] even fails to be a monotone function of var[cost].

2.1 The Profit Model

As mentioned above, maximizing the newsboy's expected utility amounts to a tradeoff between the mean and variance of profits (see (2)). We begin by deriving $E_\pi(Q)$ and $V_\pi(Q)$.

Since

$$E[(Q-D)^+] = \int_0^Q (Q-x)dF(x) = \int_0^Q F(x)dx$$

and

$$E[(Q-D)^+]^2 = \int_0^Q (Q-x)^2 dF(x) = \int_0^Q 2(Q-x)F(x)dx = 2Q\int_0^Q F(x)dx - 2\int_0^Q xF(x)dx$$

we have from (1)

$$E_\pi(Q) = (s-c)Q - (s-v)\int_0^Q F(x)dx \tag{8}$$

and

$$V_\pi(Q) = (s-v)^2 \left\{ 2Q\int_0^Q F(x)dx - 2\int_0^Q xF(x)dx - [\int_0^Q F(x)dx]^2 \right\}. \tag{9}$$

Since $E''_\pi(Q) = -(s-v)f(Q) \le 0$, $E_\pi(\cdot)$ is concave. Note also that

$$V'_\pi(Q) = 2(s-v)^2[1-F(Q)]\int_0^Q F(x)dx \ge 0.$$

Thus $V_\pi(\cdot)$ is nondecreasing. Moreover, we can rewrite $V_\pi(Q)$ as

$$\begin{aligned} V_\pi(Q) &= (s-v)^2 \left\{ -(Q - \int_0^Q F(x)dx)^2 + Q^2 - \int_0^Q F(x)dx^2 \right\} \\ &= (s-v)^2 \left\{ -(\int_0^Q [1-F(x)]dx)^2 + Q^2[1-F(Q)] + \int_0^Q x^2 dF(x) \right\}. \end{aligned}$$

Since D has a finite second moment, $\lim_{x\to\infty} x^2[1-F(x)] = 0$. (To see this, first note that since $\int_0^\infty y^2 f(y)dy < \infty$, $\lim_{x\to\infty}\int_x^\infty y^2 f(y)dy = 0$. Therefore, $\lim_{x\to\infty} x^2[1-F(x)] = 0$ follows because $0 \le x^2[1-F(x)] \le \int_x^\infty y^2 f(y)dy$ for all x.) Thus

$$\lim_{Q\to\infty} V_\pi(Q) = (s-v)^2\sigma^2.$$

Proposition 1 *(a) $E_\pi(\cdot)$ is concave and asymptotically linear with a slope $(v-c) < 0$. (b) $V_\pi(\cdot)$ is nondecreasing. Moreover, $V_\pi(Q) \le (s-v)^2\sigma^2$, $\forall\ Q \ge 0$, with $\lim_{Q\to\infty} V_\pi(Q) = (s-v)^2\sigma^2$.*

It follows from part (a) that $E_\pi(Q)$ is maximized at a finite, positive point, the so-called newsboy point/solution Q^*. (Note that $E'_\pi(0) = s-c > 0$.) From the first-order condition, it satisfies

$$F(Q^*) = \frac{s-c}{s-v}.$$

But to maximize expected utility (see (2)), we must consider both the mean and variance of the total profit. We say that Q is dominated if and only if there exists a value Q' so that $E_\pi(Q') \geq E_\pi(Q)$ and $V_\pi(Q') \leq V_\pi(Q)$ where at least one of the inequalities is strict. Clearly, if Q is dominated then it does not maximize the expected utility. From Proposition 1, it is immediate that the order quantities that are greater than Q^* are all dominated and do not need to be considered. Therefore,

Proposition 2 *The optimal order quantity (that maximizes the expected utility) is less than or equal to the newsboy point.*

From Proposition 1, as Q increases in $[0, Q^*]$, both the mean and the variance of the total profit increase. Therefore, none of the order quantities in the interval is dominated. The optimal solution can be obtained via a one-dimensional search.

Note that each order quantity is mapped to a point in the mean-variance space. The image of the order quantities in $[0, Q^*]$ is the efficient frontier, which summarizes the mean-variance tradeoff for the inventory manager. Therefore, the optimization problem can be re-stated in mean-variance space:

$$\begin{array}{ll} \max & U(E) - bV \\ s.t. & (E, V) \in \text{efficient frontier.} \end{array}$$

The optimal solution is obtained by finding the highest indifference curve for the inventory manager that touches the efficient frontier.

Example 1: Suppose D is uniformly distributed on some interval, without loss of generality, the interval $[0, 1]$. Thus, $F(x) = x$ for $0 \leq x \leq 1$ and $F(x) = 1$ for $x > 1$. We restrict Q to be on $[0, 1]$. The newsboy point is

$$Q^* = \frac{s-c}{s-v}.$$

From (8) and (9), one can easily show that

$$E_\pi(Q) = (s-c)Q - (s-v)\frac{Q^2}{2} \quad \text{and} \quad V_\pi(Q) = (s-v)^2\left(\frac{Q^3}{3} - \frac{Q^4}{4}\right).$$

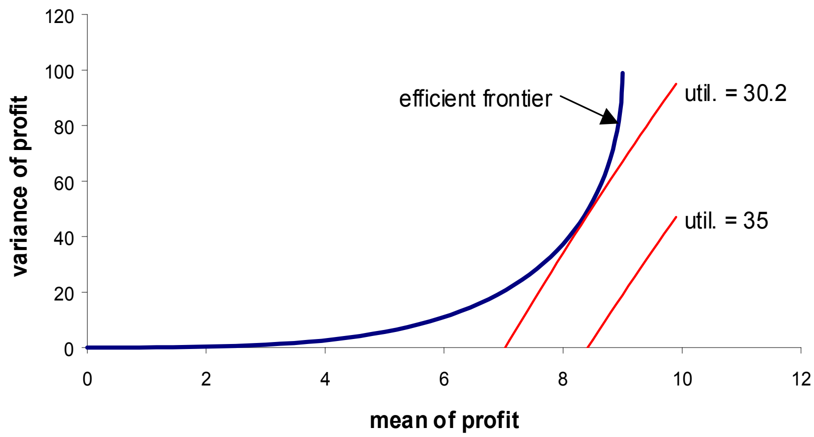

Figure 1: Efficient Frontier and Indifference Curves for Newsboy Problem with Profits

Note that $V_\pi(1) = (s-v)^2/12$ and $\sigma^2 = 1/12$. Also note that

$$\frac{V_\pi''(Q)}{(s-v)^2} = (2Q - 3Q^2).$$

Therefore, $V_\pi(Q)$ is convex for $0 \le Q \le 2/3$ and concave for $2/3 < Q \le 1$. Figure 1 depicts the efficient frontier as well as the newboy's indifference curves with $s = 100$, $c = 70$, $v = 50$, $a = 5$, and $b = 0.1$. The optimal solution arises where the highest indifference curve intersects the efficient frontier.

2.2 The Cost-I Model

Recall that this cost model is based on the sum of overage and underage costs, $\phi(Q)$, and represents the most prevalent way to analyze the newsboy problem in the literature. Under the quadratic disutility function, minimizing the expected disutility reduces to a mean-variance analysis. We thus begin with a derivation of $E_\phi(Q)$ and $V_\phi(Q)$. Since $x^- = x^+ - x$, we can rewrite $\phi(Q)$ (see (3)) as

$$\phi(Q) = (c_o + c_u)(Q - D)^+ - c_u(Q - D).$$

Therefore,

$$E_\phi(Q) = (c_o + c_u)\int_0^Q F(x)dx - c_u(Q - \mu). \tag{10}$$

The second moment of $\phi(Q)$ can be expressed as

$$E[\phi^2(Q)] = c_o^2 E[(Q-D)^+]^2 + c_u^2 E[(Q-D)^-]^2$$

since $x^+x^- \equiv 0$. It then follows that

$$V_\phi(Q) = c_o^2 \int_0^Q (Q-x)^2 dF(x) + c_u^2 \int_Q^\infty (x-Q)^2 dF(x) - (E_\phi(Q))^2. \tag{11}$$

It is well known and easily verified that $E_\phi(\cdot)$ is convex, and its minimum is achieved at Q^* which satisfies the first-order condition $E'_\phi(Q) = 0$. In other words, the profit- and cost-formulations lead to the same optimal order quantity for a risk-neutral newsboy (who optimizes based on expected profits or costs).

The behavior of $V_\phi(\cdot)$ is however more complex, in particular when compared with that of its counterpart $V_\pi(\cdot)$, the variance of profits. As before, Q is dominated (under the objective function (4)) if and only if there exists a value Q' so that $E_\phi(Q) \geq E_\phi(Q')$ and $V_\phi(Q) \geq V_\phi(Q')$ where at least one of the inequalities is strict. The mean-variance pairs of undominated order quantities trace out the efficient frontier. Thus, the optimization problem can be written as

$$\begin{aligned} \min \quad & u(E) + bV \\ s.t. \quad & (E, V) \in \text{efficient frontier}. \end{aligned}$$

Consider the class of power distribution functions, $F(x) = x^k$ for $x \in [0,1]$, with $k > 0$. The case $k = 1$ corresponds to the uniform distribution; if $k > (<)1$ the density function is increasing (decreasing). It is easily verified from (10) and (11) that

$$V'_\phi(Q) = \frac{2(c_o + c_u)Q}{k+1}[(c_o + c_u)Q^k(1 - Q^k) - c_u k Q^{k-1} + c_u k Q^k].$$

For $Q \in (0,1)$, the sign of $V'_\phi(Q)$ is the same as the sign of

$$W_\phi(Q) \stackrel{def}{=} (c_o + c_u)Q(1 - Q^k) - c_u k + c_u k Q.$$

Note that

$$W'_\phi(Q) = (c_o + c_u)(1 - Q^k - kQ^k) + c_u k$$

which is clearly decreasing in Q. Therefore $W_\phi(\cdot)$ is concave. Note further that $W_\phi(0) = -c_u k < 0$, $W_\phi(1) = 0$, and $W'_\phi(1) = -c_o k < 0$. Therefore, the sign of $W_\phi(Q)$ and hence $V'_\phi(Q)$ changes exactly once in the unique root Q^0 and the sign change is from negative to positive. The following proposition shows that the set of efficient order quantities is always an interval bordered by Q^* and Q^0.

Proposition 3 *Consider the cost measure ϕ. For the class of power demand distributions:*
(a) If $k < 1$, $Q^0 > Q^*$ and the set of efficient order quantities is $[Q^*, Q^0]$;
(b) If $k = 1$, $Q^0 = Q^*$ and Q^* is the only efficient order quantity;
(c) If $k > 1$, $Q^0 < Q^*$ and the set of efficient order quantities is $[Q^0, Q^*]$.

Proof: Since $F(Q^*) = (Q^*)^k = c_u/(c_o + c_u)$,

$$W_\phi(Q^*) = c_o Q^* - c_u k + c_u k Q^* = Q^*(c_o + c_u k) - c_u k.$$

Therefore, $Q^* < (=, >) Q^0 \Leftrightarrow W_\phi(Q^*) < (=, >) 0 \Leftrightarrow Q^* < (=, >) \frac{c_u k}{c_o + c_u k} \Leftrightarrow \frac{c_u}{c_o + c_u} < (=, >) \left(\frac{c_u k}{c_o + c_u k}\right)^k \Leftrightarrow k < (=, >) 1$, where the last equivalency follows from the function $\left(\frac{c_u k}{c_o + c_u k}\right)^k$ being decreasing in k. To verify the latter, take the derivative of the logarithm of the function and use the inequality $\ln(1 - x) + x \leq 0$ for $0 < x < 1$. To identify the set of efficient order quantities, note that $E'_\phi(Q)$ and $V'_\phi(Q)$ have opposite signs only if Q lies between Q^* and Q^0. $\square$

Thus for $k < 1$, a risk-averse inventory manager focusing on or being evaluated on the basis of cost measure ϕ, will choose an order quantity $\hat{Q}$ in access of the newsboy solution Q^*, but not beyond Q^0. The more risk averse the manager (i.e., the larger b), the larger the chosen excess $\hat{Q} - Q^*$, which we refer to as the *risk correction.* For $k = 1$, the demand distribution is uniform. In this case, the efficient frontier for the ϕ measure reduces to a single point. It is interesting to contrast this with the efficient frontier under the profit measure π

(see Figure 1). There, all Q values less than or equal to Q^* generate a point on the efficient frontier, i.e., all such Q values may arise as the optimal solution depending on the degree of risk aversion of the decision maker. On the other hand, if the decision maker's utility is based on the cost measure ϕ and $k \leq 1$, an order quantity less than Q^* will never emerge as the optimal solution regardless of the parameters a and b (i.e., the degree of risk aversion). In this case, the two performance measures point to potentially very different solutions when the manager is risk averse.

For $k > 1$, a risk-averse manager focusing on the cost measure ϕ will choose an order quantity below Q^*, applying a *negative risk correction*. The (absolute value of the) risk correction increases with his degree of risk aversion but is never larger than $Q^* - Q^0$. In contrast, a risk-averse manager focusing on the profit measure π may well apply a risk correction in excess of $Q^* - Q^0$.

For general demand distributions, the following is an efficient procedure to generate the efficient frontier. Consider an arbitrary set of grid points for the order quantity Q. Number the grid points $\{Q_1, Q_2, \cdots\}$ in nondecreasing order of their $E_\phi(Q)$-value. (Since $E_\phi(Q)$ is convex, the k smallest E-values are achieved by a consecutive set of Q-values, containing the newsboy solution. Thus, after determining $(Q_1, \cdots, Q_k)$, Q_{k+1} is either the point to the left of the smallest of these Q-values or the one to the right of the largest one.) Thus Q_1 is the newsboy solution and the point $(V_\phi(Q_1), E_\phi(Q_1))$ is on the efficient frontier. Assume the first n grid points have been considered as to whether the correspoinding mean-variance point is on the efficient frontier. Observe that $(V_\phi(Q_{n+1}), E_\phi(Q_{n+1}))$ is a point on the frontier if and only if $V_\phi(Q_{n+1}) \leq \min_{i=1,\cdots,n} V_\phi(Q_i)$. (It is noteworthy that if $V_\phi(Q)$ has multiple local optima, the Q-values contributing to the efficient frontier may be contained in several disjoint intervals. This follows from the fact that, if, on a given interval, $E_\phi(Q)$ and $V_\phi(Q)$ are both decreasing (increasing), only the right (left) most point on the interval needs to be considered.)

Figure 2 depicts the efficient frontier for a newsboy problem with a binomial demand dis-

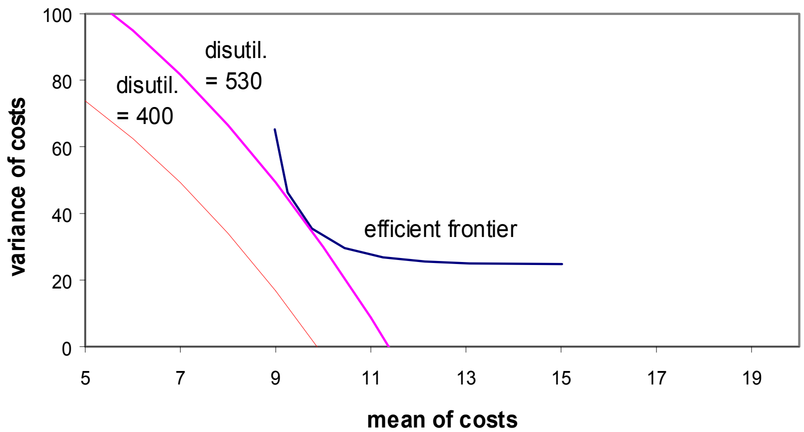

Figure 2: Efficient Frontier and Indifference Curves for Newsboy Problem with Costs

tribution (100, 0.5), $c_o = 1$, and $c_u = 10$. The newsboy solution is $Q^* = 57$, but the efficient frontier consists of the mean-variance combinations associated with $Q = 57, 58, \cdots, 75$. As with the power distributions with $k \leq 1$, these Q-values are all greater than the newsboy solution, whereas the opposite is true under the profit measure. In this example, $\mu = 50$, the safety stock consists of 7 units ($= Q^* - \mu$), and the risk correction can be as high as 18 ($= 75 - 57$), which is more than twice the safety stock. Figure 2 also displays an indifference curve that touches the efficient frontier when the disutility function has the above quadratic form with parameters $a = 1$ and $b = 4$.

2.3 The Cost-II Model

This model based on the total cost $\psi(Q)$ (see (7)) is quite similar to the previous one based on $\phi(Q)$. Recall that $\psi(Q) = -v(Q - D)^+ + s(Q - D)^- + cQ$. Since $x^- = x^+ - x$, $\psi(Q) = (s - v)(Q - D)^+ - s(Q - D) + cQ$. Therefore,

$$E_\psi(Q) = (s - v) \int_0^Q F(x)dx - s(Q - \mu) + cQ. \tag{12}$$

Thus $E_\psi(\cdot)$ is convex and is, once again, minimized at Q^*, the unique solution to the first-order condition $E'_\psi(Q) = 0$.

Now consider $V_\psi(Q)$. Note that the variance of $\psi(Q)$ is the same as the variance of $-v(Q-D)^+ + s(Q-D)^-$, which is the same as the expression for $\phi(Q)$ with c_o and c_u replaced by $-v$ and s respectively. Therefore

$$V_\psi(Q) = v^2 \int_0^Q (Q-x)^2 dF(x) + s^2 \int_Q^\infty (x-Q)^2 dF(x) - (E_\psi(Q) - cQ)^2. \tag{13}$$

Consider again the class of power distributions, $F(x) = x^k$ for $x \in [0,1]$ and $k > 0$. We have from (12) and (13) that

$$V'_\psi(Q) = \frac{2(s-v)Q}{k+1}[(s-v)Q^k(1-Q^k) - skQ^{k-1} + skQ^k].$$

For $Q \in [0,1]$, the sign of $V'_\psi(Q)$ is the same as the sign of

$$W_\psi(Q) \stackrel{def}{=} (s-v)Q(1-Q^k) - sk + skQ$$

with $W'_\psi(Q) = (s-v)(1-Q^k-kQ^k) + sk$. As with $W_\phi(\cdot)$, $W_\psi(\cdot)$ is concave. Furthermore, $W_\psi(0) = -sk < 0$, $W_\psi(1) = 0$, and $W'_\psi(1) = vk$.

Proposition 4 *Consider the cost measure ψ. For the class of power demand distributions:*
(a) If $v \geq 0$, $[Q^*, 1]$ is the set of efficient order quantities;
(b) If $v < 0$, $V_\psi(\cdot)$ has a unique minimum Q^0 on $(0,1)$. There exists a critical value k^* with $1 \leq k^* \leq \infty$ such that (i) If $k < k^*$, $Q^0 > Q^*$ and the set of efficient order quantities is $[Q^*, Q^0]$; (ii) If $k = k^*$, $Q^0 = Q^*$ and Q^* is the only efficient order quantity; (iii) If $k > k^*$, $Q^0 < Q^*$ and the set of efficient order quantities is $[Q^0, Q^*]$.

Proof: (a) In this case, $W'_\psi(1) \geq 0$ so that $W_\psi(Q)$ and hence $V'_\psi(Q)$ are nonpositive on the complete interval $[0,1]$. Therefore, $E'_\psi(Q)$ and $V'_\psi(Q)$ have opposite signs only if Q lies in $[Q^*, 1]$.

(b) Since $F(Q^*) = (Q^*)^k = (s-c)/(s-v)$, $W_\psi(Q^*) = (c-v)Q^* - sk + skQ^* = Q^*(c-v+sk) - sk$. Thus $W_\psi(Q^*) < (=,>) 0 \Leftrightarrow Q^* < (=,>) \frac{sk}{c-v+sk} \Leftrightarrow \frac{s-c}{s-v} < (=,>) \left(\frac{sk}{c-v+sk}\right)^k$. The function $\left(\frac{sk}{c-v+sk}\right)^k$ decreases from 1 to $e^{-\frac{c-v}{s}}$ as k increases from 0 to ∞. This implies the existence of a value $k^* \leq \infty$, such that $Q^* < (=,>) Q^0 \Leftrightarrow W_\psi(Q^*) < (=,>) 0 \Leftrightarrow k < (=,>) k^*$. Note $k^* \geq 1$ since $\frac{s-c}{s-v} \leq \frac{s}{c-v+s}$. Part (b) now follows from the observation that $E'_\psi(Q)$ and $V'_\psi(Q)$ have opposite signs only if Q is between Q^* and Q^0. Note that $k^* < \infty$ arises if, e.g., $v = -1$, $c = 0$, and $s = 1$ since in this case $0.5 = \frac{s-c}{s-v} > \left(\frac{sk}{c-v+sk}\right)^k$ for $k \geq 2$. □

Thus in the arguably most prevalent case with $v \geq 0$ (or even when v is negative but sufficiently small in absolute value, in which case one can show that $k^* = \infty$), the set of efficient order quantities is an interval to the right of the newsboy solution. Recall that the same applies to the cost measure ϕ only when $k < 1$ (i.e., the density function is decreasing). Note, however, that the set of efficient order quantities now extends to $Q = 1$, i.e., the maximum possible demand value. A risk-averse manager focusing on the cost measure ψ will order in excess of the newsboy solution; the more risk averse he is, the larger the risk correction he will apply. Under ψ, the chosen order quantity may extend up to the maximum demand value, whereas under ϕ, the efficient order quantities are bounded above by $Q^0 < 1$ irrespective of the manager's degree of risk aversion. Observe also that the above efficient order quantities and those under the profit measure π, or the cost measure ϕ with $k > 1$, are on opposite sides of the newsboy solution.

For more general demand distributions, one can also use the algorithm described in the context of the Cost-I Model to construct the efficient frontier.

3 The Base-Stock Model

Consider the following single-location, sinlge-product inventory model. Customers arrive according to a Poisson process with rate λ. Each customer demands one unit of the product. When there is on-hand inventory, demand is satisfied immediately; otherwise, demand is backlogged. A base-stock policy is followed: orders are placed to keep the inventory position (on-hand inventory plus outstanding orders minus backorders) at a constant level S, the base-stock level. Therefore, each customer arrival triggers an order (for one unit), i.e., one-for-one replenishment. We restrict our attention to nonnegative base-stock levels. (The case with negative base-stock levels can be handled similarly.) Let L, a nonnegative constant, be the replenishment leadtime: an order placed at time t arrives at time $t + L$.

The waiting time W that a customer experiences until his demand is satisfied is an important performance measure. A standard approach is to impose an upper bound on the *expected* waiting time $E(W)$. But in many real-world settings, customers also dislike the variability associated with the waiting time as measured by its variance $V(W)$. Therefore, the cost or disutility of waiting can be captured by a function $H(E(W), V(W))$ with H increasing in both of its arguments. [1]

Another performance measure is the steady-state holding costs incurred in the system. Let $G(S)$ be the expected value of these steady-state costs as a function of the base-stock level S. Clearly, $G(\cdot)$ is increasing. Below, we show how a tradeoff between the two measures of performance, $H(E(W), V(W))$ and $G(S)$, can be made efficiently. Our results continue to hold when G is replaced by any other increasing function. Even more generally, in parallel to the treatment of the newsboy problem, the inventory manager's concerns with the financial performance may be expressed by a quadratic utility function of the steady-state cost rate or the steady-state profit rate. The resulting mean-variance analyses of these scenarios are

[1] One can give this function a specific form under the following scenario. Suppose the demands in the above system come from a downstream firm, which satisfies demands from the end users. Assume that the end-user demands in different periods are independent and identically distributed. Assume further that the demand in each period is normal with mean μ and standard deviation σ. The replenishment leadtime for the downstream firm is W, measured in numbers of periods. It is well known that the minimum holding and backorder costs incurred by the downstream firm is an increasing function of the variance of the end-user demands during the replenishment leadtime W. Assuming that W is independent of the end-user demand process, the variance of the leadtime demand is $E(W)\sigma^2 + \mu^2 V(W)$. In this case, the above H function is linear in $E(W)$ and $V(W)$ with positive coefficients.

similar to the one developed below. We omit the details for the sake of brevity.

Since both $E(W)$ and $V(W)$ depend on the base-stock level, we hereafter use $E_S(W)$ and $V_S(W)$ to denote them. To determine these functions of S, first note that $E_0(W) = L$ and $V_0(W) = 0$, i.e., when $S = 0$, each customer waits exactly L units of time. Now assume $S > 0$. Recall that a customer arriving at time t triggers an order, which arrives at time $t+L$. Note that this order is used to satisfy the demand of the Sth customer after time t. Axsater (1990) calls this the 'assigned' demand for the order. The assigned demand arrives at time $t+T$, where T is an Erlang random variable with parameters S and λ. If $t+L \le t+T$, i.e., the supply arrives before its corresponding demand, the customer does not wait; otherwise, the customer waits $(L-T)$ time units. Therefore,

$$W = (L-T)^{+}. \tag{14}$$

Since $T \le x$ if and only if the number of customer arrivals in x units of time is at least S,

$$F_S(x) \stackrel{def}{=} Pr(T \le x) = \sum_{n=S}^{\infty} \frac{e^{-\lambda x}(\lambda x)^n}{n!}, \quad x \ge 0. \tag{15}$$

From (14),

$$Pr(W=0) = Pr(T \ge L) \quad \text{and} \quad Pr(W \le w) = Pr(T \ge L-w), \quad 0 < w \le L.$$

One can directly use this waiting-time distribution to compute $E_S(W)$ and $V_S(W)$. Below, we present a more efficient procedure to compute these functions.

Let $f_S(\cdot)$ be the density function of T, i.e.

$$f_S(x) = \lambda \frac{e^{-\lambda x}(\lambda x)^{S-1}}{(S-1)!}.$$

By definition $\int_0^L f_S(x)dx = F_S(L) \stackrel{def}{=} F_S$. From (15), we have $F_S(x) - F_{S+1}(x) = \frac{1}{\lambda} f_{S+1}(x) \ge 0$ for any $x \ge 0$, and in particular

$$F_S - F_{S+1} = \frac{1}{\lambda} f_{S+1}(L) \ge 0. \tag{16}$$

First, consider $E_S[W]$. Since

$$E_S[W] = \int_0^L (L-x)dF_S(x) = \int_0^L F_S(x)dx, \tag{17}$$

we have

$$E_{S+1}[W] - E_S[W] = \int_0^L [F_{S+1}(x) - F_S(x)]dx = \int_0^L -\frac{1}{\lambda} f_{S+1}(x)dx = -F_{S+1}/\lambda. \tag{18}$$

Since F_{S+1} is nonnegative and nonincreasing in S, $E_S[W]$ is convex and nonincreasing in S. Also, since $\lim_{S\to\infty} F_S(x) = 0$ for any fixed x, we have $\lim_{S\to\infty} E_S[W] = 0$ from (17). Note that (18), together with the initial value $E_0(W) = L$, can be used to compute $E_S[W]$ efficiently.

To compute $V_S[W]$, it now suffices to determine $E_S[W^2] \stackrel{def}{=} E[W^2]$. Note that $E_0[W^2] = L^2$. Now assume $S > 0$. Since

$$E_S[W^2] = \int_0^L (L-x)^2 dF_S(x) = \int_0^L 2(L-x)F_S(x)dx, \tag{19}$$

we have

$$\begin{aligned}
E_{S+1}[W^2] - E_S[W^2] &= -\frac{1}{\lambda}\int_0^L 2(L-x)f_{S+1}(x)dx && (20)\\
&= -\frac{2L}{\lambda}\int_0^L f_{S+1}(x)dx + \frac{2(S+1)}{\lambda^2}\int_0^L f_{S+2}(x)dx \\
&= \frac{2}{\lambda^2}\{-\lambda L F_{S+1} + (S+1)F_{S+2}\} && (21)
\end{aligned}$$

where the second equality follows since $xf_{S+1}(x) = (S+1)f_{S+2}(x)/\lambda$. From (20), $E_S[W^2]$ is nonincreasing in S. Using (21), we have the following second-order difference

$$\begin{aligned}
&(E_{S+1}[W^2] - E_S[W^2]) - (E_S[W^2] - E_{S-1}[W^2]) \\
&= \frac{2}{\lambda^2}\{-\lambda L[F_{S+1} - F_S] + (S+1)F_{S+2} - SF_{S+1}\} \\
&= \frac{2}{\lambda^2}\{-\lambda L[F_{S+1} - F_S] + (S+1)[F_{S+2} - F_{S+1}] + F_{S+1}\} \\
&= \frac{2}{\lambda^2}[Lf_{S+1}(L) - (S+1)f_{S+2}(L)/\lambda + F_{S+1}] \\
&= \frac{2}{\lambda^2}F_{S+1} \geq 0
\end{aligned}$$

where the third equality follows from (16) and the last one from $Lf_{S+1}(L) = (S+1)f_{S+2}(L)/\lambda$. Therefore, $E_S[W^2]$ is convex in S. Also, since $\lim_{S\to\infty} F_S(x) = 0$ for any fixed x, we have from (19) that $\lim_{S\to\infty} E_S[W^2] = 0$. Note that (21), together with the initial value $E_0[W^2] = L^2$, can be used to compute $E_S[W^2]$ recursively. Therefore, $V_S[W] = E_S[W^2] - (E_S[W])^2$ can also be computed recursively. Note that $\lim_{S\to\infty} V_S[W] = 0$. We conclude:

Proposition 5 *For the base-stock model, we have*

i) $E_S[W]$ is convex and nonincreasing in S with $\lim_{S\to\infty} E_S[W] = 0$.

ii) $E_S[W^2]$ is convex and nonincreasing in S with $\lim_{S\to\infty} E_S[W^2] = 0$.

iii) $V_S[W]$ can be recursively updated by using (18) and (21) as S is increased. Note that $V_S(W)$ fails to be monotone in S since $V_0(W) = 0$ and $\lim_{S\to\infty} V_S[W] = 0$.

The efficient frontier between the H- and the G-measure can be constructed by a procedure similar to that employed in §2.2 (for general demand distributions). Clearly, the choice $S = 0$ minimizes the measure $G(\cdot)$. This implies that the point $(G(0), H(E_0(W), V_0(W))) = (G(0), H(L, 0))$ is on the efficient frontier. Now evaluate sequentially, for $S = 1, 2, \cdots$, whether the corresponding point $(G(S), H(E_S(W), V_S(W)))$ is on the efficient frontier. Assume this has been done for $S = 0, \cdots, n$. Observe that the (G, H)-point associated with $S = n+1$ is on the efficient frontier if and only if $H(E_{n+1}(W), V_{n+1}(W)) < \min_{0\le S\le n} H(E_S(W), V_S(W))$.

Figure 3 is an example with $\lambda = 1$ and $L = 10$. The performance measures used are: $H(E, V) = E + 3V$ and $G(S)$ is the expected steady-state on-hand inventory. Note that the base-stock levels $S = 1, 2, \cdots, 9$ are all dominated by $S = 0$, and the efficient frontier consists of the performance combinations associated with $S = 0, 10, 11, \cdots$. In other words, if the variance of the waiting time plays a significant role, a make-to-stock system (with $S > 0$) is efficient only if a relatively large base stock level is adopted. This perhaps explains why the optimal base-stock levels in some multi-echelon inventory systems are zero (see, e.g., Axsater 1996). [2]

[2] Continuing the example given in footnote 1, if the coefficient of variation (i.e., σ/μ) of the customer

Figure 3: Waiting Time and Inventory Tradeoff for Base stock Model

4 The (R,nQ) Model

Consider the following single-location, single-product, periodic-review inventory model. Demands in different periods are independent, identically distributed random variables. Stockouts are fully backlogged. Inventory is replenished from an outside supplier with ample supply. Replenishment orders are generated by an (r, nQ) policy: whenever the inventory position (on-hand inventory plus outstanding orders minus backorders) is at or below a reorder point r, order a minimum integer multiple of Q to raise the inventory position to above r. An order placed at the beginning of period t arrives at the beginning of period $t + L$ with L a given nonnegative integer. A fixed cost K is incurred for each batch of Q units ordered, and holding and backorder costs accrue at the end of each period at rates h and p respectively. The planning horizon is infinite. The decision variables are r and Q, and the inventory manager would like to minimize the long-run average total cost incurred in the system as well as some measure of variability to be specified later. It is the goal of this section to show how a tradeoff between these, often conflicting, objectives can be made.

demand process is relatively small and thus the role of $V(W)$ in $H(E(W), V(W)) = E(W)\sigma^2 + \mu^2 V(W)$ relatively large, we are likely to see large optimal base-stock levels.

We begin by reviewing some well-known results about the (r,nQ) model. Define

$IP(t)$	=	inventory position at the beginning of period t
$IL(t)$	=	inventory level (on-hand inventory minus backorders) at the end of period t
D	=	one-period demand, a discrete random variable, with $E[D] = \mu$, probability mass function $f(\cdot)$, and cumulative distribution function $F(\cdot)$
D_{L+1}	=	demand over $L+1$ periods, with pmf $f_{L+1}(\cdot)$ and cdf $F_{L+1}(\cdot)$
$C(r,Q)$	=	long-run average total (setup, holding, and backorder) cost.

Also, let IP and IL denote the inventory position and inventory level in steady state respectively. The following is the well-known inventory balance equation

$$IL(t+L) = IP(t) - D(t, t+L) \tag{22}$$

where $D(t, t+L)$ represents the total demand in periods $t, t+1, \cdots, t+L$, which is distributed as D_{L+1}. Define

$$G(y) = E[h(y - D_{L+1})^+ + p(y - D_{L+1})^-]$$

which represents the expected holding and backorder costs in period $t + L$ given $IP(t) = y$. Since the steady-state distribution of $IP(t)$ is uniform on the interval $[r+1, r+Q]$ (see, e.g., Hadley and Whitin (1961)) and the long-run average setup cost is $\mu K/Q$,

$$C(r,Q) = \frac{\mu K + \sum_{y=r+1}^{r+Q} G(y)}{Q}$$

which is known to be jointly convex in r and Q (Zipkin 2000).

We consider two different measures of variability. One is the variance of on-hand inventory at the end of an arbitrary period. This measure is of interest if the inventory manager is primarily concerned with the fluctuations in the capital tied up in inventory. The other is the variance of total costs incurred in an arbitrary period, a useful measure if the inventory manager dislikes variations in the operating expenses needed to run the system. Below, we show how a systematic tradeoff can be made between the long-run average cost and each of the above risk measures.

4.1 Variance of on-hand inventory

Let $V_I(r,Q)$ be the variance of IL^+. Consider the following problem:

$$\begin{aligned} \textbf{P1} \qquad & \min_{r,Q}\ C(r,Q) \\ & s.t.\ V_I(r,Q) \le v \end{aligned}$$

with $v > 0$ a pre-specified constant. Varying the value of v in the above problem permits us to generate the efficient frontier.

From the inventory balance equation (see (22)) in steady state, i.e., $IL = IP - D_{L+1}$, and the fact that IP is independent of D_{L+1}, we have

$$E[IL^+|IP=y] = u_1(y) \quad \text{and} \quad E[(IL^+)^2|IP=y] = u_2(y)$$

where

$$u_1(y) = \sum_{x=0}^{y}(y-x)f_{L+1}(x) \quad \text{and} \quad u_2(y) = \sum_{x=0}^{y}(y-x)^2 f_{L+1}(x).$$

Let $u_1(y) = u_2(y) = 0$ for any integer $y < 0$. Since $u_1(y) = yF_{L+1}(y) - \sum_{x=0}^{y} x f_{L+1}(x)$, we have

$$\begin{aligned} u_1(y+1) - u_1(y) &= (y+1)F_{L+1}(y+1) - yF_{L+1}(y) - (y+1)f_{L+1}(y+1) \\ &= F_{L+1}(y). \end{aligned} \tag{23}$$

On the other hand, since $u_2(y+1) = \sum_{x=0}^{y+1}(y+1-x)^2 f_{L+1}(x) = \sum_{x=0}^{y}(y+1-x)^2 f_{L+1}(x)$, we have

$$\begin{aligned} u_2(y+1) - u_2(y) &= \sum_{x=0}^{y}[(y+1-x)^2 - (y-x)^2]f_{L+1}(x) \\ &= \sum_{x=0}^{y}[(y+1-x) + (y-x)]f_{L+1}(x) \\ &= u_1(y+1) + u_1(y). \end{aligned} \tag{24}$$

Since IP is uniformly distributed on the interval $[r+1, r+Q]$,

$$\begin{aligned} V_I(r,Q) &= E[(IL^+)^2] - (E[IL^+])^2 \\ &= \frac{1}{Q}\sum_{y=r+1}^{r+Q} u_2(y) - \left(\frac{1}{Q}\sum_{y=r+1}^{r+Q} u_1(y)\right)^2. \end{aligned} \tag{25}$$

Lemma 1 *For any fixed Q, $V_I(r,Q)$ in nondecreasing in r.*

Proof: Since

$$\begin{aligned}\sum_{y=r+2}^{r+1+Q} u_1(y) + \sum_{y=r+1}^{r+Q} u_1(y) &= \sum_{y=r+1}^{r+Q} [u_1(y+1) + u_1(y)] \\ &\overset{(24)}{=} \sum_{y=r+1}^{r+Q} [u_2(y+1) - u_2(y)] \\ &= u_2(r+1+Q) - u_2(r+1)\end{aligned}$$

we have

$$\left(\frac{1}{Q}\sum_{y=r+2}^{r+1+Q} u_1(y)\right)^2 - \left(\frac{1}{Q}\sum_{y=r+1}^{r+Q} u_1(y)\right)^2 = \frac{u_2(r+1+Q) - u_2(r+1)}{Q} \cdot \frac{u_1(r+1+Q) - u_1(r+1)}{Q}.$$

Therefore, from (25),

$$\begin{aligned}& V_I(r+1,Q) - V_I(r,Q) \\ &= \frac{u_2(r+Q+1) - u_2(r+1)}{Q} - \left(\frac{1}{Q}\sum_{y=r+2}^{r+1+Q} u_1(y)\right)^2 + \left(\frac{1}{Q}\sum_{y=r+1}^{r+Q} u_1(y)\right)^2 \\ &= \frac{u_2(r+1+Q) - u_2(r+1)}{Q}\left[1 - \frac{u_1(r+1+Q) - u_1(r+1)}{Q}\right] \\ &\geq 0\end{aligned}$$

where the inequality follows since $u_2(\cdot)$ is nondecreasing (see (24)) and since $u_1(r+1+Q) - u_1(r+1) = \sum_{y=r+1}^{r+Q}[u_1(y+1) - u_1(y)] \overset{(23)}{=} \sum_{y=r+1}^{r+Q} F_{L+1}(y) \leq Q$. □

For any fixed Q, let $C(r,Q)$ be minimized at $r = r_e(Q)$. Define $C(Q) = C(r_e(Q),Q)$. Let $r_v(Q) = \max\{r : V_I(r,Q) \leq v\}$. Let (r_0,Q_0) be any feasible solution to **P1** with $C_0 = C(r_0,Q_0)$. Define $\overline{Q} = \max\{Q : C(Q) \leq C_0\}$. Let (r^*,Q^*) be an optimal solution to **P1**.

Lemma 2

i) For any fixed Q, $r_v(Q) \geq -Q$.

ii) For any fixed Q, $r = \min\{r_e(Q), r_v(Q)\}$ solves **P1**.

iii) $Q^* \leq \overline{Q}$.

Proof: i) follows from Lemma 1 and the fact that if $r = -Q$, the inventory position is always less than or equal to zero so that the on-hand inventory is always zero. ii) follows from Lemma 1 and the fact that $-C(r, Q)$ is unimodal in r (Federgruen and Zheng 1992). iii) follows since for any feasible solution (r, Q) to **P1** with $Q > \overline{Q}$, we have by definition $C(r, Q) \geq C(Q) > C_0$. □

The following algorithm solves **P1**. (The initial solution is $r = -1$ and $Q = 1$. This is feasible because $V_I(-1, 1) = 0 < v$. The corresponding objective function value is $C(-1, 1) = \mu K + G(0)$.)

$r^* = -1$; $Q^* = 1$; $C^* = \mu K + G(0)$;
determine an upper bound on Q: $\overline{Q}$;
for $Q = 1$ to $\overline{Q}$ do
begin
    $r = -Q$;
    while $V_I(r + 1, Q) \leq v$ and $r + 1 \leq r_e(Q)$ do $r = r + 1$;
    if $C(r, Q) < C^*$ then
    begin
        $r^* = r$; $Q^* = Q$; $C^* = C(r, Q)$;
        update the upper bound $\overline{Q}$ with C^* as the new feasible cost;
    end;
end.

We applied the above algorithm to an example with a binomial demand distribution with parameters 10 and 0.5 and the following cost/leadtime parameters: $K = 10$, $h = 1$, $p = 10$, $L = 2$. The optimal solution without the variance constraint is $(r^0, Q^0) = (15, 11)$ with the corresponding long-run average cost at 11.88 per period. By solving **P1** with a progressively larger value of v, we determined the efficient frontier in mean-variance space (Figure 4). We note that all the points on the efficient frontier are generated by (r,nQ) policies whose reorder points and order quantities are both smaller than r^0 and Q^0 respectively. In other

Figure 4: Efficient Frontier for (R,nQ) Model

words, the constraint on the variance of on-hand inventory results in smaller values of r and Q.

4.2 Variance of costs

Take any period t in steady state. Let C_o be the ordering costs incurred at the beginning of period $t+L+1$, and C_{hb} the holding and backorder costs incurred at the end of period $t+L$. Call C_o+C_{hb} the total cost in a period. Let $V_c(r,Q)$ be the variance of the one-period cost. Consider the following problem:

$$\textbf{P2} \qquad \begin{aligned} &\min_{r,Q}\ V_c(r,Q) \\ &s.t.\ C(r,Q) \leq e \end{aligned}$$

where $e > 0$ is a pre-specified constant.

We begin by deriving $V_c(r,Q)$. Since the expected value of the one-period cost is $C(r,Q)$,

$$\begin{aligned} V_c(r,Q) &= E[C_o + C_{hb}]^2 - [C(r,Q)]^2 \\ &= E[C_o]^2 + E[C_{hb}]^2 + 2E[C_oC_{hb}] - [C(r,Q)]^2. \end{aligned} \tag{26}$$

It remains to derive $E[C_o]^2$, $E[C_{hb}]^2$ and $E[C_o C_{hb}]$.

The following notation is useful. Let $IP = IP(t)$, $IP' = IP(t+L)$, D_L the total demand in periods $t, \cdots, t+L-1$, and D the demand in period $t+L$. Let $D_{L+1} = D_L + D$.

First, consider C_o. Note that $r < IP' \le r+Q$ and $IP' = IP - D_L + mQ \stackrel{def}{=} O[IP - D_L]$, where m is the unique (nonnegative) integer so that IP' is in $[r+1, r+Q]$, i.e., m is the number of batches ordered in periods $t+1, \cdots, t+L$. The inventory position at the end of period $t+L$ is $IP' - D$. If this inventory position is higher than r, then no order is placed at the beginning of period $t+L+1$. If it is at or below r but higher than $r-Q$, then Q units are ordered, and so on. Since each batch (of Q units) incurs an ordering cost of K, we have

$$\begin{aligned} C_o &= \sum_{n=0}^{\infty} (nK)\mathbf{1}(r < IP' - D + nQ \le r+Q) && (27) \\ &= \sum_{n=0}^{\infty} (nK)\mathbf{1}(r < O[IP - D_L] - D + nQ \le r+Q) && (28) \end{aligned}$$

where $\mathbf{1}(\cdot)$ is the indicator function. Since IP' is uniformly distributed from $r+1$ to $r+Q$ and is independent of D, we have from (27)

$$\begin{aligned} E[C_o]^2 &= \sum_{n=0}^{\infty} (nK)^2 Pr(r < IP' - D + nQ \le r+Q) \\ &= \sum_{n=0}^{\infty} \frac{(nK)^2}{Q} \sum_{y=r+1}^{r+Q} [F(y+nQ-r-1) - F(y+nQ-r-Q-1)]. && (29) \end{aligned}$$

Now consider C_{hb}. From (22),

$$\begin{aligned} C_{hb} &= h(IP - D_{L+1})^+ + p(IP - D_{L+1})^- && (30) \\ &= h(IP - D_L - D)^+ + p(IP - D_L - D)^-. && (31) \end{aligned}$$

Using (30) and the fact that $x^+ x^- \equiv 0$, we have

$$\begin{aligned} E[C_{hb}]^2 &= h^2 E[(IP - D_{L+1})^+]^2 + p^2 E[(IP - D_{L+1})^-]^2 \\ &= h^2 \frac{1}{Q} \sum_{y=r+1}^{r+Q} \sum_{x=0}^{y} (y-x)^2 f_{L+1}(x) + p^2 \frac{1}{Q} \sum_{y=r+1}^{r+Q} \sum_{x=y+1}^{\infty} (x-y)^2 f_{L+1}(x). && (32) \end{aligned}$$

Finally, from (28), (31) and the fact that IP, D_L, and D are independent, we have

$$E[C_o C_{hb}] = \frac{K}{Q} \sum_{y=r+1}^{r+Q} \sum_{x=0}^{\infty} \sum_{z=0}^{\infty} n(y,x,z)[h(y-x-z)^+ + p(y-x-z)^-] f_L(x) f(z) \tag{33}$$

where $n(y,x,z)$ is the unique integer that satisfies

$$r < O[y-x] - z + n(y,x,z)Q \le r + Q$$

and $f_L(\cdot)$ is the probability mass function of D_L. Using (29), (32) and (33) in (26), we have $V_c(r,Q)$.

We can now solve **P2**. Recall that $C(r,Q)$ is jointly convex in r and Q. Thus the feasible region of **P2** is a convex set which is easily determined. A complete search over this feasible region results in an optimal solution. By systematically increasing the value of e and solving the corresponding problem **P2**, we can trace out the efficient frontier that represents the tradeoff between the expected value and the variance of costs.

For the example considered in the previous subsection, i.e., demand is binomial with parameters 10 and 0.5, and $K = 10$, $h = 1$, $p = 10$, $L = 2$, the optimal solution minimizing expected costs is again $(r^0, Q^0) = (15, 11)$ and the minimum expected costs are 11.88 per period. The efficient frontier in the mean-variance space is depicted in Figure 5. It is interesting that the entire efficient frontier corresponds to (r,nQ) policies with reorder points larger than r^0 and order quantities smaller than Q^0.

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Figure 5: Efficient Frontier for (R,nQ) Model

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