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OPTIMAL INVENTORY POLICY¹

By KENNETH J. ARROW, THEODORE HARRIS, AND JACOB MARSCHAK²

Optimal inventory policy is first derived for a simple model in which the future (and constant) demand flow and other relevant quantities are known in advance. This is followed by the study of uncertainty models—a static and a dynamic one—in which the demand flow is a random variable with a known probability distribution. The best maximum stock and the best reordering point are determined as functions of the demand distribution, the cost of making an order, and the penalty of stock depletion.

1. INTRODUCTION

WE propose to outline a method for deriving optimal rules of inventory policy for finished goods. The problem of inventories exists not only for business enterprises but also for nonprofit agencies such as governmental establishments and their various branches. Moreover, the concept of inventories can be generalized so as to include not only goods but also disposable reserves of manpower as well as various stand-by devices. Also, while inventories of finished goods present the simplest problem, the concept can be extended to goods which can be transformed, at a cost, into one or more kinds of finished goods if and when the need for such goods arises. The following notes prepare the way for a more general future analysis of “flexible planning.”

We shall call “net utility” the quantity that the policymaker seeks to maximize. In the case of profit-making enterprises this is conveniently approximated by profit: the difference between gross money revenue and money cost. A nonprofit agency such as a hospital may often be able to compute directly its money cost, and has to assign an approximate monetary value to the “gross utility” of the performance of its tasks; it corresponds to the “gross revenue” of an enterprise run for profit.³

¹ This paper was prepared in the summer of 1950 at the Logistics Conference of The RAND Corporation, Santa Monica, California. It will be reprinted as Cowles Commission Paper, New Series, No. 44.

² The authors express gratitude for remarks and criticism of staff members of the Cowles Commission for Research in Economics and, in particular, for detailed and helpful comments and suggestions by Gerard Debreu. Criticisms by Herbert A. Simon, Carnegie Institute of Technology, and discussions with Allen Newell, of The RAND Corporation, and with Joyce Friedman, Joseph Kruskal, and C. B. Tompkins, of the Office of Naval Research Logistics Project, George Washington University, have also proved stimulating.

The authors regret that the important work of Pierre Massé [1] did not come to their attention before this article was completed.

³ The head of a nonprofit organization, just like the head of a household, has to arrange the outcomes of alternative actions in the order of his preferences.

The net utility to any policymaker is, in general, a random variable depending on certain conditions (i.e., on variables or on relations between variables). Some of these conditions he can control, others he cannot. The former are policy means (rules of action, strategies). The noncontrolled conditions are, in general, defined by a joint probability distribution of certain variables. Rational policy consists in fixing the controlled conditions so as to maximize the expected value of net utility, given the probability distribution of noncontrolled conditions. When this probability distribution degenerates into a set of nonrandom variables we have the case of "certainty." In this limiting case, net utility itself is a nonrandom variable, to be maximized by the policymaker.

In the present paper, the policymaker is the holder of inventories. At most one of the noncontrolled conditions will be regarded as a random one: the rate of demand for the policymaker's product. Other noncontrolled conditions will be regarded as constants, or as relations with constant parameters: the pipeline time; the cost of making an order; the relation between storage cost and the size of inventory; the price paid or its relation to the size of order (the "supply function"); the gross revenue (or, more generally, the gross utility) obtained. (Speculative inventories are thus excluded from consideration.)

As to controlled conditions, we shall assume that the policymaker can control only the size of the orders he makes (at discrete points of

Moreover, if choices can be made between alternative "lotteries," each characterized by a different probability distribution of outcomes, then—as proved by von Neumann and Morgenstern [1]—numerical "utilities" can be assigned to the outcomes in such a way that the chosen lottery is the one with the highest expected utility.

An important problem arises as to the decisions of a sub-agency of a profit or nonprofit organization. The head of a sub-agency has to take decisions that would maximize expected utility or revenue not to himself but to the superior organization. He cannot calculate the effect of his action upon this utility or revenue because he does not know the actions of other sub-agencies. However, the superior organization can, instead, inform its subordinates of a certain set of "intrinsic prices." This set has the following property: If each sub-agency maximizes its "net revenue" computed in those "prices" (i.e., if it maximizes the algebraic sum of inputs and outputs, present or future, each multiplied by its "price"), then the utility for the superior organization is also maximized. The ratios between any two "intrinsic prices" are equal to the ratios at which the corresponding in- or outputs can be substituted for each other without making the superior organization worse off (its resources and technology being given). See Koopmans [1, 2], Debreu [1].

Whenever direct monetary calculation appears not to be feasible, we shall, throughout this paper, use the words "utility" and "revenue" as interchangeable; the "revenue" of a sub-agency being understood to be computed at the "intrinsic prices" set up by the superior organization.

time). This eliminates, for example, such policy means as the fixing of the selling price, or the use of advertising, to influence demand, and any bargaining with buyer(s) or competitor(s).

We believe our specialized formulation is a workable first approximation. By regarding the order size as the only controlled condition, and the demand as the only random noncontrolled condition, we do take account of most of the major questions that have actually arisen in the practice of business and nonprofit organizations.⁴

2. THE CASE OF CERTAINTY

2:A. Let x be the known constant rate of demand for the product of the organization, per unit of time. Let the gross utility (i.e., utility before deducting cost) obtained by the organization through satisfying this demand be, per unit of time,

$$ax + a_0.$$

With a nonprofit organization, a_0 expresses the value of its "being" (word coined for the British Navy). If the organization is a commercial firm, a is the selling price; otherwise a is the value to the organization of a unit of its operations. In general, a is a function of x . It will be sufficient, for our purposes, to assume a constant, and $a_0 = 0$. Denote by $b = b(q)$ the purchasing price of one unit when the size of order is q . We shall assume that $q \cdot b(q)$ is an increasing function of q , and that $b'(q) \leq 0$ (possible economy of large scale orders). Let K be the cost of handling an order, regardless of its size. Let the cost of carrying a stock z over one unit of time be

$$c_0 + 2cz,$$

where c_0 is the overhead cost of storage. In general, c_0 varies with the maximum amount stocked, and c varies with the current stock z and also (because of spoilage, leakage, and obsolescence) with the prices paid. However, we shall assume c_0 and c constant.

2:B. With K positive a continuous flow of orders would be infinitely expensive. Hence orders will be given at discrete time intervals. Let the first ordering date be 0. Let the length of the i th time interval be θ_i . Then the delivery during that interval is $x\theta_i$. See Figure 1a, where the common slope of all slanting lines is the demand flow, x .

We shall show that, under certain conditions, optimal policy will be

⁴ Before formulating the problem, a study was made of the existing business literature on inventory control, using freely the comprehensive bibliographies that were compiled by T. H. Whitin [1] of Princeton University, and by Louise B. Haack [1] of George Washington University, for projects of the Office of Naval Research at those universities.

as shown in Figure 1b: the intervals, possibly excepting the first one, will have the same optimal length and the same optimal highest and lowest stock levels.

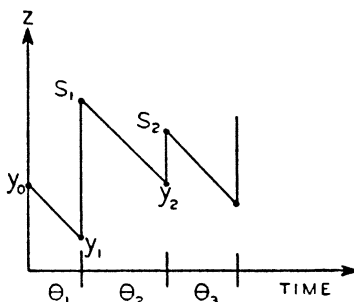


FIGURE 1a

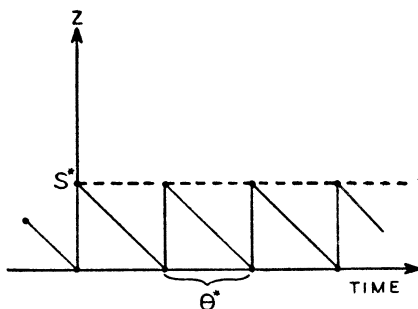


FIGURE 1b

We shall first assume that orders are fulfilled immediately. Then the amount ordered at the beginning of the i th interval,

$$q_i = S_{i-1} - y_{i-1} \geq 0,$$

where S_{i-1} and y_{i-1} denote, respectively, the stocks available at the beginning of the i th interval after and before the replenishment. Since the delivery during that interval is

$$x\theta_i = S_{i-1} - y_i,$$

therefore

$$(2.1) \quad q_i = x\theta_i + y_i - y_{i-1} \quad (i = 1, 2, \dots),$$

while the average stock during the i th interval is

$$(2.2) \quad \bar{z}_i = (S_{i-1} + y_i)/2 = (x\theta_i + 2y_i)/2 \quad (i = 1, 2, \dots).$$

The net utility achieved during the i th interval (not allowing for a time discount) is

$$(2.3) \quad u(\theta_i) = ax\theta_i - q_i \cdot b(q_i) - 2c\bar{z}_i\theta_i - K.$$

By (2.1), (2.2), this is a decreasing function of $y_i \geq 0$. Hence, for given θ_i and y_{i-1} , $u(\theta_i)$ has its maximum when $y_i = 0$. Further, we can put $y_0 = 0$: if the agency begins its operations with a stock $y' > 0$, its best policy is not to place orders till the stock runs down to zero, y'/x time units later; and this time point can be regarded as the origin. Then, by (2.3), $u(\theta_1)$ is largest, for a given θ_1 , when $y_1 = 0$; and for any given

sequence of interval lengths $\theta_1, \dots, \theta_n$, the sum $U = \sum_1^n u(\theta_i)$ will have its maximum at $y_1 = \dots = y_n = 0$.

Suppose the agency maximizes the sum of utilities over a certain given time T , neglecting any discounting for time. That is, it maximizes $U = \sum_1^n u(\theta_i)$ (where $\sum_1^n \theta_i = T$), and therefore maximizes the average utility over time, $U/T = \sum_1^n \theta_i v(\theta_i) / \sum \theta_i$, where $v(\theta_i) = u(\theta_i)/\theta_i$. We have seen that this requires $y_i = 0$ ($i = 1, 2, \dots$) for any given sequence of the θ_i . Furthermore U/T , being the weighted average of the $v(\theta_i)$, reaches its maximum when every $v(\theta_i)$ is equal to $\max_{\theta_i} v(\theta_i) = v(\theta^*)$, say. But, by (2.1)–(2.3) (with $y_i = 0, i \geq 0$),

$$(2.4) \quad v(\theta) = ax - [xb(x\theta) + cx\theta + K/\theta] = ax - C(\theta),$$

the expression in square brackets being the total cost per time unit, $C(\theta)$. If $v(\theta)$ has a maximum and $C(\theta)$ has a minimum at $\theta = \theta^*$, then

$$(2.5) \quad C'(\theta^*) = 0 = x^2 b'(x\theta^*) + cx - K/(\theta^*)^2.$$

The optimal interval θ^* between orders can thus be computed as depending on the cost parameters, c and K , and the purchasing price function, $b(q)$ —provided the policymaker maximizes the sum of utilities over time, U , without any discounting for futurity during which the initial stock will last.

We obtain thus (as in Figure 1b), for the case in which orders are fulfilled immediately (pipeline time = 0), a periodically repeated change from maximum stocks

$$(2.6) \quad S^* = x\theta^*$$

to minimum stocks

$$y = 0,$$

where the period, θ^* , satisfies (2.5).

2:C. We shall assume the purchase price function linear, so that $b''(q) = 0$ identically and

$$b(q) = b_0 - b_1 q,$$

say, with $b_1 \geq 0$. Then, by (2.5),

$$(2.7) \quad \theta^* = \sqrt{K/x(c - b_1 x)}.$$

It is seen from the second-order conditions for maximum $v(\theta)$ in (2.4) that the expression under the root sign is always positive (i.e., there would be no positive and finite optimum storage period if the ordering of one more unit decreased the price of the commodity by more than it

increased the cost of storing one unit). Using (2.6), the optimal maximum stock is

$$(2.8) \quad S^* = \sqrt{Kx/(c - b_1x)}.$$

Hence, as should be expected, the optimal order size, and therefore the optimal ordering interval, is larger, the larger the cost K of handling an order, the smaller the unit (marginal) storage cost c , and the larger the effect b_1 of the size of order upon the unit price.

We believe this is, in essence, the solution advanced by R. H. Wilson [1-4], formerly of the Bell Telephone Company, and also by other writers; see Alford and Banks [1]. We have proved the validity of Figure 1b (usually accepted intuitively) and have shown how to evaluate the optimal storage period.

2:D. If we now introduce a constant pipeline time, $\tau > 0$, elapsing between order and delivery, this will not affect S^* or θ^* , but the time of issuing the order will be shifted τ time units ahead. The order will be issued when the stock is reduced, not to zero, but to $x\tau$ units.

2:E. The policymaker may not have full control of the length of the time interval between any two successive orders. Transportation schedules or considerations of administrative convenience may be such as to make ordering impossible at intervals of length other than, say, $\theta^0 \neq \theta^*$. For example, θ^0 may be one business day or week, or it may be the period between two visits of a mail boat to an island depot. θ^0 is thus the "scheduled" or "smallest feasible" period between two non-zero orders. Denote the "best feasible period" by θ' , an unknown multiple integer of θ^0 . As before, θ^* is the best (but possibly a non-feasible) period. By considering the expression $C(\theta)$ defined in (2.4) as total cost, one finds easily that: (1) if $\theta^0 > \theta^*$, then $\theta' = \theta^0$; (2) if $\theta^* > \theta^0$, then $\theta' = \theta^*$, provided θ^* is an integer multiple of θ^0 ; (3) if θ^* is larger than θ^0 but is not an integer multiple of θ^0 , then define the integer $\bar{n} < \theta^*/\theta^0 < \bar{n} + 1$; the best feasible period θ' is, in this case, either $\bar{n}\theta^0$ or $(\bar{n} + 1)\theta^0$, whichever of the two results in a smaller cost when θ' is substituted in (2.4). In our paper, Arrow *et al.* [1, Section 2:E-F], this was treated in more detail, and an extension was made to the case in which ordering at nonscheduled times is not impossible but merely more costly than ordering at scheduled times.

For reasons of space we omit here the problem of aggregation, also treated in that paper [1, Section 2:G-I] and, from a more general viewpoint, in Marschak [2]. We assume that there is only one commodity, or that the characteristic parameters for all commodities are such as to yield the same optimal period θ^* for all. We also assume that there is only one giver of orders (depot) and one receiver of orders (manufacturer); on this, see Tompkins [1].

3. A STATIC MODEL WITH UNCERTAINTY

3:A. Suppose an organization wants to choose the level $S \geq 0$ that the stock of a certain commodity should have at the beginning of a given period, in order to provide for the demand (requirements) that will occur during that period. We shall choose the time unit to be equal to the length of this period and shall use the notations of Section 2. Thus $x \geq 0$ will denote the demand during the period. However, x will now be regarded as a random variable. We shall suppose that the organization knows the cumulative distribution of demand, $F(x)$. The gross utility, to the organization, of delivering ξ units of its product will be

$$(3.1) \quad a\xi + a_0 \quad (a, a_0 \text{ constant}).$$

The delivery during the period is a random variable: ξ equals x or S , whichever is smaller. Hence the expected gross utility is

$$(3.2) \quad aS[1 - F(S)] + a \int_0^S x dF(x) + a_0.$$

We shall assume that the amount to be spent in purchasing S units is

$$(3.3) \quad S(b_0 - b_1 S) + K; \quad b_0 > 0, \quad b_1 \geq 0,$$

so that, as in Section 2:C, the purchase price is either constant or linearly decreasing with the amount purchased. As before, the cost of handling an order is denoted by K but this term will not play any further role in the static model. We assume here that the whole stock S is to be purchased and that no utility is derived from satisfying demand after the period's end. Finally, the cost of carrying over our period the stock which has level S at the beginning of the period will be assumed to be

$$(3.4) \quad \text{const.} + cS.$$

Then, apart from a "depletion penalty," which we shall introduce in Section 3:B, the net expected loss (the negative of net expected utility) is

$$(3.5) \quad \text{const.} + S(c + b_0 - b_1 S) - aS[1 - F(S)] - a \int_0^S x dF(x).$$

3:B. We now define π , the depletion penalty, as follows: If $x \leq S$, there is no unsatisfied demand, and $\pi = 0$; but if $x > S$, the organization would be willing to pay an amount $\pi > 0$ to satisfy the excess, $x - S$, of demand over available stock.

We assume the penalty function as given. The organization—whether commercial or noncommercial—has a general idea of the value it would attach to the damage that would be caused by the nonavailability of an item. It knows the cost and the poorer performance of emergency substitutes. The penalty for depleted stocks may be very high: “A horse, a horse, my kingdom for a horse,” cried defeated Richard III.

3:C. Note that, in the case of a commercial enterprise, an independent penalty function, $\pi = \pi(x - S)$, need not be introduced. It can be replaced by considerations of “losing custom,” as in the following model. Let F_t be a Poisson distribution of demand for the period $(t, t + 1)$, with the following interpretation. Its mean, μ_t , is proportional (a) to the probability that a member of a large but finite reservoir of customers will want to buy during that period, and (b) to the number of customers. μ_t equals μ_{t-1} if the demand during $(t - 1, t)$ was satisfied. But, if that demand was in excess of the then available stock, μ_t is smaller than μ_{t-1} by an amount proportional to the unsatisfied demand, as some of the disappointed customers will drop out of the market. The problem is to maximize total expected utility over a sequence of periods $(0, 1), (1, 2), \dots$, if the initial distribution F_0 is given. (Such a dynamic model would be different from the one we are going to treat in Sections 4-7.)

3:D. We shall assume

$$\begin{aligned}\pi &= A + B(x - S) \text{ if } x > S, \\ \pi &= 0 \text{ otherwise,}\end{aligned}$$

where A, B are nonnegative constants, not both zero. Then π is a random variable, with expectation

$$(3.6) \quad (A - BS)[1 - F(S)] + B \int_S^\infty x dF(x).$$

Accordingly, the expected net loss, taking account of the expected penalty, is the sum of expressions (3.5) and (3.6) and equals, apart from a constant,

$$(3.7) \quad \begin{aligned} &S(c + b_0 - b_1 S) + A[1 - F(S)] - (B + a)S[1 - F(S)] \\ &\quad - (B + a) \int_0^S x dF(x) = L(S), \end{aligned}$$

say. The stock level $S = S^*$ is optimal if $L(S^*) \leq L(S)$ for every S . Suppose the distribution function $F(x)$ possesses a differentiable density function, $f(x) \equiv dF(x)/dx$. If the absolute minimum of L is not at $S = 0$, it will be at some point satisfying the relations

$$dL(S^*)/dS = 0, \quad d^2L(S^*)/dS^2 > 0,$$

which imply that

$$(3.8) \quad [c + b_0 - 2b_1S^*] - Af(S^*) - (B + a)[1 - F(S^*)] = 0,$$

$$(3.9) \quad -2b_1 - Af'(S^*) + (B + a)f(S^*) > 0.$$

3:E. In the economist's language, the first bracketed term in (3.8) is the "marginal cost" (of buying and carrying an additional unit in stock); the remaining two terms yield the "marginal expected utility."

It is seen from (3.8) that the optimal stock S^* is determined by the following "noncontrolled" parameters: (1) the demand distribution function, $F(x)$; (2) certain utility and cost parameters, $(c + b_0)$, b_1 , A , and $(B + a)$. If, in particular, $b_1 = 0$ (i.e., the economy of big-lot purchases is negligible), these parameters reduce to two: $A/(c + b_0)$ and $(B + a)/(c + b_0)$. To simplify further, for the sake of illustration, suppose also that $B = a = 0$, (that is, the penalty is either zero or A , independent of the size of the unsatisfied demand) and that utility derived from the functioning of the organization does not depend on the amounts delivered. Then (3.8), (3.9) become

$$(3.10) \quad f(S^*) = (c + b_0)/A, \quad f'(S^*) < 0.$$

A graphical solution for this case is shown in Figure 2. (Note that $f'(S^*) < 0$ but $f'(S') > 0$; S^* is the best stock level, but S' is not.)

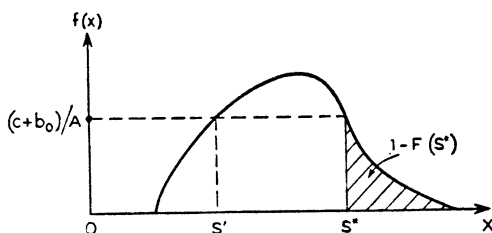


FIGURE 2

3:F. In some previous literature (Fry [1], Eisenhart [1]) the decision on inventories was related, not to utility and cost considerations, but to a preassigned probability $[1 - F(S)]$ that demand will not exceed stock. The choice of the probability level $1 - F(S)$ depends, of course, on some implicit evaluations of the damage that would be incurred if one were unable to satisfy demand. In the present paper, these evaluations are made explicit. On the other hand, since parameters such as A , B , a can be estimated only in a broad way (at least outside of a purely commercial organization, where utility equals dollar profit and where models such as that of Section 3:C can be developed), it is a

welcome support of one's judgment to check these estimates by referring to the corresponding level of probability for stock depletion. For example, if the distribution in Figure 2 were approximately normal, then to assume that penalty A is 100 times the marginal cost $c + b_0$ would be approximately equivalent to prescribing that the shaded area measuring the depletion probability should be 0.3%; to assume that $A = 10(c + b_0)$ would be approximately equivalent to making depletion probability equal to 5%, etc.

3:G. In the more general case, when $B + a > 0$ (but still $b_1 = 0$), a given optimal stock level S^* , and consequently a given probability of depletion $1 - F(S^*)$, is consistent with a continuous set of values of the pair of parameters, $A/(c + b_0) = A'$, $(B + a)/(c + b_0) = B'$, such as would satisfy the linear equation (3.8). For example, if $F(x)$ is approximately normal, then an optimal stock exceeding the average demand by two standard deviations of demand (and, consequently, a depletion probability of approximately 2.3%) will be required by any pair of values of A' , B' lying on the straight line intersecting the A' -axis at 13 and intersecting the B' -axis at 44; while an optimal stock exceeding the average demand by three standard deviations (and, consequently, a depletion probability of 0.1%) will correspond to a straight line intersecting those axes at 228 and 740, respectively. Thus a set of contour lines helps to choose an interval of optimal stock values consistent with a given region of plausible values of parameters describing penalty and gross utility.

4. A DYNAMIC MODEL WITH UNCERTAINTY: PROBLEM

4:A. The model described in Section 3 may be called a static one. We shall now present a dynamic model. In this model the commodity can be reordered at discrete instants $0, \theta_0, \dots, t\theta_0, \dots$, where θ_0 is a fixed constant. We can therefore use θ_0 as a time unit. Let x_t (t integer) be the demand over the interval $(t, t + 1)$. Assume the probability distribution of demand $F(x)$ to be independent of t . Denote by y_t the stock available at instant t , not including any replenishment that may arrive at this instant. Denote by z_t the stock at t including the replenishment. Denote by q_t the amount ordered at time t . Let the time between the ordering and the receiving of goods (pipeline time) be τ , an integer. Then

$$(4.1) \quad y_t = \max(z_{t-1} - x_{t-1}, 0) \quad (t = 1, 2, \dots),$$

$$(4.2) \quad z_{t+\tau} = y_{t+\tau} + q_t \quad (t = 0, 1, \dots).$$

In general, τ is a nonnegative random variable. We shall, however, assume $\tau = 0$ to simplify the analysis at this stage. Then (4.2) becomes

$$(4.3) \quad z_t = y_t + q_t.$$

Choose two numbers S and s , $S > s > 0$, and let them define the following rule of action:

$$(4.4) \quad \begin{aligned} &\text{If } y_t > s, q_t = 0 \text{ (and hence } z_t = y_t); \\ &\text{if } y_t \leq s, q_t = S - y_t \text{ (and hence } z_t = S). \end{aligned}$$

S and s are often called, respectively, the maximum stock and the reordering point (provided $\tau = 0$).

Figure 3 shows the sort of curve that might be obtained for stock level as a function of time if such a rule is adopted. Figure 4 shows z_t as a function of y_t .

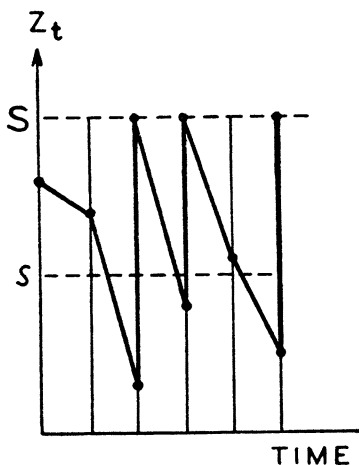


FIGURE 3

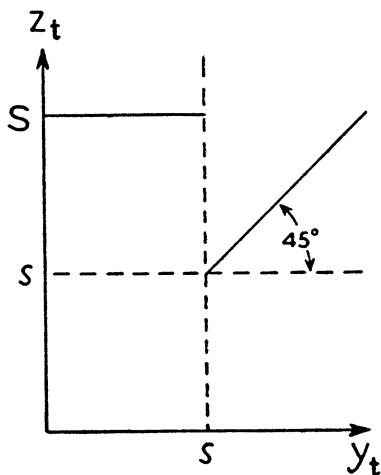


FIGURE 4

4:B. We shall assume (as we have done in Sections 2 and 3) that the cost of handling an order does not depend on the amount ordered. Let this cost be K , a constant. Let the depletion penalty be A , a constant [compare Section 3:D, with $B = 0$]. Let the marginal cost of carrying stock during a unit of time be c , as in (3.4). Assume the purchasing price per unit of commodity to be independent of the amount bought and equal to the marginal utility of one unit (i.e., in the notation of Section 3:A, $b_1 = 0$, $b_0 = a$). That is, the utility of operations of the agency, in excess of the expenses paid for these operations, is assumed constant, apart from the cost of storage and of handling orders. In the notation of Section 3:A, this constant is a_0 , while K and c denote, respectively, the cost of handling an order (of any size) and the marginal cost of storage. Our assumption is an admissible approximation in the

case of some nonprofit agencies. It would certainly be both more general and more realistic to make the marginal utility of an operation differ from its purchasing price, as was the case in our static model. But this will require further mathematical work (see Section 7:A).

4:C. If y_0 is given and values S and s are chosen, the subsequent values y_t form a random process which is "Markovian"; see Feller [2, Chapter XV]. That is, the probability distribution of y_{t+1} , given the value of y_t , is independent of y_{t-1}, \dots, y_0 . During the period $(t, t+1)$ a certain loss will be incurred whose conditional expectation, for a fixed value of y_t , we denote by $l(y_t)$. Under the simplifying assumptions of Section 4:B,

$$(4.5) \quad l(y_t) = \begin{cases} cy_t + A[1 - F(y_t)] & \text{for } y_t > s, \\ cS + A[1 - F(S)] + K & \text{for } y_t \leq s. \end{cases}$$

Thus the function $l(y_t)$ involves S and s as parameters and is constant for $y_t \leq s$. Note that

$$(4.6) \quad l(0) = l(S) + K.$$

The unconditional expectation of the loss during $(t, t+1)$, that is, the expectation of $l(y_t)$, with y_t as a random variable, will be denoted by

$$(4.7) \quad l_t = l_t(y_0).$$

We shall write $l_t(y_0)$ rather than l_t only when we need to emphasize the dependence of l_t on the initial stock level. Clearly $l_0(y) = l(y)$ for every value y of y_0 .

Figure 5 shows a possible type of graph for $l(y_t)$.

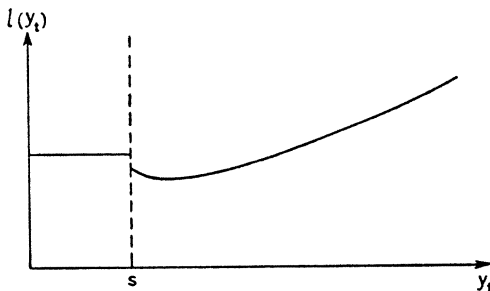


FIGURE 5

4:D. We now introduce the concepts of a discount factor, α , and of the "present value" of a loss. If the value of y_{t_0} is given, the present value at time t_0 of the expected loss incurred in the interval $(t_0 + t,$

$t_0 + t + 1$) is $\alpha^t \mathcal{E}[l(y_{t_0+t})] = \alpha^t l_t(y_{t_0})$. When maximizing expected utility, the policymaker takes into account the present values of losses, not their values at the time when they are incurred. In commercial practice, α is equal to $1/(1 + \rho)$, where ρ is an appropriate market rate of interest. In nonprofit practice, α would have to be evaluated separately (see also footnote 3). Later, however, it will be shown (see Section 5:B) that, under certain conditions, the optimal values of the parameters S , s can be found for α essentially equal to 1.

If we now define the function

$$L(y) = l_0(y) + \alpha l_1(y) + \alpha^2 l_2(y) + \cdots,$$

we see from definition (4.7) that $L(y_t)$ is the present value at time t of the total expected loss incurred during the period $(t, t + 1)$ and all subsequent periods when y_t is given. By definition, $L(y)$ involves the parameters S and s ; and the policymaker fixes these parameters so as to minimize $L(y_0)$.

4:E.⁵ Now suppose y_0 is given. For a fixed value of y_1 , the present value of the total expected loss over all periods is

$$(4.8) \quad l(y_0) + \alpha l(y_1) + \alpha^2 \mathcal{E}_{y_1}[l(y_2)] + \alpha^3 \mathcal{E}_{y_1}[l(y_3)] + \cdots,$$

where we have used $\mathcal{E}_{y_1}[l(y_r)]$ to denote the conditional expectation of $l(y_r)$, given the fixed value y_1 . Now

$$\mathcal{E}_{y_1}[l(y_r)] = l_{r-1}(y_1) \quad (r = 1, 2, \cdots)$$

because of the fact that, if y_1 is fixed, the subsequent value y_r ($r = 1, 2, \cdots$) is connected with y_1 in the same manner that y_{r-1} is connected with y_0 if y_1 is not specified. Therefore, expression (4.8) is equal to

$$(4.9) \quad \begin{aligned} & l(y_0) + \alpha l_0(y_1) + \alpha^2 l_1(y_1) + \alpha^3 l_2(y_1) + \cdots \\ & = l(y_0) + \alpha [l_0(y_1) + \alpha l_1(y_1) + \alpha^2 l_2(y_1) + \cdots] = l(y_0) + \alpha L(y_1). \end{aligned}$$

The total expected loss over all periods from the beginning, which by

⁵ The following intuitive summary of the argument of Section 4:E has been kindly suggested by a referee: During the first period, the expected loss is $l(y)$ and demand is x with probability $dF(x)$; the stock remaining will be $S - x$ or $y - x$ as the case may be, with a forward-looking expected loss of, respectively, $L(S - x)$ or $L(y - x)$. In the former case, the expected future loss at the end of the first period is $\int_0^S L(S - x) dF(x) + L(0)[1 - F(S)]$, all of which needs only to be multiplied by α to be discounted back to the beginning of the first period where it can be added to the original $l(y)$. This yields equation (4.11). The companion equation (4.12) is obtained similarly for the case when the stock remaining at the end of the first period is $y - x$.

definition is $L(y_0)$, is the expectation of the expression in (4.9), with y_1 regarded as a random variable. Hence

$$(4.10) \quad L(y_0) = l(y_0) + \alpha \mathcal{E}[L(y_1)].$$

To express the expected value of $L(y_1)$ as a function of y_0 we note that, if $y_0 \leq s$, then $z_0 = S$ and $y_1 = \max(S - x_0, 0)$; while, if $y_0 > s$, then $z_0 = y_0$ and $y_1 = \max(y_0 - x_0, 0)$. Thus

$$(4.10') \quad \begin{aligned} \mathcal{E}[L(y_1)] &= \int_{0^-}^S L(S - x) dF(x) + L(0)[1 - F(S)] \quad \text{for } y_0 \leq s, \\ \mathcal{E}[L(y_1)] &= \int_{0^-}^{y_0} L(y_0 - x) dF(x) + L(0)[1 - F(y_0)] \quad \text{for } y_0 > s. \end{aligned}$$

[Notice that from the way we have defined the rule of action, $L(y)$ is constant for $0 \leq y \leq s$ so that $L(0)$ is unambiguously defined.] Putting $y_0 = y$ we obtain from (4.10) and (4.10') the equations

$$(4.11) \quad L(y) = l(y) + \alpha \int_{0^-}^S L(S - x) dF(x) + \alpha L(0)[1 - F(S)] \quad \text{if } y \leq s,$$

$$(4.12) \quad L(y) = l(y) + \alpha \int_{0^-}^y L(y - x) dF(x) + \alpha L(0)[1 - F(y)] \quad \text{if } y > s.$$

Our problem is to find the function $L(y)$ that satisfies (4.11), (4.12) and to minimize $L(y_0)$ with respect to S, s .

5. A DYNAMIC MODEL: METHOD OF SOLUTION

5:A. In treating equations (4.11) and (4.12) we drop for the time being the assumption that $F(x)$ has a density function and assume only that the random variable x cannot take negative values. In order to take care of the possibility that $F(x)$ has a discontinuity at $x = 0$ (i.e., a positive probability that $x = 0$), we adopt the convention that Stieltjes integrals of the form $\int_0^{(\cdot)} (\cdot) dF(x)$ will

be understood to have 0^- as the lower limit. We continue to assume that $l(y)$ is given by (4.5), but it is clear that a similar treatment would hold for any non-negative function $l(y)$ that is constant for $0 \leq y \leq s$ and satisfies certain obvious regularity conditions.

Since $l(y)$, and therefore also $L(y)$, is independent of y for $0 \leq y \leq s$, equation (4.11) tells us simply that

$$(5.1) \quad L(0) = l(0) + \alpha \int_0^S L(S - x) dF(x) + \alpha L(0)[1 - F(S)],$$

while putting $y = S$ in (4.12) gives

$$(5.2) \quad L(S) = l(S) + \alpha \int_0^S L(S - x) dF(x) + \alpha L(0)[1 - F(S)].$$

Subtracting (5.2) from (5.1) we obtain, using (4.6),

$$(5.3) \quad L(0) - L(S) = K,$$

an expression which is in fact obvious since if the initial stock is 0 we immediately order an amount S at a cost K for ordering. We shall solve equation (4.12) for the function $L(y)$, considering $L(0)$ as an unknown parameter, and then use (5.3) to determine $L(0)$.

On the right side of (4.12) we make the substitution

$$(5.4) \quad \int_0^y L(y-x) dF(x) = \int_0^{y-s} L(y-x) dF(x) + L(0) \int_{y-s}^y dF(x);$$

the last term follows from the fact that $L(y-x) = L(0)$ when $0 \leq y-x \leq s$. Now make the change of variables,

$$(5.5) \quad \begin{aligned} y-s &= \eta, \\ L(y) &= L(\eta+s) = \lambda(\eta). \end{aligned}$$

Putting (5.4) and (5.5) in (4.12) gives

$$(5.6) \quad \lambda(\eta) = l(\eta+s) + \alpha L(0)[1-F(\eta)] + \alpha \int_0^\eta \lambda(\eta-x) dF(x), \quad \eta > 0.$$

Equation (5.6) is in the standard form of the integral equation of renewal theory; see, for example, Feller's paper [1]. The solution of (5.6) can be expressed as follows. Define distribution functions $F_n(x)$ ($n = 1, 2, \dots$) [the convolutions of $F(x)$] by

$$(5.7) \quad \begin{aligned} F_1(x) &= F(x), \\ F_{n+1}(x) &= \int_0^x F_n(x-u) dF(u). \end{aligned}$$

Define the function $H_\alpha(x)$:

$$(5.8) \quad H_\alpha(x) = \sum_{n=1}^{\infty} \alpha^n F_n(x), \quad 0 \leq \alpha \leq 1.$$

It is obvious that the series converges if $0 \leq \alpha < 1$, and in fact it can be seen from Feller's article [1] that it converges if $\alpha = 1$, a fact we shall need in the sequel. Putting

$$(5.9) \quad R(\eta) = l(\eta+s) + \alpha L(0)[1-F(\eta)],$$

we can write the solution of (5.6) as

$$(5.10) \quad \begin{aligned} \lambda(\eta) &= R(\eta) + \int_0^\eta R(\eta-x) dH_\alpha(x) \\ &= R(\eta) + \sum_{n=1}^{\infty} \alpha^n \int_0^\eta R(\eta-x) dF_n(x). \end{aligned}$$

This is the only solution which is bounded on every finite interval. In terms of L and l , (5.10) gives

$$(5.11) \quad \begin{aligned} L(y) &= l(y) + \alpha L(0)[1 - F(y - s)] \\ &+ \int_0^{y-s} \{l(y-x) + \alpha L(0)[1 - F(y-x-s)]\} dH_\alpha(x), \quad y > s. \end{aligned}$$

From (5.3) and (5.11) we have

$$(5.12) \quad \begin{aligned} L(0) - K &= l(S) + \int_0^{S-s} l(S-x) dH_\alpha(x) \\ &+ \alpha L(0) \left\{ 1 - F(S-s) + \int_0^{S-s} [1 - F(S-s-x)] dH_\alpha(x) \right\}. \end{aligned}$$

In (5.12) we have a linear equation which we can solve for the unknown quantity $L(0)$ which has, as we shall show, a nonvanishing coefficient in (5.12) as long as $\alpha < 1$. This gives us the value of $L(y)$ for $y \leq s$, and we can obtain $L(y)$ for $y > s$ from (5.11) since every term on the right side of that equation is now known.

The coefficient of $L(0)$ in (5.12) is

$$(5.13) \quad \begin{aligned} &1 - \alpha \left\{ 1 - F(S-s) + \int_0^{S-s} [1 - F(S-s-x)] dH_\alpha(x) \right\} \\ &= 1 - \alpha \left\{ 1 - F(S-s) + H_\alpha(S-s) - \int_0^{S-s} F(S-s-x) dH_\alpha(x) \right\} \\ &= 1 - \alpha \left\{ 1 - F(S-s) + \sum_{n=1}^{\infty} \alpha^n F_n(S-s) - \sum_{n=1}^{\infty} \alpha^n F_{n+1}(S-s) \right\} \\ &= (1 - \alpha)[1 + H_\alpha(S-s)]. \end{aligned}$$

Using (5.13) we obtain

$$(5.14) \quad L(0) = \frac{K + l(S) + \int_0^{S-s} l(S-x) dH_\alpha(x)}{(1 - \alpha)[1 + H_\alpha(S-s)]}.$$

Knowing $L(y)$ from (5.11) and (5.14), the next step is to find, for a given initial stock y_0 , the values of s and S which minimize $L(y_0)$. We shall consider only the minimization of $L(0)$, although the procedure could be worked out to minimize $L(y_0)$ for any initial stock y_0 . The procedure of minimizing $L(0)$ is not quite so special as it may appear. Suppose that for a given y_0 the values of s and S which minimize $L(y_0)$ are denoted by $s^*(y_0)$ and $S^*(y_0)$. If $s^*(0) > 0$ and if $s^*(y_0)$ and $S^*(y_0)$ are uniquely determined continuous functions of y_0 (a point which we have not investigated mathematically), then $s^*(y_0) = s^*(0)$, $S^*(y_0) = S^*(0)$ for sufficiently small y_0 . To see this we write

$$L(y) = L(y; s, S)$$

to indicate the dependence of L on s and S . Let a be a number such that $0 < a < s^*(0)$. Suppose y_0 , $0 < y_0 < a$, is sufficiently small so that $s^*(y_0) > a$. Then

$$L(y_0; s^*(y_0), S^*(y_0)) = \min_{s > a, S \geq s} L(y_0; s, S) = K + \min_{s > a, S \geq s} L(S; s, S),$$

which is minimized independently of y_0 ; Q.E.D.

In Section 5:B an optimization criterion will be given which is independent of the initial level y_0 .

We now reintroduce the assumption that $F(x)$ has a probability density which is continuously differentiable,

$$F(x) = \int_0^x f(t) dt.$$

We recall from (4.5) that $l(y)$, for $y > s$, is given by

$$l(y) = A[1 - F(y)] + cy.$$

Consider the minimization of (5.14) with respect to s and S . First we consider the case where $S - s$ is fixed. The denominator of (5.14) involves S and s only as a function of $S - s$. We therefore have to minimize the numerator of (5.14) with respect to S , subject to the constraint that S is at least as great as the fixed value of $S - s$. If the minimum value does not occur for $S = S - s$ (i.e., $s = 0$), it occurs at a value of S for which the conditions

$$(5.15) \quad c - Af(S) + \int_0^{S-s} [c - Af(S-x)] dH_a(x) = 0,$$

$$(5.16) \quad -Af'(S) - \int_0^{S-s} Af'(S-x) dH_a(x) > 0,$$

hold. It should be noted that K does not enter into (5.15) and (5.16).

If we drop the requirement that $S - s$ be fixed, then s^* and S^* , provided they satisfy the condition $0 < s^* < S^*$, occur at a point where equation (5.15) holds, together with the equation obtained by setting the derivative of (5.14) with respect to $S - s$ equal to 0 and taking the appropriate second-order conditions into account. We also need here the assumption that $H_a(x)$ is the integral of a function $h_a(x)$,

$$H_a(x) = \int_0^x h_a(t) dt.$$

Then differentiation of (5.14) with respect to $S - s$ gives, setting the derivative equal to 0 and integrating by parts,

$$(5.17) \quad A[F(S) - F(s)] = c(S - s) + K + \int_0^{S-s} [c - Af(S-x)] H_a(x) dx.$$

Presumably the minimization of (5.14) would be accomplished in practice by numerical methods.

5:B. So far we have considered α as an arbitrary parameter. It is clear that if we let $\alpha \rightarrow 1$, keeping s and S fixed, the quantity $L(0)$ becomes infinite. However, as we shall see, the quantity $(1 - \alpha)L(0)$ approaches a finite limiting value whose significance can be explained as follows. Suppose that levels s and S have been fixed and that y_0 is given. We have mentioned that the quantities y_t then form a Markovian random process. Moreover, under assumptions on $F(x)$ [which are not of practical importance] the probability distribution of y_t , as $t \rightarrow \infty$, approaches a fixed limiting distribution which is independent of y_0 . When $F(x)$ is a step function,⁶ we are dealing with a Markov chain with a denumerable number of states. If $F(x)$ is not a step function this theory can still be applied indirectly. The "age" of the stock at any given time (i.e., the length of time since the last order was placed) has a distribution of the discrete type which approaches a limit, and from this it follows that y_t has a limiting distribution. This implies that l_t , the expected loss in the interval $(t, t + 1)$ approaches a limiting value l_∞ which is independent of y_0 . (The losses during successive time intervals form a sequence of bounded random variables.) As we shall see, we can find the value of l_∞ . Then if we do not want to use a discount factor α , one way to proceed is to pick s and S so as to minimize l_∞ . This is almost equivalent to minimizing the total expected loss over a long finite time interval.

Another way to look at the situation is as follows. The limiting distribution of y_t for large t is a "stationary distribution"; i.e., if y_0 has this distribution, instead of being fixed, then y_t has the same distribution for every t . The expected loss during $(t, t + 1)$, if y_t has this distribution, is just l_∞ .⁷

Since

$$L(0) = l_0(0) + \alpha l_1(0) + \alpha^2 l_2(0) + \dots$$

and $l_t(0) \rightarrow l_\infty$ as $t \rightarrow \infty$, we have

$$(5.18) \quad L(0)(1 - \alpha) = l_0(0) + \alpha[l_1(0) - l_0(0)] + \alpha^2[l_2(0) - l_1(0)] + \dots$$

The series

$$l_0(0) + [l_1(0) - l_0(0)] + [l_2(0) - l_1(0)] + \dots$$

converges to the value l_∞ and therefore, by a standard result of analysis, we have, from (5.18),

$$\lim_{\alpha \rightarrow 1} L(0)(1 - \alpha) = l_\infty.$$

In order to determine l_∞ , we can then multiply the right side of (5.14) by $(1 - \alpha)$ and let $\alpha \rightarrow 1$, obtaining

$$(5.19) \quad l_\infty = \frac{K + l(S) + \int_0^{S-s} l(S-x) dH(x)}{1 + H(S-s)},$$

where $H(x)$ is defined by

$$H(x) = \lim_{\alpha \rightarrow 1} H_\alpha(x) = \sum_{n=1}^{\infty} F_n(x).$$

⁶ See Feller [2, Chapter XV], for the case when $F(x)$ is a step function.

⁷ This stationary distribution can be found explicitly and, as pointed out by H. Simon, gives an alternative means of finding l_∞ .

(It is not hard to see that the step

$$\lim_{\alpha \rightarrow 1} \int_0^{S-s} l(S-x) dH_\alpha(x) = \int_0^{S-s} l(S-x) dH(x)$$

is justified.)

We can then minimize the function in (5.19) with respect to s and S . It should be noted that l_∞ is, of course, independent of the initial stock y_0 .

6. A DYNAMIC MODEL: EXAMPLES

We consider now some examples for a particular function $F(x)$. It is advantageous to use a function whose convolutions can be written explicitly. From this point of view, functions of the form

$$(6.1) \quad F(x) = \frac{\beta^k}{(k-1)!} \int_0^x u^{k-1} e^{-\beta u} du, \quad k > 0, \quad \beta > 0,$$

are convenient [$(k-1)!$ is $\Gamma(k)$ if k is not an integer] since by proper choice of β and k we can give any desired values to the mean and variance,

$$\bar{x} = k/\beta, \quad \bar{x}^2 - (\bar{x})^2 = k/\beta^2,$$

and since $F_n(x)$ is then given by

$$F_n(x) = \frac{\beta^{nk}}{(nk-1)!} \int_0^x u^{nk-1} e^{-\beta u} du.$$

The function $H_\alpha(x)$ is then given by

$$(6.2) \quad H_\alpha(x) = \int_0^x e^{-\beta u} \left\{ \sum_{n=1}^{\infty} \frac{\beta^{nk} \alpha^n u^{nk-1}}{(nk-1)!} \right\} du.$$

If k is an integer, the summation in (6.2) can be performed explicitly, giving

$$(6.3) \quad H_\alpha(x) = \frac{\beta \alpha^{1/k}}{k} \int_0^x e^{-\beta u} \left(\sum_{j=1}^k \omega_j e^{\omega_j \alpha^{1/k} \beta u} \right) du,$$

where $\omega_1, \dots, \omega_k$ are the k th roots of unity. For example, if $k = 2$, we have $\omega_1 = -1, \omega_2 = 1$, so that

$$H_\alpha(x) = \frac{\beta \sqrt{\alpha}}{2} \int_0^x e^{-\beta u} (e^{\beta \sqrt{\alpha} u} - e^{-\beta \sqrt{\alpha} u}) du.$$

It is instructive to find the value of l_∞ for the simple case $f(x) = e^{-x}$. In this case, $\beta = k = 1$; $F = 1 - e^{-x}$; and, from (6.3),

$$H(x) = \int_0^x e^{-u} (e^u) du = x$$

and we have

$$\begin{aligned}
 l_{\infty} &= \frac{K + l(S) + \int_0^{S-s} l(S-x) dx}{1 + S - s} \\
 &= \frac{K + cS + Ae^{-s} + \int_0^{S-s} [c(S-x) + Ae^{-s+x}] dx}{1 + S - s} \\
 &= \frac{K + cS + Ae^{-s} + cS(S-s) - \frac{c}{2}(S-s)^2 + Ae^{-s}(e^{S-s} - 1)}{1 + S - s} \\
 &= \frac{K + cS + Ae^{-s} + \frac{c}{2}(S^2 - s^2)}{1 + S - s}.
 \end{aligned}$$

Letting $S - s = \Delta$, we see that this expression, for a fixed value of Δ , has its minimum (unless it occurs when $s = 0$) when

$$S = \log_e(A/c) - \log_e(1 + \Delta) + \Delta.$$

7. FURTHER PROBLEMS AND GENERALIZATIONS

To make the dynamic model more realistic certain generalizations are necessary. We shall register them in the present section as a program for further work.

7:A. Of the several cost and utility parameters used in the certainty model of Section 2 and in the static uncertainty model of Section 3, we have retained in the dynamic uncertainty model only three: c , the marginal cost of storage; K , the constant cost of handling an order; and A , the constant part of the depletion penalty. We have thus dropped the parameters a , b_0 , b_1 , and B . The meaning of the first three of these was discussed in Section 4:B. It can be presumed from equation (3.8) of the more developed static model that if we similarly developed the dynamic model, c could be easily replaced by $(c + b_0)$ but that $(B + a)$ would form an additional parameter altogether excluded from our simple dynamic model. Difficulties of another kind will occur when $b_1 > 0$, i.e., when there are economies of big-lot buying, which are due, not to the advantage of handling one order instead of many, but to the cheapness of transporting (and producing) large quantities. This will obviously modify the rule of action (4.4), as the loss that we intend to minimize will depend on $(S - y_i)$, the size of the replenishment order.

7:B. We have assumed the distribution $F(x)$ of demand per unit

period to be known, presumably having been estimated from previous samples. Actual estimations of this distribution were carried out by Fry and Wilson for the Bell Telephone Company, and by Kruskal and Wolf [1, 2] with the material of the medical branch of the U. S. Navy.

Instead of estimating the distribution $F(x)$, once and for all, and fixing constant values for S and s , one may vary S and s as new observations on demand are obtained. The problem is one of expressing the best values of S and s for the time t as functions of the sequence of observations available up to that time—say $S_t(x_1, \dots, x_{t-1})$ and $s_t(x_1, \dots, x_{t-1})$. More generally, one has to find a sequence of functions $q_t(x_1, \dots, x_{t-1})$ giving the best amounts to be ordered, a sequence not necessarily restricted by conditions (4.4). We do not propose to attack this problem here.⁸

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⁸ See Wald [1]; Arrow, Blackwell, and Girshick [1]. If judgment were available on the a priori probabilities of alternative distributions $F(x)$, the expected loss could be averaged over all possible $F(x)$ and minimized with respect to the unknown functions q_t or S_t , s_t . In the absence of such judgment, one might think of minimizing the maximum expected loss. In an appropriately modified form suggested by L. Savage [1], the latter principle was applied to a problem in investment decisions by Marschak [1], and its applicability discussed by Modigliani and Tobin. See also Niehans [1]. Other criteria have been proposed by Hurwicz [1].

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