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New structural properties of (s, S) policies for inventory models with lost sales

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ABSTRACT

We revisit the classical inventory model for which (s, S) policies are optimal. We consider the finite and infinite horizon versions of the lost sales problem. New structural properties are developed for the optimal policy and cost function. These properties are then used to develop computational schemes for the lost sales problem with Erlang demands.

1. Introduction

We consider finite and infinite horizon cases of the lost sales variant of the classical stochastic inventory problem: At the start of each period the inventory is reviewed to determine if it should be replenished. If an order is placed, the cost consists of a fixed ordering cost K, and a unit variable cost c. In each period demand that cannot be filled is lost at a unit cost l; if demand is less than the available inventory, the leftover stock is carried to the next period at a unit holding cost h. The objective is to find the inventory policy that minimizes the expected discounted cost. It is well known that this cost is minimized by following an optimal (s, S) policy.

Under an (s, S) policy, the inventory position of a product is reviewed regularly, and, if it is found to be below a threshold s an order is placed to bring the inventory position up to the level S. The choice of s and S is made to minimize the expected cost while taking into account the fixed ordering cost, inventory holding cost, and the cost associated with stockouts. The fundamental work on (s, S) policies is due to Arrow et al. [2] and Karlin [10]. However, the optimality of such a policy, under general conditions, for the finite horizon case with backorders was first established by Scarf [12] using the novel properties of K-convexity. The application of Scarf's seminal result to the infinite horizon case with stationary demand and cost parameters is due to Iglehart [9]. Subsequently, Veinott [14] and Porteus [11] provided alternative proofs under different conditions. Recently, Beyer and Sethi [6] added technical details missing in [9]. Analogous results on the structure of the optimal policy for the lost sales case are due to Shreve [13]; many details are also provided by Bertsekas [4] and Bensoussan et al. [3].

When it is appropriate to model demand as being discrete, effective computation schemes are available for the infinite horizon case with backorders. Of special interest are the efficient algorithms due to Federgruen and Zipkin [7] and Zheng and Federgruen [16]; the latter essentially reduces the problem to that of evaluating a single policy. Additional insights into such algorithms are offered by Feng and Xiao [8]. While these papers develop algorithms to find the optimal policy that minimizes average cost per unit time for the backorder case when demand is discrete, we compute the optimal policy for the complementary case that minimizes the expected discounted cost for the lost sales case when demand is continuous.

We begin our development in Section 2 by formulating the finite horizon version of the model and developing some new structural properties that are presented in Theorem 1 for the finite horizon case and in Theorem 2 for the infinite horizon case. While these properties hold without any distributional assumption on demand, additional analysis, as presented in Theorem 3 of Section 3, reveals that when demand is modeled as having an Erlang distribution, the cost function has appealing special structures. These structures lead to a computational scheme for finding the optimal policy for the infinite horizon problem with lost sales.

2. New structural properties

In the finite horizon model there are T periods with the last period labeled as 1 and the first period labeled as T. The demand in each period is described by a continuous random variable that

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is independently and identically distributed. In developing the model, we will use the following notation:

Cost and model parameters
K = fixed setup cost, c = unit variable cost, h = unit inventory holding cost
l = unit lost sales cost ($l > c$), b = unit backorder cost
α = discount factor ($0 < \alpha < 1$), T = time horizon
Demand information
ξ_t = random observation of demand in period t, $t = 1, 2, \ldots, T$
$f(.)$ = probability density function (PDF) of demand in each period (we assume $f(x) > 0$ for $x > 0$)
$F(.)$ = cumulative distribution function (CDF) of demand in each period
Decision variables
s_t = optimal reorder level in period t
S_t = optimal order-up-to level in period t
Cost functions
$L(.)$ = one period inventory holding and shortage penalty cost function
$V_t(x)$ = total minimal expected cost from period t onwards ($t-1, \ldots, 2, 1$), given that the on-hand inventory at the beginning of period t is x
$G_t(y)$ = total expected cost from period t onwards after inventory level is increased to y
$G(y\|s, S)$ = total expected discounted cost for an infinite horizon problem with stationary reorder point s and order-up-to level S after inventory level is increased to y
Other useful functions
$\delta(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \end{cases}$, the indicator function for ordering decisions
$x^+ = \max\{x, 0\}$

Since the planning horizon consists of T periods, it is convenient to include a mechanism to "settle accounts" at the end of the planning horizon. Since all excess demand is lost, the state variable x at time 0 must be non-negative. Specifically, we assume that leftover inventory is sold at unit value w ($w \le c$); if w is positive the inventory is salvaged, and, if w is negative, disposal costs are incurred. Hence, we can conclude that:

$$V_0(x) = -wx \quad (x \ge 0). \tag{1}$$

As in the classical treatment of the (s, S) inventory model, we assume instantaneous delivery of orders, which makes it convenient to define the one period cost, which is time independent because all parameters are stationary. Given that an order is placed and received to bring the inventory level up to y, it is given by:

$$L(y) = h\int_0^y (y-\xi)f(\xi)\mathrm{d}\xi + l\int_y^{+\infty} (\xi - y)f(\xi)\mathrm{d}\xi. \tag{2}$$

And, given that an optimal policy is followed for the remaining periods, the expected discounted cost with y units of inventory after ordering in period t is given by:

$$G_t(y) = cy + L(y) + \alpha E[V_{t-1}(y-\xi_t)^+], \quad t = 1, 2, \ldots, T. \tag{3}$$

Hence, $G_t(y)$ refers to the total expected cost from period t onwards, when the inventory level is increased to y. Thus,

$$V_t(x) = -cx + \min\{K\delta(y-x) + G_t(y) : y \ge x\}, \tag{4}$$

represents the minimal total discounted cost from period t to the last period when the initial on-hand inventory equals x. From Shreve [13], we know that both $G_t(.)$ and $V_t(.)$ are K-convex; hence there exist (s_t, S_t) which minimize $V_t(x)$: if the initial on-hand inventory $x \le s_t$, the retailer will order $S_t - x$; if $x > s_t$, no order is placed.

In order to find an upper bound for reorder levels, we define

$$G_0(y) = cy + L(y) - \alpha c\int_0^y (y-\xi)f(\xi)\mathrm{d}\xi, \tag{5}$$

which can be interpreted as a single-period newsvendor model, where all leftovers are fully refunded with purchase cost c. Since $G_0(y)$ is strictly convex for $y > 0$, there exists a unique number S_0 (>0) at which $G_0(.)$ attains its local minimum. As shown in the following theorem, S_0 provides an upper bound for $\tilde{s}_t = \max\{s_t, s_{t-1}, \ldots, s_1\}$, the maximum of reorder levels:

Theorem 1. *In a T-period problem, for period t, $t = 1, 2, \ldots, T$, we have*
(1.1) $s_t \ge s_1, \forall t \ge 2$;
(1.2) $\tilde{s}_t < S_0$;
(1.3) $G_t'(y) < 0, \forall y \in (0, \tilde{s}_t]$;
(1.4) $S_t > \tilde{s}_{t-1}$;
(1.5) $G_t'(y) \ge c - l, \forall y > 0$.

Proof. Before proving the theorem, we derive a few relationships that will facilitate the proofs. Note that

$$\begin{aligned} G_1(y) &= cy + L(y) + \alpha E\left[V_0\left((y-\xi)^+\right)\right] \\ &= cy + L(y) - \alpha w\int_0^y (y-\xi)f(\xi)\mathrm{d}\xi, \end{aligned} \tag{6}$$

which is the cost function of a single-period newsvendor model, hence is convex; and

$$\begin{aligned} G_1'(y) &= c - l + (h + l - \alpha w)F(y) = 0 \Rightarrow S_1 \\ &= F^{-1}\left(\frac{l-c}{h+l-\alpha w}\right). \end{aligned} \tag{7}$$

Also, for $G_0(y)$ defined in (5), we have

$$\begin{aligned} G_0'(y) &= c - l + (h + l - \alpha c)F(y) = 0 \Rightarrow S_0 \\ &= F^{-1}\left(\frac{l-c}{h+l-\alpha c}\right). \end{aligned} \tag{8}$$

Comparing (7) and (8), we have

$$\begin{aligned} &G_0'(y) - G_1'(y) = \alpha(w-c)F(y) \le 0, \\ &\text{i.e., } G_0'(y) \le G_1'(y) \text{ since } w \le c. \end{aligned} \tag{9}$$

Moreover, since $F^{-1}(\cdot)$ is monotone increasing, we have $S_1 \le S_0$, and $s_1 < S_1 \le S_0$.

Second, from the properties of K-convexity, we know

$$V_{m+1}(x) = \begin{cases} -cx + G_{m+1}(S_{m+1}) + K & \text{if } 0 \le x \le s_{m+1}, \\ -cx + G_{m+1}(x) & \text{if } x > s_{m+1}; \end{cases}$$
$$\text{for } m = 0, 1, 2, \ldots, T-1.$$

Moreover, if $0 \le y \le s_m$,

$$\begin{aligned} G_{m+1}(y) &= cy + L(y) + \alpha E\left[V_m\left((y-\xi)^+\right)\right] \\ &= cy + L(y) + \alpha\left[K + G_m(S_m) - c\int_0^y (y-\xi)f(\xi)\mathrm{d}\xi\right] \\ &= G_0(y) + \alpha[K + G_m(S_m)]. \end{aligned} \tag{10}$$

((10) follows from the fact that: if $y \le s_m$, then $(y-\xi)^+ \le s_m$; consequently, an order of $S_m - (y-\xi)^+$ would be placed in period m.)

And, if $y \ge s_m$, then

$$\begin{aligned} G_{m+1}(y) &= cy + L(y) + \alpha E\left[V_m\left((y-\xi)^+\right)\right] \\ &= cy + L(y) + \alpha\int_0^{y-s_m} [G_m(y-\xi) - c(y-\xi)]f(\xi)\mathrm{d}\xi \\ &\quad + \alpha\left[(K + G_m(S_m))\int_{y-s_m}^y f(\xi)\mathrm{d}\xi - c\int_{y-s_m}^y (y-\xi)f(\xi)\mathrm{d}\xi\right] \\ &\quad + \alpha\left[(K + G_m(S_m))\int_y^{+\infty} f(\xi)\mathrm{d}\xi\right] \\ &= G_0(y) + \alpha\int_0^{y-s_m} G_m(y-\xi)f(\xi)\mathrm{d}\xi \\ &\quad + \alpha\,(K + G_m(S_m))\,(1 - F(y-s_m)). \end{aligned} \tag{11}$$

((11) is derived from three contingencies: (1) if $s_m \le (y-\xi)$, then no order will be placed in period m; (2) if $0 < (y-\xi) < s_m$, an order of $S_m - (y-\xi)$ will be placed in period m; and (3) if $(y-\xi) \le 0$, or $(y-\xi)^+ = 0$, an order of S_m will be placed in period m).

Subsequently, taking derivatives of (10) and (11), we get

$$G'_{m+1}(y) = G'_0(y), \quad \text{if } 0 \le y \le s_m, \tag{12}$$

$$G'_{m+1}(y) = G'_0(y) + \alpha \int_0^{y-s_m} G'_m(y-\xi) f(\xi)\mathrm{d}\xi, \quad \text{if } y \ge s_m. \tag{13}$$

Now we will prove Theorem 1.

(1.1) First, we want to show that $G'_t(y) \le G'_1(y)$ for $y \le S_1$ (hence $S_t \ge S_1$) for all $t = 2, 3, \dots, T$. For $t = 2$, if $y > s_1$, from (13) we know that $G'_2(y) = G'_0(y) + \alpha \int_0^{y-s_1} G'_1(y-\xi) f(\xi)\mathrm{d}\xi$. Since $G'_0(y) \le G'_1(y)$ from (9) and $G_1(y)$ is strictly convex, for $s_1 < y \le S_1$ we have $G'_2(y) < G'_0(y) \le G'_1(y) \le 0$, which implies that $S_2 > S_1$.

Now, to use induction we assume that $G'_t(y) \le G'_1(y)$ for $y \le S_1$ (hence $S_t \ge S_1$), for any $t = 3, 4, \dots, T-1$. Then in period $t+1$, we compare s_t with S_1: If $s_t \le S_1$, from (12) and (9) we know that $G'_{t+1}(y) = G'_0(y) \le G'_1(y) \le 0$ for $y \le s_t$; and from (13) and (9) we know that $G'_{t+1}(y) = G'_0(y) + \alpha \int_0^{y-s_t} G'_t(y-\xi) f(\xi)\mathrm{d}\xi \le G'_0(y) + \alpha \int_0^{y-s_t} G'_1(y-\xi) f(\xi)\mathrm{d}\xi \le G'_0(y) \le G'_1(y) \le 0$ for $s_t \le y \le S_1$, where the first inequality is due to the induction hypothesis. And, if $s_t > S_1$, for $y \le S_1$, from (12) and (9) we know $G'_{t+1}(y) = G'_0(y) \le G'_1(y) \le 0$. Thus, in all cases we have $G'_{t+1}(y) \le G'_0(y) \le G'_1(y) \le 0$ for $y \le S_1$, hence $S_{t+1} \ge S_1$. Therefore, we conclude that $G'_t(y) \le G'_1(y)$ for $y \le S_1$ (hence $S_t \ge S_1$) for all $t = 2, 3, \dots, T$. Consequently, we have $\int_{s_1}^{S_1} G'_t(y)\mathrm{d}y \le \int_{s_1}^{S_1} G'_1(y)\mathrm{d}y$, i.e., $G_t(S_1) - G_t(s_1) \le G_1(S_1) - G_1(s_1) = -K$, i.e., $G_t(s_1) - G_t(S_1) \ge K$. Since $S_t \ge S_1$, and $G_t(s_t) - G_t(S_t) = K$, therefore, we must have $s_t \ge s_1$.

(1.2) (By induction) We start the analysis with the last period.

For $t = 1$, from the comparison of (7) and (8), we know that $s_1 < S_1 \le S_0$.

Now suppose Theorem 1.2 holds for $t = m$, that is, $\tilde{s}_m < S_0$, where $m = 2, 3, \dots, T-1$. Then for $t = m+1$, if we show $s_{m+1} < S_0$, then we are done. Suppose not, that is, $s_{m+1} \ge S_0$, then if the inventory-on-hand at the beginning of period $m+1$ equals s_{m+1} (the break-even point) exactly, then we could order up to S_{m+1}, and the cost associated with this decision would be

$$\begin{aligned}
&V_{m+1}(s_{m+1}) = K + c(S_{m+1} - s_{m+1}) + L(S_{m+1}) \\
&\quad + \alpha E\left[V_m\left((S_{m+1} - \xi_{m+1})^+\right)\right] \\
&= K + c(S_{m+1} - s_{m+1}) + L(S_{m+1}) \\
&\quad + \alpha \int_0^{S_{m+1}-s_m} [G_m(S_{m+1} - \xi_{m+1}) \\
&\quad - c(S_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha \int_{S_{m+1}-s_m}^{S_{m+1}} [K + G_m(S_m) - c(S_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha \int_{S_{m+1}}^{\infty} [K + G_m(S_m)] f(\xi_{m+1})\,\mathrm{d}\xi \\
&= K - c \cdot s_{m+1} + G_0(S_{m+1}) \\
&\quad \times \Bigg(\text{since } G_0(S_{m+1}) = c \cdot S_{m+1} + L(S_{m+1}) \\
&\quad - \alpha c \int_0^{S_{m+1}} (S_{m+1} - \xi_{m+1}) f(\xi_{m+1})\,\mathrm{d}\xi\Bigg) \\
&\quad + \alpha \int_0^{S_{m+1}-s_m} [G_m(S_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha [K + G_m(S_m)] (1 - F(S_{m+1} - s_m)).
\end{aligned}$$

On the other hand, if we do not place this order, the cost would be

$$\begin{aligned}
&\bar{V}_{m+1}(s_{m+1}) = L(s_{m+1}) + \alpha E\left[V_m\left((s_{m+1} - \xi_{m+1})^+\right)\right] \\
&= L(s_{m+1}) + \alpha \int_0^{s_{m+1}-s_m} [G_m(s_{m+1} - \xi_{m+1}) \\
&\quad - c(s_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha \int_{s_{m+1}-s_m}^{s_{m+1}} [K + G_m(S_m) - c(s_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha \int_{s_{m+1}}^{\infty} [K + G_m(S_m)] f(\xi_{m+1})\,\mathrm{d}\xi \\
&= L(s_{m+1}) - \alpha c \int_0^{s_{m+1}} (s_{m+1} - \xi_{m+1}) f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha \int_0^{s_{m+1}-s_m} [G_m(s_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha [K + G_m(S_m)] (1 - F(s_{m+1} - s_m)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&V_{m+1}(s_{m+1}) - \bar{V}_{m+1}(s_{m+1}) = K - c \cdot s_{m+1} + G_0(S_{m+1}) \\
&\quad + \alpha \int_0^{S_{m+1}-s_m} [G_m(S_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha [K + G_m(S_m)] (1 - F(S_{m+1} - s_m)) \\
&\quad - \left[L(s_{m+1}) - \alpha c \int_0^{s_{m+1}} (s_{m+1} - \xi_{m+1}) f(\xi_{m+1})\,\mathrm{d}\xi\right] \\
&\quad - \Bigg[\alpha \int_0^{s_{m+1}-s_m} [G_m(s_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad + \alpha [K + G_m(S_m)] (1 - F(s_{m+1} - s_m))\Bigg] \\
&= K + [G_0(S_{m+1}) - G_0(s_{m+1})] \\
&\quad \Bigg(\text{since } G_0(s_{m+1}) = c \cdot s_{m+1} + L(s_{m+1}) \\
&\quad - \alpha c \int_0^{s_{m+1}} (s_{m+1} - \xi_{m+1}) f(\xi_{m+1})\,\mathrm{d}\xi\Bigg) \\
&\quad + \alpha \int_0^{S_{m+1}-s_m} [G_m(S_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad - \alpha \int_0^{s_{m+1}-s_m} [G_m(s_{m+1} - \xi_{m+1})] f(\xi_{m+1})\,\mathrm{d}\xi \\
&\quad - \alpha [K + G_m(S_m)] (F(S_{m+1} - s_m) - F(s_{m+1} - s_m)) \\
&= K - \alpha K \cdot F(S_{m+1} - s_m) + [G_0(S_{m+1}) - G_0(s_{m+1})] \\
&\quad \text{(insert } K \text{ into the next two integrations and minus it here)} \\
&\quad + \alpha \int_0^{s_{m+1}-s_m} [K + G_m(S_{m+1} - \xi_{m+1}) - G_m(s_{m+1} - \xi_{m+1})] \\
&\quad \times f(\xi_{m+1})\,\mathrm{d}\xi + \alpha \int_{s_{m+1}-s_m}^{S_{m+1}-s_m} [K + G_m(S_{m+1} - \xi_{m+1}) \\
&\quad - K - G_m(S_m)] f(\xi_{m+1})\,\mathrm{d}\xi \\
&> 0.
\end{aligned}$$

[In the last equation above, $K - \alpha K \cdot F(S_{m+1} - s_m) \ge 0$; $G_0(S_{m+1}) - G_0(s_{m+1}) > 0$ is from the assumption of $S_0 \le s_{m+1} < S_{m+1}$, where S_0 is the minimize of $G_0(.)$ which is strictly convex; $K + G_m(S_{m+1} - \xi_{m+1}) - G_m(s_{m+1} - \xi_{m+1}) \ge 0$ follows from the property of K-convexity, as is given in Lemma 4.2.1(d) of Bertsekas [5]; $G_m(S_{m+1} - \xi_{m+1}) \ge G_m(S_m)$ is from the fact that S_m minimizes $G_m(.)$].

But, $V_{m+1}(s_{m+1}) - \bar{V}_{m+1}(s_{m+1}) > 0$ is a contradiction because $V_{m+1}(s_{m+1})$ is the minimal cost. Therefore, $s_{m+1} < S_0$. This completes the induction proof.

(1.3) (By induction) For $t = 1$, this holds because $G_1(y)$ is convex with unique minimizer at S_1 and $s_1 < S_1$. Assume $G'_m(y) < 0$ for $y \le \tilde{s}_m$, where $m = 2, 3, \ldots, T-1$.

From (12) and (13), we conclude that $G'_{m+1}(y) \le G'_0(y)$ for $y \le \tilde{s}_m$ (the inequality follows from (13) by the induction hypothesis and the fact that $s_m \le \tilde{s}_m$). Besides, $G'_{m+1}(y) < 0$ on $(0, s_{m+1}]$ is well known for K-convex functions (see Lemma 4.2.1 in [5]). Thus, if $s_{m+1} > \tilde{s}_m$, then $s_{m+1} = \tilde{s}_{m+1}$, and we are done. Otherwise, if $s_{m+1} \le \tilde{s}_m$, then $\tilde{s}_{m+1} = \tilde{s}_m$, and $y \in (s_{m+1}, \tilde{s}_{m+1}]$ involves values of y where either (12) or (13) applies. In both of these cases, $G'_{m+1}(y) \le G'_0(y) < G'_0(S_0) = 0$ since $\tilde{s}_{m+1} < S_0$ by Theorem 1.2. Thus $G'_{m+1}(y) < 0$ on $(0, \tilde{s}_{m+1}]$ is again true.

(1.4) Since $\tilde{s}_t \ge \tilde{s}_{t-1}$, the result follows immediately from Theorem 1.3 above.

(1.5) (By induction) For $t = 1$, $G'_1(y) = c - l + (h + l - \alpha w)F(y) \ge c - l$, since $l > c \ge w$.

Assume $G'_t(y) \ge c - l$ for $t = m$, where $m = 2, 3, \ldots, T-1$. Then to prove the result for $t = m+1$, we need to consider two cases: (5a) $y \le s_m$; and (5b) $y > s_m$.

In case (5a), by (12), $G'_{m+1}(y) = G'_0(y) = c - l + (h + l - \alpha c)F(y) \ge c - l$; and in case (5b), from (13) we get

$$
\begin{aligned}
&G'_{m+1}(y)\\
&\quad = c - l + (h + l - \alpha c)F(y) + \alpha \int_0^{y-s_m} [G'_m(y-\xi)]f(\xi)\mathrm{d}\xi\\
&\quad \ge c - l + (h + l - \alpha c)F(y) + \alpha \int_0^{y-s_m} (c-l)f(\xi)\mathrm{d}\xi\\
&\qquad \text{(by induction hypothesis)}\\
&\quad = c - l + (h + l - \alpha c)F(y) + \alpha(c-l)\left[F(y) - \int_{y-s_m}^{y} f(\xi)\mathrm{d}\xi\right]\\
&\quad = c - l + [h + l(1-\alpha)]F(y) + \alpha(l-c)\int_{y-s_m}^{y} f(\xi)\mathrm{d}\xi\\
&\quad \ge c - l. \quad \square
\end{aligned}
$$

Theorem 1.1 offers a lower bound for s_t, indicating that for a T-period problem, the reorder point of the last period is always the smallest one. Theorem 1.2 shows that the optimal policy S_0 of a single-period newsvendor model in which all leftovers are refunded with purchase cost c, is an upper bound for $\tilde{s}_t$; this result is used to establish Theorem 1.3 which implies that $G_t(y)$ is strictly decreasing in $(0, \tilde{s}_t]$. Therefore, any local minimum of $G_t(y)$ has to be greater than $\tilde{s}_t$ and hence $\tilde{s}_{t-1}$, which is Theorem 1.4, so $\tilde{s}_{t-1}$ is a lower bound for S_t. More importantly, this property enables us to compare $G_t(.)$ with $G_{t-1}(.), \ldots, G_1(.)$ in the common domain $[\tilde{s}_{t-1}, +\infty)$. Theorem 1.5 gives a lower bound on the slope of $G'_t(y)$ as $-(l-c)$, which is negative since it represents the incremental cost of one unit of lost sales. Using this bound we establish a new lower bound on the optimal order-up-to level S^* for the infinite horizon problem. This is stated next.

Theorem 2. *In an infinite horizon problem, the optimal order-up-to level $S^* > S_0$.*

Proof. For any stationary (s, S) policy, we have

$$
\begin{aligned}
G(y|s,S) &= cy + L(y) + \alpha\left[\int_0^{y-s} G(y-\xi|s,S)f(\xi)\mathrm{d}\xi\right.\\
&\quad \left. + (K + G(S|s,S))\,(1 - F(y-s)) - c\int_0^{y}(y-\xi)f(\xi)\mathrm{d}\xi\right].
\end{aligned}
$$

Now, if the (s, S) policy satisfies $G(s|s,S) = K + G(S|s,S)$, then

$$
\begin{aligned}
G'(y|s,S) &= c + hF(y) - l[1 - F(y)]\\
&\quad + \alpha\int_0^{y-s} G'(y-\xi|s,S)f(\xi)\mathrm{d}\xi - \alpha c F(y)\\
&= G'_0(y) + \alpha\int_0^{y-s} G'(y-\xi|s,S)f(\xi)\mathrm{d}\xi\\
&= G'_0(y) + \alpha\int_0^{y-s}\left[G'_0(y-\xi)\right.\\
&\quad \left. + \alpha\int_0^{y-\xi-s} G'(y-\xi-\eta|s,S)f(\eta)\mathrm{d}\eta\right]f(\xi)\mathrm{d}\xi\\
&= G'_0(y) + \alpha\int_0^{y-s} G'_0(y-\xi)f(\xi)\mathrm{d}\xi\\
&\quad + \alpha^2\int_0^{y-s}\left[\int_0^{y-\xi-s} G'(y-\xi-\eta|s,S)f(\eta)\mathrm{d}\eta\right]f(\xi)\mathrm{d}\xi\\
&= G'_0(y) + \alpha\int_0^{y-s} G'_0(y-\xi)f(\xi)\mathrm{d}\xi\\
&\quad + \alpha^2\int_0^{y-s}\left[\int_0^{y-\xi-s}\left[G'_0(y-\xi-\eta)\right.\right.\\
&\quad \left. + \alpha\int_0^{y-\xi-\eta-s} G'(y-\xi-\eta-\zeta|s,S)f(\zeta)\mathrm{d}\zeta\right]\\
&\quad \left.\times f(\eta)\mathrm{d}\eta\right]f(\xi)\mathrm{d}\xi\\
&= G'_0(y) + \alpha\int_0^{y-s} G'_0(y-\xi)f(\xi)\mathrm{d}\xi\\
&\quad + \alpha^2\int_0^{y-s}\left[\int_0^{y-\xi-s} G'_0(y-\xi-\eta)f(\eta)\mathrm{d}\eta\right]f(\xi)\mathrm{d}\xi\\
&\quad + \alpha^3\int_0^{y-s}\left[\int_0^{y-\xi-s}\left[\int_0^{y-\xi-\eta-s} G'(y-\xi-\eta-\zeta|s,S)\right.\right.\\
&\quad \left.\left.\times f(\zeta)\mathrm{d}\zeta\right]f(\eta)\mathrm{d}\eta\right]f(\xi)\mathrm{d}\xi\\
&= \cdots\\
&= G'_0(y) + \alpha\int_0^{y-s} G'_0(y-\xi)f(\xi)\mathrm{d}\xi + \cdots\\
&\quad + \alpha^n\int_0^{y-s} G'_0(y-\xi)f_n(\xi)\mathrm{d}\xi\\
&\quad + \alpha^{n+1}\int_0^{y-s} G'(y-\xi|s,S)f_{n+1}(\xi)\mathrm{d}\xi\\
&\quad \text{(here } f_n(\cdot) \text{ represents the } n\text{-fold convolution of } f(\cdot))\\
&= G'_0(y) + \sum_{i=1}^{n}\alpha^i\int_0^{y-s} G'_0(y-\xi)f_i(\xi)\mathrm{d}\xi\\
&\quad + \alpha^{n+1}\int_0^{y-s} G'(y-\xi|s,S)f_{n+1}(\xi)\mathrm{d}\xi.
\end{aligned}
$$

Now, note that from Theorem 1.5, we have $G'_t(y) \ge c - l, \forall y > 0$. Therefore, it is true that for the infinite horizon case, $G'(y|s,S) \ge c - l, \forall y > 0$; and it is straightforward to show that $G'(\infty|s,S) = c + h/(1-\alpha)$. Hence, $G'(y|s,S)$ is bounded.

Now since

$$
\begin{aligned}
G'(y|s,S) &= G'_0(y) + \sum_{i=1}^{n}\alpha^i\int_0^{y-s} G'_0(y-\xi)f_i(\xi)\mathrm{d}\xi\\
&\quad + \alpha^{n+1}\int_0^{y-s} G'(y-\xi|s,S)f_{n+1}(\xi)\mathrm{d}\xi,
\end{aligned}
$$

as n tends to infinity, $\alpha^{n+1}\int_0^{y-s} G'(y-\xi|s,S)f_{n+1}(\xi)\mathrm{d}\xi$ tends to zero, so that

$$G'(y|s,S) = G_0'(y) + \sum_{i=1}^{\infty}\alpha^i \int_0^{y-s} G_0'(y-\xi)f_i(\xi)\mathrm{d}\xi.$$

Since it is obvious that for $y \le S_0$,

$$G'(y|s,S) = G_0'(y) + \sum_{i=1}^{\infty}\alpha^i \int_0^{y-s} G_0'(y-\xi)f_i(\xi)\mathrm{d}\xi < 0,$$

therefore, we have that the optimal $S^* > S_0$. □

For the infinite horizon problem, the facts that the optimal reorder point $s^* < S_0$ (a consequence of Theorem 1.2) while the optimal order-up-to level $S^* > S_0$ can be explained by noting that when $K = 0$, the optimal policy reduces to the single number base stock policy with target level S_0. And, as K becomes positive, s^* falls below this target while S^* increases above this target. Such splitting of the domain helps to separate out the search space for the optimal s^* and S^* into two non-overlapping intervals that can yield effective computational schemes for the infinite horizon problem as we will demonstrate later for demands with Erlang distribution.

Theorems 1.1 and 1.2 provide new bounds for the optimal reorder point s_t as $s_1 \le s_t \le \tilde{s}_t < S_0$. The result that $s_t \ge s_1$ for the reorder point given in Theorem 1.1 is the counterpart of the result $S_t \ge S_1$ for the order-up-to level given in Lemma 1 of Iglehart [9]. Additionally, our $G_t'(y) < 0\ \forall y \in (0, \tilde{s}_t]$ given in Theorem 1.3 is a generalization of $G_t'(y) < 0\ \forall y \in (0, s_t]$ given in Lemma 4.2.1 of Bertsekas [5]. Theorem 1.4 provides $\tilde{s}_{t-1}$ as a dynamic lower bound for the optimal order-up-to level S_t. Hence, it provides a different lower bound for S_t than that in (6) or (6′) of Veinott [14]. In conjunction with this lower bound, we can use Iglehart's [9] result for an upper bound of S_t to accelerate computation. Thus, for our model which assumes similar conditions to that of Scarf [12], we have bounds for the optimal (s_t, S_t) policy that are different but comparable to the bounds in (6) of Veinott [14]. While Veinott assumed a different set of conditions to derive (6), he obtained weaker bounds (6′) than (6) under Scarf's [12] conditions. Later, Veinott and Wagner [15] established bounds of the form (6) under Scarf's assumptions except with $c_{T+1} = c$ (Scarf assumed $c_{T+1} = 0$). Note that since in our model, the salvage value $c_{T+1} = w \le c$, we relax the assumptions in both [12] and [15]. While Theorem 1.5 itself is a new result, using it we further establish Theorem 2 which is also a new result. In the following section, we will use Theorems 1 and 2 to study the infinite horizon problem with Erlang demands.

3. The lost sales problem with Erlang demands

In general, a random variable X is said to have an Erlang-r distribution, when: $f(x) = \frac{\lambda^r x^{r-1}}{(r-1)!}\mathrm{e}^{-\lambda x}$, $\lambda > 0$, r positive integer; and $F(x) = 1 - \sum_{j=0}^{r-1}\mathrm{e}^{-\lambda x}\frac{(\lambda x)^j}{j!}$. Under an Erlang-$r$ demand distribution, the cost function $G_t(.)$ has a special structure that makes it possible to conveniently compare it with $G_t'(.)$. In particular, we will show in the following lemma that $G_t(y) = B_t + a_t y + \tilde{A}_t(y)\mathrm{e}^{-\lambda y}$. Here a_t depends on t only, B_t is a constant whose value depends on the optimal controls s and S for periods $t-1, \dots, 2, 1$, that have been previously computed using backward dynamic program. Furthermore, since B_t is a constant, it disappears in $G_t'(y)$; hence S_t is independent of B_t, so is s_t given that $G_t(s_t) = K + G_t(S_t)$. It is also the case that the coefficients of $\tilde{A}_t(y)$, a polynomial of degree tr, depend on these pre-computed values of s and S. The result, which is proved in an Addendum (available from the authors), is formally stated as:

Lemma 1. *In a T-period problem, if demand in each period follows Erlang-r distribution, then in any period t, $t = 1, 2, \dots, T$, for $y \in [\tilde{s}_{t-1}, +\infty)$,*

(1.1) *$G_t(y) = B_t + a_t y + \tilde{A}_t(y)\mathrm{e}^{-\lambda y}$, where $\tilde{A}_t(y)$ is a polynomial of degree tr;*

(1.2) *$G_t'(y) = a_t + \mathrm{e}^{-\lambda y}[\tilde{A}_t'(y) - \lambda\tilde{A}_t(y)] = \mathrm{e}^{-\lambda y}\left[a_t\mathrm{e}^{\lambda y} + A_{tr}(y)\right]$, with*

$$\begin{cases} a_t = (c - \alpha^t w) + (1 + \alpha + \alpha^2 + \dots + \alpha^{t-1})h, \\ A_{tr}(y) = a_{tr\cdot 1} + a_{tr\cdot 2}y + \dots + a_{tr\cdot tr}y^{tr-1}, \\ [A_{tr}(y)]^{(mr)} = [\alpha\lambda^r]^m A_{(t-m)r}(y) \quad \text{for } m = 1, 2, \dots, t-1; \end{cases}$$

where $a_{tr\cdot j}$, $1 \le j \le tr$, are constants and the superscript (mr) represents the mrth derivative.

Since the coefficients for period t depend only on optimal controls from problems with shorter planning horizons, Lemma 1 easily leads to a recursive computational scheme for finite horizon problems, provided $G_t'(y)$ is well behaved, a fact which follows from the finite horizon analog of Theorem 3 below. Some more work is needed to apply Lemma 1 to the infinite horizon case. We proceed by writing $G_t'(y)$ in terms of an infinite series, but we appropriately drop the subscript t to reflect the stationary nature of the policy. Subsequently, we express the term $A_{Tr}(y)$ as an infinite series. Then, as T tends to infinity, from Lemma 1 and Theorem 2, we have:

$$G'(y|s,S) = b_0 + \mathrm{e}^{-\lambda y}\sum_{i=1}^{\infty} b_i y^{i-1}, \quad \text{for any } s \in (s_1, S_0),$$
$$S \in (S_0, +\infty) \text{ and } y \in [S_0, +\infty), \tag{14}$$

where the coefficients, b_i's, depend on s and S for any stationary policy (s, S). Moreover,

$$G(y|s,S) = cy + L(y) + \alpha\left[\int_0^{y-s} G(y-\xi|s,S)f(\xi)\mathrm{d}\xi \right.$$
$$\left. + (K + G(S|s,S))(1 - F(y-s)) - c\int_0^y (y-\xi)f(\xi)\mathrm{d}\xi\right]. \tag{15}$$

Besides, if for any stationary reorder point s, the order-up-to level S is chosen in a way such that $G(s|s,S) = K + G(S|s,S)$ holds, then (15) reduces to

$$G(y|s) = cy + L(y) + \alpha\left[\int_0^{y-s} G(y-\xi|s)f(\xi)\mathrm{d}\xi \right.$$
$$\left. + G(s|s)(1 - F(y-s)) - c\int_0^y (y-\xi)f(\xi)\mathrm{d}\xi\right], \tag{16}$$

since at the optimal values of s and S, $G(s|s,S) = K + G(S|s,S)$ must hold.

And for general s and S, we have that

$$\begin{aligned} G'(y|s,S) &= c + hF(y) - l[1 - F(y)] \\ &\quad + \alpha\int_0^{y-s} G'(y-\xi|s,S)f(\xi)\mathrm{d}\xi - \alpha cF(y) \\ &\quad + \alpha[G(s|s,S) - K - G(S|s,S)]f(y-s) \\ &= c(1-\alpha) + h - (h + l - \alpha c)\sum_{m=0}^{r-1}\frac{(\lambda y)^m \mathrm{e}^{-\lambda y}}{m!} \\ &\quad + \alpha[G(s|s,S) - K - G(S|s,S)]f(y-s) \\ &\quad + \sum_{m=0}^{r-1}\frac{[\lambda(y-s)]^m \mathrm{e}^{-\lambda y}}{m!} \\ &\quad + \alpha\int_0^{y-s}\left[b_0 + \mathrm{e}^{-\lambda(y-\xi)}\sum_{i=1}^{+\infty} b_i(y-\xi)^{i-1}\right]\frac{\lambda^r\xi^{r-1}\mathrm{e}^{-\lambda\xi}}{(r-1)!}\mathrm{d}\xi \\ &= c(1-\alpha) + h - (h + l - \alpha c)\sum_{m=0}^{r-1}\frac{(\lambda y)^m \mathrm{e}^{-\lambda y}}{m!} \end{aligned}$$

$$
\begin{aligned}
&+\alpha b_0\left[1-\sum_{m=0}^{r-1}\frac{\lambda^m(y-s)^m e^{-\lambda(y-s)}}{m!}\right]\\
&+\alpha\frac{\lambda^r e^{-\lambda y}}{(r-1)!}\int_0^{y-s}\left[\sum_{i=1}^{+\infty}b_i(y-\xi)^{i-1}\xi^{r-1}\right]d\xi\\
&+\alpha[G(s|s,S)-K-G(S|s,S)]\sum_{m=0}^{r-1}\frac{\lambda^m(y-s)^m e^{-\lambda(y-s)}}{m!}\\
=&\,[c(1-\alpha)+h+\alpha b_0]-(h+l-\alpha c)\sum_{m=0}^{r-1}\frac{(\lambda y)^m e^{-\lambda y}}{m!}\\
&+\alpha[G(s|s,S)-K-G(S|s,S)]\sum_{m=0}^{r-1}\frac{e^{\lambda s}\lambda^m(y-s)^m e^{-\lambda y}}{m!}\\
&+\alpha\lambda^r e^{-\lambda y}\sum_{i=1}^{+\infty}b_i\\
&\times\left[\frac{(i-1)!y^{i+r-1}}{(i+r-1)!}-\sum_{j=1}^{r}\frac{(i-1)!s^{i-1+j}(y-s)^{r-j}}{(i-1+j)!(r-j)!}\right]. \qquad (17)
\end{aligned}
$$

Since (14) must equal (17), by matching the coefficients we get

$$b_0=c+h/(1-\alpha); \qquad (18)$$

$$
\begin{aligned}
b_m =&-(h+l-\alpha c)\frac{\lambda^{m-1}}{(m-1)!}\\
&+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\\
&\times\sum_{k=m-1}^{r-1}\binom{k}{m-1}\frac{\lambda^k(-s)^{k+1-m}}{k!}\\
&-\alpha\lambda^r\sum_{i=1}^{+\infty}b_i\sum_{j=m-1}^{r-1}\binom{j}{m-1}\frac{(i-1)!(-1)^{j+1-m}s^{i+r-m}}{(i-1+r-j)!j!};\\
&(1\le m\le r) \qquad (19)
\end{aligned}
$$

$$b_{tr+m}=\frac{(\alpha\lambda^r)^t(m-1)!b_m}{(tr+m-1)!}\quad (t\ge 1,\ 1\le m\le r). \qquad (20)$$

Therefore, after repeated substitutions,

$$
\begin{aligned}
G'(y|s,S)&=b_0+e^{-\lambda y}\sum_{i=1}^{+\infty}b_i y^{i-1}\\
&=b_0+e^{-\lambda y}\sum_{m=0}^{+\infty}\left[\frac{b_1(\alpha\lambda^r)^m y^{mr}}{(mr)!}+\frac{b_2(\alpha\lambda^r)^m y^{mr+1}}{(mr+1)!}\right.\\
&\quad\left.+\cdots+\frac{b_r(r-1)!(\alpha\lambda^r)^m y^{mr+r-1}}{(mr+r-1)!}\right].
\end{aligned}
$$

Now let $Q_i(y)=\sum_{m=0}^{\infty}\frac{(i-1)!\alpha^m\lambda^{mr}y^{mr+i-1}}{(mr+i-1)!}$, for $i=1,2,\ldots,r$, and $Q(y)=\sum_{i=1}^{r}b_iQ_i(y)$ so that

$$G'(y|s,S)=b_0+e^{-\lambda y}Q(y). \qquad (21)$$

It is easy to verify that

$$Q^{(r)}(y)=\alpha\lambda^r Q(y), \qquad (22)$$

which is a system of r differential equations. Solving (22), we get the general solutions as:

$$
\begin{cases}
Q(y)=\phi e^{\lambda\sqrt[r]{\alpha}y}\\
\qquad+\displaystyle\sum_{j=1}^{(r-1)/2}e^{\gamma_j y}[u_j\cos\beta_j y+v_j\sin\beta_j y] \quad \text{if } r \text{ is odd},\\
Q(y)=\phi_1 e^{\lambda\sqrt[r]{\alpha}y}+\phi_2 e^{-\lambda\sqrt[r]{\theta}y}\\
\qquad+\displaystyle\sum_{j=1}^{(r-2)/2}e^{\gamma_j y}[u_j\cos\beta_j y+v_j\sin\beta_j y] \quad \text{if } r \text{ is even},
\end{cases} \qquad (23)
$$

where $(\beta_j,\gamma_j)=\left(\sqrt[r]{\alpha}\lambda\cdot\sin\frac{2j\pi}{r},\sqrt[r]{\alpha}\lambda\cdot\cos\frac{2j\pi}{r}\right)$, each of r points which uniformly divide the circle with radius $\sqrt[r]{\alpha}\lambda$ into r parts; u_j, v_j, ϕ, ϕ_1 and ϕ_2 are some constants that depend on b_i, $1\le i\le r$. To solve for the values of u_j, v_j, ϕ, ϕ_1 and ϕ_2 we must solve the following group of r equations:

$$Q(0)=b_1,\qquad Q'(0)=b_2,\ldots,\qquad Q^{(r-1)}(0)=b_r. \qquad (24)$$

In general, there will be a finite but indeterminate number of local minima (or maxima) that satisfy $G'(y|s,S)=0$. However, in some special cases there is a unique solution. This is formally stated as:

Theorem 3. *For any reorder point $s<S_0$, in the interval $[S_0,\infty)$,*

(3.1) *If the demand distribution is exponential (Erlang-1), then $G(y|s,S)$ is strictly convex for values of s and S satisfying $G(s|s,S)=K+G(S|s,S)$;*
(3.2) *if the demand distribution is Erlang-2, then $G(y|s,S)$ is unimodal; and,*
(3.3) *if the demand distribution is Erlang-r $(r>2)$, then $G(y|s,S)$ has a finite number of local minima.*

Proof. (3.1) For $r=1$, from (18)–(20), we have

$$
\begin{aligned}
&b_0=c+h/(1-\alpha),\\
&b_1=-(h+l-\alpha c)+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\\
&\qquad-\alpha\lambda\sum_{i=1}^{+\infty}b_i\frac{s^i}{i},\\
&b_{m+1}=\frac{(\alpha\lambda)^m b_1}{m!}\quad (m\ge 1).
\end{aligned}
$$

Hence we can write b_1 as

$$
\begin{aligned}
b_1=&\{-(h+l-\alpha c)+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\}\\
&-\alpha\lambda\sum_{i=1}^{+\infty}b_i\frac{s^i}{i}\\
=&\{-(h+l-\alpha c)+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\}\\
&-b_1\left[\frac{\alpha\lambda s}{1!}+\frac{(\alpha\lambda s)^2}{2!}+\cdots+\frac{(\alpha\lambda s)^m}{m!}+\cdots\right]\\
=&\{-(h+l-\alpha c)+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\}\\
&-b_1\left(e^{\alpha\lambda s}-1\right)\\
\Rightarrow b_1=&\frac{-(h+l-\alpha c)+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}}{e^{\alpha\lambda s}}.
\end{aligned}
$$

Then, from (14) we get

$$
\begin{aligned}
G'(y|s,S)&=b_0+e^{-\lambda y}\sum_{i=1}^{+\infty}b_i y^{i-1}\\
&=b_0+b_1e^{-\lambda y}\left[1+\frac{\alpha\lambda y}{1!}+\frac{(\alpha\lambda y)^2}{2!}+\cdots+\frac{(\alpha\lambda y)^m}{m!}+\cdots\right]\\
&=c+h/(1-\alpha)\\
&\quad+\frac{\{-(h+l-\alpha c)+\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\}}{e^{\alpha\lambda s}}\\
&\quad\times e^{\lambda y(\alpha-1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&G''(y|s,S)\\
&=\frac{(1-\alpha)\lambda\{(h+l-\alpha c)-\alpha[G(s|s,S)-K-G(S|s,S)-b_0]e^{\lambda s}\}}{e^{\alpha\lambda s}}\\
&\quad\times e^{\lambda y(\alpha-1)}.
\end{aligned}
$$

For values of s and S satisfying $G(s|s,S)=K+G(S|s,S)$, we have

$$G''(y|s,S) = \frac{(1-\alpha)\lambda[(h+l-\alpha c)+\alpha b_0 e^{\lambda s}]}{e^{\alpha\lambda s}} e^{\lambda y(\alpha-1)} > 0,$$

so that $G(y|s,S)$ is strictly convex.

Moreover, for such values of s and S,

$$G'(y|s,S) = 0 \Rightarrow y = \frac{\ln\left[(1-\alpha)\{(h+l-\alpha c)+\alpha b_0 e^{\lambda s}\}\right] - \ln[h+c(1-\alpha)] - \alpha\lambda s}{(1-\alpha)\lambda}. \quad (25)$$

(3.2) When $r=2$, from (19), we know

$$b_1 = -(h+l-\alpha c) + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]e^{\lambda s} \times (1-\lambda s) + \alpha\lambda^2 \sum_{i=1}^{+\infty} b_i \left(\frac{s^{i+1}}{i+1}\right),$$

$$b_2 = -(h+l-\alpha c)\lambda + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s} + \alpha\lambda^2 \sum_{i=1}^{+\infty} b_i \left(\frac{s^i}{i}\right).$$

Separating even and odd powers of s, we can write

$$\begin{aligned}
b_1 &= -(h+l-\alpha c) + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]e^{\lambda s} \times (1-\lambda s) + \alpha\lambda^2 \sum_{i=1}^{+\infty} b_i \left(\frac{s^{i+1}}{i+1}\right) \\
&= -(h+l-\alpha c) \\
&\quad + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]e^{\lambda s}(1-\lambda s) \\
&\quad + \alpha\lambda^2 \left[b_1 \frac{s^2}{2} + \frac{\alpha\lambda^2 b_1}{2!} \times \frac{s^4}{4} + \frac{[\alpha\lambda^2]^2 b_1}{4!} \times \frac{s^6}{6} + \cdots + \frac{[\alpha\lambda^2]^m b_1}{2m!} \times \frac{s^{2m+2}}{(2m+2)} + \cdots\right] \\
&\quad + \alpha\lambda^2 \left[b_2 \frac{s^3}{3} + \frac{\alpha\lambda^2 b_2}{3!} \times \frac{s^5}{5} + \frac{[\alpha\lambda^2]^2 b_2}{5!} \times \frac{s^7}{7} + \cdots + \frac{[\alpha\lambda^2]^m b_2}{(2m+1)!} \times \frac{s^{2m+3}}{(2m+3)} + \cdots\right] \\
&= -(h+l-\alpha c) \\
&\quad + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]e^{\lambda s}(1-\lambda s) \\
&\quad + \alpha\lambda^2 b_1 \left[\frac{2 + \left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} - \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2\alpha\lambda^2}\right] \\
&\quad + \alpha\lambda^2 b_2 \left[\frac{\left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} + \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2\alpha\lambda^2}\right]. \quad (26)
\end{aligned}$$

Now, from (26), we can derive that

$$\begin{aligned}
b_1 &= -\{(h+l-\alpha c) - \alpha[G(s|s,S) - K - G(S|s,S) - b_0] \times e^{\lambda s}(1-\lambda s)\} \\
&\quad + b_1 \left[\frac{2 + \left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} - \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2}\right] \\
&\quad + b_2 \left[\frac{\left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} + \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2}\right] \\
\Rightarrow b_1 &\left[\frac{\left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} - \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2}\right] \\
&\quad + b_2 \left[\frac{\left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} + \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2}\right] \\
&\quad - \{(h+l-\alpha c) - \alpha[G(s|s,S) - K - G(S|s,S) - b_0] \times e^{\lambda s}(1-\lambda s)\} = 0. \quad (27)
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_2 &= -(h+l-\alpha c)\lambda + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s} + \alpha\lambda^2 \sum_{i=1}^{+\infty} b_i \left(\frac{s^i}{i}\right) \\
&= -(h+l-\alpha c)\lambda + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s} \\
&\quad + \alpha\lambda^2 \left[b_1 \frac{s}{1} + \frac{\alpha\lambda^2 b_1}{2!} \times \frac{s^3}{3} + \frac{[\alpha\lambda^2]^2 b_1}{4!} \times \frac{s^5}{5} + \cdots + \frac{[\alpha\lambda^2]^m b_1}{2m!} \times \frac{s^{2m+1}}{(2m+1)} + \cdots\right] \\
&\quad + \alpha\lambda^2 \left[b_2 \frac{s^2}{2} + \frac{\alpha\lambda^2 b_2}{3!} \times \frac{s^4}{4} + \frac{[\alpha\lambda^2]^2 b_2}{5!} \times \frac{s^6}{6} + \cdots + \frac{[\alpha\lambda^2]^m b_2}{(2m+1)!} \times \frac{s^{2m+2}}{(2m+2)} + \cdots\right] \\
&= -(h+l-\alpha c)\lambda + \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s} \\
&\quad + \alpha\lambda^2 b_1 \left[\frac{e^{\sqrt{\alpha}\lambda s} - e^{-\sqrt{\alpha}\lambda s}}{2\sqrt{\alpha}\lambda}\right] + \alpha\lambda^2 b_2 \left[\frac{e^{\sqrt{\alpha}\lambda s} + e^{-\sqrt{\alpha}\lambda s} - 2}{2\alpha\lambda^2}\right] \\
\Rightarrow b_2 &= -\{(h+l-\alpha c)\lambda - \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s}\} + \sqrt{\theta}\lambda b_1 \left[\frac{e^{\sqrt{\alpha}\lambda s} - e^{-\sqrt{\alpha}\lambda s}}{2}\right] \\
&\quad + b_2 \left[\frac{e^{\sqrt{\alpha}\lambda s} + e^{-\sqrt{\alpha}\lambda s} - 2}{2}\right] \\
\Rightarrow &\sqrt{\alpha}\lambda b_1 \left[\frac{e^{\sqrt{\alpha}\lambda s} - e^{-\sqrt{\alpha}\lambda s}}{2}\right] + b_2 \left[\frac{e^{\sqrt{\alpha}\lambda s} + e^{-\sqrt{\alpha}\lambda s}}{2} - 2\right] \\
&\quad - \{(h+l-\alpha c)\lambda - \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s}\} \\
&= 0. \quad (28)
\end{aligned}$$

Let us define

$$\begin{aligned}
C_1(s) &= \{(h+l-\alpha c) - \alpha[G(s|s,S) - K - G(S|s,S) - b_0] \times e^{\lambda s}(1-\lambda s)\}, \\
C_2(s) &= \{(h+l-\alpha c)\lambda - \alpha[G(s|s,S) - K - G(S|s,S) - b_0]\lambda e^{\lambda s}\}, \\
A_1(s) &= \left[\frac{\left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} - \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2}\right], \\
A_2(s) &= \sqrt{\alpha}\lambda \left[\frac{e^{\sqrt{\alpha}\lambda s} - e^{-\sqrt{\alpha}\lambda s}}{2}\right], \\
B_1(s) &= \left[\frac{\left(\sqrt{\alpha}\lambda s - 1\right)e^{\sqrt{\alpha}\lambda s} + \left(\sqrt{\alpha}\lambda s + 1\right)e^{-\sqrt{\alpha}\lambda s}}{2}\right], \\
B_2(s) &= \left[\frac{e^{\sqrt{\alpha}\lambda s} + e^{-\sqrt{\alpha}\lambda s}}{2} - 2\right].
\end{aligned}$$

Then solving (27) and (28), we can write

$$b_1 = \frac{C_1(s)B_2(s) - C_2(s)B_1(s)}{A_1(s)B_2(s) - A_2(s)B_1(s)} \quad \text{and}$$

$$b_2 = \frac{C_1(s)A_2(s) - C_2(s)A_1(s)}{B_1(s)A_2(s) - B_2(s)A_1(s)}.$$

Then, from (14) we get

$$\begin{aligned}G'(y|s,S) &= b_0 + e^{-\lambda y}\sum_{i=1}^{+\infty} b_i y^{i-1}\\ &= c + h/(1-\alpha) + b_1 e^{-\lambda y}\\ &\quad\times\left[1+\frac{\alpha(\lambda y)^2}{2!}+\frac{\alpha^2(\lambda y)^4}{4!}+\cdots+\frac{\alpha^m(\lambda y)^{2m}}{2m!}+\cdots\right]\\ &\quad+ b_2 e^{-\lambda y}\left[y+\frac{\alpha\lambda^2 y^3}{3!}+\frac{\alpha^2\lambda^4 y^5}{5!}+\cdots+\frac{\alpha^n\lambda^{2m}y^{2m+1}}{(2m+1)!}+\cdots\right]\\ &= c + h/(1-\alpha) + b_1 e^{-\lambda y}\left[\frac{e^{\sqrt{\alpha}\lambda y}+e^{-\sqrt{\alpha}\lambda y}}{2}\right]\\ &\quad+ b_2 e^{-\lambda y}\left[\frac{e^{\sqrt{\alpha}\lambda y}-e^{-\sqrt{\alpha}\lambda y}}{2\sqrt{\alpha}\lambda}\right]\\ &= c + h/(1-\alpha) + e^{(\sqrt{\alpha}-1)\lambda y}\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\\ &\quad+ e^{-(\sqrt{\alpha}+1)\lambda y}\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right].\end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned}G''(y|s,S) &= \left(\sqrt{\alpha}-1\right)\lambda\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]e^{(\sqrt{\alpha}-1)\lambda y}\\ &\quad-\left(\sqrt{\alpha}+1\right)\lambda\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right]e^{-(\sqrt{\alpha}+1)\lambda y}\\ &= \lambda e^{-(\sqrt{\alpha}+1)\lambda y}\left\{\left(\sqrt{\alpha}-1\right)\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]e^{2\sqrt{\alpha}\lambda y}\right.\\ &\quad\left.-\left(\sqrt{\alpha}+1\right)\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\right\}.\end{aligned}$$

Now we will prove the result by contradiction. Suppose $G(y|s,S)$ has more than one local minimum in $(s,+\infty)$. Let y_1 and y_2 be the first and second of those minima. Then we have

$$\begin{cases} G''(y_1|s,S) = \lambda e^{(-\sqrt{\alpha}-1)\lambda y_1}\left\{\left(\sqrt{\alpha}-1\right)\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\right.\\ \qquad\left.\times\, e^{2\sqrt{\alpha}\lambda y_1}-\left(\sqrt{\alpha}+1\right)\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\right\} > 0,\\ G''(y_2|s,S) = \lambda e^{(-\sqrt{\alpha}-1)\lambda y_2}\left\{\left(\sqrt{\alpha}-1\right)\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\right.\\ \qquad\left.\times\, e^{2\sqrt{\alpha}\lambda y_2}-\left(\sqrt{\alpha}+1\right)\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\right\} > 0.\end{cases}$$

Since $G(y|s,S)$ must attain a local maximum at some point y_0 within the interval (y_1,y_2), we also have

$$\begin{aligned}G''(y_0|s,S) &= \lambda e^{(-\sqrt{\alpha}-1)\lambda y_0}\left\{\left(\sqrt{\alpha}-1\right)\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]e^{2\sqrt{\alpha}\lambda y_0}\right.\\ &\quad\left.-\left(\sqrt{\alpha}+1\right)\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right]\right\} < 0.\end{aligned}$$

This is a contradiction, because

$$\left(\sqrt{\alpha}-1\right)\left[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}\right]e^{2\sqrt{\alpha}\lambda y}-\left(\sqrt{\alpha}+1\right)\left[\frac{b_1}{2}-\frac{b_2}{2\sqrt{\alpha}\lambda}\right]$$

is either non-decreasing (if $(\sqrt{\alpha}-1)[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}] \ge 0$), or non-increasing (if $(\sqrt{\alpha}-1)[\frac{b_1}{2}+\frac{b_2}{2\sqrt{\alpha}\lambda}] \le 0$), and therefore, it is not possible to have $G''(y_1|s,S) > 0$, $G''(y_0|s,S) < 0$ for $y_1 < y_0 < y_2$, and $G''(y_2|s,S) > 0$. Hence, we cannot have more than one local minimum. Therefore, $G(y|s,S)$ is unimodal.

(3.3) When r is odd, $G'(y|s,S) = b_0 + e^{-\lambda y}Q(y) = c + h/(1-\alpha) + \phi e^{\lambda(\sqrt[r]{\alpha}-1)y} + \sum_{j=1}^{(r-1)/2} e^{(\gamma_j-\lambda)y}[u_j\cos\beta_j y + v_j\sin\beta_j y]$.

Note that $0 \le \alpha < 1$, and hence, $\gamma_j = \sqrt[r]{\alpha}\lambda\cdot\cos\frac{2j\pi}{r} < \lambda$. Also note that $\left|u_j\cos\beta_j y + v_j\sin\beta_j y\right| \le \sqrt{u_j^2+v_j^2}$. So there exists a large positive M such that $G'(y|s,S) > 0$ for $y \ge M$. And for $y\in[0,M]$, $u_j\cos\beta_j y + v_j\sin\beta_j y$ can change sign at most $[M\beta_j/\pi]$ times, therefore, the sign changes of $G'(y|s,S)$ in $[0,M]$ can only be a finite number of times. The proof for even r is similar. □

To understand the significance of Theorem 3, start by assuming that the optimal s is known. Then for any larger reorder point, for the cases in the first two parts of Theorem 3, the infinite horizon problem has only one minimum which must also be the global minimum. And, if the shape parameter exceeds 2, as in the third case, it follows from Theorem 3.3 that there may be multiple minima to evaluate. But, in general, we expect that there will not be too many local minima. To see this intuition, consider the case with $r=3$. In this case, $G'(y)=0$ is equivalent to

$$\begin{aligned}&b_0 + e^{-\lambda y}Q(y) = 0 \Rightarrow c + h/(1-\alpha) + e^{-\lambda y}\left[\phi e^{\lambda\sqrt[3]{\alpha}y}\right.\\ &\quad\left.+ e^{-\lambda\sqrt[3]{\alpha}y/2}\left(u_1\cos\frac{\sqrt{3}\lambda\sqrt[3]{\alpha}y}{2}+v_1\sin\frac{\sqrt{3}\lambda\sqrt[3]{\alpha}y}{2}\right)\right] = 0\\ &\Rightarrow c + h/(1-\alpha) + \left[\phi e^{-\lambda(1-\sqrt[3]{\alpha})y} + e^{-\lambda(1+\sqrt[3]{\alpha}/2)y}\right.\\ &\quad\left.\times\left(u_1\cos\frac{\sqrt{3}\lambda\sqrt[3]{\alpha}y}{2}+v_1\sin\frac{\sqrt{3}\lambda\sqrt[3]{\alpha}y}{2}\right)\right] = 0.\end{aligned} \tag{30}$$

Hence, the number of roots depends on the number of times the sum of the damped sinusoidal curves (inside the square bracket, which tends to zero as y goes to infinity) intersects b_0. For $r>3$, the number of zeroes depends on the number of times a weighted average of such sinusoidal curves intersects b_0. Hence, the search will terminate when the trial value of y first reaches that value of y at which the maximum amplitude falls below b_0.

Now that we have established that when demand has an Erlang distribution, the function $G(y)$ has a finite number of minima, the optimal policy can be computed easily. To proceed, notice that (1) even though the optimal s is unknown, from Theorems 1.1 and 1.2 we have, $s_1 \le s \le S_0$, so that Theorem 3 can be invoked; and (2) the coefficients b_i's in expressions (18)–(20) can be computed at any assumed stationary point (s,S) satisfying $G(s|s,S) = K + G(S|s,S)$ since $\alpha[G(s|s,S) - K - G(S|s,S) - b_0]$ could be reduced to $-\alpha b_0$ and hence $G(y|s,S)$ would become $G(y|s)$ [see (15) and (16)]. Therefore, if we define $\delta = (S_0-s_1)/N$ as a discretization step, then it leads to the following:

Algorithm.

For $n = 0, N$ do
  $s_n = s_1 + n\delta$;
  $y_n = \min\{y : y > S_0 \text{ and } G(s_n|s_n) = K + G(y|s_n)\}$;
  end;
  $n^* = \arg\min_{\{n=0,1,\ldots,N\}}\{G(y_n|s_n)\}$;
  $s^* = s_1 + n^*\delta$; $S^* = y_{n^*}$;
end.

From the expressions of the first order condition $G'(y|s,S)=0$ in the proof of Theorem 3, it is evident that the inputs needed for the above algorithm are the cost and demand distribution parameters, and s_1 and S_0 which are easily found. In particular, to find y_n in this algorithm, notice that for a given s_n, the unique y ($>S_0$) is found by solving for it using (25) for exponential demand and (29) for Erlang-2 demand. For Erlang-r demands ($r > 2$), due to the finiteness of the number of local minima together with the sinusoidal equations (21)–(24), it is possible to find the optimal policy by implementing a search procedure as elucidated by (30).

To summarize, our algorithm is essentially a line search for S for a given value of s in $[s_1, S_0]$ which is guaranteed to contain the optimal reorder point, and then varying s, until we get the pair of (s, S) which gives the lowest cost; the stopping rule derived from Theorem 3 assures that there are only a finite number of candidate pairs whose costs have to be to compared. Thus, our algorithm can be thought of as the continuous analog to that of Zheng and Federgruen [16] who developed a simple computation scheme for the periodic-review discrete demand infinite horizon case with backorders. Interestingly, our approach differs from theirs in the sense that their method searches for s for a given value of S, and then varies S. Finally, our algorithm for the periodic-review continuous Erlang-r demands with the infinite horizon expected discounted cost criterion for the lost sales model complements that of Archibald [1] who developed a two-dimensional grid search for the continuous-review discrete compound Poisson demand with the long-run average cost criterion.

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