OPTIMALITY OF (*s*, *S*) POLICIES IN THE INFINITE HORIZON DYNAMIC INVENTORY PROBLEM*

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The optimal ordering policy for a n-period dynamic inventory problem in which the ordering cost is linear plus a fixed reorder cost and the other one-period costs are convex is characterized by a pair of critical numbers, (s_n, S_n); see Scarf, [4]. In this paper we give bounds for the sequences $\{s_n\}$ and $\{S_n\}$ and discuss their limiting behavior. The limiting (s, S) policy characterizes the optimal ordering policy for the infinite horizon problem. Similar results are obtained if there is a time-lag in delivery.

1. Introduction

Our model will consider a single commodity. A sequence of ordering decisions is to be made periodically, e.g., at the beginning of each week. These decisions may result in a replenishment of the inventory of the commodity. Consumption during these time intervals may cause a depletion of the inventory. The cumulative demands in successive periods are assumed to form a sequence of independent identically distributed random variables. Three costs are incurred during each period which influence the ordering decisions. There is an ordering cost $c(z)$, where z is the amount purchased; a holding cost $h(\cdot)$ is charged for inventories on hand (the cumulative excess of supply over demand); and finally, a shortage cost $p(\cdot)$ is associated with the failure to meet demands. Holding and shortage costs are charged at the end of each period. We assume initially that when an order is placed delivery of the commodity is instantaneous. (This assumption will be relaxed in Section 6.) Furthermore, we assume that excess demands, i.e., demands which cannot be immediately filled, are backlogged. When backlogging is permitted, the inventory level may take on negative values. A negative level should be interpreted as an amount of the commodity owed to consumption. Our objective is to calculate the optimal ordering policy which is defined as the policy which minimizes the total expected cost over the duration of the process.

Suppose the inventory immediately after an order is delivered is y, then the expected holding and shortage costs incurred in that period are given by

$$L(y) = \begin{cases} \int_0^y h(y-\xi)\phi(\xi)\,d\xi + \int_y^\infty p(\xi - y)\phi(\xi)\,d\xi & y \geqq 0 \\ \int_0^\infty p(\xi - y)\phi(\xi)\,d\xi & y < 0, \end{cases}$$

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where $\phi(\cdot)$ is the density of the demand distribution. We shall assume that the mean of the demand distribution exists together with sufficient moments so that all integrals of $h(\cdot)$ and $p(\cdot)$ also exist. Let $c(z)$ be given by

$$c(z) = \begin{cases} 0 & z = 0 \\ K + c \cdot z & z > 0 \end{cases}$$

with K and c positive constants. Let $f_n(x)$ denote the expected value of the discounted costs for a n-period inventory problem, where the initial inventory is x units and an optimal ordering rule is used at each purchasing opportunity. Then $f_n(x)$ satisfies the functional equation

$$(1) \qquad f_n(x) = \min_{y \geq x} \left\{ c(y - x) + L(y) + \alpha \int_0^\infty f_{n-1}(y - \xi)\phi(\xi)\, d\xi \right\},$$

where $\alpha(0 < \alpha < 1)$ is the discount factor and $f_0(x) \equiv 0$.

If $L(y)$ is convex, Scarf has proved in [4] that the optimal policy for the n-period problem is characterized by a pair of critical numbers, (s_n, S_n), $s_n < S_n$. If $x \leqq s_n$, order $(S_n - x)$; and if $x > s_n$, do not order. S_n is the value of y which minimizes the function

$$G_n(y) = cy + L(y) + \alpha \int_0^\infty f_{n-1}(y - \xi)\phi(\xi)\, d\xi,$$

and s_n is defined as the smallest value less than S_n such that

$$G_n(s_n) = G_n(S_n) + K.$$

If S_n is not a unique minimum, we choose the smallest such value. In the proof of this result, Scarf introduced the notion of a K-convex function. The notion is defined in the following

Definition. Let $K \geqq 0$, and let $f(x)$ be a differentiable function. We say that $f(x)$ is K-convex if

$$K + f(x + a) - f(x) - af'(x) \geqq 0, \quad \text{for all } a > 0 \text{ and all } x.$$

If differentiability is not assumed, then the appropriate definition of K-convexity is

$$K + f(x + a) - f(x) - a\left[\frac{f(x) - f(x - b)}{b}\right] \geqq 0,$$

$$\text{for all } a > 0, b > 0, \text{ and all } x.$$

To show that the policy (s_n, S_n) is optimal for the n-period problem, Scarf shows that $G_n(y)$ is K-convex. The function $G_n(y)$ may have many maxima and minima; however, the fact that it is K-convex implies that the oscillations are never large enough to cause a deviation from the (s_n, S_n) policy. In this proof Scarf also shows that the functions $f_n(x)$ are K-convex.

2. Some Bounds on the Sequences $\{s_n\}$ and $\{S_n\}$

In this section we shall find upper and lower bounds for the sequences $\{s_n\}$ and $\{S_n\}$, $n = 2, 3, \cdots$. It is easy to verify that

$$f_n(x) = \begin{cases} -cx + G_n(s_n) & x \leqq s_n \\ -cx + G_n(x) & s_n < x \end{cases}$$

and that each function of the sequence $\{f_n(x)\}$ is a continuous function of x for $-\infty < x < \infty$. If we assume $L(y)$ is differentiable, then

$$G'_n(y) = c + L'(y) + \alpha \int_0^\infty f'_{n-1}(y - \xi)\phi(\xi)\, d\xi. \tag{2}$$

The interchange of differentiation and integration is easily justified by the Lebesgue dominated convergence theorem. Each function $f_n(x)$ is differentiable except at the point $x = s_n$ where bounded left and right-hand derivatives exist. Clearly, we have

$$f'_n(x) = \begin{cases} -c & x < s_n \\ -c + G'_n(x) & x > s_n \end{cases} \tag{3}$$

We shall assume $G_1(x)$, which is actually convex, attains its minimum. (In the contrary event, the optimal policy is to never order.) If $p(\cdot)$ is linear, this condition obtains when $p > c$. Under these hypotheses we obtain

Lemma 1. $S_1 \leqq S_n$ for $n = 2, 3, \cdots$.

Proof: We shall prove inductively that $G'_n(x) < 0$ for $x < S_1$. The definition of S_n then gives us the result immediately. Since $G_1(x)$ is convex and attains its minimum at S_1 (the smallest value for which the minimum is obtained), $G'_1(x) < 0$ for $x < S_1$. Assume $G'_n(x) < 0$ for $x < S_1$. Then from (3) $f'_n(x) \leqq -c$ for $x < S_1$. Now using (2) we obtain $G'_{n+1}(x) \leqq G'_1(x) - \alpha c < 0$ for $x < S_1$.

Next we obtain a lower bound for the sequence $\{s_n\}$ in

Lemma 2. $2s_1 - S_1 < s_n$ for $n = 2, 3, \cdots$.

Proof: The result is obtained by proving inductively that $G'_n(x) \leqq K/-(S_1 - s_1) < 0$ for $x \leqq s_1$. The cord connecting the two points $(s_1, G_1(s_1))$ and $(S_1, G_1(S_1))$ will have slope $K/-(S_1 - s_1)$. Since $G_1(x)$ is convex, $G'_1(x) \leqq K/-(S_1 - s_1)$ for $x \leqq s_1$. Assume $G'_n(x) \leqq K/-(S_1 - s_1)$ for $x \leqq s_1$. Then since $f'_n(x) \leqq -c$ for $x \leqq s_1$ (see proof of *Lemma* 1) it follows that

$$G'_{n+1}(x) \leqq G'_1(x) - \alpha c < K/-(S_1 - s_1) \text{ for } x \leqq s_1.$$

Since from *Lemma* 1 $S_n \geqq S_1$ and by definition $G_n(s_n) = K + G_n(S_n)$, we see that $s_1 - (S_1 - s_1) < s_n$ for $n = 2, 3, \cdots$.

In [3] we showed that the sequence $\{S_n\}$ is uniformly bounded from above for the case $\alpha = 1$. The same proof will give the corresponding result for our present case, $0 < \alpha < 1$. Hence we have

Lemma 3. $S_n \leqq M < \infty$ for $n = 1, 2, \cdots$.

3. The Cost Function for the Infinite Horizon Problem

In this section we use the bounds on the sequences $\{s_n\}$ and $\{S_n\}$ together with the method of successive approximations given by Bellman, Glicksberg and Gross in [2] to obtain the convergence of the sequence $\{f_n(x)\}$. More specifically we prove

Theorem 1. *The* $\lim_{n\to\infty} f_n(x)$ *exists and the convergence is uniform for all* x *in any finite interval. The limit function* $f(x)$ *is, of course, continuous.*

Proof: Following the notation used in [2] let

$$T(y, x, f) = c(y - x) + L(y) + \alpha \int_0^\infty f(y - \xi)\phi(\xi) \, c$$

and $y_n(x)$ be the minimizing value of y, as a function of x, in equation (1). Then (1) becomes

$$f_n(x) = \min_{y \geqq x} \{T(y, x, f_{n-1})\} = T(y_n, x, f_{n-1}).$$

Using the optimality properties of y_n and y_{n+1} we obtain

$$T(y_{n+1}, x, f_n) - T(y_{n+1}, x, f_{n-1}) \leqq f_{n+1}(x) - f_n(x)$$
$$\leqq T(y_n, x, f_n) - T(y_n, x, f_{n-1})$$

or expressed more conveniently

$$|f_{n+1}(x) - f_n(x)| \leqq \max \left\{ \begin{matrix} |T(y_n, x, f_n) - T(y_n, x, f_{n-1})|, \\ |T(y_{n+1}, x, f_n) - T(y_{n+1}, x, f_{n-1})| \end{matrix} \right\}.$$

Expanding $T(\cdot, \cdot, \cdot)$ and cancelling terms yields

$$|f_{n+1}(x) - f_n(x)|$$
$$\leqq \max \left\{ \begin{matrix} \left| \alpha \int_0^\infty [f_n(y_n - \xi) - f_{n-1}(y_n - \xi)]\phi(\xi) \, d\xi \right|, \\ \left| \alpha \int_0^\infty [f_n(y_{n+1} - \xi) - f_{n-1}(y_{n+1} - \xi)]\phi(\xi) \, d\xi \right| \end{matrix} \right\}. \tag{4}$$

Now choose two positive constants A and B such that $-A \leqq 2s_1 - S_1$ and $B \geqq M$ (of Lemma 3). Then from (4) it follows that

$$\max_{-A \leqq x \leqq B} \{|f_{n+1}(x) - f_n(x)|\}$$
$$\leqq \alpha \max_{-A \leqq x \leqq B} \left\{ \int_0^\infty |f_n(x - \xi) - f_{n-1}(x - \xi)| \, \phi(\xi) \, d\xi \right\}$$

upon using the fact that $-A \leqq y_n(x) \leqq B$ for $-A \leqq x \leqq B$ and $n = 1, 2, \cdots$. From Lemma 2 we know that $s_n > -A$ for $n = 1, 2, \cdots$. Therefore each function of the sequence $\{f_n(x)\}$ is linear with slope $-c$ for $x \leqq -A$. This fact enables us to deduce that

$$\max_{-A \leqq x \leqq B} \{|f_{n+1}(x) - f_n(x)|\} \leqq \alpha \max_{-A \leqq x \leqq B} \{|f_n(x) - f_{n-1}(x)|\}.$$

Iterating this inequality we obtain

$$\max_{-A \leqq x \leqq B} \{ |f_{n+1}(x) - f_n(x)| \} \leqq \alpha^n \max_{-A \leqq x \leqq B} \{ |f_1(x)| \} \quad n = 1, 2, \cdots .$$

Hence the series $\sum_{n=0}^{\infty} [f_{n+1}(x) - f_n(x)]$ converges absolutely and uniformly for all x in the interval $-A \leqq x \leqq B$. Since $\{f_n(x)\}$ is a non-decreasing sequence in n for each fixed x and A and B may be chosen arbitrarily large, we have shown that $f_n(x)$ converges monotonely and uniformly for all x in any finite interval. The functions $f_n(x)$ are continuous and the convergence uniform, thus the limit function $f(x)$ is also continuous.

Having established the convergence of the sequence $\{f_n(x)\}$ we now prove

Theorem 2. The limit function $f(x)$ *satisfies equation* (1), *the functional equation of inventory theory.*

Proof: Since convergence to $f(\cdot)$ is monotone

$$f_n(x) = \min_{y \geqq x} \{T(y, x, f_{n-1})\} \leqq \min_{y \geqq x} \{T(y, x, f)\}.$$

Letting n tend to infinity yields

$$f(x) \leqq \min_{y \geqq x} \{T(y, x, f)\}. \tag{5}$$

On the other hand

$$f(x) \geqq \min_{y \geqq x} \{T(y, x, f_n)\}. \tag{6}$$

Now since $S_n \leqq M < \infty$ from Lemma 3, we can choose a constant N such that $N \geqq M$ and $N \geqq x$ and add it as an upper constraint on y in the minimization without changing our equation. Hence (6) becomes

$$f(x) \geqq \min_{N \geqq y \geqq x} \{T(y, x, f_n)\}.$$

The limit of the right-hand side exists since it is a bounded monotone sequence. For a fixed x $\{T(y, x, f_n)\}$ is a sequence of continuous functions of y for $N \geqq y > x$. The discontinuities of the functions $\{T(y, x, f_n)\}$, considered as a function of y, at the point $y = x$ are jumps of an amount K. A straightforward, but tedious, argument allows us to interchange the limiting and minimum operations. Thus

$$f(x) \geqq \min_{N \geqq y \geqq x} \left\{ \lim_{n \to \infty} T(y, x, f_n) \right\}.$$

The interchange of limit and integral is valid by virtue of the Lebesgue monotone convergence theorem. This yields

$$f(x) \geqq \min_{N \geqq y \geqq x} \{T(y, x, f)\} = \min_{y \geqq x} \{T(y, x, f)\}$$

and shows in conjunction with (5) that

$$f(x) = \min_{y \geqq x} \{T(y, x, f)\}. \tag{7}$$

4. Limiting Behavior of the Sequences $\{s_n\}$ and $\{S_n\}$

We consider now the sequence of functions $\{G_n(x)\}$ for $-A \leqq x \leqq B$ (A and B chosen as in Theorem 1). The convergence of this sequence gives us

Theorem 3. *The sequences* $\{s_n\}$ *and* $\{S_n\}$ *contain convergent subsequences. Every limit point of the sequence* $\{S_n\}$ *is a point at which the function*

$$G(x) = cx + L(x) + \alpha \int_0^\infty f(x - \xi)\phi(\xi)\, d\xi$$

attains its minimum. If $G(x)$ *has a unique minimum point, the sequence* $\{S_n\}$, *of course, converges. Furthermore,* $G(x)$ *is K-convex and any limit point* s *of the sequence* $\{s_n\}$ *satisfies* $G(S) + K = G(s)$, *where* S *is a limit point of the sequence* $\{S_n\}$. *The optimal ordering policy for the infinite horizon problem is of* (s, S) *type.*

Proof: Lemmas 1–3 show that the two sequences $\{S_n\}$ and $\{s_n\}$ are bounded. Thus the existence of convergent subsequences follows immediately. The Lebesgue monotone convergence theorem allows us to assert that

$$\lim_{n\to\infty} G_n(x) = G(x). \tag{8}$$

The proof of *Theorem* 1 shows that this convergence is monotone and uniform in any finite interval. Choose an $\epsilon > 0$. Since the convergence in (8) is uniform, we can find a N_ϵ such that for $n, m \geqq N_\epsilon$

$$|G_n(S_m) - G_m(S_m)| < \epsilon.$$

Using the minimizing property of S_m we obtain

$$G_n(S_m) < G_m(S_m) + \epsilon \leqq G_m(x) + \epsilon.$$

We now fix $m \geqq N_\epsilon$ and choose $n > m$. Monotone convergence of the sequence $\{G_n(x)\}$ gives

$$G_n(S_m) < G_m(x) + \epsilon \leqq G_n(x) + \epsilon.$$

Finally, letting n tend to infinity we obtain

$$G(S_m) \leqq G(x) + \epsilon.$$

Since ϵ was arbitrary, we see that any limit point of the sequence $\{S_n\}$ minimizes $G(x)$. If $G(x)$ has a unique minimum, it will be the only limit point of the sequence $\{S_n\}$ and hence the sequence will converge. Using the second definition of K-convexity and (8) we see immediately that $G(x)$ is K-convex, since it is the limit of K-convex functions. Since s_n is defined by $G_n(s_n) = G_n(S_n) + K$, any limit point s of the sequence $\{s_n\}$ will satisfy $G(s) = G(S) + K$, where S is a point at which $G(x)$ attains its minimum. The fact that $G(x)$ is K-convex guarantees that the optimal policy for the infinite horizon problem is of (s, S) type. This completes the proof of the theorem.

We have not been able to find a sufficient condition for $G(x)$ to have a unique minimum, which by the theorem will guarantee convergence of the sequence $\{S_n\}$. It seems likely that strict convexity of $L(x)$ might be such a condition, however a proof eludes us.

5. Uniqueness of $f(x)$

The uniqueness problem was studied in [2] for the case of a linear ordering cost and no backlogging of excess demand. There the method of Theorem 1 was applied to show that the limit function is the unique solution of the functional equation of inventory theory which is bounded for x in any finite interval. In the present case we have only been able to obtain the weaker result

Theorem 4.[1] *If* $F(x)$ *is a solution of equation* (7) *which is bounded in any finite interval and is such that for some* y $\inf_{x \leqq y} F(x) > -\infty$, *then* $F(x) = f(x)$.

We shall show that $\lim_{n\to\infty} f_n(x) = F(x)$ uniformly in any finite interval $[A, B]$, say. We may assume that $A \leqq 2s_1 - S_1$ and $B \geqq M$. In proving this theorem we use the following lemma.

Lemma 4. *If* $F(x)$ *satisfies the hypotheses of Theorem* 4, *then* $\bar{y}(x)$, *the minimizing value of* y *in equation* (7), *satisfies*

(a) $\bar{y}(x) \leqq D$ for all $x \leqq D$, for some $D \geqq B$, and

(b) $\bar{y}(x) = T$ for $x \leqq -C$, where $-C < T \leqq D$.

Proof: The existence of $\bar{y}(x)$ is guaranteed by the assumption that $F(x)$ is a solution of equation (7). To demonstrate (a) we observe that there are two possibilities. Either $\bar{y}(x) \leqq B$ for all $x \leqq B$ or $\bar{y}(x') > B$ for some $x' \leqq B$. In the former event (a) holds when we put $D = B$. In the latter event if we let

$$\tilde{G}(y) = cy + L(y) + \alpha \int_0^\infty F(y - \xi)\phi(\xi)\, d\xi,$$

we must have

$$F(x') = K - cx' + \tilde{G}(\bar{y}(x')),$$

where

$$\tilde{G}(\bar{y}(x')) = \min_{y \geqq x'} \tilde{G}(y).$$

For every $x \leqq \bar{y}(x')$ for which $x < \bar{y}(x)$, we have

$$\tilde{G}(\bar{y}(x)) = \min_{y \geqq x} \tilde{G}(y) \leqq \tilde{G}(\bar{y}(x'))$$

so that $\bar{y}(x) \leqq \bar{y}(x')$ for all $x \leqq \bar{y}(x')$ for which $x < \bar{y}(x)$ by virtue of the definition of $\bar{y}(x')$. Thus we may let $D = \bar{y}(x')$, thereby assuring that (a) holds in this case also.

To demonstrate (b) we first show that there exists a constant E such that $F(x) \geqq E - cx$ for $x \leqq D$. Since $F(x)$ satisfies (7), we obtain

$$\begin{aligned} F(x) &\geqq c\bar{y}(x) + L(\bar{y}(x)) - cx + \alpha \int_0^\infty F(\bar{y}(x) - \xi)\phi(\xi)\, d\xi \\ &\geqq G_1(S_1) - cx + \alpha \int_0^\infty F(\bar{y}(x) - \xi)\phi(\xi)\, d\xi. \end{aligned}$$

[1] The author is indebted to the referee for this revised statement, and its proof, of the author's original uniqueness theorem.

By hypothesis, $\inf_{y \leqq D} F(y) > -\infty$. Hence

$$F(x) \geqq G_1(S_1) + \alpha \inf_{y \leqq D} F(y) - cx = E - cx \quad \text{for } x \leqq D. \tag{9}$$

We now exploit (9) to show that there exists a constant $x_0 (\leqq A)$ such that

$$\bar{y}(x) > x \quad \text{for} \quad x \leqq x_0. \tag{10}$$

For suppose the contrary; i.e., for every $x_0 (\leqq A)$ there exists an infinite sequence of numbers $\{x_i\}$ for which $\lim_{i \to \infty} x_i = -\infty$ and $\bar{y}(x_i) = x_i \leqq x_0$, $i = 1, 2, \cdots$. In this event we have from (7) that

$$\begin{aligned} F(x_i) &= L(x_i) + \alpha \int_0^\infty F(x_i - \xi)\phi(\xi)\, d\xi \\ &\geqq L(x_i) + E - \alpha c x_i + \alpha c \mu, \end{aligned}$$

where $\mu = \int_0^\infty \xi \phi(\xi)\, d\xi$. On the other hand we also know that

$$F(x_i) \leqq K + cA - cx_i + L(A) + \alpha \int_0^\infty F(A - \xi)\phi(\xi)\, d\xi.$$

Combining these inequalities we obtain

$$[L(x_i) + cx_i] - \alpha c x_i + R \leqq 0 \quad \text{for } i = 1, 2, \cdots, \tag{11}$$

where R is a properly chosen constant. However, $\lim_{i \to \infty} -\alpha c x_i = +\infty$ and $\lim_{i \to \infty} [L(x_i) + cx_i] = +\infty$, since $G'_1(x) < 0$ for $x < S_1$. Thus we arrive at a contradiction to (11), proving the existence of a $x_0 \leqq A$ such that $\bar{y}(x) > x$ for $x \leqq x_0$.

Since $\underline{\lim}_{x \to -\infty} G_1(x) = +\infty$ and $\underline{\lim}_{x \to -\infty} F(x) = +\infty$, it follows that $\underline{\lim}_{x \to -\infty} \tilde{G}(x) = +\infty$ also. Thus, there exists a number $-C$ $(\leqq x_0)$ for which $\inf_{y \leqq -C} \tilde{G}(y) > \inf_{y \geqq -C} \tilde{G}(y)$. Hence from (a) and (10) we have $-C < \bar{y}(x) \leqq D$ for $x \leqq -C$. In fact since for $x \leqq -C$ $\min_{y \geqq x} \tilde{G}(y) = \min_{y \geqq -C} \tilde{G}(y)$, we may assume that $\bar{y}(x) = T$ for $x \leqq -C$, where $-C < T \leqq D$ and $\tilde{G}(T) = \min_{y \geqq -C} \tilde{G}(y)$. This completes the proof of (b). Observe that the restrictions placed on $\bar{y}(x)$ by this lemma permit policies in a much wider class than simply (s, S) policies.

We return now to the proof of *Theorem* 4.

Proof: Using the method and notation of *Theorem* 1 we have

$$\sup_{-C \leqq x \leqq D} \{|F(x) - f_n(x)|\} \leqq \alpha \sup_{-C \leqq x \leqq D} \left\{ \int_0^\infty |F(x - \xi) - f_{n-1}(x - \xi)| \phi(\xi)\, d\xi \right\},$$

where C and D are the constants of *Lemma* 4. We write sup instead of max here since $F(x)$ is not assumed to be continuous. We have used the fact that $-C \leqq \bar{y}(x), \bar{y}_n(x) \leqq D$ for $-C \leqq x \leqq D$. Since $\bar{y}(x) = T$ for $x \leqq -C$, $F(x)$ is linear with slope $-c$ for $x \leqq -C$. Hence we obtain

$$\sup_{-C \leqq x \leqq D} \{|F(x) - f_n(x)|\} \leqq \alpha \sup_{-C \leqq x \leqq D} \{|F(x) - f_{n-1}(x)|\}.$$

Iterating this inequality, we see that $\lim_{n\to\infty} f_n(x) = F(x)$ uniformly for x in any finite interval. Thus $F(x) = f(x)$.

6. The Case of a Time Lag in Delivery

When there is a time lag in delivery and excess demand is backlogged, Scarf has also shown in [4] that the optimal policy for a n-period problem is of (s_n, S_n) form. Following Scarf's notation let the time lag be λ periods; i.e., an order placed at the beginning of a period is delivered at the beginning of the period λ periods later. Let x be the initial inventory and x_j the stock to be delivered j periods later, where $j = 1, 2, \cdots, \lambda - 1$. Then $f_n(x, x_1, \cdots, x_{\lambda-1})$, the minimum expected cost, satisfies the functional equation

$$
\begin{aligned}
&f_n(x, x_1, \cdots, x_{\lambda-1}) \\
(12)\qquad &= \min_{z\geq 0}\left\{c(z) + L(x) + \alpha \int_0^\infty f_{n-1}(x + x_1 - \xi, x_2, \cdots, z)\phi(\xi)\, d\xi\right\}.
\end{aligned}
$$

If $L(x)$ is convex, the optimal ordering policy is characterized by the pair (s_n, S_n). When $x + x_1 + \cdots + x_{\lambda-1} > s_n$, do not order; when $x + x_1 + \cdots + x_{\lambda-1} \leqq s_n$, order $S_n - (x + x_1 + \cdots + x_{\lambda-1})$. Scarf uses an argument from [1, p. 159] to show that (12) can be written for $n \geqq \lambda$ as

$$
\begin{aligned}
f_n(x, x_1, \cdots, x_{\lambda-1}) &= L(x) + \alpha \int_0^\infty L(x + x_1 - \xi)\phi(\xi)\, d\xi \\
&+ \cdots + \alpha^{\lambda-1} \int_0^\infty \cdots \int_0^\infty L\left(x + \cdots + x_{\lambda-1} - \sum_{i=1}^{\lambda-1} \xi_i\right)\phi(\xi_1)\cdots\phi(\xi_{\lambda-1})\, d\xi_1 \cdots d\xi_{\lambda-1} \\
&+ g_n(x + x_1 + \cdots + x_{\lambda-1}),
\end{aligned}
$$

where $g_n(u)$ satisfies the functional equation

$$
\begin{aligned}
g_n(u) = \min_{z\geq 0}\Bigg\{c(z) + \alpha^\lambda \int_0^\infty \cdots \int_0^\infty L\left(u + z - \sum_{i=1}^{\lambda} \xi_i\right)\phi(\xi_1) \\
(13)\qquad \cdots \phi(\xi_\lambda)\, d\xi_1 \cdots d\xi_\lambda + \alpha \int_0^\infty g_{n-1}(u + z - \xi)\phi(\xi)\, d\xi.
\end{aligned}
$$

Clearly, equation (13) is of the same form as equation (1). Accordingly, we may apply the same analyses to (13) as we did to (1) to show the results corresponding to Section 2–5 hold also for the case of a time lag in delivery.

References

1. Arrow, K. J., S. Karlin, and H. Scarf, *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford, California, 1958.
2. Bellman, R., I. Glicksberg, and O. Gross, "On the Optimal Inventory Equation," *Management Science*, Vol. 2, No. 1 (October 1955), pp. 83–104.
3. Iglehart, D. L., "Dynamic Programming and Stationary Analyses of Inventory Problems," to appear in the first Office of Naval Research-The Institute of Management Sciences monograph on *Mathematical Methods in Logistics*.
4. Scarf, H., "The Optimality of (S,s) Policies in the Dynamic Inventory Problem," *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, California, 1960.