INVENTORY CONTROL WITH AN EXPONENTIAL UTILITY CRITERION

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A base-stock policy is shown to be optimal when a dynamic version of the "news vendor" model is optimized with respect to an exponential utility criterion.

Multiperiod stochastic inventory models have invariably been optimized with respect to risk-neutral criteria. However, a plethora of experimental data confirm that human behavior is often discrepant with risk neutrality. The exponential utility function has appealing decision-theoretic properties and it is natural to consider its use in an investigation of the effects of sensitivity to risk. Exponential utility functions have been studied in general multiperiod stochastic decision models by Howard and Matheson (1963), Porteus (1975), Eagle (1975), Jaquette (1976), Denardo and Rothblum (1979), Whittle (1981), and Chung and Sobel (1986). However, Bouakiz, (1986) which concerns a replacement model, and this paper appear to be the first results on the impact of exponential utility functions on optimal policies in structured classes of models.

The "news vendor" inventory model is a prototype of many decision problems in operations research. It addresses the balance between excess supply and excess demand. A one-period news vendor model was optimized with respect to a "target-level" criterion by Lau (1980). Risk-neutral multiperiod versions of the news vendor model have been studied extensively by Bellman, Glicksberg and Gross (1955), Veinott (1965), and others. The central results are sufficient conditions for a base-stock policy to be optimal. A base-stock policy specifies an order quantity of $(y^* - x)^+$ units when x is the inventory level (and $(a)^+$ denotes $\max\{a, 0\}$ for a scalar a). The base-stock level, y^*, may depend on various model parameters, including the period number, but not on x.

We show that a base-stock policy is optimal when a multiperiod news vendor model is optimized with an exponential utility criterion. That is, the objective is to maximize the expected utility of the present value of the time stream of net profits; the utility function is $u(z) = -\exp(-\mu z)$ with $\mu > 0$. Our model contains negative net benefits, i.e., costs; so maximizing expected utility corresponds to

$$\text{minimize } E\{\exp[\mu B(N)]\}, \qquad (0)$$

where larger values of $\mu > 0$ connote greater sensitivity to risk and $B(N)$ is the present value of costs incurred during an N-period planning horizon. Let $0 \leq \beta < 1$ be the single-period discount factor.

The finite horizon problem with a time-invariant structure is formulated in Section 1 and analyzed in Sections 2 and 3. Under assumptions that include (as a special case) linear inventory-related costs and backordering (or losing) excess demand, we show that the infinite horizon version of (0) has an optimal policy with the following structure. There is a function $y(\cdot)$ such that the base-stock levels in successive periods are $y(\mu)$, $y(\beta\mu)$, $y(\beta^2\mu)$, ... After sufficiently many periods, $y(\beta^{i-1}\mu)$ has the same numerical value as the optimal base-stock level in the corresponding risk-neutral model. Optimal risk-sensitive behavior differs from optimal risk-neutral behavior only for a limited number of periods. The requisite number of periods is an increasing function of μ, the risk sensitivity parameter.

Notation

The left-hand derivative of a convex function on R, the set of real numbers, always exists. When the

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function depends on the exponential risk parameter μ, say $L(y, \mu)$ with $L(\cdot, \mu)$ convex on R for each $\mu > 0$, we write $L'(y, \mu)$ for

$\lim_{\epsilon\downarrow 0}[L(y, \mu) - L(y - \epsilon, \mu)]/\epsilon$.

Similarly, if $M(y, \mu) = E[L(y - D, \mu)]$ for some random variable D, then $M'(y, \mu) = E[L'(y - D, \mu)]$ can be justified by the monotone convergence theorem.

1. FORMULATION

The following notation describes a periodic review inventory model in which ordered goods are delivered immediately. If excess demand is backlogged, Scarf (1960) showed that a model having a fixed delivery lag corresponds to a model with immediate delivery of ordered goods. In period n, let x_n and y_n be the respective inventory levels before and after additional goods are ordered (and delivered). Thus, $0 \leqslant y_n - x_n$ is the quantity ordered. It is assumed that: i) the demand in period n, D_n, is unknown when y_n is selected, and ii) $D, D_1, D_2, \ldots$ are independent and identically distributed nonnegative random variables.

The ordering cost in period n is $c \cdot (y_n - x_n)$ and the inventory-related cost is $g(y_n, D_n)$. For example, a model with linear holding and penalty costs corresponds to

$$g(y, d) = h \cdot (y - d)^+ + p \cdot (d - y)^+.$$

The dynamical equation is

$$x_{n+1} = v(y_n, D_n) \tag{1}$$

which encompasses backordering excess demand

$$[v(y, d) = y - d]$$

and losing excess demand

$$[v(y, d) = (y - d)^+].$$

In a model with an n-period planning horizon, the salvage value is modeled as $c \cdot v(y_n, D_n)$. If excess demand is backlogged, Veinott observes that this form corresponds to each unit of stock having a salvage value of c (each unit of excess demand must be supplied at a cost of c).

The present value of costs during the N-period planning horizon, net of salvage value, is

$$B(N) := \sum_{i=1}^{N} \beta^{i-1}[c(y_i - x_i) + g(y_i, D_i)] - \beta^N c \cdot v(y_n, D_n).$$

Substitution of (1) yields

$$B(N) = \sum_{i=1}^{N} \beta^{i-1} w(y_i, D_i) - cx_1, \tag{2}$$

where

$$w(y, d) := cy + g(y, d) - \beta cv(y, d). \tag{3}$$

We assume for each $d \geqslant 0$ that $w(\cdot, d)$ is a convex function on R and is minimized on R at d. This assumption is reasonable, as we will illustrate.

If excess demand is lost and $g(y, d)$ is consistent with linear holding and penalty costs,

$$\begin{aligned} w(y, d) &= cy + h(y - d)^+ + p(d - y)^+ - \beta c(y - d)^+ \\ &= (p + h - \beta c)(y - d)^+ - (p - c)y + pd. \end{aligned}$$

Therefore, $w'(d, d) = -(p - c)$ and $w'(d + \gamma, d) = h + c(1 - \beta)$ if $\gamma > 0$. As a result, $p \geqslant c \geqslant 0$, $h \geqslant 0$, and $\beta < 1$ imply that $w(\cdot, d)$ is convex and uniquely minimized at d.

If excess demand is backlogged and $g(y, d)$ is consistent with linear holding and penalty costs,

$$\begin{aligned} w(y, d) &= cy + h(y - d)^+ + p(d - y)^+ - \beta c(y - d) \\ &= (h + p)(y - d)^+ - [p - c(1 - \beta)]y + pd. \end{aligned}$$

Therefore, $w'(d, d) = -[p - c(1 - \beta)]$ and $w'(d + \theta, d) = h + c(1 - \beta)$ if $\theta > 0$. As a result, $p > c(1 - \beta) > 0$ and $h \geqslant 0$ imply that $w(\cdot, d)$ is convex and uniquely minimized at d.

2. FINITE-HORIZON MODEL

In this section, we prove that the N-period problem for each $N = 1, 2, \ldots$ has an optimal base-stock policy. Specifically, there is a sequence of functions $y_1(\cdot)$, $y_2(\cdot)$, $\ldots$ such that the optimal base-stock level in period i is $y_i(\beta^{i-1}\mu)$, $i = 1, \ldots, N$.

For $\mu > 0$, let

$$v_N(x, \mu) := \inf E\{\exp[\mu B(N)] \mid x_1 = x\}, \tag{4}$$

where the infimum is taken with respect to all nonanticipative decision policies. From (2),

$$v_N(x, \mu) = \exp(-c\mu x) \cdot f_N(x, \mu), \tag{5}$$

where

$$f_N(x, \mu) := \inf E\left\{\exp\left[\mu \sum_{i=1}^{N} \beta^{i-1} w(y_i, D_i)\right]\right\} \tag{6}$$

and, again, the infimum is taken with respect to all nonanticipative decision policies. It follows from (5) that a policy is optimal for (4) if and only if it is optimal for (6). We analyze (6) in this paper.

As in Jaquette (1976), and Chung and Sobel (1986), let $f_0(\cdot\,,\,\cdot) \equiv 1$ and

$$f_n(x, \mu) = \inf\{J_n(y, \mu)\colon y \geqslant x\} \tag{7}$$

$$J_n(y, \mu) := E\{\exp[\mu w(y, D)]f_{n-1}(y - D, \beta\mu)\} \tag{8}$$

for $n = 1, 2, \ldots$. The value of (6) satisfies (7) with $n = N$. We assume for all y, $n \geqslant 1$, and $\mu > 0$ that the expectation in (8) exists and is finite, and $J_n(\cdot\,, \mu)$ achieves its global minimum. Reasonable sufficient conditions for these assumptions to be valid will emerge from the analysis in Section 3.

Let $y_n(\mu)$ be the largest global minimizer of $J_n(\cdot\,, \mu)$:

$$y_n(\mu) := \sup\{z\colon J_n(z, \mu) \leqslant J_n(y, \mu) \quad \text{for all } y \in R\}. \tag{9}$$

For each finite horizon n and risk parameter $\mu > 0$, there is an optimal base-stock policy.

Theorem 1. *Suppose for each $d \geqslant 0$, that $w(\cdot\,, d)$ is a convex funtion on R and minimized at d. Then for all $n \geqslant 1$ and $\mu > 0$, $f_n(\cdot\,, \mu)$ and $J_n(\cdot\,, \mu)$ are convex functions and there exists $y_n(\mu) \geqslant 0$ such that*

$$y = \max\{x, y_n(\mu)\} \tag{10a}$$

achieves (7), *i.e.,*

$$f_n(x, \mu) = \begin{cases} J_n(y_n(\mu), \mu) & \text{if } x \leqslant y_n(\mu) \\ J_n(x, \mu) & \text{if } x > y_n(\mu) \end{cases}. \tag{10b}$$

Proof. In order to initiate an inductive proof, (8) and $f_0 \equiv 1$ yield

$$J_1(y, \mu) = E\{\exp[\mu w(y, D)]\}$$

so $\mu > 0$ implies convexity of $J_1(\cdot\,, \mu)$. Hence, (9) and (10a, b) are valid at $n = 1$. Also, $w'(0, d) \leqslant 0$ for all $d \geqslant 0$ (because $w(\cdot\,, d)$ is convex on R and minimized at d) implies that the global minimum of $J_1(\cdot\,, \mu)$ is achieved on $[0, \infty)$, i.e., $y_1(\mu) \geqslant 0$.

Suppose for some $n \geqslant 1$ and all $\mu > 0$ that $f_n(\cdot\,, \mu)$ is convex, that (10a, b) is valid, and $y_n(\mu) \geqslant 0$. From (8),

$$\begin{aligned} J'_{n+1}(y, \mu) &= E\{\exp[\mu w(y, D)] \cdot [\mu w'(y, D) f_n(y - D, \beta\mu) \\ &\qquad + f'_n(y - D, \beta\mu)]\}. \end{aligned} \tag{11}$$

For $d \geqslant 0$, let $W(y) = \mu w(y, d)$ and $F(y) = f_n(y - d, \beta\mu)$. Then $J'_{n+1}(\cdot\,, \mu)$ is nondecreasing, hence $J_{n+1}(\cdot\,, \mu)$ is convex, if

$$\exp[W(y)][W'(y)F(y) + F'(y)] \tag{12}$$

is nondecreasing in y. Let $\Theta > 0$ and

$$\begin{aligned} \Delta := {} & \exp[W(y + \Theta)] \\ & \cdot [W'(y + \Theta)F(y + \Theta) + F'(y + \Theta)] \\ & - \exp[W(y)][W'(y)F(y) + F'(y)]. \end{aligned}$$

Thus, $\Delta \geqslant 0$ would imply that (12) is nondecreasing.

Lemma 1. $\Delta \geqslant 0$.

Proof. If $W'(y + \Theta) < 0$, then convexity of $W(\cdot)$ implies $W'(y) \leqslant W'(y + \Theta) < 0$. Also,

$$F'(y + \Theta) = F'(y) = 0 \quad \text{and } F(y + \Theta) = F(y) \tag{13}$$

for the following reasons. Since $W(y) = \mu w(y, d)$ and $w(\cdot\,, d)$ is convex and minimized at d, $W'(y + \Theta) < 0$ implies $y + \Theta \leqslant d$. But $F(y + \Theta) = f_n(y + \Theta - d, \beta\mu)$ and $y_n(\beta\mu) \geqslant 0$; so (10b) implies (13). Therefore,

$$\begin{aligned} \Delta &= F(y)\{\exp[W(y + \Theta)]W'(y + \Theta) \\ &\qquad - \exp[W(y)]W'(y)\} \\ &\geqslant F(y)\exp[W(y)][W'(y + \Theta) - W'(y)] \geqslant 0. \end{aligned}$$

If $W'(y) \geqslant 0$, then $0 \leqslant W'(y) \leqslant W'(y + \Theta)$ and

$$\begin{aligned} \Delta &\geqslant F'(y + \Theta)\{\exp[W(y + \Theta] - \exp[W(y)]\} \\ &\quad + W'(y)\{\exp[W(y + \Theta)F(y + \Theta) \\ &\qquad - \exp[W(y)]F(y)\} \\ &\geqslant F'(y + \Theta)\exp[W(y)]\{\exp[\Theta W'(y)] - 1\} \\ &\quad + W'(y)\{\exp[W(y + \Theta)]F(y + \Theta) \\ &\qquad - \exp[W(y)]F(y)\} \\ &\geqslant W'(y)\{\exp[W(y + \Theta)]F(y + \Theta) \\ &\qquad - \exp[W(y)]F(y)\} \\ &\geqslant 0. \end{aligned}$$

If $W'(y) < 0 \leqslant W'(y + \Theta)$, then $F'(y) = 0$ and

$$\begin{aligned} \Delta &= \exp[W(y + \Theta)] \\ &\quad \cdot [W'(y + \Theta)F(y + \Theta) + F'(y + \Theta)] \\ &\quad - \exp[W(y)]W'(y)F(y) \\ &\geqslant \exp[W(y + \Theta)]W'(y + \Theta)F(y + \Theta) \\ &\quad - \exp[W(y)]W'(y)F(y) \\ &\geqslant \exp[W(y)]\{\exp[\Theta W'(y)]W'(y + \Theta)F(y + \Theta) \\ &\quad - \exp[W(y)]W'(y)F(y)\} \geqslant 0. \end{aligned}$$

Therefore, $J_{n+1}(\cdot\,, \mu)$ is a convex function on R. Hence, for each $\mu > 0$, (9) and (10a, b) are valid with $n+1$ replacing n, and $f_{n+1}(\cdot\,, \mu)$ is a convex function on R. The induction and the theorem's proof are completed by establishing that $J'_{n+1}(0, \mu) \leqslant 0$; so $y_{n+1}(\mu) \geqslant 0$. From (8),

$$J'_{n+1}(0, \mu) = E\{\exp[\mu w(0, D)][\mu w'(0, D) f_n(-D, \beta\mu) + f'_n(-D, \beta\mu)]\}.$$

However, $y_n(\beta\mu) \geqslant 0$ implies $f'_n(-D, \beta\mu) \equiv 0$. Also, $w'(0, d) \leqslant 0$ for all $d \geqslant 0$ because $w(\cdot\,, d)$ is convex and minimized at d. Therefore, $J'_{n+1}(0, \mu) \leqslant 0$.

3. THE INFINITE-HORIZON PROBLEM

It is apparent from (7) and (8) that the sequence of states in an infinite-horizon model is (x_1, μ), $(x_2, \beta\mu)$, $(x_3, \beta^2\mu)$, ... So the base-stock policy that corresponds to (10a) is

$$y_n = \max\{x_n, y(\beta^{n-1}\mu)\} \tag{14}$$

for an appropriate function $y(\cdot)$. In this section, we give sufficient conditions for (14) to be optimal.

The sequence of base-stock levels that corresponds to (14) is $y(\mu)$, $y(\beta\mu)$, $y(\beta^2\mu)$, ... At the end of the section we explain why (under sufficient conditions) for sufficiently large n, $y(\beta^{n-1}\mu)$ is an optimal risk-neutral base-stock level.

To establish the optimality of (14), we employ several definitions. The infinite-horizon present value of costs that corresponds to (2) is

$$B := \sum_{i=1}^{\infty} \beta^{i-1} w(y_i, D_i) - cx_1.$$

The minimal expected disutility is

$$\inf E[\exp(\mu B)] = \exp(-c\mu x) f(x, \mu), \tag{15}$$

where

$$f(x, \mu) := \inf E\left\{\exp\left[\mu \sum_{i=1}^{\infty} \beta^{i-1} w(y_i, D_i)\right]\right\}. \tag{16}$$

It is intuitive that the functions f_n and y_n that are defined in (7) and (9) converge to their infinite-horizon counterparts, i.e., $f_n(x, \mu) \to f(x, \mu)$ and $y_n(\mu) \to y(\mu)$ as $n \to \infty$. We confirm this intuition, but we only sketch proofs because the line of argument has been employed in several risk-neutral infinite-horizon models.

The principal steps are: i) the existence of a function F such that $f_n(x, \mu) \to F(x, \mu)$; ii) the convergence of $y_n(\mu)$ to $y(\mu)$; and iii) $f = F$. Recall the definition $f_0(\cdot\,, \mu) \equiv 1$ for all $\mu > 0$.

Lemma 2. *Let $q(x, \mu) := E[exp[\mu w(x, D)]\}$ and suppose that*

$$\prod_{i=1}^{\infty} q(x, \beta^{i-1}\mu) < \infty \tag{17}$$

for all $x \in R$ and $\mu > 0$, $w(d, d) \geqslant 0$, and the assumptions of Theorem 1 are valid. Then $f_n(x, \mu) \leqslant f_{n+1}(x, \mu)$ for all n, x, and μ, and there exists

$$F(x, \mu) := \lim_{n\to\infty} f_n(x, \mu). \tag{18}$$

Proof. Pointwise monotonicity of f_n can be established inductively with $f_0 \equiv 1$, (7), (8), and $w(d, d) \geqslant 0$ for $d \geqslant 0$. The monotone sequence $f_0(x, \mu) \leqslant f_1(x, \mu) \leqslant \ldots$ is bounded by the expected utility of the base-stock policy which specifies $y_n = x_1$ for all n. So (7), (8), and (17) yield

$$\begin{aligned} f_n(x, \mu) &= \inf E\{\exp[\mu w(y, D)] f_{n-1}(y-D, y-x) : y \geqslant x\} \\ &\leqslant E\{\exp[\mu w(x, D)] f_{n-1}(x-D, \beta\mu)\} \\ &\leqslant f_{n-1}(x, \beta\mu) q(x, \mu) \leqslant f_{n-2}(x, \beta^2\mu) q(x, \beta\mu) q(x, \beta) \\ &\leqslant \ldots \leqslant \prod_{i=1}^{n} q(x, \beta^{i-1}\mu) \end{aligned}$$

because (10b) and the convexity of $f_n(\cdot\,, \beta)$ imply monotonicity of $f_n(\cdot\,, \mu)$. Thus, (17) implies that the monotone sequence is bounded; so there exists a limit.

Assumption 17 is not particularly restrictive. For example, if D is a bounded random variable, say $P\{0 \leqslant D \leqslant m\} = 1$, then

$$\prod_{i=1}^{\infty} q(x, \beta^{i-1}\mu) = E\left[\exp\left(\mu \sum_{i=1}^{\infty} \beta^{i-1} D_i\right)\right] \leqslant \exp\left(\mu \sum_{i=1}^{\infty} \beta^{i-1} m\right) = \exp[\mu m/(1-\beta)] < \infty.$$

The next result establishes convergence in policy and employs the definitions:

$$J(y, \mu) := E\{\exp[\mu w(y, D)] F(y-D, \beta\mu)\}$$

$$y(\mu) := \sup\{y : J'(y, \mu) \leqslant 0\}.$$

Lemma 3. *Under the assumptions of Theorem 1 and Lemma 2, if*

$$E\{w'(y, D)\exp[\mu w(y, D)]\} < 0, \quad y < y(\mu) \tag{19}$$

then $y(\mu)$ minimizes $J(\cdot\,, \mu)$ on R, and

$$y(\mu) = \lim_{n\to\infty} y_n(\mu). \tag{20}$$

Proof. The claims follow from straightforwardly modifying Theorem 8-15 in Heyman and Sobel (1984) to refer to sequential decision processes with multiplicative rewards instead of additive rewards. However, to invoke this theorem, $y_n(\mu) < \infty$ for each $\mu > 0$ and n are necessary. A sufficient condition is $J_n'(y, \mu) > 0$ for sufficiently large y.

Let $\Phi(\cdot)$ be the distribution function of D, and let

$$A(y, z) := [\mu w'(y, z) f_{n-1}(y - z, \beta\mu) + f_{n-1}'(y - z, \beta\mu)]\exp[\mu w(y, z)].$$

From (8), (11), $y(\mu) \geqslant 0$, (10b), and $w(\cdot\,, d)$ convex on R and minimized at d,

$$\begin{aligned} J_n'(y, \mu) &= \int_0^y A(y,z)\,d\Phi(z) + \int_y^\infty A(y,z)\,d\Phi(z) \\ &\geqslant f_{n-1}(0, \beta\mu)\mu \int_0^y w'(y,z)\exp[\mu w(y,z)] \quad d\Phi(z). \end{aligned} \quad (21)$$

Since $w'(y, z) \leqslant (\geqslant)\ 0$ if $y \leqslant (\geqslant)\ z$, (19) implies that (21) is positive for sufficiently large y.

The optimality of $y_n = \max\{x_n, y(\beta^{n-1}\mu)\}$ follows from Lemmas 2 and 3 if $f = F$. A sufficient condition for $f = F$ is that y_n be confined to a compact set (for each x). That is, we now assume that $x_n \leqslant y_n \leqslant x_n + M$, where M is an upper bound on the quantity ordered.

Lemma 4. *If the constraints $x_n \leqslant y_n \leqslant x_n + M$, $n = 1, 2, \ldots$. are imposed, and the assumptions of Lemma 3 are valid, then $f = F$ and*

$$f(x, \mu) = \inf\{J(y, \mu): x \leqslant y \leqslant x + M\}.$$

Proof. Modify Theorems 8-14 and 8-15 in Heyman and Sobel to refer to sequential decision processes with multiplicative rewards instead of additive rewards.

The principal result in this section is that a base-stock inventory policy is optimal for the infinite-horizon model. The next result is a consequence of Lemmas 2, 3, and 4.

Theorem 2. *Suppose that $x_n \leqslant y_n \leqslant x_n + M$ is imposed for all n, (17) and (19) are valid, and for each d, $w(\cdot\,, d)$ is convex on R and minimized at d, and $w(d, d) \geqslant 0$. Then*

$$y_n = \max\{x_n, y(\beta^{n-1}\mu)\} \quad n = 1, 2, \ldots \quad (22)$$

is optimal, i.e.,

$$f(x, \mu) = \exp(c\mu x)E[\exp(\mu B) \mid (22) \text{ and } x_1 = x].$$

Jaquette shows that a stationary policy is "ultimately" stationary in every finite discounted Markov decision process (MDP) with the exponential utility criterion. Our model is a finite MDP if x_1 is an integer, D is integer-valued and bounded, and y_n is confined to $\{(x_n)^+, (x_n)^+ + 1, \ldots, M\}$ for each n. It follows from $y_n(\mu) \geqslant 0$ for all n and μ that $(x)^+ \leqslant y$ is without loss of optimality.

Theorem 3. *If $y_n \in \{(x_n)^+, (x_n)^+ + 1, \ldots, M\}$ for all n, D is an integer-valued bounded random variable, x is an integer and $x \leqslant M$, (17) and (19) are valid, and for each d, $w(\cdot\,, d)$ is convex on R and minimized at d, and $w(d, d) \geqslant 0$, then there exists N such that $y(\beta^N\mu)$ is constant for all $n > N$.*

Proof. See Jaquette (1976).

4. CONCLUSION

We have shown that the optimal ordering policy is given by a sequence of critical numbers if the ordering costs are linear and the penalty and holding costs are convex. These critical numbers depend on the time horizon, the discount rate, and the parameter μ. The infinite-horizon policy is ultimately stationary and approaches the risk-neutral policy as n gets large.

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