

**COMPUTING OPTIMAL (s, S) INVENTORY POLICIES**

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**ARTHUR F. VEINOTT, JR. AND HARVEY M. WAGNER**

**TECHNICAL REPORT NO. 1**

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# COMPUTING OPTIMAL $(s,S)$ INVENTORY POLICIES

Arthur F. Veinott, Jr.\* and Harvey M. Wagner\*\*  
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## Summary

A complete computational approach for finding optimal  $(s,S)$  inventory policies is developed. The method is an efficient and unified approach for all values of the model parameters, including a non-negative set-up cost, a discount factor  $0 \leq \alpha \leq 1$ , and a lead time. The method is derived from renewal theory and stationary analysis, generalized to permit the unit interval range of values for  $\alpha$ . Careful attention is given to the problem associated with specifying a starting condition (when  $\alpha < 1$ ); a resolution is found that guarantees an  $(s,S)$  policy optimal for all starting conditions is produced by the computations. The special case of linear holding and penalty costs is treated in detail. In the final section, a model in which there is a minimum guaranteed demand in each period is studied, and a simplified method of solution is developed.

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## 1. Introduction

We consider the familiar dynamic inventory model in which demands for a single product in each of an unbounded sequence of periods are independent, identically distributed discrete random variables. There is a constant lead time, a discount factor  $0 \leq \alpha \leq 1$ , a fixed set-up cost, a linear purchase cost, a convex expected holding and penalty cost function, and (usually) total backlogging of unfilled demand. The objective is to choose from among the class of all ordering policies one which minimizes the long run "equivalent" average cost per period; such a policy is called optimal. Under these assumptions, it is known [9, 12, 13a, 15] that there is an optimal policy of the  $(s, S)$  type.

Our principal objective in this paper is to develop an efficient computing procedure for finding an optimal  $(s, S)$  policy. We formulate the problem in detail and discuss several possible computational approaches in Section 2. In Section 3 we develop formulas for computing the equivalent average cost per period,  $a_\alpha(x|s, S)$ , associated with a given  $(s, S)$  policy and a fixed initial amount  $x$  of stock on hand and on order. The formulas are developed in two closely related ways--by the aid of renewal theory [1c, 2, 3, 7] and by means of a generalized (to the case  $\alpha < 1$ ) form of stationary analysis [1b, 2, 7, 14]. The formulas are new for the case  $\alpha < 1$ ,  $x \geq s$ . We construct an algorithm

for searching the  $(s,S)$  policies to find one that simultaneously minimizes  $a_\alpha(x|s,S)$  for all  $x$  in Section 4. The procedure involves two steps. The first is to find the collection  $\mathcal{J}$  of all  $(s,S)$  policies that minimize  $a_\alpha(x|s,S)$  for some suitably small and fixed value of  $x$ . The second is to search the policies in  $\mathcal{J}$  to find one that minimizes  $a_\alpha(x|s,S)$  for every  $x$ . This second step is necessary when  $\alpha < 1$  because some of the policies in  $\mathcal{J}$  may not minimize  $a_\alpha(x|s,S)$  for every  $x$ , as was first recognized in [2]. An example demonstrating this phenomenon is given in Section 5, where a special case in which there is a minimal guaranteed demand is studied. This case is of independent interest because the computations required to find an optimal  $(s,S)$  policy then reduce to that needed for a one period model.

Several types of ordering rules that are characterized by a small number of parameters have been proposed in the inventory theory literature as alternatives to the  $(s,S)$  policy. These papers typically suggest selecting from among a specified class of policies one that performs best. Although this is a meaningful approach if  $\alpha = 1$ , it is ordinarily impossible when  $\alpha < 1$ . The reason is that when  $\alpha < 1$  the policy that is best will in general depend upon the particular initial amount of stock on hand and on order. Consequently, in sharp contrast to the  $(s,S)$  case, no one of these alternative policies will in general be uniformly best for all initial conditions.

## 2. Model Formulation

### Basic Definitions

We assume that the demands  $\xi_1, \xi_2, \dots$  for a single item in

periods  $1, 2, \dots$  are independent, non-negative, discrete random variables with common probability distribution  $\varphi$ ,  $\varphi(k) = \Pr(\xi_t = k)$ , ( $k = 0, 1, \dots$ ;  $t = 1, 2, \dots$ ). At the beginning of each period the stock on hand and on order is reviewed. An order may then be placed for any positive integral amount of stock. An order placed in period  $t (= 1, 2, \dots)$  is delivered at the beginning of period  $t + \lambda$ , where  $\lambda$  is a known non-negative integer. When the demand during a period exceeds the inventory on hand after receipt of incoming orders in the period, the excess demand is backlogged until it is subsequently filled by a delivery.

In Figure 1 we exhibit three successive periods, and indicate the precise sequence of events. We utilize the following notation for the random variables of interest:

$X_t$  = inventory on hand plus on order prior to placing any order in period  $t$

$Y_t$  = inventory on hand plus on order subsequent to placing any order but before the demand occurs in period  $t$

$W_t$  = inventory on hand after period  $t$  demand occurs.

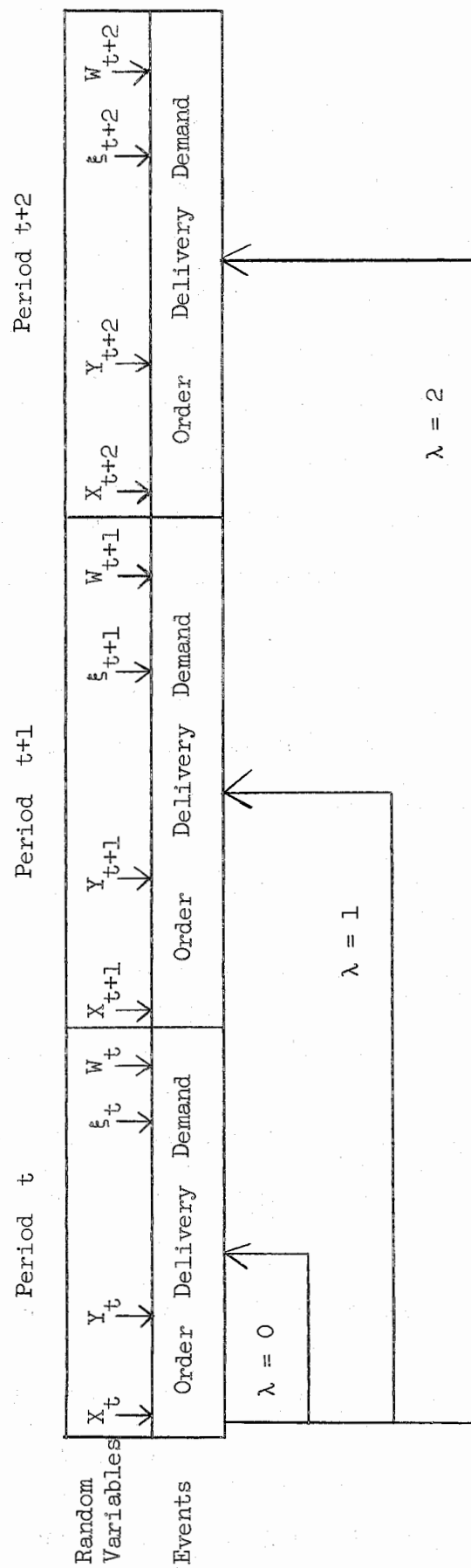
Since full backlogging of unfilled demand is assumed, each of the above random variables may take on negative values indicating the existence of a backlog.

The economic parameters employed are described as

$K\delta(z) + c \cdot z$  = ordering cost incurred at the time of delivery, i.e., in period  $t + \lambda$ , of an order for  $z (\geq 0)$  units placed in period  $t$ , where  $K \geq 0$ ,  $c \geq 0$ , and  $\delta(0) = 0$ ,  $\delta(z) = 1$  if  $z > 0$ .

$L_{t+\lambda}$  = holding and penalty cost incurred in period  $t + \lambda$ .

FIGURE 1



$$L(y) = E(L_{t+\lambda} | Y_t = y).$$

$\alpha$  = single period discount factor,  $0 \leq \alpha \leq 1$ .

We assume that  $L(y)$  is convex and that  $L(y)$  and  $c \cdot y + L(y)$  approach  $\infty$  as  $|y| \rightarrow \infty$ .

A few words of explanation about these economic parameters are in order. As defined above, all the economic consequences immediately attributable to an ordering decision in period  $t$  are to be measured in period  $t+\lambda$ . If, instead, the entire ordering cost is actually incurred in period  $t$ , and if  $\lambda > 0$  and  $\alpha > 0$ , then  $K\delta(z) + c \cdot z$  should be replaced by  $\alpha^{-\lambda}[K\delta(z) + c \cdot z]$ . Of course, other timing assumptions are also possible, and should be handled by discounting the cost of placing an order in period  $t$  to period  $t+\lambda$ . Generally  $\alpha = 1/(1+i)$  where  $i$  is the (decimal) interest rate per period; when  $\alpha = 0$ , the model is equivalent to a single period inventory problem.

Frequently the holding and penalty costs for period  $t+\lambda$  are postulated as

$$\begin{aligned} & hW_{t+\lambda} \quad \text{for } W_{t+\lambda} \geq 0 \\ & -pW_{t+\lambda} \quad \text{for } W_{t+\lambda} < 0, \end{aligned}$$

where  $h > 0$  and  $p > 0$ . In this instance, since  $W_{t+\lambda} = Y_t - \xi_t - \dots - \xi_{t+\lambda}$ ,

$$(1) \quad L(y) = \begin{cases} h \sum_{k=0}^y (y-k) \phi^{\lambda+1}(k) + p \sum_{k=y+1}^{\infty} (k-y) \phi^{\lambda+1}(k) & y \geq 1 \\ p \sum_{k=0}^{\infty} (k-y) \phi^{\lambda+1}(k) & y \leq 0, \end{cases}$$



## Inclusion of the Ordering Cost in the Holding and Penalty Cost Function

Before beginning the formal analysis, it is convenient to make a useful simplification which permits including the cost  $c$  with the holding and penalty cost function.

Suppose that  $0 \leq \alpha < 1$ , that  $X_1 = x$ , and that the  $Y_t$  are uniformly bounded [as would be the case when an  $(s, S)$  policy is followed]. Then the total holding, penalty, and ordering cost (excluding set-ups) discounted to the beginning of period  $\lambda+1$  is

$$\begin{aligned} & \sum_{t=1}^{\infty} \alpha^{t-1} L_{t+\lambda} + c \sum_{t=1}^{\infty} \alpha^{t-1} (Y_t - X_t) \\ &= \sum_{t=1}^{\infty} \alpha^{t-1} L_{t+\lambda} + c(Y_1 - x) + c \sum_{t=2}^{\infty} \alpha^{t-1} (Y_t - Y_{t-1} + \xi_{t-1}) \\ &= \sum_{t=1}^{\infty} \alpha^{t-1} [L_{t+\lambda} + (1-\alpha)cY_t] + c \left[ \sum_{t=1}^{\infty} \alpha^t \xi_t - x \right]. \end{aligned}$$

Since the second bracketed term above is independent of the ordering policy followed, it suffices to consider only the first bracketed term. Thus it is convenient to work with  $L_{t+\lambda} + (1-\alpha)cY_t$  as a composite of the cost due to  $c$  and the holding and penalty cost in period  $t+\lambda$ .

Now

$$(3) \quad E[L_{t+\lambda} + (1-\alpha)cY_t | Y_t = y] = L(y) + (1-\alpha)cy \equiv G_{\alpha}(y).$$

Therefore, hereafter we consider the revised model with the cost function  $G_{\alpha}(y)$ , which we will refer to as the conditional expected holding and penalty cost function, and the fixed set-up cost  $K$ .<sup>2/</sup>

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<sup>2/</sup> The simplification in this subsection is based on a suggestion by Martin Beckman. The possibility of such a simplification was noted in his paper [3].

When  $\alpha = 1$  the average cost per period (excluding set-ups) is

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \sum_{t=1}^n L_{t+\lambda} + \frac{1}{n} c \sum_{t=1}^n (Y_t - X_t) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \sum_{t=1}^n L_{t+\lambda} + \frac{1}{n} c(Y_1 - x) + \frac{c}{n} \sum_{t=2}^n (Y_t - Y_{t-1} + \xi_{t-1}) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \sum_{t=1}^n L_{t+\lambda} + \frac{c}{n} \sum_{t=1}^n \xi_t \right] \end{aligned}$$

so that we may use (3) in this event too. Of course, here  $G_1(y) = L(y)$ , so that the revised model in this case simply involves omission of the ordering cost.

### 3. Equivalent Average Cost Per Period

#### Renewal Approach

The key relation to be derived in this section is a formula yielding the equivalent average cost per period  $a_\alpha(x|s, S)$ , ( $0 \leq \alpha \leq 1$ ) resulting from a given  $(s, S)$  policy [in terms of  $S$  and  $D \equiv S-s$ ] and starting condition  $X_1 = x$ . Two closely related mathematical approaches for this purpose are renewal theory [1c, 2, 3] and generalized stationary analysis [1b, 2, 14]. We have found that the renewal approach seems to provide the required results with less expository effort, and so we will use that method to derive the relationships of interest. On the other hand, a generalized stationary analysis seemingly has merit in providing more readily comprehensible interpretations of the results. Therefore, in the next subsection we will provide an explanatory bridge between the two views. The reader who is mainly interested in how to use the formulas developed in this section can skip immediately to Section 4.

Let  $T(d)$  be the first period in which the cumulative demand exceeds the non-negative integer  $d$ ; that is,  $T(d)$  is the smallest integer for which  $\xi_1 + \dots + \xi_{T(d)} > d$ . If  $X_1 = S$  and  $d = D$ , then period  $t = T(D) + 1$  will be the first time  $X_t < s$ , and consequently the first order will be placed at the start of that period.

Assume  $X_1 = x$  and no new order is placed until the period after the cumulative demand exceeds  $d$ , that is, no new order is placed in periods  $1, 2, \dots, T(d)$ . We want to obtain the conditional expected holding and penalty cost incurred during the interval  $\lambda+1, \dots, \lambda+T(d)$  given that  $X_1 = x$  and  $Y_i = X_i$ ,  $i = 1, \dots, T(d)$ . These costs are discounted to the beginning of period  $\lambda+1$ . In taking this expectation, we must account for  $T(d)$  being a random variable. Since no order is placed in period 1,  $G_\alpha(x)$  is the contribution to this expectation in period  $\lambda+1$ . If  $\xi_1 \leq d$ , no order is placed in period 2. Then  $X_2 = Y_2 = x - \xi_1$ , and the contribution to this expectation in period  $\lambda+2$  is the discounted average of  $G_\alpha(x - \xi_1)$ , namely,  $\alpha \sum_{k=0}^d G_\alpha(x-k) \phi(k)$ . In general, if  $\xi_1 + \dots + \xi_i \leq d$ , no order is placed in period  $i+1$ . Then  $X_{i+1} = Y_{i+1} = x - \xi_1 - \dots - \xi_i$ , and the contribution to this expectation in period  $\lambda+i+1$  is the discounted average of

$G_\alpha(x - \xi_1 - \dots - \xi_i)$ , namely,  $\alpha^i \sum_{k=0}^d G_\alpha(x-k) \phi^i(k)$ . Therefore,

letting  $L_\alpha(x, d)$  be the desired conditional expected holding and penalty cost during the interval  $\lambda+1, \dots, \lambda+T(d)$ , we have

$$(4) \quad L_\alpha(x, d) = G_\alpha(x) + \sum_{i=1}^{\infty} \sum_{k=0}^d \alpha^i G_\alpha(x-k) \phi^i(k) \quad (x = \dots, -1, 0, 1, \dots; d = 0, 1, \dots).$$

We define

$$(5) \quad m_{\alpha}(k) \equiv \sum_{i=1}^{\infty} \alpha^i \phi^i(k) \quad (k = 0, 1, \dots)$$

$$(6) \quad M_{\alpha}(k) \equiv \sum_{i=1}^{\infty} \alpha^i \phi^i(k) = \sum_{j=0}^k m_{\alpha}(j) \quad (k = 0, 1, \dots)$$

We call  $M_{\alpha}$  the discount renewal function. We can interpret  $M_{\alpha}(k)$  as the discounted number of periods before the cumulative demand exceeds  $k$ .<sup>3/</sup> It is known [2] that  $M_{\alpha}(k)$  [and hence  $m_{\alpha}(k)$ ] is finite if  $\alpha\phi(0) < 1$ , an assumption that we impose to avoid trivialities. (For completeness we prove this finiteness property in Section 1 of the Appendix.) It follows that the series (4) converges absolutely. If we interchange the order of summation in (4) we find

$$(7) \quad L_{\alpha}(x, d) = G_{\alpha}(x) + \sum_{j=0}^d G_{\alpha}(x-j) m_{\alpha}(j) \quad (x = \dots, -1, 0, 1, \dots; d = 0, 1, \dots)$$

This formula is useful because, as we shall see in Section 4, there is a simple recursive procedure for calculating  $m_{\alpha}(k)$ .

Continuing the analysis under the assumption that an order occurs in the period  $T(d) + 1$ , we proceed to find the expected set-up cost of this order. The order is delivered in period  $T(d) + 1 + \lambda$ , and the appropriate set-up cost discounted to the beginning of period  $\lambda + 1$  is

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<sup>3/</sup> This interpretation may be justified as follows. Let  $I(t) = \alpha^t$  if  $\xi_1 + \dots + \xi_t \leq k$  and let  $I(t) = 0$  otherwise. Then

$$E\left[\sum_{t=1}^{\infty} I(t)\right] = \sum_{t=1}^{\infty} E[I(t)] = \sum_{t=1}^{\infty} \alpha^t \phi^t(k) = M_{\alpha}(k)$$

$K\alpha^{T(d)}$ . Therefore, we seek the expected value of  $\alpha^{T(d)}$ . Let  $\tau_\alpha(d) \equiv E[\alpha^{T(d)}]$ . Now

$$\Pr[T(d) = i] = \Phi^{i-1}(d) - \Phi^i(d), \quad (i = 1, 2, \dots; d = 0, 1, \dots)$$

where  $\Phi^0(d) \equiv 1$ .

Thus by (6) we obtain the result implicit in [2]

$$\begin{aligned} \tau_\alpha(d) &= \sum_{i=1}^{\infty} \alpha^i [\Phi^{i-1}(d) - \Phi^i(d)] \\ (8) \quad &= [\alpha - (1-\alpha) M_\alpha(d)] \quad (d = 0, 1, \dots) \end{aligned}$$

Assume  $\alpha < 1$  and that a fixed  $(s, S)$  policy is followed. Let  $f(x|s, S)$  denote the total expected cost incurred in periods  $\lambda+1, \lambda+2, \dots$  discounted to the beginning of period  $\lambda+1$  when  $X_\lambda = x$ . Where no ambiguity will result, we write  $f(x)$  for brevity instead of  $f(x|s, S)$ . If  $X_\lambda = S$ , then no order will be placed in periods  $1, 2, \dots, T(D)$ , where  $D \equiv S-s$ . In the following period the  $(s, S)$  policy ensures that  $Y_{T(D)+1} = S$ , so that a renewal of the process takes place. Making use of this observation and the results (7) and (8) above, we have<sup>4/</sup>

$$f(S) = L_\alpha(S, D) + K\tau_\alpha(D) + f(S) \tau_\alpha(D),$$

so that

$$(9) \quad f(S) = \frac{L_\alpha(S, D) + K\tau_\alpha(D)}{1 - \tau_\alpha(D)}.$$

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<sup>4/</sup> This approach was suggested by similar arguments on related problems in [1c].

It follows immediately that if  $X_1 = x < s$ , then

$$f(x) = K + f(S) \quad x < s .$$

Now if  $X_1 = x \geq s$ , no order will be placed in periods 1, 2, ...,  $T(x-s)$  and  $Y_{T(x-s)+1} = S$ , at which time a renewal occurs. Therefore

$$f(x) = L_\alpha(x, x-s) + K\tau_\alpha(x-s) + f(S)\tau_\alpha(x-s) \quad x \geq s .$$

Substituting (9) in the above formulas for  $f(x)$  gives

$$(10) \quad f(x) = \begin{cases} \frac{L_\alpha(S, D) + K}{1 - \tau_\alpha(D)} & x < s \\ L_\alpha(x, x-s) + \left[ \frac{L_\alpha(S, D) + K}{1 - \tau_\alpha(D)} \right] \tau_\alpha(x-s) & x \geq s . \end{cases}$$

This formula was first given in essence in [2] for  $x < s$ , although the derivation there is different.

It is convenient to convert the total discounted cost  $f(x|s, S)$  to an equivalent cost per period  $a_\alpha(x|s, S)$ , that is,  $a_\alpha(x|s, S)$  is chosen so that  $f(x|s, S) \equiv \sum_{i=1}^{\infty} \alpha^{i-1} a_\alpha(x|s, S)$ . Thus  $a_\alpha(x|s, S) = (1-\alpha) f(x|s, S)$ . Again we write  $a_\alpha(x)$  for brevity instead of  $a_\alpha(x|s, S)$ . From (8) we have

$$(1-\alpha)[1 - \tau_\alpha(D)]^{-1} = [1 + M_\alpha(D)]^{-1} .$$

Thus we obtain from (10)

$$(11) \quad a_{\alpha}(x) = \begin{cases} \frac{L_{\alpha}(S,D) + K}{1 + M_{\alpha}(D)} & x < s \\ (1-\alpha) L_{\alpha}(x, x-s) + \left[ \frac{L_{\alpha}(S,D) + K}{1 + M_{\alpha}(D)} \right] \tau_{\alpha}(x-s) & x \geq s . \end{cases}$$

Observe that if we define  $a_1(x) = \lim_{\alpha \rightarrow 1} a_{\alpha}(x)$ , then since  $\tau_1(x-s) = 1$ , we conclude from (11) that

$$(12) \quad a_1(x) = \frac{L_1(S,D) + K}{1 + M_1(D)} \equiv a$$

which is completely independent of  $x$ . As was noted in [2], (12) gives the long run average cost per period.

#### Generalized Stationary Analysis Approach<sup>5/</sup>

The above derivations can also be accomplished by means of stationary probability analysis [1b, 2, 13a, 14] appropriately generalized to encompass  $0 \leq \alpha \leq 1$ . Without going into all the details of the approach, we outline the fundamental concepts and their relation to the renewal quantities previously established.

For brevity we restrict our discussion to the case where the value of  $X_1$  is less than  $s$ , and consequently  $X_t \leq S$  for all  $t$ . Define<sup>6/</sup>

$$p_{X_t}(x) \equiv \Pr(X_t = x) .$$

<sup>5/</sup> The development of the algorithm for finding an optimal  $(s,S)$  policy in Section 4 is not dependent upon this subsection.

<sup>6/</sup> Note that throughout this subsection, we use the symbol  $x$  to represent a value of  $X_t$  and not just the specific value of  $X_1$ , unless we state otherwise.

For  $0 \leq \alpha < 1$ , let

$$\begin{aligned}
 \pi_X(x) &\equiv \sum_{t=1}^{\infty} \alpha^{t-1} p_{X_t}(x) \\
 (13) \quad &= p_{X_1}(x) + \alpha \sum_{t=1}^{\infty} \alpha^{t-1} p_{X_{t+1}}(x) .
 \end{aligned}$$

We can interpret  $\pi_X(x)$  as the discounted expected number of periods  $t$  in which  $X_t = x$  over the infinite horizon. It is easily verified that  $\sum_{x=-\infty}^S \pi_X(x) = (1-\alpha)^{-1}$ , which can be interpreted as the total discounted number of periods. We normalize by letting

$$(14) \quad P_X(x) = (1-\alpha) \pi_X(x) .$$

$P_X(x)$  is the discounted fraction of the time that  $X_t = x$ .

It follows from the transition law for  $X_t$  that

$$(15) \quad p_{X_{t+1}}(x) = \begin{cases} \varphi(S-x) \sum_{j=-\infty}^{s-1} p_{X_t}(j) + \sum_{j=s}^S \varphi(j-x) p_{X_t}(j) & x < s , \\ \varphi(S-x) \sum_{j=-\infty}^{s-1} p_{X_t}(j) + \sum_{j=x}^S \varphi(j-x) p_{X_t}(j) & s \leq x \leq S . \end{cases}$$

Substituting (15) into (13), multiplying (13) by  $(1-\alpha)$ , and noting that  $p_{X_1}(x) = 0$  for  $x \geq s$ , we obtain

$$(16) \quad P_X(x) = \begin{cases} (1-\alpha)p_{X_1}(x) + \alpha \left[ \varphi(S-x) \sum_{j=-\infty}^{s-1} P_X(j) + \sum_{j=s}^S \varphi(j-x) P_X(j) \right] & x < s , \\ \alpha \left[ \varphi(S-x) \sum_{j=-\infty}^{s-1} P_X(j) + \sum_{j=x}^S \varphi(j-x) P_X(j) \right] & s \leq x \leq S . \end{cases}$$

This equation is also valid for  $\alpha = 1$ ; in that event the  $P_X(x)$  are ordinary stationary probabilities.



Let

$$\sigma = \sum_{j=-\infty}^{s-1} P_X(j) = \text{discounted fraction of the periods in which an order is placed.}$$

Let  $k = S - x$  for  $s \leq x \leq S$ , that is,  $k = 0, 1, \dots, D$ . Then

(16) for  $s \leq x \leq S$  can be written as

$$(17) \quad \frac{P_X(S-k)}{\sigma} = \alpha \left\{ \varphi(k) + \sum_{j=0}^k \varphi(k-j) \left[ \frac{P_X(S-j)}{\sigma} \right] \right\} \quad (k = 0, 1, \dots, D).$$

It is shown in Section 1 of the Appendix that (17) is a renewal equation, and

$$\frac{P_X(S-k)}{\sigma} = m_\alpha(k),$$

where  $m_\alpha(k)$  is defined by (5), so that

$$(18) \quad P_X(S-k) = m_\alpha(k) \sigma \quad (k = 0, 1, \dots, D).$$

Now by the definition of  $\sigma$  and the fact that the  $P_X(x)$  sum to 1

$$\sum_{j=0}^D P_X(S-j) = 1 - \sigma.$$

Therefore from (18)

$$\sigma = \left[ 1 + \sum_{j=0}^D m_\alpha(j) \right]^{-1} = [1 + M_\alpha(D)]^{-1}.$$

Substituting this formula into (18) and then (18) into (16) yields, for

$X_1$  less than  $s$ ,

$$(19) \quad P_X(x) = \begin{cases} (1-\alpha)P_{X_1}(x) + \frac{\alpha \left[ \phi(S-x) + \sum_{j=0}^D \phi(S-j-x)m_\alpha(j) \right]}{1 + M_\alpha(D)}, & x < s \\ \frac{m_\alpha(S-x)}{1 + M_\alpha(D)} & s \leq x \leq S. \end{cases}$$

This formula was established in [1b] for  $\alpha = 1$ .

Letting  $X_1 = x < s$  and using (19), we can write  $a_\alpha(x)$  in (11) as

$$(20) \quad a_\alpha(x) = G_\alpha(S)\sigma + \sum_{j=s}^S G_\alpha(j) P_X(j) + K\sigma.$$

When the cost function  $L(y)$  has the form (1), we may write (20) in a more intuitive (but computationally less convenient) alternative form. To this end, let

$$p_{W_t}(w) = \Pr(W_t = w) \quad \text{and} \quad P_W(w) = (1-\alpha) \sum_{t=1}^{\infty} \alpha^{t-1} p_{W_{t+\lambda}}(w).$$

Clearly  $P_W(w)$  is the discounted fraction of the periods in which the end of period inventory on hand is  $w$ . By elementary algebraic manipulation it can be demonstrated that for  $X_1 = x < s$

$$a_\alpha(x) + \alpha c \mu - (1-\alpha)cx = K\sigma + c \sum_{k=-\infty}^{s-1} (S-k)P_X(k) + h \sum_{w=0}^S wP_W(w) - p \sum_{w=-\infty}^{-1} wP_W(w) \quad x < s.$$

#### 4. Computation of an Optimal $(s, S)$ Policy

We now develop an algorithm for finding an optimal  $(s, S)$  policy, say  $(s^*, S^*)$ . We emphasize at the outset that  $(s^*, S^*)$  simultaneously minimizes the (equivalent) average cost per period  $a_\alpha(x|s, S)$

[see (11) above] over the class of all  $(s, S)$  policies for every fixed (integer) value of  $x$ . We shall carefully distinguish between  $(s^*, S^*)$  and an  $(s, S)$  policy that is optimal for only certain values of  $x$ . Specifically if  $\bar{X}$  is a set of integers, we say that  $(s', S')$  is optimal for  $\bar{X}$  if for each fixed  $x \in \bar{X}$ ,  $(s', S')$  minimizes  $a_\alpha(x|s, S)$  over the class of all  $(s, S)$  policies.

Unfortunately  $a_\alpha(x|s, S)$  is not a convex function of  $(s, S)$ , nor, in general, are all of its local minima [with respect to  $(s, S)$ ] global minima. However, from (11) it is clear that for  $x < s = S - D$ ,  $a_\alpha(x|S - D, S)$  equals a function of  $S$  and  $D$  only, say  $\mathcal{L}_\alpha(S, D)$ . In addition, since  $G_\alpha(\cdot)$  is convex,  $a_\alpha(x|S - D, S)$  is convex in  $S$  for fixed  $x$  and  $D$  satisfying  $x < S - D$ . We exploit these facts in a three step algorithm for finding  $(s^*, S^*)$ ; we first outline the algorithm briefly below and then develop it in detail.

Step i Determine bounds on  $s^*$  and  $S^*$ .

In [9, 13a] it is shown that there are easily computed numbers (integers in our case)  $\underline{s}$ ,  $S_1$ ,  $\bar{S}$  such that  $\underline{s} \leq s^*$  and  $S_1 \leq S^* \leq \bar{S}$ . The definitions of these constants are given in the first subsection below. One immediate consequence is that  $D^* \equiv S^* - s^* \leq \bar{S} - \underline{s}$ .

Step ii Find  $(s, S)$  policies optimal for  $x < \underline{s}$ .

Since  $(s^*, S^*)$  is optimal, it is certainly optimal for  $x < \underline{s}$ . Thus,  $(s^*, S^*)$  minimizes  $\mathcal{L}_\alpha(S, D)$  over the class of all  $(s, S)$  policies. In this step we find the collection  $\mathcal{S}$  of all  $(s, S)$  policies that minimize  $\mathcal{L}_\alpha(S, D)$  over the

class of  $(s, S)$  policies falling within the bounds defined in Step i. From what we have said above,  $\mathcal{J}$  contains  $(s^*, S^*)$ . Thus, each policy in  $\mathcal{J}$  is optimal for  $x < \underline{s}$ . Detailed procedures for minimizing  $\mathcal{J}_\alpha(S, D)$  are given in the second subsection below.

Step iii Choose an optimal  $(s, S)$  policy from  $\mathcal{J}$ .

When  $\alpha = 1$ , every policy in  $\mathcal{J}$  is optimal since  $a_1(x|s, S) = \mathcal{J}_1(S, D)$  for all  $x$ . However, if  $\alpha < 1$  and if  $\mathcal{J}$  contains more than one policy, some of the policies in  $\mathcal{J}$  may not be optimal. (We give a significant example of this phenomenon in Section 5.) Therefore, in this step we give methods of identifying which policies in  $\mathcal{J}$  are optimal. These techniques are described in the third subsection below.

#### Bounds on $s^*$ and $S^*$

Let  $\bar{y}$  be the smallest integer that minimizes the function  $G_\alpha(y)$ . Since  $\lim_{|y| \rightarrow \infty} G_\alpha(y) = \infty$ ,  $\bar{y}$  exists. Specifically  $\bar{y}$  is the smallest integer  $y$  satisfying

$$\Delta G_\alpha(\bar{y}-1) \leq 0 \leq \Delta G_\alpha(\bar{y}) ,$$

where  $\Delta G_\alpha(y) = G_\alpha(y+1) - G_\alpha(y)$ . We can interpret  $\bar{y}$  as the optimal value for  $S$  if  $K = 0$ . When the cost function  $L(y)$  has the form (1),  $\bar{y}$  is determined from

$$\phi^{\lambda+1}(\bar{y}-1) \leq \frac{p - (1-\alpha)c}{p+h} \leq \phi^{\lambda+1}(\bar{y}) .$$

Let  $\bar{S}$  be the largest integer such that

$$(21) \quad G_{\alpha}(\bar{S}) \leq G_{\alpha}(\bar{y}) + K.$$

Since  $\lim_{y \rightarrow \infty} G_{\alpha}(y) = \infty$ ,  $\bar{S}$  exists, and it is shown in [13a] that

$$(22) \quad S^* \leq \bar{S}.$$

Let  $S_1$  be the smallest integer that minimizes the function

$$G_0(y) = cy + L(y),$$

and  $s_1$  the smallest integer such that

$$G_0(s_1) \leq K + G_0(S_1).$$

Since  $\lim_{|y| \rightarrow \infty} G_0(y) = \infty$ , both  $S_1$  and  $s_1$  exist. We can interpret

$(s_1, S_1)$  as the optimal policy when the horizon is a single period, and  $S_1$  can be determined analogously to  $\bar{y}$  where we let  $\alpha = 0$  in the preceding formulas. It is shown in [9] that

$$(23) \quad \underline{s} \equiv 2s_1 - S_1 \leq s^* \quad \text{and} \quad S_1 \leq S^*.$$

Note that  $\underline{s}$  as well as  $s_1$  may be negative.

#### (s, S) Policies Optimal for $x < \underline{s}$

In this section we describe in detail how the minimization of

$$(24) \quad \mathcal{L}_{\alpha}(S, D) = \frac{L_{\alpha}(S, D) + K}{1 + M_{\alpha}(D)}$$

in Step ii of our algorithm may be carried out. Our approach is to first find the values of  $S$  that minimize  $\mathcal{L}_{\alpha}(S, D)$  for each fixed

$D \in \{0, \dots, \bar{S}-s\}$ . This enables us to tabulate the function

$\mathcal{L}_\alpha(D) \equiv \min_S \mathcal{L}_\alpha(S, D)$ . We then minimize  $\mathcal{L}_\alpha(D)$  over  $D \in \{0, \dots, \bar{S}-s\}$ .

To carry out these computations it is convenient to recall from Section 3 that

$$L_\alpha(S, D) = G_\alpha(S) + \sum_{j=0}^D G_\alpha(S-j) m_\alpha(j)$$

and

$$M_\alpha(k) = \sum_{j=0}^k m_\alpha(j) .$$

In Section 1 of the Appendix we show that  $m_\alpha(k)$  may be calculated recursively by means of the known formula [7]

$$(25) \quad m_\alpha(k) = \begin{cases} \frac{\alpha\varphi(0)}{1 - \alpha\varphi(0)} & (k = 0) \\ \frac{\alpha \left[ \varphi(k) + \sum_{j=0}^{k-1} \varphi(k-j) m_\alpha(j) \right]}{1 - \alpha\varphi(0)} & (k = 1, 2, \dots). \end{cases}$$

Since  $\mathcal{L}_\alpha(S, D)$  is convex in  $S$ , an integer  $S$  minimizes  $\mathcal{L}_\alpha(S, D)$  for fixed  $D$  if and only if

$$\Delta_1 \mathcal{L}_\alpha(S-1, D) \leq 0 \leq \Delta_1 \mathcal{L}_\alpha(S, D)$$

where  $\Delta_1$  signifies the first difference of  $\mathcal{L}_\alpha$  with respect to  $S$ .

The above inequality is equivalent to

$$(26) \quad \Delta G_\alpha(S-1) + \sum_{j=0}^D \Delta G_\alpha(S-1-j) m_\alpha(j) \leq 0 \leq \Delta G_\alpha(S) + \sum_{j=0}^D \Delta G_\alpha(S-j) m_\alpha(j) .$$

Notice that (26) does not involve  $K$ .

When the cost function  $L(y)$  has the form (1), (26) is equivalent to <sup>7/</sup>

$$(27) \frac{\phi^{\lambda+1}(S-1) + \sum_{j=0}^D \phi^{\lambda+1}(S-1-j)m_{\alpha}(j)}{1 + M_{\alpha}(D)} \leq \frac{p-(1-\alpha)c}{p+h} \leq \frac{\phi^{\lambda+1}(S) + \sum_{j=0}^D \phi^{\lambda+1}(S-j)m_{\alpha}(j)}{1 + M_{\alpha}(D)}.$$

The inequality (27) has the following interpretation suggested by an equivalent result for the case  $\alpha = 1$  in [14]. Choose  $S$  just large enough so that the expected discounted number of periods between two successive orders during which there is no shortage of stock [the numerator of the right side of (27)] does not fall below a fraction  $\frac{p-(1-\alpha)c}{p+h}$  of the expected discounted number of periods between two successive orders  $[1 + M_{\alpha}(D)]$ .

The inequality (27) can be interpreted more directly by proceeding as follows. Using the definition of  $P_W(w)$ , (13), (14), (19), the fact

that  $\sum_{x=-\infty}^{s-1} P_X(x) = [1 + M_{\alpha}(D)]^{-1}$ , and assuming that  $X_1 < s$ , we have

$$\begin{aligned} P_W(w) &= (1-\alpha) \sum_{t=1}^{\infty} \alpha^{t-1} \left[ \sum_{k=-\infty}^{s-1} \phi^{\lambda+1}(S-w) p_{X_t}(k) + \sum_{k=s}^S \phi^{\lambda+1}(k-w) p_{X_t}(k) \right] \\ &= \frac{\phi^{\lambda+1}(S-w) + \sum_{j=0}^D \phi^{\lambda+1}(S-j-w)m_{\alpha}(j)}{1 + M_{\alpha}(D)} \quad w \leq S. \end{aligned}$$

<sup>7/</sup> This inequality is a generalization of a result given in [3]. Note that  $\phi^{\lambda+1}(k) = \phi^{\lambda+1}(k) = 0$  for  $k < 0$ , so that in fact the indicated upper limit of summation of the index  $j$  on the right of (27) can be expressed as  $\min(S, D)$ .

Thus (27) takes the alternative simple form that was first given in [14] for the case  $\alpha = 1$

$$(27)' \quad \sum_{w=1}^S P_W(w) \leq \frac{p - (1-\alpha)c}{p + h} \leq \sum_{w=0}^S P_W(w) .$$

If  $K = 0$ , we know from [1a, 1d, 5] that there exists an optimal policy of the form  $(S, S)$ . In this event (27) reduces to

$$\phi^{\lambda+1}(S-1) \leq \frac{p - (1-\alpha)c}{p + h} \leq \phi^{\lambda+1}(S) .$$

If  $\lambda = 0$  and unsatisfied demand is lost, then the ratio  $\frac{p-(1-\alpha)c}{p+h}$  in (27) should be replaced by  $\frac{p-c}{p+h-\alpha c}$ , which is known from [5].

We recall from the outline of step i that we need consider only those  $(s, S)$  policies for which

$$\underline{S}(D) \equiv \max(\underline{s} + D, S_1) \leq S \leq \bar{S} .$$

These limits can be utilized in solving (26) [or (27)] for  $S$ . Let  $R(S)$  be the right side of (26). First compute  $R[\underline{S}(D)-1]$  and  $R(\bar{S})$ . If it is not true that

$$(28) \quad R[\underline{S}(D) - 1] \leq 0 \leq R(\bar{S})$$

then the value of  $S$ , say  $S'$ , that minimizes  $\mathcal{L}_\alpha(S, D)$  does not lie between  $\underline{S}(D)$  and  $\bar{S}$ . Thus we can immediately reject  $D$  as a possible value of  $D^*$ . If (28) does hold, then the familiar technique of interval bisection may be employed with the limits  $\underline{S}(D)$  and  $\bar{S}$  as the starting interval. Specifically, at iteration  $i$  suppose we have  $\underline{r}^i \leq S \leq \bar{r}^i$ . Let  $S^i = \text{integer part } [.5(\underline{r}^i + \bar{r}^i)]$ . Then



$$\underline{r}^{i+1} = \underline{r}^i \quad \text{and} \quad \bar{r}^{i+1} = S^i \quad \text{if} \quad R(S^i) > 0$$

$$\underline{r}^{i+1} = S^i \quad \text{and} \quad \bar{r}^{i+1} = \bar{r}^i \quad \text{if} \quad R(S^i) \leq 0$$

until, for  $i = k$ ,  $\bar{r}^k = \underline{r}^{k+1}$ , when

$$(29) \quad \text{Optimal } S = \begin{cases} \underline{r}^k & \text{if } R(\underline{r}^k) > 0 \\ \bar{r}^k & \text{if } R(\underline{r}^k) \leq 0 . \end{cases}$$

An obvious modification of the above procedure enables us to solve (27) in the same way as we solve (26).

The bisection method is a more efficient search procedure than directly minimizing  $\mathcal{L}_\alpha(S, D)$  with respect to  $S$  using, say, a Fibonacci approach [10]. With the bisection method, no more than  $n$  evaluations need be made if  $S$  is known to lie in an interval of  $2^{n-1}$  consecutive integers. Note that in general the bisection procedure above does not yield every optimal value for  $S$  when more than one exists, but it does locate the largest.<sup>8/</sup> In general it is necessary to find all optimal values of  $S$  to guarantee that in Step iii an optimal  $(s, S)$  policy will be obtained. As we indicate in the next subsection, there are important special cases where it will suffice only to locate the largest value of  $S$  for each  $D$ .

The above computations provide us with one or more values of  $S$  satisfying (28) that minimize  $\mathcal{L}_\alpha(S, D)$  for each fixed  $D \in \{0, \dots, \bar{S} - s\}$ .

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<sup>8/</sup> To locate the smallest value, replace the conditions " $R(S^i) >$ " and " $R(S^i) \leq$ " by " $R(S^i) \geq$ " and " $R(S^i) <$ ", respectively.

Thus we may compute  $\mathcal{L}_\alpha(D)$  for each such  $D$ , compare the results, and select the values of  $D$  that minimize  $\mathcal{L}_\alpha(D)$ . This procedure provides us with the desired set  $\mathcal{S}$  of  $(s, S)$  policies that are optimal for  $X_1 = x < \underline{s}$ .

In the remainder of this subsection we outline a procedure that substantially reduces the computations needed in Step ii, but does not guarantee that an optimal policy will be found. However, the second author has conducted extensive tests for the case  $\alpha = 1$  (which will be reported elsewhere) that have shown that the method often does yield an optimal policy and so far has always produced a policy that is nearly optimal as measured in terms of average cost per period.

The method proposed is to apply the Fibonacci search procedure to minimize  $\mathcal{L}_\alpha(D)$  even though  $\mathcal{L}_\alpha(D)$  is not unimodal as it should be for theoretical validity. To illustrate the idea, this approach was applied to the model in Table 1 and the results are summarized in the final columns. Notice that the approximation prematurely reduced  $D$  at  $\mu = 52$  resulting in an expected cost 6.6% above optimal. It probably would be possible to detect that such an error has occurred by the substantial jump evident in the expected cost function.<sup>9/</sup>

The attractive feature of the Fibonacci search procedure, which is explained in detail in [4, 10], is its speed of convergence. If we can assert that an optimal  $D$  lies in an interval of width  $R$ , say  $\bar{D}+1 \leq D \leq \bar{D}+R$ , then the Fibonacci search method requires testing only  $N_R$  values of  $D$ , as given in Table 2. Thus if  $R = 54$ , we need test only 8 values of  $D$ .

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<sup>9/</sup> There are a variety of other safeguard features that can be employed. For example, split the interval for  $D$  into two (or more) parts and perform the Fibonacci search on each part.

TABLE 1

## Example of Fibonacci Approximation

$\phi(k) = \text{Poisson}$     Holding Cost = 1/unit     $K = 64$   
 $E(\xi_t) = \mu$     Penalty Cost = 9/unit     $\alpha = 1$   
 $\lambda = 0$

$\mu$	Optimal Policy				Approximate Policy				$\frac{a_1(\cdot s,S)100}{a_1(\cdot s^*,S^*)}$
	$s^*$	$S^*$	D	$a_1(\cdot s^*,S^*)$	s	S	D	$a_1(\cdot s,S)$	
21	15	65	50	50.40590	15	65	50	50.40590	100
22	16	68	52	51.63222	16	68	52	51.63222	100
23	17	52	35	52.75658	17	52	35	52.75658	100
24	18	54	36	53.51777	18	54	36	53.51777	100
51	43	110	67	71.61085	43	110	67	71.61085	100
52	44	112	68	72.24602	44	61	17	77.01544	106.6
55	47	118	71	74.14860	45	65	20	77.38106	104.4
59	51	126	75	76.67902	49	69	20	77.82948	101.5
61	52	131	79	77.92867	50	71	21	78.05713	100.2
63	54	73	19	78.28676	50	73	23	78.28676	100
64	55	74	19	78.40221	50	74	24	78.40221	100

TABLE 2

## Fibonacci Search

Interval Length R	1	2	4	7	12	20	33	54	88	143
Number of Tests $N_R$	1	2	3	4	5	6	7	8	9	10

We digress to remark here that Table 1 shows that the optimal  $D$  is not a monotone function of  $\mu$ . A complete table for  $1 \leq \mu \leq 64$  would demonstrate that the graph of optimal values of  $D$  dips twice (at the points indicated in Table 1) in the interval  $1 \leq \mu \leq 64$ .

### Choosing an Optimal $(s, S)$ Policy from $\mathcal{S}$

The computations in Step ii determine a set  $\mathcal{S} \equiv \{(s^i, S^i) | i=1, \dots, n; s^1 \leq \dots \leq s^n\}$  of  $(s, S)$  policies, each of which is optimal for  $x < s$ , and at least one of which is an optimal  $(s, S)$  policy (for all  $x$ ). In this subsection we develop methods of determining which of the policies in  $\mathcal{S}$  are optimal. We assume that  $\alpha < 1$  and  $n > 1$ , for otherwise, as we have already noted, our task would be trivial. We start by giving a universally applicable procedure, and then discuss simplifications for important special cases.

#### Theorem 1

If  $0 \leq \alpha < 1$ , if the two policies  $(s, S)$ ,  $(s', S')$  are such that  $s \leq s'$ , and if  $a_\alpha(x|s, S) = a_\alpha(x|s', S')$  for  $x < s'$ , then  $a_\alpha(x|s, S) = a_\alpha(x|s', S')$  for all  $x$ .

This theorem, which is proved in Section 2 of the Appendix, may be applied as follows to determine the policies in  $\mathcal{S}$  that are optimal. First observe that by hypothesis  $a_\alpha(x|s^{n-1}, S^{n-1}) = a_\alpha(x|s^n, S^n)$  for  $x < s^{n-1}$ . Next for each  $x$ ,  $s^{n-1} \leq x < s^n$ , compute  $a_\alpha(x|s^{n-1}, S^{n-1})$  and compare with  $a_\alpha(x|s^n, S^n) = \mathcal{L}_\alpha(S^n, D^n)$ , ( $D^n \equiv S^n - s^n$ ). As soon as one policy proves less favorable than the other, eliminate it. If both policies are equally favorable for all  $x$  in this interval, then by the theorem they are equally favorable for all  $x$ . In this event, select

one arbitrarily, say  $(s^{n-1}, S^{n-1})$ . Repeat the procedure with  $(s^{n-2}, S^{n-2})$  and with the policy remaining. In general, policies  $(s^i, S^i)$  and  $(s^j, S^j)$ ,  $i < j$ , are to be compared by testing the equality of  $a_\alpha(x|s^i, S^i)$  and  $a_\alpha(x|s^j, S^j)$  for  $s^i \leq x < s^j$ . Continue the test procedure until one policy remains, which is then optimal.

One special application of Theorem 1 is:

Corollary 1.1

If  $s^1 = \dots = s^n$ , then each  $(s^i, S^i)$ ,  $(i = 1, \dots, n)$  is optimal.

The general procedure given above for choosing an optimal policy from  $\mathcal{S}$  can frequently be substantially simplified. For this purpose we require a definition. Given a specific  $(s, S)$  policy, we say that the value  $x'$  is accessible from  $X_1 = x$  if there exists a  $t > 1$  such that

$$\Pr(X_t = x' | X_1 = x) > 0.$$

We have the following result which is proved in Section 2 of the Appendix.

Lemma 1

If  $(s, S)$  is optimal for  $X_1 = x$  and if  $0 < \alpha < 1$ , then  $(s, S)$  is optimal for every  $x'$  accessible from  $x$ .

This lemma together with Theorem 1 enables us to establish the following useful theorem.

### Theorem 2

If  $0 < \alpha < 1$ , if  $(s^i, S^i)$  is optimal, and if each  $x'$  for which  $\min(s^i, s^j) \leq x' < \max(s^i, s^j)$  is accessible from  $S^j$  under  $(s^j, S^j)$ , then  $(s^j, S^j)$  is optimal.

### Proof:

By construction of  $\mathcal{S}$ ,  $a_\alpha(x|s^i, S^i) = a_\alpha(x|s^j, S^j)$  for  $x < \min(s^i, s^j)$ . Thus, since  $(s^i, S^i)$  is optimal,  $(s^j, S^j)$  is optimal for  $x < \min(s^i, s^j)$ . Now every  $x'$  accessible from  $S^j$  under  $(s^j, S^j)$  is also accessible from  $x < \min(s^i, s^j)$  under  $(s^j, S^j)$ . Therefore, by an hypothesis of the theorem and Lemma 1,  $(s^j, S^j)$  is optimal for  $x < \max(s^i, s^j)$ . Then by Theorem 1,  $(s^j, S^j)$  is optimal.

Since  $s^1 \leq \min(s^i, s^j)$  and  $\max(s^i, s^j) \leq s^n$ , we have the following immediate corollaries of practical significance.

### Corollary 2.1

If  $0 < \alpha < 1$  and if each  $x'$  for which  $s^1 \leq x' < s^n$  is accessible from  $S^j$  under  $(s^j, S^j)$ , then  $(s^j, S^j)$  is optimal.

### Corollary 2.2

If  $0 < \alpha < 1$  and if  $\varphi(k) > 0$  for  $k = 1, \dots, s^n - s^1$ , then  $(s^n, S^n)$  is optimal.

Obviously the hypothesis on  $\varphi(k)$  in Corollary 2.2 is satisfied for a Poisson or Negative Binomial distribution. Observe that if  $\varphi(k) > 0$  for  $k = 1, \dots, \bar{S} - \underline{s}$ , then it follows from Corollary 2.2 that  $(s^n, S^n)$  is optimal, since  $s^n - s^1 \leq \bar{S} - \underline{s}$ . In this situation, it is only necessary to compute  $\bar{S}$  and  $\underline{s}$  to verify the hypothesis, and there

is no need to develop the entire set  $\mathcal{S}$ ; consequently the procedure in Steps ii and iii of our algorithm is as simple for  $\alpha < 1$  as for  $\alpha = 1$ . In Step ii, for each  $D$  considered, it suffices to select the largest  $S$  that minimizes  $\mathcal{L}_\alpha(S, D)$  by the bisection procedure as outlined above.

In the event that the value  $X_1 = x$  is actually specified and that an optimal policy for this particular value is being sought, the above general procedure can be shortened. Every policy  $(s^i, S^i)$  where  $x < s^i$  is equally favorable; evaluate  $a_\alpha(x|s^i, S^i)$  by (11) for any one of these policies. Then also evaluate  $a_\alpha(x|s^j, S^j)$  for each  $j$  for which  $s^j \leq x$ . Finally, select a policy having the minimum value for  $a_\alpha(x|\cdot, \cdot)$ .

#### Function Limits and Tabled Quantities

Steps ii and iii of the algorithm as we have expressed it require only finite summations. We summarize here the limits of the functions which must be evaluated and stored for the general case in Steps ii and iii. It is necessary to have available  $m_\alpha(k)$  and  $M_\alpha(k)$  for  $k = 0, 1, \dots, \bar{S} - \underline{s}$ , and  $G_\alpha(y)$  and  $\Delta G_\alpha(y)$  for  $\underline{s} \leq y \leq \bar{S}$ .

When the cost function  $L(y)$  has the form (1), it is possible to exploit the special formulas for this case as given above. Specifically, the following sequence of computations can be performed which will provide all the tabled information required for Steps i, ii, and iii.

Calculation 1: Starting at  $k = 0$ , generate and store  $\phi^{\lambda+1}(k)$  and  $G_\alpha(k)$ , obtaining in the process  $\bar{y}$  and  $S_1$ , until the condition for  $k = \bar{S}$  is satisfied. Record  $S_1$  and  $\bar{S}$ .

Calculation 2: Determine  $s_1$  and  $s$ . If  $s < 0$ , generate and store  $G_\alpha(y)$  for  $y = -1, -2, \dots, s$ .

Calculation 3: Generate and store  $m_\alpha(k)$  and  $M_\alpha(k)$  for  $k = 0, 1, \dots, \bar{s} - s$ .

Notice in this case it is not necessary to compute  $\Delta G_\alpha(y)$ ,  $\phi(k)$ ,  $\Phi(k)$ , and there is no need to store  $\phi^{\lambda+1}(k)$ .

The computation of  $L(y)$  needed for Calculations 1 and 2 may be simplified with the aid of the following formulas.

$$\begin{aligned} L(y) &= p[(\lambda+1)\mu - y] + (h+p) \sum_{k=0}^y (y-k)\phi^{\lambda+1}(k) \\ &= p[(\lambda+1)\mu - y] + (h+p) \sum_{k=0}^{y-1} \phi^{\lambda+1}(k), \quad y \geq 1. \\ L(y) &= p[(\lambda+1)\mu - y] \quad y \leq 0. \end{aligned}$$

This representation can also be viewed recursively

$$L(y) = L(y-1) + \Delta L(y-1) \quad y \geq 1$$

where

$$\begin{aligned} \Delta L(y-1) &= (h+p)\phi^{\lambda+1}(y-1) - p \\ &= \Delta L(y-2) + (h+p)\phi^{\lambda+1}(y-1) \quad y \geq 2, \\ \Delta L(0) &= (h+p)\phi^{\lambda+1}(0) - p \quad y = 0. \end{aligned}$$

Further reductions in computations are possible if  $\phi(k)$  is either a Poisson or Negative Binomial distribution:



$$\phi^n(k) = e^{-n\mu} (n\mu)^k / k! , \quad \sum_{k=0}^{\infty} k\phi^n(k) = n\mu \quad \text{Poisson ,}$$

$$\phi^n(k) = \binom{nr-1+k}{nr-1} q^{nr} (1-q)^k , \quad \sum_{k=0}^{\infty} k\phi^n(k) = n\mu = nr(1-q)/q \quad \begin{array}{l} \text{Negative} \\ \text{Binomial} \end{array} .$$

Then since  $\phi^n(k) = (n\mu/k)\phi^n(k-1)$  for the Poisson distribution

$$L(y) = p[(\lambda+1)\mu-y] + (h+p)[y\phi^{\lambda+1}(y-1) - (\lambda+1)\mu\phi^{\lambda+1}(y-2)] , \quad y \geq 1 .$$

Similarly since  $\phi^n(k) = [(1-q)(nr-1+k)/k]\phi^n(k-1)$  for the Negative Binomial distribution,

$$L(y) = p[(\lambda+1)\mu-y] + (h+p)[y\phi^{\lambda+1}(y-1) - (\lambda+1)\mu\phi^{\lambda+1}(y-2) \\ + (y-1)(1-q)\phi^{\lambda+1}(y-1)/q] , \quad y \geq 1 .$$

### 5. A Special Case of Guaranteed Demand

In this section we examine a case having twofold interest: the demand assumption is an important special case leading to an extreme simplification of the computational requirements for determining an optimal policy, and the example illustrates the need for Step iii of our algorithm. The following theorem is proved in Section 3 of the Appendix.

#### Theorem 3

If  $\phi(k) = 0$ ,  $k = 0, 1, \dots, \bar{S}-s^*$ , where  $s^*$  is the smallest integer such that  $G_{\alpha}(s^*) \leq G_{\alpha}(\bar{y})+K$ , then  $(s^*, \bar{y})$  is optimal (for all values of  $X_1$ ).

When the hypothesis of the theorem applies, the demand in each period is guaranteed to exceed  $\bar{S}-s^*$ . The computation of an optimal policy is then of the same order of magnitude as finding an optimal policy in a one period model. Once a period occurs in which an order is placed, an order will be placed in every succeeding period.

As an example, suppose that the demand occurring during a period is uniformly distributed between  $\beta$  and  $\beta+\gamma$  ( $\beta > 0$ ,  $\gamma > 0$ ). For simplicity we discuss the case where there is a density function  $\phi(k)$  of demand and where the order quantities need not be integral. Thus

$$\phi(k) = \begin{cases} 1/\gamma & \beta \leq k \leq \beta + \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Assume the cost function  $L(y)$  has the form (1) and  $\lambda = 0$ .

Let

$$\theta = [p - (1-\alpha)c]/(p+h).$$

Elementary calculations show that

$$\bar{S} = \frac{K + .5\gamma(p+h)(1-\theta^2)}{h + (1-\alpha)c} + \beta,$$

$$s^* = \theta\gamma - \sqrt{2K\gamma/(p+h)} + \beta,$$

$$\bar{S}-s^* = \frac{K + .5\gamma(p+h)(1-\theta^2)}{h + (1-\alpha)c} - \theta\gamma + \sqrt{2K\gamma/(p+h)},$$

$$\bar{y} = \beta + \theta\gamma,$$

$$D^* = \bar{y} - s^* = \sqrt{2K\gamma/(p+h)},$$

if  $\theta \geq 0$ ,  $\bar{S} \geq \beta+\gamma$ , and  $s^* \geq \beta$ . (Under our assumptions these conditions usually hold. If they do not hold, the formulas are easily

modified.) Observe that

$$\bar{s} - s^* \geq K/[h+(1-\alpha)c] - \gamma + \sqrt{2K\gamma/(p+h)},$$

which gives some indication of how large the guaranteed demand  $\beta$  must be in order to apply the theorem.

Theorem 3 provides us with a simple example of a situation in which several  $(s, S)$  policies are optimal for sufficiently small values of  $X_1$  while not all of them are optimal (for all values of  $X_1$ ). Specifically, for any  $s$  such that  $\varphi(k) = 0$ ,  $k = 0, 1, \dots, \bar{y} - s$ ,  $(s, \bar{y})$  is optimal for all  $X_1 = x < \min(s, s^*)$  but only  $(s^*, \bar{y})$  is in general optimal (for all values of  $X_1$ ). To see this, observe that since  $x < \min(s, s^*)$ , both policies call for an order of  $\bar{y} - x$  units to be placed in period 1. In every period thereafter, the same size order will be placed with both policies. Consequently they incur the same costs. But if  $\min(s, s^*) \leq x < \max(s, s^*)$ , the two policies will not agree in the first period and therefore  $(s, \bar{y})$  will not be optimal in general (for all values of  $X_1$ ).

## APPENDIX

Here we provide proofs of several propositions appearing in the text.

### 1. Some Renewal Formulas

We recall from Section 3 the definition

$$(6) \quad M_{\alpha}(k) = \sum_{i=1}^{\infty} \alpha^i \Phi^i(k) .$$

We show now that this series converges absolutely for  $\alpha\phi(0) < 1$ . It is convenient to consider two cases,  $\alpha < 1$  and  $\alpha = 1$ . In the former event

$$M_{\alpha}(k) \leq \sum_{i=1}^{\infty} \alpha^i \leq \frac{1}{1-\alpha} < \infty .$$

In the latter event it follows that  $\phi(0) < 1$ . But this means that  $\Phi^{k+1}(k) < 1$  for each non-negative integer  $k$ . Now<sup>10/</sup>

$$\Phi^{n(k+1)}(k) \leq [\Phi^{k+1}(k)]^n, \quad (n = 1, 2, \dots).$$

Also  $\Phi^t(k)$  is non-increasing in  $t$ . Hence, letting  $\Phi^0(k) \equiv 1$ , we have

$$\begin{aligned} M_1(k) &= \sum_{i=0}^{\infty} \Phi^i(k) - 1 = \sum_{n=0}^{\infty} \sum_{t=0}^k \Phi^{n(k+1)+t}(k) - 1 \\ &\leq (k+1) \sum_{n=0}^{\infty} \Phi^{n(k+1)}(k) - 1 \leq (k+1) \sum_{n=0}^{\infty} [\Phi^{k+1}(k)]^n \leq \frac{k+1}{1-\Phi^{k+1}(k)} - 1 < \infty \end{aligned}$$

which completes the proof.

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<sup>10/</sup> We use the fact that if  $U$  and  $V$  are independent non-negative random variables, then

$$\Pr(U + V \leq t) \leq \Pr(U \leq t, V \leq t) = \Pr(U \leq t) \Pr(V \leq t) .$$

Next we establish that  $m_{\alpha}(k)$  satisfies the renewal equation

$$(A1) \quad m_{\alpha}(k) = \alpha \left[ \varphi(k) + \sum_{j=0}^k \varphi(k-j) m_{\alpha}(j) \right] .$$

Now from (5)

$$\begin{aligned} m_{\alpha}(k) &= \alpha \varphi(k) + \alpha \sum_{i=1}^{\infty} \alpha^i \varphi^{i+1}(k) \\ &= \alpha \left[ \varphi(k) + \sum_{i=1}^{\infty} \alpha^i \sum_{j=0}^k \varphi(k-j) \varphi^i(j) \right] \\ &= \alpha \left[ \varphi(k) + \sum_{j=0}^k \varphi(k-j) m_{\alpha}(j) \right] . \end{aligned}$$

The equation (A1) can be solved recursively as follows:

$$m_{\alpha}(0) = \alpha[\varphi(0) + \varphi(0) m_{\alpha}(0)] \text{ so that}$$

$$m_{\alpha}(0) = \frac{\alpha \varphi(0)}{1 - \alpha \varphi(0)} .$$

For  $k > 0$ ,

$$\begin{aligned} m_{\alpha}(k) &= \alpha \left[ \varphi(k) + \sum_{j=0}^k \varphi(k-j) m_{\alpha}(j) \right] \\ &= \alpha \left[ \varphi(k) + \sum_{j=0}^{k-1} \varphi(k-j) m_{\alpha}(j) \right] + \alpha \varphi(0) m_{\alpha}(k) \text{ so that} \\ m_{\alpha}(k) &= \frac{\alpha \left[ \varphi(k) + \sum_{j=0}^{k-1} \varphi(k-j) m_{\alpha}(j) \right]}{1 - \alpha \varphi(0)} . \end{aligned}$$

This establishes the recursion (25).

## 2. Proofs of Theorem 1 and Lemma 1

### Proof of Theorem 1:

Let

$$b(d, t, k) \equiv \Pr[T(d) = t, \sum_{r=1}^t \xi_r = k].$$

Then by hypothesis, for  $x \geq s'$

$$\begin{aligned} \frac{a_\alpha(x|s, S)}{1 - \alpha} &= f(x|s, S) \\ &= L_\alpha(x, x-s') + \sum_{k > x-s'} \sum_{t=1}^{\infty} \alpha^t b(x-s', t, k) f(x-k|s, S) \\ &= L_\alpha(x, x-s') + \sum_{k > x-s'} \sum_{t=1}^{\infty} \alpha^t b(x-s', t, k) f(x-k|s', S') \\ &= f(x|s', S') = \frac{a_\alpha(x|s', S')}{1 - \alpha}, \end{aligned}$$

which proves Theorem 1.

### Proof of Lemma 1:

Let  $(s^*, S^*)$  be an optimal policy. Let  $t \geq 1$  be the smallest integer such that  $X_{t+1} = x'$ , and let  $t = +\infty$ , otherwise.

Let  $q$  denote the total discounted cost incurred during periods  $\lambda+1, \lambda+2, \dots, \lambda+t$  when using  $(s, S)$  and let  $Q = E q$ . Since  $x'$  is accessible from  $x$  under  $(s, S)$  and  $0 < \alpha < 1$ , we have  $0 < E(\alpha^t) \equiv A < 1$ . Also

$$Q + A f(x'|s^*, S^*) \geq f(x|s^*, S^*) = f(x|s, S) = Q + A f(x'|s, S),$$

where the inequality follows from the fact that using  $(s, S)$  in periods  $1, 2, \dots, t$  and  $(s^*, S^*)$  in periods  $t+1, t+2, \dots$ , cannot be

better than using  $(s^*, S^*)$  in every period. Since  $0 < A < \infty$  and  $|Q| < \infty$ , the inequality implies that

$$f(x'|s, S) \leq f(x'|s^*, S^*) .$$

But the reverse inequality is also true by the optimality of  $(s^*, S^*)$ . Consequently equality holds, proving the lemma.

Proof of Theorem 3:

Consider first the case  $\alpha < 1$ . We also assume  $\alpha > 0$ , since the result is well-known for  $\alpha = 0$  without any restriction on  $\varphi(k)$ . We use the familiar dynamic programming recursions.

$$f_n(x) = \min_{y \geq x} \{K\delta(y-x) + G^n(y)\} - cx \quad n \geq \lambda + 1$$

$$G^n(y) = cy + L(y) + \alpha \sum_{k=0}^{\infty} f_{n-1}(y-k) \varphi(k) .$$

As we shall see, it is convenient to let

$$f_{\lambda}(x) = A - cx$$

where

$$A = [G_{\alpha}(\bar{y}) + K + \alpha cu] \frac{1}{1-\alpha}$$

$$G_{\alpha}(y) = (1-\alpha)cy + L(y) .$$

It is well-known from [12, 15] that if  $f_{\lambda}(x)$  is any K-convex function and if there are  $n$  periods to the end of the horizon, one optimal policy is  $(s_n, S_n)$  where  $S_n$  is the smallest integer at which  $G^n$  achieves its minimum and  $s_n$  is the smallest integer such that  $G^n(s_n) \leq G^n(S_n) + K$ . We seek to show that  $s_n = s^*$  and  $S_n = \bar{y}$

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It is well-known from [12, 15] that if  $f_{\lambda}(x)$  is any K-convex function and if there are  $n$  periods to the end of the horizon, one optimal policy is  $(s_n, S_n)$  where  $S_n$  is the smallest integer at which  $G^n$  achieves its minimum and  $s_n$  is the smallest integer such that  $G^n(s_n) \leq G^n(S_n) + K$ . We seek to show that  $s_n = s^*$  and  $S_n = \bar{y}$

for all  $n$ . For then it follows from the discrete demand version of Theorem 3, Section 4 of [9] that  $(s^*, \bar{y})$  is an optimal  $(s, S)$  policy for the infinite stage problem.

We start with  $n = \lambda + 1$ .

$$\begin{aligned} G^{\lambda+1}(y) &= cy + L(y) + \alpha \sum_{k=0}^{\infty} [A - c(y-k)] \varphi(k) \\ &= G_{\alpha}(y) + \alpha A + \alpha cu . \end{aligned}$$

Consequently  $s_{\lambda+1} = s^*$  and  $S_{\lambda+1} = \bar{y}$ .

Now suppose  $s_{n-1} = s^*$ ,  $S_{n-1} = \bar{y}$ , and  $f_{n-1}(x) = A - cx$  for  $x < s^*$ . By the proof in Section 5 of [13a], it follows that  $s_n \leq \bar{S}$ . But for  $y \leq \bar{S}$  we have by an hypothesis of the theorem and the inductive assumption that

$$\begin{aligned} G^n(y) &= cy + L(y) + \alpha \sum_{k=\bar{S}-s^*+1}^{\infty} f_{n-1}(y-k) \varphi(k) \\ &= cy + L(y) + \alpha \sum_{k=\bar{S}-s^*+1}^{\infty} [A - c(y-k)] \varphi(k) \\ &= G_{\alpha}(y) + \alpha A + \alpha cu . \end{aligned}$$

Thus  $s_n = s^*$  and  $S_n = \bar{y}$  as was to be shown. Then for  $x < s^*$ ,

$$\begin{aligned} f_n(x) &= K + G^n(\bar{y}) - cx \\ &= K + G_{\alpha}(\bar{y}) + \alpha A + \alpha cu - cx \\ &= (1-\alpha)A + \alpha A - cx = A - cx , \end{aligned}$$

which completes the induction.

We now turn to the case  $\alpha = 1$ . By definition of  $s^*$  and  $\bar{y}$  for this case,

$$\Delta L(\bar{y}-1) < 0 \leq \Delta L(\bar{y})$$

$$L(s^*) \leq L(\bar{y}) + K < L(s^*-1) .$$

Using these inequalities and the fact that  $c \geq 0$ ,  $s^* \leq \bar{y}$ , we have for all sufficiently large  $\alpha < 1$

$$(1-\alpha)c + \Delta L(\bar{y}-1) < 0 \leq (1-\alpha)c + \Delta L(\bar{y})$$

$$(1-\alpha)cs^* + L(s^*) \leq (1-\alpha)c\bar{y} + L(\bar{y}) + K < (1-\alpha)c(s^*-1) + L(s^*-1) .$$

Thus  $(s^*, \bar{y})$  is optimal for all sufficiently large  $\alpha < 1$ , as we have shown above.

Now let  $(s, S)$  be another policy. Then for all sufficiently large  $\alpha < 1$

$$a_\alpha(x|s^*, \bar{y}) \leq a_\alpha(x|s, S) .$$

But  $a_\alpha$  is continuous in  $\alpha$  from the left at  $\alpha = 1$ , so that

$$a_1(x|s^*, \bar{y}) \leq a_1(x|s, S) ,$$

which completes the proof.

## References

- [1] K. J. Arrow, S. Karlin, and H. Scarf, Studies in the Mathematical Theory of Inventory and Production, Stanford University Press, Stanford, 1958.
  - a. S. Karlin, "Optimal Inventory Policy for the Arrow-Harris-Marschak Dynamic Model," Chapter 9.
  - b. S. Karlin, "Steady State Solutions," Chapter 14.
  - c. S. Karlin, "The Application of Renewal Theory to the Study of Inventory Policies," Chapter 15.
  - d. S. Karlin and H. Scarf, "Inventory Models of the Arrow-Harris-Marschak Type with Time Lag," Chapter 10.
- [2] K. J. Arrow, T. Harris, and J. Marschak, "Optimal Inventory Policy," Econometrica 19, 250-272 (1951).
- [3] M. Beckmann, "An Inventory Model for Arbitrary Interval and Quantity Distributions of Demands," Management Science 8, 35-57 (October 1961).
- [4] R. Bellman and S. Dreyfus, Applied Dynamic Programming, Princeton University Press, Princeton, 1962.
- [5] R. Bellman, I. Glicksberg, and O. Gross, "On the Optimal Inventory Equation," Management Science 2, 83-104 (1955).
- [6] F. d'Epenoux, "A Probabilistic Production and Inventory Problem," Management Science 10, 98-108 (1963). (This is a translation of an article that appeared in Revue Francaise de Recherche Operationelle, 4, 3, 1960.)
- [7] W. Feller, An Introduction to Probability Theory and its Applications, (2nd ed.), John Wiley, New York, 1957.
- [8] R. Howard, Dynamic Programming and Markov Processes, John Wiley, New York, 1960.
- [9] D. L. Iglehart, "Optimality of (S,s) Policies in the Infinite Horizon Dynamic Inventory Problem," Management Science 9, 259-267 (1963).
- [10] J. Kiefer, "Sequential Minimax Search for a Maximum," Proc. Amer. Math. Soc., 4, 502-506 (1953).
- [11] A. Manne, "Linear Programming and Sequential Decisions," Management Science 6, 259-267 (1960).

- [12] H. Scarf, "The Optimality of  $(S,s)$  Policies in the Dynamic Inventory Problem," in Mathematical Methods in the Social Sciences, K. J. Arrow, S. Karlin, and P. Suppes (eds.) Stanford University Press, Stanford, 1960.
- [13] H. Scarf, D. Gilford, and M. Shelly (eds.), Multistage Inventory Models and Techniques, Stanford University Press: Stanford, 1963.
- a. D. L. Iglehart, "Dynamic Programming and Stationary Analysis of Inventory Problems," Chapter 1.
  - b. H. Scarf, "A Survey of Analytic Techniques in Inventory Theory," Chapter 7.
  - c. A. F. Veinott, Jr., "Optimal Stockage Policies with Non-Stationary Stochastic Demands," Chapter 4.
- [14] H. M. Wagner, Statistical Management of Inventory Systems, John Wiley, New York, 1962, Chapter 2.
- [15] E. Zabel, "A Note on the Optimality of  $(S,s)$  Policies in Inventory Theory," Management Science 9, 123-125 (1962).