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ON THE OPTIMALITY OF (s, S) INVENTORY POLICIES: NEW CONDITIONS AND A NEW PROOF*

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Abstract. Scarf [6] has shown that the (s, S) policy is optimal for a class of discrete review dynamic nonstationary inventory models. In this paper a new proof of this result is found under new conditions which do not imply and are not implied by Scarf's hypotheses. We replace Scarf's hypothesis that the one period expected costs are convex by the weaker assumption that the negatives of these expected costs are unimodal. On the other hand we impose the additional assumption not made by Scarf that the absolute minima of the one period expected costs are (nearly) rising over time. For the infinite period stationary model, this last hypothesis is automatically satisfied. Thus in this case our hypotheses are weaker than Scarf's. The bounds on the optimal parameter values given by Veinott and Wagner [12] are established for the present case. The bounds in a period are easily computed, and depend only upon the expected costs for that period. Moreover, simple conditions are given which ensure that the optimal parameter values in a given period equal their lower bounds. When there is no fixed charge for ordering, this reduces to earlier results of Karlin [5] and Veinott [9], [10], [11] for the nonstationary case. The above result is exploited to extend the planning horizon theorem of Veinott [9] to the case where there is a fixed charge for ordering.

1. Model formulation. We consider a single product dynamic inventory model in which the demands $D_1, D_2, \cdots$, for a single product in periods $1, 2, \cdots$, are independent random variables with distributions $\Phi_1, \Phi_2, \cdots$. Assume $\{\eta_i\}$ are given constants such that[1] $D_i \geqq \eta_i$ for all i. At the beginning of each period the system is reviewed. An order may be placed for any nonnegative quantity of stock. An order placed at the beginning of period i is delivered at the beginning of period $i + \lambda$, where λ is a known nonnegative integer.

Let x_i denote the stock on hand and on order prior to placing any order in period i. Let y_i denote the stock on hand and on order after ordering in period i. It is possible for x_i and y_i to be negative indicating the existence of a backlog. We assume that the amount of stock on hand and on order at the end of period i is a specified Borel function $v_i(y_i, D_i)$ of y_i and D_i.

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[1] Actually the main results of the paper given in §2 also hold in the more general case where D_i is a random vector. All that is required is to let $\mathfrak{D}_i$ be the Borel set of possible values of D_i and replace the interval $[\eta_i, \infty)$ of possible values of D_i everywhere by $\mathfrak{D}_i$. This more general formulation allows for consideration of several classes of demands, random deterioration rates, random departures of backlogged demand, and random prices, for example, by suitable interpretation of the components of D_i.

Thus $x_{i+1} = v_i(y_i, D_i)$. If $\lambda > 0$ we assume that all unsatisfied demand is backlogged so $v_i(y_i, D_i) = y_i - D_i$.

When $\lambda = 0$ our formulation provides for the possibilities of deterioration of stock in storage (perishable goods) and partial backlogging of unsatisfied demand [11, p. 766]. For example suppose that whenever $y_i < D_i$, then a fraction b $(0 \leqq b \leqq 1)$ of the unsatisfied demand is backlogged and the remainder leaves immediately. If instead $y_i > D_i$, then a fraction $1 - a$ $(0 \leqq a \leqq 1)$ of the inventory on hand spoils and is not available for future use. These assumptions imply that $v_i(\cdot, \cdot)$ takes the form

$$v_i(y_i, D_i) = \begin{cases} a \cdot (y_i - D_i) & \text{if } y_i \geqq D_i, \\ b \cdot (y_i - D_i) & \text{if } y_i \leqq D_i. \end{cases}$$

Note that if $a = 0$ we have the case of perishable goods while if $a = 1$ we have the case of nonperishable goods. If $b = 0$ we have the lost sales case while if $b = 1$ we have the backlog case. In the literature these last two cases are usually discussed only where $a = 1$.

At the beginning of period i, the inventory manager is assumed to have observed the vector

$$H_i = (x_1, \cdots, x_i, y_1, \cdots, y_{i-1}, D_1, \cdots, D_{i-1}),$$

representing the history of the process up to the beginning of period i. He bases his ordering decision in period i upon H_i.

An ordering policy for period i is a real valued Borel function $\bar{Y}_i(\cdot)$ to be used as follows. At the beginning of period i, after having observed the past history H_i, the manager orders $\bar{Y}_i(H_i) - x_i$ which is assumed to be nonnegative of course. Also let $\tilde{Y}_i = (\bar{Y}_i, \cdots, \bar{Y}_n)$ denote a sequence of ordering policies for periods $i, \cdots, n$.

Three types of costs are considered: ordering, holding, and shortage. Assume that the cost of ordering z units in period i is $K_i\delta(z) + c_iz$, where $K_i \geqq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. The cost is incurred at the time of delivery of the order. Let $g_i(y, D_{i+\lambda})$ denote the holding and shortage cost in period $i + \lambda$ when y is the amount of stock actually on hand after receipt of orders to be delivered before the end of period $i + \lambda$. We assume that $g_i(\cdot, \cdot)$ is a real valued Borel function.

Let $\alpha_i(\geqq 0)$ be the discount factor for period $i + \lambda$. That is, α_i is the value at the beginning of period $i + \lambda$ of one cost unit at the beginning of period $i + \lambda + 1$. Let $\beta_1 = 1$ and $\beta_i = \prod_{j=1}^{i-1} \alpha_j$ for $i > 1$.

For the case $\lambda = 0$ let

$$W_i(y, t) = c_iy + g_i(y, t) - \alpha_ic_{i+1}v_i(y, t).$$

For the case $\lambda > 0$ let

$$W_i(y, t) = c_i y + \int_{-\infty}^{\infty} g_i(y - z, t)\, d\Phi_i^{\lambda}(z) - \alpha_i c_{i+1}(y - E(D_i)),$$

where $\Phi_i^{\lambda}(\cdot)$ is the distribution of $D_i + \cdots + D_{i+\lambda-1}$.

Now for $\lambda \geqq 0$ let

$$G_i(y) = \int_{-\infty}^{\infty} W_i(y, t)\, d\Phi_{i+\lambda}(t).$$

We assume that all integrals given above exist and are finite.

We suppose that each unit of stock left over after $\lambda + n$ periods can be discarded with a return of c_{n+1}. Similarly, each unit of backlogged demand remaining after $\lambda + n$ periods is satisfied at a cost c_{n+1}. In the literature it has often been assumed that $c_{n+1} = 0$.

Thus, the expected discounted cost incurred in periods $\lambda + 1, \cdots, \lambda + n$ when following the policy $\tilde{Y}_1$ in periods $1, \cdots, n$ is

$$E\left\{\sum_{i=1}^{n} \beta_i \left[K_i\delta(y_i - x_i) + c_i(y_i - x_i) + g_i\left(y_i - \sum_{j=i}^{i+\lambda-1} D_j, D_{i+\lambda}\right)\right] - \beta_{n+1} c_{n+1}\left(x_{n+1} - \sum_{i=n+1}^{n+\lambda} D_i\right)\right\}.$$

By substituting $x_i = v_{i-1}(y_{i-1}, D_{i-1})$ into the above formula we get as in [10], [12],

$$\sum_{i=1}^{n} \beta_i E[K_i\, \delta(y_i - x_i) + G_i(y_i)] - \left[c_1 x_1 - \beta_{n+1} c_{n+1} \sum_{i=n+1}^{n+\lambda} E(D_i)\right].$$

Since the second bracketed term is not affected by the choice of $\tilde{Y}_1$, it is convenient to omit it from the analysis. Thus we may define the conditional expected discounted cost incurred in periods $\lambda + i, \cdots, \lambda + n$ when following $\tilde{Y}_i$ in periods $i, \cdots, n$ given the observed history H_i as

$$f_i(\tilde{Y}_i \mid H_i) = \sum_{j=i}^{n} \beta_j E_{H_i}[K_j\delta(y_j - x_j) + G_j(y_j)]. \tag{1}$$

We seek a policy $\tilde{Y}_1^* = (\bar{Y}_1^*, \cdots, \bar{Y}_n^*)$, called optimal, which satisfies

$$f_i(\tilde{Y}_i^* \mid H_i) \leqq f_i(\tilde{Y}_i \mid H_i), \qquad i = i, \cdots, n, \tag{2}$$

for all H_i and $\tilde{Y}_i$, where of course $\tilde{Y}_i^* = (\bar{Y}_i^*, \cdots, \bar{Y}_n^*)$. It is easy to show by induction on i (starting with $i = n$) that if there is an optimal policy, then $f_i(\tilde{Y}_i^* \mid H_i)$ depends upon H_i only through x_i, so we may write

$$f_i(\bar{Y}_i^* \mid H_i) = f_i(x_i), \qquad i = 1, \cdots, n, \tag{3}$$

where the f_i satisfy $(f_{n+1}(x) \equiv 0)$

$$f_i(x) = \inf_{y \geqq x} \{K_i\delta(y - x) + G_i(y) + \alpha_i E f_{i+1}(v_i(y, D_i))\} \tag{4}$$

for $i = 1, \cdots, n$ and all x with the infimum being attained for each x. Conversely if there is a sequence of functions $\{f_i\}$ which satisfy (4), with the infimum being attained for each x, then there exists an optimal policy $\bar{Y}_1^*$. Moreover, $\bar{Y}_i^*(H_i)$ is any value of y which minimizes the expression in braces on the right side of (4) subject to $y \geqq x$ where $x = x_i$. Since the minimizing value of y is a function only of x_i, it follows that $\bar{Y}_i^*(H_i) \equiv \bar{Y}_i^*(x_i)$ depends upon H_i only through x_i. For notational convenience, we define

$$J_i(y) = G_i(y) + \alpha_i E f_{i+1}(v_i(y, D_i)). \tag{5}$$

In what follows we shall have occasion to impose one or more of the following assumptions for each i $(K_{n+1} \equiv 0)$:

(i) $G_i(y)$ and $v_i(y, t)$ are continuous in y for each $t \geqq \eta_i$;

(ii) $\underline{\lim}_{y \to \infty} G_i(y) > \inf_y G_i(y) + \alpha_i K_{i+1}$;

(iii) $\underline{\lim}_{y \to -\infty} G_i(y) > \inf_y G_i(y) + K_i$;

(iv) $-G_i(y)$ is unimodal in y;

(v) $v_i(y, t)$ is nondecreasing in y for each $t \geqq \eta_i$; moreover, $v_i(y, t)$ is bounded above in t on $[\eta_i, \infty)$ for each fixed y;

(vi) $K_i \geqq \alpha_i K_{i+1}$.

If (i)–(iii) hold, there are a number $\underline{S}_i$ which minimizes $G_i(y)$ on $(-\infty, \infty)$ and numbers $\underline{s}_i (\leqq \underline{S}_i)$ and $\bar{S}_i (\geqq \underline{S}_i)$ such that

$$G_i(\bar{S}_i) = G_i(\underline{S}_i) + \alpha_i K_{i+1}$$

and

$$G_i(\underline{s}_i) = G_i(\underline{S}_i) + K_i.$$

If in addition (vi) holds, there is a number $\bar{s}_i$, $\underline{s}_i \leqq \bar{s}_i \leqq \underline{S}_i$, such that

$$G_i(\bar{s}_i) = G_i(\underline{S}_i) + (K_i - \alpha_i K_{i+1}).$$

2. The optimality of the (s, S) **policy.** In this section we shall show that that if (i)–(vi) hold and if

(vii) $\quad v_i(\underline{S}_i, t) \leqq \underline{S}_{i+1} \quad$ for $\quad t \geqq \eta_i \quad$ and $\quad i = 1, 2, \cdots, n-1,$

then there is an optimal policy which is an (s, S) policy. By this we mean that there is a sequence $\{(s_i, S_i)\}$ of pairs of numbers such that $(s_i \leqq S_i)$,

$$\bar{Y}_i^*(H_i) = \begin{cases} S_i & \text{if } x_i < s_i, \\ x_i & \text{if } x_i \geqq s_i, \end{cases}$$

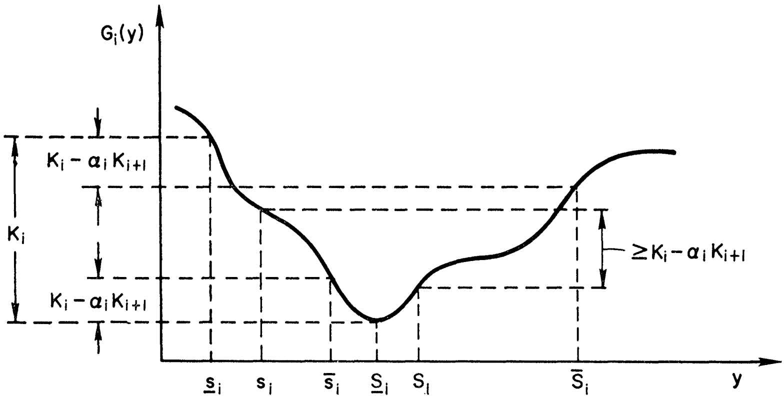

FIG. 1

for all i and H_i. Moreover the numbers satisfy

$$\underset{\sim}{s}_i \leqq s_i \leqq \bar{s}_i \leqq \underset{\sim}{S}_i \leqq S_i \leqq \bar{S}_i \tag{6}$$

and

$$G_i(s_i) \geqq G_i(S_i) + (K_i - \alpha_i K_{i+1}) \tag{7}$$

for all i. The bounds in (6) and (7) are depicted in Fig. 1. The inequality (7) has the interpretation that if it is optimal to order in period i with the initial inventory level x, then the reduction in the immediate expected costs due to $G_i(\cdot)$ must be at least $K_i - \alpha_i K_{i+1}$. In the special case where $\eta_i = 0$ and $v_i(y, t) = y - t$ for all i, then (vii) reduces to the simpler form: $\underset{\sim}{S}_i \leqq \underset{\sim}{S}_{i+1}$, $i = 1, \cdots, n - 1$.

The first proof that the (s, S) policy is optimal under reasonably general conditions is due to Scarf [6]. (See also [13].) Scarf assumes that $G_i(y)$ is convex in y with $G_i(y) \to \infty$ as $|y| \to \infty$, that $v_i(y, t) = y - t$, and that (vi) holds.[2] These assumptions imply (i)–(vi) and are in fact quite a bit stronger. To elaborate on this point we remark that if $W_i(y, t)$ is convex in y, then $G_i(y)$ is convex in y for any distribution $\Phi_{i+\lambda}$. However, if $\Phi_{i+\lambda}$ has a density $\phi_{i+\lambda}$ with $\phi_{i+\lambda}(t - \theta)$ having a monotone likelihood ratio with respect to θ, then $-G_i(y)$ will be unimodal under conditions where $W_i(y, t)$ is not convex in y. Specifically, $-G_i(y)$ is unimodal if $W_i(y, t) = W_i^1(y - t) + W_i^2(t)$ for some functions W_i^1 and W_i^2 where $-W_i^1(z)$ is unimodal in z, [3]. See [4] for a discussion of the utility of this assumption.

Although Scarf imposes stronger assumptions than (i)–(vi), he does not require that (vii) hold. Thus his results are not implied by ours nor conversely.

[2] Scarf also assumed that the cost functions and demand distributions do not change over time although that is not essential to his proof [8, p. 200].

As an example to illuminate the significance of (vii) suppose $\lambda = 0$; $D_1, \cdots, D_n$ are identically distributed; and $c_i = c$, $\alpha_i = \alpha$, $g_i(y, t) = g(y, t)$, $v_i(y, t) = y - t$, and $\eta_i = 0$ for $i = 1, 2, \cdots, n$. Then $G_i(y) = G(y)$ so $\underset{\sim}{S}_i = \underset{\sim}{S}$ for $i = 1, 2, \cdots, n - 1$. If in addition $c_{n+1} = c$, then $G_n(y) = G(y)$ also, so $\underset{\sim}{S}_n = \underset{\sim}{S}$ and (vii) holds. On the other hand if $c_{n+1} = 0$ (as is assumed, for example, by Scarf [6] and others), then $G_n(y) = G(y) + \alpha c[y - E(D_n)]$. Thus if $\alpha c > 0$, we should ordinarily expect $\underset{\sim}{S} > \underset{\sim}{S}_n$ in which case (viii) fails to hold for $i = n - 1$. Both of the above definitions of c_{n+1} are reasonable formulations of a "stationary" version of our model. It is of some interest then that the first assumption assures that (vii) holds while the second does not. Of course in infinite horizon models [1], [2], the difference between the two stationary formulations vanishes. Thus our hypotheses are actually weaker than Scarf's for the infinite horizon stationary models.

Bounds on s_i and S_i were first established by Iglehart in [1], [2] under Scarf's hypotheses, the assumption that the cost functions and demand distributions do not change over time, and $c_{n+1} \equiv 0$. Under the above assumptions except $c_{n+1} = c$, Veinott and Wagner [12] have established bounds of the form (6). In their analysis $G_i(y) = G(y)$, so the bounds in (6) are independent of i even though this is not true of s_i and S_i. Our present analysis shows that the bounds remain valid under the weaker hypotheses imposed here.

The principal tool of Scarf's proof is the fact that if $J_i(y)$ is K_i-convex (see [6] for a definition), then so is $f_i(x)$. This method of proof fails under our hypotheses because $J_i(y)$ need not be K_i-convex.[3] Our proof is based instead upon the following two lemmas which establish properties of functions satisfying (4), (5).

LEMMA 1.

$$f_i(x) \leqq f_i(x') + K_i, \qquad x \leqq x', \qquad i = 1, \cdots, n. \tag{8}$$

Moreover, if (v) *holds, then*

$$J_i(y') - J_i(y) \geqq G_i(y') - G_i(y) - \alpha_i K_{i+1}, \qquad y \leqq y', \qquad i = 1, \cdots, n. \tag{9}$$

Proof.

From (4) and (5) we have for $x \leqq x'$ that

$$f_i(x) \leqq K_i + \inf_{y \geqq x} J_i(y) \leqq K_i + \inf_{y \geqq x'} J_i(y) \leqq K_i + f_i(x'),$$

which establishes (8).

[3] As an illustration, if $K_i = K < 1$, $G_i(y) = G(y) = \min(1, |y|)$, $\eta_i \geqq 0$, $v_i(y, t) = y - t$, and $\alpha_i = \alpha \leqq 1$, then (i)–(vii) hold. However, $J_n(y) = G(y)$ which is not K_n-convex.

For $y \leqq y'$, we have from (v) that $v_i(y, D_i) \leqq v_i(y', D_i)$. Thus from (4), (5), and (8) we get

$$\begin{aligned} J_i(y') - J_i(y) \\ &= G_i(y') - G_i(y) + \alpha_i E[f_{i+1}(v_i(y', D_i)) - f_{i+1}(v_i(y, D_i))] \\ &\geqq G_i(y') - G_i(y) - \alpha_i K_{i+1}, \end{aligned}$$

which completes the proof.

The proofs of (8) and (9) are purely analytic. An alternative proof of (8) may be constructed by using the following argument which may be made rigorous. If the initial inventory on hand and on order at the beginning of period i is x but one orders in each period j ($\geqq i$) so as to bring the inventory level after ordering to the level which would be optimal if the initial inventory level in period i were instead x' ($\geqq x$), then the associated expected discounted cost would not exceed $f_i(x') + K_i$. But since the policy just described cannot be better than the optimal policy, (8) must hold. A similar kind of argument may be used to establish (9).

LEMMA 2. *If* (v) *holds, if* $\{a_j\}$ *is a sequence of numbers for which* $v_j(a_j, t) \leqq a_{j+1}$ *for* $t \geqq \eta_j$ *and* $j \geqq i$, *and if* $G_j(y)$ *is nonincreasing in* y *on* $(-\infty, a_j]$ *for* $j \geqq i$, *then*

$$J_j(y') - J_j(y) \leqq G_j(y') - G_j(y) \leqq 0, \qquad y \leqq y' \leqq a_j, \tag{10}$$

and

$$f_j(x') - f_j(x) \leqq 0, \qquad x \leqq x' \leqq a_j, \tag{11}$$

for $j \geqq i$.

Proof. The proof is by induction on j. Suppose (10), (11) hold for $j + 1$ ($>i$). By (v), $v_j(y, D_j) \leqq v_j(y', D_j) \leqq v_j(a_j, D_j) \leqq a_{j+1}$. Hence using (11) for $j + 1$ we get

$$\begin{aligned} J_j(y') - J_j(y) \\ &= G_j(y') - G_j(y) + \alpha_j E[f_{j+1}(v_j(y', D_j)) - f_{j+1}(v_j(y, D_j))] \\ &\leqq G_j(y') - G_j(y) \leqq 0, \end{aligned}$$

which proves (10) for j.

It follows from (10) for the integer j that

$$\begin{aligned} f_j(x) &= \min \{J_j(x), K_j + \inf_{y>x} J_j(y)\} \\ &\geqq \min \{J_j(x'), K_j + \inf_{y>x'} J_j(y)\} = f_j(x'), \end{aligned}$$

which proves (11) for the integer j. The same arguments suffice to establish (10), (11) for $j = n$ which starts the induction and completes the proof.

The proofs of (10) and (11) are purely analytic. An alternative proof of (11) may be devised by using the following argument which can be made rigorous. Suppose the initial inventory level in period j is x'. Suppose also that one orders so as to bring the initial inventory level after ordering in each period k $(\geqq j)$ as close as possible to the level which would be optimal if the initial inventory level in period j were x $(\leqq x')$. This policy incurs expected discounted costs which are no greater than $f_j(x)$. But the policy also must incur expected discounted costs at least as large as $f_j(x')$. Combining these remarks proves (11). The inequality (10) can be justified in a similar way.

THEOREM 1. *If* (i)–(vii) *hold, there exists an optimal policy which is an* (s, S) *policy. Moreover, the parameters of that policy satisfy* (6), (7).

Proof. The proof is constructive and proceeds in several steps. To begin with suppose $J_i(y)$ is continuous in y.

(a) $J_i(y)$ is nonincreasing on $(-\infty, \underline{S}_i]$.

To see this recall from (iv) that $G_j(y)$ is nonincreasing in y on $(-\infty, \underline{S}_j]$ for $j \geqq i$. Thus by (vii) and Lemma 2, (a) holds.

Since $J_i(y)$ is continuous, there is an S_i which minimizes $J_i(y)$ on $[\underline{S}_i, \bar{S}_i]$. Thus S_i satisfies (6). Moreover,

(b) $\min_y J_i(y) = J_i(S_i)$.

To see this observe from (a) that S_i minimizes $J_i(y)$ on $(-\infty, \bar{S}_i]$. Also by Lemma 1, (iv), and the definition of $\bar{S}_i$ we have for $y > \bar{S}_i$ that

$$\begin{aligned} J_i(y) - J_i(\underline{S}_i) &\geqq G_i(y) - G_i(\underline{S}_i) - \alpha_i K_{i+1} \\ &\geqq G_i(\bar{S}_i) - G_i(\underline{S}_i) - \alpha_i K_{i+1} = 0. \end{aligned}$$

Thus (b) holds.

(c) There exists a number s_i satisfying (6), (7) and

$$J_i(S_i) + K_i - J_i(s_i) = 0. \tag{12}$$

In order to prove this assertion we observe from Lemma 2, (b), and the definitions of $\underline{S}_i$ and $\underline{s}_i$ that

$$\begin{aligned} J_i(S_i) + K_i - J_i(\underline{s}_i) &\leqq J_i(\underline{S}_i) + K_i - J_i(\underline{s}_i) \\ &\leqq G_i(\underline{S}_i) + K_i - G_i(\underline{s}_i) = 0. \end{aligned} \tag{13}$$

On the other hand by Lemma 1 and the definitions of $\underline{S}_i$ and $\bar{s}_i$ we have

$$\begin{aligned} J_i(S_i) + K_i - J_i(\bar{s}_i) &\geqq G_i(S_i) - G_i(\bar{s}_i) + K_i - \alpha_i K_{i+1} \\ &\geqq G_i(\underline{S}_i) - G_i(\bar{s}_i) + K_i - \alpha_i K_{i+1} = 0. \end{aligned} \tag{14}$$

From (13), (14), and the continuity of $J_i(y)$, it follows that there is an s_i satisfying (6) and (12). Moreover (7) holds also since by Lemma 1 and (12) we have

$$0 = J_i(S_i) + K_i - J_i(s_i) \geqq G_i(S_i) - G_i(s_i) + K_i - \alpha_i K_{i+1}$$

which completes the proof of (c).

(d) The value of y which minimizes the right side of (4) is determined by

$$y = \begin{cases} S_i & \text{if } x < s_i\,, \\ x & \text{if } x \geqq s_i\,. \end{cases}$$

To prove (d), observe from (a), (b), and (c) that for $x < s_i$,

$$J_i(x) \geqq J_i(s_i) = J_i(S_i) + K_i = \min_y J_i(y) + K_i\,,$$

so $y = S_i$ minimizes the right side of (4). Now for $s_i \leqq x \leqq S_i$, the same arguments give

$$J_i(x) \leqq J_i(s_i) = \min_y J_i(y) + K_i\,,$$

so $y = x$ minimizes the right side of (4). Finally for $\underline{S}_i < x < y$, we have from Lemma 1, (iv), and (vi) that

$$J_i(y) + K_i - J_i(x) \geqq G_i(y) - G_i(x) + K_i - \alpha_i K_{i+1} \geqq 0,$$

so $y = x$ minimizes the right side of (4). This completes the proof of (d).

It remains only to verify our assumption that

(e) $J_i(y)$ is continuous in y.

We prove (e) by induction on i. The assertion is trivial for $i = n$ since $J_n(y) = G_n(y)$. Suppose now (e) holds for the integer $i + 1$. Then by (d),

$$f_{i+1}(x) = \begin{cases} K_{i+1} + J_{i+1}(S_{i+1}) & \text{if } x < s_{i+1}\,, \\ J_{i+1}(x) & \text{if } x \geqq s_{i+1}\,. \end{cases} \tag{15}$$

Since $J_{i+1}(y)$ is continuous and (12) holds for $i+1$, $f_{i+1}(x)$ is evidently continuous. Since $G_i(y)$ is continuous by (i), $J_i(y)$ will be continuous if

$$Ef_{i+1}(v_i(y, D_i)) \equiv q(y)$$

is continuous in y on any arbitrary interval, $[a, b]$, say. Since by (i) and the continuity of f_{i+1}, the composite function $f_{i+1}(v_i(y, t))$ is continuous in y, $q(y)$ will be continuous on $[a, b]$ if the composite function is uniformly bounded for $y \in [a, b]$ and all t ($\geqq \eta_i$) by virtue of the dominated convergence theorem. We now show that there is a number B such that

$$J_{i+1}(S_{i+1}) \leqq f_{i+1}(v_i(y, t)) \leqq f_{i+1}(B) + K_{i+1} \tag{16}$$

for all t ($\geqq \eta_i$) and $y \in [a, b]$, which gives the desired bounds. The left-hand inequality follows from (15) and (b). Since by (v), $v_i(y, t) \leqq v_i(b, t) \leqq B$ for some B, the right-hand inequality follows from Lemma 1.

The proof is now complete since we have constructed a solution to (4) with the infimum being attained for each x.

As we have remarked before, if Scarf's hypotheses ($G_i(y)$ convex, $G_i(y) \to \infty$ as $|y| \to \infty$, $v_i(y, t) = y - t$, and (vi)) are substituted for (i)–(vii), then there exists an optimal policy which is an (s, S) policy. However, the lower bounds in (6) for s_i and S_i are no longer valid when (vii) fails to hold. The reason for this is clear upon reflection. For example, suppose $D_{n-1} \geqq 0$ and $P(D_{n-1} < \underline{S}_{n-1} - \underline{S}_n) > 0$ so (vii) does not hold. In this event it is apparent from (1) that one would not want to order up to $\underline{S}_{n-1}$ (or more) in period $n - 1$ if $G_n(y)$ increased sufficiently rapidly on the interval $[\underline{S}_n, \underline{S}_{n-1}]$. The reason for this is, of course, that the relatively low expected costs in period $n - 1$ would be more than offset by the extremely high expected costs in period n. For a concrete illustration see footnote 4 below.

Let

$$\underset{\sim}{S}_i = \begin{cases} \underline{S}_n & \text{if } i = n, \\ \min(\underline{S}_i, \underset{\sim}{S}_{i+1} + \eta_i) & \text{if } i = 1, 2, \cdots, n-1. \end{cases}$$

Let $\underset{\sim}{s}_i$ ($\leqq \min(\underline{s}_i, \underset{\sim}{S}_i)$) be chosen so that

$$G_i(\underset{\sim}{s}_i) = G_i(\underset{\sim}{S}_i) + K_i .$$

THEOREM 2. *Under Scarf's hypotheses, there is an optimal policy* $\{(s_i, S_i)\}$ *which satisfies* (7) *and*

$$(6') \qquad \underset{\sim}{s}_i \leqq s_i \leqq \bar{s}_i \quad \text{and} \quad \underset{\sim}{S}_i \leqq S_i \leqq \bar{S}_i, \quad i = 1, 2, \cdots, n.$$

Proof. We only sketch the proof, leaving the details to the reader. The upper bounds on s_i and S_i and the inequality (7) are established by applying Lemma 1 in exactly the same way as in Theorem 1. The lower bounds on s_i and S_i may be established by applying (10) with $a_j = \underset{\sim}{S}_j$ for all j in a manner similar to that employed in proving Theorem 1.

We remark that if (vii) holds then $\underset{\sim}{S}_i = \underline{S}_i$ and $\underset{\sim}{s}_i = \underline{s}_i$ for all i so that (6′) reduces to (6).

3. Planning horizons and special cases. The next result tells us that if S_k is sufficiently small in comparison with s_{k+1}, then $(\underline{s}_k, \underline{S}_k)$ is optimal for period k. Observe that this is the policy that is optimal for period k when considered by itself or as the final period of a k-period model. Moreover, if $S_i, \cdots, S_k$ are sufficiently small in comparison with s_{k+1}, an optimal policy for periods $i, \cdots, k$ may be determined without evaluating $f_{k+1}(x)$ for any x. In this sense, period k is a planning horizon. The actual calculations are carried out using (4) recursively where $f_{k+1}(x) \equiv 0$ for all x. The theorem generalizes some results in [9] to the case where there is a setup cost for placing orders.

THEOREM 3. *Suppose* (i)–(vii) *or Scarf's hypotheses hold, and that* $\{(s_j, S_j)\}$ *is an optimal policy.*

(a) *If* $\{a_j\}$ *is a collection of numbers for which*

$$\underline{S}_k \leqq a_k, \tag{17[4]}$$

$$a_{k+1} \leqq s_{k+1}, \tag{18}$$

$$v_j(a_j, t) \leqq a_{j+1} \quad \text{for} \quad t \geqq \eta_j, \qquad \text{and} \tag{19}$$

$$S_j \leqq a_j \tag{20}$$

for $j = k$, *then* $(\underline{s}_k, \underline{S}_k)$ *is optimal for period* k.

(b) *If* (18)–(20) *hold for* $j = i, i + 1, \cdots, k$, *then one optimal policy for periods* $i, i + 1, \cdots, k$, *is independent of* $f_{k+1}(\cdot)$.

Proof. We begin by proving part (a). From Theorem 1, Scarf's theorem, and (18),

$$f_{k+1}(x) = K_{k+1} + J_{k+1}(S_{k+1}) \equiv Q, \qquad x \leqq a_{k+1}. \tag{21}$$

It follows from (v) and (19) that for $t \geqq \eta_k$ and $y \leqq a_k$, $v_k(y, t) \leqq v_k(a_k, t) \leqq a_{k+1}$. Combining this remark and (21) we get

$$J_k(y) = G_k(y) + \alpha_k E f_{k+1}(v_k(y, D_k)) = G_k(y) + \alpha_k Q, \quad y \leqq a_k. \tag{22}$$

Now by (20), $J_k(y)$ achieves its minimum on $(-\infty, \infty)$ in $(-\infty, a_k]$. Since this is so it follows from (22) and (17) that $\underline{S}_k$ minimizes $J_k(y)$ on $(-\infty, \infty)$. Moreover, again by (22) we have

$$J_k(\underline{s}_k) = G_k(\underline{s}_k) + \alpha_k Q = K_k + G_k(\underline{S}_k) + \alpha_k Q = K_k + J_k(\underline{S}_k).$$

Hence, by Theorem 1 and Scarf's theorem, $(\underline{s}_k, \underline{S}_k)$ is optimal for period k, which establishes part (a).

In order to prove part (b) we observe from Theorem 1, Scarf's theorem, and (20) that the optimal policy in period j, $i \leqq j \leqq k$, can be determined provided only that we can evaluate $J_j(y)$ for $y \leqq a_j$. Now from (v), (19), and (20) it follows easily by induction on j that $J_j(y)$ may be evaluated for $y \leqq a_j$ without evaluating $f_{k+1}(x)$ for $x > a_{k+1}$, i.e., for $y \leqq a_j$, $J_j(y)$ depends upon f_{k+1} only through the constant Q. (We have already shown this for $j = k$ which starts the induction.)

We may exhibit the dependence of $J_j(y)$ upon Q by writing $J_j^Q(y)$. It is easy to show by induction on j that

$$J_j^Q(y) = J_j^0(y) + Q \prod_{t=j}^{k} \alpha_t, \quad j = i, i + 1, \cdots, k. \tag{23}$$

[4] The hypothesis (17) is easily seen to be satisfied if (18)–(20) hold for $j = k$ and either (1) (i)–(vii) hold or (2) Scarf's assumptions are fulfilled and $K_{k+1} > 0$ or $S_k < a_k$. The following example shows that (17) cannot be dispensed with under Scarf's hypotheses. Assume $n = k + 1 = 2$, $G_1(y) = | y - 2 |$, $G_2(y) = 2 | y - 1 |$, $K_1 = K_2 = 0$, $P(D_1 = 0) = P(D_2 = 0) = 1$, and $\eta_1 = \eta_2 = 0$. Then $s_1 = S_1 = s_2 = S_2 = 1$ and $\underline{s}_1 = \underline{S}_1 = 2$. Now let $a_1 = a_2 = 1$. Then (18)–(20) hold, but $(\underline{s}_1, \underline{S}_1)$ is not optimal for period 1.

(Again we have already done this for $j = k$ which starts the induction.) Thus if a policy is optimal for period j, $i \leqq j \leqq k$, for some Q, that same policy is optimal for all Q. Hence if we assume $Q = 0$ and determine the optimal policy for periods $i, i + 1, \cdots, k$ in the usual way (taking account of (20)), that policy is optimal for the original problem where (21) holds. We have thus shown that the optimal policy in periods $i, i + 1, \cdots, k$ is independent of $f_{k+1}(\cdot)$ as required.

As an illustration of the application of Theorem 3(a), we have the following result.

COROLLARY 1. *If* (i)–(vii) *or Scarf's hypotheses hold, and if*

$$v_k(\bar{S}_k, t) \leqq \underset{\sim}{s}_{k+1} \quad \text{for} \quad t \geqq \eta_k, \tag{24}$$

then $(\underset{\sim}{s}_k, \underset{\sim}{S}_k)$ *is optimal for period* k.

Proof. Let $a_k = \bar{S}_k$ and $a_{k+1} = \underset{\sim}{s}_{k+1}$. Then apply Theorem 3(a).

We remark that if (24) holds, then by (v) and the definitions of $\underset{\sim}{S}_k$, $\bar{S}_k$, $\underset{\sim}{s}_{k+1}$, $\underset{\sim}{S}_{k+1}$ (recall $\underset{\sim}{s}_{k+1} = s_{k+1}$ if (i)–(vii) hold),

$$v_k(\underset{\sim}{S}_k, t) \leqq v_k(\bar{S}_k, t) \leqq \underset{\sim}{s}_{k+1} \leqq \underset{\sim}{S}_{k+1},$$

so (vii) holds for the integer k. It follows therefore that if (24) holds for all k, then (vii) necessarily holds also. In this event we can replace $\underset{\sim}{s}_{k+1}$ in (24) by s_{k+1}.

Example 1. Suppose unsatisfied demand is backlogged in period k so that $v_k(y, t) = y - t$. Then (24) reduces to $\bar{S}_k - \underset{\sim}{s}_{k+1} \leqq \eta_k$. Thus if the minimal demand in period k is at least $\bar{S}_k - \underset{\sim}{s}_{k+1}$, then $(\underset{\sim}{s}_k, \underset{\sim}{S}_k)$ is optimal for period k by Corollary 1. A special case of this result is established in [12, p. 545] for the stationary case.

The next example illustrates the application of Theorem 3(b).

Example 2. Suppose unsatisfied demands are backlogged so that $v_j(y, t) = y - t$. Also let $a_{k+1} = \underset{\sim}{s}_{k+1}$ and let $\{a_j\}$ be defined recursively by $a_j - \eta_j = a_{j+1}$ for $j \leqq k$. Thus $a_i = \underset{\sim}{s}_{k+1} + \sum_{j=i}^{k} \eta_j$ for $i \leqq k$. Now let i be an integer for which

$$\bar{S}_j \leqq \underset{\sim}{s}_{k+1} + \sum_{t=j}^{k} \eta_t, \qquad j = i, i + 1, \cdots, k. \tag{25}$$

Then the hypotheses of Theorem 3(b) are evidently satisfied. In particular, if $\eta_t = 0$ for $i \leqq t \leqq k$, then the optimal policy in periods $i, i + 1, \cdots, k$ may be determined without evaluating $f_{k+1}(\cdot)$ if $\bar{S}_j \leqq \underset{\sim}{s}_{k+1}$ for $i \leqq j \leqq k$.

We remark that if (i)–(vii) hold and if $K_i = 0$ for all i, then we choose $\underset{\sim}{s}_i$, $\bar{s}_i$, and $\bar{S}_i$ equal to $\underset{\sim}{S}_i$ so $s_i = S_i = \underset{\sim}{S}_i$ for all i by Theorem 1. In this event the hypothesis (vii) is equivalent to the hypothesis that (24) holds for all k. This observation together with Corollary 1 establishes the next result which is known from [10], [11].

COROLLARY 2. *If* (i)–(vii) *hold and if* $K_i = 0$ *for all* i, *then* $\{(\underline{S}_i, \underline{S}_i)\}$ *is an optimal policy.*

4. Applications and extensions. In this section we discuss some applications and extensions of our results.

Applications of the basic lemmas. Lemmas 1 and 2 are useful in establishing bounds on the ordering regions and order quantities even where $-G_i(y)$ is not unimodal. We shall illustrate this point under the assumption that $K_i = 0$ for all i, leaving the other case to the reader. Suppose $G_i(y)$ appears as in Fig. 2. The domain of $G_i(y)$ is divided into six regions labeled 1, 2, $\cdots$, 6. If period i were considered by itself, and if the initial inventory in period i fell in an odd-numbered region, it would be optimal to order to the upper bound of that region, viz., to U_1, U_3, or U_5 as appropriate. If the initial inventory in period i fell in an even-numbered region, no order should be placed. If $i < n$, then the above policy need not be optimal for the n-period model. However, suppose y lies in an even-numbered region. Then by Lemma 1,

$$J_i(y') - J_i(y) \geqq G_i(y') - G_i(y) \geqq 0 \tag{26}$$

for all $y' \geqq y$ provided (v) holds so it is optimal not to order in period i for the n-period model. Notice also that the above inequality tells us that if the initial inventory level in period i is a and if it is optimal to order, then the inventory level after ordering must lie in the interval $[b, c]$.

If (v) holds, if $v_j(U_1, t) \leqq U_1$ for $t \geqq \eta_j$, and if $G_j(y)$ is nonincreasing in y on $(-\infty, U_1]$ for $j \geqq i$, then by Lemma 2, $J_i(y)$ is minimized on $(-\infty, U_1]$ at $y = U_1$. Similarly from (26), $J_i(y)$ is minimized on $[U_1, \infty)$ at $y = U_1$. Combining these remarks we see that $J_i(y)$ is minimized on $(-\infty, \infty)$ at $y = U_1$. Hence, in region 1 it is optimal to order up to U_1.

Variation of the bounds over time. It is of interest to determine how s_i

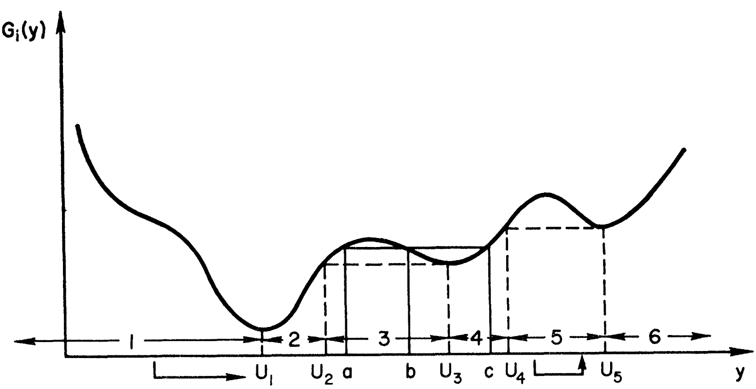

FIG. 2

and S_i vary over time in relation to the variation of the cost functions and demand distributions. Although this appears to be quite difficult, we can instead examine how the bounds on s_i and S_i vary over time. Such studies are of interest in their own right and because they provide us with a tool for determining conditions under which the hypotheses (vii), (17)–(20), (24), (25) of our several results hold. Throughout this subsection we shall assume for simplicity that sufficient regularity conditions are imposed to permit differentiation and interchange of differentiation with integration where required.

As a preliminary we record several lemmas from [11]. Let I be a subset of the integers $1, \cdots, n$.

LEMMA 3. *If $\partial W_i(y, t)/\partial y$ is nonincreasing in $t \geqq \eta_i$ and $i \in I$ for each y, and if $\Phi_i(t) \geqq \Phi_j(t)$ for all t and $i, j \in I$, $i < j$, then*

$$(27) \qquad G_i'(y) \geqq G_j'(y), \qquad i, j \in I, \quad i < j, \quad \text{and all } y.$$

LEMMA 4. *If $W_i(y, t) = W_i^1(y - t) + W_i^2(t)$ for some functions W_i^1, W_i^2, if $dW_i^1(z)/dz$ is nonincreasing in $i \in I$ and nondecreasing in z, and if $\Phi_i(t) \geqq \Phi_j(t - b_{ij})$ for all t and $i, j \in I$, $i < j$, and some numbers b_{ij}, then*

$$(28) \qquad G_i'(y) \geqq G_j'(y - b_{ij}), \qquad i, j \in I, \quad i < j, \quad \text{and all } y.$$

LEMMA 5. *If $\lambda = 0$, if $W_i(y, t) = W^1(y - t) + W_i^2(t)$ for some functions W^1, W_i^2, and if $\Phi_i(t) = \Phi(t - \eta_i)$ for all t and $i \in I$ for some distribution Φ, then*

$$(29) \qquad G_i(y) = G(y - \eta_i) + Q_i, \qquad i \in I, \quad \text{and all } y,$$

where Q_i is a constant and $G(y) = \int_{-\infty}^{\infty} W(y - t)\, d\Phi(t)$.

In the remainder of this subsection we assume for simplicity that $K_i = K$ and $\alpha_i = \alpha$ ($\leqq 1$) for all i, and $\lambda = 0$. Our methods can be applied without these hypotheses, but not without expanding the exposition. See in particular [11] for conditions under which the hypotheses of Lemmas 3 and 4 are satisfied when $\lambda > 0$. There is also an analog of Lemma 5 when $\lambda > 0$.

Let $\mathcal{S}_i = (\underline{s}_i, \bar{s}_i, \underline{S}_i, \bar{S}_i)$, $B_{ij} = (b_{ij}, b_{ij}, b_{ij}, b_{ij})$, and $H_i = (\eta_i, \eta_i, \eta_i, \eta_i)$. Let $\mathcal{S} = (\underline{s}, \bar{s}, \underline{S}, \bar{S})$, where $\underline{s}$, $\bar{s}$, $\underline{S}$, $\bar{S}$ are defined for the function $G(\cdot)$ (see Lemma 5) in the usual way. For definiteness where $\mathcal{S}_i$ is not uniquely defined we choose it as follows. First pick the smallest possible $\underline{S}_i$. Then pick the smallest $\underline{s}_i$, $\bar{s}_i$, and $\bar{S}_i$. Do the same for $\mathcal{S}$. The following theorem, which is an easy consequence of Lemmas 3–5, describes how $\mathcal{S}_i$ varies over time in relation to the variation of W_i and Φ_i (as reflected in G_i) over time.

THEOREM 4. *Suppose* (i)–(iv) *hold.*

(a) *If* (27) *holds, then* $\mathcal{S}_i \leqq \mathcal{S}_j$ *for* $i, j \in I$, $i < j$.

(b) *If* (28) *holds, then* $\mathcal{S}_i - B_{ij} \leqq \mathcal{S}_j$ *for* $i, j \in I$, $i < j$.

(c) *If* (29) *holds, then* $\mathcal{S}_i - H_i = \mathcal{S}$ *for* $i \in I$.

Satisfying the hypotheses of the main results. In this subsection we use Theorem 4 to give conditions under which the important hypotheses (vii) and (24) of Theorem 1 and Corollary 1 respectively are satisfied. We begin by giving conditions under which (vii) holds.

It will be convenient in what follows to assume that there is an extended real number θ such that

$$(30)\quad v_i(y, t) \leqq \max(\theta, y - \eta_i), \quad \text{for} \quad t \geqq \eta_i, \quad \text{all } i, \quad \text{and all } y.$$

As an example, if $\eta_i \geqq 0$ and unsatisfied demands are backlogged so $v_i(y, t) = y - t$, then (30) holds with $\theta \geqq -\infty$. Alternatively if $\eta_i \geqq 0$ and if unsatisfied demands are lost, so $v_i(y, t) = \max(y - t, 0)$, then (30) holds with $\theta \geqq 0$. In applications θ should be chosen as small as possible. Thus $\theta = -\infty$ in the backlog case and $\theta = 0$ in the lost sales case.

The following result is a simple consequence of Theorem 4.

COROLLARY 3. *Suppose* (i)–(iv) *and* (30) *hold,* $I = \{1, 2, \cdots, n\}$, *and* $\underline{S}_i \geqq \theta$ *for all* $i > 1$.

(a) *If* (27) *holds and* $\eta_i \geqq 0$ *for all* $i < n$, *then* (vii) *holds.*

(b) *If* (28) *holds and* $\eta_i - b_{i,i+1} \geqq 0$ *for all* $i < n$, *then* (vii) *holds.*

(c) *If* (29) *holds and* $\eta_i \geqq 0$ *for all* $i \leqq n$, *then* (vii) *holds.*

The next corollary gives a condition ensuring that the hypothesis (24) of Corollary 1 holds.

COROLLARY 4. *If* (i)–(iv), (vii), (30) *hold, if* (28) *holds with* $I = \{k, k+1\}$, *if* $\mathcal{S}_{k+1} \geqq \theta$, *and if*

$$(31)\qquad \bar{S}_k - \mathcal{S}_k \leqq \eta_k - b_{k,k+1},$$

then (24) *holds.*

Proof. $v_k(\bar{S}_k, t) \leqq \max(\theta, \bar{S}_k - \eta_k) \leqq \max(\theta, \mathcal{S}_k - b_{k,k+1}) \leqq \max(\theta, \mathcal{S}_{k+1}) = \mathcal{S}_{k+1} = \mathcal{S}_{k+1}$.

Stationary infinite horizon models. This paper is primarily concerned with a finite horizon model. If the model is stationary, i.e., G_i, K_i, α_i, Φ_i are independent of i, then it is convenient to consider an infinite period version of the model. In this case fairly obvious modifications of Iglehart's results and proofs [1], [2] (see also [12, pp. 530–531]) for $0 \leqq \alpha \leqq 1$ show that if (i)–(vii) hold, there is an optimal (s, S) policy with the optimal choice of parameters being independent of time and satisfying (6), (7). Methods for computing these parameters are discussed in [8] and [12].

Restrictions on inventory levels. In some applications it may be desirable to limit the choice of the inventory y_i on hand and on order after ordering in period $i(= 1, 2, \cdots, n)$ to an interval $[\underline{y}_i, \bar{y}_i]$ say. The upper bound $\bar{y}_i$ might reflect limitations on storage space while the lower bound $\underline{y}_i$ could reflect a desire to limit the size of the backlogged demand. As a specific illustration, suppose demands occur over only the first $n - 1$ periods, so $D_i = 0$, $i \geqq n$. Then we may wish to require that no unsatisfied demand exist at the end of period $n + \lambda$.[5] This may be accomplished by setting $\underline{y}_n = 0$ so $y_n \geqq 0$. This implies $y_{n+\lambda} \geqq 0$ if $v_i(y, 0) \geqq 0$ for $y \geqq 0$ and $n \leqq i$.

In other applications it is natural to suppose that the demands are integers. Of course this restriction is already allowed in our formulation. However, in such cases it is usually necessary to impose the additional restriction that the order quantities and stock levels be integers. We shall now generalize our original model and results to provide for such integer restrictions and for bounds on the stock levels.

Let Y_i denote the nonempty set of admissible stock levels y_i on hand and on order after ordering in period i. Let $\underline{y}_i = \inf Y_i$ and $\bar{y}_i = \sup Y_i$. Let $\mathfrak{D}_i$ denote the (Borel) set of possible values of the demand D_i in period i. Let X_{i+1} denote the nonempty set of possible values of the stock on hand and on order before ordering in period $i + 1$. We naturally impose the consistency condition that $v_i(y, t) \in X_{i+1}$ for all $y \in Y_i$ and $t \in \mathfrak{D}_i$. In addition we suppose that if the stock on hand and on order before ordering in period i is at least $\underline{y}_i$ in period i, then it is possible *not* to order in period i. Formally, we assume that $x \in X_1$ and $\underline{y}_i \leqq x$ imply $x \in Y_i$, $i = 1, 2, \cdots, n$, where $X_1 \equiv \{x_1\}$. We also suppose that the domains of $f_i(\cdot)$, $G_i(\cdot)$, $J_i(\cdot)$, and $v_i(\cdot, \cdot)$ are respectively X_i, Y_i, Y_i, and $Y_i \times \mathfrak{D}_i$. Moreover, we shall replace (i)–(iii) respectively by:

(i′) (i) holds and Y_i is closed;
(ii′) either (ii) holds or $\bar{y}_i < \infty$;
(iii′) either (iii) holds or $-\infty < \underline{y}_i$.

If (i′) holds, we may define

$$G_i^+(y) = G_i(\inf\{z \mid z > y, z \in Y_i\})$$

for $y < \bar{y}_i$ and $G_i^+(\bar{y}_i) = \infty$ if $\bar{y}_i < \infty$. Also

$$G_i^-(y) = G_i(\sup\{z \mid z < y, z \in Y_i\})$$

for $\underline{y}_i < y$ and $G_i^-(\underline{y}_i) = \infty$ if $-\infty < \underline{y}_i$. For example, if $Y_i = (-\infty, \infty)$, then $G_i^-(y) = G_i(y) = G_i^+(y)$, while if Y_i is the set of integers, then $G_i^-(y) = G_i(y - 1)$ and $G_i^+(y) = G_i(y + 1)$ for $y \in Y_i$.

If (i′)–(iii′) hold, there are a number $\underline{S}_i \in Y_i$ that minimizes $G_i(\cdot)$ on Y_i and numbers $\underline{s}_i$ $(\leqq \underline{S}_i)$ and $\bar{S}_i$ $(\geqq \underline{S}_i)$ such that

$$G_i(\bar{S}_i) \leqq G_i(\underline{S}_i) + \alpha_i K_{i+1} \leqq G_i^+(\bar{S}_i)$$

---

[5] I am indebted to G. Lieberman for a discussion of this application.

and

$$G_i(\underline{s}_i) \leqq G_i(\underline{S}_i) + K_i \leqq G_i^-(\underline{s}_i).$$

If in addition (vi) holds, there is a number $\bar{s}_i$, $\underline{s}_i \leqq \bar{s}_i \leqq \underline{S}_i$, such that

$$G_i(\bar{s}_i) \leqq G_i(\underline{S}_i) + K_i - \alpha_i K_{i+1} \leqq G_i^-(\bar{s}_i).$$

It is easy to check that the statements and proofs of Lemmas 1 and 2 remain valid in our new setup. Also Theorem 1 holds provided we replace (i)–(iii) by (i′)–(iii′) and replace (7) by

$$G_i^-(s_i) \geqq G_i(S_i) + (K_i - \alpha_i K_{i+1}). \tag{7′}$$

Only obvious modifications of the proof of Theorem 1 are required.

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