A NOTE ON THE OPTIMALITY OF (S, s) POLICIES IN INVENTORY THEORY*

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In this paper a proof of Herbert Scarf's on the optimality of (S,s) policies in inventory theory is extended to cases where differentiability of cost functions may not be available. In addition to an ordering cost composed of a unit cost plus a reorder cost all that is needed in the proof is that the expected holding and shortage cost function be convex.

In a notable paper Herbert Scarf proved the optimality of (S, s) policies in inventory problems with an ordering cost composed of a unit cost plus a reorder cost and other usual conditions such as linear holding and shortage costs [1]. The stratagem used in the proof was the notion of a K-convex function as applied to the G functions familiar in these problems. As Scarf assumed differentiability of the G functions, the purpose of this note is to generalize his proof to cases where differentiability may not be available.[1] To facilitate comparisons the notation and method of presentation closely follow Scarf's paper.

Let the ordering cost function be

$$(1) \qquad c(z) = \begin{cases} 0 & z = 0 \\ K + c\cdot z & z > 0 \end{cases}$$

where z is the order, c the unit ordering cost and K the reorder cost and assume delivery is immediate. Suppose the holding cost $h(\cdot)$ is a function of the excess of supply over demand at the end of a period and the shortage cost $p(\cdot)$ is a function of the excess of demand over supply at the end of a period. The expected holding and shortage costs for a period are then given by

$$(2) \qquad L(y) = \begin{cases} \int_0^y h(y-\xi)\varphi(\xi)\,d\xi + \int_y^\infty p(\xi-y)\varphi(\xi)\,d\xi & y \geqq 0 \\ \int_0^\infty p(\xi-y)\varphi(\xi)\,d\xi & y < 0 \end{cases}$$

where φ is the density function of demand and y is the stock after delivery of an order. Assume that $L(y)$ is convex.

Suppose the initial stock is x and the horizon has n periods. Let $C_n(x)$ be the minimum present expected value of costs over the horizon so that

$$(3) \qquad C_n(x) = \min_{y \geqq x} \left\{ c(y-x) + L(y) + \alpha \int_0^\infty C_{n-1}(y-\xi)\varphi(\xi)\,d\xi \right\}$$

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[1] In an exchange of letters the author learned that Scarf, independently, has obtained a proof identical to the one offered in this note. The author is also indebted to Herbert Scarf for helpful comments and suggestions.

where $0 < \alpha < 1$ is the discount factor. Now define

$$G_n(y) = c \cdot y + L(y) + \alpha \int_0^\infty C_{n-1}(y - \xi)\varphi(\xi)\, d\xi. \tag{4}$$

In his paper Scarf shows that if $G_n(y)$ is K-convex the optimal policy is (S_n, s_n). He then proves that $G_n(y)$ is K-convex in cases where $G_n(y)$ is differentiable.

If differentiability is not assumed the definition of K-convexity is the following. Let $K \geqq 0$. Then $f(x)$ is K-convex if

$$K + f(a + x) - f(x) - a\left[\frac{f(x) - f(x - b)}{b}\right] \geqq 0 \tag{5}$$

for all positive a and b and all x. We now prove inductively that $G_n(y)$ is K-convex and thus that the optimal policy is (S_n, s_n).

$G_1(y) = c \cdot y + L(y)$ is 0-convex and therefore K-convex. Assume $G_2, \cdots, G_n$ are K-convex. To prove the K-convexity of $G_{n+1}(y)$ it is sufficient, from (4), to show that

$$\int_0^\infty C_n(y - \xi)\varphi(\xi)\, d\xi$$

is K-convex and using properties of K-convexity, not noted here, it is sufficient to prove that $C_n(x)$ is K-convex.

By the assumption that G_n is K-convex the optimal policy for the n-period problem is (S_n, s_n) so that

$$C_n(x) = \begin{cases} K - c \cdot x + G_n(S_n) & x < s_n \\ \quad - c \cdot x + G_n(x) & x \geqq s_n . \end{cases} \tag{6}$$

Using (6) and the definition of K-convexity we evaluate four cases:

Case 1. $x - b > s_n$. In this region $C_n(x)$ is a linear function plus a K-convex function and therefore is K-convex.

Case 2. $x - b < s_n < x$. In this region

$$\begin{aligned} K + C_n(x + a) - C_n(x) - \frac{a}{b}\Big\{C_n(x) - C_n(x - b)\Big\} \\ = K + G_n(x + a) - G_n(x) - \frac{a}{b}\Big\{G_n(x) - K - G_n(s_n)\Big\} \\ = K + G_n(x + a) - G_n(x) - \frac{a}{b}\Big\{G_n(x) - G_n(s_n)\Big\} = V. \end{aligned}$$

Now let

$$K + G_n(x + a) - G_n(x) - \frac{a}{x - s_n}\Big\{G_n(x) - G_n(s_n)\Big\} = V^*.$$

Note that $K + G_n(x + a) - G_n(x) \geqq 0$ since otherwise it would pay to order from x to $x + a$, which is excluded by the (S_n, s_n) policy, and if $G_n(x) - G_n(s_n) \leqq 0$, that $V \geqq 0$ and $V^* \geqq 0$ since $x - s_n > 0$. However, if $G_n(x) - G_n(s_n) \geqq 0$, which may be true for some $x > S_n$, then $V \geqq V^*$ since $b > x - s_n$. Thus if $V^* \geqq 0$ then $V \geqq 0$ for all permissible a, b, and x. However, $V^* \geqq 0$ since $G_n(x)$ is K-convex and as a consequence, $C_n(x)$ is K-convex.

Case 3. $x < s_n < x + a$. In this region

$$K + C_n(x + a) - C_n(x) - \frac{a}{b}\left\{ C_n(x) - C_n(x - b) \right\}$$

$$= G_n(x + a) - G_n(S_n) \geqq 0$$

and $C_n(x)$ is K-convex. The inequality holds since S_n yields the minimum of G_n.

Case 4. $x + a < s_n$. In this region $C_n(x)$ is linear and therefore K-convex. The proof that the (S_n, s_n) policy is optimal is now complete.

A similar argument applies as well to the case of a time lag in delivery.

Finally, it is important to note that a continuous density of demand is not necessary in the proof. Since all we require is that the sum of expected holding and shortage costs be convex, the proof may be adapted to any general demand distribution satisfying this requirement for any holding cost and shortage cost functions.

Reference

1. Scarf, Herbert, "The Optimality of (S,s) Policies in the Dynamic Inventory Problem," in Arrow, Karlin, and Suppes, *Mathematical Methods in the Social Sciences 1959*, Stanford University Press, 1960, pp. 196–202.