Homework set 6

A *hyperplane* in a normed space V is a set of the form $H = x_0 + H_0$, where $H_0 \subset V$ is a subspace such that $\operatorname{codim} H_0 := \dim V/H_0 = 1$.

Problem 1. Let V be a normed space. Show that H is a closed hyperplane if and only if there exists a bounded linear functional $f \in V^\vee$ such that $H = f^{-1}(c)$ for some $c \in \mathbf{R}$. Then show that $V \setminus H$ consists of two disjoint connected components.

Solution. Let $H = x_0 + H_0$ be a closed hyperplane. Then $H_0 \subset V$ is a closed subspace, so there exists a bounded linear functional $f \in V^\vee$ such that $H_0 \subset \ker f$. We then have the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H_0 \arrow[r] \arrow[d, "j"] & V \arrow[r, "q"] \arrow[d, "\mathrm{id}"] & V/H_0 \arrow[r] & 0 \\
0 \arrow[r] & \ker f \arrow[r] & V \arrow[r, "f"] & \mathbf{R} \arrow[r] & 0
\end{tikzcd}$$

from which we define a linear map $p : V/H_0 \to \mathbf{R}$ by $[x] \mapsto f(x)$, where $x \in [x] \in V/H_0$. This map is linear and nonzero, and thus an isomorphism. It follows that j is also an isomorphism. Therefore $H_0 = \ker f$, so $H = f^{-1}(f(x_0))$.

Let now $f \in V^\vee$. If x is such that $f(x) = c$ then $f^{-1}(c) = x + \ker f$, so it is a hyperplane, and closed because $\ker f = f^{-1}(0)$ is closed.

The complement is

$$V \setminus H = f^{-1}((-\infty, c) \cup (c, \infty)) = f^{-1}((-\infty, c)) \cup f^{-1}((c, \infty)) =: V_- \cup V_+$$

which are two open disjoint sets. If $x, y \in V_+$ then

$$f((1-t)x + ty) = (1-t)f(x) + tf(y) > (1-t)c + tc = c$$

so the segment $\{(1-t)x + ty \mid 0 \le t \le 1\} \subset V_+$. The set is thus path connected. $\square$

Problem 2. Let V be a normed space and let

$$S(r) = \{x \in V \mid \|x\| = r\}, \quad B(r) = \{x \in V \mid \|x\| < r\},$$

be the sphere and ball of radius r. Show that for any $x_0 \in S(r)$ there exists a hyperplane $H \subset V$ that contains x_0 such that the open ball $B(r)$ is entirely contained in one component of $V \setminus H$.

Solution. Let $S = \mathbf{R}x_0$ and define $f : S \to \mathbf{R}$ by $f(x_0) = r$, where $r = |x_0|$, and extend by linearity. Then

$$|f| = \sup_{|\lambda x_0|=1} |f(\lambda x_0)| = \sup_{|\lambda| r=1} |\lambda| r = 1.$$

By Hahn–Banach we can extend this to $f \in V^\vee$ such that $|f| = 1$. For any $x \in B(r)$ we then have

$$|f(x)| \leq |f||x| < r,$$

so $B(r) \subset \{x \in V \mid f(x) < r\}$ while $x_0 \in \{x \in V \mid f(x) = r\}$. □

The *annihilator* of a set M in a normed space V is the set

$$M^\perp = \{f \in V^\vee \mid f(x) = 0 \text{ for all } x \in M\}.$$

Problem 3. Let V be a normed space and $S \subset V$ a subspace. Show that $S^\vee$ is isometric to $V^\vee/S^\perp$.

Solution. We have short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\rho & \longrightarrow & V^\vee & \xrightarrow{\rho} & S^\vee & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & S^\perp & \longrightarrow & V^\vee & \longrightarrow & V^\vee/S^\perp & \longrightarrow & 0,
\end{array}$$

where the upper row is exact because of Hahn–Banach. The map $\rho : V^\vee \to S^\vee$ is just the restriction map $f \mapsto f_{|S}$. Its kernel is

$$\ker\rho = \{f \in V^\vee \mid f_{|S} = 0\} = S^\perp.$$

We thus get a well-defined linear map $\psi : S^\vee \to V^\vee/S^\perp$ by $\psi(f) = [\hat{f}]$, where $\hat{f}$ is an extension of f.

This map is injective, because if $\psi(f) = [\hat{f}] = 0$ then $\hat{f} \in S^\perp$ and $f = \rho(\hat{f}) = 0$. It is also surjective: Let $[g] \in V^\vee/S^\perp$, where $g \in V^\vee$. Set $f = g_{|S}$. Then $\psi(f) = [g]$.

Note that every $x \in V$ defines a linear functional $e_x \in (V^\vee)^\vee$ by $e_x(f) = f(x)$. We have $S^\perp = \bigcap_{x\in S} \ker e_x$, so $S^\perp$ is the intersection of closed sets and thus closed. Therefore $V^\vee/S^\perp$ is a normed space. For $f \in S^\vee$ we have

$$|\psi(f)| = |[\hat{f}]| = \inf_{\hat{f}_{|S}=f} |\hat{f}| = |f|,$$

where the last equality is by Hahn–Banach. □

Problem 4. Let V be a normed space and $S \subset V$ a closed subspace. Show that $(V/S)^\vee$ is isometric to $S^\perp$.

Solution. Since S is closed the quotient V/S is a normed space. We define a map $\psi : S^\perp \to (V/S)^\vee$ by

$$\psi(f)([x]) = f(x).$$

We have to show this is a well-defined linear map and that $\psi(f)$ is bounded. By definition we have $f_{|S} = 0$, so the map is well-defined and linear.

This map is injective: If $\psi(f)([x]) = 0$ for all $[x]$ then $f(x) = 0$ for all x, so $f = 0$. The map is also surjective: Let $g \in (V/S)^\vee$. Define a map $f \in V^\vee$ by $f(x) = g([x])$. Then $f_{|S} = 0$ so $f \in S^\perp$ and $\psi(f) = g$.

Now

$$|\psi(f)| = \sup_{[x]=1} |\psi(f)([x])|.$$

We have $|[x]| = \inf_{y\in S} |x - y|$, so if $|[x]| = 1$ there exists $x \in V$ and (y_n) in S such that $|x - y_n| \to 1$ as $n \to \infty$ (and $|x - y| \geq 1$ for all $y \in S$). For such an x we have

$$|\psi(f)([x])| = |f(x)| = \frac{|f(x)|}{|x|}|x| \leq |f|\,|x| \leq |f|(|y| + |x - y|)$$

for $y \in S$. Taking $\inf$ over $y \in S$ we get $|\psi(f)([x])| \leq |f|$ when $|[x]| = 1$, so $|\psi(f)| \leq |f|$.

For any x we have $|[x]| \leq |x|$, so

$$|\psi(f)| = \sup_{|[x]|\neq 0} \frac{|\psi(f)([x])|}{|[x]|} \geq \sup_{x\notin S} \frac{|f(x)|}{|x|} = \sup_{x\neq 0} \frac{|f(x)|}{|x|} = |f|$$

because $f_{|S} = 0$, so $|\psi(f)| = |f|$. $\square$