Homework set 3

Due by 15:00 on Monday, September 11, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`. You may quote problems from older homework sets and results we've read in the textbook if you feel like they help.

Recall that in last week's problems we defined a subset $E \subset V$ of a metrizable vector space to be *bounded* if for any neighborhood U of 0 there exists a $\lambda > 0$ such that $E \subset \mu U$ for any $\mu \geq \lambda$.

Problem 1. Let V be a normed vector space. Show that $E \subset V$ is bounded in the metrizable sense if and only if it is bounded in the usual sense, that is, there exists $r > 0$ such that $|x| \leq r$ for all $x \in E$.

Problem 2. Let V be finite-dimensional and let $|\cdot|_1$ and $|\cdot|_2$ be two norms on V. Show that the norms are equivalent, meaning that there exist constants c and C such that

$$c|x|_1 \leq |x|_2 \leq C|x|_1$$

for all $x \neq 0$.

A set $E \subset V$ is *convex* if for any $x, y \in E$ the vectors $\lambda x + (1-\lambda)y$ are also in E for $0 \leq \lambda \leq 1$.

Problem 3.

1. Show that the unit ball in a normed vector space is convex.
2. Show that a linear subspace $E \subset V$ is convex.
3. If $f : V \to W$ is linear and $E \subset V$ is convex, show that $f(E)$ is convex.

Problem 4. Let V and W be normed vector spaces and let $f : V \to W$ be linear. Show that f is continuous if and only if it is bounded, in the sense that there exists a constant $C > 0$ such that $|f(x)| \leq C|x|$ for all x.

Problem 5. Let V be a normed vector space. We are going to prove that V can be embedded in a Banach space.

1. Define X to be the set of Cauchy sequences in V. Show that X may be given the structure of a vector space.
2. Show that $(x_n) \mapsto \lim_{n\to\infty} |x_n|$ defines a seminorm on X; that is a function that satisfies the conditions to be norm except $|x| = 0$ does not imply $x = 0$.

3. Show that $N := \{x \in X \mid |x| = 0\}$ is a closed subspace of X, and that the seminorm on X induces a norm on the quotient space X/N.

4. Show that X/N is complete, and thus a Banach space.

5. Show that the map $f : V \to X/N$ that sends x to the image of the sequence $(x, x, \ldots)$ under the quotient map is linear, injective, and continuous.

6. Show that X/N satisfies the following universal property: If Y is a Banach space and $f : V \to Y$ is a continuous linear map, then there is a unique continuous linear map $\hat{f} : X/N \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d] \arrow[dr, "f"] & \\ X/N \arrow[r, "\hat{f}"] & Y \end{tikzcd}$$