Homework set 2

Due by 15:00 on Monday, September 4, 2023.

Please select three problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

A *metrizable vector space* is a vector space V that is equipped with a metric d for which the addition map $(x, y) \mapsto x+y$ and multiplication map $(\lambda, x) \mapsto \lambda x$ are continuous. We suppose here that V is defined over a field k such that $\mathbf{R} \subset k$ (for example, the real or complex numbers[1]).

Problem 1. Let $T_a(x) = x + a$ and $M_\lambda(x) = \lambda x$ for $a \in V$ and $\lambda \in k$. Show that T_a and M_λ are homeomorphisms of V with itself; that is, they are continuous maps that have a continuous inverse.

Problem 2. Let V and W be metrizable vector spaces and $f : V \to W$ a linear map. Show that f is continuous if and only if it is continuous at 0.

A subset $E \subset V$ of a metrizable vector space is *bounded* if for any neighborhood U of 0 there exists a $\lambda > 0$ such that $\mu E \subset U$ for any $\mu \geq \lambda$.

Note that this is *not* the same notion of boundedness we get from the metric on V; it can be defined if we only have a topology and not a metric. Sets can be bounded in one notion and not the other.

Problem 3. Let U be an open set that contains 0. Let (r_j) be an increasing sequence of positive real numbers such that $r_j \to \infty$. Show that $V = \bigcup_{j=1}^{\infty} r_j U$. Conclude that if $K \subset V$ is compact, then K is bounded.

Problem 4. Let $E \subset V$ be a set. Show that the following are equivalent:

1. E is bounded.
2. If (x_n) is a sequence in E and (λ_n) is a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$, then $\lambda_n x_n \to 0$ as $n \to \infty$.

A map $f : V \to W$ between metrizable vector spaces is *bounded* if it maps bounded sets to bounded sets.

Problem 5. Let V and W be metrizable vector spaces and $f : V \to W$ a linear map. Show that if f is continuous then it is bounded.[2] Find spaces V and W and a linear function $f : V \to W$ that is not continuous.

[1] Or the field $\mathbf{C}(X)$ of rational functions over $\mathbf{C}$ if you want to get weird.

[2] The converse is true if we assume the metric is translation invariant, that is, it satisfies $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in V$.