Homework set 1

Due by 12:00 on August 28, 2023.

We haven't read anything in the course yet so this is a warmup set. Please select two problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`. You are allowed to collaborate on the solutions, just note whether you did so.

The functional analysis we will cover is linear, meaning it takes place in vector spaces and deals with linear functions. We assume everyone knows what a vector space over a field is. In the course we will only consider vector spaces over the real or complex numbers.

Problem 1 (3 points)**.** Let S be a set and let k be a field. Recall that a function $\phi : S \to k$ has *finite support* if the set $\operatorname{supp} \phi := \{s \in S \mid \phi(s) \neq 0\}$ is finite. Show that the set

$$\mathbf{F}(S) := \{\phi : S \to k \mid \operatorname{supp} \phi \text{ is finite}\}$$

can be given the structure of a vector space over k. Find a basis for the space. Find its dimension.

Recall that if V and W are vector spaces over a field k, then a map $f : V \to W$ is *linear* if $f(x+y) = f(x) + f(y)$ for all $x, y \in V$ and $f(\lambda x) = \lambda f(x)$ for all $\lambda \in k$ and $x \in V$.

Problem 2 (3 points)**.** Let S and R be nonempty sets and let $f : S \to R$ be a function. Show that there is an induced linear map $\mathbf{F}(f) : \mathbf{F}(R) \to \mathbf{F}(S)$. When is $\mathbf{F}(f)$ injective or surjective?

Problem 3 (3 points)**.** Give three examples of infinite-dimensional vector spaces that are not isomorphic. (Prove they are not isomorphic.)

Recall that the *dual space* of a vector space V over a field k is the space V^* of linear maps $f : V \to k$.

Problem 4 (3 points)**.** Let V and W be vector spaces over k and let $f : V \to W$ be a linear function. Show that there is an induced linear function $f^* : W^* \to V^*$. When is f^* injective or surjective?

Problem 5 (3 points)**.** Show that the map

$$\iota : V \to V^{**}, \quad x \mapsto (f \mapsto f(x))$$

is linear and injective. Conclude that if V is finite-dimensional, then V and V^{**} are isomorphic.

Problem 6 (3 points)**.** Let V be a vector space over k and let $S \subset V$ be a subspace. The quotient space V/S is defined to be the set

$$V/S := \{[v] \mid v \in V, [v] = [w] \text{ if } v - w \in S\}.$$

Show that V/S can be given the structure of a vector space over k and that the obvious map $q : V \to V/S$ is linear and surjective. Prove the following universal property of the quotient space:

If W is a vector space over k and $f : V \to W$ is a linear function that maps S to 0, then there is a unique linear map $g : V/S \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[d, "q"] \arrow[dr, "f"] & \\ V/S \arrow[r, "g"] & W \end{tikzcd}$$

commutes.