Homework set 8

Due by 15:00 on Monday, October 16, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

Problem 1. Let V be a Hilbert space. Show that $V^\vee$ is a Hilbert space when given the inner product

$$\langle f, \bar{g} \rangle = \langle y, \bar{x} \rangle,$$

where $x, y \in V$ and $x \mapsto f$ and $y \mapsto g$ under the Riesz representation theorem.

Problem 2. Recall that the annihilator of a subset $M \subset V$ of a normed space is the set

$$\operatorname{ann} M = \{f \in V^\vee \mid f(x) = 0 \text{ for all } x \in M\}.$$

Discuss the relationship between $M^\perp$ and $\operatorname{ann} M$ in a Hilbert space V.

Problem 3. Let V be a Hilbert space and let $S \subset V$ be a closed subspace. Show that the quotient space V/S is also a Hilbert space. Show that $S^\perp$ is isometric to V/S.

Problem 4. Let V and W be Hilbert spaces and (f_n) a sequence of bounded operators from V to W such that $f_n \to f$. Show that $f_n^* \to f^*$.

Problem 5. Let V and W be Hilbert spaces and $f : V \to W$ a bounded operator. Show that:

1. $\operatorname{im} f^* \subset (\ker f)^\perp$.
2. $(\operatorname{im} f)^\perp \subset \ker f^*$.
3. $\ker f = (\operatorname{im} f^*)^\perp$.

Problem 6. Let V and W be Hilbert spaces and $f : V \to W$ a bounded operator. Show that $f^* f : V \to V$ is a bounded self-adjoint operator and that $|f| = \sqrt{|f^* f|}$.