Homework set 3

Problem 1. Let V be a normed vector space. Show that $E \subset V$ is bounded in the metrizable sense if and only if it is bounded in the usual sense, that is, there exists $r > 0$ such that $|x| \leq r$ for all $x \in E$.

Solution. Suppose E is bounded in the metrizable sense. Take $U = B(1)$. There is a $\lambda > 0$ such that $E \subset \mu B(1)$ for any $\mu \geq \lambda$. But then $|x| \leq |\lambda|$ for any $x \in E$.

Suppose then that E is bounded in the norm sense. Then there exists $r > 0$ such that $E \subset B(r)$. Let U be an open neighborhood around 0. Then there exists $\varepsilon > 0$ such that $0 \in B(\varepsilon) \subset U$. It follows that $E \subset (\mu/\varepsilon)B(\varepsilon) \subset (\mu/\varepsilon)U$ for any $\mu \geq r$. □

Problem 2. Let V be finite-dimensional and let $|\cdot|_1$ and $|\cdot|_2$ be two norms on V. Show that the norms are equivalent, meaning that there exist constants c and C such that

$$c|x|_1 \leq |x|_2 \leq C|x|_1$$

for all $x \neq 0$.

Solution. Let's show that the map $x \mapsto |x|_1$ is continuous with respect to the metric defined by $|\cdot|_2$. Let $\varepsilon > 0$. We want to show that there is a $\delta > 0$ such that $|x|_1 < \varepsilon$ if $|x|_2 < \delta$.

Let's do this by constructing an increasing chain of subspaces of V on which this is true. First let $v \neq 0$ and let $L = \mathbf{R}v$ be a line. If $x, x_0 \in L$ then $x = \lambda v$ and $x_0 = \lambda_0 v$ for some $\lambda, \lambda_0 \in \mathbf{R}$, so

$$|x - x_0|_1 = |\lambda - \lambda_0||v|_1 = |\lambda - \lambda_0|\frac{|v|_1}{|v|_2}|v|_2 = \frac{|v|_1}{|v|_2}|x - x_0|_2.$$

If $\varepsilon > 0$ is given we pick $\delta > 0$ such that $|x - x_0|_2 < \varepsilon|v|_2/|v|_1$, and get that $|x - x_0|_1 < \varepsilon$.

Suppose then that $S \subset V$ is a proper subspace on which $|\cdot|_1$ is continuous and let $v \in V \setminus S$. We want to show $|\cdot|_1$ is continuous on $S + \mathbf{R}v$. Any vector therein can be written as $x + \lambda v$ with $x \in S$ and $\lambda \in \mathbf{R}$. Let $v_0 = x_0 + \lambda_0 v$ and $v = x + \lambda v$. Then

$$\begin{aligned}|v - v_0|_1 = |(x - x_0) + (\lambda - \lambda_0)v|_1 &\leq |x - x_0|_1 + |\lambda - \lambda_0||v|_1 \\ &= |x - x_0|_1 + |\lambda - \lambda_0|\frac{|v|_1}{|v|_2}|v|_2 \\ &= |x - x_0|_1 + \frac{|v|_1}{|v|_2}|\lambda v - \lambda_0 v|_2.\end{aligned}$$

Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $|x - x_0|_1 < \varepsilon/2$ and $|\lambda v - \lambda_0 v|_2 < \varepsilon|v|_2/(2|v|_1)$ if $|v - v_0|_2 < \delta$. Then $|v - v_0|_1 < \varepsilon$, so $|\cdot|_1$ is continuous at v_0.

We can thus construct an increasing flag $0 \subset L_1 \subset L_2 \subset \cdots$ of subspaces of V on which $|\cdot|_1$ is continuous. As V is finite dimensional, we eventually have $L_n = V$, so $|\cdot|_1$ is continuous.

Having proven this, we note that the unit sphere is compact since V is finite dimensional. As $x \mapsto |x|_1$ is continuous it attains a minimum c and maximum C on the unit sphere. If $x \in V$ there exists u in the unit sphere such that $x = |x|_2 u$. Then

$$|x|_1 = |x|_2|u|_1 \leq C|x|_2$$

for all x, and similarly $|x|_1 \geq c|x|_2$. □

Problem 3.

1. Show that the unit ball in a normed vector space is convex.
2. Show that a linear subspace $E \subset V$ is convex.
3. If $f : V \to W$ is linear and $E \subset V$ is convex, show that $f(E)$ is convex.

Solution. All obvious. □

Problem 4. Let V and W be normed vector spaces and let $f : V \to W$ be linear. Show that f is continuous if and only if it is bounded, in the sense that there exists a constant $C > 0$ such that $|f(x)| \leq C|x|$ for all x.

Solution. Suppose f is continuous. Then there is a $\delta > 0$ such that $f(B(\delta)) \subset B(1)$, so $f(B(1)) \subset B(1/\delta)$ by linearity. If $x \neq 0$ then $(1/|x| + \varepsilon)x \in B(1)$ for any $\varepsilon > 0$, so

$$\frac{1}{|x| + \varepsilon}|f(x)| = \left| f\left(\frac{1}{|x| + \varepsilon} x \right) \right| \leq \frac{1}{\delta}$$

and taking limits we get $|f(x)| \leq |x|/\delta$.

Suppose that f is bounded. We know that f is continuous if and only if it is continuous at 0. Let $\varepsilon > 0$ and let $\delta < \varepsilon/C$; then

$$|f(x)| \leq C|x| < \varepsilon$$

for any $x \in B(\delta)$, so f is continuous at 0. □

Problem 5. Let V be a normed vector space. We are going to prove that V can be embedded in a Banach space.

1. Define X to be the set of Cauchy sequences in V. Show that X may be given the structure of a vector space.

2. Show that $(x_n) \mapsto \lim_{n\to\infty} |x_n|$ defines a seminorm on X; that is a function that satisfies the conditions to be norm except $|x| = 0$ does not imply $x = 0$.

3. Show that $N := \{x \in X \mid |x| = 0\}$ is a subspace of X, and that the seminorm on X induces a norm on the quotient space X/N.

4. Show that X/N is complete, and thus a Banach space.

5. Show that the map $f : V \to X/N$ that sends x to the image of the sequence $(x, x, \ldots)$ under the quotient map is linear, injective, and continuous.

6. Show that X/N satisfies the following universal property: If Y is a Banach space and $f : V \to Y$ is a continuous linear map, then there is a unique continuous linear map $\hat{f} : X/N \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d] \arrow[dr, "f"] & \\ X/N \arrow[r, "\hat{f}"] & Y \end{tikzcd}$$

Solution. 1. We have $X = \{(x_n) \mid x_n \in V \text{ is Cauchy}\}$. Define $\lambda x = (\lambda x_n)$ and $x + y = (x_n + y_n)$. The first is clearly well defined and the second is also because

$$|x_n + y_n - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m|$$

so $(x_n + y_m)$ is Cauchy if (x_n) and (y_n) are. Then X is a vector space as it is a subspace of the space of all sequences in V, which is just $\prod_{\mathbf{N}} V$.

2. First off this is well-defined because if (x_n) is Cauchy then so is $(|x_n|)$ and $\mathbf{R}$ is complete. This behaves correctly with respect to scaling and satisfies the triangle inequality. However $|x| = 0$ only implies that (x_n) converges to 0, not that it is zero, so this is a seminorm.

3. Since $|\cdot|$ is a seminorm we see that $N = \{x \in X \mid |x| = 0\}$ is a linear subspace. We attempt to define a norm on X/N by $|[x]| = |x|$. Let $[y] = 0$. Then

$$|[x + y]| = |x + y| \leq |x| + |y| = |x| = |[x]|$$

and

$$|[x]| = |x| = |x + y - y| \leq |x + y| + |y| = |x + y| = |[x + y]|$$

so $|\cdot|$ is well-defined on X/N.

This satisfies $|\lambda[x]| = |\lambda||[x]|$ and

$$|[x]+[y]| = |[x+y]| = |x+y| \leq |x|+|y| = |[x]|+|[y]|.$$

If $|[x]| = 0$ then $|x| = 0$ so $x \in N$ and $[x] = 0$. We thus have a norm.

4. Let (x_n) be a Cauchy sequence in X/N. Then there are x_{nm} in V such that $x_n = x_{mn}$, and each $(x_{mn})_m$ is a Cauchy sequence. Let $y = (x_{nn})$. Then

$$x_{nn} - x_{mm} = (x_{nn} - x_{nl}) + (x_{nl} - x_{ml}) + (x_{ml} - x_{mm})$$

each of which we can make $< \varepsilon/3$, so (x_{nn}) is Cauchy. Similarly we see that $(x_n) \to y$, so X is complete.

5. The map is clearly linear and injective. We have $|(x, x, \ldots)| = |x|$, so it is bounded and thus continuous.

6. Let Y be a Banach space and let $f : V \to Y$ be continuous. We define $\hat{f} : X/N \to Y$ by

$$\hat{f}([x]) = \lim_{n\to\infty} f(x_n),$$

where $[x] = (x_n)$ is the image of a Cauchy sequence in V. This is well-defined because (x_n) is a Cauchy sequence in V and f is continuous, so $(f(x_n))$ is a Cauchy sequence in Y and thus has a limit as Y is Banach. Further, if $x_n \to 0$ then $f(x_n) \to 0$, so this maps all of N to 0. We see that $\hat{f}$ is a linear map.

As $f : V \to Y$ is continuous there is a $C > 0$ such that $|f(x)| \leq C|x|$ for all $x \in V$. Let $[x] \in X/N$ and let (x_n) be a Cauchy sequence that represents $[x]$. Then $|f(x_n)| \leq C|x_n|$ for all n, so

$$|\hat{f}([x])| = \left|\lim_{n\to\infty} f(x_n)\right| = \lim_{n\to\infty} |f(x_n)| \leq C \lim_{n\to\infty} |x_n| = C|[x]|$$

where we can pull the limit out because both limits exist. Then $\hat{f}$ is bounded and thus continuous. □