Homework set 8

Due by 15:00 on Monday, October 16, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

Problem 1. Let V be a Hilbert space. Show that $V^\vee$ is a Hilbert space when given the inner product

$$\langle f, \bar{g} \rangle = \langle y, \bar{x} \rangle,$$

where $x, y \in V$ and $x \mapsto f$ and $y \mapsto g$ under the Riesz representation theorem.

Solution. Let's denote this correspondence by $x = \rho(f)$. Then we have $f(z) = \langle z, \bar{x} \rangle = \langle z, \overline{\rho(f)} \rangle$ for all $z \in V$. We claim that $\langle f, \bar{g} \rangle = \langle \rho(g), \overline{\rho(f)} \rangle$ defines an inner product: It is additive in each variable because ρ is additive, and we have $\rho(\lambda f) = \bar{\lambda} \rho(f)$, so it is sesquilinear. Further $|f|^2 = |\rho(f)|^2 = 0$ if and only if $\rho(f) = 0$, which happens if and only if $f = 0$ by Riesz.

Suppose then that (f_n) is a Cauchy sequence in $V^\vee$, and set $x_n = \rho(f_n)$. As $|f_n - f_m| = |x_n - x_m|$ by definition, it follows that (x_n) is Cauchy, so its limit x exists. Set $f(y) := \langle y, \bar{x} \rangle$. Then $|f - f_n| = |x - x_n|$ so $f_n \to f$. □

Problem 2. Recall that the annihilator of a subset $M \subset V$ of a normed space is the set

$$\operatorname{ann} M = \{ f \in V^\vee \mid f(x) = 0 \text{ for all } x \in M \}.$$

Discuss the relationship between $M^\perp$ and $\operatorname{ann} M$ in a Hilbert space V.

Solution. There is a linear map

$$\theta : M^\perp \to \operatorname{ann} M, \quad y \mapsto (x \mapsto \langle x, \bar{y} \rangle).$$

It is injective by construction, and bounded as $|\theta(x)| = |x|$ for all x. If $f \in \operatorname{ann} M$ there is a $y \in V$ such that $f(x) = \langle x, \bar{y} \rangle$ for all x by Riesz. But for $x \in M$ we have $0 = f(x) = \langle x, \bar{y} \rangle$, so $y \in M^\perp$, and θ is thus surjective. □

Problem 3. Let V be a Hilbert space and let $S \subset V$ be a closed subspace. Show that the quotient space V/S is also a Hilbert space. Show that $S^\perp$ is isometric to V/S.

Solution. By Homework 6, Problem 4 we know that $(V/S)^\vee \cong \operatorname{ann} S$. Problems 1 and 2 here then give us a diagram

$$\begin{tikzcd}
V/S \arrow[d] & S^\perp \arrow[d] \\
(V/S)^\vee \arrow[r] & \operatorname{ann} S
\end{tikzcd}$$

where all the arrows are isometries, so we win an isometry $V/S \to S^\perp$. $\square$

Problem 4. Let V and W be Hilbert spaces and (f_n) a sequence of bounded operators from V to W such that $f_n \to f$. Show that $f_n^* \to f^*$.

Solution. First let $f : V \to W$ be an arbitrary bounded operator. For any $x \in V$ and $y \in W$ we have

$$\langle f(x), \bar{y} \rangle = \langle x, \overline{f^*(y)} \rangle$$

and we then see that

$$|f^*(y)|^2 = \langle f^*(y), \overline{f^*(y)} \rangle = \langle f(f^*(y)), \bar{y} \rangle.$$

Now

$$|\langle f(f^*(y)), \bar{y} \rangle|^2 \leq |f(f^*(y))|^2 |y|^2 \leq |f|^2 |f^*(y)|^2 |y|^2$$

so taking square roots and remembering that the inner product term was nonnegative we see that

$$|f^*(y)| \leq |f||y|$$

for all y and thus $|f^*| \leq |f|$.

Now, by linearity we may assume that $f = 0$, and then $f^* = 0$. By the above we then get $|f_n^*| \leq |f_n| \to 0$. $\square$

Problem 5. Let V and W be Hilbert spaces and $f : V \to W$ a bounded operator. Show that:

1. $\operatorname{im} f^* \subset (\ker f)^\perp$.
2. $(\operatorname{im} f)^\perp \subset \ker f^*$.
3. $\ker f = (\operatorname{im} f^*)^\perp$.

Solution. 1. Let $y \in W$ and $x \in \ker f$. Then

$$0 = \langle f(x), \bar{y} \rangle = \langle x, \overline{f^*(y)} \rangle$$

so $f^*(y) \in (\ker f)^\perp$.

2. Let $x \in V$ and $y \in (\operatorname{im} f)^\perp$. Then

$$0 = \langle f(x), \bar{y} \rangle = \langle x, \overline{f^*(y)} \rangle.$$

But x was arbitrary so we can take $x = f^*(y)$ and see that $f^*(y) = 0$, so $y \in \ker f^*$.

3. From 1 we see that $\ker f \subset (\operatorname{im} f^*)^\perp$. Applying 2 to f^* we get that $(\operatorname{im} f^*)^\perp \subset \ker(f^*)^* = \ker f$. Together we get the result. $\square$

Problem 6. Let V and W be Hilbert spaces and $f : V \to W$ a bounded operator. Show that $f^* f : V \to V$ is a bounded self-adjoint operator and that $|f| = \sqrt{|f^* f|}$.

Solution. Since f is bounded then so is f^*, and the composition of bounded operators is bounded. Now

$$|f| = \sup_{x \neq 0} \frac{|f(x)|}{|x|} = \sup_{x \neq 0} \frac{\sqrt{\langle f(x), \overline{f(x)} \rangle}}{|x|} = \sup_{x \neq 0} \frac{\sqrt{\langle f^* f(x), \bar{x} \rangle}}{|x|} \leq \sqrt{|f^* f|}$$

by Cauchy–Schwarz. On the other hand we have $|f^* f| \leq |f^*||f| \leq |f|^2$, so $|f^* f| = |f|^2$. $\square$