Homework set 5

Due by 15:00 on Monday, September 25, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`. You may quote problems from older homework sets and results we've read in the textbook if you feel like they help.

Recall that the (algebraic) dual of a vector space V is the space of linear maps $V^* = \mathrm{Hom}(V, \mathbf{R})$. For a normed space, we will usually want to talk about the bounded linear functionals on the space. We will denote the set of those by $V^\vee := \{f \in V^* \mid f \text{ is bounded}\}$.

Problem 1. If V is a normed space and $S \subset V$ a subset, we define the distance from a point x to S by $d(x, S) = \inf_{y \in S} |x - y|$. Let $f : V \to \mathbf{R}$ be a bounded linear functional and let

$$H_f = \{x \in V \mid f(x) = 1\}.$$

Show that

$$|f| = \frac{1}{d(0, H_f)}.$$

Problem 2. Let $V = \mathbf{R}^2$ and let $f : V \to V$ be the operator defined by the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Calculate $|A|$ when V has each of the norms below:

1. $|(x, y)|_1 = |x| + |y|$.
2. $|(x, y)|_2 = \sqrt{x^2 + y^2}$.
3. $|(x, y)|_\infty = \max\{|x|, |y|\}$.

If this is too difficult we may also accept bounds in terms of the other norms or numerical approximations.

Problem 3. Let V and $W \neq 0$ be normed spaces. Suppose that $\dim V = \infty$. Show that there exists an unbounded linear operator $f : V \to W$. Conclude that $V^\vee \neq V^*$. (Hint: Every vector space has a basis.)

If X is a set and f a function on a subset $S \subset X$, we say that a function $\hat{f}$ on X is an extension of f if $\hat{f}(x) = f(x)$ for all $x \in S$.

Theorem (Hahn–Banach)**.** *Let V be a normed space and let $S \subset V$ be a subspace. If $f \in S^\vee$ there exists a bounded extension $\hat{f} \in V^\vee$ of f such that $|\hat{f}| = |f|$.*

Problem 4 (Proof of Hahn–Banach, part 1)**.** Let V be a normed space.

1. Let $S \subset V$ be a linear subspace. Let $x_1 \in V \setminus S$ and let $S_1 = S + \mathbf{R}x_1$. Show that S_1 is a linear subspace that contains S, and that every vector in S_1 can be written uniquely as $x = x_0 + \lambda x_1$, where $x_0 \in S$ and $\lambda \in \mathbf{R}$.

2. Let $f \in S^\vee$. Show that $f_1(x) = f(x_0)$ is a bounded extension of f from $S^\vee$ to $S_1^\vee$, and that $|f_1| = |f|$.

A partial order on a set S is a binary relation $\prec$ on S that satisfies:

- Reflexivity: $x \prec x$.
- Antisymmetry: $x \prec y$ and $y \prec x$ imply $x = y$.
- Transitivity: $x \prec y$ and $y \prec z$ imply $x \prec z$.

As an example, consider the inclusion $U \subset V$ of subsets of S.

A subset $T \subset S$ is totally ordered if for every x, y in T we have either $x \prec y$ or $y \prec x$. An element y is an upper bound for T if $x \prec y$ for every $x \in T$. Finally an element $y \in S$ is maximal if $x \prec y$ implies $x = y$.

Zorn's lemma says that if $(S, \prec)$ is a partially ordered set such that every totally ordered subset contains an upper bound, then $(S, \prec)$ contains at least one maximal element.

Problem 5 (Proof of Hahn–Banach, part 2)**.** Let V be a normed space and $S \subset V$ a subspace. Let also $f \in S^\vee$ be a bounded linear functional. Denote by $\mathcal{L}$ the set of all bounded extensions (M, g) of f to a subspace M such that $|g| = |f|$.

1. Show that $(S, f) \in \mathcal{L}$, so it is not empty.

2. We write $(M, g) \prec (M', g')$ if $M \subset M'$ and $g'(x) = g(x)$ for all $x \in M$. Show that this is a partial order on $\mathcal{L}$.

3. Suppose that $\mathcal{F}$ is a totally ordered subset of $\mathcal{L}$. Define a set $W = \bigcup_{(M,f)\in\mathcal{F}} M$. Show that W is in fact a vector subspace of V.

4. Suppose that $\mathcal{F}$ is a totally ordered subset of $\mathcal{L}$. Define W as above and define $h : W \to \mathbf{R}$ by $h(x) = g(x)$ for any (M, g) such that $x \in M$. Show that this is well-defined, and that h is an extension of f.

5. Conclude that there exists a maximal extension $h : W \to \mathbf{R}$ of f, and use part 1 to conclude that we must have $W = V$.