Homework set 6

Due by 15:00 on Monday, October 2, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`. You may quote problems from older homework sets and results we've read in the textbook if you feel like they help.

A *hyperplane* in a normed space V is a set of the form $H = x_0 + H_0$, where $H_0 \subset V$ is a subspace such that $\operatorname{codim} H_0 := \dim V/H_0 = 1$.

Problem 1. Let V be a normed space. Show that H is a closed hyperplane if and only if there exists a bounded linear functional $f \in V^\vee$ such that $H = f^{-1}(c)$ for some $c \in \mathbf{R}$. Then show that $V \setminus H$ consists of two disjoint connected components.

Problem 2. Let V be a normed space and let

$$S(r) = \{x \in V \mid \|x\| = r\}, \quad B(r) = \{x \in V \mid \|x\| < r\},$$

be the sphere and ball of radius r. Show that for any $x_0 \in S(r)$ there exists a hyperplane $H \subset V$ that contains x_0 such that the open ball $B(r)$ is entirely contained in one component of $V \setminus H$.

The *annihilator* of a set M in a normed space V is the set

$$M^\perp = \{f \in V^\vee \mid f(x) = 0 \text{ for all } x \in M\}.$$

We say that a normed space V is *isometric* to a normed space W if there exists a linear isomorphism $f : V \to W$ that preserves their norms, that is, such that $\|f(x)\| = \|x\|$ for all $x \in V$.

Recall that if

$$0 \longrightarrow S \longrightarrow V \longrightarrow V/S \longrightarrow 0$$

is a short exact sequence of vector spaces, then

$$0 \longrightarrow (V/S)^* \longrightarrow V^* \longrightarrow S^* \longrightarrow 0$$

is also exact. The next two problems establish a similar duality between sub- and quotient spaces for normed spaces and bounded linear functionals.

Problem 3. Let V be a normed space and $S \subset V$ a subspace. Show that $S^\vee$ is isometric to $V^\vee/S^\perp$.[1]

Problem 4. Let V be a normed space and $S \subset V$ a closed subspace. Show that $(V/S)^\vee$ is isometric to $S^\perp$.

[1] Why does the quotient space have a norm and not just a seminorm?