# Homework set 11

Due by 15:00 on Monday, November 6, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

**Problem 1.** Let $X$ and $Y$ be normed spaces and $f : X \to Y$ a linear operator. Show that the graph $\Gamma_f$ of $f$ is closed in $X \oplus Y$ if and only if the operator satisfies: If $x_n \to 0$ in $X$ and $f(x_n) \to y$, then $y = 0$.

**Problem 2.** Let $X$ and $Y$ be Banach spaces and $f : X \to Y$ a bounded operator. Show that there exists a $c > 0$ such that $\|f(x)\| \geq c\|x\|$ if and only if $\ker f = 0$ and $\operatorname{im} f$ is closed.

**Problem 3.** Prove a converse of the closed graph theorem: If $f : X \to Y$ is a bounded linear operator between normed spaces, then the graph $\Gamma_f$ of $f$ is closed in $X \oplus Y$.

**Problem 4.** Let $X$ be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ norms on $X$ such that $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ are both complete. Show that the following are equivalent:

- If $\|x_n\|_1 \to 0$ then $\|x_n\|_2 \to 0$.
- If $(x_n)$ converges in $X_1$ then it converges in $X_2$.

If either condition holds, show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Recall that a set $U$ in a metric space $X$ is *open* if there exist open balls $B(x_i, r_i)$ for $i \in I$ such that $U = \bigcup_{i \in I} B(x_i, r_i)$. The collection $\tau(X)$ of open sets of a metric space $X$ is called a *topology* on $X$.

**Problem 5.** Let $X$ be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ norms on $X$ such that $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ are both complete. Suppose that $\tau(X_1) \subset \tau(X_2)$. Show that $\tau(X_1) = \tau(X_2)$.