Homework set 5

Problem 1. If V is a normed space and $S \subset V$ a subset, we define the distance from a point x to S by $d(x, S) = \inf_{y \in S} |x - y|$. Let $f : V \to \mathbf{R}$ be a bounded linear functional and let

$$H_f = \{x \in V \mid f(x) = 1\}.$$

Show that

$$|f| = \frac{1}{d(0, H_f)}.$$

Solution. If $|f| = 0$ then $f = 0$ and $H_f = \varnothing$ and $\inf$ of the empty set is ∞, so that works. We may thus assume that $f \neq 0$.

If $f(x) = 0$ then $|f(x)|/|x| = 0$, so

$$\begin{aligned} |f| = \sup_{x \neq 0} \frac{|f(x)|}{|x|} = \sup_{f(x) \neq 0} \frac{|f(x)|}{|x|} &= \sup_{f(x) \neq 0} \frac{1}{|x/f(x)|} \\ &= \frac{1}{\inf\limits_{f(x) \neq 0} |x/f(x)|} = \frac{1}{d(0, H_f)}. \end{aligned}$$ □

Problem 2. Let $V = \mathbf{R}^2$ and let $f : V \to V$ be the operator defined by the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Calculate $|A|$ when V has each of the norms below:

1. $|(x, y)|_1 = |x| + |y|$.
2. $|(x, y)|_2 = \sqrt{x^2 + y^2}$.
3. $|(x, y)|_\infty = \max\{|x|, |y|\}$.

If this is too difficult we may also accept bounds in terms of the other norms or numerical approximations.

Solution. Case by case analysis shows the answer in 1 is 6, and the answer in 3 is 7. The answer in 2 is the largest square root of an eigeinvalue of $A^t A$, or approximately 5.465 if we don't want to calculate that. □

Problem 3. Let V and $W \neq 0$ be normed spaces. Suppose that $\dim V = \infty$. Show that there exists an unbounded linear operator $f : V \to W$. Conclude that $V^\vee \neq V^*$. (Hint: Every vector space has a basis.)

Solution. Let $(e_\alpha)_{\alpha \in J}$ be a basis for V. As $\dim V = \infty$ the set J contains a countable subset J_0 that we identify with $\mathbf{N}$. Pick a nonzero $y \in W$ of norm 1 and define $f : V \to W$ by $f(e_\alpha) = ny$ if $n \mapsto \alpha$ and 0 otherwise, and extend by linearity to all of V. Then f is an unbounded linear operator. We can in particular do this when $W = \mathbf{R}$, so there always exists an unbounded linear functional on an infinite-dimensional space. □

Theorem (Hahn–Banach)**.** *Let V be a normed space and let $S \subset V$ be a subspace. If $f \in S^\vee$ there exists a bounded extension $\hat{f} \in V^\vee$ of f such that $|\hat{f}| = |f|$.*

Problem 4 (Proof of Hahn–Banach, part 1)**.** Let V be a normed space.

1. Let $S \subset V$ be a linear subspace. Let $x_1 \in V \setminus S$ and let $S_1 = S + \mathbf{R}x_1$. Show that S_1 is a linear subspace that contains S, and that every vector in S_1 can be written uniquely as $x = x_0 + \lambda x_1$, where $x_0 \in S$ and $\lambda \in \mathbf{R}$.

2. Let $f \in S^\vee$. Show that $f_1(x) = f(x_0)$ is a bounded extension of f from $S^\vee$ to $S_1^\vee$, and that $|f_1| = |f|$.

Solution. 1. Obvious because $S_1 = S \oplus \mathbf{R}x_1$.

2. Define $f_1(x + \lambda x_0) = f(x) + \lambda c$, where c will be chosen later. This functional is clearly an extension of f to S_1. We would like it to satisfy $|f_1| = |f|$, and may assume that $|f| = 1$. We would then like to have

$$-|x + \lambda x_0| \leq f(x) + c\lambda \leq |x + \lambda x_0|$$

for any x and λ. This holds for $\lambda = 0$ by hypothesis. For $\lambda \neq 0$ we can rewrite this as

$$-|x/\lambda + x_0| - f(x/\lambda) \leq c \leq |x/\lambda + x_0| - f(x/\lambda),$$

or equivalently

$$-|y + x_0| - f(y) \leq c \leq |y + x_0| - f(y)$$

for $y \in S$. For $y, z \in S$ we have

$$f(z) - f(y) = f(z - y) \leq |z - y| \leq |z + x_0| + |y + x_0|$$

by $|f| = 1$, so in fact

$$-|y + x_0| - f(y) \leq |z + x_0| - f(z)$$

for all $y, z \in S$. Then

$$a := \sup_{y \in S} -|y + x_0| - f(y), \quad b := \inf_{z \in S} |z + x_0| - f(z)$$

are finite and satisfy $a \leq b$, so any $c \in [a, b]$ will do. □

A partial order on a set S is a binary relation $\prec$ on S that satisfies:

- Reflexivity: $x \prec x$.
- Antisymmetry: $x \prec y$ and $y \prec x$ imply $x = y$.
- Transitivity: $x \prec y$ and $y \prec z$ imply $x \prec z$.

As an example, consider the inclusion $U \subset V$ of subsets of S.

A subset $T \subset S$ is totally ordered if for every x, y in T we have either $x \prec y$ or $y \prec x$. An element y is an upper bound for T if $x \prec y$ for every $x \in T$. Finally an element $y \in S$ is maximal if $x \prec y$ implies $x = y$.

Zorn's lemma says that if $(S, \prec)$ is a partially ordered set such that every totally ordered subset contains an upper bound, then $(S, \prec)$ contains at least one maximal element.

Problem 5 (Proof of Hahn–Banach, part 2)**.** Let V be a normed space and $S \subset V$ a subspace. Let also $f \in S^\vee$ be a bounded linear functional. Denote by $\mathcal{L}$ the set of all bounded extensions (M, g) of f to a subspace M such that $|g| = |f|$.

1. Show that $(S, f) \in \mathcal{L}$, so it is not empty.
2. We write $(M, g) \prec (M', g')$ if $M \subset M'$ and $g'(x) = g(x)$ for all $x \in M$. Show that this is a partial order on $\mathcal{L}$.
3. Suppose that $\mathcal{F}$ is a totally ordered subset of $\mathcal{L}$. Define a set $W = \bigcup_{(M,f)\in\mathcal{F}} M$. Show that W is in fact a vector subspace of V.
4. Suppose that $\mathcal{F}$ is a totally ordered subset of $\mathcal{L}$. Define W as above and define $h : W \to \mathbf{R}$ by $h(x) = g(x)$ for any (M, g) such that $x \in M$. Show that this is well-defined, and that h is an extension of f.
5. Conclude that there exists a maximal extension $h : W \to \mathbf{R}$ of f, and use part 1 to conclude that we must have $W = V$.

Solution. $\square$