Homework set 2

Due by 15:00 on Monday, September 4, 2023.

Please select three problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

A *metrizable vector space* is a vector space V that is equipped with a metric d for which the addition map $(x, y) \mapsto x+y$ and multiplication map $(\lambda, x) \mapsto \lambda x$ are continuous. We suppose here that V is defined over a field k such that $\mathbf{R} \subset k$ (for example, the real or complex numbers.

Problem 1. Let $T_a(x) = x + a$ and $M_\lambda(x) = \lambda x$ for $a \in V$ and $\lambda \in k$. Show that T_a and M_λ are homeomorphisms of V with itself; that is, they are continuous maps that have a continuous inverse.

Solution. The restriction of a continuous function to a subspace is continuous. If $p(x, y) = x + y$ is the addition map then T_a is its restriction to $\{(x, a) \in V \times V \mid x \in V\}$, so T_a is continuous. The inverse of T_a is T_{-a}, which is also continuous, so it is a homeomorphism.

Similarly M_λ is the restriction of the multiplication map to $\{(\lambda, x) \in k \times V \mid x \in V\}$, so it is continuous. (We have to assume there's a metric also on k we can use, so let's do that.) If $\lambda \neq 0$ the inverse of M_λ is $M_{1/\lambda}$, which is also continuous. $\square$

Problem 2. Let V and W be metrizable vector spaces and $f : V \to W$ a linear map. Show that f is continuous if and only if it is continuous at 0.

Solution. If f is continuous it is clearly continuous at 0.

Suppose then that f is continuous at 0 and let's show it is continuous at x. A function on a metric space is continuous at x if and only if $f(x_n) \to f(x)$ for every sequence (x_n) that converges to x. If (x_n) is a sequence that converges to x, then $(x_n - x)$ is a sequence that converges to 0, and

$$f(x_n) - f(x) = f(x_n - x) \to 0$$

by linearity and continuity at 0, so $f(x_n) \to f(x)$. $\square$

A subset $E \subset V$ of a metrizable vector space is *bounded* if for any neighborhood U of 0 there exists a $\lambda > 0$ such that $E \subset \mu U$ for any $\mu \geq \lambda$.

Note that this is *not* the same notion of boundedness we get from the metric on V; it can be defined if we only have a topology and not a metric. Sets can be bounded in one notion and not the other.

Problem 3. Let U be an open set that contains 0. Let (r_j) be an increasing sequence of positive real numbers such that $r_j \to \infty$. Show that $V = \bigcup_{j=1}^{\infty} r_j U$. Conclude that if $K \subset V$ is compact, then K is bounded.

Solution. Pick $x \in V$. Multiplication by a scalar is continuous and $0 \cdot x = 0 \in U$, so there is some $\lambda > 0$ such that $\lambda x \in U$. If we pick j such that $r_j > 1/\lambda$ then $x \in r_j U$. Therefore $V = \bigcup_{j=1}^{\infty} r_j U$.

Let $K \subset V$ be compact and let U be an open neighborhood of 0. There exists an $\varepsilon > 0$ such that $0 \in B(\varepsilon) \subset U$. By the above, $(r_j B(\varepsilon))$ is an open covering of V, so it contains a finite subcover of K. Therefore there is an $r > 0$ such that $K \subset rB(\varepsilon) = B(r\varepsilon)$. If $\mu \geq r$ then $K \subset B(r\varepsilon) \subset B(\mu\varepsilon) \subset \mu U$, so K is bounded. $\square$

Problem 4. Let $E \subset V$ be a set. Show that the following are equivalent:

1. E is bounded.
2. If (x_n) is a sequence in E and (λ_n) is a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$, then $\lambda_n x_n \to 0$ as $n \to \infty$.

Solution. Suppose first that E is bounded. Let (x_n) be a sequence in E and let (λ_n) be a sequence of scalars that tends to 0. Let $\varepsilon > 0$ and consider $B(\varepsilon)$. As E is bounded there is a $\lambda > 0$ such that $E \subset \mu B(\varepsilon)$ for every $\mu \geq \lambda$. There is an $n(\lambda)$ such that $1/\lambda_n \geq \lambda$ for $n \geq n(\lambda)$. Then $\lambda_n x_n \in B(\varepsilon)$ for all $n \geq n(\lambda)$, so $\lambda_n x_n \to 0$.

Suppose now that E is not bounded and let U be an open neighborhood of 0. For every n there is then an element $x_n \in E \setminus nU$. But then (x_n) is a sequence of elements in E and $\lambda_n = 1/n$ a sequence of scalars that tends to zero such that x_n/n does not tend to 0, so the sequence condition does not hold. $\square$

A map $f : V \to W$ between metrizable vector spaces is *bounded* if it maps bounded sets to bounded sets.

Problem 5. Let V and W be metrizable vector spaces and $f : V \to W$ a linear map. Show that if f is continuous then it is bounded. Find spaces V and W and a linear function $f : V \to W$ that is not continuous.

Solution. Let $E \subset V$ be a bounded set, let $U \subset W$ be an open neighborhood of 0. As f is continuous and linear, then $f^{-1}(U) \subset V$ is an open neighborhood of 0. Therefore there is a $\lambda > 0$ such that $E \subset \mu f^{-1}(U)$ for any $\mu \geq \lambda$. But then $f(E) \subset f(\mu f^{-1}(U)) = \mu f(f^{-1}(U)) = \mu U$ for any $\mu \geq \lambda$ by linearity.

Let $V = W = \mathcal{C}^\infty([0,1])$ and let $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Then V is a metrizable vector space. Consider the linear map $f \mapsto f'$. The set $E = \{e^{-nx}\}_{n\geq 0} \subset V$ is bounded: Let $U \subset V$ be an open neighborhood around 0, and let $B(\varepsilon) \subset U$. We have $d(e^{-nx}, 0) = 1$ for any n, so $E \subset \mu B(\varepsilon) \subset \mu U$ for any $\mu > 1/\varepsilon$.

However, let $x_n := (e^{-nx})' = -ne^{-nx}$ define a sequence of points in the image of E. Let $\lambda_n = 1/\sqrt{n}$. Then $\lambda_n \to 0$ but

$$d(\lambda_n x_n, 0) = d(-\sqrt{n}e^{-nx}, 0) = \sqrt{n} \to \infty$$

so $(\lambda_n x_n)$ does not tend to 0, and the image of E is thus not bounded. Therefore the map $f \mapsto f'$ is not continuous. $\square$