Homework set 9

Due by 15:00 on Monday, October 23, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

Problem 1. Let $f : V \to V$ be a self-adjoint bounded operator on a Hilbert space.

1. Suppose that λ is an eigenvalue of f. Show that λ is real.
2. Show that the eigenvectors corresponding to two different eigenvalues are orthogonal.

Problem 2. Let $V = L^2([0,1])$ and let $T : V \to V$ be the operator $T(f)(x) = xf(x)$. Show that T is self-adjoint, but that T has no eigenvalues.

Problem 3. Let V be a Hilbert space and $A : V \to V$ a bounded operator. Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a power series with radius of convergence $R > 0$. If $|A| < R$, show that there exists a bounded operator T on V such that for all $x, y \in V$ we have

$$\langle T(x), \bar{y}\rangle = \sum_{n\geq 0} a_n \langle A^n x, \bar{y}\rangle.$$

(The operator T is usually denoted $f(A)$.)

Problem 4. Calculate the adjoint of the right-shift operator $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ on ℓ^2.

Problem 5. Calculate $\exp S$ of the right-shift operator S on ℓ^2.