Homework set 7

Due by 15:00 on Monday, October 9, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

Recall that if X is an inner product space and $Y \subset X$ is a complete subspace then for any $x \in X$ there exists a unique $y \in Y$ such that $|x - y| = d(x, Y)$.[1] Also recall that if $M \subset X$ is a set, we denote

$$M^\perp = \{x \in X \mid \langle x, y \rangle = 0 \text{ for all } y \in M\}.$$

Problem 1. Let X be a Hilbert space and let $Y \subset X$ be a closed subspace. Define $P : X \to X$ by $x \mapsto y$, where $y \in Y$ is the unique element such that $|x - y| = d(x, Y)$.

1. Show that P is linear.
2. Show that P is bounded.
3. Show that $P^2 = P$.
4. Show that $\ker P = Y^\perp$ and $\operatorname{im} P = Y$.
5. Conclude that $X = Y \oplus Y^\perp$.

Problem 2. Let X be a Hilbert space and $M \subset X$ a subset.

1. Show that $M^\perp$ is a closed subspace of X.
2. Show that $M \subset (M^\perp)^\perp$.
3. Show that $M = (M^\perp)^\perp$ if M is a closed subspace.

Problem 3. Let X be a Hilbert space and $Y \subset X$ a subspace (not necessarily closed). Show that Y is dense in X if and only if $Y^\perp = \{0\}$.

Problem 4. Let X be an inner product space and fix $y \in X$. Show that $f(x) = \langle x, \bar{y} \rangle$ is a bounded linear functional on X and that $|f| = |y|$.

Problem 5. Let X be a Hilbert space and let $f \in X^\vee$ be a bounded linear functional.

1. Show that $X = \ker f \oplus \ker f^\perp$.
2. Show that $\dim \ker f^\perp = 1$.
3. Prove the Riesz representation theorem: There exists a unique $y \in X$ such that $f(x) = \langle x, \bar{y} \rangle$ for all x.

[1] Kreyszig, Lemma 3.3-2.