Homework set 10

Due by 15:00 on Monday, October 30, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

Problem 1 (Banach–Steinhaus)**.** Let V and W be Banach spaces and (T_n) a sequence in $B(V, W)$ such that for every $x \in V$ there is a $y \in W$ such that $\|T_n(x) - y\| \to 0$. Show that there exists a bounded $T \in B(V, W)$ such that $\|T_n(x) - T(x)\| \to 0$ and that $\sup \|T_n\| < \infty$.

Problem 2. Consider the space $C([a, b])$ of continuous functions on the interval $[a, b]$ equipped with the $\sup$ norm. Show that weak convergence implies pointwise convergence; that is, $f_n \xrightarrow{w} f$ implies $f_n(x) \to f(x)$ for all $x \in [a, b]$.

Problem 3. Let V be a normed space. Show that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$ imply that $x_n \to x$.

Problem 4. Let V be a Banach space and W a normed space. Let (T_n) be a sequence in $B(V, W)$. Show that if (T_n) is strongly operator convergent with limit T, then $T \in B(V, W)$.

Problem 5. Let V and W be Banach spaces. Show that a sequence (T_n) in $B(X, Y)$ is strongly operator convergent if and only if the sequence $(\|T_n\|)$ is bounded and the sequence $(T_n(x))$ is Cauchy for every $x \in M$, where $\operatorname{span} M$ is dense in X.