# Homework set 4

Due by 15:00 on Monday, September 18, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`. You may quote problems from older homework sets and results we've read in the textbook if you feel like they help.

If $f : V \to W$ is a linear map we denote its kernel by $\ker f = \{x \in V \mid f(x) = 0\}$.

**Problem 1.** Let $f : V \to V$ be a linear map.

1. Suppose $V$ is finite-dimensional. Show that $f$ is bijective if and only if $\ker f = 0$.

2. Consider $l^\infty$, the space of bounded sequences $(x_n)$ of real numbers. Let $f : l^\infty \to l^\infty$ be the right shift operator $f(x_n) = (0, x_1, x_2, \ldots)$. Show that $\ker f = 0$ but that $f$ is not bijective.

**Problem 2.** Let $f : V \to W$ be a surjective bounded operator from a normed space to a normed space. Suppose that there is a $c > 0$ such that $|f(x)| \geq c|x|$ for all $x$. Show that $f^{-1} : W \to V$ exists and is bounded.

**Problem 3.** Let $l^\infty$ be the space of bounded sequences $(x_n)$ of real numbers.

1. Show that $f : l^\infty \to l^\infty$ defined by $f(x_n) = (x_n/n)$ is linear and bounded.

2. Show that $f^{-1}$ exists and is linear but not bounded.

**Problem 4.** Let $V = C^0([0,1])$ be the space of real-valued continuous functions on $[0,1]$. Let $T : V \to \mathbf{R}$ be defined by $T(f) = f(0)$.

1. Show that $T$ is a linear functional.

2. Consider the norm $|f| = (\int_0^1 |f(x)|^p dx)^{1/p}$ for $1 \leq p < \infty$. Show that $T$ is not bounded with respect to this norm.

3. Consider the norm $|f| = \sup_{0 \leq x \leq 1} |f(x)|$. Show that $T$ is bounded with respect to this norm and calculate its norm.