Homework set 13

Due by 15:00 on Monday, November 21, 2023.

Please select **three** problems to solve and hand in written solutions either in person or to `gunnar@magnusson.io`.

Problem 1. Let A be a complex Banach algebra with identity and $x \in A$ such that $x^n = 0$ for some integer $n > 0$. Show that $\sigma(x) = \{0\}$.

Problem 2. Let A be a complex Banach algebra. Let $x \in A$ be invertible and y be such that $\|yx^{-1}\| < 1$. Show that $x - y$ is invertible and that

$$(x-y)^{-1} = \sum_{n \geq 0} x^{-1}(yx^{-1})^n.$$

Problem 3. Let X be a complex Banach space. If $U \subset \mathbf{C}$ is open, we say that $f : U \to X$ is *holomorphic* if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in U$ and $f' : U \to \mathbf{C}$ is continuous. We say that f is *weakly holomorphic* if the function $\phi \circ f : U \to \mathbf{C}$ is holomorphic for all $\phi \in X^\vee$. Show that f is holomorphic if and only if it is weakly holomorphic.

If $\gamma : [a, b] \to U$ is a curve in an open set $U \subset \mathbf{C}$ and $f : U \to X$ a continuous function into a Banach space X, we define $\int_\gamma f(z)dz$ like the usual Riemann integral: as the limit in X of sums of the form

$$\sum_{j=1}^{n} (\gamma(t_j) - \gamma(t_{j-1}))f(\gamma(t_j)),$$

where $a = t_0 \leq \cdots \leq t_n = b$ is a partition of $[a, b]$. Then $\int_\gamma f(z)dz \in X$ and if $\phi \in X^\vee$ we get $\phi(\int_\gamma f(z)dz) = \int_\gamma \phi(f(z))dz$ by linearity.

Problem 4 (Cauchy's theorem)**.** Let X be a complex Banach space and $f : U \to X$ be holomorphic, where $U \subset \mathbf{C}$ is simply connected. Let γ be a simple closed curve in U. Then

$$\int_\gamma f(z)\, dz = 0.$$

Problem 5 (Cauchy's integral formula)**.** Let X be a complex Banach space and $f : U \to X$ be holomorphic. Let $z_0 \in U$ and let γ be a closed simple curve in U such that $z_0 \in \operatorname{int} \gamma$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$