Homework set 7

Recall that if X is an inner product space and $Y \subset X$ is a complete subspace then for any $x \in X$ there exists a unique $y \in Y$ such that $|x - y| = d(x, Y)$.

Problem 1. Let X be a Hilbert space and let $Y \subset X$ be a closed subspace. Define $P : X \to X$ by $x \mapsto y$, where $y \in Y$ is the unique element such that $|x - y| = d(x, Y)$.

1. Show that P is linear.
2. Show that P is bounded.
3. Show that $P^2 = P$.
4. Show that $\ker P = Y^\perp$ and $\operatorname{im} P = Y$.
5. Conclude that $X = Y \oplus Y^\perp$.

Solution. 1. We know that if $y = P(x)$ then $x - y$ is orthogonal to Y. The converse is also true: If $y \in Y$ is such that $x - y \perp Y$ then $y = P(x)$. This holds because for such a y and any $z \in Y$ we have

$$|x - z| = |(x - y) + (y - z)| = |x - y| + |y - z| \geq |x - y|.$$

Suppose now that $x_1, x_2 \in X$ and let $y_j = P(x_j)$. For $\lambda, \mu \in \mathbf{C}$ we then have

$$\langle \lambda x_1 + \mu x_2 - (\lambda y_1 + \mu y_2), \bar{y} \rangle = \lambda \langle x_1 - y_1, \bar{y} \rangle + \mu \langle x_2 - y_2, \bar{y} \rangle = 0$$

for any $y \in Y$. Therefore $\lambda P(x_1) + \mu P(x_2) = P(\lambda x_1 + \mu x_2)$.

2. Note that $d(x, Y) \leq |x|$ for any $x \in X$. Then

$$|P(x)| = |P(x) - x + x| \leq |P(x) - x| + |x| \leq 2|x|.$$

3. It is enough to show that $P(y) = y$ for any $y \in Y$. But $d(y, Y) = 0$ for $y \in Y$, so $P(y) = y$.

4. If $y \in Y$ then $P(y) = y$, so $\operatorname{im} P = y$. Let $x \in Y^\perp$. Then $0 \in Y$ is such that $x - 0 = x \perp Y$, so $P(x) = 0$. Now let $x \in \ker P$. Then $x - P(x) = x$ is orthogonal to Y, so $x \in Y^\perp$.

5. For any $P : X \to X$ with $P^2 = P$ we have $X = \ker P \oplus \operatorname{im} P = Y \oplus Y^\perp$. $\square$

Problem 2. Let X be a Hilbert space and $M \subset X$ a subset.

1. Show that $M^\perp$ is a closed subspace of X.

2. Show that $M \subset (M^\perp)^\perp$.

3. Show that $M = (M^\perp)^\perp$ if M is a closed subspace.

Solution. 1. $M^\perp$ is a subspace by linearity of the inner product. It is closed because $M^\perp = \bigcap_{y \in M} \ker(x \mapsto \langle x, \bar{y} \rangle)$.

2. Let $x \in M$ and $y \in M^\perp$. Then $\langle y, \bar{x} \rangle = \overline{\langle x, \bar{y} \rangle} = 0$ so $x \in (M^\perp)^\perp$.

3. By Problem 1 we have $X = M \oplus M^\perp = M^\perp \oplus (M^\perp)^\perp$ and $M \subset (M^\perp)^\perp$. We get exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & M^\perp & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (M^\perp)^\perp & \longrightarrow & X & \longrightarrow & M^\perp & \longrightarrow & 0 \end{array}$$

where the second and third downward arrows are isomorphisms, so the first one is as well by general theory. □

Problem 3. Let X be a Hilbert space and $Y \subset X$ a subspace (not necessarily closed). Show that Y is dense in X if and only if $Y^\perp = \{0\}$.

Solution. Suppose Y is dense. If $x \in Y^\perp$ then there are y_n in Y such that $y_n \to x$. But then $|x|^2 = 0$ by continuity.

Now suppose that $Y^\perp = \{0\}$. It's clear that if $Y \subset Z$ then $Z^\perp \subset Y^\perp$, so $\overline{Y}^\perp \subset Y^\perp = \{0\}$. As $\overline{Y}$ is a closed subspace we then get

$$\overline{Y} = (\overline{Y}^\perp)^\perp = \{0\}^\perp = X.$$ □

Problem 4. Let X be an inner product space and fix $y \in X$. Show that $f(x) = \langle x, \bar{y} \rangle$ is a bounded linear functional on X and that $|f| = |y|$.

Solution. This is a linear functional because the inner product is linear in its first variable, and Cauchy–Schwarz gives

$$|f(x)| = |\langle x, \bar{y} \rangle| \leq |x||y|,$$

so f is bounded and $|f| \leq |y|$. By taking a multiple of y we get equality, so $|f| = |y|$. □

Problem 5. Let X be a Hilbert space and let $f \in X^\vee$ be a bounded linear functional.

1. Show that $X = \ker f \oplus \ker f^\perp$.
2. Show that $\dim \ker f^\perp = 1$.
3. Prove the Riesz representation theorem: There exists a unique $y \in X$ such that $f(x) = \langle x, \bar{y} \rangle$ for all x.

Solution. 1. As f is bounded then $\ker f$ is a closed subspace, so by Problem 1 we have $X = \ker f \oplus \ker f^\perp$.

2. We do have to assume that $f \neq 0$. Let $y, z \in \ker f^\perp$ and suppose that both are nonzero. By scaling we may assume that $f(y) = f(z) = 1$. Then $y - z \in \ker f$, so $0 = \langle y, \overline{y - z} \rangle = |y|^2 - \langle y, \bar{z} \rangle$ and $\langle y, \bar{z} \rangle = |y|^2$. Similarly we find that $\langle z, \bar{y} \rangle = |z|^2$. Multiplying together we get

$$|\langle y, \bar{z} \rangle|^2 = |y|^2 |z|^2,$$

which only holds when $z = \lambda y$ by Cauchy–Schwarz. Therefore (y) is a basis of $\ker f^\perp$.

3. We first show that there exists a nonzero $y \in \ker f^\perp$ such that $f(y) = |y|^2$. Pick any nonzero $y \in \ker f^\perp$. Multiplying it by $e^{i\theta}$ we may assume that $f(y) > 0$. Then multiplying by a real $t > 0$ we want to solve $t f(y) = t^2 |y|^2$ for t, which we can.

Now every $x \in X$ can be written as $x = z + \lambda y$, where $z \in \ker f$. Then

$$f(x) = \lambda f(y) = \lambda |y|^2 = \langle x, \bar{y} \rangle.$$

If z is another such element, then we get $\langle x, \bar{y} \rangle = f(x) = \langle x, \bar{z} \rangle$ for all x, so $\langle x, \overline{y - z} \rangle = 0$ for all x. Then $|y - z|^2 = 0$, so $z = y$. $\square$