Heat Mass Transfer (2007) 43:1151–1156
DOI 10.1007/s00231-006-0200-0

ORIGINAL

An improved lumped analysis for transient heat conduction by using the polynomial approximation method

P. Keshavarz · M. Taheri

Received: 21 September 2005 / Accepted: 16 October 2006 / Published online: 11 November 2006

Abstract In this study, unsteady state one-dimensional heat conduction is analyzed using a polynomial approximation method. As a classical lumped model is only applicable for use with Biot numbers of less than 0.1, and additionally, it cannot be used for high-temperature gradients within the region, an improved lumped model is implemented for a typical long slab, long cylinder and sphere. It has been shown that in comparison to a finite difference solution, the improved model is able to calculate average temperature as a function of time for higher value of Biot numbers. The comparison also shows that the presented model has better accuracy when compared with others recently developed models. The simplified relations obtained in this study can be used for engineering calculations in many conditions.

List of symbols

A	Surface area
B	Dimensionless number in equation (7)
Bi	Biot Number
Fo	Fourier Number
k	Thermal conductivity
h	Heat transfer coefficient
m	Order of the geometry
r	Coordinate
R	Maximum coordinate
S	Shape factor
t	Time
T	Temperature
V	Volume
x	Dimensionless coordinate
FD	Finite difference
PA	Polynomial approximation
SL	Simple lumped
SP	Singular perturbation

Greek symbols

α	Thermal diffusivity
τ	Dimensionless time
θ	Dimensionless temperature
$\overline{\theta}$	Dimensionless average temperature

Subscripts

°	Initial
p	Polynomial
F	Finite difference solution
∞	Infinite

P. Keshavarz (✉) · M. Taheri
Department of Chemical and Petroleum Engineering,
School of Engineering, Shiraz University, Shiraz, Iran
e-mail: keshavz@shirazu.ac.ir

1 Introduction

The lumped parameter model has been widely used in the analysis of transient heat conduction, especially in engineering applications. For example, in the thermo-hydraulic analysis of nuclear reactors, this classical approach is extremely useful and sometimes mandatory when a simplified formulation of transient heat conduction is needed. Recently, the dynamics of chaotic instabilities in boiling water nuclear reactors have aroused interest. In such studies, the lumped parameter approach has been the only available option in the fuel dynamics model [12]. However, a simple lumped model is only valid for very low Biot numbers. In this preliminary model, solid resistance can be ignored in comparison with fluid resistance, and so the solid has a uniform temperature that is simply a function of time. The criterion for the Biot number is about 0.1, which is applicable just for either small solids or for solids with high thermal conductivity. In other words, the simple lumped model is valid for moderate to low temperature gradients [3, 5, 6].

In many engineering applications, the Biot number is much higher than 0.1, and so the condition for a simple lumped model is not satisfied. Additionally, the moderate to low temperature gradient assumption is not reasonable in such applications, thus more accurate models should be adopted. Lots of investigations have been done to use or modify the lumped model. The purpose of modified lumped parameter models is to establish simple and more precise relations for higher values of Biot numbers and large temperature gradients. For example, if a model is able to predict average temperature for Biot numbers up to 10, such a model can be used for a much wider range of materials with lower thermal conductivity. Chang and Lahey [2] used one dimensional homogeneous assumption for adiabatic two phase flow, a one node lumped parameter approach for heated wall dynamics, and neutron point kinetics for the consideration of nuclear feedback in a boiling water reactor loop. They found that a boiling channel coupled with a riser could experience chaotic oscillations. Alhama and Campo [1] have studied the application of a simple lumped model for the cooling of a long slab with different Biot numbers in each side. The question this raised was under what circumstances can the transient cooling of a long slab by asymmetric heat conduction be treated with a lumped model? Their results show that a simple lump model is qualified to handle the general distribution model without incurring temperature errors that exceed 5% as long as the combination of the Biot numbers are confined to the area circumscribed by a curvilinear rectangle that has its upper right vertex at 0.1075 for both sides. Cotta and Mikhailov [4] proposed an improved lumped parameter model based on Hermit approximation for integral that defined average temperature and heat flux. Regis et al. [7] have shown that the Cotta model can be used to predict average temperature in a nuclear fuel rod for Biot numbers up to 20. Su and Cotta [11] have presented a higher order lumped parameter formulation for simplified light water reactor thermohydraulic analysis. The Hermit approximation method was also used by Su [10] for transient heat conduction in long slabs with different Biot numbers in each side. His results compared to finite difference solution yield significant improvement of average temperature prediction over the classical lumped model. Finally, Sadat proposed another modified lumped model for one dimensional transient heat conduction in a long slab, long cylinder and sphere by using a Singular Perturbation Method [9]. His result is the same as [10] in case of the long slab.

In this work, a new improved lumped parameter model is deployed, based on the Polynomial Approximation Technique. Unsteady state, one dimensional temperature distribution for long slabs, long cylinders and spheres are analyzed following this method. The obtained results of the presented model are compared to the simple lumped model as well as to the Sadat model [9]. Also, the numerical solution to the problem by finite difference is used as a control reference in order to check the accuracy of each model.

2 Mathematical formulation

Unsteady state one dimensional temperature distribution of a long slab, long cylinder or sphere can be expressed by the following partial differential equation.

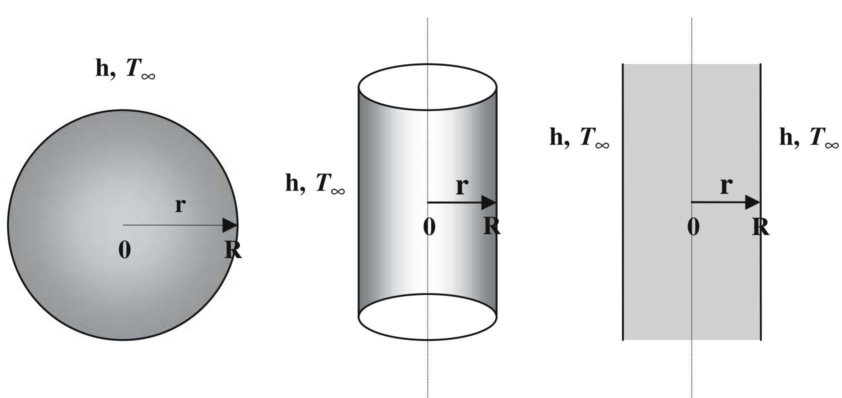

Fig. 1 Symmetric scheme for the transient heat conduction problems

Heat transfer coefficient is assumed to be constant for all geometries, as illustrated in Fig. 1.

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^m} \frac{\partial}{\partial r}\left(r^m \frac{\partial T}{\partial r}\right) \tag{1}$$

where $m = 0$ for slab, 1 for cylinder and 2 for sphere. Boundary conditions are:

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = 0 \tag{2}$$

$$-k\frac{\partial T}{\partial r} = h(T - T_\infty) \quad \text{at } r = R \tag{3}$$

And initial condition:

$$T = T_0 \quad \text{at } t = 0 \tag{4}$$

In the derivation of Eq. (1), it is assumed that thermal conductivity is independent of temperature. If not, temperature dependence must be applied, but the same procedure can be followed. For simplicity, Eq. (1) and boundary conditions can be rewritten in dimensionless form:

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{x^m} \frac{\partial}{\partial x}\left(x^m \frac{\partial \theta}{\partial x}\right) \tag{5}$$

$$\frac{\partial \theta}{\partial x} = 0 \quad \text{at } x = 0 \tag{6}$$

$$\frac{\partial \theta}{\partial x} = -B\theta \quad \text{at } x = 1 \tag{7}$$

$$\theta = 1 \quad \text{at } \tau = 0 \tag{8}$$

It should be noted that for a long slab with the same Biot number in both sides, temperature distribution is the same for each half, and so just one half can be considered. Dimensionless parameters are defined as follows:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty} \quad B = \frac{hR}{k} \quad \tau = \frac{\alpha t}{R^2} \quad x = \frac{r}{R}$$

3 Polynomial approximation method

Polynomial approximation method is one of the simplest, and in some cases, one of the most accurate methods to estimate partial differential equations [6, 8]. The method involves two steps: first, selection of the proper polynomial with time dependent coefficients, and second, to convert a partial differential equation into an integral equation. This integral can then be converted into an ordinary differential equation, where the dependent variable is average temperature and independent variable is time. The steps are applied on Eq. (5) as follows.

3.1 First step

According to the nature of parabolic partial differential equations, a regular series would be the best for its approximation [8]. It leads to the following polynomial:

$$\theta_p = a_0(\tau) + a_1(\tau)x + a_2(\tau)x^2 \tag{9}$$

This polynomial must satisfy all boundary and initial conditions. Application of the first boundary condition in the polynomial leads to a_1 $(\tau) = 0$. Then the polynomial becomes:

$$\theta_p = a_0(\tau) + a_2(\tau)x^2 \tag{10}$$

a_0 (τ) and a_2 (τ) will be evaluated in the second step.

3.2 Second step

Average temperature for long slab, long cylinder and sphere can be written as:

$$\overline{\theta} = \frac{\int_V \theta \mathrm{d}V}{\int \mathrm{d}V} = \frac{\int_0^1 \theta\, x^m\, \mathrm{d}x}{\int_0^1 x^m\, \mathrm{d}x} = (m+1)\int_0^1 x^m \theta\, \mathrm{d}x \tag{11}$$

$m = 0$ for slab, 1 and 2 for cylinder and sphere, respectively. Integration of Eq. (10) yield:

$$\overline{\theta} = a_0 + \frac{m+1}{m+3} a_2 \tag{12}$$

Integration of both sides of Eq. (5) also gives:

$$\int_0^1 x^m \frac{\partial \theta}{\partial \tau} \mathrm{d}x = \int_0^1 \frac{\partial}{\partial x}\left(x^m \frac{\partial \theta}{\partial x}\right) \mathrm{d}x \tag{13}$$

Then, by using the definition of average and some calculations, one finally obtains:

$$\frac{\partial \overline{\theta}}{\partial \tau} = (m+1)\left.\frac{\partial \theta}{\partial x}\right|_{x=1} \tag{14}$$

On the other hand, the derivation of Eq. (9) and applying the second boundary condition leads to:

$$\left.\frac{\partial \theta_p}{\partial x}\right|_{x=1} = 2a_2 \tag{15}$$

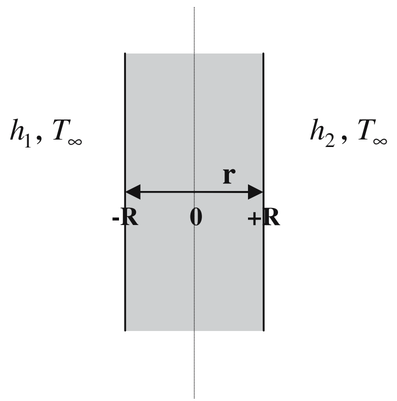

Fig. 2 Scheme of the transient heat conduction in a long slab with different heat transfer coefficients in each side

$$-2a_2 = B(a_0 + a_2) \tag{16}$$

Substitution of a_0 from Eq. (15) in Eqs. (12) and (16) gives:

$$\left.\frac{\partial \theta_p}{\partial x}\right|_{x=1} = -\frac{B(m+3)}{m+B+3}\overline{\theta_p} \tag{17}$$

Then from Eq. (14) and Eq. (17) we have:

$$\frac{\mathrm{d}\overline{\theta_p}}{\mathrm{d}\tau} = -\frac{B(m+1)(m+3)}{m+B+3}\overline{\theta_p} \tag{18}$$

This simple, first order ordinary differential equation can be solved by using the initial condition:

$$\overline{\theta_p} = \exp\left(-\frac{B(m+1)(m+3)}{m+B+3}\tau\right) \tag{19}$$

To be able to compare Eq. (19) with the simple lumped equation, this equation should be expressed in terms of Biot and Fourier numbers:

$$Bi = \frac{hs}{k}, \quad Fo = \frac{\alpha t}{s^2}, \quad s = \frac{V}{A}$$

So for a long slab, long cylinder and sphere, we have:

$$Bi = \frac{hR}{(m+1)k} \tag{20}$$

$$Fo = (m+1)^2\frac{\alpha t}{R^2} \tag{21}$$

Applying these definitions in Eq (19) gives:

$$\overline{\theta_p} = \exp\left(-\frac{1}{\frac{m+1}{m+3}Bi+1}BiFo\right) \tag{22}$$

The Simple Lumped equation is usually written in the following form [5]:

$$\overline{\theta} = \exp(-BiFo) \tag{23}$$

Application of equation (22) is as simple as the simple lumped equation. The accuracy of this equation and comparison with other models will be discussed later.

Another case for long slab can be studied, where the heat transfer coefficients in both sides of the slab are different as shown in Fig. 2. In this case, $a_1\ (\tau)\ \neq\ 0$, because of no symmetry in the center. Applying the same sequence (first step and second step), which are not repeated here, one finally obtains:

$$\overline{\theta_p} = \exp\left(-\frac{3(B_1+B_2+2B_1B_2)}{2(3+2B_1+2B_2+B_1B_2)}\tau\right) \tag{24}$$

And in terms of Biot and Fourier numbers, we have:

$$\overline{\theta_p} = \exp\left(-\frac{3(Bi_1+Bi_2+2Bi_1Bi_2)}{2(3+2Bi_1+2Bi_2+Bi_1Bi_2)}Fo\right) \tag{25}$$

4 Results and discussion

In order to check the accuracy of approximation methods, a reference solution must be used. To do this, Eqs. 5–8 are solved numerically by using the finite difference method. Then, the results of numerical solution are used as a reference. In the case of a long slab, the average temperature of the presented model is the same as that given by the improved model of Sadat [9], which used a singular perturbation method, and Su [10] who used hermit approximation for integral. But the results for the cylinder and sphere are different. So in the following, only the results for a long cylinder and sphere have been discussed.

Figure 3 presents the obtained results from the ‘‘polynomial approximation model’’ (P. A.), ‘‘simple lumped model’’ (S. L.), ‘‘singular perturbation model [9]’’ (S. P.) and ‘‘numerical solution by finite difference’’ (F. D.) for very low Biot numbers. In this figure, all the models have good agreement with the reference solution, and so accuracy of all curves are good and the same as each other. It is due to very low Biot numbers, as discussed before.

Figures 4, 5 and 6 present the model results for the higher value of Biot numbers for the sphere and long cylinder, respectively. Figure 4 shows the deviation of simple lump and singular perturbation models from the numerical solution, while there is still no deviation for the presented model (P. A.). It can be seen from other figures that the deviation of all models from the

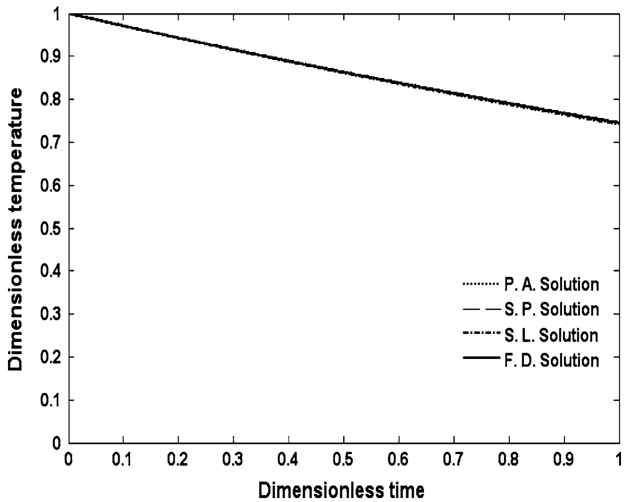

Fig. 3 Average dimensionless temperature (θ) versus dimensionless time (τ) for sphere, B = 0.1, Bi = 0.1/3

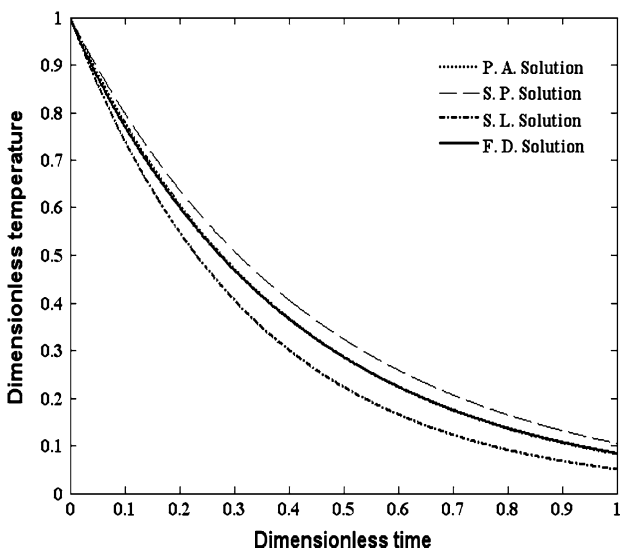

Fig. 4 Average dimensionless temperature (θ) versus dimensionless time (τ) for sphere, B = 1, Bi = 1/3

reference solution become higher for higher Biot numbers. But it is also shown clearly that the agreement of the presented model with the referenced solution is much better than others, especially for higher Biot numbers.

It is also desirable to evaluate the amount of error between the model and the numerical solution. Relative error can be defined as:

$$E = \left| \frac{\overline{T_F} - \overline{T_P}}{\overline{T_F}} \right| \quad (26)$$

To replace average temperatures in Eq. (26) with dimensionless forms, $\overline{\theta_F}$ and $\overline{\theta_P}$ are defined as follows:

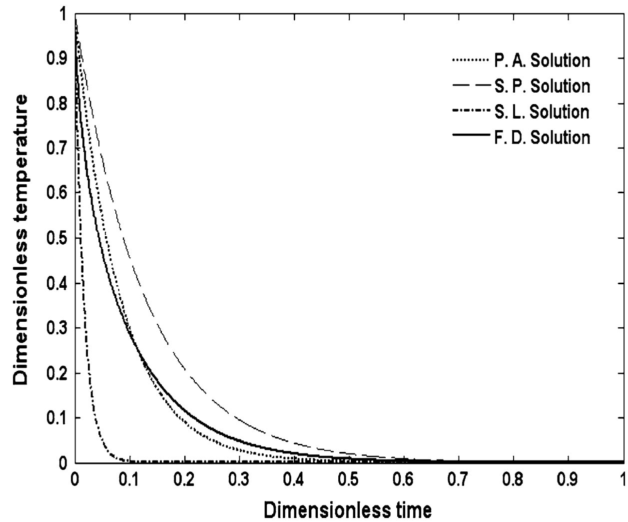

Fig. 5 Average dimensionless temperature (θ) versus dimensionless time (τ) for sphere, B = 20, Bi = 20/3

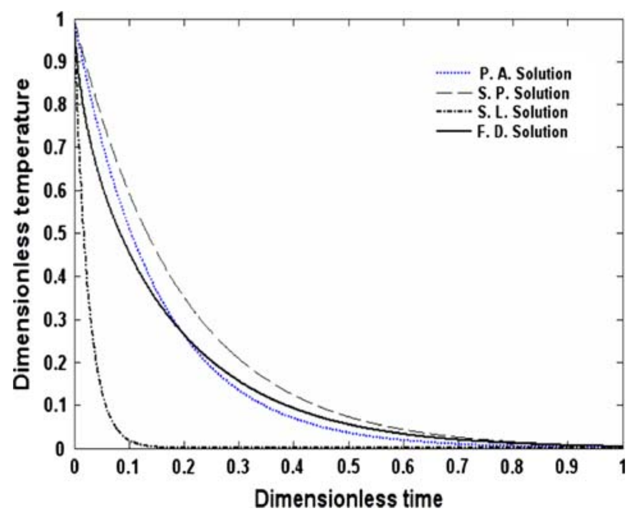

Fig. 6 Average dimensionless temperature (θ) versus dimensionless time (τ) for cylinder, B = 20, Bi = 20/2

$$\overline{\theta_F} = \frac{\overline{T_F} - T_\infty}{T_0 - T_\infty} \quad (27)$$

$$\overline{\theta_P} = \frac{\overline{T_P} - T_\infty}{T_0 - T_\infty} \quad (28)$$

A combination of these three equations gives:

$$E = \left| \frac{\overline{\theta_p} - \overline{\theta_F}}{\overline{\theta_F} + \frac{T_\infty}{T_0 - T_\infty}} \right| \quad (29)$$

It should be noted that when $t \to \infty$ then $\theta \to 0$, therefore it is not correct to use dimensionless temperatures to calculate relative error directly.

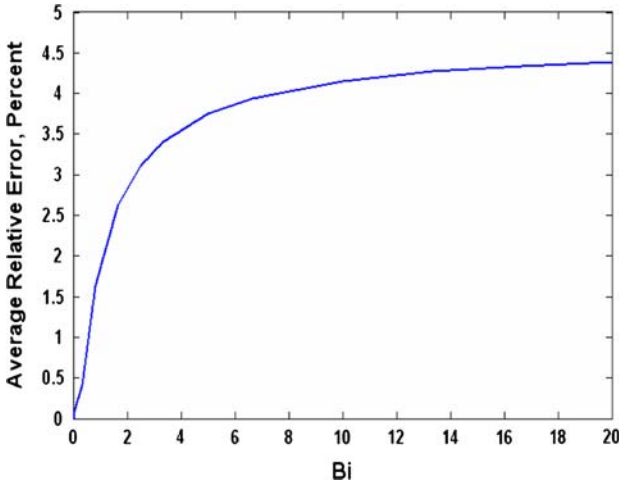

Fig. 7 Average relative error versus Biot number, T_0 = 2,000 K and T_∞ = 300 K

The average temperature in Eq. (26) is a function of time and as a result, the relative error is also time dependent. Average relative error for a whole domain of time can be defined as:

$$E_{\mathrm{av}} = \frac{1}{N}\sum_{i=0}^{N} E_i \tag{30}$$

N is the number of time segments in numerical analysis. The average relative error is independent of time and can be evaluated for different Biot numbers. On the other hand, Eq. (29) shows that the average relative error is not only a function of $(\overline{\theta_F} - \overline{\theta_p})$, which is itself a function of Biot numbers, but also is a function of $(T_0 - T_\infty)$.

Figure 7 shows the average relative error versus time for a special case of sphere cooling, where T_0 = 2,000 K and T_∞ = 300 K. As discussed before, when the Biot number increases, the average error also increases, but as shown in the figure, the average error is lower than 5% for a Biot lower than 20.

5 Conclusion

An improved lumped parameter model is applied to the transient heat conduction in a long slab, long cylinder and sphere. By using the polynomial approximation approach, a simple model is implemented, which can be used to calculate average temperature for a higher value of Biot numbers with better accuracy in comparison with other models. The relations can be used for engineering calculations and estimations in many conditions.

References

1. Alhama F, Campo A (2001) The connection between the distributed and lumped models for asymmetric cooling of long slabs by heat convection. Int Comm Heat Mass Transf 28:127–137
2. Chang CJ, Lahey RT (1997) Analysis of chaotic instabilities in boiling systems. Nucl Eng Des 167:307–334
3. Corra EJ, Cotta RM (1998) Enhanced lumped-differential formulations of diffusion Problems. Appl Math Modeling 22:137–152
4. Cotta RM, Mikhailov MD (1997) Heat conduction—lumped analysis, integral transform, symbolic computation. Wiley, Chichester
5. Holman JP (1981) Heat Transfer, 5th edn, McGraw-Hill, New York
6. Necati Ozisik M (1980) Heat conduction, 1st edn, Wiley, New York
7. Regis CR, Cotta RM, Su J (2001) Improved lumped analysis of transient heat conduction in a nuclear fuel rod. Int Comm Heat Mass Transfer 27:357–366
8. Rice RG, Do DD (1994) Applied mathematics and modeling for chemical engineers, 3rd edn. Wiley, New York
9. Sadat H (2005) A general lumped model for transient heat conduction in one dimensional geometries. Appl Therm Eng 25:567–576
10. Su J (2001) Improved lumped models for asymmetric cooling of a long slab by heat convection. Int Comm Heat Mass Transf 28:973–983
11. Su J, Cotta RM (2001) Improved lumped parameter formulation for simplified LWR thermohydraulic analysis. Ann Nucl Energy 28:1019–1031
12. Zanobetti D (1989) Power station simulators, 1st edn. Elsevier, Amsterdam