Decentralized Many-to-One Matching with Random Search

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Abstract

I analyze a canonical many-to-one matching market within a decentralized search model with frictions, where a finite number of firms and workers meet randomly until the market clears. I compare the stable matchings of the underlying market and equilibrium outcomes when time is nearly costless. In contrast to the case where each firm has just a single vacancy, I show that stable matchings are not obtained as easily. In particular, there may be no Markovian equilibrium that uniformly implements either the worker- or the firm-optimal stable matching in every subgame. The challenge results from the firms' ability to withhold capacity strategically. Yet, this is not the case for markets with vertical preferences on one side, and I construct the equilibrium strategy profile that leads to the unique stable matching almost surely. Moreover, multiple vacancies enable firms to implicitly collude and achieve unstable but firm-preferred matchings, even under Markovian equilibria. Finally, I identify one sufficient condition on preferences to rule out such opportunities.

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1 Introduction

In the absence of a central planner, many applications of many-to-one matchings evolve over time. For instance, consider the labor market for junior positions, where a finite number of job candidates and job vacancies interact over time. The labor market initially comprises a pool of job seekers and available positions. However, with the gradual execution of contracts between some agents, the market shrinks, with the number of vacancies and available workers decreasing as positions are filled.

This unfolding job search process highlights a fundamental trade-off for job candidates and employers. Job seekers must decide whether to accept the current job offer whenever they receive one or continue searching for potentially more favorable alternatives. On the other side of the market, employers face a similar trade-off. They must decide whether to accept the available candidates or wait for potential candidates who might be a better fit for their organizations. Moreover, since firms usually have multiple vacancies, they must consider the whole team they will have at the end of the employment procedure. Of course, it is not for sure for either side that they will encounter better alternatives in the future. One can observe similar dynamics at graduate school admissions with exploding offers or when families are looking for schools for their children in competitive markets[1].

Unlike decentralized search, when many-to-one matchings are centralized, a central planner collects preferences from both sides and imposes an allocation for the entire economy. A minimal requirement that we would expect from those assignments is *stability*, which ensures agents have no incentives to deviate from the proposed allocation. In a many-to-one matching context, this would mean agents do not prefer the outside option of not participating in the allocation, and no firm and worker pairs prefer each other over their assignments. When there are no complementarities between the workers, we know stable matchings exist by Roth (1985). Nevertheless, it is unclear whether agents can find stable matchings of the underlying market without the intervention of a central planner.

The contribution of this paper is to study whether equilibrium behavior by the agents in a search game leads to stable (or unstable) matchings. Doing so, I restrict attention to simple strategies, in which agents condition their behavior on the current state of the market only *(Markovian strategies)*, and consider the game outcomes that are obtained almost surely *(enforced matchings)*. I show that the worker- or firm-optimal stable matchings of the underlying market may not be enforceable in any subgame under equilibrium behavior. On the other hand, for some markets, there are equilibria that enforce unstable matchings.

[1] For an analysis of childcare assignments in Germany, see Reischmann, Klein, and Giegerich (2021).

These results differ from the one-to-one counterpart of the model analyzed by Wu (2015), where firms hire only one worker. The difference shows that, unlike centralized markets, there is a significant disparity between finite one-to-one and many-to-one markets when analyzed within a search model with random meetings. A substantial portion of this discrepancy is attributed to the nature of the firms to employ multiple workers over time and stay in the game until they fill their capacities.

This paper studies a finite many-to-one market with general preferences à la Gale and Shapley (1962) where the matching evolves without a central planner, and there are search frictions as in Smith (2006). Firms hire multiple workers up to their capacity, whereas each worker can work for at most one firm. I analyze the finite market within a search and matching framework, in which bilateral meetings between the agents are random and time is nearly costless[2]. The market is common knowledge to the agents. Upon meeting, agents decide whether to accept or reject each other. Mutual acceptance results in leaving the market, which means irreversibly leaving the market for the worker and losing one vacancy for the firm. Therefore, as many-to-one matching evolves within the search game, the initial market of available agents shrinks over time. In the analysis, I consider the matchings that are attained almost surely in equilibrium[3].

First, I show that firms might find it favorable to unilaterally deviate from a strategy profile that would enforce the worker- or firm-optimal stable matching for any subgame. The one-step deviation occurs in the sense that firms hire less preferred workers than their stable partners in early periods and continue the search game with fewer vacancies. By doing so, they might ensure more favorable sets of workers, making the one-step deviation profitable. Interestingly, this result about the decentralized search game follows a similar pattern from the centralized many-to-one matching literature, i.e., when firms of the centralized market play a capacity revelation game. Konishi and Ünver (2006) show that firms might have incentives to misreport their capacities when this information is unknown to the central planner. For instance, firms may trigger rejection cycles and increase their overall welfare by underreporting their capacities. Here, by employing some workers in early periods, firms mimic capacity underreporting (hence, the rejection cycles) for subsequent periods, even though every component is common knowledge to the agents.

Nevertheless, no rejection cycle can be triggered when preferences on at least one side of the market are perfectly aligned (a consensus among workers about which firm is better

[2]When time is too costly, the agents become so impatient that they accept all individually rational partners instead of waiting for potentially better matches.

[3]Wu (2015) shows how unstable matchings can arise with positive probability for one-to-one and every one-to-one matching is also a many-to-one matching.

or vice versa), as shown by Balinski and Sönmez (1999). Therefore, firms can only be worse off by such one-step deviations. Hence, under vertical preferences, there is an equilibrium strategy profile that enforces the unique stable matching of the underlying market. I show this by constructing the strategy profile, in which workers accept the firms they prefer at least as much as those they are matched with under the stable matching and reject all others. With a slight difference, firms accept the workers they prefer at least as much as their least preferred worker under the stable matching. The strategy profile is simple in the sense that agents "follow" the unique stable matching for any initial or remaining market, which applies to both on- and off-the-equilibrium path.

Second, I demonstrate that unstable matchings may be enforceable as equilibrium outcomes when waiting is costless. In contrast to the unilateral deviation incentives in the previous part preventing stable matchings from being enforceable in every subgame, firms form collusions to enforce unstable matchings. By collectively and credibly committing to avoid forming blocking pairs, they can switch their stable partners and achieve more favorable outcomes. However, for the firms to engage in such a strategic action, at least some firms should be willing to exchange some workers of the stable matching[4]. Naturally, some alignment in preferences prevents such scenarios. One such preference structure is exemplified by the Sequential Preference Condition, which not only ensures the existence of a unique stable matching in the market but also prevents unstable matchings from being enforced in equilibria.

Crucially, both of the main results are different if we assume firms have a capacity of one. This environment, traditionally referred to as a *one-to-one* or *marriage* market, is analyzed by Wu (2015) within a search and matching framework with similar features. In his paper, Wu shows that for any stable matching of the underlying market, there is a Markovian strategy profile that enforces the stable matching. Similarly, he shows that there is no equilibrium Markovian strategy profile under which an unstable matching is enforced. It turns out these results are restricted to one-to-one environments where every firm has at most one vacancy and are not robust to multiple vacancies.

These theoretical results suggest some policy implications as well. For instance, in college admission problems, the primary objective is often to ensure that better students are assigned to more preferred colleges and make it undesirable for agents, particularly colleges, to engage in strategic manipulation that could hinder *fair* outcomes. Such manipulative behavior would be counterproductive, given that colleges typically serve as public goods,

[4] In many-to-one markets, there might be individually rational matchings that all firms prefer to the firm-optimal stable matching as shown by Roth (1985).

and their pursuit of self-interest through strategies undermines the collective welfare they are meant to provide. However, the results of this paper suggest that, in equilibrium, colleges could collude to secure better outcomes for themselves, which comes at the expense of leaving students in a disadvantaged position. From a policy standpoint, these findings emphasize the importance of having a central clearinghouse to counteract and prevent such behavior from emerging and safeguard the interests of all parties involved, especially when agents on one side of the market have multiple capacities.

Furthermore, I analyze how the equilibrium outcomes of the decentralized many-to-one search model differentiate from the dynamic stability notion, studied by Doval (2022) for one-to-one markets and by Altınok (2019) for the many-to-one case. In dynamic stability, workers arrive over time, and firms consider the endgame implemented by evolving matchings. Despite the similarity, workers arriving over time translates into binary meeting probabilities, either 0 or 1. Thus, dynamically stable matchings may not be enforceable via the equilibria of the decentralized many-to-one search model with random search.

The remainder of the paper is structured as follows: After discussing the related literature, I introduce the model in Section 2. In Section 3, I compare the outcomes that are enforced in equilibrium with stable and unstable matchings of the underlying market. I show a sufficient condition on preferences that prevents the enforceability of unstable matchings in equilibria in Section 4. In Section 5, I compare the decentralized many-to-one search model with random search to the notion of dynamic stability by Altınok (2019). I conclude in Section 6.

Literature Review

The paper connects to multiple strands in both centralized matching and search literature. Naturally, the first strand is the traditional finite many-to-one market literature that determines the market I am working with (as in Gale and Shapley (1962), Roth (1985), Roth and Sotomayor (1989), Roth and Sotomayor (1992)). Many results of these papers are significant when the market is considered in a search and matching framework. For instance, Roth (1985) shows that, unlike one-to-one markets, there might be individually rational matchings that are preferred to the firm-optimal stable matching by all firms. In this paper, this fact opens up collusion opportunities for the firms in equilibrium. Moreover, Konishi and Ünver (2006) considers the many-to-one market within the framework of a capacity revelation game. They show that firms might have incentives to misreport their capacities. Therefore, neither the worker- nor the firm-optimal stable matching is necessarily implementable. In this paper, even though the market is common knowledge for agents, firms can mimic capacity underreporting by hiring in the early periods of the search game. Hence, stable matchings may not be enforceable as search equilibria.

Before discussing other strands, current work is the most closely related to Wu (2015), in the sense that Wu considers the one-to-one counterpart of the model analyzed in this paper. Similar to the title of Roth (1985), *the decentralized finite many-to-one matching with random search is not equivalent to its one-to-one counterpart.* In the many-to-one case, stable matchings may not be enforceable in any subgame in equilibrium, and unstable matchings can be enforced more easily. These results are attributed to firms employing multiple workers over time, differentiating the current paper from Wu (2015).

The second strand of related literature analyzes finite matching markets from a dynamic perspective with directed offers. Among those, Pais (2008) considers where acceptance is deferred in line with the Gale and Shapley (1962) algorithm. However, the current paper models a different trade-off. First, an agent can still improve within the game when acceptance is deferred. However, when acceptance is permanent, it eliminates the chance of meeting someone better. Hence, this paper considers the potential regret upon accepting someone. Furthermore, unlike the deferred acceptance algorithm, the meeting technology in this paper still allows the separated agents to meet again in future periods as long as they are still on the market. In addition to Wu (2015), Haeringer and Wooders (2011) and Niederle and Yariv (2009) analyze a more similar trade-off to this paper in which they consider irreversible market exit upon match with one-to-one markets. Similarly, Alcalde, Pérez-Castrillo, and Romero-Medina (1998) and Alcalde and Romero-Medina (2000) analyze a game of only two stages for many-to-one markets, and Alcalde and Romero-Medina (2005) consider the sequential proposal version of the latter paper. For a comparison between deferred and immediate acceptance for one-to-one markets, see Alcalde (1996).

Another approach in the dynamic finite markets strand is the *dynamic stability* notion, studied by Doval (2022) for one-to-one markets and by Altınok (2019) for many-to-one markets. In both papers, agents arrive, and matchings evolve over time, with the condition that there should not be blocking pairs in *intertemporal* matchings as well as in the final matching. In the last section of the paper, I show that dynamically stable matchings are not necessarily enforceable. This is mainly due to the difference in meeting technology of both models. In fact, I consider scenarios where every pair has a positive probability of meeting, but the dynamically stable matching of a market may heavily depend on how agents arrive over time - which requires some meeting probabilities being exactly 0. Similar to dynamic stability, Blum, Roth, and Rothblum (1997) analyze markets that are initially stable but destabilize through new entrants to the job market or through people who leave.

The third strand that this paper connects to is the study of steady states of a search and matching game with a continuum of agents under nontransferable utility. When comparing the steady states with the stable matchings of the underlying market, Burdett and Coles (1997), Eeckhout (1999), and Smith (2006) assume vertical preferences that result in a unique stable matching, whereas Adachi (2003) and Lauermann and Nöldeke (2014) consider general preferences. In Adachi (2003), the stock of searching agents is exogenously given, whereas Lauermann and Nöldeke (2014) consider an endogenous steady-state, endogeneity being naturally the case for a finite market. Even though mutual acceptance has the same consequence of leaving the market, rejecting a potential partner and they marrying someone else has different implications in a continuum because there are still replicas of the same agent in the market that would be available in the future. For search models under transferable utility and vertical agents, see Becker (1973) and Shimer and Smith (2000), and Chade, Eeckhout, and Smith (2017) for a detailed survey of this strand.

Importantly, all these papers are in a one-to-one setting. The steady-state analysis for a many-to-one market would allow the firms to meet the same agent repeatedly. Clearly, having multiples of the same worker is not defined in the preferences of any underlying matching market and would require technical manipulation in preferences similar to Hatfield and Kominers (2015), which is not a realistic approach within a search and matching framework.

Search problems are widely studied not only on micro levels but also on macro levels to answer bigger-scale questions and better understand the economies as a whole. Most famous examples of the search literature include Mortensen (1982) and Pissarides (1985) search models. These early papers have ex-ante homogeneous agents and focus on wage bargaining and unemployment dynamics in the economy.

2 Model

2.1 Environment

The search environment is the same as a standard many-to-one matching market, with a finite set of firms and a finite set of workers that are bilaterally and randomly searching for each other. The set of firms is denoted by $F = \{f_1, ..., f_n\}$ and the set of workers is $W = \{w_1, ...w_m\}$, where $(f, w) \in F \times W$ denotes a generic firm-worker pair.

Workers have complete and strict preferences over firms and remaining unemployed.

The utility that worker w receives is denoted by $v(i, w) \in \mathbb{R}$, where $i \in F \cup \{w\}$. This means workers are indifferent between the vacancies of the same firm and their coworkers. The many-to-one structure implies that firms can employ many workers, whereas a worker can be matched to one firm at most. The capacity vector $q = (q_{f_1}, ..., q_{f_n})$ specifies the maximum number of workers a firm can employ. Consequently, firms have complete and strict preferences over sets of workers, which is denoted by $u(f, \Omega) \in \mathbb{R}$, where $\Omega \subset W$. The utility of not being in a match is normalized for both parties: $u(f, \emptyset) = v(w, w) = 0$. A pair $(f, w) \in F \times W$ is an acceptable pair at u, v if w is *acceptable* for f, and f is acceptable for w, that is, $u(f, \{w\}) > 0$ and $v(f, w) > 0$[5].

An underlying preference relation over individual workers induces firms' preferences over sets of workers. In other words, if two sets differ in only one worker, the firm prefers the set with the more preferred worker. This condition is referred to as *q-responsive preferences* in the existing literature. Formally:

Definition 1. *The preferences of firms over 2^W are q-responsive if they satisfy the following conditions:*

1. *For all $\Omega \subset W$ such that $|\Omega| > q_f$, we have $u(f, \Omega) < 0$.*
2. *For all $\Omega \subset W$ such that $|\Omega| < q_f$ and $w \notin \Omega$, $u(f, \Omega \cup \{w\}) > u(f, \Omega)$ if and only if $u(f, \{w\}) > u(f, \emptyset) = 0$.*
3. *For all $\Omega \subset W$ such that $|\Omega| < q_f$ and $w, w' \notin \Omega$, $u(f, \Omega \cup \{w\}) > u(f, \Omega \cup \{w'\})$ if and only if $u(f, \{w\}) > u(f, \{w'\})$*

A many-to-one search market is represented by $M = (F, W, q, u, v)$ and all components are common knowledge to all agents. The summary of the assumptions on the market for a search game to start is the following:

1. Both parties have strict preferences.
2. The utility of being single is normalized to zero for both parties.
3. Firms have q-responsive preferences over sets of workers.
4. The market is finite: $|F| < \infty$, $|W| < \infty$, and $q_f < \infty \quad \forall f$.
5. The initial market is nontrivial in the sense that there are some acceptable pairs.

[5]For notational convenience, $\Omega \succ_f \Omega'$ is often used instead of $u(f, \Omega) > u(f, \Omega')$, and same for $f \succ_w f'$ instead of $v(f, w) > v(f', w)$

2.2 The Search Game

The game starts at $t = 0$ with the initial market M and continues for an indefinite amount of time. Each day, a random pair (f, w) meets randomly. I describe the meeting process in detail in the following subsection 2.3. Upon meeting, w first decides whether to apply to f or not. Then, if w applies, f decides whether to accept w. If w does not apply or f rejects w, they separate and return to the market to keep searching. If w applies and f accepts, w leaves the market, and f loses one of its vacancies. If f has more vacancies, it stays in the market but leaves if q_f is full after hiring w.

Upon hiring, w receives a one-time payoff of $v(f, w)$. The firm also receives a one-time payoff. However, this depends on the already hired workers. Namely, suppose f has already hired $\Omega \subset W$ before meeting w. In that case, the one-time payoff it gets after hiring w is $u(f, \Omega \cup \{w\}) - u(f, \Omega)$, i.e., it enjoys the additional utility it derives from hiring w. The instant utility captures the immediate utility gain of extending the labor force[6]. The meeting and the decisions take place on the same day t, and if the meeting concludes with hiring, the agents' utilities are discounted by δ^t. The common discount factor refers to the cost of time and is the first source of friction in the model.

Leaving the market upon mutual acceptance is permanent. That is, workers cannot quit, and the firms cannot fire workers. As a result, the market weakly shrinks over time. Any *submarket* or *remaining market* is then denoted by $M' = (F', W', q', u, v)$, where $F' \subset F$ denotes the remaining firms in the market, $W' \subset W$ the remaining workers, and $q' \subset Q$ the remaining capacities of the firms F'. The capacities of the firms will decrease over time, therefore $q' \leq q$ necessarily.

2.3 The Contact Function

The second source of friction in the decentralized many-to-one search game is the random meeting process defined by the contact function. For any day, the contact function $C(f, w, M')$ defines the probability that the pair (f, w) will meet given that the remaining market at that day is M'. The assumptions on the contact function are the following:

1. The pair (f, w) meets with positive probability only if f has not filled its capacity and w is not matched to any firm: $C(f, w, M') = 0$ if $f \notin M'$ or $w \notin M'$.
2. A meeting occurs each day: $\sum_{(f,w)\in M'} C(f, w, M') = 1$.

[6] I will revisit this assumption in Section 3.2

3. Every meeting between the pairs in the remaining market is somewhat possible: $\exists \epsilon > 0$ s.t. $C(f, w, M') \geq \epsilon$ if $(f, w) \in M'$.
4. By definition, the contact function does not depend on the history itself. The pair (f, w) has the same meeting probability whenever the remaining market is the same along different histories.

The game ends if either one side of the market is fully matched or there are no mutually acceptable pairs remaining in the market, that is for any given submarket M', $\nexists (f, w)$ such that $u(f, \Omega \cup \{w\}) \geq u(f, \Omega)$ for $f \in M'$ holding Ω and $v(f, w) \geq 0$ for $w \in M'$ [7]. Workers search until they are matched to a firm, or the game ends. Similarly, firms search until they fill their capacity or the game ends.

The search game is represented by the tuple $\Gamma = (F, W, q, u, v, C, \delta)$, and components of the search game are common knowledge.

2.4 Equilibria

Due to complete information and the finite and dynamic structure of the many-to-one bilateral search game, the appropriate equilibrium concept for analysis is subgame perfect equilibrium. The set of all histories is denoted by $\mathcal{H}$, where $h \in \mathcal{H}$ and $\hat{\mathcal{H}}$ is the set of all non-terminal histories, after which the game continues with a new pair meeting. Additional to the initial search game Γ, any subgame that follows after history h is $\Gamma(h) = (F', W', q', u, v, C, \delta)$, and $M'(h)$ is the remaining market. A strategy profile is denoted by σ, and $\sigma | \Gamma(h)$ is its restriction to the subgame $\Gamma(h)$. Furthermore, let μ_h denote the instantaneous matching at history h, i.e., all the meetings that have ended with employment.

Along with the regular subgame perfect equilibrium analysis, the paper considers some refinements to the information structure. The baseline scenario, called *full-awareness*, assumes that agents possess the capability to observe and base their decisions on every aspect of the game's history. The *private-dinner condition* refers to environments where meetings between the agents take place in private settings. Agents can observe what meetings realize and their conclusions, but in case of separation, they do not observe the reason for the split, i.e., who rejected whom. However, when a meeting concludes with employment, it indirectly signals mutual acceptance to all agents.

Taking the refinement one step further, we reach the most stringent, and the most commonly used refinement in the literature is the *Markov condition*, under which agents

[7] Due to q-responsive preferences, $u(f, \Omega \cup \{w\}) \geq u(f, \Omega)$ is equivalent to $u(f, w) \geq 0$

condition their behavior only on the current state of the market but not on any other component of the history. The current state of the market consists of an instantaneous matching for every history, which also implies a remaining market[8]. This progression of information structure refinements enriches the analysis and sheds light on the intricate dynamics of the economic model under scrutiny.

Independent of the information structure, the game dictates the following for optimal pure strategy behavior upon meeting:

1. A worker w applies to f if the expected utility upon application exceeds the expected utility of continuing the search.
2. Any firm f accepts a w for one of its vacancies if the instant utility gain and the expected utility gain for the remaining vacancies exceeds the expected utility gain for continuing search without accepting w.

A strategy profile σ constitutes a subgame perfect equilibrium of the search game $(F, W, q, u, v, C, \delta)$ if σ is optimal for every agent in every subgame[9].

3 Connection to Centralized Many-to-One Markets à la Roth

In this section, I analyze the equilibrium outcomes attained almost surely in equilibrium. Specifically, I compare them to stable and unstable matchings of the underlying many-to-one market. Comparing search outcomes and static allocations differs from the one-to-one case. This divergence is primarily attributed to the dynamic evolution of firms' engagement with multiple workers as the game unfolds.

3.1 Evolving Many-to-One Matching

At each history h, the instantaneous matching μ_h collects all the meetings that have ended with recruitment so far. In compliance with the centralized matching literature, the rules of the search game ensure that any instantaneous matching is naturally a many-to-one matching and has the following properties:

[8]The Markov condition in Wu (2015) considers conditioning on the remaining market only, which does not change any of the results presented in this paper.

[9]The conditions get stronger as we move from full awareness to Markov condition. Therefore, the set of equilibrium strategies weakly shrinks in the same direction.

1. $|\mu_h(w)| = 1$ and if $\mu_h(w) \neq w$, then $\mu_h(w) \subset F$.
2. $\mu_h(f) \in 2^W$ and $|\mu_h(f)| \leq q_f$.
3. $\mu_h(w) = f$ if and only if $w \in \mu_h(f)$.

Verbally, workers are matched to only one firm at most, and if they are not single, they are matched to a firm. Firms are matched to subsets of workers that do not exceed their capacity. Lastly, a worker is matched to a firm if and only if that firm employs the worker, so the matching process is bilateral.

Observe that any instantaneous matching μ_h implies a remaining market such that already matched agents and seats of the firms are removed from the initial market. Similarly, any contact function $C(f, w, M')$ together with a strategy profile σ induces a probability mass function on outcome matchings, details of which are to be found in the appendix. A matching μ *arises* if it appears as an outcome matching with positive probability under σ. If μ obtains almost surely under σ, I say that σ *enforces* μ. [10]

The prevailing solution concept in centralized matching theory is stability, which assesses the sustainability of a matching. In essence, it ensures that agents lack incentives to disrupt the proposed matching, whether through individual or bilateral actions, thus making it both individually rational and unblocked. Formally, a matching μ is individually rational if all the matched pairs are acceptable. A matching μ is blocked by the pair (f, w) if they are not matched under μ but prefer each other over their matches under μ. That is, μ is blocked by the pair (f, w) if at least one of the following conditions hold:

1. If $|\mu(f)| \leq q_f$ and $\mu(w) \neq f$,
   $v(f, w) > v(\mu(w), w)$ and $u(f, w) > u(f, w')$ for some $w' \in \mu(f)$.
2. If $|\mu(f)| < q_f$ and $\mu(w) \neq f$,
   $v(f, w) > v(\mu(w), w)$ and $u(f, w) > 0$

If μ is individually rational and unblocked, it is a stable matching. Since the firms' preferences are q-responsive, the set of stable matchings is non-empty for any initial market. Furthermore, by the famous Rural Hospital Theorem by Roth (1984), the set of matched agents is the same under every stable matching, and each firm that does not fill its quota has the same set of agents matched under every stable matching.

[10] If σ enforces μ, the set of matchings that arise under σ may contain other elements than μ, all of which have zero probability of arising.

3.2 Enforcing Stable Matchings

In the first analysis section, the main question is the relation between the subgame perfect equilibrium outcomes of the search game and the stable outcomes of the underlying many-to-one market.

Firstly, small values of δ reflect high costs of time. If waiting is sufficiently costly, the workers would apply to acceptable firms, and firms would accept every acceptable worker (partners that give positive utility). Therefore, the rather interesting question is for larger δ, in fact, for $\delta \to 1$, that is called as *limit equilibria*.

Definition 2. *A strategy profile σ is a* ***limit equilibrium*** *of the many-to-one search environment (F, W, q, u, v, C) if there exists some $d < 1$ such that σ is an SPE of the many-to-one search game $(F, W, q, u, v, C, \delta)$ for all $\delta > d$.*

In the one-to-one component of the search model, Wu (2015) demonstrates that for any stable matching μ^* of any underlying market, there is a strategy profile σ^* that satisfies the Markov condition, is a limit equilibrium, and enforces μ^*. In addition, he constructs this strategy profile. As one might expect similarities with the many-to-one model (at least with responsive preferences), in the following, I show that such a Markovian strategy profile may not exist in the many-to-one search game - even with additively separable utility over the workers[11]. The crucial distinction is that firms have a collective structure and stay on the market until they fill their capacities.

In this paper, I consider limit equilibria where waiting is almost costless and enforced matchings while assuming responsive preferences. Therefore, for all results presented in this paper, the assumption about how firms derive utility from expanding their employment set can be replaced with another, simpler one: that firms only care about the end-outcome of the search game[12]. This specific focus allows us to abstract from other strategic problems that can arise to prolong or expedite the game on the firm's side and target the differences caused by cumulative employment only.

Enforcing the Worker-Optimal Stable Matching via Markov Equilibria

According to the strategy profile defined in Wu, for any given stable matching, agents accept partners they prefer at least as much as their stable partners, which is a limit equilibrium and eventually leads to the stable matching of interest. Off-the-equilibrium path, agents play

[11] The utility function satisfies: $u(f, \Omega \cup \{w\}) = u(f, \Omega) + u(f, \{w\})\ \ \forall \Omega \subset W$ such that $w \notin \Omega$

[12] Details can be found in the Appendix for Proposition 1.

the same strategy according to the firm-optimal stable matching, enforcing FOSM for off-the-equilibrium path remaining markets. In the following, I test this in many-to-one markets in the following way: Is there a many-to-one counterpart of the strategy profile by Wu that would enforce stable matchings for any subgame of the initial game? A natural starting point here is to start with extremal matchings, i.e., the worker-optimal and the firm-optimal stable matchings.

For any initial many-to-one market (with responsive preferences), there is a worker-optimal stable matching μ^W (WOSM), corresponding to the outcome of the worker-proposing deferred acceptance algorithm. In the following proposition, I show that if there is a Markovian strategy profile that enforces the WOSM for any subgame, it is not a limit equilibrium for some initial markets.

Proposition 1. *There is no Markovian strategy profile that is a limit equilibrium and enforces the worker-optimal stable matching in any subgame of any many-to-one search game* $(F, W, q, u, v, C, \delta)$.

The proposition can be proved by constructing a market for which a Markovian strategy profile enforcing the WOSM in any subgame is not a limit equilibrium.

Proof. Suppose otherwise, i.e., there is a Markovian strategy profile σ^* that is a limit equilibrium and that enforces the worker-optimal stable matching in any subgame of any many-to-one search game $(F, W, q, u, v.C, \delta)$. Note that σ^* restricted to any subgame $\Gamma(h)$ would enforce the WOSM of the remaining market $M'(h)$.

Now, consider the following example with $F = \{f_1, f_2, f_3\}$, $q = \{2, 1, 1\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$. In addition to the following preference profile, suppose $\{w_3, w_5\} \succ_{f_1} \{w_1, w_4\}$, which does not violate responsive preferences. Other than that, any q-responsive completion of firm preferences is admissible.

$f_1 : w_3 \succ w_4 \succ w_1 \succ w_2 \succ w_5$

$f_2 : w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5$

$f_3 : w_2 \succ w_3 \succ w_1 \succ w_4 \succ w_5$

$w_1 : f_1 \succ f_2$

$w_2 : f_2 \succ f_3$

$w_3 : f_3 \succ f_1$

$w_4 : f_1 \succ f_2 \succ f_3$

$w_5 : f_1 \succ f_2 \succ f_3$

In this example, the WOSM is $\mu^W = \{(f_1; w_1, w_4), (f_2; w_2), (f_3; w_3)\}$. First, observe that for μ^W to be enforceable, f_1 should reject w_5 upon meeting because knowing that f_1 will accept, w_5 will apply to f_1 in any SPE. Second, since by assumption σ^* enforces the WOSM for any many-to-one search game, it enforces the worker-optimal stable matching of any submarket of the initial market. Third, in another many-to-one market where all agents and preferences are the same, but f_1 has a capacity of 1, $\mu^W = \{(f_1; w_3), (f_2; w_1), (f_3; w_2)\}$[13].

Consider the subgame where the meeting between (f_1, w_5) results in employment. By assumption, the worker-optimal stable matching is enforced in the remaining subgame by σ^* restricted to that subgame. In the remaining market, all firms have a capacity of 1 and $\mu^W = \{(f_1; w_3), (f_2; w_1), (f_3; w_2)\}$ as noted above. Therefore, a one-step deviation to accept w_5 leads f_1 to $\{w_3, w_5\}$, instead of $\{w_1, w_4\}$. Since $\{w_3, w_5\} \succ_{f_1} \{w_1, w_4\}$, f_1 finds it plausible to employ w_5 upon application. Since every meeting between the pairs has a positive probability, the profitable one-shot deviation by f_1 to accept w_5 conflicts with σ^* enforcing the WOSM.

In fact, $\mu^W = \{(f_1; w_3), (f_2; w_1), (f_3; w_2)\}$ is the unique stable matching of the remaining market. Therefore, Wu's off-the-path conjecture of playing according to the firm-optimal stable matching when an off-path acceptance occurs would also not work.

$\square$

The preceding example illustrates how firms are willing to accept unstable partners initially to secure a more advantageous group of workers ultimately, which is not possible in a one-to-one search game. This one-step deviation by f_1, involving filling one capacity with a less desirable candidate in the early stages of the game, aligns with the *capacity manipulation game* analyzed in Konishi and Ünver (2006). In their research, they demonstrate that firms have incentives to misrepresent their capacities as a means to enhance their overall welfare under both the worker-proposing and firm-proposing deferred acceptance algorithm. In the aforementioned example, f_1 accepting w_5 would mimic capacity underreporting in subsequent rounds, ultimately benefiting the firm with a more favorable set of workers.

Enforcing the Firm-Optimal Stable Matching via Markov Equilibria

Given that it is a firm that engages in a one-step deviation by misreporting its capacities in the previous part, and Wu uses the firm-optimal stable matching for off-path, one might naturally contemplate that this impossibility result could be overcome if we consider the

[13]Firm f_1 rejecting w_1 in the first step of DA results in a rejection chain, resulting f_1 ending up with w_3.

firm-optimal stable matching μ^F (FOSM). However, again, as demonstrated by Konishi and Ünver (2006), similar incentives for misreporting capacities may persist even when employing the firm-proposing deferred acceptance algorithm—a scenario reflected in the many-to-one search game by the hiring unstable workers in early periods.

Below, I present the FOSM counterpart of the previous proposition, along with an illustrative example that highlights an initial market configuration for which no Markovian strategy profile can enforce the FOSM in every subgame.

Proposition 2. *There is no Markovian strategy profile that is a limit equilibrium and enforces the firm-optimal stable matching in any subgame of any many-to-one search game* $(F, W, q, u, v, C, \delta)$.

Proof. Suppose otherwise, i.e., there is a Markovian strategy profile that is a limit equilibrium and that enforces firm-optimal stable matching in any subgame of any many-to-one search game $(F, W, q, u, v.C, \delta)$. Now, consider the following example with $F = \{f_1, f_2\}$, $q = \{3, 3\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. In addition to the following preference profile, suppose $\{w_2, w_4, w_6\} \succ_{f_1} \{w_4, w_5\}$ and any q-responsive completion of firm preferences.

$$f_1 : w_1 \succ w_2 \succ w_4 \succ w_3 \succ w_5 \succ \emptyset \succ w_6$$
$$f_2 : w_4 \succ w_5 \succ w_1 \succ w_3 \succ w_2 \succ \emptyset \succ w_6$$

$$w_1 : f_2 \succ f_1$$
$$w_2 : f_2 \succ f_1$$
$$w_3 : f_2 \succ f_1$$
$$w_4 : f_1 \succ f_2$$
$$w_5 : f_1 \succ f_2$$
$$w_6 : f_1 \succ f_2$$

In this example, $\mu^F = \{(f_1; w_4, w_5), (f_2; w_1, w_2, w_3)\}$. Similar to above, by employing w_6 in early periods (who is even an unacceptable worker for the firm), f_1 can mimic capacity underreporting. In the following subgame, the firm-optimal stable matching is $\mu^F = \{(f_1; w_2, w_4), (f_2; w_1, w_3, w_5)\}$ and f_1 ensures a more favorable outcome overall. □

A Remedy to Enforce Stable Matchings: Vertical Preferences

Both propositions above show how firms can profitably deviate from a strategy profile that would enforce a stable matching. Even though one might expect different incentives under the worker-optimal or firm-optimal matching, they share a common characteristic of involving rejection chains in the deferred acceptance outcome, which firms initiate by underreporting their capacities. This cascading effect is mimicked by employing less favorable workers and ultimately yields a benefit for the deviating firm.

It's well-established in the literature that such profitable chains do not occur when one side of the market possesses identical preferences for the other side — in other words, when preferences are vertical on at least one side of the market. In this scenario, a unique stable matching μ^* emerges as the outcome of both worker-proposing and firm-proposing deferred acceptance algorithms.

In this part, given a matching μ for the market (F, W, q, u, v) let $\mathcal{M}_\mu$ denote the set of all markets (F', W', q', u, v) such that $\cup_{F \setminus F'} \mu(f) = W \setminus W'$ and $q'_f = q_f - |\mu(f)| \quad \forall f \in F$. In other words, $\mathcal{M}_\mu$ denotes all the markets that remain after some pairs of μ are matched.

In the following, I construct the Markovian strategy profile that is a limit equilibrium and enforces the unique stable matching μ^* for any initial many-to-one market with vertical preferences. The strategy profile enforces the unique stable matching for any history, both off- and on the equilibrium path. On the equilibrium path, all the remaining markets will be elements of $\mathcal{M}_{\mu^*}$, leading to μ^* eventually. Observe that if $M \in \mathcal{M}_{\mu^*}$, μ^* restricted to M is also the unique stable matching for M.

Definition 3. *For any given market (F, W, q, u, v) with vertical preferences on at least one side (either firms share the same preferences up to their capacity, or workers have the same preferences over firms), σ_{μ^*} denotes the strategy profile in which the agents behave in the following way upon meeting:*

1. *Workers apply only to firms such that $v(f, w) \geq v(\mu^*(w), w)$ in any submarket.*
2. *Firms accept workers such that $u(f, w) \geq \min(u(f, w'))$ for $w' \in \mu^*(f)$, reject others.*

In other words, σ_{μ^*} prescribes that workers apply to firms that they weakly prefer to their allocation under μ^*, and firms accept workers whom they prefer to their least favorite worker under μ^*.

Theorem 1. *For any given market (F, W, q, u, v) with vertical preferences on at least one side, σ_{μ^*} is a limit equilibrium for any search game $(F, W, q, u, v, C, \delta)$ and it enforces the unique stable matching μ^*.*

The idea of the proof is first to show σ^* being a limit equilibrium and then to show it indeed enforces μ^*. The second step follows from the fact that a pair accepts each other if and only if they are stable partners. For the first step, I show that any one-step deviation yields a worse payoff for the agents. First of all, since the strategy profile is Markovian, any deviation to not accepting a partner does not change the remaining market, hence, we are still on the equilibrium path. The only profitable deviation could arise in accepting a partner that is not accepted under σ_{μ^*}, which was the case for the propositions above. However, since preferences are vertical on the one side, such rejection cycles do not occur by misreporting capacities, and the outcome of the search game only changes by one worker. Since preferences are responsive, this difference is reflected in the overall preference, which concludes that no profitable deviation is possible. The detailed proof can be found in the appendix.

Intuitively, if agents are patient enough in a many-to-one market with vertical preferences, it is possible to obtain no other outcome but the unique stable matching in equilibrium. Even though the equilibrium strategy profile depends on the stable matching of the underlying market, the strategy profile is feasible to sustain with complete information. [14]

3.3 Enforcing Unstable Matchings

Knowing that the stable matchings of the underlying many-to-one market can be sustained as search equilibria, the natural follow-up question would be about the enforceability of unstable matchings. As Wu (2015) shows, unstable matchings may also be enforced in full-awareness equilibria and may arise with positive probability as outcome matchings. Those possibility results apply to our framework since every one-to-one matching is essentially a many-to-one matching, with the specification that $q_f = 1 \quad \forall f$.

On the other hand, Wu (2015) also shows that the only way to enforce unstable matchings under limit equilibria is by "Reward and Punishment Schemes", akin to those in Rubinstein and Wolinsky (1990), which is the main discussion object of this section. In Wu's one-to-one model, the reward and punishment scheme works as follows for any blocking pair (f, w): On the equilibrium path, the w is punished for initiating the block by applying, and f is credibly rewarded for not obliging. Only that way, the blocking pair does not realize, and an unstable matching can be enforced as an equilibrium outcome. In the following, I show that in a many-to-one search model, there exists an increased potential for implementing such schemes by utilizing the remaining capacities of firms.

[14]Furthermore, if we instead consider a many-to-many environment where workers can also work for multiple firms at the same time and have responsive preferences over firms (and one side has vertical preferences), replacing their strategy profile with "Workers apply only to firms such that $v(f, w) \geq \min(f', w)$ such that $f' \in \mu(w)$ in any submarket", the strategy profile would yield the stable many-to-many matching μ^*.

Crucially, in a one-to-one world, implementing such schemes requires knowledge about who rejected whom in past meetings. Consequently, when agents cannot condition their behavior on who rejected whom in past failed events, such schemes are not implementable. That information restriction is in line with what was defined as "private-dinner equilibria", where meetings take place in private environments, and others can only observe (and condition their behavior on) the outcome of meetings. Under the private-dinner condition, the successful meetings indirectly convey the information of mutual acceptance. However, in case of an unsuccessful meeting, it remains unknown to the agents who rejected whom. Since this restriction deactivates the reward and punishment schemes, no unstable matching can be enforced in a private-dinner equilibrium.

The following result shows that this result is not robust to transitioning to a many-to-one model, i.e., allowing for some firms to have $q_f > 1$. Introducing multiple capacities creates opportunities for firms to employ alternative forms of strategic manipulation and ensure more favorable outcomes for themselves. Inevitably, this has adverse implications for workers since responsive preferences preserve the lattice structure of the matchings Roth (1985). This aspect will become even more apparent when examining the example presented following the proposition.

Proposition 3. *In a many-to-one finite decentralized search model where waiting is costless, unstable matchings can be enforced by equilibria that satisfy the Markov condition.*

Proof. I prove this proposition by an example, which will also help illustrate the intuition behind the essential difference between many-to-one and one-to-one models.

Example 1. *Suppose there are two firms $F = \{f_1, f_2\}$ such that $q_1 = 3$ and $q_2 = 2$ and four workers $W = \{w_1, w_2, w_3, w_4\}$ and the ordinal preference relation derived from the preferences of the agents is as follows*[15]*:*

$$w_1 : f_1 \succ f_2 \qquad\qquad f_1 : w_4 \succ w_1 \succ w_2 \succ w_3$$
$$w_2 : f_1 \succ f_2 \qquad\qquad f_2 : w_1 \succ w_2 \succ w_3 \succ w_4$$
$$w_3 : f_1 \succ f_2$$
$$w_4 : f_2 \succ f_1$$

Now, consider the following strategy profile σ^U such that, at every history, w_1, w_2, w_3 only apply to f_1 as long as it is in the market (after f_1 leaves, apply to f_2), w_4 only applies

[15] Any completion for the firm side as long as responsiveness is ensured

to f_2 if is ranked within the remaining capacity in the remaining market for the f_2 or apply to both firms upon meeting. On the firm side, f_1 accepts the top q_1' of the remaining workers, and f_2 waits until f_1 leaves the market (only accepts w_1, w_2, w_3 -who do not accept- until f_1 fulfills its capacity), then only accepts the top q_2' of the remaining workers. In the off-path subgames, agents only accept candidates they like at least as much as their stable partners.

The strategy-profile σ^U is an equilibrium when $\delta = 1$ and depends only on the remaining market for every history, which is even more restricted than the private-dinner condition. Additionally, σ^U enforces $\mu^U = \{(f_1; w_1, w_2, w_4), (f_2; w_3)\}$ which is unstable due to the blocking pair (f_2, w_4).

□

It is easy to show that σ^U enforces μ^U since only pairs that end with pairing with positive probability are as prescripted in μ^U. Moreover, under σ^U, f_1 achieves its favorite employment set (due to responsive preferences). Similarly, w_1 and w_2 are employed by their favorite firm. Therefore, those agents have no incentive to deviate from σ^U at any point in the game. Similarly, w_3 not applying to f_2 or f_2 not accepting does not change the remaining market and, therefore, is no candidate for a profitable one-step deviation. When f_1 is still in the game, w_3 is indifferent between applying to f_2 or not when waiting is costless. Let's take the only candidate pair for a profitable deviation, the blocking pair $(f_2; w_4)$, and suppose they meet on the equilibrium path. The worker w_4 is already applying to f_2, and not applying will not be a profitable deviation either.

It is particularly noteworthy how f_2 rejects w_4 upon his application. Since having a successful meeting with w_1 and w_2 is impossible, f_2 would like to hire both w_3 and w_4. However, if f_2 deviates to accepting w_4 in any subgame following that history, w_3 will apply only to f_1, and f_1 will hire him in return. Therefore, deviating to accepting w_4 before f_1 fulfills its capacity makes f_2 lose w_3 in return, which is less favorable. In other words, since f_2 is forward-looking, it understands the downside of engaging in a block with w_4.

Intuitively, f_2 accepting w_4 starts a rejection cycle similar to Kojima and Pathak (2009). With f_1 starting accepting w_3, w_3 starts rejecting f_2. Both firms prefer μ^U over μ^*, and they can credibly switch w_3 and w_4 in the limit equilibrium by using their remaining capacities as a commitment device. The remaining capacities enable the reward and punishment schemes to be implemented even though the information is restricted to the remaining market.

This difference between one-to-one and many-to-one search models relies on the difference in the underlying static markets. In one-to-one markets, there is no individually rational matching, which is preferred to the firm-optimal stable matching by all firms. In the many-to-one case, the enforced matching μ^U is preferred to μ^* by both firms, and the firms can implement that as long as they can credibly signal each other that blocking pairs will not realize. In return, workers w_3 and w_4 are worse off from being switched by the firms.

Last, but not the least, observe that there is a unique stable matching in the underlying many-to-one market, that is $\mu^* = \{(f_1; w_1, w_2, w_3), (f_2; w_4)\}$. As previously elucidated in the literature review, the uniqueness of the centralized stable outcome usually averts unstable limit equilibrium outcomes, yet, in this particular instance, such prevention does not apply.

To summarize the findings above, in a many-to-one search model, firms can collude strategically, allowing them to implement outcomes more favorable to themselves than the stable matching, even though it may be unique. Unfortunately, such strategic behavior comes at the detriment of workers, who end up worse off as a result. From a policy proposal perspective, the normative question then arises as to whether providing firms with this room for strategic maneuvering is desirable or not. This concern is particularly evident in a setting with colleges and students, where colleges are viewed as public goods, and it becomes imperative to prevent them from developing strategies that benefit themselves at the expense of rendering students worse off. The same applies to the school choice and residency matching problems.

Comparing one-to-one and many-to-one models, we see that enforcing stable matchings with equilibrium strategies is more challenging in a many-to-one search model, whereas unstable matchings are more easily enforceable. These observations emphasize that the role of a central planner becomes essential when it comes to many-to-one matching. The central planner's intervention might be crucial as she can impose a stable allocation that not only ensures fairness but also eliminates any blocking pairs that could lead to undesirable outcomes. By doing so, the central planner might help maintain the integrity of the matching process and protect the interests of all parties involved, promoting a more equitable allocation in many-to-one markets.

4 A Related One-to-One Search Model à la Wu

The fact that stable matchings are more challenging to enforce in many-to-one matchings, as well as unstable matchings are more easily enforceable, is particularly striking when we consider the centralized matching literature. We know that when preferences are responsive, many-to-one and one-to-one matching markets can be seamlessly mapped onto each other as demonstrated by Roth and Sotomayor (1992).

The mapping is done via a "related marriage market", which is obtained by replicating each firm by their capacities and treating this new replica market as a one-to-one market. When firms' preferences are responsive in the original market, we know that a many-to-one matching is stable if and only if the corresponding one-to-one matching in the related marriage market is stable. This compelling finding significantly simplifies the analysis of centralized many-to-one matchings.

Propositions 1, 2, and 3 already hint at the many-to-one search model differing fundamentally from its one-to-one counterpart. Furthermore, this difference cannot be eliminated by imposing strict regularity conditions on within-firm preferences (such as additively separable utility). Instead, the fact that the end matching evolves through time allows for strategic behavior among firms based on their remaining capacities.

To gain more concrete insights into the distinctive search behavior in a many-to-one market, it becomes imperative to delineate a corresponding one-to-one search model for comparative analysis. By examining the differences between the many-to-one and one-to-one search models, we can comprehensively understand the unique characteristics and dynamics inherent in each setting. This analytical approach will shed light on the complexities of the many-to-one search models, paving the way for valuable insights into how search behavior evolves in such contexts, which is the objective of this section. To streamline the discussion and prevent redundancy, any aspects left unspecified in this context can be assumed to remain analogous to the original many-to-one search model.

4.1 The Related One-to-One Search Game

The *related* one-to-one search model will be a translation of our many-to-one search model onto a one-to-one environment. The related one-to-one market is obtained where each firm is replicated as many times as its capacity. In other words, each seat of the firms is individually present in the search market. Furthermore, the seats are individually searching for workers and hence become competitors.

When we replicate firms, we obtain q_f identical seats. Namely, the set of the seats in the related one-to-one market is $S = \{s_{11}, ..., s_{1q_1}, ..., s_{n1}, ..., s_{nq_n}\}$, where s_{ij} is the jth seat of firm i. Recall that firms' preferences in the original many-to-one market are complete and responsive. This means that we can deduct the firms' preferences over individuals. Each seat has the same preference over individuals as the firm. Regarding the environment, the replication is the same as Roth and Sotomayor (1989). The search model requires an additional adjustment with the contact function.

On the workers' side, there is no replication. Each worker from $W = \{w_1, ...w_m\}$ is still searching for himself. Workers are indifferent between the seats of the same firm, and each worker i prefers a seat in firm j over a seat in firm k if and only if he prefers firm j over firm k in the many-to-one market. Formally, $v(s_{jn}, w_i) < v(s_{km}, w_i)$ whenever $v(f_j, w_i) < v(f_k, w_i)$. For simplicity, from now on, I break the indifferences in workers' preferences such that they prefer the seat with a smaller index within the same firm: $v(s_{j1}, w_i) > ... > v(s_{jq_j}, w_i)$[16]. A related one-to-one market is then the tuple $M_R = (S, W, u, v)$. Similarly, any submarket (remaining market) is $M'_R = (S', W', u, v)$ with $S' \subset S$ and $W' \subset W$ [17].

The contact function is mildly adjusted such that for each submarket, the sum over the probabilities of w and s meeting in M'_R and s is replicated from f of M' is equal to $C(f, w, M')$ in the original market. All other assumptions on the contact function remain the same. Most importantly, all pairs of seats and workers have a positive probability of meeting in line with Wu (2015).

The related one-to-one game is denoted by the tuple $\Gamma_R = (S, W, u, v, C, \delta)$.

Recall that Wu (2015) shows that no unstable matching can be enforced in a private-dinner equilibrium, whereas Proposition 3 shows otherwise, even under a more restrictive information criterion. Therefore, the equilibria of one-to-one and many-to-one search models are generically not equivalent. In the following, I will provide a sufficient condition for the underlying many-to-one market that will restore the projectability between many-to-one and one-to-one models and ensure that no unstable matching will be enforced in a private-dinner equilibrium.

Definition 4. *The pair (s, w) is called a top-pair for any related (sub)market if: Among the seats that find w acceptable, s is the best for w, and among the workers that find s acceptable, w is the best for s.*

Following Wu, a related marriage market satisfies the Sequential Preference Condition (SPC) if there is an ordering of the seats and workers and a positive integer k such that:

1. *For any $i \leq k$, (s_i, w_i) is a top-pair in the (sub)market.*
2. *Discarding the top-pairs results in a trivial market with non-acceptable pairs.*

[16] This tie breaking rule reduces the number of stable matchings in the related one-to-one market, yet does not affect the analysis, see Appendix.

[17] Any instantaneous matching μ translates onto the one-to-one replica such that $\mu_R(w) = s_{in} \Rightarrow \mu(w) = f_i$.

In a one-to-one market, the top-pairs always mutually accept each other upon meeting. This is almost trivial since the top-pairs cannot hope for a better alternative to be matched in future periods. Once the top-pair leaves the market, the same applies to another because the market satisfies SPC. Since every pair has a positive probability of meeting, there is an equilibrium path that matches the top-pairs, which happens with positive probability, which ensures no unstable matching can be enforced in a limit equilibrium. In the following, I adjust this condition to many-to-one markets.

Definition 5. *A many-to-one market satisfies the Sequential Preference Condition if the following two conditions hold:*

1. *The related one-to-one market satisfies SPC.*
2. *Firm preferences are lexicographic for top pairs: If (s_{ij}, w_i) is a top pair in the remaining related market, $u(f_i, \Omega_i) > u(f_i, \Omega_{-i})$ for all (Ω_i, Ω_{-i}) such that $w_i \in \Omega_i$ but $w_i \notin \Omega_{-i}$.*

Proposition 4. *If the initial many-to-one market satisfies the Sequential Preference Condition, no unstable matching can be enforced in a limit equilibrium.*

Proof. In a market where preferences satisfy SPC, there is a unique matching that allocates top pairs to each other. In the original many-to-one search game, the top pairs would always accept each other (which, on the firm side, is ensured via lexicographic preferences). Since every pair has a positive probability of meeting, there is an equilibrium path in which top pairs meet each other in the SPC ordering, and this happens with positive probability. Therefore, enforced matching cannot be unstable when the preferences of the initial market satisfy SPC. □

Corollary 1. *Suppose the initial many-to-one market satisfies the Sequential Preference Condition. In that case, the only enforceable matching of the many-to-one market in a limit equilibrium corresponds to the only enforceable matching of the related one-to-one market in a limit equilibrium.*

Even though we need a modification for the Sequential Preference Condition for many-to-one markets, the result and the proof method apply from Wu (2015). The unique stable set of workers will apply to the firm, and the firm will accept those. Thus, they cannot credibly threaten each other with rejection, resulting in employment upon first meeting. Furthermore, no unstable matching can be enforced in the related market, and a many-to-one matching is stable if and only if the related one-to-one matching is stable in the related market. Consequently, the equilibrium outcomes of both models are equivalent when the contact function is realized according to the SPC ordering, as well as the remaining markets are related. Thus, SPC rebuilds the connection from the many-to-one search model to the one-to-one static market regarding enforced matchings.

Note that by preventing unstable matchings from being enforced, SPC provides a solution to the discussion at the end of Section 3.3 about the potential necessity of a central planner for more general preferences to prevent firms from manipulating the search outcome. However, unstable matchings can still arise with positive probability even under SPC, as shown in Wu (2015).

5 Connection to Dynamic Stability à la Doval

On the way from centralized matching with clearinghouses that impose an allocation on an economy to a decentralized search model, one natural stepping stone to consider would be the "dynamic stability" concept, introduced by Doval (2022) for one-to-one environments and then later incorporated into many-to-one by Altınok (2019). In this section, I compare the search outcomes to dynamically stable matchings. Since we are in a many-to-one environment, the definitions and examples below are based on Altınok (2019).

Intuitively, the dynamic stability concept also incorporates the time component, or at least the sequential acceptance of agents, even though waiting is almost costless. Formally, what differs from the current model is that all employers are around in all periods, whereas candidates arrive over time and W_t denotes the workers that arrive at period t, and the agents form matches over exogenously given T periods, $t = 1, 2, ..., T$.

For the rest of this section, I am restricting attention to $T = 2$, that is, workers arrive over two periods. Then, a history is the empty set for the first period and the 1st-period matching in period 2, and the strategies map histories into the matchings of the same period.

Definition 6. *A t-period matching μ_t is a mapping from the set of candidates that have arrived until t to the set of employers; that is, for each t,*

$$\mu_t : \bigcup_{\tau=1}^{t} W_t \to F \cup \{\emptyset\}$$

and satisfies the following properties:

1. *Capacity constraints are always respected: $|\mu_t^{-1}(f)| \leq q_f$ for each f for $t = 1, 2$.*

2. First-period matchings are irreversible: $\mu_2(w) = \mu_1(w)$ *if* $\mu_1(w) \neq \emptyset$*,*

Similarly, M_t is the set of $t-$period matchings for $t = 1, 2$, $h_t := (h_\tau)_1^t$ a history of matchings at t where H_t denotes all possible period-t histories, where $H_1 = \emptyset$ and $H_2 = M_1$. A strategy profile is then (s_1, s_2), where $s_t : H_t \to M_t$ for $t = 1, 2$. The second period is identical to a static market, and a first-period block refers to a blocking coalition that exists in the first period and that can implement a better second-period matching for them by forming a coalition in the first period.

Consider the following example by Altınok to illustrate the difference between the static stability vs the dynamic one:

Example 2. *Suppose there are two firms* $F = \{f_1, f_2\}$ *such that* $q_1 = 3, q_2 = 2$ *and six workers* $W = \{w_1, ..., w_6\}$ *and the preferences are as below:*

$$
\begin{aligned}
&w_1 : f_1 \succ f_2 & \qquad & f_1 : w_6 \succ w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5 \\
&w_2 : f_1 \succ f_2 & & f_2 : w_1 \succ w_2 \succ w_3 \succ w_6 \\
&w_3 : f_1 \succ f_2 & & \\
&w_4 : f_1 \succ f_2 & & \\
&w_5 : f_1 \succ f_2 & & \\
&w_6 : f_2 \succ f_1 & &
\end{aligned}
$$

In addition, suppose $\{w_4, w_5, w_6\} \succ \{w_1, w_2, w_3\}$ *for* f_1*, which is still in line with responsiveness but depicts that* f_1 *has extreme preferences, in the sense that it prefers combining extreme workers rather than the average ones.*

In this market, the unique stable matching is $\mu^* = \{(f_1; w_1, w_2, w_3), (f_2; w_6), (\emptyset, w_4), (\emptyset, w_5)\}$, which in fact is not dynamically stable in this particular market: Suppose instead of everybody being in the market at once, workers arrive in 2 periods, such that workers w_4 and w_5 arrive in period 1. Then, f_1 would form a period-1 matching with w_4 and w_5 (1st period block), enters the second-period with $q_1' = 1$. The unique stable μ in period 2: f_1 matches with w_6, f_2 is matched with $\{w_1, w_2\}$, so the dynamically stable matching is $\mu^* \neq \mu^D = \{(f_1; w_4, w_5, w_6), (f_2; w_1, w_2), (\emptyset, w_3)\}$

Both firms prefer the outcome μ^D to μ^*, which is not stable because of the blocking pair (f_1, s_1). What happens with dynamic stability is that the firms in a sense exchange w_6 with w_1, w_2. To make this exchange credible, f_1 fills its capacity with w_4, w_5 in the first period.

The straightforward intuition places dynamic stability between centralized and completely decentralized search models, with the number of periods given and the central planner still imposing the matching on the economy somehow. Furthermore, the dynamic stability concept and the decentralized search model share a similar flavor in the strategic commitment and manipulation room they provide to firms with multiple capacities. Nevertheless, the following proposition will show that (contrary to the connection to completely centralized markets) dynamically stable matchings may not be enforced as search equilibria.

Proposition 5. *Dynamically stable matchings may not be enforceable by limit equilibria.*

Proof. Take the example by Altınok (2019) one more time, and suppose the dynamically stable $\mu^D = \{(f_1; w_4, w_5, w_6), (f_2; w_1, w_2), (\emptyset, w_3)\}$ is enforced in a limit equilibrium. Since every meeting has a positive probability in the decentralized search model, suppose f_1 and w_6 meet the first day. If w_6 applies to f_1: If f_1 accepts, firms have the same preferences over remaining workers, f_1 employs $\{w_1, w_2, w_6\}$. The best f_1 can hope for: f_1 accepts w_6. Therefore, w_6 applies to f_1 even with small waiting costs upon meeting. Since every pair has a positive probability of meeting in the initial market, $\mu^D = \{(f_1; w_4, w_5, w_6), (f_2; w_1, w_2), (\emptyset, w_3)\}$ cannot be enforced by limit equilibria. □

In fact, as the proof of this proposition already suggests, the dynamic stability concept depends heavily on which workers arrive in which period, which resembles the meeting probability being exactly 0 and exactly 1 for some of the candidates. The concept is not robust to more general probabilities such as that are defined by the contact function, as well as the workers becoming strategic agents as well.

6 Conclusion

In this paper, I describe and analyze a finite decentralized many-to-one bilateral search model. The finite search model is the many-to-one counterpart of Wu (2015) that allows firms to employ more than one worker. The outcomes of subgame perfect equilibria are compared to the stable matchings of the underlying many-to-one market. Unlike the one-to-one finite search model, the stable matchings of the underlying market may not be enforceable by simple strategies. On the other hand, when one side of the market has vertical preferences, there is a Markovian strategy profile that enforces the unique stable matching.

Unlike the centralized matching models, the many-to-one search model presents another fundamental distinction in its projection onto one-to-one environments. In one-to-one settings, enforcing unstable matchings necessitates the application of reward and punishment schemes, which is only feasible once agents can observe the details of failed meetings. On the contrary, in the many-to-one search model, firms hold the capacity to implement reward and punishment strategies through their remaining capacities, enabling them to commit to credible strategies that yield a more advantageous matching outcome for themselves compared to the stable matching. This capability stems solely from the fact that firms possess multiple capacities, making it impractical to attempt resolving the manipulation incentives on the firm side by imposing stringent assumptions on within-firm preferences, such as additively separable utility.

The transition from one-to-one to many-to-one search models presents new challenges in enforcing stable matchings and the relative ease of enforcing unstable ones. This underscores the significance of employing a centralized clearinghouse in many-to-one markets such as student admissions and labor markets compared to their one-to-one counterparts.

Dynamically stable matchings may not be enforceable in limit equilibria either. Indeed, the concept hinges on the specific sequence of worker arrivals over time, which is not necessarily robust to non-binary meeting probabilities.

Nevertheless, when preferences of all firms satisfy the sequential preference condition, i.e., there are top pairs of firms and workers that mutually prefer each other over all other available options, the unique stable matching is the only enforceable matching in a limit equilibrium. Furthermore, the related matching in the related one-to-one market is the only enforceable outcome in any limit equilibrium in a one-to-one search model à la Wu (2015), and the instantaneous matchings of many-to-one and one-to-one models are equivalent.

The paper investigates the intricacies of finite decentralized many-to-one matching, offering a comprehensive understanding of the underlying dynamics that govern agents' behaviors. Notably, showing how many-to-one matching markets differ from their one-to-one counterparts when considered in a search model bridges the gap between centralized many-to-one and decentralized one-to-one models. Discrepancies are present with both models, even though firm preferences are responsive and agents are sufficiently patient. In general, the presence of multiple capacities within this framework introduces opportunities for strategic manipulation by firms, either by sacrificing some vacancies or through collusion. Nonetheless, it is worth noting that the characterization of matchings that can be enforced through limit equilibria remains a topic that necessitates further exploration and investigation in future research.

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Appendix

Theorem 1:

Proof.

- σ^* enforces the stable μ^*: μ^* obtains almost surely on the equilibrium path. For any meeting function realization, another outcome than μ^* arising from σ^* has a probability of 0. Suppose f and w are not matched under μ^* but they end up together under σ^* for some meeting function realization. Mutual acceptance requires $u(f,w) \geq \min(u(f,w'))$ such that $w' \in \mu(f)$ and $v(f,w) \geq v(\mu^*(w),w)$ which contradicts with μ^* being stable under responsive preferences. This part concludes all pairs in the outcome are consistent with μ^*.

  Also, note that all existing agents meet with some positive probability. Since the pairs under μ^* accept each other, and the probability of a rejecting pair (or a rejection pair cycle) occurring has a probability of 0, μ^* obtains almost surely on the equilibrium path.

- σ^* constitutes a limit equilibrium of any search game. For any firm f, the expected utility of σ^* is $u(f,\mu^*(f))$ and for any worker w the expected utility of σ^* is $v(\mu^*(w),w)$. Note that a one-step deviation to reject a partner that is accepted under μ^* does not change the submarket, therefore we are still on-the-equilibrium-path. The only deviation by an agent that would yield a switch to off-path is accepting a partner from the other side who would not be accepted μ^*.

  Now consider a one-step deviation by w such that w accepts f instead of rejecting as under σ^*. Rejection under σ^* implies $v(f,w) < v(\mu^*(w),w)$. If f rejects w, the subgame does not change and the expected utility of w does not change. If f accepts, w receives a lower utility than $v(\mu^*(w),w)$. Therefore, it is not beneficial for w to accept f.

  A similar logic applies to f and a one-step deviation towards rejecting, even though f has multiple capacities. We know that when one side has vertical preferences, the deferred-acceptance outcome is the same as the serial dictatorship outcome. Suppose firms have the same preferences over individual workers, i.e. we are in a college admissions model. Then, there is a unique stable matching mu^* that can be achieved both by firm-proposing DA and the worker-proposing DA, as well as the serial dictatorship where the workers are ranked according to the vertical preferences of firms.

  Furthermore, this applies to any submarket of the initial market. In any subgame of the initial game that is on-the-equilibrium-path, the expected utility under μ^* is $u(f,\mu^*(f))$.

Consider a one-shot deviation where f has already hired $\Omega \subset W$ and accepts a worker w that has applied to f and that it prefers less than its least preferred stable partner: i.e: $u(f, w) < \min(u(f, w'))$ such that $w' \in \mu^*(f)$. Denote the subgame where f rejects w as Γ and the one with the deviation Γ'. In Γ under μ^*, $q'_f = q_f - |\Omega|$, and the firm receives $u(f, \mu^*(f))$, where $\mu^*(f) = \Omega \cup \Omega'$.

If $\Omega' = \emptyset$, either f has filled its capacity or $u(f, w) < 0$, ensuring $u(f, \mu^*(f) \cup \{w\}) < u(f, \mu^*(f))$ in either case. If $\Omega' \neq \emptyset$, the first $q' - 1$ workers that f employs under Γ and Γ' are the same by the serial dictatorship representation, and the end outcome differs only by one worker. Denote the worker that is employed under Γ but not under Γ' as w^*. The expected utility with one-shot deviation is $u(f, \Omega \cup \Omega' \setminus \{w^*\} \cup \{w\})$. We know that $u(f, w) < \min(u(f, w'))$ such that $w' \in \mu^*(f)$, therefore $u(f, w) < u(f, w^*)$. With responsive preferences, $u(f, \Omega \cup \Omega' \setminus \{w^*\} \cup \{w\}) < u(f, \mu^*(f))$, ensuring the one-shot deviation being not profitable.

Verbally, any deviation to accepting a worker that is not acceptable under μ^* only changes that worker with one of the stable workers. With responsive preferences, it is ensured that the difference between the specific workers is reflected in the overall preference, preventing deviation.

$\square$

Proposition 1: How can we restrict attention to end-outcomes? 12

Take the same example from the Proposition, and once again, suppose there is a Markovian strategy profile σ^* that is a limit equilibrium and enforces the WOSM in any many-to-one search game. Under σ^*, f_1 either meets with w_1 first and then w_4, depending on the contact function realization. Since σ^* enforces WOSM, the expected utility gain when $\mu_h(f_1) = \emptyset$ is $u(f_1, \{w_1, w_4\})$. When f_1 deviates to accepting w_5, it first gets $u(f_1, w_5)$. Then, since σ^* is an equilibrium in every subgame by assumption, the expected utility gain is the additional utility from adding w_3. Therefore, the overall utility from employing w_5 is $U = u(f_1, w_5) + \delta^{tau}(u(f_1, \{w_3, w_5\}) - u(f_1, w_5))$ (τ referring to the expected day the match will conclude), which converges to $u(f_1, \{w_3, w_5\})$ as δ converges to 1. The same logic applies to other propositions.

More on Equivalent Equilibria of Many-to-One and One-to-One

Proposition 4 can be analyzed with a different perspective for related markets with indifferences, in which ties between seats are not broken. In that case, SPC ensures that the equilibria of the original many-to-one and the related one-to-one market are *equivalent*. Intuitively, the equilibria of the many-to-one search model and its related one-to-one search model are equivalent if the workers adapt the same acceptance strategy for the seats of a firm as the strategy they use for the firm, and the firms' seats use the same acceptance strategy for each worker as the firm itself.

Definition 7. *For any many-to-one market and its related market, the equilibria of the many-to-one search game and its related one-to-one search game are equivalent if under the same information restriction and for every related subgame:*

1. *Workers of the related market accept the seats that belong to the firms they apply to in the original market and reject others.*
2. *Seats of the related market accept the same workers as their mother firm.*

The following lemma additionally shows that we can track equilibrium equivalence from remaining markets, even though we cannot observe agents' strategy profiles.

Lemma 1. *For each realization of the contact function C, the equilibria of the many-to-one search model and its related one-to-one search model are equivalent if and only if the remaining markets are related for each history.*

Proof. Let C be any realization of the contact function. I will prove the lemma by proving the if statements from both directions.

1. The equilibria of many-to-one search and the related one-to-one search are equivalent $\Rightarrow$ remaining markets are related for each history.
   Easily proven by induction. Start with the initial market M. Equivalent acceptance strategies imply:
   $$\begin{aligned} f \text{ accepts } w &\iff s \text{ accepts } w \\ w \text{ accepts } f &\iff w \text{ accepts } s \end{aligned}$$
   This means, for the same realized related contact function, M' after the first day is the same. Apply this to every step, the first part of the lemma concludes.
2. Remaining markets are related for each history $\Rightarrow$ The equilibria of both search models are equivalent.
   Suppose s is a seat of f. If when s, w and f, w meet after h at the related remaining markets, and the remaining market after this is also the same s, f use the same acceptance strategies.

If this holds for each remaining market and history, the equilibria are equivalent.

□

The above lemma establishes the equivalence between the equilibrium strategies and the remaining markets. Recall that Wu (2015) shows that no unstable matching can be enforced in a private-dinner equilibrium, whereas Proposition 3 shows otherwise, even under a more restrictive information criterion. Therefore, the equilibria of one-to-one and many-to-one search models are generically not equivalent, which, on the other hand, is ensured if the initial market satisfies the sequential preference condition.

7 Matchings of the Centralized Market à la Search Game

This subsection takes a quick detour into the existing literature of many-to-one matchings and describes the adaptations that will translate the existing results into a many-to-one search environment. In order to find the set of stable matchings, a linear programming approach is developed. As Vate (1989) and Rothblum (1992) characterize stable matchings in a marriage market as extreme points of a convex polytope, Baïou and Balinski (2000) extend the results to a many-to-one matching market. They show that simply replacing the parameters of a marriage market with their many-to-one counterparts does not extend the results of Rothblum, but needs a slight differentiation. Neme and Oviedo (2020) adapt their approach as well, so am I:

Given a matching μ, an assignment matrix $x^B \in \mathbb{R}^{|F|\times|W|}$ (B for Baïou and/or Balinski) is defined where all its elements are denoted by $x^B(f,w)$ where $x^B(f,w) \in \{0,1\}$ and $x^B(f,w) = 1$ if and only if $\mu(w) = f$.

Following Baïou and Balinski, let CP denote the convex polytope generated by the following linear inequalities:

$$\sum_{j\in W} x^B_{f,j} \leq q_f \qquad \forall f \in F \tag{1}$$

$$\sum_{i\in F} x^B_{i,w} \leq 1 \qquad \forall w \in W \tag{2}$$

$$x^B_{f,w} \geq 0 \qquad \forall (f,w) \in F \times W \tag{3}$$

$$x^B_{f,w} = 0 \qquad \text{for unacceptable pairs } (f,w) \tag{4}$$

The integer solutions to (1)-(3) are assignment matrices of simple many-to-one matchings. A matching, where some entries $x^B(f,w)$ are non-integers in the interval $(0,1)$ is called a fractional matching. We can interpret the fractional matchings as probabilities that the agents are matched to one another as well as the timeshares the respective agents spend with each other.

An example of a many-to-one assignment matrix with 2 firms $\{f_1, f_2\}$ and 2 workers $\{w_1, w_2\}$ would look like as follows:

	f_1	f_2
w_1	$x^B(f_1, w_1)$	$x^B(f_2, w_1)$
w_2	$x^B(f_1, w_2)$	$x^B(f_2, w_2)$

where all the entries are nonnegative and:

$$\sum_F x^B(f,w) \leq 1 \;\text{ for both workers}$$
$$\sum_W x^B(f,w) \leq q_f \;\text{ for both firms}$$

Adding (4) imposes the individual rationality constraint, that the match is at least as good as the outside option. As Baïou and Balinski show, adding another linear inequality to the CP system:

$$\sum_{u(f,j)>u(f,w)} x^B_{f,j} + q_f \sum_{v(i,w)>v(f,w)} x^B_{i,w} + q_f x^B_{f,w} \geq q_f \qquad \forall (f,w) \in A \tag{5}$$

defines the stable convex polytope SCP and the integer solutions to SCP define stable simple matchings, which are individually rational and pairwise stable.

In Kojima and Manea (2010) and Kesten and Ünver (2015), "The School-Choice Birkhoff-von Neumann Theorem states that any fractional matching can be represented as a lottery (not necessarily unique) over simple matchings", which allows us to interpret a fractional matching in a third way. Nevertheless, the intuition about the stable matchings turns out to be incorrect and the non-integer solutions of inequalities (1)-(5) do not immediately give us stability when it comes to fractional matchings.

In fact, as shown by Baïou and Balinski Baïou and Balinski (2000) and elaborated further in Neme and Oviedo Neme and Oviedo (2020), the non-integer solutions to the SCP might be blocked in a *fractional way*, by a firm and worker, who want to increase their timeshare together, at the expense of those they like less at a non-integer solution to *SCP*. Formally:

Definition 8. *A matching is blocked by the firm-worker pair* (f, w) *in a fractional way, when* $x^B(f, w) < 1$, $v(f, w) > v(f', w)$ *for some* f' *such that* $x^B(f', w) > 0$ *and* $u(f, w) > u(f, w')$ *for some* w' *such that* $x^B(f, w') > 0$.

Example: An example from Baïou and Balinski (2000) and Neme and Oviedo (2020) which shows an assignment matrix, which is a solution to *SCP* and blocked in a fractional way by is as follows:

	f_1	f_2
w_1	1	0
w_2	0.5	0.5
w_3	0.5	0.5
w_4	0	1

where the preferences (over individuals, derived from the preferences over sets) are such that:

$$\begin{aligned}
&f_1 : u(f_1, w_1) > u(f_1, w_2) > u(f_1, w_3) > u(f_1, w_4) \quad \text{and} \quad q_{f_1} = 2 \\
&f_2 : u(f_2, w_4) > u(f_2, w_3) > u(f_2, w_2) > u(f_2, w_1) \quad \text{and} \quad q_{f_2} = 2 \\
&w_1 : v(f_2, w_1) > v(f_1, w_1) \\
&w_2 : v(f_2, w_2) > v(f_1, w_2) \\
&w_3 : v(f_2, w_3) > v(f_1, w_3) \\
&w_4 : v(f_1, w_4) > v(f_2, w_4)
\end{aligned}$$

In the example above, it can easily be checked that the numbers solve the linear problem *SCP*. However, w_3 likes f_2 better than f_1, but his time is shared equally between the firms. In addition, f_2 likes w_3 better than w_2 but one seat is shared equally between those workers. In such a case, the pair (f_2, w_3) blocks the assignment above in a fractional way so that they can increase their time spent together in exchange for their other partners in the matching, f_1, and w_2.

As it is not mentioned in either of the papers, the lottery interpretation of the fractional matchings helps us understand the underlying misfunction in this example. Although the lottery over simple matchings which represents a fractional matching is generically not unique, in this example it actually is unique. The fractional matching is a lottery over the following two simple matchings with equal probability 0.5:

	f_1	f_2
w_1	1	0
w_2	1	0
w_3	0	1
w_4	0	1

	f_1	f_2
w_1	1	0
w_2	0	1
w_3	1	0
w_4	0	1

The reason why there is a blocking pair in a fractional way can be observed in the lottery as well. Although the first simple matching of the lottery is stable, the second one is not. Furthermore, the fact that it is not stable pins down the fractional blocking pair: The second simple matching is blocked by the pair (f_2, w_3).

Neme and Oviedo (2020) refer to matchings in which there are no incentives to block (neither as in the usual way nor in the fractional interpretation) as "strong stable matchings" and prove that they can be found adding the additional constraint to SCP:

Definition 9. *Let (F, W, q, u, v) be a many-to-one matching market. A fractional matching is strongly stable if for each acceptable pair (f, w), x satisfies the strong stability condition:*

$$\left[q_f - \sum_{u(f,j) \geq u(f,w)} x^B_{f,j}\right] \cdot \left[1 - \sum_{v(i,w) \geq v(f,w)} x^B_{i,w}\right] = 0 \tag{6}$$

Observe that the assignment matrix of a simple stable matching fulfills (6). This follows from the simple fact that if f and w are matched, the second multiplier is 0. If they are not matched with each other, at least one of them is consuming its own capacity.

For fractional matchings, when (6) does not hold for some (f, w), $q_f > \sum_{u(f,j) \geq u(f,w)} x^B_{f,j}$ and $1 > \sum_{v(i,w) \geq v(f,w)} x^B_{i,w}$, it means that there are f' and w' such that $u(f, w) > u(f, w')$, $v(f, w) > v(f', w)$ and $x(f, w') > 0$, $x(f', w) > 0$ and $x(f, w) < 1$. In such a scenario, the (f, w) would block the assignment and increase their time shared together.

Insightful Theorem 1 from Neme and Oviedo (2020) concludes: "If x^B is a strongly stable fractional matching, it can be represented as a convex combination of stable matchings. Furthermore, a lottery over simple stable matchings is strongly stable as well." This establishes the lottery interpretation as in Lauermann and Nöldeke (2014).

Both Baïou and Balinski (2000) and Neme and Oviedo (2020) constructed the matching as an assignment matrix with two sides in rows and columns, respectively. This approach is quite useful for visualizing and pointing out simple stable matchings. However, the structure in Lauermann and Nöldeke (2014) requires a different approach. The main difference is that the many-to-one papers of Baïou and Balinski and Neme and Oviedo use an ordinal utility approach, whereas Lauermann and Nöldeke employ a cardinal utility in their model, which allows them to calculate the expected utilities for the agents as well. As discussed before, cardinal utility adaptation is vital for a search structure.

If $u(f, \Omega)$ is a linear function of individual values, i.e. additively separable such that $u(f, \Omega) = \sum_{i \in \Omega} u(f, w_i)$, we could still use the assignment matrix with the agents in rows and columns to calculate expected utilities. In order to employ such a structure in a many-to-one framework and be able to calculate expected utilities at the same time, we would need a different assignment matrix $x \in \mathbb{R}^{|F| \times |2^W|}$, satisfying:

$$\sum_{F} \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f, \Omega) \leq 1 \qquad \forall w \tag{7}$$

$$\sum_{\Omega \subset 2^W} x(f, \Omega) \leq 1 \qquad \forall f \tag{8}$$

$$x(f, \Omega) = 0 \qquad \forall |\Omega| > q_f \tag{9}$$

$$x(f, \Omega) \geq 0 \qquad \forall (f, \Omega) \in F \times 2^W \tag{10}$$

The above-described assignment matrix has firms in the columns and all possible subsets of the workers in the rows. In a many-to-one matching, a specific worker cannot be matched to two firms. Hence, considering a worker would now require considering each subset that this worker appears in.

The expected utilities from a matching x then can be calculated as follows:

$$U(f; x) = \sum_{\Omega} x(f, \Omega) u(f, \Omega)$$
$$V(w; x) = \sum_{F} \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f, \Omega) v(f, w)$$

Example: An assignment matrix of a many-to-one matching market with 2 workers $\{w_1, w_2\}$ and 2 firms $\{f_1, f_2\}$ would be as follows according to the latter description:

	f_1	f_2
$\{w_1, w_2\}$	$x(f_1, \{w_1, w_2\})$	$x(f_2, \{w_1, w_2\})$
w_1	$x(f_1, w_1)$	$x(f_2, w_1)$
w_2	$x(f_1, w_2)$	$x(f_2, w_2)$

where all the entries are nonnegative and:

$$
\begin{aligned}
&x(f_1, \{w_1, w_2\}) + x(f_1, w_1) + x(f_2, \{w_1, w_2\}) + x(f_2, w_1) \leq 1 \\
&x(f_1, \{w_1, w_2\}) + x(f_1, w_2) + x(f_2, \{w_1, w_2\}) + x(f_2, w_2) \leq 1 \\
&x(f, \{w_1, w_2\}) + x(f, w_1) + x(f, w_2) \leq 1 \ \text{ for both firms} \\
&x(f, \Omega) = 0 \quad \text{if} \quad |\Omega| > q_f \ \text{ for all subsets}
\end{aligned}
$$

With expected utilities:

$$
\begin{aligned}
U(f; x) &= x(f, \{w_1, w_2\})u(f, \{w_1, w_2\}) + x(f, w_1)u(f, w_1) + x(f, w_2)u(f, w_2) \\
V(w_1; x) &= \sum_{f \in F} \left[x(f, \{w_1, w_2\}) + x(f, w_1)\right] v(f, w_1) \\
V(w_2; x) &= \sum_{f \in F} \left[x(f, \{w_1, w_2\}) + x(f, w_2)\right] v(f, w_2)
\end{aligned}
$$

Proposition 6. *The assignment matrix described before by Baïou-Balinski and Neme-Oviedo, which has the workers instead of sets of workers in the rows can easily be calculated from the matrix described above, by setting* $x^B(f, w) = \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f, \Omega)$.

Similarly, the entries of the assignment matrix x *can be calculated if all the entries of* x^B *are integers, i.e.* $x^B(f, w) = \{0, 1\} \ \ \forall (f, w)$*. This does not necessarily hold if some entries of* $x^B \in (0, 1)$.

Proof. The first part of the lemma is trivial with the given equality. For the second part, the example below illustrates the calculation of the assignment matrix x from x^B for simple matchings. Furthermore, the second part of the example serves as a proof that x calculated from x^B is not necessarily unique for fractional matchings.

Example: Consider again the example of a many-to-one matching market with 2 workers $\{w_1, w_2\}$ and 2 firms $\{f_1, f_2\}$. In the first scenario, let us take a simple matching with all entries are either 0 or 1. In that case, if $u(f, H)$ and $v(h, w)$ are known, the expected utilities can easily be calculated because the assignment matrix x^B implies a unique x.

x^B	f_1	f_2
w_1	1	0
w_2	1	0

x	f_1	f_2
$\{w_1, w_2\}$	1	0
w_1	0	0
w_2	0	0

However, if x^B is a fractional matching, the corresponding x is not necessarily unique. Under different x representations of x^B, the expected utility of the workers will be equal, whereas the expected utilities of the firms might differ. There is an example of a fractional matching x^B with multiple x representations below, where both x_1 and x_2 imply x^B, which serves as a proof to the Proposition 6 above.

x^B	f_1	f_2
w_1	0.5	0.5
w_2	0.3	0.4

x_1	f_1	f_2
$\{w_1, w_2\}$	0.3	0
w_1	0.2	0.5
w_2	0	0.4

x_2	f_1	f_2
$\{w_1, w_2\}$	0.1	0
w_1	0.4	0.5
w_2	0.2	0.4

$\square$

When we consider the finite decentralized many-to-one search game, the acceptance decisions of the agents represent a stopping agreement. Therefore, we need to be able to calculate the expected utilities from continuing the search and compare them to the gains from an immediate acceptance decision. As the next subsection shows, this new definition of a matching matrix will enable us to do such comparisons.

From Equilibria to Assignment Matrices

In pursuance of the analysis of how the subgame perfect equilibria of the many-to-one search model relate to the stable matchings of the centralized many-to-one market, I will describe the assignment matrix that can be obtained from the search equilibrium. In fact, any equilibrium of the search model implies an assignment matrix. Intuitively, the assignment matrix will show the probability of a firm and a subset of workers being matched in equilibrium. In the following revisions, this methodology will be implemented to show whether equilibrium matchings are stable.

In order to calculate the assignment matrix from an equilibrium of the search game, we look at the terminal histories. For any finite terminal history $h \in \mathcal{H} \backslash \hat{\mathcal{H}}$ that takes T periods, let $C_h^t(f, w, M_h^t) \in \{0, 1\}$ denote the meeting function realization for any $t \in \{0, ..., T\}$, where M_h^t is the remaining market implied by μ_h^t, the instantaneous matching in the beginning of period t. [18] The meeting function realization is such that $C_h^t(f, w, M_h^t) = 1$ for firm f and worker w who meet at period t along h and $C_h^t(f, w, M_h^t) = 0$ for all other pairs. The pair (f, w) such that $C_h^t(f, w, M_h^t) = 1$ will be referred to as i_h^t, since they are the agents of the stage game.

Similarly, $a_h^t(i_h^t, \mu_h^t) \in \{A, R\}^2$ denotes the action realization for i_h^t.[19] After the action profile of t realizes, the instantaneous matching is updated to μ_h^{t+1} and remaining market is M_h^{t+1}. Since the market does not change if any of the parties reject, $M_h^{t+1} = M_h^t$ unless $a_h^t(i_h^t, \mu_h^t) = (A, A)$. If both parties accept, the worker leaves the market and the firm leaves the market if the capacity is full.

With this formulation, we can now calculate the probability of a terminal history h occurring on the equilibrium path of the search game. The probability of meeting is simply determined by the choice function, $\mathbb{P}(C_h^t(f, w, M_h^t) = 1) = C(f, w, M_h^t)$.

In order to reach day 1 on the equilibrium path of h, the agents who meet on day 0 should be aligned with h and they should decide accordingly as well. The probability of reaching day 1 under history h, denoted by $\mathbb{P}(\mu_h^1)$ and satisfies $\mathbb{P}(\mu_h^1) = C(i_h^0, M_h^0)\mathbb{P}(a_h^0(i_h^0, \mu_h^0))$. By induction, the probability of reaching any day $t \leq T$ along history h can be calculated by multiplying the probability of agents meeting and behaving according to the history along the equilibrium path of h:

$$\mathbb{P}(\mu_h^t) = \prod_{k=1}^{t} C(i_h^{k-1}, M_h^{k-1})\mathbb{P}(a_h^{k-1}(i_h^{k-1}, \mu_h^{k-1}))$$

Subsequently, for any given many-to-one search game $(F, W, q, u, v, C, \delta)$ once the equilibrium acceptance strategies of the agents are calculated, we can restrict attention to terminal histories and easily calculate the probability of any f, Ω being matched at the end of the game by simply adding up the probabilities of different terminal histories at the end of which f and Ω are together. By simply taking this probability equal to $x(f, \Omega)$, a matching matrix can be constructed.[20]

[18] Clearly, $M_h^0 = M$ for any history.

[19] Recall that we restrict attention to pure strategies.

[20] The game ends almost surely, and it can easily be shown that this matrix satisfies the properties of a many-to-one matching.