Decentralized Many-to-One Matching with Random Search

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Abstract

In this paper, I introduce and analyze a finite decentralized many-to-one search model, where firms and workers meet randomly and time is nearly costless. Different than the existing literature, I show that stable matchings of the many-to-one market may not be enforceable under Markovian search equilibria even if agents are sufficiently patient. Nevertheless, for markets with vertical preferences, I construct the equilibrium strategy profile that enforces the unique stable matching. Moreover, in many-to-one search, firms collect workers in a cumulative manner, where they expand their employment set over time. This distinct characteristic affects the search process fundamentally by allowing firms to strategically utilize their seats over time, leading to more favorable outcomes for themselves. Therefore, unlike centralized markets, there is no straightforward related-market connection between one-to-one and many-to-one search models. I establish one sufficient condition on the preferences of agents to retrieve the connection, which also prevents the enforcement of unstable matchings in equilibrium.

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1 Introduction

The paper considers a two-sided many-to-one matching market, such as the labor market, in which firms can hire multiple workers up to their capacity, whereas each worker can be employed by only one firm. Such markets have garnered significant attention and investigation ever since earliest works on matching theory. However, a noteworthy characteristic of the analyses in this domain is their reliance on a central clearinghouse. The clearinghouse collects preferences from both sides of the market and subsequently imposes an allocation on the entire economy. This approach, not only is theoretically insightful, but also finds practical applications in various real-world scenarios. Notable instances include the National Residency Matching Program in the United States and college admissions processes in countries such as Brazil, Hungary, India, and Turkey. These applications wield profound influence, directly impacting the lives of hundreds of thousands each year, thereby emphasizing the invaluable contributions this research has made.

Conversely, numerous many-to-one matching scenarios evolve within decentralized markets, operating over time without the intermediation of a central clearinghouse. Consider, for instance, the process of students applying to graduate schools. In this context, students apply to schools, and graduate school committees autonomously determine whether to extend acceptance offers. As finitely many students and graduate schools engage in mutual acceptance decisions, the initial market gradually shrinks over time, eventually concluding as a matching for the next academic year. A parallel dynamic can be observed in entry-level job markets, where firms actively recruit senior year students to fill their vacancies, a process that continues until the students graduate.

In this paper, I study such decentralized many-to-one search models. Naturally, such markets are not frictionless and the two key frictions agents face in the search model are random meetings in the market and the associated costs of waiting. Random meetings can be considered as the worker hearing about the firm. Subsequently, upon meeting, a consequential decision ensues for agents - whether to accept the current option or to continue the search for potentially more favorable partners. Sequentially, the worker is the first to decide whether to apply to the firm. Should the worker decide to apply, the firm then decides whether to add the worker to its team. It is crucial to note that matches are irreversible, leading to a gradual reduction in the initial market over time. The central focus of this investigation lies in the end-matching, which denotes the culmination of the matching process until the end of the game.

Naturally, if the waiting costs are too high for the agents, they become so impatient that leads them to accept any individually rational option they encounter, rather than waiting for potentially more suitable partners. Consequently, our analysis is primarily directed towards scenarios characterized by sufficiently low waiting costs, where the agents are more inclined to engage in a discerning evaluation of their options and seek out more favorable matches rather than settling for the first opportunity that comes their way.

In exploring the dynamics of decentralized many-to-one matching with random search, two distinct approaches naturally emerge. The first approach draws inspiration from the realm of centralized many-to-one markets, where one would contemplate the unfolding of economic interactions without the intervention of a central planner tasked with aggregating preferences from both sides and enforcing the allocation. For instance, Roth and Sotomayor (1992) posits that even in the absence of a central planner, the search outcomes would maintain stability, provided that search frictions remain relatively small. According to them, sufficiently knowledgeable agents would then refrain from engaging in less favorable matchings for themselves. This perspective was already challenged by many papers in various contexts, some of which are further described in the literature review.

The second approach takes cues from one-to-one environments. In the centralized framework, many results obtained in one-to-one settings can be directly applied to one-to-one scenarios, contingent upon mild rationality assumptions regarding firm preferences. As such, the current paper also serves as a robustness check to the benchmark decentralized one-to-one model established by Wu (2015) and offers valuable insights into the transition from one-to-one finite search models to many-to-one settings. This dual approach enables us to thoroughly explore the intricacies of decentralized many-to-one matching, bridging the gap between centralized many-to-one and decentralized one-to-one models, and uncovering the underlying dynamics that govern the agents' behaviors.

The first striking difference the decentralized many-to-one search model comes forward when we consider the sustainability of stable matchings in equilibrium. Roth's conjecture posits that any unilateral or bilateral deviations that lead to more plausible outcomes could find a place on the equilibrium path. Consequently, at least a portion of the equilibrium outcomes should consist of stable matchings, which inherently lack incentives for such deviations. This concept aligns with Wu's findings, which demonstrate that stable matchings in one-to-one markets can be sustained through relatively straightforward strategy profiles.

However, a pivotal departure from this pattern becomes apparent in the context of the many-to-one search model. In this setting, I reveal that sustaining stable matchings is not as straightforward. The deviation from the strategy profile that would enforce the stable matching occurs in the sense that firms opt to allocate some of their vacancies to less favorable workers. By effectively reducing the number of vacancies, the deviation might serve to ensure the formation of a more advantageous team towards the end of the game, introducing a nuanced dynamic that distinguishes the many-to-one search model from its one-to-one counterpart.

Nevertheless, it's worth noting that the potential deviation mentioned earlier does not confer any advantage to firms when at least one side of the market shares identical preferences over the other, a scenario often referred to as "vertical preferences." This situation is not a rare occasion and arises in instances where there is a centralized exam score determining college admissions or when students unanimously agree on the superiority of certain colleges. In such cases, I construct the equilibrium strategy profile that ensures the unique stable matching as the ultimate outcome, which pairs the most preferred firm (worker) with its (his) corresponding most preferred worker (firm), which concludes the discussion of enforcing stable matchings in the many-to-one search model.

Turning our attention to the enforcement of *unstable* matchings, the transition from one-to-one to many-to-one models ushers in noteworthy differences. Enforcing an unstable matching in equilibrium implies the presence of a firm-worker pair who mutually prefer each other to their current partners. Wu's one-to-one analysis reveals that such unstable matchings can be enforced in equilibrium, but this enforcement requires a specific mechanism: the incentives for this particular pair to mutually opt out of the suggested matching must be penalized, while the decision not to opt out is rewarded in return. These so called *reward and punishment schemes* are essential for sustaining unstable outcomes. In the special case where everyone can match with a single agent from the other side, such schemes are hindered when agents are unable to observe the reasons behind failed meetings. This lack of observation prevents the reinforcement of unstable outcomes.

On the other hand, the fact that firms can employ many workers opens up collusion opportunities for the firms. By collectively and credibly committing to avoid forming blocking pairs, firms can switch their stable partners and achieve more favorable outcomes. This behavior becomes even more pronounced when we consider the application of many-to-one matchings with colleges and students. In student placement problems, the primary objective is often to ensure fairness and make it undesirable for agents, particularly colleges, to engage in strategic manipulation that could hinder *fair* outcomes. Additionally, such manipulative behavior would be counterproductive, given that colleges typically serve as public goods, and their pursuit of self-interest through strategies undermines the collective welfare they are meant to provide.

However, the many-to-one search model suggests that, in equilibrium, colleges could potentially collude to secure better outcomes for themselves, but this would come at the expense of leaving students in a disadvantaged position. From a policy standpoint, these findings underscore the importance of having a central clearinghouse to counteract and prevent such behavior from emerging and safeguard the interests of all parties involved.

It is worth noting that the strategic manipulation room requires switching incentives for the firms. That is, for the firms to engage in such a strategic action, at least some firms should be willing to exchange some workers under the stable matching. Naturally, some alignment in preferences prevents such scenarios. One such preference structure is exemplified by the Sequential Preference Condition, which not only ensures the existence of a unique stable matching in the market but also prevents unstable matchings from being enforced in limit equilibria.

Last, but not the least, I analyze how the equilibrium outcomes of the decentralized many-to-one search model differentiate from a semi-dynamic and semi-decentralized notion of *dynamic stability*. In dynamic stability, agents are also able to consider the endgame that would be implemented through matchings that evolve through time via the remaining market. Different than the statically stable matchings being easily enforceable, this might not be the case for the dynamically stable matchings, because of the differences in meeting technology in both models.

The remainder of the paper is structured as follows: After discussing the related literature in the subsequent subsection, I introduce the model in Section 2. In Section 3, I show how stable matchings may not be easily enforceable in a many-to-one model and describe how the unique stable matching can be enforced as search equilibrium outcome when preferences are vertical. In Section 4, I discuss how unstable matchings can be enforced and continue the discussion about how the many-to-one model departs from its one-to-one counterpart when we allow firms to employ more than one worker. With the differences being already hinted at, I build the related one-to-one market and show a sufficient condition on preferences that prevents unstable matchings from being enforced in limit equilibria. In Section 6, I compare the decentralized many-to-one search model to dynamic stability. The detailed proofs, as well as how a matching matrix can be constructed from equilibrium strategies are to be found in the appendix.

Literature Review

In two-sided matchings, the agents belong to one of two disjoint sets and are to be matched to the agents on the other set. The theory is accepted to have started with the famous Gale and Shapley (1962) paper and has revealed a lot about the structure of two-sided matching markets so far. The existence of a stable matching was shown via construction, in which the deferred acceptance algorithm was introduced, and it was proven that the resulting matching of the algorithm is stable. Following this rather early start-off, the theory has deepened slowly but with sound steps through other famous works by Roth in the 1980s (such as Roth (1985)) and Roth and Sotomayor (1989). Finally, the basic results were collected in a book, *Two-sided matching*, by Roth and Sotomayor (1992). These works constitute the benchmark in most of the papers in the literature, as well as the current paper.

There is also wide literature on the connection between search and matching models, where the stability of outcomes that arise as search equilibria are analyzed. One of the earliest examples of such papers is Eeckhout (1999), where agents have vertical preferences over the other side of the market. The paper shows that as the Poisson meeting process becomes instantaneous, the search model equilibria are equal to the stable matchings. Later, Adachi (2003) considers general preferences with an exogenous distribution of agents, where the search is costly in terms of time. Similarly, as search costs disappear, search model equilibria converge to stable matchings in the underlying marriage market. Lauermann and Nöldeke (2014) contribute to this literature in the sense that they also allow for an endogenously evolving stock of agents. In their more general model, search model equilibria converge to stable outcomes if there is a unique stable matching in the underlying market.

However, current work is the most closely related to Wu (2015), in the sense that Wu considers a **finite** one-to-one search market. All the papers discussed before analyze a continuum of both sides of agents and focus on the steady-state. However, the very nature of the many-to-one problem, the steady-state analysis with a continuum of agents allows for meeting the same agent over and over again with time for the firms. Clearly, replicas of the same agent are not defined in the preferences of any underlying matching market. For this reason, a finite market as in Wu (2015) is a more realistic fit for the many-to-one markets. The equilibrium concept is then not a steady-state analysis but a subgame perfect equilibrium of the finite game. Furthermore, one would expect the paper by Wu to be easily extendable to a many-to-one model, which is the case for markets with a central planner when preferences are responsive. However, I show that this is not the case for search models and many-to-one search model described in this paper has essential differences from Wu's one-to-one model.

One other paper that considers a finite market is to be found in Niederle and Yariv (2009), which is also a one-to-one market with aligned preferences, and the search is directed in the sense that firms make directed offers to workers. The current analysis is more general in the sense that I consider general preferences and a many-to-one structure, and the results often do not depend on the number of stable matchings in the underlying market. In fact, even without aligned preferences, large markets tend to have a small number of stable matchings as in Immorlica and Mahdian (2003), Kojima and Pathak (2009), and Ashlagi, Kanoria, and Leshno (2017), which does not alter the main results of the paper.

Another closely related work would be the many-to-one dynamic stability paper by Altınok (2019), which is a generalization of one-to-one Doval (2022). In this working paper, Altınok implements the existing many-to-one matching into a centralized model with two periods. His work is closely related to this paper in the sense that he analyzes how this slightly dynamized many-to-one market relates to the static version as well as its one-to-one counterpart. However, this paper is different from Altınok's dynamic but still centralized setup in the sense that it does not put any constraints on the number of periods and the market is fully decentralized.

Search problems are widely studied not only on micro levels to understand the structure but also on macro levels to answer bigger questions and have a better understanding of the economies. Most famous examples of the search literature include Ken Burdett and Coles (1997), Kenneth Burdett and Coles (1999), Mortensen (1982), and Pissarides (1985) search models. These papers do not consider the preferences of the agents but rather focus on wage bargaining and unemployment dynamics.

Therefore, to the best of my knowledge, this is the first paper that considers a finite many-to-one search model in a dynamic fashion, and that is completely decentralized.

2 Model

2.1 Environment

The search environment is the same as a standard many-to-one matching market. There is a finite set of firms and a finite set of workers that are bilaterally and randomly searching for each other. The set of firms is denoted by $F = \{f_1, ..., f_n\}$ and the set of workers is $W = \{w_1, ...w_m\}$, where $(f, w) \in F \times W$ denotes a generic firm-worker pair.

Workers have complete and strict preferences over firms and remaining unemployed. The utility that worker w receives is denoted by $v(i, w) \in \mathbb{R}$, where $i \in F \cup \{w\}$. Workers are indifferent between the seats of firms and their coworkers. The many-to-one structure implies that firms can employ many workers, whereas a worker can be matched to one firm at most. The capacity vector $q = (q_{f_1}, ..., q_{f_n})$ specifies the maximum number of workers a firm can employ. Consequently, firms have complete and strict preferences over sets of workers, which is denoted by $u(f, \Omega) \in \mathbb{R}$, where $\Omega \subset W$. The utility of not being in a match is normalized for both parties: $u(f, \emptyset) = v(w, w) = 0$. A pair $(f, w) \in F \times W$ is an acceptable pair at u, v if w is *acceptable* for f , and f is acceptable for w, that is, $u(f, \{w\}) > 0$ and $v(f, w) > 0$[1].

Firms have preferences over sets of workers that actually are induced by an underlying preference relation over individual workers. In other words, if two sets differ in only one worker, the firm prefers the set with the more preferred worker. This condition is referred to as *q-responsive preferences* in the existing literature. Formally:

Definition 1. *The preferences of firms over 2^W are q-responsive if they satisfy the following conditions:*

1. *For all $\Omega \subset W$ such that $|\Omega| > q_f$, we have $u(f, \Omega) < 0$.*
2. *For all $\Omega \subset W$ such that $|\Omega| < q_f$ and $w \notin \Omega$, $u(f, \Omega \cup \{w\}) > u(f, \Omega)$ if and only if $u(f, \{w\}) > u(f, \emptyset) = 0$.*
3. *For all $\Omega \subset W$ such that $|\Omega| < q_f$ and $w, w' \notin \Omega$, $u(f, \Omega \cup \{w\}) > u(f, \Omega \cup \{w'\})$ if and only if $u(f, \{w\}) > u(f, \{w'\})$*

A many-to-one search market is represented by $M = (F, W, q, u, v)$ and all components are common knowledge to all agents. The summary of the assumptions on the market for a search game to start are the following:

1. Both parties have strict preferences.
2. The utility of being single is normalized to zero for both parties.
3. Firms have q-responsive preferences over sets of workers.
4. The market is finite: $|F| < \infty$, $|W| < \infty$, and $q_f < \infty \quad \forall f$.
5. The initial market is nontrivial in the sense that there are some acceptable pairs.

2.2 The Search Game

The game starts at $t = 0$ with the initial market M and continues for an indefinite amount of time. Each day, a random pair (f, w) meets randomly. I describe the meeting process in

[1]For notational convenience, $\Omega \succ_f \Omega'$ is often used instead of $u(f, \Omega) > u(f, \Omega')$, and same for $f \succ_w f'$ instead of $v(f, w) > v(f', w)$

detail in the following subsection 2.3. Upon meeting, w first decides whether to apply to f or not. Then, if w applies, f decides whether to hire w. If w does not apply or f rejects w, they separate and return to the market to keep searching. If w applies and f accepts, w leaves the market, and f loses one of its vacancies. If f has more vacancies, it stays in the market but leaves if q_f is full after hiring w.

Upon hiring, w receives a one-time payoff of $v(f, w)$. The firm also receives a one-time payoff. However, this depends on the already hired workers. Namely, if f has already hired $\Omega \subset W$ before meeting w, the one-time payoff it gets after hiring w is $u(f, \Omega \cup \{w\}) - u(f, \Omega)$, i.e. it enjoys the additional utility it derives from hiring w. The instant utility captures the immediate utility gain of extending the labor force[2]. The meeting and the decisions take place on the same day t and if the meeting concludes with hiring, the agents' utilities are discounted by δ^t. The common discount factor refers to the cost of time and is the first source of friction in the model.

Leaving the market upon mutual acceptance is permanent. That is, workers can not quit and the firms can not fire workers. As a result, the market weakly shrinks over the course of time. Any *submarket* or *remaining market* is then denoted by $M' = (F', W', q', u, v)$, where $F' \subset F$ denotes the remaining firms in the market, $W' \subset W$ the remaining workers, and $q' \subset Q$ the remaining capacities of the firms F'. The capacities of the firms will decrease over time, therefore $q' \leq q$ necessarily.

2.3 The Contact Function

The second source of friction in the decentralized many-to-one search game is the random meeting process that is defined by the contact function.

For any day, the contact function $C(f, w, M')$ defines the probability that the pair (f, w) will meet given that the remaining market at that day is M'. There are several but humble assumptions on the contact function such as:

1. The pair (f, w) meets with positive probability only if f has not filled its capacity and w is not matched to any firm: $C(f, w, M') = 0$ if $f \notin M'$ or $w \notin M'$.
2. A meeting occurs each day: $\sum_{(f,w) \in M'} C(f, w, M') = 1$.
3. Every meeting between the pairs in the remaining market is somewhat possible: $\exists \epsilon > 0$ s.t. $C(f, w, M') \geq \epsilon$ if $(f, w) \in M'$.

[2]This assumption will be revisited in Section 3.2

4. By definition, the contact function does not depend on the history itself. Whenever the remaining market is the same along different histories, the pair (f, w) has the same meeting probability.

The game ends if either one side of the market is fully matched or there are no mutually acceptable pairs remaining in the market, that is for any given submarket M', $\nexists(f, w)$ such that $u(f, \Omega \cup \{w\}) \geq u(f, \Omega)$ for $f \in M'$ holding Ω and $v(f, w) \geq 0$ for $w \in M'$ [3]. Workers search until they are matched to a firm or the game ends. Similarly, firms search until they fill their capacity or the game ends.

The search game is represented by the tuple $\Gamma = (F, W, q, u, v, C, \delta)$.

2.4 Equilibria

Every component of the search game defined until now is commonly known by every agent. Due to complete information, and the finite and dynamic structure of the many-to-one bilateral search game, the appropriate equilibrium concept for analysis is subgame perfect equilibrium. The set of all histories is denoted by $\mathcal{H}$, where $h \in \mathcal{H}$ and $\hat{\mathcal{H}}$ is the set of all non-terminal histories, after which the game continues with a new pair meeting. Additional to the initial search game Γ, any subgame that follows after history h is $\Gamma(h) = (F', W', q', u, v, C, \delta)$, and $M'(h)$ is the remaining market. A strategy profile is denoted by σ and $\sigma|\Gamma(h)$ is its restriction to the subgame $\Gamma(h)$. Furthermore, let μ_h denote the instantaneous matching at history h, i.e. all the meetings that have ended with employment so far.

Along with the regular subgame perfect equilibrium analysis, the paper considers some refinements to the information structure. To begin, the baseline scenario called *full-awareness*, assumes that agents possess the capability to observe and base their decisions on every aspect of the game's history. The *private-dinner condition* refers to environments where meetings between the agents take place in private settings. Agents can observe what meetings realize and their conclusions, but in case of separation, they do not observe the reason for the split, i.e. who rejected whom. However, when a meeting concludes with employment, it indirectly signals mutual acceptance to all agents.

Taking the refinement one step further, we reach the most stringent, and the most commonly used refinement in the literature is the *Markov condition*, under which agents can condition their behavior only on the current state of the market, but not on any other com-

[3] Due to q-responsive preferences, $u(f, \Omega \cup \{w\}) \geq u(f, \Omega)$ is equivalent to $u(f, w) \geq 0$

ponent of the history. The current state of the market consists of an instantaneous matching for every history, which also implies a remaining market[4]. This progression of information structure refinements enriches the analysis and sheds light on the intricate dynamics of the economic model under scrutiny.

Independent of the information structure, the game dictates the following for optimal pure strategy behavior upon meeting:

1. A worker w applies to f if the expected utility upon application exceeds the expected utility of continuing the search.
2. Any firm f accepts a w for one of its vacancies if the instant utility gain and the expected utility gain for the remaining vacancies exceeds the expected utility gain for continuing search without accepting w.

A strategy profile σ constitutes a subgame perfect equilibrium of the search game $(F, W, q, u, v, C, \delta)$ if σ is optimal for every agent in every subgame[5].

3 Connection to Centralized Many-to-One Markets à la Roth

3.1 Evolving Many-to-One Matching

At each history h, the instantaneous matching μ_h collects all the meetings that have ended with recruitment so far. In compliance with the centralized matching literature, and the rules of the search game ensure that any instantaneous matching is naturally a many-to-one matching and has the following properties:

1. $|\mu_h(w)| = 1$ and if $\mu_h(w) \neq w$, then $\mu_h(w) \subset F$.
2. $\mu_h(f) \in 2^W$ and $|\mu_h(f)| \leq q_f$.
3. $\mu_h(w) = f$ if and only if $w \in \mu_h(f)$.

Verbally, workers are matched to only one firm at the same time, and if they are not single, they are matched to a firm. Firms are matched to subsets of workers that do not

[4] The Markov condition in Wu (2015) considers conditioning on the remaining market only, which does not change any of the results presented in the paper.

[5] The conditions get stronger as we move from full awareness to Markov condition. Therefore, the set of equilibrium strategies weakly shrinks in the same direction.

exceed their capacity. Lastly, a worker is matched to a firm if and only if the worker is employed by that firm, so the matching process is bilateral.

Observe that any instantaneous matching μ_h implies a remaining market such that already matched agents and seats of the firms are removed from the initial market. Similarly, any contact function $C(f, w, M')$ together with a strategy profile σ induces a probability mass function on outcome matchings, details of which are to be found in the appendix. A matching μ *arises* if it appears as an outcome matching with positive probability under σ. If μ obtains almost surely under σ, I say that σ *enforces* μ. If σ enforces μ, the set of matchings that arise under σ may contain other elements than μ, all of which have zero probability of arising.

The prevailing equilibrium concept in centralized matching theory is stability, which assesses the sustainability of a matching. In essence, it ensures that agents lack incentives to disrupt the proposed matching, whether through individual or bilateral actions, thus making it both individually rational and unblocked. Formally, a matching μ is individually rational if all the matched pairs are acceptable. A matching μ is blocked by the pair (f, w) if they are not matched under μ but prefer each other over their matches under μ. That is, μ is blocked by the pair (f, w) if at least one of the following conditions hold:

1. If $|\mu(f)| \leq q_f$ and $\mu(w) \neq f$,
   $v(f, w) > v(\mu(w), w)$ and $u(f, w) > u(f, w')$ for some $w' \in \mu(f)$.
2. If $|\mu(f)| < q_f$ and $\mu(w) \neq f$,
   $v(f, w) > v(\mu(w), w)$ and $u(f, w) > 0$

If μ is individually rational and unblocked, it is a stable matching. Since the preferences of the firms are q-responsive, the set of stable matchings is non-empty for any initial market. Furthermore, by the famous Rural Hospital Theorem of Roth (1984), the set of matched agents is the same under every stable matching, and each firm that does not fill its quota has the same set of agents matched under every stable matching.

3.2 Enforcing Stable Matchings

In the first analysis section, the main question is the relation between the subgame perfect equilibrium outcomes of the search game and the stable outcomes of the underlying many-to-one market.

Firstly, small values of δ reflect high costs of time. If waiting is sufficiently costly, the workers would apply to acceptable firms and firms would accept every acceptable worker (partners that give positive utility). Therefore, the rather interesting question is for larger δ, in fact for $\delta \to 1$, that is called as *limit equilibria*.

Definition 2. *A strategy profile σ is a* ***limit equilibrium*** *of the many-to-one search environment (F, W, q, u, v, C) if there exists some $d < 1$ such that σ is an SPE of the many-to-one search game $(F, W, q, u, v, C, \delta)$ for all $\delta > d$.*

In the one-to-one component of the search model, Wu (2015) demonstrates that for any stable matching μ^* of the underlying market, there is a strategy profile σ^*, that satisfies the Markov condition, is a limit equilibrium, and enforces μ^*. In addition, he constructs the strategy profile for any initial market. As one might expect similarities with the many-to-one model (at least with responsive preferences), in the following, I show that such a Markovian strategy profile that enforces the stable matchings may not exist in the many-to-one search - even with additively separable utility over the workers[6]. The crucial distinction here lies in the fact that firms have a collective structure and stay on the market until they fill their capacities.

In this paper, I consider limit equilibria where waiting is almost costless and enforced matchings while assuming responsive preferences. Therefore, for all results presented in this paper, the assumption about how firms derive utility from expanding their employment set can be replaced with another, simpler one: that firms only care about the end-outcome of the search game[7]. This specific focus allows us to abstract from other strategic problems that can arise to prolong or expedite the game on the firm's side and target the differences caused by the cumulative employment only.

Enforcing the Worker-Optimal Stable Matching via Markov Equilibria

For any initial many-to-one market (with responsive preferences), there is a worker-optimal stable matching μ^W (WOSM), which corresponds to the outcome of the worker-proposing deferred acceptance algorithm. The following proposition shows that, for some initial markets, μ^W may not be enforceable in a limit equilibrium that satisfies the Markov condition.

Proposition 1. *There exist initial markets (F, W, q, u, v) for which there is no Markovian strategy profile that is a limit equilibrium and enforces the worker-optimal stable matching in the many-to-one search game $(F, W, q, u, v, C, \delta)$.*

[6] $u(f, \Omega \cup \{w\}) = u(f, \Omega) + u(f, \{w\}) \;\; \forall \Omega \subset W$ such that $w \notin \Omega$

[7] Details can be found in the Appendix for Proposition 1.

Proof. Suppose otherwise, i.e. there is a Markovian strategy profile σ^* that is a limit equilibrium and that enforces the worker-optimal stable matching in any many-to-one search game $(F, W, q, u, v.C, \delta)$. Now, consider the following example with $F = \{f_1, f_2, f_3\}$, $q = \{2, 1, 1\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$. In addition to the following preference profile, suppose $\{w_3, w_5\} \succ_{f_1} \{w_1, w_4\}$, which does not violate responsive preferences. Other than that, any q-responsive completion of firm preferences is admissible.

$f_1 : w_3 \succ w_4 \succ w_1 \succ w_2 \succ w_5$	$w_1 : f_1 \succ f_2$
$f_2 : w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5$	$w_2 : f_2 \succ f_3$
$f_3 : w_2 \succ w_3 \succ w_1 \succ w_4 \succ w_5$	$w_3 : f_3 \succ f_1$
	$w_4 : f_1 \succ f_2 \succ f_3$
	$w_5 : f_1 \succ f_2 \succ f_3$

In this example, the WOSM is $\mu^W = \{(f_1; w_1, w_4), (f_2; w_2), (f_3; w_3)\}$. First of all, observe that for μ^W to be enforceable, f_1 should reject w_5 upon meeting, because knowing that f_1 will accept, w_5 will apply to f_1 in any SPE. Second, since by assumption σ^* enforces the WOSM for any many-to-one search game, it enforces the worker-optimal stable matching in any submarket of the initial market. Third, in another many-to-one market where all agents and preferences are the same, but f_1 has a capacity of 1, $\mu^W = \{(f_1; w_3), (f_2; w_1), (f_3; w_2)\}$[8].

Now, consider the subgame where the meeting between (f_1, w_5) results in employment. By assumption, the worker-optimal stable matching is enforced in the remaining subgame by σ^* restricted to that subgame. In the remaining market, $q' = (1, 1, 1)$ and $\mu^W = \{(f_1; w_3), (f_2; w_1), (f_3; w_2)\}$. Since $\{w_3, w_5\} \succ_{f_1} \{w_1, w_4\}$, f_1 finds it plausible to employ w_5 upon application. Since every meeting between the pairs has a positive probability, the profitable one-shot deviation by f_1 to accept w_5 conflicts with σ^* enforcing the WOSM.

□

The preceding example illustrates how firms are willing to accept unstable partners initially to ultimately secure a more advantageous group of workers. This one-step deviation by f_1, involving filling one capacity with a less desirable candidate in the early stages of the game, aligns with the *capacity manipulation game* analyzed in Konishi and Ünver (2006). In their research, they demonstrate that firms have incentives to misrepresent their capacities

[8] f_1 rejecting w_1 in the first step of DA results in a rejection chain, resulting f_1 ending up with w_3.

as a means to enhance their overall welfare, under both the worker-proposing and firm-proposing deferred acceptance algorithm. In the aforementioned example, f_1 accepting w_5 would mimic capacity underreporting in subsequent rounds, which ultimately benefits the firm with a more favorable set of workers.

Enforcing the Firm-Optimal Stable Matching via Markov Equilibria

Given that it is a firm that can reasonably contemplate a one-step deviation by misreporting its capacities in the previous part, one might naturally contemplate that this impossibility result could be overcome if we consider the firm-optimal stable matching μ^F (FOSM). However, again as demonstrated by Konishi and Ünver (2006), similar incentives for misreporting capacities may persist even when employing the firm-proposing deferred acceptance algorithm—a scenario reflected in the many-to-one search game by the hiring of unstable workers in the initial rounds.

Below, I present the FOSM counterpart of the previous proposition, along with an illustrative example that highlights an initial market configuration for which no Markovian strategy profile can enforce the FOSM.

Proposition 2. *There exist initial markets (F, W, q, u, v) for which there is no Markovian strategy profile that is a limit equilibrium and enforces the firm-optimal stable matching in the many-to-one search game $(F, W, q, u, v, C, \delta)$.*

Proof. Suppose otherwise, i.e. there is a Markovian strategy profile that is a limit equilibrium and that enforces firm-optimal stable matching in any many-to-one search game $(F, W, q, u, v.C, \delta)$. Now, consider the following example with $F = \{f_1, f_2\}$, $q = \{3, 3\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. In addition to the following preference profile, suppose $\{w_2, w_4, w_6\} \succ_{f_1} \{w_4, w_5\}$ and any q-responsive completion of firm preferences.

$f_1 : w_1 \succ w_2 \succ w_4 \succ w_3 \succ w_5 \succ \emptyset \succ w_6$

$f_2 : w_4 \succ w_5 \succ w_1 \succ w_3 \succ w_2 \succ \emptyset \succ w_6$

$w_1 : f_2 \succ f_1$

$w_2 : f_2 \succ f_1$

$w_3 : f_2 \succ f_1$

$w_4 : f_1 \succ f_2$

$w_5 : f_1 \succ f_2$

$w_6 : f_1 \succ f_2$

In this example, $\mu^F = \{(f_1; w_4, w_5), (f_2; w_1, w_2, w_3)\}$. Similar to above, by employing w_6 in early periods (who is even an unacceptable worker for the firm), f_1 can mimic capacity underreporting. In the following subgame, the firm-optimal stable matching is $\mu^F = \{(f_1; w_2, w_4), (f_2; w_1, w_3, w_5)\}$ and f_1 ensures a more favorable outcome overall. $\square$

A Remedy to Enforce Stable Matchings: Vertical Preferences

Both propositions above show how firms can profitably deviate from a strategy profile that enforces a stable matching. Even though one might expect different incentives under the worker-optimal or firm-optimal matching, they share a common characteristic of involving rejection chains, which firms initiate by underreporting their capacities. This cascading effect ultimately yields a benefit for the firm engaging in capacity misreporting.

It's well-established in the literature that such profitable chains do not occur when one side of the market possesses identical preferences for the other side — in other words when preferences are vertical on at least one side of the market. In this scenario, a unique stable matching μ^* emerges as the outcome of both worker-proposing and firm-proposing deferred acceptance algorithms.

In this part, given a matching μ for the market (F, W, q, u, v) let $\mathcal{M}_\mu$ denote the set of all markets (F', W', q', u, v) such that $\cup_{F \setminus F'} \mu(f) = W \setminus W'$ and $q'_f = q_f - |\mu(f)| \quad \forall f \in F$. In other words, $\mathcal{M}_\mu$ denotes all the markets that remain after some pairs of μ are matched.

In the following, I construct the Markovian strategy profile that is a limit equilibrium and enforces the unique stable matching μ^* for any initial many-to-one market with vertical preferences. On the equilibrium path, all the remaining markets will be elements of $\mathcal{M}_{\mu^*}$, leading to μ^* eventually. Observe that if $M \in \mathcal{M}_{\mu^*}$, μ^* restricted to M is also the unique stable matching for M.

Definition 3. *For any given market (F, W, q, u, v) with vertical preferences on at least one side (i.e. either firms share the same preferences up to their capacity, or workers have the same preferences over firms.), σ_{μ^*} denotes the strategy profile in which the agents behave in the following way upon meeting:*

1. *Workers apply only to firms such that $v(f, w) \geq v(\mu^*(w), w)$ in any submarket.*
2. *Firms accept workers such that $u(f, w) \geq \min(u(f, w'))$ for $w' \in \mu^*(f)$, reject others.*

In other words, σ_{μ^*} prescribes that workers apply to firms that they weakly prefer to their allocation under μ^*, and firms accept workers whom they prefer to their least favorite worker under μ^*.

Theorem 1. *For any given market (F, W, q, u, v) with vertical preferences on at least one side, σ_{μ^*} is a limit equilibrium for any search game $(F, W, q, u, v, C, \delta)$ and it enforces the unique stable matching μ^*.*

Proof. The idea of the proof is first to show σ^* being a limit equilibrium and then to show it indeed enforces μ^*. The second step follows from the fact that a pair accepts each other if and only if they are stable partners. For the first step, I show that any one-step deviation yields a worse payoff for the agents. First of all, since the strategy profile is Markovian, any deviation to not accepting a partner does not change the remaining market, hence we are still on the equilibrium path. The only profitable deviation could arise in accepting a partner that is not accepted under σ_{μ^*}, which was the case for the propositions above. However, since preferences are vertical on the one side, such rejection cycles do not occur by misreporting capacities, and the outcome of the search game only changes by one worker. Since preferences are responsive, this difference is reflected in the overall preference, which concludes that no profitable deviation is possible. The detailed proof can be found in the appendix. $\square$

Intuitively, if agents are patient enough in a many-to-one market with vertical preferences, it is possible to obtain no other outcome but stable matching as the equilibrium outcome. Even though the equilibrium strategy profile depends on the stable matching of the underlying market, the strategy profile is feasible to sustain with complete information. [9]

4 Enforcing Unstable Matchings

Knowing that the stable matchings of the underlying many-to-one market can be sustained as search equilibria, the natural follow-up question would be about the enforceability of unstable matchings. As Wu (2015) shows, unstable matchings may also be enforced in full-awareness equilibria and may arise with positive probability as outcome matchings. Those possibility results are simply applicable to our framework since every one-to-one matching is essentially a many-to-one matching, with the specification that $q_f = 1 \quad \forall f$.

[9] Furthermore, if we instead consider a many-to-many environment where workers can also work for multiple firms at the same time and have responsive preferences over firms (and one side has vertical preferences), replacing their strategy profile with "Workers apply only to firms such that $v(f, w) \geq \min(f', w)$ such that $f' \in \mu(w)$ in any submarket", the strategy profile would yield the stable many-to-many matching μ^*.

On the other hand, Wu (2015) also shows that, the only way to enforce unstable matchings under limit equilibria is by "Reward and Punishment Schemes", akin to those in Rubinstein and Wolinsky (1990), which is the main discussion object of this section. In Wu's one-to-one model, the reward and punishment scheme works as follows for any blocking pair (f, w): On the equilibrium path, the w is punished for initiating the block by applying, and f is credibly rewarded for not obliging. Only that way, the blocking pair does not realize and an unstable matching can be enforced as an equilibrium outcome. In the following, I show that in a many-to-one search model, there exists an increased potential for implementing such schemes by utilizing the remaining capacities of firms.

Crucially, in a one-to-one world, implementing such schemes requires knowledge about who rejected whom in past meetings. Consequently, when agents cannot condition their behavior on who rejected whom in past failed events, such schemes are not implementable. That information restriction is in line with what was defined as "private-dinner equilibria", where meetings take place in private environments and others can only observe (and condition their behavior on) the outcome of meetings. Under the private-dinner condition, the successful meetings indirectly convey the information of mutual acceptance, however in case of an unsuccessful meeting, it remains unknown to the agents who rejected whom. Since this restriction deactivates the reward and punishment schemes, no unstable matching can be enforced in a private-dinner equilibrium.

The following result shows that this result is not robust to transitioning to a many-to-one model, i.e. allowing for $\exists f$ to have $q_f > 1$. The introduction of multiple capacities creates opportunities for firms to employ alternative forms of strategic manipulation, and ensure more favorable outcomes for themselves. Inevitably, this has adverse implications for workers, since responsive preferences preserve the lattice structure of the matchings Roth (1985). This aspect will become even more apparent when examining the example presented following the proposition.

Proposition 3. *In a many-to-one finite decentralized search model, unstable matchings can be enforced by limit equilibria even when information is restricted to the remaining market.*

Proof. I prove this proposition by an example, which will also help illustrate the intuition behind the essential difference between many-to-one and one-to-one models.

Example 1. *Suppose there are two firms $F = \{f_1, f_2\}$ such that $q_1 = 3$ and $q_2 = 2$ and four workers $W = \{w_1, w_2, w_3, w_4\}$ and the ordinal preference relation derived from the preferences of the agents is as follows*[10]*:*

[10] Any completion for the firm side as long as responsiveness is ensured

$$w_1 : f_1 \succ f_2 \qquad\qquad f_1 : w_4 \succ w_1 \succ w_2 \succ w_3$$

$$w_2 : f_1 \succ f_2 \qquad\qquad f_2 : w_1 \succ w_2 \succ w_3 \succ w_4$$

$$w_3 : f_1 \succ f_2$$

$$w_4 : f_2 \succ f_1$$

Now, consider the following strategy profile σ^U such that, at every history, w_1, w_2 only apply to f_1 as long as it is in the market (after f_1 leaves, apply to f_2), w_3 and w_4 only apply to their top choices (f_1, f_2 respectively) if they are ranked within the remaining capacity in the remaining market for the firms or apply to both firms upon meeting. On the firm side, f_1 accepts the top q_1' of the remaining workers and f_2 waits until f_1 leaves the market (does not hire anybody until f_1 fulfills its capacity), then only accepts the top q_2' of the remaining workers.

The strategy-profile σ^U is limit equilibrium and depends only on the remaining market for every history, which is even more restricted than the private-dinner condition. Additionally, σ^U enforces $\mu^U = \{(f_1; w_1, w_2, w_4), (f_2; w_3)\}$ which is unstable due to the blocking pair (f_2, w_4).

$\square$

It is easy to show that σ^U enforces μ^U since only pairs that end with pairing with positive probability are as prescripted in μ^U. Moreover, under σ^U, f_1 achieves its own favorite employment set (due to responsive preferences). Similarly, w_1 and w_2 are employed by their favorite firm. Therefore, those agents have no incentive to deviate from σ^U at any point in the game. Similarly, w_3 not applying to f_2 or f_2 not accepting does not change the remaining market and, therefore is no candidate for a profitable one-step deviation. Let's take the only candidate pair for such a deviation, that is, the blocking pair $(f_2; w_4)$, and suppose they meet on the equilibrium path. The worker w_4 is already applying to f_2 and not applying will not be a profitable deviation either.

It is particularly noteworthy how f_2 rejects w_4 upon his application. Since having a successful meeting with w_1 and w_2 is not possible, f_2 would like to hire both w_3 and w_4. However, if f_2 deviates to accepting w_4, in any subgame following that history, w_3 will apply only to f_1 and f_1 will hire him in return. Therefore, deviating to accepting w_4 before f_1 fulfills its capacity makes f_2 lose w_3 in return, which is less favorable.

Intuitively, f_2 accepting w_4 starts a rejection cycle similar to Kojima and Pathak (2009). With f_1 starting accepting w_3, w_3 starts rejecting f_2. Both firms prefer μ^U over μ^* and they can credibly switch w_3 and w_4 in the limit equilibrium, by using their remaining capacities as a commitment device. The remaining capacities enable the reward and punishment schemes to be implemented even though the information is restricted to the remaining market.

This difference between one-to-one and many-to-one search models relies on the difference in the underlying static markets. In one-to-one markets, there is no individually rational matching which is preferred to the firm-optimal stable matching by all firms. In the many-to-one case, the enforced matching μ^U is preferred to μ^* by both firms and the firms can implement that as long as they can credibly signal each other that blocking pairs will not realize. In return, workers w_3 and w_4 are worse off from being switched by the firms.

Last, but not the least, observe that there is a unique stable matching in the underlying many-to-one market, that is $\mu^* = \{(f_1; w_1, w_2, w_3), (f_2; w_4)\}$. As previously elucidated in the literature review, the uniqueness of the centralized stable outcome usually averts unstable limit equilibrium outcomes, yet, in this particular instance, such prevention does not apply.

To summarize the findings above, in a many-to-one search model, firms possess the ability to collude strategically, allowing them to implement outcomes that are more favorable to themselves than the stable matching, even though it may be unique. Unfortunately, such strategic behavior comes at the detriment of workers, who end up worse off as a result. From a policy proposal perspective, the normative question then arises as to whether providing firms with this room for strategic maneuvering is desirable or not. This concern is particularly evident in a setting with colleges and students, where colleges are viewed as public goods, and it becomes imperative to prevent them from developing strategies that benefit themselves at the expense of rendering students worse off. The same applies to the school choice and residency matching problems.

Comparing one-to-one and many-to-one models, we see that enforcing stable matchings with equilibrium strategies is harder in a many-to-one search model, whereas unstable matchings are more easily enforceable. These observations emphasize that the role of a central planner becomes more essential when it comes to many-to-one matching. The central planner's intervention might be crucial as she can impose a stable allocation that not only ensures fairness but also eliminates any blocking pairs that could lead to undesirable outcomes. By doing so, the central planner might help maintain the integrity of the matching process and protect the interests of all parties involved, promoting a more equitable allocation in many-to-one markets.

5 Connection to Decentralized One-to-One à la Wu

The fact that stable matchings are harder to enforce in many-to-one matchings as well as unstable matchings are more easily enforceable is particularly striking when we consider the centralized matching literature. We know that when preferences are responsive, many-to-one and one-to-one matching markets can be seamlessly mapped onto each other as demonstrated by Roth and Sotomayor (1992).

The mapping is done via a "related marriage market", which is obtained by replicating each firm by their capacities and treating this new replica market as a one-to-one market. When preferences of firms are responsive in the original market, we know that a many-to-one matching is stable if and only if the corresponding one-to-one matching in the related marriage market is stable. This compelling finding significantly simplifies the analysis of centralized many-to-one matchings.

Propositions 1, 2, and 3 already hint at the fact that the many-to-one search model differs from its one-to-one counterpart fundamentally. Furthermore, this difference cannot be eliminated by imposing strict regularity conditions on within-firm preferences (such as additively separable utility). Instead, the fact that the end matching evolves through time allows for strategic behavior among firms based on their remaining capacities.

To gain more concrete insights into the distinctive search behavior in a many-to-one market, it becomes imperative to delineate a corresponding one-to-one search model for comparative analysis. By examining the differences between the many-to-one and one-to-one search models, we can develop a comprehensive understanding of the unique characteristics and dynamics inherent in each setting. This analytical approach will shed light on the complexities of the many-to-one search models, paving the way for valuable insights into how search behavior evolves in such contexts, which is the objective of this section. To streamline the discussion and prevent redundancy, any aspects left unspecified in this context can be assumed to remain analogous to the original many-to-one search model.

5.1 The Related One-to-One Search Game

The *related* one-to-one search model will be a translation of our many-to-one search model onto a one-to-one environment. The related one-to-one market is obtained where each firm is replicated as many times as its capacity. In other words, each seat of the firms is individually present in the search market. Furthermore, the seats are individually searching for workers and hence become competitors.

When we replicate firms, we obtain q_f identical seats. Namely, the set of the seats in the related one-to-one market is $S = \{s_{11}, ..., s_{1q_1}, ..., s_{n1}, ..., s_{nq_n}\}$, where s_{ij} is the jth seat of firm i. Recall that the preferences of firms in the original many-to-one market are complete and responsive. This means that we can deduct the firms' preferences over individuals. Each seat has the same preference over individuals as the firm. In terms of the environment, the replication is exactly the same as Roth and Sotomayor (1989). The search model requires an additional adjustment with the contact function.

On the workers' side, there is no replication. Each worker from $W = \{w_1, ...w_m\}$ is still searching for himself. Workers are indifferent between the seats of the same firm and each worker i prefers a seat in firm j over a seat in firm k if and only if he prefers firm j over firm k in the many-to-one market. Formally, $v(s_{jn}, w_i) < v(s_{km}, w_i)$ whenever $v(f_j, w_i) < v(f_k, w_i)$. For simplicity, from now on, I break the indifferences in workers' preferences such that they prefer the seat with a smaller index within the same firm: $v(s_{j1}, w_i) > ... > v(s_{jq_j}, w_i)$[11]. A related one-to-one market is then the tuple $M_R = (S, W, u, v)$. Similarly, any submarket (remaining market) is $M'_R = (S', W', u, v)$ with $S' \subset S$ and $W' \subset W$ [12].

The contact function is mildly adjusted such that for each submarket, the sum over the probabilities of w and s meeting in M'_R and s is replicated from f of M' is equal to $C(f, w, M')$ in the original market. All other assumptions on the contact function remain the same, most importantly, all pairs of seats and workers have a positive probability of meeting in line with Wu (2015).

The related one-to-one game is denoted by the tuple $\Gamma_R = (S, W, u, v, C, \delta)$.

Recall that Wu (2015) shows that no unstable matching can be enforced in a private-dinner equilibrium, whereas Proposition 3 shows otherwise, even under a more restrictive information criterion. Therefore, the equilibria of one-to-one and many-to-one search models are generically not equivalent. In the following, I will provide a sufficient condition for the underlying many-to-one market, that will not just restore the projectability between many-to-one and one-to-one models, but also ensure that no unstable matching will be enforced in a private-dinner equilibrium.

Definition 4. *The pair (s, w) is called a top-pair for any related (sub)market if: Among the seats that find w acceptable, s is the best for w, and among the workers that find s acceptable, w is the best for s.*

Following Wu, a related marriage market satisfies the Sequential Preference Condition

[11] This tie breaking rule reduces the number of stable matchings in the related one-to-one market, yet does not affect the analysis, see Appendix.

[12] Any instantaneous matching μ translates onto the one-to-one replica such that $\mu_R(w) = s_{in} \Rightarrow \mu(w) = f_i$.

(SPC) if there is an ordering of the seats and workers and a positive integer k such that:

1. *For any $i \leq k$, (s_i, w_i) is a top-pair in the (sub)market.*
2. *Discarding the top-pairs results in a trivial market with non-acceptable pairs.*

In a one-to-one market, the top-pairs always mutually accept each other upon meeting. This is almost trivial since the top-pairs can not possibly hope for a better alternative to be matched in future periods. Once the top-pair leaves the market, the same applies to another because the market satisfies SPC. Since every pair has a positive probability of meeting, there is an equilibrium path that matches the top-pairs which happens with positive probability, which ensures no unstable matching can be enforced in a limit equilibrium. In the following, I adjust this condition to many-to-one markets.

Definition 5. *A many-to-one market satisfies the Sequential Preference Condition if the following two conditions hold:*

1. *The related one-to-one market satisfies SPC.*
2. *Firm preferences are lexicographic for top pairs: If (s_{ij}, w_i) is a top pair in the remaining related market, $u(f_i, \Omega_i) > u(f_i, \Omega_{-i})$ for all (Ω_i, Ω_{-i}) such that $w_i \in \Omega_i$ but $w_i \notin \Omega_{-i}$.*

Proposition 4. *If the initial many-to-one market satisfies the Sequential Preference Condition, no unstable matching can be enforced in a limit equilibrium.*

Proof. In a market where preferences satisfy SPC, there is a unique matching that allocates top pairs to each other. In the original many-to-one search game, the top pairs would always accept each other (which, on the firm side is ensured via lexicographic preferences). Since every pair has a positive probability of meeting, there is an equilibrium path in which top pairs meet each other in the SPC ordering, and this happens with positive probability. Therefore, enforced matching cannot be unstable when the preferences of the initial market satisfy SPC. □

Corollary 1. *If the initial many-to-one market satisfies the Sequential Preference Condition, the only enforceable matching of the many-to-one market in a limit equilibrium corresponds to the only enforceable matching of the related one-to-one market in a limit equilibrium.*

Even though we need a modification for the Sequential Preference Condition for many-to-one markets, the result and the proof method apply from Wu (2015). The unique stable set of workers will apply to the firm, and the firm will accept those. Thus they cannot credibly threaten each other with rejection, which then results in employment upon first meeting. Furthermore, no unstable matching can be enforced in the related market and a many-to-one matching is stable if and only if the related one-to-one matching is stable in the related market. Consequently, the equilibrium outcomes of both models are equivalent when the contact function realizes according to the SPC ordering, as well as the remaining markets are related. Thus, SPC rebuilds the connection from the many-to-one search model to the one-to-one static market in terms of enforced matchings.

Note that by preventing unstable matchings from being enforced, SPC provides a solution to the discussion at the end of Section 4 about the potential necessity of a central planner for more general preferences to prevent firms from manipulating the search outcome. However, unstable matchings can still arise with positive probability even under SPC, as shown in Wu (2015).

6 Connection to Dynamic Stability à la Doval

On the way from centralized matching with clearinghouses that impose an allocation on an economy to a decentralized search model, where agents randomly meet each other and decide whether to accept and leave the market or wait for better partners, one natural stepping stone to consider would be the "dynamic stability" concept, introduced by Doval (2022) for one-to-one environments and then later incorporated into many-to-one by Altınok (2019). In this section, I compare the search outcomes to dynamically stable matchings. Since we are also in a many-to-one environment, the definitions and examples below will be based on Altınok (2019).

Intuitively, the dynamic stability concept also incorporates the time component, or at least the sequential acceptance of agents even though waiting is almost costless. Formally, what differs from the current model is that all employers are around in all periods, whereas candidates arrive over time and W_t denotes the workers that arrive at period t, and the agents form matches over exogenously given T periods, $t = 1, 2, ..., T$.

For the rest of this section, I am restricting attention to $T = 2$, that is, workers arrive over two periods. Then, a history is the empty set for the first period, and the 1st-period matching in period 2, and the strategies map histories into the matchings of the same period.

Definition 6. *A t-period matching μ_t is a mapping from the set of candidates that have arrived until t to the set of employers; that is, for each t,*

$$\mu_t : \bigcup_{\tau=1}^{t} W_t \to F \cup \{\emptyset\}$$

and satisfies the following properties:

1. *Capacity constraints are always respected:* $|\mu_t^{-1}(f)| \leq q_f$ *for each* f *for* $t = 1, 2$.
2. *First-period matchings are irreversible:* $\mu_2(w) = \mu_1(w)$ *if* $\mu_1(w) \neq \emptyset$,

Similarly, M_t is the set of $t-$period matchings for $t = 1, 2$, $h_t := (h_\tau)_1^t$ a history of matchings at t where H_t denotes all possible period-t histories, where $H_1 = \emptyset$ and $H_2 = M_1$. A strategy profile is then (s_1, s_2), where $s_t : H_t \to M_t$ for $t = 1, 2$. The second period is identical to a static market and a first-period block refers to a blocking coalition that exists in the first period and that can implement a better second-period matching for them by forming a coalition in the first period.

Consider the following example by Altınok to illustrate the difference between the static stability vs the dynamic one:

Example 2. *Suppose there are two firms* $F = \{f_1, f_2\}$ *such that* $q_1 = 3, q_2 = 2$ *and six workers* $W = \{w_1, ..., w_6\}$ *and the preferences are as below:*

$w_1 : f_1 \succ f_2$

$w_2 : f_1 \succ f_2$

$w_3 : f_1 \succ f_2$

$w_4 : f_1 \succ f_2$

$w_5 : f_1 \succ f_2$

$w_6 : f_2 \succ f_1$

$f_1 : w_6 \succ w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5$

$f_2 : w_1 \succ w_2 \succ w_3 \succ w_6$

In addition, suppose $\{w_4, w_5, w_6\} \succ \{w_1, w_2, w_3\}$ *for* f_1*, which is still in line with responsiveness but depicts that* f_1 *has extreme preferences, in the sense that it prefers combining extreme workers rather than the average ones.*

In this market, the unique stable matching is $\mu^* = \{(f_1; w_1, w_2, w_3), (f_2; w_6), (\emptyset, w_4), (\emptyset, w_5)\}$, which in fact is not dynamically stable in this particular market: Suppose instead of everybody being in the market at once, workers arrive in 2 periods, such that workers w_4 and w_5 arrive in period 1. Then, f_1 would form a period-1 matching with w_4 and w_5 (1st period block), enters the second-period with $q_1' = 1$. The unique stable μ in period 2:

f_1 matches with w_6, f_2 is matched with $\{w_1, w_2\}$, so the dynamically stable matching is $\mu^* \neq \mu^D = \{(f_1; w_4, w_5, w_6), (f_2; w_1, w_2), (\emptyset, w_3)\}$

Both firms prefer the outcome μ^D to μ^*, which is not stable because of the blocking pair (f_1, s_1). What happens with dynamic stability is that the firms in a sense exchange w_6 with w_1, w_2. To make this exchange credible, f_1 fills its capacity with w_4, w_5 in the first period.

The straightforward intuition places dynamic stability between centralized and completely decentralized search models, with the number of periods given and the central planner still imposing the matching on the economy somehow. Furthermore, the dynamic stability concept and the decentralized search model share a similar flavor in the strategic commitment and manipulation room they provide to firms with multiple capacities. Nevertheless, the following proposition will show that (contrary to the connection to completely centralized markets) dynamically stable matchings may not be enforced as search equilibria.

Proposition 5. *Dynamically stable matchings may not be enforceable by limit equilibria.*

Proof. Take the example by Altınok (2019) one more time, and suppose the dynamically stable $\mu^D = \{(f_1; w_4, w_5, w_6), (f_2; w_1, w_2), (\emptyset, w_3)\}$ is enforced in a limit equilibrium. Since every meeting has a positive probability in the decentralized search model, suppose f_1 and w_6 meet the first day. If w_6 applies to f_1: If f_1 accepts, firms have the same preferences over remaining workers, f_1 employs $\{w_1, w_2, w_6\}$. The best f_1 can hope for: f_1 accepts w_6. Therefore, w_6 applies to f_1 even with small waiting costs upon meeting. Since every pair has a positive probability of meeting in the initial market, $\mu^D = \{(f_1; w_4, w_5, w_6), (f_2; w_1, w_2), (\emptyset, w_3)\}$ cannot be enforced by limit equilibria. □

In fact, as the proof of this proposition already suggests, the dynamic stability concept depends heavily on which workers arrive in which period, which resembles the meeting probability being exactly 0 and exactly 1 for some of the candidates. The concept is not robust to more general probabilities such as that are defined by the contact function, as well as the workers becoming strategic agents as well.

7 Conclusion

In this paper, I describe and analyze a finite decentralized many-to-one bilateral search model. Considering the components of the model, a finite search model is described à la Wu (2015) and the outcome of subgame perfect equilibria are compared to the stable matchings of the underlying many-to-one market. The existence of an equilibrium is guaranteed by the finite structure of the game, as well as the existence of a stable matching in the underlying market. Nevertheless, different than the one-to-one finite search model, the stable matchings of the underlying market may not be enforceable by simple strategies. On the other hand, when one side of the market has vertical preferences over the other side, there is a Markovian strategy profile that enforces the unique stable matching.

Different than the centralized matching models, the many-to-one search model presents another fundamental distinction in its projection onto one-to-one environments. In one-to-one settings, enforcing unstable matchings necessitates the application of reward and punishment schemes, which is not feasible once agents are unable to observe the details of failed meetings. On the contrary, in the many-to-one search model, firms hold the capacity to implement reward and punishment strategies through their remaining capacities, enabling them to commit to credible strategies that yield a more advantageous matching outcome for themselves compared to the stable matching. This capability stems solely from the fact that firms possess multiple capacities, making it impractical to attempt resolving the manipulation incentives on the firm side by imposing stringent assumptions on within-firm preferences, such as additively separable utility.

However, when preferences of all firms satisfy the sequential preference condition, that is, there are top pairs of firms and workers that mutually prefer each other over all other available options, then, there is a unique stable matching which is the only enforceable matching in a limit equilibrium. Furthermore, the related matching in the related one-to-one market is the only enforceable outcome in any limit equilibrium in a one-to-one search model à la Wu (2015), and the equilibria, as well as the instantaneous matchings of many-to-one and one-to-one models, are equivalent.

Different than the positive result that shows the enforceability of stable outcomes, dynamically stable matchings may not be enforceable in limit equilibria. Indeed, the concept heavily hinges on the specific sequence of worker arrivals over time and lacks robustness when confronted with more general probabilities.

The paper shows how many-to-one matching markets differ from their one-to-one counterparts when considered in a search model, even though firm preferences are responsive and agents are sufficiently patient. In general, the presence of multiple capacities within this framework introduces opportunities for strategic manipulation by firms, allowing them to deviate towards more favorable outcomes. Nonetheless, it is worth noting that the characterization of matchings that can be enforced through limit equilibria remains a topic that necessitates further exploration and investigation in future research.

References

Adachi, Hiroyuki (2003). “A search model of two-sided matching under nontransferable utility”. In: *Journal of Economic Theory* 113(2), pp. 182–198.

Altınok, Ahmet (2019). *Dynamic many-to-one matching*. Tech. rep. Technical Report, Mimeo.

Ashlagi, Itai, Yash Kanoria, and Jacob D Leshno (2017). “Unbalanced random matching markets: The stark effect of competition”. In: *Journal of Political Economy* 125(1), pp. 69–98.

Baïou, Mourad and Michel Balinski (2000). “The stable admissions polytope”. In: *Mathematical programming* 87(3), pp. 427–439.

Burdett, Ken and Melvyn G Coles (1997). “Marriage and class”. In: *The Quarterly Journal of Economics* 112(1), pp. 141–168.

Burdett, Kenneth and Melvyn G Coles (1999). “Long-term partnership formation: marriage and employment”. In: *The Economic Journal* 109(456), F307–F334.

Doval, Laura (2022). “Dynamically stable matching”. In: *Theoretical Economics* 17(2), pp. 687–724.

Eeckhout, Jan (1999). “Bilateral search and vertical heterogeneity”. In: *International Economic Review* 40(4), pp. 869–887.

Gale, David and Lloyd S Shapley (1962). “College admissions and the stability of marriage”. In: *The American Mathematical Monthly* 69(1), pp. 9–15.

Immorlica, Nicole and Mohammad Mahdian (2003). “Marriage, honesty, and stability”. In.

Kesten, Onur and M Utku Ünver (2015). “A theory of school-choice lotteries”. In: *Theoretical Economics* 10(2), pp. 543–595.

Kojima, Fuhito and Mihai Manea (2010). “Incentives in the probabilistic serial mechanism”. In: *Journal of Economic Theory* 145(1), pp. 106–123.

Kojima, Fuhito and Parag A Pathak (2009). “Incentives and stability in large two-sided matching markets”. In: *American Economic Review* 99(3), pp. 608–627.

Konishi, Hideo and M Utku Ünver (2006). “Games of capacity manipulation in hospital-intern markets”. In: *Social choice and Welfare* 27(1), pp. 3–24.

Lauermann, Stephan and Georg Nöldeke (2014). “Stable marriages and search frictions”. In: *Journal of Economic Theory* 151, pp. 163–195.

Mortensen, Dale T (1982). *The matching process as a non-cooperative bargaining game, The Economics of Information and Uncertainty.*

Neme, Pablo A and Jorge Oviedo (2020). “A characterization of strongly stable fractional matchings”. In: *TOP* 28(1), pp. 97–122.

Niederle, Muriel and Leeat Yariv (2009). *Decentralized matching with aligned preferences.* Tech. rep. National Bureau of Economic Research.

Pissarides, Christopher A (1985). "Short-run equilibrium dynamics of unemployment, vacancies, and real wages". In: *The American Economic Review* 75(4), pp. 676–690.

Roth, Alvin E (1984). "The evolution of the labor market for medical interns and residents: a case study in game theory". In: *Journal of political Economy* 92(6), pp. 991–1016.

Roth, Alvin E (1985). "The college admissions problem is not equivalent to the marriage problem". In: *Journal of economic Theory* 36(2), pp. 277–288.

Roth, Alvin E and Marilda Sotomayor (1989). "The college admissions problem revisited". In: *Econometrica: Journal of the Econometric Society*, pp. 559–570.

Roth, Alvin E and Marilda Sotomayor (1992). "Two-sided matching". In: *Handbook of game theory with economic applications* 1, pp. 485–541.

Rothblum, Uriel G (1992). "Characterization of stable matchings as extreme points of a polytope". In: *Mathematical Programming* 54(1), pp. 57–67.

Rubinstein, Ariel and Asher Wolinsky (1990). "Decentralized trading, strategic behaviour and the Walrasian outcome". In: *The Review of Economic Studies* 57(1), pp. 63–78.

Vate, John H Vande (1989). "Linear programming brings marital bliss". In: *Operations Research Letters* 8(3), pp. 147–153.

Wu, Qinggong (2015). "A finite decentralized marriage market with bilateral search". In: *Journal of Economic Theory* 160, pp. 216–242.

Theorem 1:

Proof.

- σ^* enforces the stable μ^*: μ^* obtains almost surely on the equilibrium path. For any meeting function realization, another outcome than μ^* arising from σ^* has a probability of 0. Suppose f and w are not matched under μ^* but they end up together under σ^* for some meeting function realization. Mutual acceptance requires $u(f, w) \geq \min(u(f, w'))$ such that $w' \in \mu(f)$ and $v(f, w) \geq v(\mu^*(w), w)$ which contradicts with μ^* being stable under responsive preferences. This part concludes all pairs in the outcome are consistent with μ^*.

  Also, note that all existing agents meet with some positive probability. Since the pairs under μ^* accept each other, and the probability of a rejecting pair (or a rejection pair cycle) occurring has a probability of 0, μ^* obtains almost surely on the equilibrium path.

- σ^* constitutes a limit equilibrium of any search game. For any firm f, the expected utility of σ^* is $u(f, \mu^*(f))$ and for any worker w the expected utility of σ^* is $v(\mu^*(w), w)$. Note that a one-step deviation to reject a partner that is accepted under μ^* does not change the submarket, therefore we are still on-the-equilibrium-path. The only deviation by an agent that would yield a switch to off-path is accepting a partner from the other side who would not be accepted μ^*.

  Now consider a one-step deviation by w such that w accepts f instead of rejecting as under σ^*. Rejection under σ^* implies $v(f, w) < v(\mu^*(w), w)$. If f rejects w, the subgame does not change and the expected utility of w does not change. If f accepts, w receives a lower utility than $v(\mu^*(w), w)$. Therefore, it is not beneficial for w to accept f.

  A similar logic applies to f and a one-step deviation towards rejecting, even though f has multiple capacities. We know that when one side has vertical preferences, the deferred-acceptance outcome is the same as the serial dictatorship outcome. Suppose firms have the same preferences over individual workers, i.e. we are in a college admissions model. Then, there is a unique stable matching mu^* that can be achieved both by firm-proposing DA and the worker-proposing DA, as well as the serial dictatorship where the workers are ranked according to the vertical preferences of firms.

  Furthermore, this applies to any submarket of the initial market. In any subgame of the initial game that is on-the-equilibrium-path, the expected utility under μ^* is $u(f, \mu^*(f))$. Consider a one-shot deviation where f has already hired $\Omega \subset W$ and accepts a worker w that has applied to f and that it prefers less than its least preferred stable partner:

i.e: $u(f,w) < \min(u(f,w'))$ such that $w' \in \mu^*(f)$. Denote the subgame where f rejects w as Γ and the one with the deviation Γ'. In Γ under μ^*, $q'_f = q_f - |\Omega|$, and the firm receives $u(f, \mu^*(f))$, where $\mu^*(f) = \Omega \cup \Omega'$.

If $\Omega' = \emptyset$, either f has filled its capacity or $u(f,w) < 0$, ensuring $u(f, \mu^*(f) \cup \{w\}) < u(f, \mu^*(f))$ in either case. If $\Omega' \neq \emptyset$, the first $q' - 1$ workers that f employs under Γ and Γ' are the same by the serial dictatorship representation, and the end outcome differs only by one worker. Denote the worker that is employed under Γ but not under Γ' as w^*. The expected utility with one-shot deviation is $u(f, \Omega \cup \Omega' \setminus \{w^*\} \cup \{w\})$. We know that $u(f,w) < \min(u(f,w'))$ such that $w' \in \mu^*(f)$, therefore $u(f,w) < u(f,w^*)$. With responsive preferences, $u(f, \Omega \cup \Omega' \setminus \{w^*\} \cup \{w\}) < u(f, \mu^*(f))$, ensuring the one-shot deviation being not profitable.

Verbally, any deviation to accepting a worker that is not acceptable under μ^* only changes that worker with one of the stable workers. With responsive preferences, it is ensured that the difference between the specific workers is reflected in the overall preference, preventing deviation.

□

Proposition 1: How can we restrict attention to end-outcomes? 7

Take the same example from the Proposition, and once again, suppose there is a Markovian strategy profile σ^* that is a limit equilibrium and enforces the WOSM in any many-to-one search game. Under σ^*, f_1 either meets with w_1 first and then w_4, depending on the contact function realization. Since σ^* enforces WOSM, the expected utility gain when $\mu_h(f_1) = \emptyset$ is $u(f_1, \{w_1, w_4\})$. When f_1 deviates to accepting w_5, it first gets $u(f_1, w_5)$. Then, since σ^* is an equilibrium in every subgame by assumption, the expected utility gain is the additional utility from adding w_3. Therefore, the overall utility from employing w_5 is $U = u(f_1, w_5) + \delta^{tau}(u(f_1, \{w_3, w_5\}) - u(f_1, w_5))$ (τ referring to the expected day the match will conclude), which converges to $u(f_1, \{w_3, w_5\})$ as δ converges to 1. The same logic applies to other propositions.

More on Equivalent Equilibria of Many-to-One and One-to-One

Proposition 4 can be analyzed with a different perspective for related markets with indifferences, in which ties between seats are not broken. In that case, SPC ensures that the equilibria of the original many-to-one and the related one-to-one market are *equivalent*. Intuitively, the equilibria of the many-to-one search model and its related one-to-one search model are equivalent if the workers adapt the same acceptance strategy for the seats of a firm as the strategy they use for the firm, and the firms' seats use the same acceptance strategy for each worker as the firm itself.

Definition 7. *For any many-to-one market and its related market, the equilibria of the many-to-one search game and its related one-to-one search game are equivalent if under the same information restriction and for every related subgame:*

1. *Workers of the related market accept the seats that belong to the firms they apply to in the original market and reject others.*
2. *Seats of the related market accept the same workers as their mother firm.*

The following lemma additionally shows that we can track equilibrium equivalence from remaining markets, even though we can not observe agents' strategy profiles.

Lemma 1. *For each realization of the contact function C, the equilibria of the many-to-one search model and its related one-to-one search model are equivalent if and only if the remaining markets are related for each history.*

Proof. Let C be any realization of the contact function. I will prove the lemma by proving the if statements from both directions.

1. The equilibria of many-to-one search and the related one-to-one search are equivalent $\Rightarrow$ remaining markets are related for each history.
   Easily proven by induction. Start with the initial market M. Equivalent acceptance strategies imply:
   $$\begin{aligned} f \text{ accepts } w \quad &\Longleftrightarrow \quad s \text{ accepts } w \\ w \text{ accepts } f \quad &\Longleftrightarrow \quad w \text{ accepts } s \end{aligned}$$
   This means, for the same realized related contact function, M' after the first day is the same. Apply this to every step, the first part of the lemma concludes.
2. Remaining markets are related for each history $\Rightarrow$ The equilibria of both search models are equivalent.
   Suppose s is a seat of f. If when s, w and f, w meet after h at the related remaining markets, and the remaining market after this is also the same s, f use the same acceptance strategies.
   If this holds for each remaining market and history, the equilibria are equivalent.

$\square$

The above lemma establishes the equivalence between the equilibrium strategies and the remaining markets. Recall that Wu (2015) shows that no unstable matching can be enforced in a private-dinner equilibrium, whereas Proposition 3 shows otherwise, even under a more restrictive information criterion. Therefore, the equilibria of one-to-one and many-to-one search models are generically not equivalent, which, on the other hand, is ensured if the initial market satisfies the sequential preference condition.

A Matchings of the Centralized Market à la Search Game

This subsection takes a quick detour into the existing literature of many-to-one matchings and describes the adaptations that will translate the existing results into a many-to-one search environment. In order to find the set of stable matchings, a linear programming approach is developed. As Vate (1989) and Rothblum (1992) characterize stable matchings in a marriage market as extreme points of a convex polytope, Baïou and Balinski (2000) extend the results to a many-to-one matching market. They show that simply replacing the parameters of a marriage market with their many-to-one counterparts does not extend the results of Rothblum, but needs a slight differentiation. Neme and Oviedo (2020) adapt their approach as well, so am I:

Given a matching μ, an assignment matrix $x^B \in \mathbb{R}^{|F|\times|W|}$ (B for Baïou and/or Balinski) is defined where all its elements are denoted by $x^B(f,w)$ where $x^B(f,w) \in \{0,1\}$ and $x^B(f,w) = 1$ if and only if $\mu(w) = f$.

Following Baïou and Balinski, let CP denote the convex polytope generated by the following linear inequalities:

$$\sum_{j\in W} x^B_{f,j} \leq q_f \qquad \forall f \in F \tag{1}$$

$$\sum_{i\in F} x^B_{i,w} \leq 1 \qquad \forall w \in W \tag{2}$$

$$x^B_{f,w} \geq 0 \qquad \forall (f,w) \in F \times W \tag{3}$$

$$x^B_{f,w} = 0 \qquad \text{for unacceptable pairs } (f,w) \tag{4}$$

The integer solutions to (1)-(3) are assignment matrices of simple many-to-one matchings. A matching, where some entries $x^B(f,w)$ are non-integers in the interval $(0,1)$ is called a fractional matching. We can interpret the fractional matchings as probabilities that the agents are matched to one another as well as the timeshares the respective agents spend with each other.

An example of a many-to-one assignment matrix with 2 firms $\{f_1, f_2\}$ and 2 workers $\{w_1, w_2\}$ would look like as follows:

	f_1	f_2
w_1	$x^B(f_1, w_1)$	$x^B(f_2, w_1)$
w_2	$x^B(f_1, w_2)$	$x^B(f_2, w_2)$

where all the entries are nonnegative and:

$$\sum_F x^B(f,w) \leq 1 \text{ for both workers}$$
$$\sum_W x^B(f,w) \leq q_f \text{ for both firms}$$

Adding (4) imposes the individual rationality constraint, that the match is at least as good as the outside option. As Baïou and Balinski show, adding another linear inequality to the *CP* system:

$$\sum_{u(f,j)>u(f,w)} x^B_{f,j} + q_f \sum_{v(i,w)>v(f,w)} x^B_{i,w} + q_f x^B_{f,w} \geq q_f \qquad \forall (f,w) \in A \tag{5}$$

defines the stable convex polytope *SCP* and the integer solutions to *SCP* define stable simple matchings, which are individually rational and pairwise stable.

In Kojima and Manea (2010) and Kesten and Ünver (2015), "The School-Choice Birkhoff-von Neumann Theorem states that any fractional matching can be represented as a lottery (not necessarily unique) over simple matchings", which allows us to interpret a fractional matching in a third way. Nevertheless, the intuition about the stable matchings turns out to be incorrect and the non-integer solutions of inequalities (1)-(5) do not immediately give us stability when it comes to fractional matchings.

In fact, as shown by Baïou and Balinski Baïou and Balinski (2000) and elaborated further in Neme and Oviedo Neme and Oviedo (2020), the non-integer solutions to the *SCP* might be blocked in a *fractional way*, by a firm and worker, who want to increase their timeshare together, at the expense of those they like less at a non-integer solution to *SCP*. Formally:

Definition 8. *A matching is blocked by the firm-worker pair* (f, w) *in a fractional way, when* $x^B(f, w) < 1$*,* $v(f, w) > v(f', w)$ *for some* f' *such that* $x^B(f', w) > 0$ *and* $u(f, w) > u(f, w')$ *for some* w' *such that* $x^B(f, w') > 0$*.*

Example: An example from Baïou and Balinski (2000) and Neme and Oviedo (2020) which shows an assignment matrix, which is a solution to *SCP* and blocked in a fractional way by is as follows:

	f_1	f_2
w_1	1	0
w_2	0.5	0.5
w_3	0.5	0.5
w_4	0	1

where the preferences (over individuals, derived from the preferences over sets) are such that:

$$f_1 : u(f_1, w_1) > u(f_1, w_2) > u(f_1, w_3) > u(f_1, w_4) \quad \text{and} \quad q_{f_1} = 2$$
$$f_2 : u(f_2, w_4) > u(f_2, w_3) > u(f_2, w_2) > u(f_2, w_1) \quad \text{and} \quad q_{f_2} = 2$$
$$w_1 : v(f_2, w_1) > v(f_1, w_1)$$
$$w_2 : v(f_2, w_2) > v(f_1, w_2)$$
$$w_3 : v(f_2, w_3) > v(f_1, w_3)$$
$$w_4 : v(f_1, w_4) > v(f_2, w_4)$$

In the example above, it can easily be checked that the numbers solve the linear problem *SCP*. However, w_3 likes f_2 better than f_1, but his time is shared equally between the firms. In addition, f_2 likes w_3 better than w_2 but one seat is shared equally between those workers. In such a case, the pair (f_2, w_3) blocks the assignment above in a fractional way so that they can increase their time spent together in exchange for their other partners in the matching, f_1, and w_2.

As it is not mentioned in either of the papers, the lottery interpretation of the fractional matchings helps us understand the underlying misfunction in this example. Although the lottery over simple matchings which represents a fractional matching is generically not unique, in this example it actually is unique. The fractional matching is a lottery over the following two simple matchings with equal probability 0.5:

	f_1	f_2
w_1	1	0
w_2	1	0
w_3	0	1
w_4	0	1

	f_1	f_2
w_1	1	0
w_2	0	1
w_3	1	0
w_4	0	1

The reason why there is a blocking pair in a fractional way can be observed in the lottery as well. Although the first simple matching of the lottery is stable, the second one is not. Furthermore, the fact that it is not stable pins down the fractional blocking pair: The second simple matching is blocked by the pair (f_2, w_3).

Neme and Oviedo (2020) refer to matchings in which there are no incentives to block (neither as in the usual way nor in the fractional interpretation) as "strong stable matchings" and prove that they can be found adding the additional constraint to SCP:

Definition 9. *Let (F, W, q, u, v) be a many-to-one matching market. A fractional matching is strongly stable if for each acceptable pair (f, w), x satisfies the strong stability condition:*

$$\left[q_f - \sum_{u(f,j)\geq u(f,w)} x^B_{f,j}\right] \cdot \left[1 - \sum_{v(i,w)\geq v(f,w)} x^B_{i,w}\right] = 0 \tag{6}$$

Observe that the assignment matrix of a simple stable matching fulfills (6). This follows from the simple fact that if f and w are matched, the second multiplier is 0. If they are not matched with each other, at least one of them is consuming its own capacity.

For fractional matchings, when (6) does not hold for some (f, w), $q_f > \sum_{u(f,j)\geq u(f,w)} x^B_{f,j}$ and $1 > \sum_{v(i,w)\geq v(f,w)} x^B_{i,w}$, it means that there are f' and w' such that $u(f, w) > u(f, w')$, $v(f, w) > v(f', w)$ and $x(f, w') > 0$, $x(f', w) > 0$ and $x(f, w) < 1$. In such a scenario, the (f, w) would block the assignment and increase their time shared together.

Insightful Theorem 1 from Neme and Oviedo (2020) concludes: "If x^B is a strongly stable fractional matching, it can be represented as a convex combination of stable matchings. Furthermore, a lottery over simple stable matchings is strongly stable as well." This establishes the lottery interpretation as in Lauermann and Nöldeke (2014).

Both Baïou and Balinski (2000) and Neme and Oviedo (2020) constructed the matching as an assignment matrix with two sides in rows and columns respectively. This approach is quite useful to visualize and point out simple stable matchings. However, the structure in Lauermann and Nöldeke (2014) requires a different approach. The main difference is that the many-to-one papers of Baïou and Balinski and Neme and Oviedo use an ordinal utility approach, whereas Lauermann and Nöldeke employ a cardinal utility in their model, which allows them to calculate the expected utilities for the agents as well. As discussed before, cardinal utility adaptation is vital for a search structure.

If $u(f,\Omega)$ is a linear function of individual values, i.e. additively separable such that $u(f,\Omega) = \sum_{i\in\Omega} u(f,w_i)$, we could still use the assignment matrix with the agents in rows and columns to calculate expected utilities. In order to employ such a structure in a many-to-one framework and be able to calculate expected utilities at the same time, we would need a different assignment matrix $x \in \mathbb{R}^{|F|\times|2^W|}$, satisfying:

$$\sum_{F} \sum_{\substack{\Omega\subset 2^W \\ w\in\Omega}} x(f,\Omega) \leq 1 \qquad \forall w \tag{7}$$

$$\sum_{\Omega\subset 2^W} x(f,\Omega) \leq 1 \qquad \forall f \tag{8}$$

$$x(f,\Omega) = 0 \qquad \forall |\Omega| > q_f \tag{9}$$

$$x(f,\Omega) \geq 0 \qquad \forall (f,\Omega) \in F \times 2^W \tag{10}$$

The above-described assignment matrix has firms in the columns and all possible subsets of the workers in the rows. In a many-to-one matching, a specific worker can not be matched to two firms. Hence, considering a worker would now require considering each subset that this worker appears in.

The expected utilities from a matching x then can be calculated as follows:

$$U(f;x) = \sum_{\Omega} x(f,\Omega)u(f,\Omega)$$
$$V(w;x) = \sum_{F} \sum_{\substack{\Omega\subset 2^W \\ w\in\Omega}} x(f,\Omega)v(f,w)$$

Example: An assignment matrix of a many-to-one matching market with 2 workers $\{w_1,w_2\}$ and 2 firms $\{f_1,f_2\}$ would be as follows according to the latter description:

	f_1	f_2
$\{w_1, w_2\}$	$x(f_1, \{w_1, w_2\})$	$x(f_2, \{w_1, w_2\})$
w_1	$x(f_1, w_1)$	$x(f_2, w_1)$
w_2	$x(f_1, w_2)$	$x(f_2, w_2)$

where all the entries are nonnegative and:

$$
\begin{aligned}
&x(f_1, \{w_1, w_2\}) + x(f_1, w_1) + x(f_2, \{w_1, w_2\}) + x(f_2, w_1) \leq 1 \\
&x(f_1, \{w_1, w_2\}) + x(f_1, w_2) + x(f_2, \{w_1, w_2\}) + x(f_2, w_2) \leq 1 \\
&x(f, \{w_1, w_2\}) + x(f, w_1) + x(f, w_2) \leq 1 \ \text{ for both firms} \\
&x(f, \Omega) = 0 \quad \text{if} \quad |\Omega| > q_f \ \text{ for all subsets}
\end{aligned}
$$

With expected utilities:

$$
\begin{aligned}
U(f; x) &= x(f, \{w_1, w_2\})u(f, \{w_1, w_2\}) + x(f, w_1)u(f, w_1) + x(f, w_2)u(f, w_2) \\
V(w_1; x) &= \sum_{f \in F} \left[x(f, \{w_1, w_2\}) + x(f, w_1)\right] v(f, w_1) \\
V(w_2; x) &= \sum_{f \in F} \left[x(f, \{w_1, w_2\}) + x(f, w_2)\right] v(f, w_2)
\end{aligned}
$$

Proposition 6. *The assignment matrix described before by Baïou-Balinski and Neme-Oviedo, which has the workers instead of sets of workers in the rows can easily be calculated from the matrix described above, by setting* $x^B(f, w) = \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f, \Omega)$.

Similarly, the entries of the assignment matrix x can be calculated if all the entries of x^B are integers, i.e. $x^B(f, w) = \{0, 1\} \ \forall (f, w)$. *This does not necessarily hold if some entries of* $x^B \in (0, 1)$.

Proof. The first part of the lemma is trivial with the given equality. For the second part, the example below illustrates the calculation of the assignment matrix x from x^B for simple matchings. Furthermore, the second part of the example serves as a proof that x calculated from x^B is not necessarily unique for fractional matchings.

Example: Consider again the example of a many-to-one matching market with 2 workers $\{w_1, w_2\}$ and 2 firms $\{f_1, f_2\}$. In the first scenario, let us take a simple matching with all entries are either 0 or 1. In that case, if $u(f, H)$ and $v(h, w)$ are known, the expected utilities can easily be calculated because the assignment matrix x^B implies a unique x.

x^B	f_1	f_2
w_1	1	0
w_2	1	0

x	f_1	f_2
$\{w_1, w_2\}$	1	0
w_1	0	0
w_2	0	0

However, if x^B is a fractional matching, the corresponding x is not necessarily unique. Under different x representations of x^B, the expected utility of the workers will be equal, whereas the expected utilities of the firms might differ. There is an example of a fractional matching x^B with multiple x representations below, where both x_1 and x_2 imply x^B, which serves as a proof to the Proposition 6 above.

x^B	f_1	f_2
w_1	0.5	0.5
w_2	0.3	0.4

x_1	f_1	f_2
$\{w_1, w_2\}$	0.3	0
w_1	0.2	0.5
w_2	0	0.4

x_2	f_1	f_2
$\{w_1, w_2\}$	0.1	0
w_1	0.4	0.5
w_2	0.2	0.4

$\square$

When we consider the finite decentralized many-to-one search game, the acceptance decisions of the agents represent a stopping agreement. Therefore, we need to be able to calculate the expected utilities from continuing the search and compare them to the gains from an immediate acceptance decision. As the next subsection shows, this new definition of a matching matrix will enable us to do such comparisons.

From Equilibria to Assignment Matrices

In pursuance of the analysis of how the subgame perfect equilibria of the many-to-one search model relate to the stable matchings of the centralized many-to-one market, I will describe the assignment matrix that can be obtained from the search equilibrium. In fact, any equilibrium of the search model implies an assignment matrix. Intuitively, the assignment matrix will show the probability of a firm and a subset of workers being matched in equilibrium. In the following revisions, this methodology will be implemented to show whether equilibrium matchings are stable.

In order to calculate the assignment matrix from an equilibrium of the search game, we look at the terminal histories. For any finite terminal history $h \in \mathcal{H} \backslash \hat{\mathcal{H}}$ that takes T periods, let $C_h^t(f, w, M_h^t) \in \{0, 1\}$ denote the meeting function realization for any $t \in \{0, ..., T\}$, where M_h^t is the remaining market implied by μ_h^t, the instantaneous matching in the beginning of period t. [13] The meeting function realization is such that $C_h^t(f, w, M_h^t) = 1$ for firm f and worker w who meet at period t along h and $C_h^t(f, w, M_h^t) = 0$ for all other pairs. The pair (f, w) such that $C_h^t(f, w, M_h^t) = 1$ will be referred to as i_h^t, since they are the agents of the stage game.

Similarly, $a_h^t(i_h^t, \mu_h^t) \in \{A, R\}^2$ denotes the action realization for i_h^t.[14] After the action profile of t realizes, the instantaneous matching is updated to μ_h^{t+1} and remaining market is M_h^{t+1}. Since the market does not change if any of the parties reject, $M_h^{t+1} = M_h^t$ unless $a_h^t(i_h^t, \mu_h^t) = (A, A)$. If both parties accept, the worker leaves the market and the firm leaves the market if the capacity is full.

With this formulation, we can now calculate the probability of a terminal history h occurring on the equilibrium path of the search game. The probability of meeting is simply determined by the choice function, $\mathbb{P}(C_h^t(f, w, M_h^t) = 1) = C(f, w, M_h^t)$.

In order to reach day 1 on the equilibrium path of h, the agents who meet on day 0 should be aligned with h and they should decide accordingly as well. The probability of reaching day 1 under history h, denoted by $\mathbb{P}(\mu_h^1)$ and satisfies $\mathbb{P}(\mu_h^1) = C(i_h^0, M_h^0)\mathbb{P}(a_h^0(i_h^0, \mu_h^0))$. By induction, the probability of reaching any day $t \leq T$ along history h can be calculated by multiplying the probability of agents meeting and behaving according to the history along the equilibrium path of h:

$$\mathbb{P}(\mu_h^t) = \prod_{k=1}^{t} C(i_h^{k-1}, M_h^{k-1})\mathbb{P}(a_h^{k-1}(i_h^{k-1}, \mu_h^{k-1}))$$

Subsequently, for any given many-to-one search game $(F, W, q, u, v, C, \delta)$ once the equilibrium acceptance strategies of the agents are calculated, we can restrict attention to terminal histories and easily calculate the probability of any f, Ω being matched at the end of the game by simply adding up the probabilities of different terminal histories at the end of which f and Ω are together. By simply taking this probability equal to $x(f, \Omega)$, a matching matrix can be constructed.[15]

[13] Clearly, $M_h^0 = M$ for any history.

[14] Recall that we restrict attention to pure strategies.

[15] The game ends almost surely, and it can easily be shown that this matrix satisfies the properties of a many-to-one matching.