Due Date : January 25 (10pm), 2019

Instructions

- *This assignment serves as a warm-up for the following assignments. You are not obliged to finish this assignment, but some of the results here might be useful for the upcoming assignments. Unless otherwise specified, you may use the results in this assignment directly in your answer in the future.*

- *Use a document preparation system such as LaTeX.*

- *You will be using Gradescope, you should have received an email to sign up, otherwise sign up for an account on gradescope.com and use course code 9EVRGV*

- *Submit this test submission on Gradescope (not necessary to complete and not marked)*

Question 1. Given any unit vector $\boldsymbol{n}$ (i.e. $||\boldsymbol{n}|| = 1$), we define the hyperplane $\mathcal{H}_{\boldsymbol{n}} := \{\boldsymbol{x} : \boldsymbol{n}^\top \boldsymbol{x} = 0\}$ for which $\boldsymbol{n}$ is known as the normal vector. For any vector $\boldsymbol{x}$, we define its projection into $\mathcal{H}_{\boldsymbol{n}}$ as $\pi_{\boldsymbol{n}}(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{x}^\top \boldsymbol{n})\boldsymbol{n}$.

1. Given two vectors $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, take $\boldsymbol{n} = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{||\boldsymbol{x}_2 - \boldsymbol{x}_1||}$. Show that $\pi_{\boldsymbol{n}}(\boldsymbol{x}_1) = \pi_{\boldsymbol{n}}(\boldsymbol{x}_2)$.

2. Let $\boldsymbol{w}$ be a vector and define $y_1 := \boldsymbol{x}_1^\top \boldsymbol{w}$ and $y_2 := \boldsymbol{x}_2^\top \boldsymbol{w}$. Show that $y_1 = y_2$ if and only if $\boldsymbol{w} \in \mathcal{H}_{\boldsymbol{n}}$.

*3. Let $\boldsymbol{X}$ be a n by p matrix whose rows $\boldsymbol{X}_{i,:}$ are all distinct. Show that there exists a vector $\boldsymbol{w}$ of length p such that the scalars $(\boldsymbol{X}\boldsymbol{w})_i$ are all distinct.

Question 2. Recall the variance of X is $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

1. Let X be a random variable with finite mean. Show $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

2. Let X and Z be random variables on the same probability space. Show that $\mathrm{Var}(X) = \mathbb{E}_Z[\mathrm{Var}(X|Z)] + \mathrm{Var}_Z(\mathbb{E}[X|Z])$. (Hint : $\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}[X|Y]]$.)

Question 3. Let $X \in \mathcal{X}$ be a random variable with density function f_X, and $g : \mathcal{X} \to \mathcal{Y}$ be continuously differentiable, where $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\mathbb{R}$. Let $Y := g(X)$, which is continuously distributed with density function f_Y.

1. Show that if g is monotonic, $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$.

2. Let $f_X(x) = \mathbf{1}_{x \in [0,1]}(x)$ and $f_Y(y) = \mathbf{1}_{y \in [0,2]}(y) \cdot \frac{y}{2}$. Find a monotonic mapping g that translates f_X and f_Y.

*3. Let $N_Y = \{y \in \mathcal{Y} : g(x) = y, g(x)' = 0 \text{ for some } x \in \mathcal{X}\}$. Show that in general if $g'(x) = 0$ at most finitely many times, for $y \in \mathcal{Y} \setminus N_Y$,

$$f_Y(y) = \sum_{x \in \{x : g(x) = y\}} \frac{f_X(x)}{|g'(x)|}$$

4. Let $X \sim \mathcal{N}(0,1)$, i.e. $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$, and $g(x) = x^2$. Determine $f_Y(y)$.

Question 4. Let Q and P be univariate normal distributions with mean and variance μ, σ^2 and m, s^2, respectively. Derive the entropy $H(Q)$, the cross-entropy $H(Q, P)$, and the KL divergence $D_{\mathrm{KL}}(Q||P)$.