Definition of Safety

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1 Language

We define our language over four sets of symbols: numerals, symbolic constants, variables, and aggregate names.

1.1 Term and Pools

We inductively define *terms*, *tuples of terms*, and *pools*:

- all numerals and variables are terms,
- $f(\boldsymbol{t})$ is a term if f is a symbolic constant and $\boldsymbol{t}$ is a pool,
- $t_1 \star t_2$ is a term if t_1, t_2 are terms and $\star \in \{+, -, \times, \div, ..\}$,
- $\langle \boldsymbol{t} \rangle$ is a term if $\boldsymbol{t}$ is a pool,
- $t_1, \dots, t_n$ is a tuple of terms if $n \geq 0$ and t_i is a term,
- $\dot{t_1}; \dots ; \dot{t_n}$ is a pool if $n \geq 1$ and each $\dot{t_i}$ is a tuple of terms.

We omit writing the parenthesis for terms of form $f()$.

We inductively define a term to be *evaluable* if

- it is a numeral, or
- it has form $t_1 \star t_2$ where t_1 and t_2 are evaluable and $\star \in \{+, -, \times, \div\}$.

We then inductively define function *eval* mapping evaluable terms to sets of numerals:

- for numerals t, we let $\mathit{eval}(t) = \{t\}$, and
- for terms of form $t_1 \star t_2$, we let

$$
\begin{aligned}
&\mathit{eval}(t_1 + t_2) = \{s_1 + s_2 \mid s_1 \in \mathit{eval}(t_1), s_2 \in \mathit{eval}(t_2)\},\\
&\mathit{eval}(t_1 - t_2) = \{s_1 - s_2 \mid s_1 \in \mathit{eval}(t_1), s_2 \in \mathit{eval}(t_2)\},\\
&\mathit{eval}(t_1 \times t_2) = \{s_1 \times s_2 \mid s_1 \in \mathit{eval}(t_1), s_2 \in \mathit{eval}(t_2)\}, \text{ and}\\
&\mathit{eval}(t_1 \div t_2) = \{s_1 \div s_2 \mid s_1 \in \mathit{eval}(t_1), s_2 \in \mathit{eval}(t_2), s_2 \neq 0\}.
\end{aligned}
$$

We say that a term t is *nonzero* if it is evaluable and $0 \notin \mathit{eval}(t)$.

1.2 Atoms and Literals

An *atom* has form $p(\boldsymbol{t})$ where p is a symbolic constant and $\boldsymbol{t}$ is a pool. We omit writing parenthesis for atoms of form $p()$.

A *comparison* has form $t_1 \prec t_2$, where t_1, t_2 are terms and $\prec \in \{<, >, \leq, \geq, =, \neq\}$. Furthermore, we let *negate* be a function to negate relation symbols: $< \mapsto \geq$, $> \mapsto \leq$, $\leq \mapsto >$, $\geq \mapsto <$, $= \mapsto \neq$ and $\neq \mapsto =$.

A *literal* is either an atom or a comparison optionally preceded by the *default negation* symbol $\neg$.

A *conditional literal* has form $l : \dot{l}$, where l is a literal and $\dot{l}$ is a (possibly empty) tuple of literals.

1.3 Aggregates

An *aggregate* has the form

$$\alpha\{\dot{t_1} : \dot{l_1}; \ldots; \dot{t_n} : \dot{l_n}\} \prec s \tag{1}$$

where

- $n \geq 0$,
- α is an aggregate name,
- each $\dot{t_i}$ is a tuple of terms,
- each $\dot{l_i}$ is a (possible empty) tuple of literals,
- $\prec \in \{<, >, \leq, \geq, =, \neq\}$, and
- s is a term.

1.4 Rules

A *rule* has form

$$a_1 \vee \cdots \vee a_m \leftarrow l_1 \wedge \cdots \wedge l_n \tag{2}$$

where $m, n \geq 0$, each a_i is an atom and each l_i is a literal, conditional literal or aggregate.

A *choice rule* has form

$$\{a_1; \ldots; a_m\} \leftarrow l_1 \wedge \cdots \wedge l_n \tag{3}$$

where $m, n \geq 0$, each a_i is an atom and each l_i is a literal, conditional literal or aggregate.

We refer to the a_i and l_i in rules of form (2) and (3) as *head atoms* and *body literals*, respectively.

2 Safety

In the following, we use function $vars(e)$ to obtain all variables occurring in an expression e. Furthermore, we say that a variable X occurs *globally* in

- a literal l if $X \in vars(l)$,
- a conditional literal $l : \dot{l}$ if $X \in vars(l) \setminus vars(\dot{l})$,
- an aggregate of form (1) if $X \in s$, and
- a rule of form (2) or (3) if it occurs globally in a head atom or body literal.

2.1 Terms

We inductively define function pt for terms, tuples of terms and pools:

- for numerals n, we let $pt(n) = \emptyset$ and $dt(n) = \emptyset$,
- for variables X, we let $pt(X) = \{X\}$ and $dt(X) = \emptyset$,
- for term tuples $\dot{t} = t_1, \dots, t_n$, we let

$$\begin{aligned} pt(\dot{t}) &= pt(t_1) \cup \dots \cup pt(t_n) \text{ and} \\ dt(\dot{t}) &= (dt(t_1) \cup \dots \cup dt(t_n)) \setminus pt(\dot{t}), \end{aligned}$$

- for pools $\boldsymbol{t} = \dot{t}_1; \dots; \dot{t}_n$, we let

$$\begin{aligned} pt(\boldsymbol{t}) &= pt(\dot{t}_1) \cap \dots \cap pt(\dot{t}_n) \text{ and} \\ dt(\boldsymbol{t}) &= (dt(t_1) \cup \dots \cup dt(t_n)) \setminus pt(\boldsymbol{t}), \end{aligned}$$

- for terms of form $f(\boldsymbol{t})$, we let $pt(f(\boldsymbol{t})) = pt(\boldsymbol{t})$ and $dt(f(\boldsymbol{t})) = dt(\boldsymbol{t})$,
- for terms of form $t_1 \star t_2$, we let

$$\begin{aligned} pt(t_1 \star t_2) &= \begin{cases} pt(t_2) & t_1 \text{ is evaluable and } \star \in \{+, -\}, \textit{or} \\ & t_1 \text{ is nonzero and } \star = \times, \\ pt(t_1) & t_2 \text{ is evaluable and } \star \in \{+, -\}, \textit{or} \\ & t_2 \text{ is nonzero and } \star = \times, \\ \emptyset & \text{otherwise} \end{cases} \\ dt(\boldsymbol{t}) &= vars(t_1 \star t_2) \setminus pt(t_1 \star t_2) \end{aligned}$$

2.2 Body Literals

Next, we define function dep for body literals:

- for an atom a of form $p(\boldsymbol{t})$, we let $dep(a) = \{(pt(\boldsymbol{t}), \emptyset), (\emptyset, dt(\boldsymbol{t}))\}$,

- for a comparison a of form $t_1 \prec t_2$ with $\prec \notin \{=\}$, we let $\mathit{dep}(a) = \{(\emptyset, \mathit{vars}(a))\}$,
- for a comparison a of form $t_1 = t_2$, we let $\mathit{dep}(a) = \{(\mathit{pt}(t_1), \mathit{vars}(t_2)), (\mathit{pt}(t_2), \mathit{vars}(t_1)), (\emptyset, \mathit{dt}(t_1) \cup \mathit{dt}(t_2))\}$,
- for a literal l of form $\neg a$ where a is an atom, we let $\mathit{dep}(l) = \{(\emptyset, \mathit{vars}(l))\}$,
- for a literal l of form $\neg t_1 \prec t_2$, we let $\mathit{dep}(l) = \mathit{dep}(t_1 \ \mathit{negate}(\prec) \ t_2)$,
- for conditional literal l, we let $\mathit{dep}(l) = \{(\emptyset, \mathit{vars}(l))\}$,[1]
- for an aggregate l of form (1) with $\prec \notin \{=\}$, we let $\mathit{dep}(l) = \{(\emptyset, \mathit{vars}(l))\}$,
- for an aggregate l of form (1) with $\prec \in \{=\}$, we let $\mathit{dep}(l) = \{(\mathit{pt}(s), \mathit{vars}(\dot{t}_1 : \dot{l}_1; \dots; \dot{t}_n : \dot{l}_n)), (\emptyset, \mathit{dt}(s))\}$.

[1] We could use $\mathit{dep}(l) = \{(\emptyset, \mathit{dt}(l_0) \cup \mathit{vars}(l_1, \dots, l_n))\}$ but would have to check that the remaining variables in l_0 are either provided elsewhere or do not occur in other conditional literals. For example, *clingo* accepts
$\leftarrow p(X, Y) : q(Y)$
and
$\leftarrow p(X, Y) : q(Y) \wedge r(X; Z)$
as safe but should discard
$\leftarrow p(X, Y) : q(Y) \wedge r(X, Y) : s(Y)$
as unsafe.
On the other hand, we could also omit this case altogether because it should not be relevant in practice at all. (In *clingo*, it is only implemented because of an internal rewriting step.)

2.3 Gathering Dependencies

Next, we define function *analyze* to gather dependencies in rules, conditional literals, and aggregate elements:

- for a rule r of form (2) or (3), we let

$$\mathit{analyze}(r) = \bigcup_{1 \leq i \leq m} \{(\emptyset, \mathit{vars}(a_i))\} \cup \bigcup_{1 \leq i \leq n} \mathit{dep}(l_i),$$

- for a conditional literal l of form $l_0 : l_1, \dots, l_n$, we let

$$\mathit{analyze}(l) = \{(\emptyset, \mathit{vars}(l_0))\} \cup \bigcup_{1 \leq i \leq n} \mathit{dep}(l_i),$$

- for an aggregate element e of form $\dot{t} : l_1, \dots, l_n$, we let

$$\mathit{analyze}(e) = \{(\emptyset, \mathit{vars}(\dot{t}))\} \cup \bigcup_{1 \leq i \leq n} \mathit{dep}(l_i).$$

Given a set PD of pairs of sets of variables and a set of variables V, we let $\mathit{PD}_{|V} = \{(P \cap V, D \cap V) \mid (P, D) \in \mathit{PD}\}$.

2.4 Safety Definition

We define operator C_{PD} parametrized with a set PD of pairs of sets of variables applied to a set V of variables as

$$C_{\mathit{PD}}(V) = \bigcup_{(P,D) \in \mathit{PD}, D \subseteq V} P.$$

Let r be a rule with global variables G. We say that a rule r is *globally safe* if G is the least fixed point of C_{PD} with $\mathit{PD} = \mathit{analyze}(r)_{|G}$. Furthermore, we say that a conditional literal or aggregate element e occurring in rule r is *locally safe* if $L = vars(e) \setminus G$ is the least fixed point of C_{PD} with $PD = analyze(e)_{|L}$.

A rule is *safe* if it is globally safe and all occurrences of conditional literals and aggregate elements in it are locally safe.

3 Other Examples

$$\begin{aligned}
dep(p(X;Y)) &= \{(pt(X;Y), \emptyset), (\emptyset, dt(X;Y))\} \\
&= \{(pt(X) \cap pt(Y), \emptyset), (\emptyset, (dt(X) \cup dtY) \setminus (pt(X) \cap pt(Y)))\} \\
&= \{(\{X\} \cap \{Y\}, \emptyset), (\emptyset, (\emptyset \cup \emptyset) \setminus (\{X\} \cap \{Y\}))\} \\
&= \{(\emptyset, \emptyset), (\emptyset, \emptyset)\}
\end{aligned}$$

$$\begin{aligned}
dep(p(X;X+Y)) &= \{(pt(X;X+Y), \emptyset), (\emptyset, dt(X;X+Y))\} \\
&= \{(pt(X) \cap pt(X+Y), \emptyset), (\emptyset, (dt(X) \cup dtX+Y) \setminus (pt(X) \cap pt(X+Y)))\} \\
&= \{(\{X\} \cap \emptyset, \emptyset), (\emptyset, (\emptyset \cup (vars(X+Y) \setminus pt(X+Y))) \setminus (\{X\} \cap \emptyset))\} \\
&= \{(\emptyset, \emptyset), (\emptyset, (\{X, Y\} \setminus \emptyset) \setminus \emptyset)\} \\
&= \{(\emptyset, \emptyset), (\emptyset, \{X, Y\})\}
\end{aligned}$$

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