MT411

$\Rightarrow$ implicit differentiation

Robotic Engineering

$P_{new} = [A] P_{old}$

$\frac{d}{dx} y^2 = 2y \frac{dy}{dx} = 2y\dot{y}$

mapping matrix

Chapter 10

$\tau = [J] F$

Velocity Kinematics – The Jacobian

mapping matrix = Jacobian

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Velocity Kinematic

To follow any prescript velocity, it is important to know the relationship between the velocity of the tool and the joint velocity.

So, we are going to find velocity relationship relating the linear and angular velocities of the end effector to the joint velocities.

The velocity relationships are determined by the Jacobian of the function. The jacobian is a matrix that generalizes the notions of the ordinary derivative of the scalar function.

$$\text{angular velocity} \quad \omega = \dot{\theta} k \quad \text{where } k \text{ is unit vector}$$

$$\text{linear velocity} \quad v = \omega \times r \quad \text{where } r \text{ is vector from origin}$$

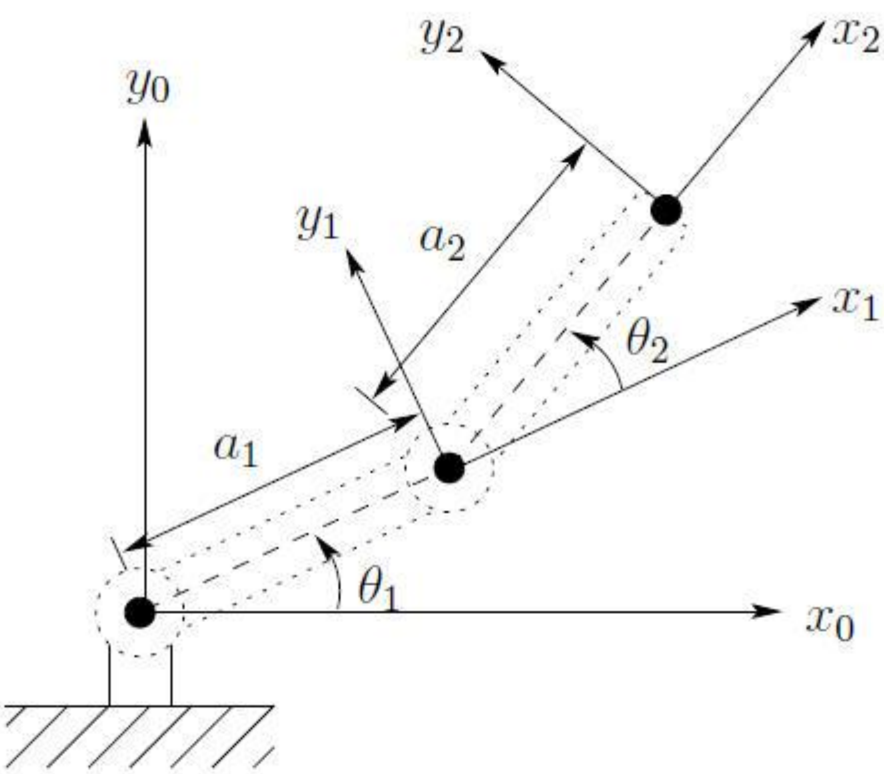
Force/Torque relationships

Interaction of manipulator with environment produces forces and moments at the end effectors or tool.

The force and torque are related by $\tau = J^{T}(q)F$

The Jacobian term is important from this aspect. So we need to know how to obtain Jacobian term.

Velocity Kinematics: 2 links manipulator



$$P_x = x_2$$
$$P_y = y_2$$

$$P_x = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$
$$P_y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

$$x_2 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$
$$y_2 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

Differentiate both side, (prove is on next page)

$$\dot{x}_2 = -a_1 \sin\theta_1 \dot{\theta}_1 - a_2 \sin(\theta_1 + \theta_2)(\theta_1 + \theta_2)$$
$$\dot{y}_2 = a_1 \cos\theta_1 \dot{\theta}_1 + a_2 \cos(\theta_1 + \theta_2)(\theta_1 + \theta_2)$$

Velocity Kinematics: 2 links manipulator

$$x_2 = a_1\cos\theta_1 + a_2\cos\theta_1\cos\theta_2 - a_2\sin\theta_1\sin\theta_2$$

$$\dot{x}_2 = -a_1\sin\theta_1\theta_1 - a_2\sin\theta_1\theta_1\cos\theta_2 - a_2\cos\theta_1\theta_1\sin\theta_2 - a_2\cos\theta_1\sin\theta_2\theta_2 - a_2\sin\theta_1\cos\theta_2\theta_2$$

$$= -a_1\sin\theta_1\theta_1 - a_2\left(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)\theta_1 - a_2\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)\theta_2$$

$$= -a_1\sin\theta_1\dot{\theta}_1 - a_2\left(\sin\left(\theta_1+\theta_2\right)\theta_1 + \sin\left(\theta_2+\theta_1\right)\theta_2\right)$$

$$= -a_1\sin\theta_1\theta_1 - a_2\sin\left(\theta_1+\theta_2\right)\left(\theta_1+\theta_2\right)$$

$$y_2 = a_1\sin\theta_1 + a_2\sin\theta_1\cos\theta_2 + a_2\cos\theta_1\sin\theta_2$$

$$\dot{y}_2 = a_1\cos\theta_1\theta_1 + a_2\cos\theta_1\theta_1\cos\theta_2 - a_2\sin\theta_1\theta_1\sin\theta_2 - a_2\sin\theta_1\sin\theta_2\theta_2 + a_2\cos\theta_1\cos\theta_2\theta_2$$

$$= a_1\cos\theta_1\theta_1 + a_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)\theta_1 + a_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)\dot{\theta}_2$$

$$= a_1\cos\theta_1\theta_1 + a_2\left(\cos\left(\theta_1+\theta_2\right)\right)\theta_1 + a_2\left(\cos\left(\theta_1+\theta_2\right)\right)\theta_2$$

$$= a_1\cos\theta_1\theta_1 + a_2\cos\left(\theta_1+\theta_2\right)\left(\theta_1+\theta_2\right)$$

Velocity Kinematics: 2 links manipulator

Put in vector form

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 \dot{\theta}_1 - a_2 \sin(\theta_1 + \theta_2)(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 \dot{\theta}_1 + a_2 \cos(\theta_1 + \theta_2)(\theta_1 + \theta_2) \end{bmatrix}$$

Rewrite as

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 & +a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ (\dot{\theta}_1 + \theta_2) \end{bmatrix}$$

Tool velocity

Joints velocity

Velocity Kinematics: 2 links manipulator

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 & +a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ (\dot{\theta}_1 + \theta_2) \end{bmatrix}$$

$$let\ \dot{x} = \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} \ and\ \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ (\dot{\theta}_1 + \theta_2) \end{bmatrix} where\ J = \begin{bmatrix} -a_1 \sin\theta_1 & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 & +a_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\boxed{\dot{x} = J\,\dot{\theta}}$$

*****J is called Jacobian of the manipulator***

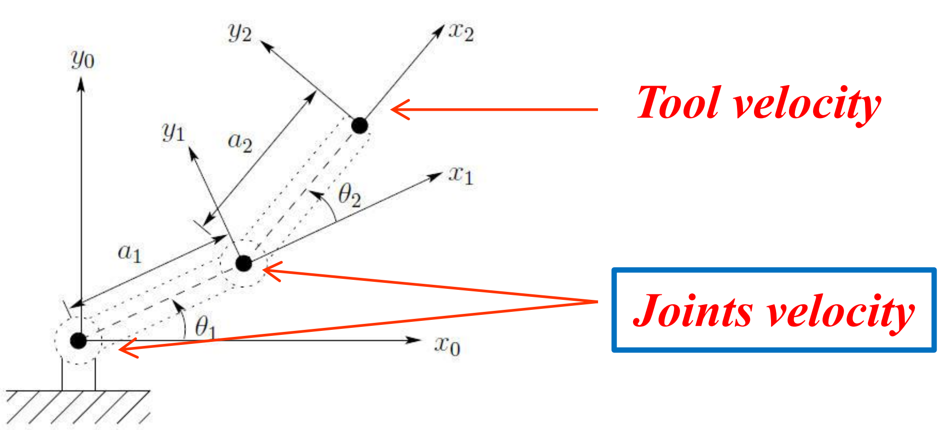
Velocity Kinematics: Tool and Joint Velocity

Tool velocity

Joints velocity

$$\dot{x} = J\,\dot{\theta}$$

In this chapter, we are going to find Jacobian matrix of the manipulator.



Derivation of the Jacobian

$$T\,(or\,H) = \left[\begin{array}{c:c} 3 & 3 \\ * & * \\ 3 & 1 \\ \hdashline 1 * 3 & 1*1 \end{array}\right] = \left[\begin{array}{c:c} Rotation & Translation \\ \hdashline 0 & 1 \end{array}\right]$$

$$T = \left[\begin{array}{c:c} R & d \\ \hdashline 0 & 1 \end{array}\right]$$

As robot moves about, both joint variables and end effector position, d, and orientation, R, will be function of time.

The objective of this section is to relate the linear and angular velocity of the end effector to the vector of joint velocities $\dot{q}(t)$ **.**

Rewrite d position as o position in function of time as:

Origin point

$$T_n^0(q) = \left[\begin{array}{c:c} R_n^0(q) & o_n^0(q) \\ \hdashline 0 & 1 \end{array}\right]$$

Derivation of the Jacobian

Linear velocity $v_n^0 = \dot{o}_n^0$ *and Angular velocity* ω_n^0

$$v_n^0 = J_v \dot{q}$$

$$\omega_n^0 = J_\omega \dot{q}$$

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

$$\boxed{\xi = J\dot{q}}$$

Body velocity

Joint velocity

Manipulator Jacobian is a 6xn matrix where n is the number of link

Angular and Linear Velocity

Angular velocity: $\omega = \dot{\theta} k$

k is a unit vector in the direction of the axis of rotation.

$\dot{\theta}$ ***is a time derivative of the*** θ ***.***

Linear velocity: $v = \omega \times r$

r is a vector from the origin to the point.

Combining Linear and Angular Velocity

Linear velocity Jacobian :

$$J_v = \begin{bmatrix} J_{v1} & J_{v2} & \ldots\ldots\ldots\ldots & J_{vn} \end{bmatrix}$$

$$J_{vi} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \textit{for revolute joint } i \\ z_{i-1} & \textit{for prismatic joint } i \end{cases}$$

Angular velocity Jacobian :

$$J_\omega = \begin{bmatrix} J_{\omega 1} & J_{\omega 2} & \ldots\ldots\ldots\ldots & J_{\omega n} \end{bmatrix}$$

$$J_{\omega i} = \begin{cases} z_{i-1} & \textit{for revolute joint } i \\ 0 & \textit{for prismatic joint } i \end{cases}$$

Angular Velocity

Angular velocity for different frames:

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_{1}^{0}\omega_{1,2}^{1} + R_{2}^{0}\omega_{2,3}^{2} + R_{3}^{0}\omega_{3,4}^{3} + \ldots\ldots + R_{n-1}^{0}\omega_{n-1,n}^{n-1}$$

$\omega_{1,2}^{1}$ ***denotes the angular velocity of frame 2 that corresponds to the changing*** R_{2}^{1} ***, express relative to the coordinate system frame 1.***

Thus the product of $R_{1}^{0}\omega_{1,2}^{1}$ ***expresses this angular velocity relative to the coordinate system frame 0.***

Angular velocity can be obtained from:

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + \omega_{1,2}^{0} + \omega_{2,3}^{0} + \omega_{3,4}^{0} + \ldots\ldots + \omega_{n-1,n}^{0}$$

Linear Velocity

Linear velocity:

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} q_i$$

This is linear velocity of the end effector that would result if $\dot{q}_i$ were equal to one and the other $\dot{q}_j$ were zero.

$$J_{vi} = \frac{\partial o_n^0}{\partial q_i} = \dot{o}_n^0$$

Combining Linear and Angular Velocity

Linear velocity Jacobian :

$$J_v = \begin{bmatrix} J_{v1} & J_{v2} & \ldots\ldots\ldots\ldots & J_{vn} \end{bmatrix}$$

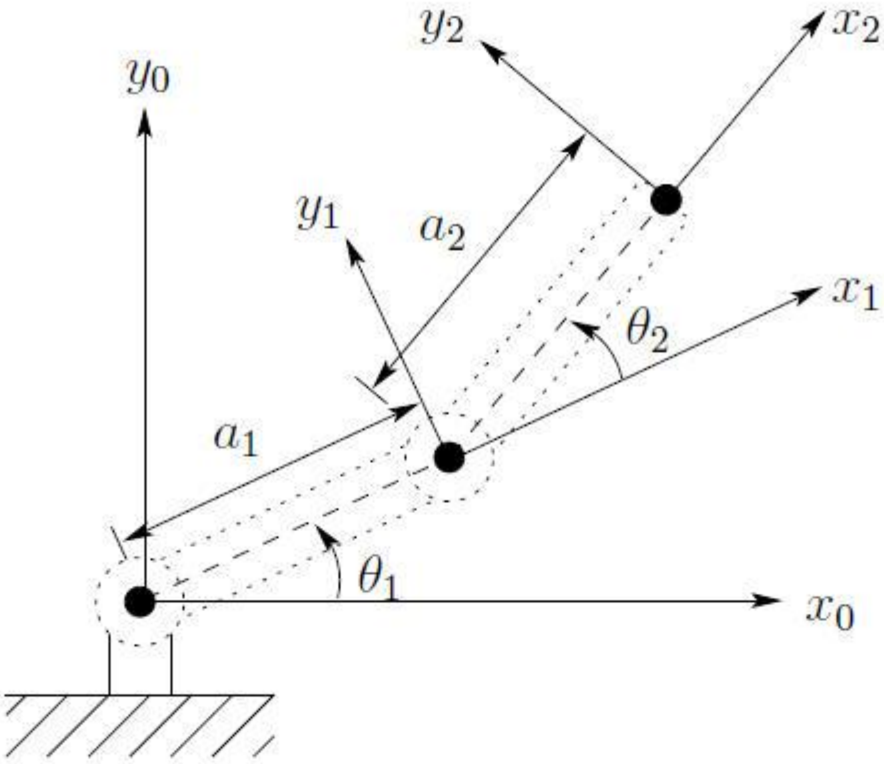
$$J_{vi} = \begin{cases} \left(o_n^0 - o_{i-1}^0\right) & \textit{for revolute joint } i \\ z_{i-1} & \textit{for prismatic joint } i \end{cases}$$

Angular velocity Jacobian :

$$J_\omega = \begin{bmatrix} J_{\omega 1} & J_{\omega 2} & \ldots\ldots\ldots\ldots & J_{\omega n} \end{bmatrix}$$

$$J_{\omega i} = \begin{cases} z_{i-1} & \textit{for revolute joint } i \\ 0 & \textit{for prismatic joint } i \end{cases}$$

Jacobian Analysis : 2 links manipulator



Tool velocity ***joint velocity***

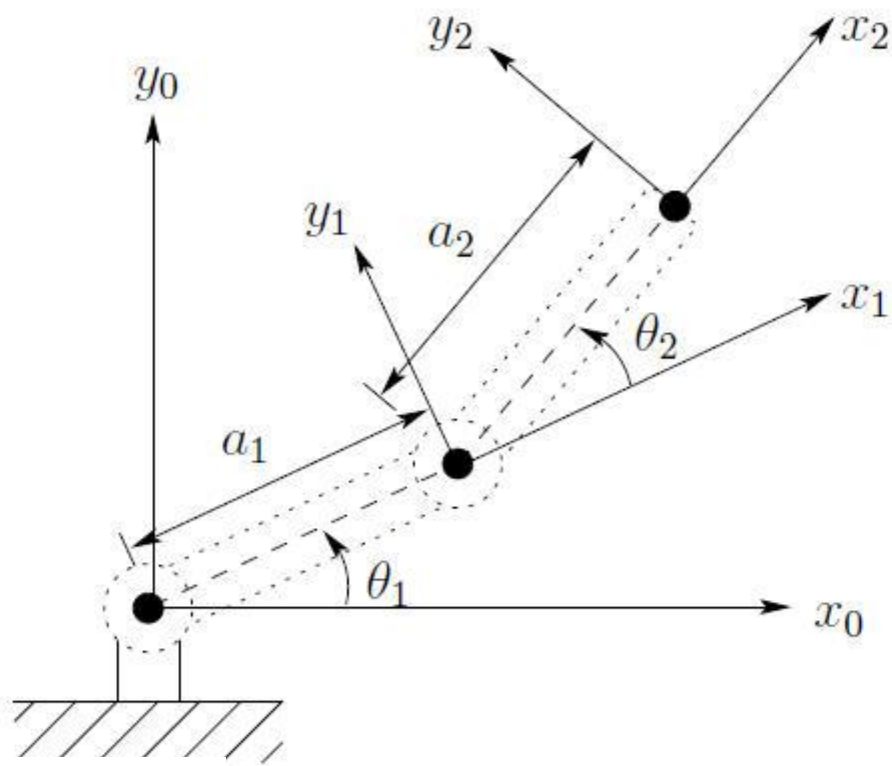
$$\dot{x} = J\,\dot{\theta}$$

$$[6\times1]=[6\times2][2\times1]$$

3 linear + 3 Revolute velocity (x,y,z)

Jacobian Analysis : 2 links manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*



$$
A_1 = \begin{array}{c} \begin{array}{cccc} & & Z_{01} & O_{01} \end{array} \\ \left[\begin{array}{cccc} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}
$$

$$
A_2 = \begin{array}{c} \begin{array}{cccc} & & Z_{12} & O_{12} \end{array} \\ \left[\begin{array}{cccc} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}
$$

$$
T_1^0 = A_1
$$

$$
T_2^0 = A_1 A_2 = \begin{array}{c} \begin{array}{cccc} \phantom{c_{12}} & \phantom{-s_{12}} & & O_{02} \end{array} \\ \left[\begin{array}{cccc} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}
$$

Jacobian Analysis : 2 links manipulator

R R robot

n=2

$$J_{vi} = \{ z_{i-1} \times (o_n - o_{i-1})$$

2 links, n =2: $J_v = \left[z_0 \times (o_2 - o_0) \quad z_1 \times (o_2 - o_1) \right]$

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \qquad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

state i=1

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_v = \left[z_0 \times (O_2 - O_0) \quad z_1 \times (O_2 - O_1) \right]$$

$$J_w = \left[z_0 \quad z_1 \right]$$

Jacobian Analysis : 2 links manipulator

$$J_v = \begin{bmatrix} z_0 \times \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix} & z_1 \times \begin{bmatrix} a_2c_{12} \\ a_2s_{12} \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial o_n^0}{\partial q_i} = \left(o_n^0 - o_{i-1}^0 \right) \end{bmatrix}$$

joint 1

$$= \begin{bmatrix} -a_1s_1 - a_2s_{12} & -a_2s_{12} \\ a_1c_1 + a_2c_{12} & a_2c_{12} \\ 0 & 0 \end{bmatrix}$$

$$J_{v2}\begin{bmatrix}\dfrac{\partial O^0_n}{\partial q_1} & \dfrac{\partial O^0_n}{\partial q_2}\end{bmatrix} = \begin{array}{cc} \theta_2 & \theta_1 \\ \left[\dfrac{d}{dq_1}\begin{pmatrix}a_1c_1+a_2c_{12}\\ a_1s_1+a_2s_{12}\\ 0\end{pmatrix}\right. & \left.\dfrac{d}{dq_2}\begin{pmatrix}a_2c_{12}\\ a_2s_{12}\\ 0\end{pmatrix}\right]\end{array}$$

joint 1 implicit

$\cos(\theta_1+\theta_2) = c_{12}$

$\cos\theta_1\cos\theta_2 = c_1c_2$

$\cos(\theta_1-\theta_2) = c_{1-2}$

$$\frac{\partial}{\partial q_1}\begin{pmatrix}a_1c_1+a_2c_1c_2+a_2s_1s_2\\ a_1s_1+a_2s_1s_2+a_2s_2c_1\\ 0\end{pmatrix} \qquad \frac{\partial}{\partial q_2}\begin{pmatrix}a_2c_1c_2-a_2s_1s_2\\ a_2s_1c_2-a_2c_1s_2\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-a_1s_1+a_2c_2(-s_1)-a_2c_1s_2 & a_2c_1(-s_2)-a_2s_1s_2\\ a_1c_1+a_2c_2c_1+a_2s_2(-s_1) & a_2s_1(-s_2)+a_2c_1c_2\\ 0 & 0\end{pmatrix}$$

$$j_v = \begin{pmatrix}-a_1s_1-a_2(s_{12}) & -a_2s_{12}\\ a_1c_1+a_2(c_{12}) & -a_2c_{12}\\ 0 & 0\end{pmatrix}$$

$$J = \begin{bmatrix}J_v\\ J_w\end{bmatrix} = \begin{bmatrix}-a_1s_1-a_2(s_{12}) & -a_2s_{12}\\ a_1c_1+a_2(c_{12}) & -a_2c_{12}\\ 0 & 0\\ \hline 0 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

$$j_w = \begin{bmatrix}z & z_1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

if $\theta_1 = x_1$

$\theta_2 = x_2$

Jacobian Analysis : 2 links manipulator

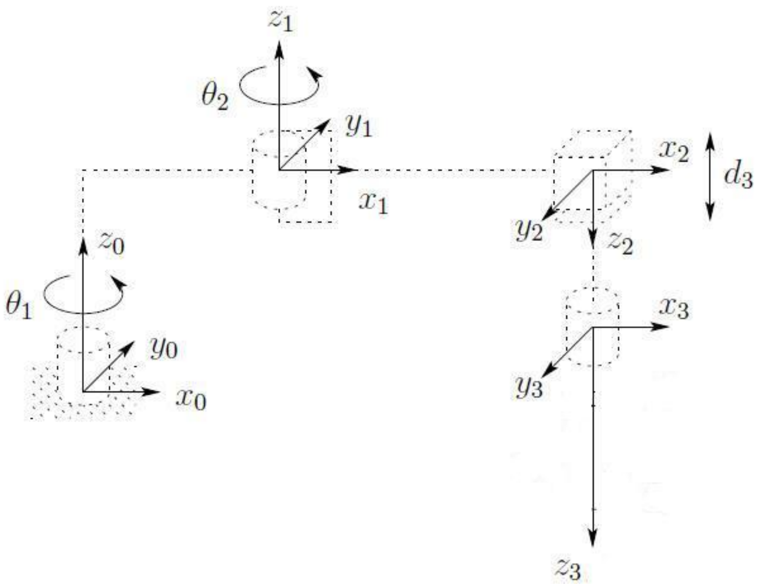
$$J_\omega = \begin{bmatrix} z_0 & z_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Jacobian Analysis : RRP (SCARA)



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^{\star}$
2	a_2	180	0	$\theta^{\star}$
3	0	0	d_3^*	0

Tool velocity *joint velocity*

$$\dot{x} = J\,\dot{\theta}$$

$$[6\times1] = [6\times3][3\times1]$$

3 linear + 3 Revolute velocity (x,y,z)

Jacobian Analysis : RRP (SCARA)

$$J_{vi} = \begin{cases} z_{i-1} \times \left(o_n - o_{i-1}\right) & (R) \\ z_{i-1} & (P) \end{cases}$$

3 links, n =3: $J_v = \left[z_0 \times \left(o_3 - o_0\right) \quad z_1 \times \left(o_3 - o_1\right) \quad z_2 \right]$

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \qquad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \qquad o_3 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ -d_3 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad z_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Jacobian Analysis : RRP (SCARA)

$$J_v = \begin{bmatrix} z_0 \times \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ -d_3 \end{bmatrix} & z_1 \times \begin{bmatrix} a_2c_{12} \\ a_2s_{12} \\ -d_3 \end{bmatrix} & z_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial o_n^0}{\partial q_i} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1s_1 - a_2s_{12} & -a_2s_{12} & 0 \\ a_1c_1 + a_2c_{12} & a_2c_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Jacobian Analysis : RRP (SCARA)

$$J_\omega = \begin{bmatrix} z_0 & z_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Jacobian Analysis : MATLAB

```
J= jacob0(Twolinks,[th1 th2])
```

Use jacob0 function
From matlab inly to find J value
Dont use Jacob0 function
To find Jequation for exam

Velocity Kinematics: Further Work

Tool velocity

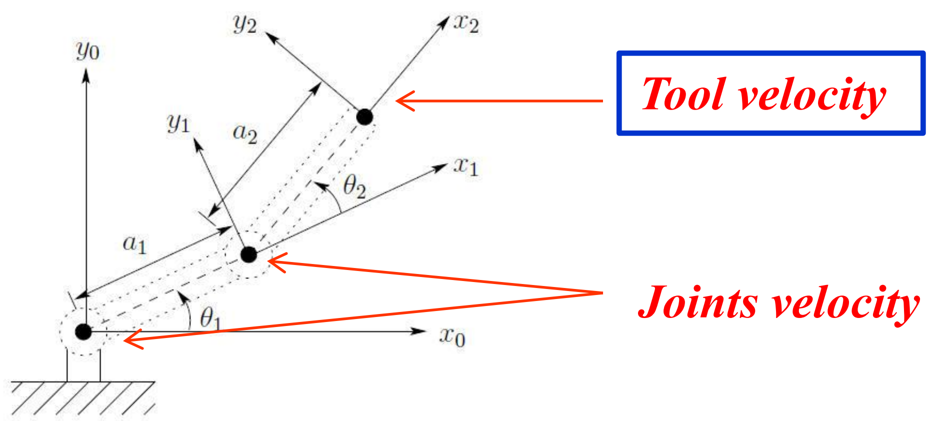
Joints velocity

$$\dot{x} = J\,\dot{\theta}$$

The joint velocities are found from the end-effector velocities via inverse Jacobian.

$$\dot{\theta} = J^{-1}\,\dot{x}$$

Our next problem is to determine what the joint velocity needed for the desired end effector velocity.



Tool Velocity

It is necessary to relate the velocity of the tool frame to the velocity of the end-effector frame.

$$\xi = J\,\dot{q}$$

$$where$$

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \quad and \quad J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

Assume frame 6 is end-effector, therefore $\xi_6^6 = \left[Mapping\right]\xi_{Tool}^{Tool}$

$$\xi_6^6 = \begin{bmatrix} R_{Tool}^6 & S(d_{Tool}^6)R_{Tool}^6 \\ 0 & R_{Tool}^6 \end{bmatrix} \xi_{Tool}^{Tool}$$

Prove of this equation can be read from Robot Modeling and Control, M. W. Spong, S. Hutchinson, M. Vidyasagar pg. 138-139