MCE4101
Robotic Engineering

Chapter 7

Inverse Kinematics (Analytic)

↳ analytic

↳ equation

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DH Table

equation

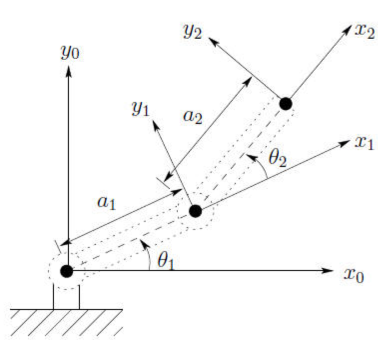
= compare

Inverse Kinematics

Inverse kinematic is to determine the joint variables of the manipulators given the position and orientation of the end effector. Simply, finding joints angles given x and y.

Solution is difficult to find and solution is not unique.

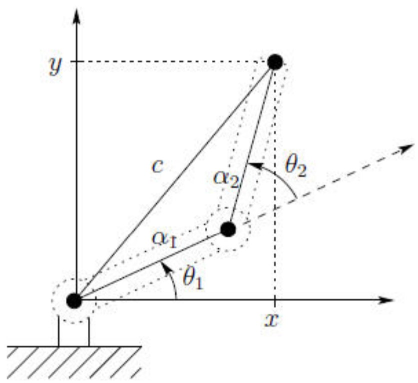
Usually there are 2 approach: Geometric and Analytic.



Inverse Kinematics

Following the figure, we wish to find joint angle θ_1 and θ_2 given x and y coordinates.

Now let go through Analytic approach.



Inverse Kinematics :Analytic

Following the figure, we wish to find joint angle θ_1 and θ_2 given x and y coordinates.

Now let go through Analytic approach.

Final orientation is given

$$T_2^0 = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & x \\ \sin(\phi) & \cos(\phi) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematic is known from DH table

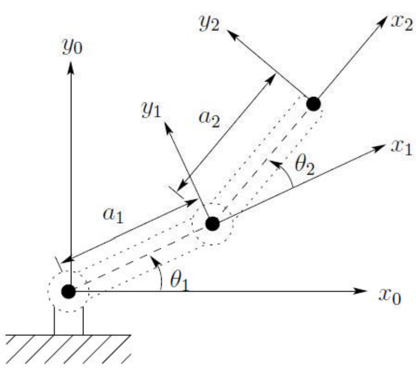
$$T_2^0 = \begin{bmatrix} \beta_1 & \beta_2 & 0 & M \\ \beta_3 & \beta_4 & 0 & N \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With comparison

$$\cos(\phi) = \beta_1, \qquad -\sin(\phi) = \beta_2$$
$$x = M\,, \quad y = N$$
$$Obtain\ \beta_1, \beta_2, M\ and\ N$$

Inverse Kinematics : Examples 2 links planar by Analytic approach

Following the figure, we wish to find joint angle θ_1 *and* θ_2 given x ,y coordinates and angle ϕ orientation (ϕ end location).



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

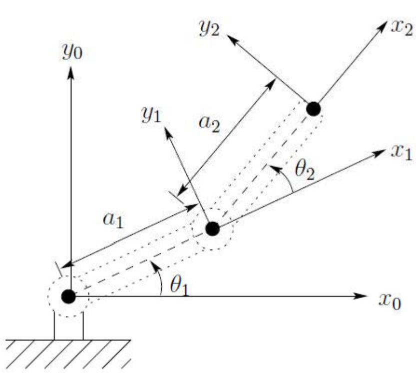
$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^0 = A_1$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics : Examples 2 links planar by Analytic approach

From Chapter 4 : Forward Kinematics, we know that



$$T_1^0 = A_1$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} \boxed{c_{12} \quad -s_{12}} & 0 & \boxed{a_1c_1 + a_2c_{12}} \\ s_{12} \quad c_{12} & 0 & \boxed{a_1s_1 + a_2s_{12}} \\ 0 \quad 0 & 1 & 0 \\ 0 \quad 0 & 0 & 1 \end{bmatrix}$$

$$Let\ \ T_2^0 = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & \boxed{x} \\ \sin(\phi) & \cos(\phi) & 0 & \boxed{y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where ϕ is end location and known
This case $\phi = \theta_1 + \theta_2$.

With comparison

$$\cos(\phi) = c_{12}, \qquad \sin(\phi) = s_{12}$$

$$x = a_1c_1 + a_2c_{12}, \quad y = a_1s_1 + a_2s_{12}$$

Inverse Kinematics : Examples 2 links planar by Analytic approach

$$From\ x = a_1c_1 + a_2c_{12},\ \ y = a_1s_1 + a_2s_{12}$$

$$\begin{aligned}
x^2 + y^2 &= (a_1c_1 + a_2c_{12})^2 + (a_1s_1 + a_2s_{12})^2 \\
&= (a_1c_1)^2 + 2a_1c_1a_2c_{12} + (a_2c_{12})^2 + (a_1s_1)^2 + 2a_1s_1a_2s_{12} + (a_2s_{12})^2 \\
&= a_1^2(c_1)^2 + a_1^2(s_1)^2 + a_2^2(c_{12})^2 + a_2^2(s_{12})^2 + 2a_1s_1a_2s_{12} + 2a_1c_1a_2c_{12} \\
&= a_1^2 + a_2^2 + 2a_1s_1a_2(s_1c_2 + c_1s_2) + 2a_1c_1a_2(c_1c_2 - s_1s_2) \\
&= a_1^2 + a_2^2 + 2a_1a_2s_1s_1c_2 + 2a_1a_2s_1s_2c_1 + 2a_1a_2c_1c_1c_2 - 2a_1a_2s_1s_2c_1 \\
&= a_1^2 + a_2^2 + 2a_1a_2(s_1s_1 + c_1c_1)c_2 \\
&= a_1^2 + a_2^2 + 2a_1a_2c_2
\end{aligned}$$

$$\frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} = c_2$$

$$\boxed{\Rightarrow \cos(\theta_2) = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}}$$

Inverse Kinematics : Examples 2 links planar by Analytic approach

$$\cos(\theta_2) = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} := D$$

$$\sin(\theta_2) = \pm\sqrt{1 - D^2}$$

$$\boxed{\theta_2 = A\tan 2(D, \pm\sqrt{1 - D^2})}$$

Inverse Kinematics : Examples 2 links planar by Analytic approach

$$\begin{aligned}
&Back\ to: \quad x = a_1c_1 + a_2c_{12} &&, \quad y = a_1s_1 + a_2s_{12} \\
&x = a_1c_1 + a_2(c_1c_2 - s_1s_2) &&, \quad y = a_1s_1 + a_2(s_1c_2 + c_1s_2) \\
&x = a_1c_1 + a_2c_1c_2 - a_2s_1s_2 &&, \quad y = a_1s_1 + a_2s_1c_2 + a_2c_1s_2 \\
&x = c_1(a_1 + a_2c_2) - (a_2s_2)s_1 &&, \quad y = s_1(a_1 + a_2c_2) + c_1(a_2s_2)
\end{aligned}$$

$Let\ k_1 = a_1 + a_2c_2 \quad and \quad k_2 = a_2s_2$ (*refer to Lecture 6 pg.39*)

$$\boxed{\Rightarrow Reform\ as \quad x = k_1c_1 - k_2s_1,\ y = k_1s_1 + k_2c_1}$$

$IF \quad r = \sqrt{k_1^2 + k_2^2} \quad so \quad \gamma = A\tan 2(k_1, k_2)$ r can refer as c in Lecture 6

$THEN \quad k_1 = r\cos\gamma \quad and \quad k_2 = r\sin\gamma$ k_1 and k_2 is known now

Inverse Kinematics : Examples 2 links planar by Analytic approach

From

$$x = k_1c_1 - k_2s_1,\ y = k_1s_1 + k_2c_1 \quad and \quad k_1 = r\cos\gamma \quad and \quad k_2 = r\sin\gamma$$

$$Now \quad x = r\cos\gamma c_1 - r\sin\gamma s_1,\ y = r\cos\gamma s_1 + r\sin\gamma c_1$$

$$x = r\cos(\gamma + \theta_1),\ y = r\sin(\gamma + \theta_1)$$

$$\therefore \quad \cos(\gamma + \theta_1) = \frac{x}{r},\ \sin(\gamma + \theta_1) = \frac{y}{r}$$

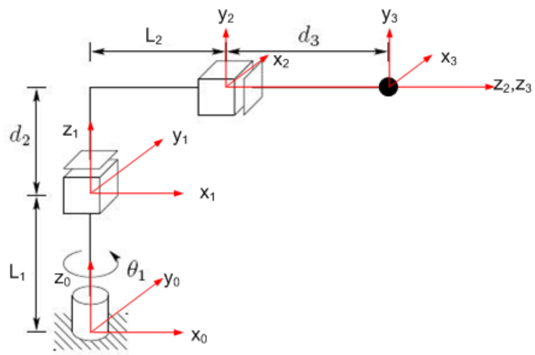
$$\gamma + \theta_1 = A\tan 2(\frac{x}{r}, \frac{y}{r}) = A\tan 2(x, y)$$

$$So \quad \theta_1 = A\tan 2(x, y) - \gamma$$

$$= A\tan 2(x, y) - A\tan 2(k_1, k_2)$$

$$\boxed{\begin{gathered}\theta_1 = A\tan 2(x, y) - A\tan 2(k_1, k_2) \quad where\ k_1 = a_1 + a_2c_2 \quad and \quad k_2 = a_2s_2 \\ \Rightarrow \theta_1 = A\tan 2(x, y) - A\tan 2(a_1 + a_2c_2, a_2s_2)\end{gathered}}$$

Inverse Kinematics : Examples 3 links RPP Analytic approach



Link	α	a	d	θ
1	0	0	L_1	θ_1^*
2	90°	0	d_2^*	90°
3	0	0	$L_2+d_3^*$	0

```
T03 =
[ -sin(th1), 0, cos(th1), cos(th1)*(L2 + d3)]
[  cos(th1), 0, sin(th1), sin(th1)*(L2 + d3)]
[         0, 1,        0,            L1 + d2]
[         0, 0,        0,                  1]
```

Inverse Kinematics : Examples 3 links RPP Analytic approach

```
T03 =
[ -sin(th1), 0, cos(th1), cos(th1)*(L2 + d3)]
[  cos(th1), 0, sin(th1), sin(th1)*(L2 + d3)]
[         0, 1,        0,            L1 + d2]
[         0, 0,        0,                  1]
```

→ P(x,y,x)

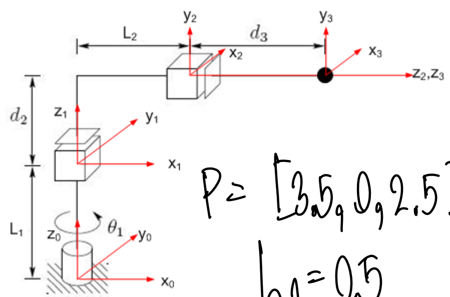
```
R03 =
[ -sin(th1), 0, cos(th1)]
[  cos(th1), 0, sin(th1)]
[         0, 1,        0]
```

→ θ_1^* value

***After obtain the variables, you should check your answer with forward kinematic process.*

Inverse Kinematics : Examples 3 links RPP
Analytic approach

Find the required values for θ_1^*, d_2^*, d_3^*, if desire position is P(3.5,0,2.5), with $L_1 = 0.5, L_2 = 1.5$



Link	α	a	d	θ
1	0	0	L_1	θ_1^*
2	90°	0	d_2^*	90°
3	0	0	$L_2 + d_3^*$	0

$P = [3.5, 0, 2.5]$

$L_1 = 0.5$

```
T03 =
[ -sin(th1), 0, cos(th1), cos(th1)*(L2 + d3)]
[  cos(th1), 0, sin(th1), sin(th1)*(L2 + d3)]
[         0, 1,        0,            L1 + d2]
[         0, 0,        0,                  1]
```

→ P(3.5,0,2.5)

from (3)
$L_1 + d_2 = 2.5$
$d_2 = 2$

from(1) $d_{L3} \cos\theta_1 = 3.5$ (x)
(2) $d_{L3} \sin\theta_1 = 0$ (y)
atan2d(3.5, 0)
0

$\theta_1 = 0°$
from(1) $d_{L3} \cos(0) = 3.5$
$d_{L3} = 3.5$
$L_2 + d_3^* = 3.5$
$d_3^* = 2$

Inverse Kinematics : Examples 3 links RPP
Analytic approach

with $L_1 = 0.5, L_2 = 1.5$

$L_1 + d_2^* = 2.5$

$d_2^* = 2.5 - L_1 = 2$

$\cos\theta_1 (L_2 + d_3^*) = 3.5$

$\sin\theta_1 (L_2 + d_3^*) = 0$

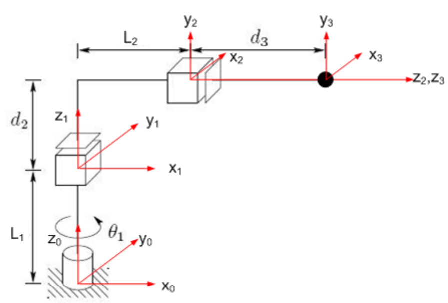
$\theta_1 = A\tan 2(3.5, 0)$

$\theta_1^* = 0°$

with $\theta_1^* = 0°$

$(L_2 + d_3^*) = 3.5$

$d_3^* = 3.5 - L_2 = 2$



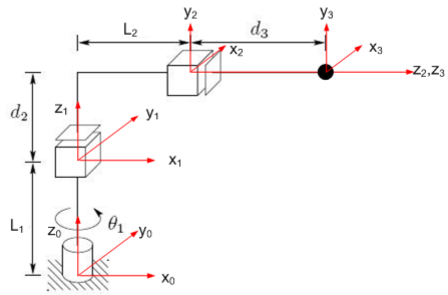
To obtain P(3.5,0,2.5), we could set $\theta_1^* = 0°$, $d_2^* = 2$, $d_3^* = 2$

**check your answer with forward kinematic process.*

Inverse Kinematics : Examples 3 links RPP Analytic approach

$with\ L_1 = 0.5, L_2 = 1.5$

Link	α	a	d	θ
1	0	0	L_1	θ_1^*
2	90°	0	d_2^*	90°
3	0	0	$L_2 + d_3^*$	0

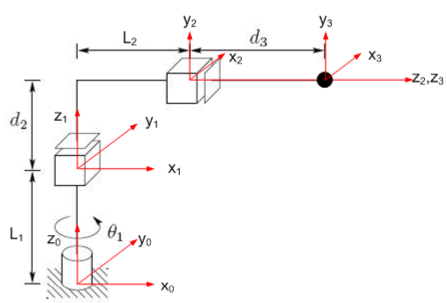


$\theta_1^* = 0°, d_2^* = 2, d_3^* = 2$

Use Forward kinematic obtain the end position

Inverse Kinematics : Examples 3 links RPP Analytic approach (Repeat for different position)

Find the required values for θ_1^*, d_2^*, d_3^*, if desire position is P(1,1.732,2), with $L_1 = 0.5, L_2 = 1.5$



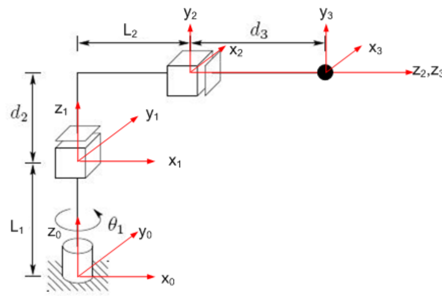
Link	α	a	d	θ
1	0	0	L_1	θ_1^*
2	90°	0	d_2^*	90°
3	0	0	$L_2 + d_3^*$	0

```
T03 =
[ -sin(th1), 0, cos(th1), cos(th1)*(L2 + d3)]
[  cos(th1), 0, sin(th1), sin(th1)*(L2 + d3)]
[         0, 1,        0,            L1 + d2]
[         0, 0,        0,                  1]
```

P(1,1.732,2)

Inverse Kinematics : Examples 3 links RPP Analytic approach

$with\ L_1 = 0.5, L_2 = 1.5$



To obtain P(1,1.732,2), we could set

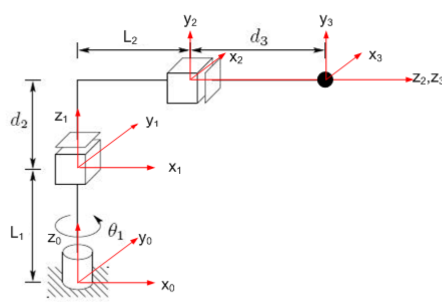
$$\theta_1^* =$$
$$d_2^* =$$
$$d_3^* =$$

**check your answer with forward kinematic process.*

Inverse Kinematics : Examples 3 links RPP Analytic approach

$with\ L_1 = 0.5, L_2 = 1.5$

Link	α	a	d	θ
1	0	0	L_1	θ_1^*
2	90°	0	d_2^*	90°
3	0	0	$L_2 + d_3^*$	0



$\theta_1^* = \ °, d_2^* = \ , d_3^* =$

Use Forward kinematic obtain the end position