MT411
Robotic Engineering

Chapter 6

Inverse Kinematics (Geometric)

drawing
get equation

geometric — chapter 6

analysis — chapter 7

programming
DH Table

Narong Aphiratsakun, D.Eng

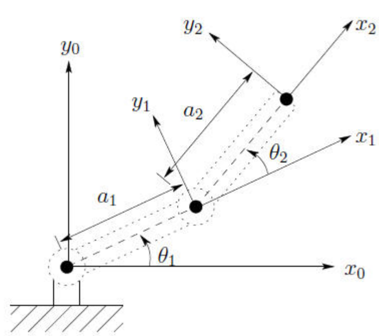
Assumption University
Faculty of Engineering

Inverse Kinematics

Inverse kinematic is to determine the joint variables of the manipulators given the position and orientation of the end effector. Simply, finding joints angles given x and y.

Solution is difficult to find and solution is not unique.

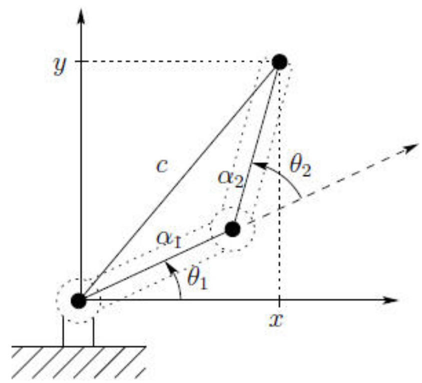
Usually there are 2 approach: Geometric and Algebraic.



Inverse Kinematics

Following the figure, we wish to find joint angle θ_1 *and* θ_2 given x and y coordinates.

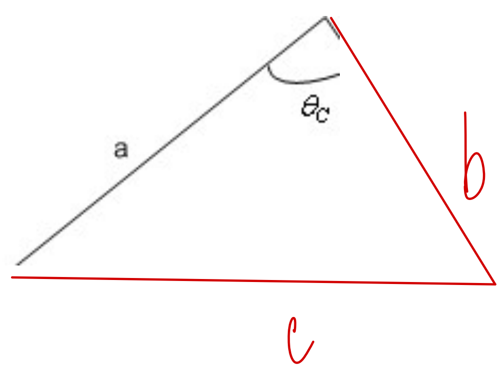
First let focus on **Geometric** approach.



Cosine Law

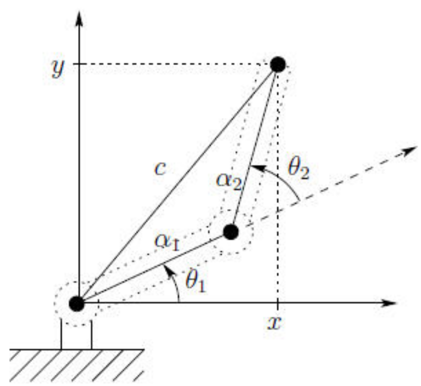
With cosine law

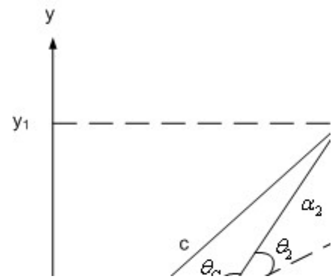
$$c^2 = a^2 + b^2 - 2ab\cos\theta_c$$



Inverse Kinematics : Examples 2 links planar by Geometric approach

Following the figure, we wish to find joint angle θ_1 *and* θ_2 given x and y coordinates.

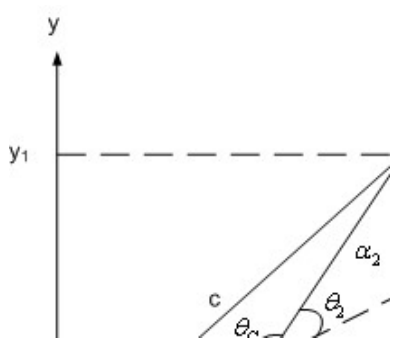




$$c^2 = \alpha_1^{\,2} + \alpha_2^{\,2} - 2\alpha_1\alpha_2\cos\theta_c$$
$$\theta_c + \theta_2 = \pi$$

**First obtain θ_2

Inverse Kinematics : Examples 2 links planar by Geometric approach



$$c^2 = \alpha_1^{\,2} + \alpha_2^{\,2} - 2\alpha_1\alpha_2\cos\theta_c$$
$$\theta_c + \theta_2 = \pi$$
$$c^2 = \alpha_1^{\,2} + \alpha_2^{\,2} - 2\alpha_1\alpha_2\cos(\pi - \theta_2)$$
$$\cos(\pi - \theta_2) = -\cos(\theta_2)$$
$$\therefore c^2 = \alpha_1^{\,2} + \alpha_2^{\,2} + 2\alpha_1\alpha_2\cos(\theta_2)$$

$$\cos(\theta_2) = \frac{c^2 - \alpha_1^{\,2} - \alpha_2^{\,2}}{2\alpha_1\alpha_2} = \frac{x^2 + y^2 - \alpha_1^{\,2} - \alpha_2^{\,2}}{2\alpha_1\alpha_2}$$

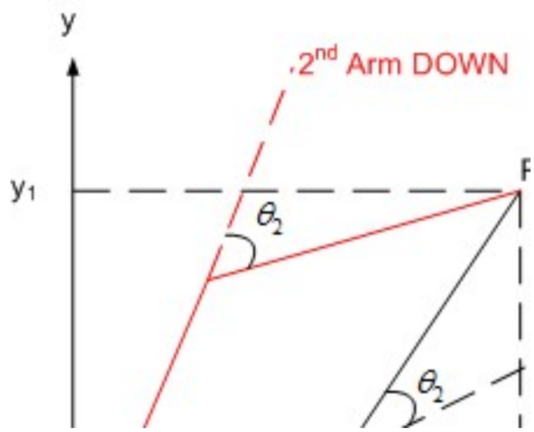
$$\cos(\theta_2) = \frac{x^2 + y^2 - \alpha_1^{\,2} - \alpha_2^{\,2}}{2\alpha_1\alpha_2} = D$$

$$\cos^2(\theta_2) + \sin^2(\theta_2) = 1$$
$$\sin(\theta_2) = \pm\sqrt{1 - D^2}$$

$$\tan(\theta_2) = \frac{\sin(\theta_2)}{\cos(\theta_2)} = \frac{\pm\sqrt{1-D^2}}{D}$$

$$\boxed{\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-D^2}}{D}}$$

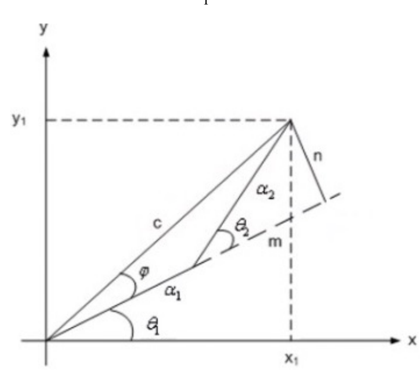
Inverse Kinematics : Examples 2 links planar by Geometric approach



$$\theta_2 = \tan^{-1}\frac{+\sqrt{1-D^2}}{D} \quad or \quad \theta_2 = \tan^{-1}\frac{-\sqrt{1-D^2}}{D}$$

Inverse Kinematics : Examples 2 links planar by Geometric approach

**Now obtain θ_1



$$\omega = \varphi + \theta_1 \qquad \therefore \theta_1 = \omega - \varphi$$

$$c\cos(\omega) = x_1$$
$$c\sin(\omega) = y_1$$

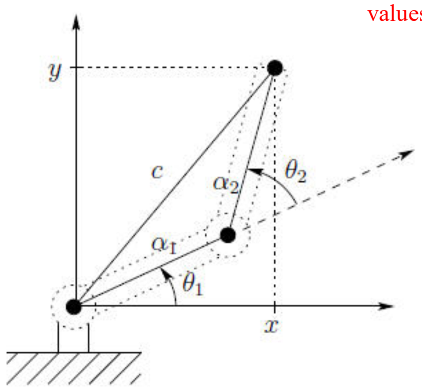
$$\tan(\omega) = \frac{y_1}{x_1}$$

$$c\cos(\varphi) = \alpha_1 + m \quad where \quad m = \alpha_2\cos(\theta_2)$$
$$c\sin(\varphi) = n \quad where \quad n = \alpha_2\sin(\theta_2)$$

$$\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)} = \frac{\alpha_2\sin(\theta_2)}{\alpha_1 + \alpha_2\cos(\theta_2)}$$

$$\boxed{\begin{aligned}\theta_1 &= \omega - \varphi \\ &= \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{\alpha_2\sin(\theta_2)}{\alpha_1 + \alpha_2\cos(\theta_2)}\right)\end{aligned}}$$

Inverse Kinematics : Examples 2 links planar by Geometric approach



Ex: Let $\alpha_1 = 3$ and $\alpha_2 = 5$, obtain the possible values of θ_1 and θ_2 for point(7.5,2).

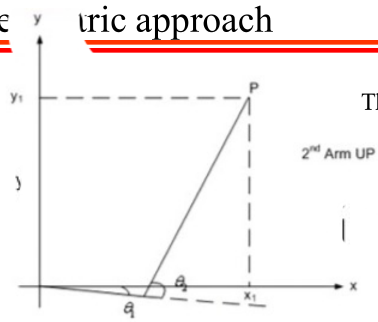
$$D = \frac{7.5^2 + 2^2 - 3^2 - 5^2}{2(3*5)} = \frac{26.25}{30} = 0.875$$

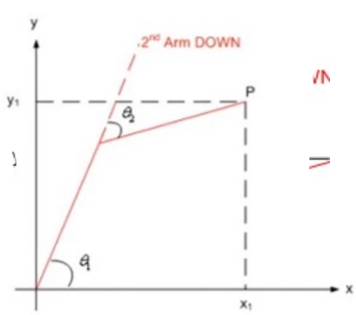
$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-0.875^2}}{0.875} = \pm 28.96$$

2 links robot to get to (7.5,2)

$\alpha_1 = 3 \quad \alpha_2 = 5$

Inverse Kinematics : Examples 2 links planar by Geometric approach





There are 2 cases: $\theta_2 = \pm 28.96°$

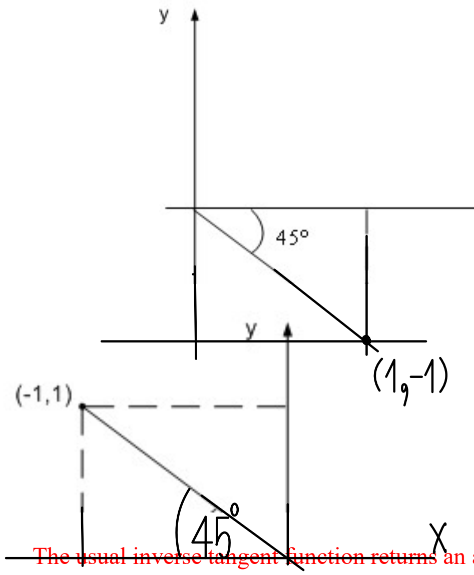
If $\theta_2 = +28.96°$ is chosen

$$\begin{aligned}\theta_1 &= \tan^{-1}\left(\frac{2}{7.5}\right) - \tan^{-1}\left(\frac{5\sin(28.96)}{3+5\cos(28.96)}\right) \\ &= \tan^{-1}(0.27) - \tan^{-1}(0.328) \\ &= 15.11° - 18.16° \\ &= -3.05°\end{aligned}$$

If $\theta_2 = -28.96°$ is chosen

$$\begin{aligned}\theta_1 &= \tan^{-1}\left(\frac{2}{7.5}\right) - \tan^{-1}\left(\frac{5\sin(-28.96)}{3+5\cos(-28.96)}\right) \\ &= \tan^{-1}(0.27) - \tan^{-1}(-0.328) \\ &= 15.11° + 18.16° \\ &= 33.27°\end{aligned}$$

The Arctangent Function: Atan



$$\tan(\theta) = \frac{-1}{1}$$

$$\theta = A\tan\frac{-1}{1} = -45°$$

$$\tan(\theta) = \frac{1}{-1}$$

$$\theta = A\tan\frac{1}{-1} = -45°$$

The usual inverse tangent function returns an angle in the range of -90° and +90°.

The Two-Argument Arctangent Function: Atan2

The usual inverse tangent function returns an angle in the range of -90° and +90°.

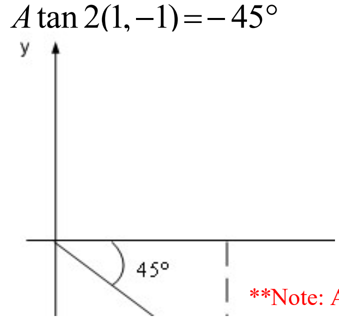
The two-argument arctangent function, ***Atan2(x,y)***, is useful to define the full range of angles (range of -180° and +180°).

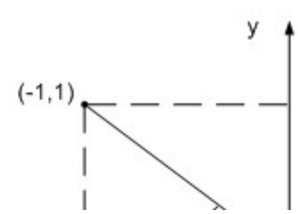
This function uses the signs of x and y to select the **appropriate quadrant** for the angle θ.

$$\boxed{\theta_2 = A\tan 2(x,y) = A\tan 2(D, \pm\sqrt{1-D^2})}$$

$A\tan 2(1,-1) = -45°$

$A\tan 2(-1,1) = 135°$





Note: Atan2 is undefined when **both x and y are zero.

The Two-Argument Arctangent Function: Atan2

In this class, we use the following syntax for Atan2:

$$\theta_2 = A\tan 2(x, y) \;= A\tan 2(D, \pm\sqrt{1-D^2})$$

matlab

In some text book, syntax for Atan2:

$$\theta_2 = A\tan 2(y, x) \;= A\tan 2(\pm\sqrt{1-D^2}, D)$$

```
^2-a1^2-a2^2)/(2*a1*a2)
((sqrt(1-D^2)),D)
```

function:

$\theta_2 = A\tan 2(D, \pm\sqrt{1-D^2})$

$-A\tan 2(a_1 + a_2\cos(\theta_2), a_2\sin(\theta_2))$

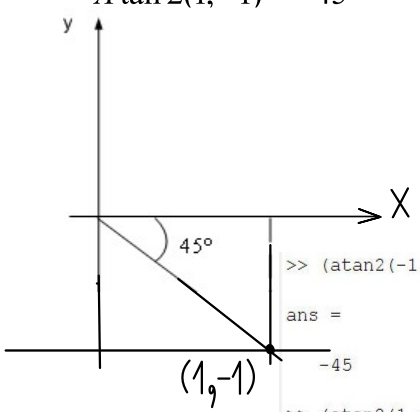
$\theta_2 = \text{Atan 2d}\,(\underset{x}{D}, \underset{y}{\pm\sqrt{1-D^2}})$

$\theta_2 = \text{Atan 2d}$. Matlab

$((\pm\sqrt{1-D^2}), D)$

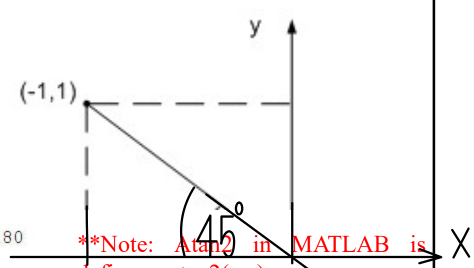
The Two-Argument Arctangent Function: Atan2

$A\tan 2(1,-1) = -45°$




```
>> (atan2(-1,1))/pi*180

ans =

   -45

>> (atan2(1,-1))/pi*180

ans =

   135
```

$A\tan 2(-1,1) = 135°$



**Note: Atan2 in MATLAB is define as atan2(y,x).

Inverse Kinematics : using Atan2

From previous

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-D^2}}{D} \quad \text{and} \quad \theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{\alpha_2\sin(\theta_2)}{\alpha_1+\alpha_2\cos(\theta_2)}\right)$$

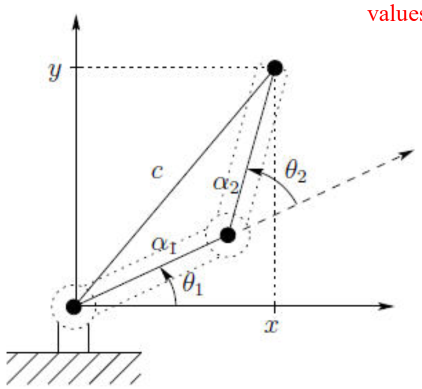
With two-argument arctangent function:

$$\boxed{\theta_2 = A\tan 2(D, \pm\sqrt{1-D^2})}$$

$$\boxed{\theta_1 = A\tan 2(x_1, y_1) - A\tan 2(\alpha_1+\alpha_2\cos(\theta_2), \alpha_2\sin(\theta_2))}$$

Inverse Kinematics : Examples 2 links planar with Atan2 by Geometric approach

Ex: Let $\alpha_1 = 3$ and $\alpha_2 = 5$, obtain the possible values of θ_1 and θ_2 for point(7.5,2).

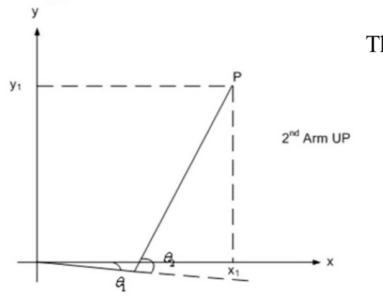


$$D = \frac{7.5^2+2^2-3^2-5^2}{2(3*5)} = \frac{26.25}{30} = 0.875$$

$$\theta_2 = A\tan 2(D, \pm\sqrt{1-D^2})$$

$$\theta_2 = A\tan 2(0.875, \pm 0.484)$$
$$= +28.95° \quad and \quad -28.95°$$

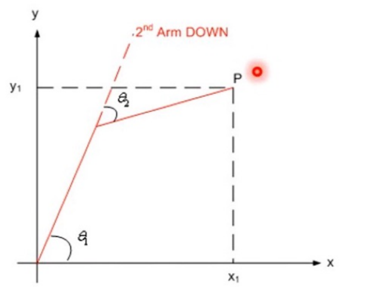
Inverse Kinematics : Examples 2 links planar with Atan2 by Geometric approach



There are 2 cases: $\theta_2 = \pm 28.96°$

If $\theta_2 = +28.96°$ is chosen

$$\theta_1 = A\tan 2(x_1, y_1) - A\tan 2(\alpha_1 + \alpha_2\cos(\theta_2), \alpha_2\sin(\theta_2))$$
$$\theta_1 = A\tan 2(7.5, 2) - A\tan 2(3 + 5\cos(28.96), 5\sin(28.96))$$
$$= 15.11° - 18.16°$$
$$= -3.05°$$

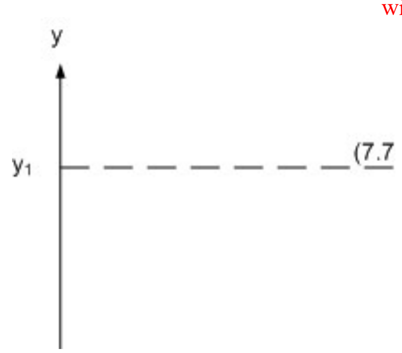


If $\theta_2 = -28.96°$ is chosen

$$\theta_1 = A\tan 2(x_1, y_1) - A\tan 2(\alpha_1 + \alpha_2\cos(\theta_2), \alpha_2\sin(\theta_2))$$
$$\theta_1 = A\tan 2(7.5, 2) - A\tan 2(3 + 5\cos(-28.96), 5\sin(-28.96))$$
$$= 15.11° + 18.16°$$
$$= 33.27°$$

Class Exercise

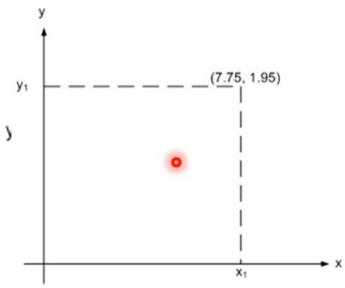
Ex: Let α_1 = 3 and α_2 = 5, obtain the possible values of θ_1 and θ_2 for point(7.75,1.95). Compute with using Atan2 function.



$$D = 0.996 \ , \ \sqrt{1 - D^2} = 0.090$$

$$\theta_2 = \pm 5.13°$$

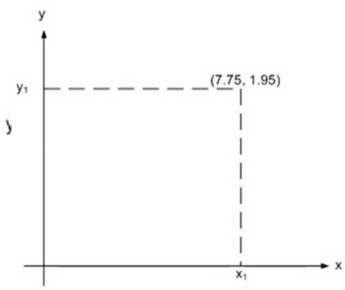
Inverse Kinematics : Examples 2 links planar



There are 2 cases: $\theta_2 = \pm 5.13°$

If $\theta_2 = +5.13°$ is chosen

$$\theta_1 = 14.123° - 3.41° = 10.712°$$

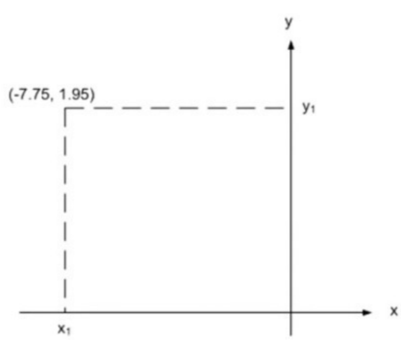


If $\theta_2 = -5.13°$ is chosen

$$\theta_1 = 14.123° + 3.41° = 17.57°$$

Class Exercise

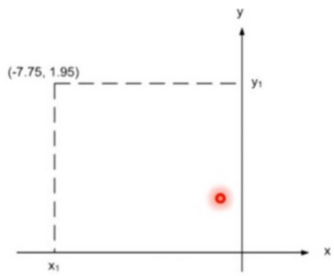
Ex: Let $\alpha_1 = 3$ and $\alpha_2 = 5$, obtain the possible values of θ_1 and θ_2 for point(-7.75,1.95). Compute with using Atan2 function.



$$D = 0.996 \ , \ \sqrt{1 - D^2} = 0.090$$

$$\theta_2 = \pm 5.13°$$

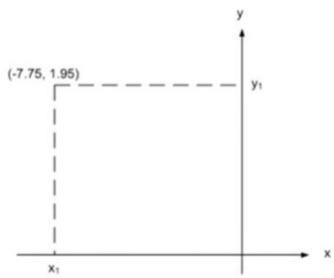
Inverse Kinematics : Examples 2 links planar



There are 2 cases:

If $\theta_2 = +5.13°$ is chosen

$$\theta_1 = -14.123° - 3.41° = -17.533°$$
$$= 162.467°$$

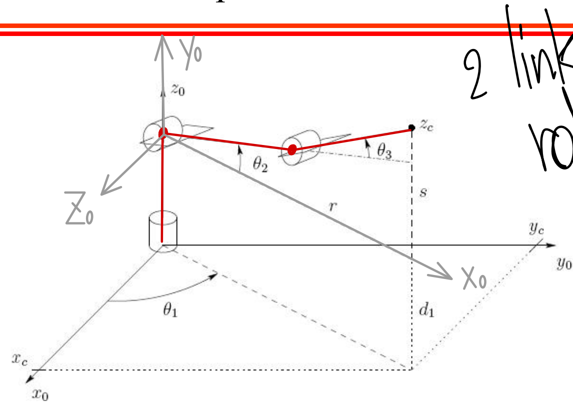


If $\theta_2 = -5.13°$ is chosen

$$\theta_1 = -14.123° + 3.41° = -10.713°$$
$$= 169.287°$$

Inverse Kinematics Examples : RRR

2 links robot

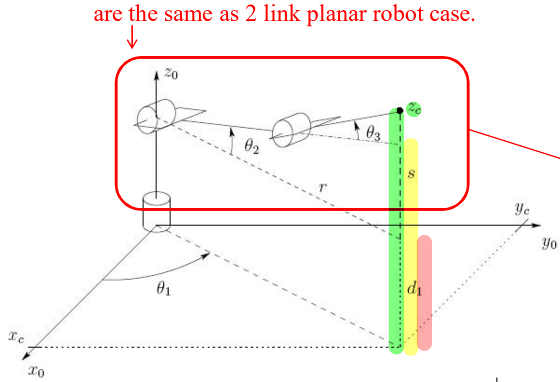


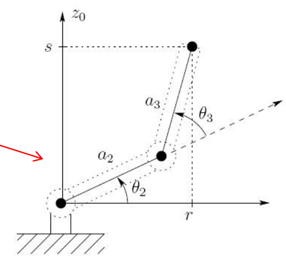
Obtain θ_1, θ_2 *and* θ_3

RRR Case 1

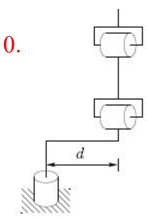
Inverse Kinematics Examples : RRR Case1 : d = 0

are the same as 2 link planar robot case.





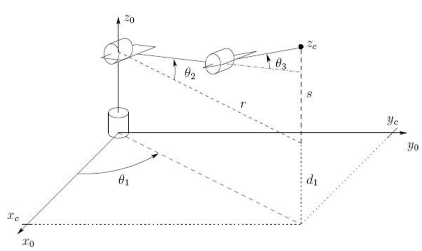
Case 1: No offset, d = 0.

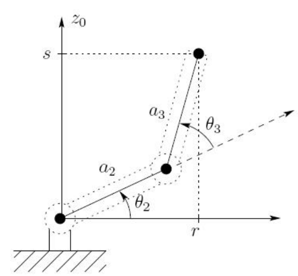


$$r = \sqrt{x_c^2 + y_c^2}$$

$$s = z_c - d_1$$

Inverse Kinematics Examples : RRR Case1 : d = 0





$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$= \frac{x_c^2 + y_c^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

$$r^2 = x_c^2 + y_c^2, \; s = z_c - d_1$$

$$\boxed{\theta_3 = A\tan 2(D, \pm\sqrt{1 - D^2})}$$ x,y → matlab th3= atan2d($\sqrt{1-D^2}$, D)

$$\boxed{\theta_2 = A\tan 2\left(\sqrt{x_c^2 + y_c^2}, z_c - d_1\right) - A\tan 2\left(a_2 + a_3\cos(\theta_3), a_3\sin(\theta_3)\right)}$$

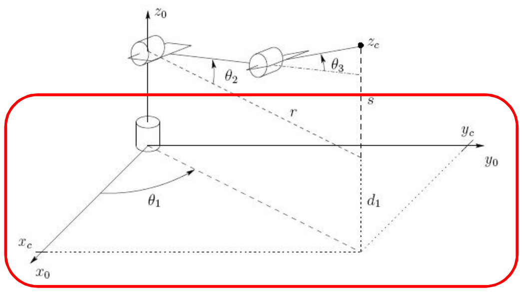
r s

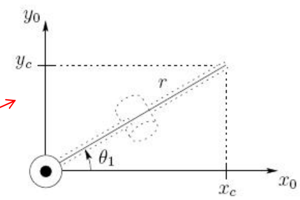
↓ matlab

th2 = atan2d(zc-d1, $\sqrt{x_c^2+y_c^2}$) -atan2d(a3sindθ3, a2+a3cosdθ3)

s r

Inverse Kinematics Examples : RRR Case1 : d = 0





$$\boxed{\theta_1 = A\tan 2(x_c, y_c)}$$

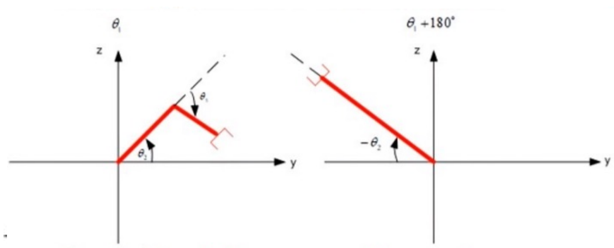
Other valid solution is $\theta_1 = A\tan 2(x_c, y_c) + \pi$

But θ_2 *and* θ_3 *will be differnt answer that previosuly defined*

Inverse Kinematics Examples : RRR Case1 : d = 0

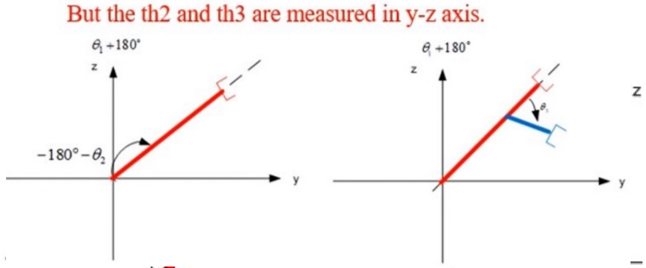
if $\boxed{\theta_1 = A\tan 2(x_c, y_c) + \pi}$

This mean Robot is turn 180 deg to the other quadrant.



$$\boxed{\theta_2 = -180° + \theta_2}$$

But the th2 and th3 are measured in y-z axis.

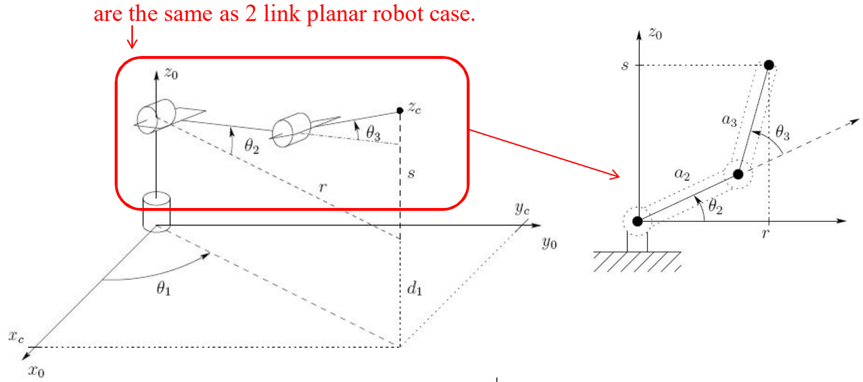


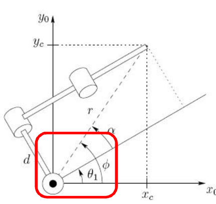
$$\boxed{\theta_3 = -1 * \theta_3}$$

RRR Case 2

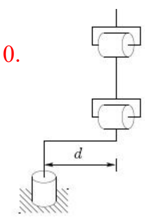
Inverse Kinematics Examples : RRR Case2 : d ≠ 0

are the same as 2 link planar robot case.





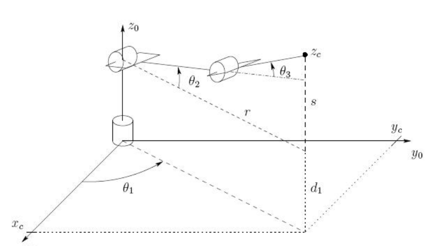
Case 2: Offset, d ≠ 0.

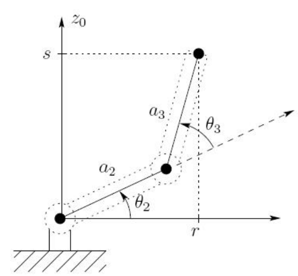


$$r = \sqrt{x_c^2 + y_c^2 - d^2}$$

$$s = z_c - d_1$$

Inverse Kinematics Examples : RRR Case2 : d ≠ 0





$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

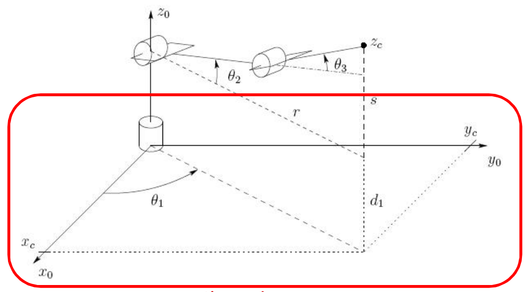
$$= \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

$$r^2 = x_c^2 + y_c^2 - d^2, s = z_c - d_1$$

$$\boxed{\theta_3 = A\tan 2(D, \pm\sqrt{1 - D^2})}$$

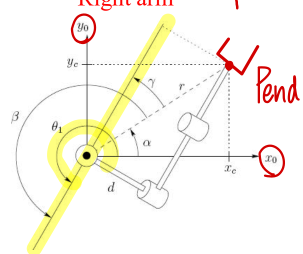
$$\boxed{\theta_2 = A\tan 2\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - A\tan 2\left(a_2 + a_3\cos(\theta_3), a_3\sin(\theta_3)\right)}$$

Inverse Kinematics Examples : RRR Case2 : d ≠ 0

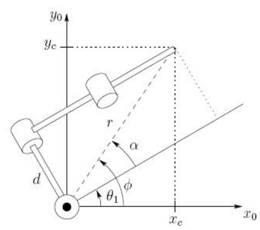


end point shift by d

Right arm

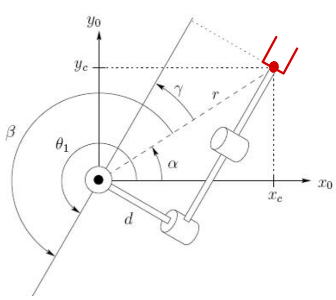


Left arm



Inverse Kinematics Examples : RRR Case2 : d ≠ 0

Right arm configuration



$$\theta_1 = \alpha + \beta$$

$$\alpha = A\tan 2(x_c, y_c)$$

$$\beta = \gamma + \pi$$

$$where \;\; \gamma = A\tan 2(\sqrt{r^2 - d^2}, d)$$

$$\therefore \beta = A\tan 2(-\sqrt{r^2 - d^2}, -d)$$

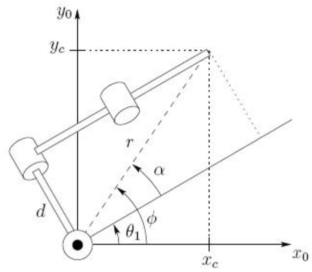
$$= A\tan 2(-\sqrt{x_c^2 + y_c^2 - d^2}, -d)$$

$$as \cos(\theta + \pi) = -\cos\theta \;\; and \;\; \sin(\theta + \pi) = -\sin\theta$$

$$\boxed{\theta_1 = A\tan 2(x_c, y_c) + A\tan 2(-\sqrt{x_c^2 + y_c^2 - d^2}, -d)}$$

Inverse Kinematics Examples : RRR Case2 : d ≠ 0

Left arm configuration



$$\theta_1 = \phi - \alpha$$

$$\phi = A\tan 2(x_c, y_c)$$

$$\alpha = A\tan 2(\sqrt{r^2 - d^2}, d)$$

$$= A\tan 2(\sqrt{x_c^2 + y_c^2 - d^2}, d)$$

$$\boxed{\theta_1 = A\tan 2(x_c, y_c) - A\tan 2(\sqrt{x_c^2 + y_c^2 - d^2}, d)}$$

case1 $d=0$

$\boxed{\begin{array}{l}\theta_{3a}, \theta_{3b} \\ \theta_{2a} \\ \theta_{1a}\end{array}} \xrightarrow{\theta_{1a}+\pi} \boxed{\begin{array}{c}\theta_{3c}, \theta_{3d} \\ \theta_{2b} \\ \theta_{1b}\end{array}}$

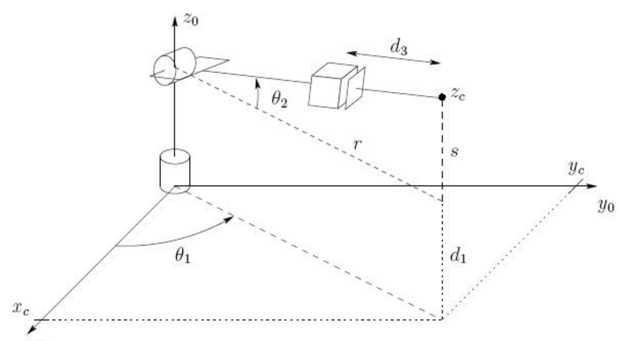
case2 $d \neq 0$

θ_{3a}, θ_{3b}

θ_2

RA θ_1 LA θ_1

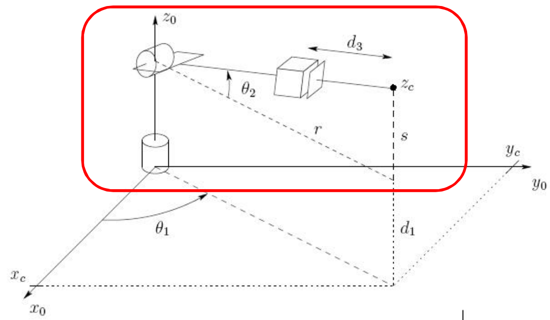
Inverse Kinematics Examples : RRP



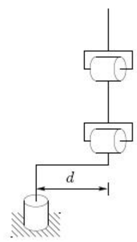
Obtain θ_1, θ_2 *and* d_3

RRP Case 1

Inverse Kinematics Examples : RRP Case1 : d = 0



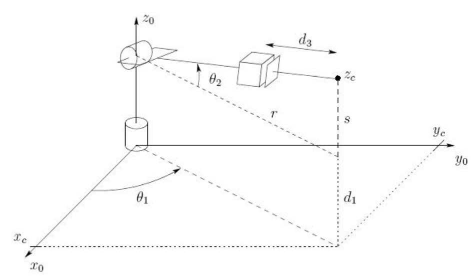
Case 1: No offset, d = 0.



$$r = \sqrt{x_c^2 + y_c^2}$$

$$s = z_c - d_1$$

Inverse Kinematics Examples : RRP Case1 : d = 0

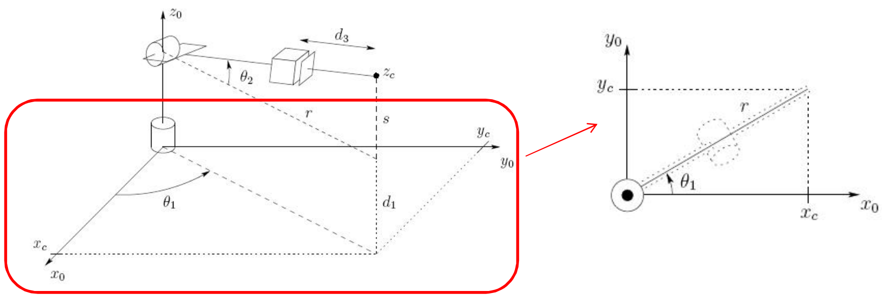


$$\boxed{\begin{aligned} d_3 + a_2 &= \sqrt{r^2 + s^2} \\ d_3 &= \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2} - a_2 \end{aligned}}$$

$$r^2 = x_c^2 + y_c^2, s = z_c - d_1$$

$$\boxed{\theta_2 = A\tan 2(r, s) = A\tan 2\left(\sqrt{x_c^2 + y_c^2}, z_c - d_1\right)}$$

Inverse Kinematics Examples : RRP Case1 : d = 0



$$\boxed{\theta_1 = A\tan 2(x_c, y_c)}$$

Other valid solution is $\theta_1 = A\tan 2(x_c, y_c) + \pi$

if $\boxed{\theta_1 = A\tan 2(x_c, y_c) + \pi}$

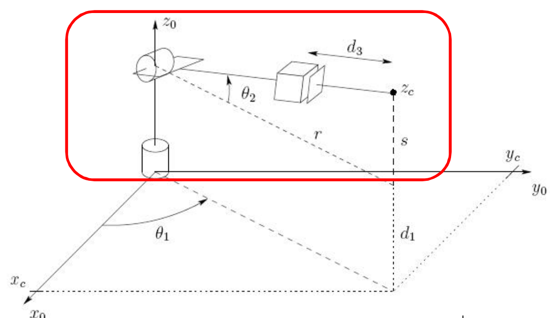
This mean Robot is turn 180 deg to the other quadrant.

$$\boxed{\theta_2 = -180° - \theta_2}$$

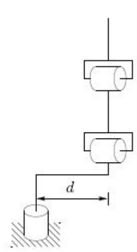
$$\boxed{d_3 = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2} - a_2}$$

RRP Case 2

Inverse Kinematics Examples : RRP Case2 : d ≠ 0

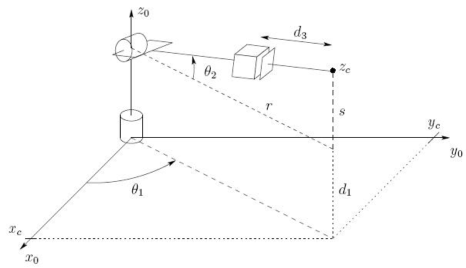


Case 2: Offset, d ≠ 0.



$$r = \sqrt{x_c^2 + y_c^2 - d^2}$$
$$s = z_c - d_1$$

Inverse Kinematics Examples : RRP Case2 : d ≠ 0



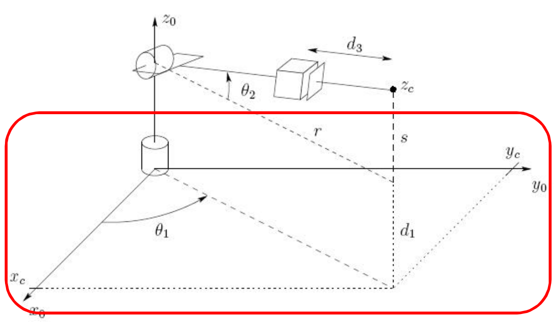
$$d_3 + a_2 = \sqrt{r^2 + s^2}$$

$$d_3 = \sqrt{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2} - a_2$$

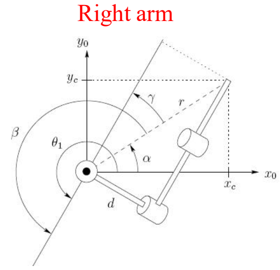
$$r^2 = x_c^2 + y_c^2 - d^2 , s = z_c - d_1$$

$$\theta_2 = A\tan 2(r, s) = A\tan 2\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right)$$

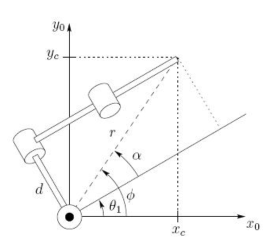
Inverse Kinematics Examples : RRP Case2 : d ≠ 0



Right arm

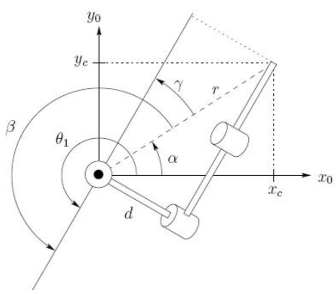


Left arm



Inverse Kinematics Examples : RRP Case2 : d ≠ 0

Right arm configuration



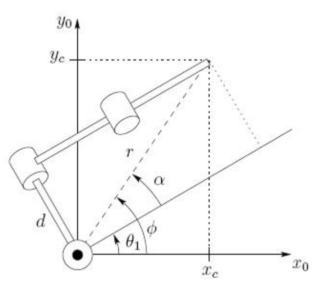
$$\theta_1 = \alpha + \beta$$
$$\alpha = A\tan 2(x_c, y_c)$$
$$\beta = \gamma + \pi$$
$$where \quad \gamma = A\tan 2(\sqrt{r^2 - d^2}, d)$$
$$\therefore \beta = A\tan 2(-\sqrt{r^2 - d^2}, -d)$$
$$= A\tan 2(-\sqrt{x_c^2 + y_c^2 - d^2}, -d)$$

$$as \cos(\theta + \pi) = -\cos\theta \quad and \quad \sin(\theta + \pi) = -\sin\theta$$

$$\boxed{\theta_1 = A\tan 2(x_c, y_c) + A\tan 2(-\sqrt{x_c^2 + y_c^2 - d^2}, -d)}$$

Inverse Kinematics Examples : RRP Case2 : d ≠ 0

Left arm configuration



$$\theta_1 = \phi - \alpha$$

$$\phi = A\tan 2(x_c, y_c)$$
$$\alpha = A\tan 2(\sqrt{r^2 - d^2}, d)$$
$$= A\tan 2(\sqrt{x_c^2 + y_c^2 - d^2}, d)$$

$$\boxed{\theta_1 = A\tan 2(x_c, y_c) - A\tan 2(\sqrt{x_c^2 + y_c^2 - d^2}, d)}$$

Change Colour

Assumption University (ABAC)
Classwork 10 (26 Aug 2021)

Name……Todsavad T……………………………**ID**……6114215……………

RRR with no offset configuration.

a. Let a_2 = 3 and a_3 = 5, d_1 = 2, obtain one set of the possible values of θ_1, θ_2 and θ_3 for point (7.75, 1.95, 1.8). Compute with using Atan2 function. Show calculation steps.

b. Draw the following configurations of RRR, specify arm up and arm down configuration.

Find θ_2, θ_3 when $\theta_1 + \pi$

(a) $$\frac{x_c^2 + y_c^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

$$D = \frac{(7.75)^2 + (1.95)^2 + (1.8-2)^2 - 3^2 - 5^2}{2(3\times5)} = 0.9968$$

$$\theta_3 = A\tan 2(D, \pm\sqrt{1-D^2})$$

$$\theta_3 = \tan^{-1}2d\left[\frac{\pm\sqrt{1-0.9968^2}}{0.9968}\right] = \pm 4.56^\circ$$

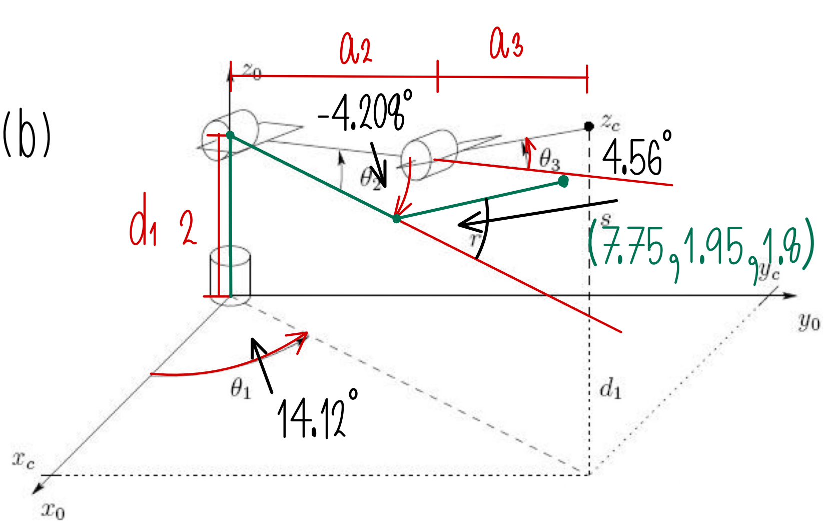
$$\theta_2 = A\tan 2\left(\sqrt{x_c^2+y_c^2}, z_c - d_1\right) - A\tan 2\left(a_2 + a_3\cos(\theta_3), a_3\sin(\theta_3)\right)$$

$$\theta_2 = \tan^{-1}2d\left[(1.8-2), \sqrt{(7.75)^2+(1.95)^2}\right] - \left[\tan^{-1}2d(5\sin 4.56, 3 + 5\cos(4.56))\right] = -4.208^\circ$$

$$\theta_1 = A\tan 2(x_c, y_c)$$

$$\theta_1 = \tan^{-1}2d\left[7.75, 1.95\right] = 14.12^\circ$$

(b)



Links	θ (Z)	d (Z)	a (X)	α (X)
1	θ_1^*	d_1	0	90°
2	θ_2^*	0	a_2	0
3	θ_3^*	0	a_3	0