Persamaan-persamaan yang sering digunakan

$$W=\prod_s \frac{[(g_s-1)+n_s]!}{(g_s-1)!n_s!} \quad (\text{BE}) \tag{1}$$

$$W = N!\prod_s\left\{\frac{g_s^{n_s}}{n_s!}\right\} \quad \text{(Boltzmann)} \tag{2}$$

$$W=\prod_s\left\{\frac{g_s^{n_s}}{n_s!}\right\} \quad \text{(semiklasik)} \tag{3}$$

$$\overline{n}_{\mathrm{FD}} = \frac{1}{e^{(\epsilon-\mu)/kT}+1} \tag{4}$$

$$\overline{n}_{\mathrm{BE}} = \frac{1}{e^{(\epsilon-\mu)/kT}-1} \tag{5}$$

$$\overline{n}_{\mathrm{boltzmann}} = e^{\mu/kT}e^{-\epsilon/kT} = e^{-(\epsilon-\mu)/kT} \tag{6}$$

$$F=-kT\ln Z \tag{7}$$

$$S=-\left(\frac{\partial F}{\partial T}\right)_{V,N}, \tag{8}$$

$$P=-\left(\frac{\partial F}{\partial V}\right)_{T,N}, \tag{9}$$

$$\mu=+\left(\frac{\partial F}{\partial N}\right)_{T,V} \tag{10}$$

$$S=k\ln W \tag{11}$$

$$\frac{\mathcal{F}(s_2)}{\mathcal{F}(s_1)}=\frac{\Omega(s_2)}{\Omega(s_1)} \tag{12}$$

$$\mathcal{P}(s)=\frac{1}{Z}e^{-E_s/kT} \tag{13}$$

$$\frac{N_s}{N}=\frac{e^{-E(s)/kT}}{Z} \tag{14}$$

$$\overline{E} = \frac{\sum_s E(s)N(s)}{N} = \sum_s E(s)\frac{N(s)}{N} = \sum_s E(s)f = \frac{1}{Z}\sum_s E(s)e^{-\beta E(s)} \tag{15}$$

Degenerasi gas fermi.
Jumlah Keadaan energi dalam rentang energi antara ϵ hingga $\epsilon + d\epsilon$ adalah

$$g(\epsilon)d\epsilon = \frac{2\pi(2m)^{3/2}\epsilon^{1/2}d\epsilon\cdot V}{h^3} \tag{16}$$

karena ada $-1/2$ dan $1/2$ maka dikali 2

$$g(\epsilon)=V\cdot4\pi\left(\frac{2m}{h^2}\right)^{3/2}\epsilon^{1/2} \tag{17}$$

Untuk $\epsilon<\epsilon_{\mathrm{F}}(0)$ maka

$$f(\epsilon)=\frac{1}{e^{-\infty}+1}=1 \tag{18}$$

untuk $\epsilon>\epsilon_{\mathrm{F}}(0)$ maka

$$f(\epsilon)=\frac{1}{e^{\infty}+1}=0 \tag{19}$$

jadi:

$$\int_0^{\epsilon_{\mathrm{F}}} V\cdot 4\pi\left(\frac{2m}{h^2}\right)^{3/2}\epsilon^{1/2}d\epsilon = N \tag{20}$$

$$\epsilon_{\mathrm{F}}(0)=\frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{3/2} \tag{21}$$

$$kT_{\mathrm{F}}(0)=\epsilon_{\mathrm{F}}(0) \tag{22}$$

$$\overline{\epsilon}(0) = \frac{\int_0^{\epsilon_{\mathrm{F}}(0)}\epsilon g(\epsilon)d\epsilon}{\int_0^{\epsilon_{\mathrm{F}}(0)}g(\epsilon)d\epsilon}=\frac{3}{5}\epsilon_{\mathrm{F}}(0) \tag{23}$$

Dengan $\epsilon=p^2/2m$, maka

$$U=N\overline{\epsilon}=\frac{3}{5}N\epsilon_{\mathrm{F}} \tag{24}$$

$$P=-\frac{\partial}{\partial V}\left[\frac{3}{5}N\frac{h^2}{2m}\left(\frac{3N}{8\pi}\right)^{3/2}\cdot V^{-3/2}\right] = \frac{2N\epsilon_{\mathrm{F}}}{5V}=\frac{2U}{3V} \tag{25}$$

Statistik bose-einstein: Tinjauan radiasi benda hitam
Jumlah mode panjang gelombang λ sampai $d\lambda$

$$g(\lambda)d\lambda=\frac{4\pi}{\lambda^4}d\lambda \tag{26}$$

Karena ada dua polarisasi

$$g(\lambda)d\lambda=\frac{8\pi}{\lambda^4}d\lambda \tag{27}$$

perhatikan: persamaan (6) setara dengan persmaan (1)
dis. BE

$$n_s=\frac{g_s}{e^{h\nu_s/kT}-1} \tag{28}$$

Sehingga jumlah foton, $n_\lambda(\lambda)d\lambda$ untuk rentang λ hingga $\lambda+d\lambda$

$$n_\lambda(\lambda)d\lambda=\frac{8\pi}{\lambda^4}d\lambda\cdot\frac{1}{e^{hc/k\lambda T}-1} \tag{29}$$

Di mana disubst. $h\nu=hc/\lambda$. Energi $E(\lambda)=n_\lambda(\lambda)h\nu=n_\lambda(\lambda)hc/\lambda$ dinyatakan oleh:

$$E(\lambda)d\lambda=\frac{8\pi hc\,d\lambda}{\lambda^5(e^{hc/k\lambda T}-1)} \tag{30}$$

Untuk panjang gelombang yang besar, $e^{hc/k\lambda T}\simeq 1+hc/k\lambda T$ sehingga

$$E(\lambda)d\lambda\simeq\frac{8\pi kT\,d\lambda}{\lambda^4} \tag{31}$$

atau formulasi Rayleigh Jeans. untuk panjang gelombang yang rendah, $e^{hc/k\lambda T}\gg1$

$$E(\lambda)d\lambda\simeq\frac{8\pi hc}{\lambda^5}e^{-hc/k\lambda T}\,d\lambda \tag{32}$$

atau formulasi Wein. Dari persamaan (11) dapat diperoleh total energi persatuan volume yang terlingkupi dalam benda hitam tersebut yakni:

$$E=\int_0^\infty E(\lambda)d\lambda = \int_0^\infty\frac{8\pi hc\,d\lambda}{\lambda^5(e^{hc/k\lambda T}-1)}$$

$$=\frac{8\pi h}{c^3}\left\{\frac{kT}{h}\right\}^4\int_0^\infty\frac{t^3\,dt}{e^t-1} \tag{33}$$

Dengan

$$\int_0^\infty\frac{t^3}{e^t-1}=6\sum_{n=1}^\infty\frac{1}{n^4}=\frac{\pi^4}{15} \tag{34}$$

Maka persamaan 33 dapat dituliskan menjadi

$$\boxed{E=\left\{\frac{8\pi^5k^4}{15h^3c^3}\right\}T^4} \tag{35}$$

Gas fonon.
Jumlah fonon $n(\nu)d\nu$ dengan frekuensi antara ν sampai $\nu+d\nu$ adalah

$$n_\nu(\nu)d\nu=\frac{g(\nu)d\nu}{e^{h\nu/kT}-1} \tag{13}$$

Aproksimasi Debye dinyatakan oleh

$$g(\nu)d\nu=C\nu^2d\nu,\qquad \nu\le\nu_{\mathrm{m}}$$

$$g(\nu)d\nu=0,\qquad \nu>\nu_{\mathrm{m}}$$

Untuk mendapatkan konstanta C maka dilakukan integrasi terhadap seluruh mode yang mungkin yang nantinya akan menghasilkan nilai $3N$ yakni

$$3N=\int_0^\infty g(\nu)d\nu=\int_0^{\nu_{\mathrm{m}}}C\nu^2d\nu = \frac{1}{3}C\nu_{\mathrm{m}}^3$$

atau

$$C_{\mathrm{m}}=\frac{9N}{v_{\mathrm{m}}^3}$$

sehingga aproksimasi debye memberikan

$$n_\nu(\nu)d\nu=\frac{9N}{\nu_{\mathrm{m}}^3}\frac{\nu^2d\nu}{e^{h\nu/kT}-1},\ \nu\le\nu_{\mathrm{m}}$$

$$=0,\qquad \nu>\nu_{\mathrm{m}}$$

kemudian

$$E=\int_0^{\nu_{\mathrm{m}}}h\nu n_\nu(\nu)d\nu = \frac{9N_Ah}{\nu_{\mathrm{m}}^3}\int_0^{\nu_{\mathrm{m}}}\frac{\nu^3d\nu}{e^{h\nu/kT}-1}$$

kemudian panas spesifik dapat diperoleh yakni

$$C_\nu=\left\{\frac{\partial E}{\partial T}\right\}=\frac{9N_Ah^2}{v_{\mathrm{m}}^3}\frac{1}{kT^2}\cdot\int_0^{\nu_{\mathrm{m}}}\frac{\nu^4e^{h\nu/kT}d\nu}{(e^{h\nu/kT}-1)^2} \tag{36}$$

Jika diadakan perubahan variabel yakni $x=h\nu/kT$ dan $h\nu_{\mathrm{m}}/k$ diganti oleh θ_{D} maka temperatur karakteristik pada Persamaan di atas dapat dinyatakan menjadi:

$$C_\nu=9R\left\{\frac{T}{\theta_{\mathrm{D}}}\right\}^3\int_0^{\theta_{\mathrm{D}}/T}\frac{x^4e^xdx}{(e^x-1)^2}$$

Untuk temperatur tinggi, di mana $\theta_{\mathrm{D}}\ll1$ maka $e^x\simeq1+x\simeq1$ sehingga

$$C_\nu\simeq9R\left\{\frac{T}{T_{\mathrm{D}}}\right\}^3\int_0^{\theta_{\mathrm{D}}/T}x^2dx=3R$$

Untuk temperatur rendah, di mana $e^{-\theta_{\mathrm{D}}/T}\ll1$ maka batas atas integrasi diganti menjadi ∞ sehingga

$$C_\nu\simeq9R\left\{\frac{T}{\theta_{\mathrm{D}}}\right\}^3\int_0^\infty\frac{x^4e^xdx}{(e^x-1)^2}$$

atau dengan melakukan ekspansi taylor terhadap e^x maka

$$\int_0^\infty\frac{x^4e^xdx}{(e^x-1)^2}=24\sum_{n=1}^\infty\frac{1}{n^4}=\frac{4\pi^4}{15}$$

sehingga

$$C\nu\simeq\frac{12}{5}\pi^4R\left\{\frac{T}{\theta_{\mathrm{D}}}\right\}^3$$

Prinsip ekuipartisi energi.
Energi kinetik dalam satu derajat kebebasan (misal x) dinyatakan oleh
$\epsilon_x=p_x^2/2m$ sehingga

$$\overline{\epsilon_x}=\frac{\int_\Gamma p_x^2/2m\,e^{-\epsilon/kT}d\Gamma}{\int_\Gamma e^{-\epsilon/kT}d\Gamma} \tag{37}$$

Jika energi dinyatakan dalam dua bagian yakni $p_x^2/2m$ dan $(\epsilon-p_x^2/2m)$ makan persamaan di atas dapat dituliskan menjadi

$$\overline{\epsilon_x}=\left(\frac{\int\exp\left(-\left(\epsilon-\frac{p_x^2}{2m}\right)/kT\right)}{\int\exp\left(-\left(\epsilon-\frac{p_x^2}{2m}\right)/kT\right)}\cdot\frac{dx\,dy\,dz\,dp_y\,dp_z}{dx\,dy\,dz\,dp_y\,dp_z}\cdot\frac{\int_{-\infty}^{\infty}\frac{p_x^2}{2m}\exp\left(-p_x^2/2mkT\right)dp_x}{\int_{-\infty}^{\infty}\exp\left(-p_x^2/2mkT\right)dp_x}\right)$$

$$=\frac{1}{2}kT$$

Di mana telah dilakukan substitusi $p_x^2/2mkT=u^2$
Untuk harmonik osilator, di mana energinya dinyatakan sebagai

$$\epsilon_x=\frac{p_x^2}{2m}+\frac{1}{2}\mu x^2 \tag{38}$$

maka energi rata-ratanya adalah:

$$\overline{\epsilon_x}=\frac{\int_\Gamma\left\{p_x^2/2m+\frac{1}{2}\mu x\right\}e^{-\epsilon/kT}d\Gamma}{\int_\Gamma e^{-\epsilon/kT}d\Gamma}$$

$$=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{p_x^2}{2m}+\frac{1}{2}\mu x^2\right\}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}}\cdot\frac{\exp\left(-\left[\frac{p_x^2}{2m}+\frac{1}{2}\mu x^2\right]/kT\right)}{\exp\left(-\left[\frac{p_x^2}{2m}+\frac{1}{2}\mu x^2\right]/kT\right)}\cdot\frac{dx\,dp_x}{dx\,dp_x} \tag{39}$$

Jika dilakukan substitusi $p_x^2/2m=r^2\sin^2\theta$ dan $\frac{1}{2}\mu x^2=r^2\cos^2\theta$ maka diperoleh

$$\overline{\epsilon_x}=\frac{\int_0^{2\pi}d\theta\int_0^\infty e^{-r^2/kT}r^3\,dr}{\int_0^{2\pi}d\theta\int_0^\infty e^{-r^2/kT}r\,dr} = kT$$

Statistik Semiklasik
Secara klasik, jumlah keadaan energi dalam rentang ϵ sampai $\epsilon+d\epsilon$ dinyatakan oleh:

$$g(\epsilon)d\epsilon=BV2\pi(2m)^{3/2}\epsilon^{1/2}d\epsilon \tag{40}$$

Sehingga fungsi partisi menjadi

$$Z=\sum_s g_se^{-\epsilon_s/kT} \equiv\int_0^\infty e^{-\epsilon/kT}g(\epsilon)d\epsilon = 2\pi BV(2m)^{3/2}\cdot\int_0^\infty\epsilon^{1/2}e^{-\epsilon/kT}d\epsilon = BV(2\pi mkT)^{3/2} \tag{41}$$

yang nantinya akan diperoleh

$$F=-NkT\ln[BV(2\pi mkT)^{3/2}] \tag{42}$$

dan

$$S=-\left\{\frac{\partial F}{\partial T}\right\} = \boxed{Nk\ln[BV(2\pi mkT)^{3/2}]+\frac{3}{2}Nk} \tag{43}$$

Dengan persamaan ini akan terdapat kenaikan entropi sebesar $2Nk\ln2$ pada pencampuran 2 volume gas. Fakta ini menuntun pada perumusan statistik semiklasik, di mana bobot konfigurasi dinyatakan oleh

$$W_{\mathrm{MB}}=\prod_s\frac{g_s^{n_s}}{n_s!} \tag{44}$$

$$W_{\mathrm{BE}}=\prod_s\frac{(n_s+g_s-1)!}{n_s!(g_s-1)!} \tag{45}$$

$$W_{\mathrm{FD}}=\prod_s\frac{g_s!}{n_s!(g_s-n_s)!} \tag{46}$$

Pada limit klasik, yakni $g_s\gg n_s\gg1$ maka dengan aproksimasi Stirling diperoleh:

$$\ln W_{\mathrm{MB}}=\sum_s(n_s\ln g_s-n_s\ln n_s+n_s) = \sum_s\left(n_s\ln\frac{g_s}{n_s}+n_s\right) \tag{47}$$

$$\ln W_{\mathrm{BE}}\simeq\sum_s[(n_s+g_s)\ln(n_s+g_s)-n_s\ln n_s-g_s\ln g_s] = \prod_s\left[n_s\ln\left\{\frac{n_s+g_s}{n_s}\right\}+g_s\ln\left\{\frac{n_s+g_s}{g_s}\right\}\right] \simeq\prod_s\left(n_s\ln\frac{g_s}{n_s}+n_s\right) \tag{48}$$

di mana telah digunakan hampiran $n_s+g_s-1\simeq(n_s+g_s)\frac{g_s+n_s}{n_s}\simeq\frac{g_s}{n_s}$. Dan $\ln\left\{\frac{n_s+g_s}{g_s}\right\}=\ln\left\{1+\frac{n_s}{g_s}\right\}\simeq\frac{n_s}{g_s}$
Kemudian

$$\ln W_{\mathrm{FD}}=\sum_s[g_s\ln g_s-n_s\ln n_s-(g_s-n_s)\ln(g_s-n_s)] = \sum_s\left[n_s\ln\left\{\frac{g_s-n_s}{n_s}\right\}-g_s\ln\left\{\frac{g_s-n_s}{g_s}\right\}\right] \simeq\sum_s\left(n_s\ln\frac{g_s}{n_s}+n_s\right) \tag{49}$$

Entropi kemudian dinyatakan oleh

$$S=Nk\ln\frac{Z}{N}+\frac{E}{T}+Nk \tag{50}$$

Sementara pernyataannya dalam kuantum mekanik adalah:

$$Z=\frac{V}{h^3}(2\pi mkT)^{3/2} \tag{51}$$

dengan demikian entropi untuk sistem semi-klasik dinyatakan oleh

$$S=Nk\left\{\ln\left[\frac{V(2\pi mkT)^{3/2}}{Nh^3}\right]+\frac{5}{2}\right\}$$

energi bebas helmoltz kemudian dinyatakan sebagai

$$F=-kT\ln\frac{Z^N}{N!} \tag{52}$$

dengan $\boldsymbol{Z}=Z^N/N!$ menyatakan fungsi partisi total untuk sistem semi-klasik.

Ensembel Kanonik

$$\boldsymbol{Z}=\sum_ie^{-E_i/kT} \tag{53}$$

$$p_i=p(0)e^{-E_i/kT}=\frac{e^{-E_i/kT}}{\boldsymbol{Z}} \tag{54}$$

$$p(0)=\frac{1}{\sum_ie^{-E_i/kT}} \tag{55}$$

Jadi

$$\sum_ip_i=1 \tag{56}$$

$$F=-kT\ln\boldsymbol{Z} \tag{57}$$

Fluktuasi energi pada Ensembel Kanonik

$$\overline{(\delta E)^2}=\overline{(E-\overline{E})^2}=\overline{E^2}-\overline{E}^2 \tag{58}$$

$$\overline{E}=\sum_ip_iE_i=\frac{1}{\boldsymbol{Z}}\frac{\partial\boldsymbol{Z}}{\partial\beta} \tag{59}$$

$$\overline{(\delta E)^2}=\frac{\partial\overline{E}}{\partial\beta}=kT^2\frac{\partial\overline{E}}{\partial T}=C_vkT^2 \tag{60}$$

$$\mathcal{F}=\left\{\frac{\overline{(\delta E)^2}}{\overline{E}^2}\right\}^{1/2}=\left\{\frac{kT^2C_v}{\overline{E}^2}\right\}^{1/2} \tag{61}$$

dengan $C_v=(3/2)Nk$ dan $\overline{E}=(3/2)NkT$ maka

$$\mathcal{F}=\left\{\frac{\frac{3}{2}Nk^2T^2}{\left(\frac{3}{2}NkT\right)^2}\right\}^{1/2}=\left(\frac{3}{2}N\right)^{-1/2} \tag{62}$$

Enesembel Grand Kanonik

$$\boldsymbol{Z}=\sum_ie^{(\mu N-E)_i/kT} \tag{63}$$

$$p_i=e^{-[pV+(\mu N_i-E_i)/kT} \tag{64}$$

$$\sum_ip_i=1 \tag{65}$$

atau (66)

$$e^{-pV/kT}\sum_ie^{(\mu N_i-E_i)/kT}=1 \tag{67}$$

$$e^{-pV/kT}=\frac{1}{\sum_ie^{(\mu N_i-E_i)/kT}}=\frac{1}{\boldsymbol{Z}} \tag{68}$$

Sehingga

$$p_i=\frac{e^{(\mu N_i-E_i)/kT}}{\boldsymbol{Z}} \tag{69}$$

$$\overline{N}=\sum_ip_iN_i = \sum_i\frac{N_ie^{(\mu N_i-E_i)/kT}}{\boldsymbol{Z}} = \frac{kT}{\boldsymbol{Z}}\left\{\frac{\partial\boldsymbol{Z}}{\partial\mu}\right\} = kT\left\{\frac{\partial\ln\boldsymbol{Z}}{\partial\mu}\right\}_{V,T} \tag{70}$$

dan

$$\overline{N^2}=\frac{(kT)^2}{\boldsymbol{Z}}\left\{\frac{\partial^2\boldsymbol{Z}}{\partial\mu^2}\right\}_{V,T} \tag{71}$$

Dari pers.68 dapat pula diperoleh

$$(pV)=kT\ln\boldsymbol{Z} \tag{72}$$

sehingga jumlah rata-rata partikel pada pers.70 dapat dinyatakan ke dalam

$$\overline{N}=\left\{\frac{\partial(pV)}{\partial\mu}\right\}_{T,V} \tag{73}$$

Perhatikan

$$d(pV)=p\,dV+S\,dT+\overline{N}\,d\mu \tag{74}$$

demikian pula

$$p=\left\{\frac{\partial(pV)}{\partial V}\right\}_{T,\mu} \tag{75}$$

$$S=\left\{\frac{\partial(pV)}{\partial T}\right\}_{V,\mu} \tag{76}$$

Untuk distribusi Bose-einstein, pers. 70 akan menghasilkan

$$\overline{N}=kT\frac{\partial}{\partial\mu}\left\{\ln\prod_j[1-e^{(\mu-\epsilon_j)kT}]^{-1}\right\}_{V,T} = -kT\sum_j\left\{\frac{\partial}{\partial\mu}\ln[1-e^{(\mu-\epsilon_j)/kT}]\right\}_{V,T} = \sum_j\frac{1}{e^{(\epsilon_j-\mu)/kT}-1} = \sum_j\overline{n_j} \tag{77}$$

Dengan cara yang sama untuk distribusi fermi-dirac diperoleh

$$\overline{N}=\sum_j\overline{n_j}=\sum_j\frac{1}{e^{(\epsilon_j-\mu)/kT}+1} \tag{78}$$

Fluktuasi Jumlah Partikel dalam ensembel grand kanonik

$$\frac{\partial}{\partial\mu}\left\{\frac{1}{\boldsymbol{Z}}\left(\frac{\partial\boldsymbol{Z}}{\partial\mu}\right)\right\}_{V,T} = \left\{\frac{1}{\boldsymbol{Z}}\frac{\partial^2\boldsymbol{Z}}{\partial\mu^2}-\frac{1}{\boldsymbol{Z}^2}\left(\frac{\partial\boldsymbol{Z}}{\partial\mu}\right)^2\right\}_{V,T} \tag{79}$$

ini akhirnya menghasilkan

$$\frac{1}{(kT)}\left\{\frac{\partial\overline{N}}{\partial\mu}\right\}_{V,T}=\frac{1}{(kT)^2}(\overline{N^2}-\overline{N}^2)$$

$$\overline{(\delta N)^2}=kT\left\{\frac{\partial\overline{N}}{\partial\mu}\right\}_{V,T} \tag{80}$$

Ekspansi Sommerfeld

$$N=\int_0^\infty g(\epsilon)\overline{n}_{\mathrm{FD}}\,d\epsilon = g_0\int_0^\infty\epsilon^{1/2}\overline{n}_{\mathrm{FD}}\,d\epsilon \tag{81}$$

$\overline{n}_{\mathrm{FD}}$ menyatakan fungsi dist. FD($f(\epsilon)$). Karen daerah yan g ditinjau hanya disekitar $\epsilon=\mu$,maka integralnya dapat dinyatakan ke dalam integral parsial

$$N=\frac{2}{3}g_0\epsilon^{3/2}\overline{n}_{\mathrm{FD}}(\epsilon)\Big|_0^\infty+\frac{2}{3}g_0\int_0^\infty\epsilon^{3/2}\left(-\frac{d\overline{n}_{\mathrm{FD}}}{d\epsilon}\right)d\epsilon \tag{82}$$

suku pertama akan habis pada kedua batas integral sementara suku kedua dinyatakan kembali melalui

$$-\frac{d\overline{n}_{\mathrm{FD}}}{d\epsilon}=-\frac{d}{d\epsilon}(e^{(\epsilon-\mu)/kT}+1)^{-1} = \frac{1}{kT}\frac{e^x}{(e^x+1)^2} \tag{83}$$

dengan $x=(\epsilon-\mu)/kT$.
Jadi

$$N=\frac{2}{3}g_0\int_0^\infty\frac{1}{kT}\frac{e^x}{(e^x+1)^2}\epsilon^{3/2}d\epsilon = \frac{2}{3}g_0\int_{-\mu/kT}^\infty\frac{e^x}{(e^x+1)^2}\epsilon^{3/2}d\epsilon \tag{84}$$

Ada dua pendekatan yang dilakukan, pertama adalah uraian Taylor terhadap $\epsilon^{3/2}$ di sekitar $\epsilon=\mu$ kedua meng-ekstends batas bawah integralnya sampai $-\infty$
Dengan demikian diperoleh

$$\epsilon^{3/2}=\mu^{3/2}+(\epsilon-\mu)\frac{d}{d\epsilon}\epsilon^{3/2}\Big|_{\epsilon=\mu}+\frac{1}{2}(\epsilon-\mu)^2\frac{d^2}{d\epsilon^2}\epsilon^{3/2}\Big|_{\epsilon=\mu}+\cdots = \mu^{3/2}+\frac{3}{2}(\epsilon-\mu)\mu^{1/2}+\frac{3}{8}(\epsilon-\mu)^2\mu^{1/2}+\cdots \tag{85}$$

sehingga

$$N=\frac{2}{3}g_0\int_{-\infty}^\infty\frac{e^x}{(e^x+1)^2}\left[\mu^{3/2}+\frac{3}{2}(\epsilon-\mu)\mu^{1/2}+\frac{3}{8}(\epsilon-\mu)^2\mu^{1/2}+\cdots\right] \tag{86}$$

Integrasi selanjutnya dapat dilakukan pada masing-masing suku yakni untuk suku pertama adalah

$$\int_{-\infty}^\infty\frac{e^x}{(e^x+1)^2}dx = \int_{-\infty}^\infty-\frac{d\overline{n}_{\mathrm{FD}}}{d\epsilon}d\epsilon = \overline{n}_{\mathrm{FD}}(-\infty)-\overline{n}_{\mathrm{FD}}(\infty) = 1-0=1 \tag{87}$$

Untuk suku kedua

$$\int_{-\infty}^\infty\frac{xe^x}{(e^x+1)^2}dx=\int_{-\infty}^\infty\frac{x}{(e^x+1)(1+e^{-x})}dx=0. \tag{88}$$

mengingat pernyataan tersebut merupakan fungsi ganjil dari x.
Suku ketiga dapat diintegrasikan secara parsial secara berurutan yang natinya akan menghasilkan

$$\int_{-\infty}^\infty\frac{x^2e^x}{(e^x+1)^2}dx=\frac{\pi^2}{3} \tag{89}$$

Dengan mengumpulkan hasil-hasil tersebut, maka nilai N selanjutnya dapat dituliskan menjadi

$$N=\frac{2}{3}g_0\mu^{3/2}+\frac{1}{4}g_0(kT)^2\mu^{-1/2}\cdot\frac{\pi^2}{3}+\cdots = N\left(\frac{\mu}{\epsilon_{\mathrm{F}}}\right)^{3/2}+N\frac{\pi^2}{8}\frac{(kT)^2}{\epsilon_{\mathrm{F}}^{3/2}\mu^{1/2}}+\cdots \tag{90}$$

Di mana telah dilakukan substitusi untuk $g_0=3N/2\epsilon_{\mathrm{F}}^{3/2}$. Dengan membagi kedua ruas dengan N diperoleh

$$\frac{\mu}{\epsilon_{\mathrm{F}}}=\left[1-\frac{\pi^2}{8}\left(\frac{kT}{\epsilon_{\mathrm{F}}}\right)^2+\cdots\right]^{2/3} = 1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_{\mathrm{F}}}\right)^2+\cdots \tag{91}$$

yang menunjukkan potensial kimia μ akan naik secara berangsur-angsur seiring naiknya T.
Hasil ini juga dapat digunakan untuk menhitung integral untuk nilai total energi yakni

$$U=\int_0^\infty\epsilon g(\epsilon)\overline{n}_{\mathrm{FD}}(e\epsilon)d\epsilon = \int_0^\infty\epsilon g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon \tag{92}$$

yakni

$$U=\frac{3}{5}N\frac{\mu^{5/2}}{\epsilon_{\mathrm{F}}^{3/2}}+\frac{3\pi^2}{8}N\frac{(kT)^2}{\epsilon_{\mathrm{F}}}+\cdots \tag{93}$$

Di mana dengan memasukkan pers. 91 diperoleh

$$U=\frac{3}{5}N\epsilon_{\mathrm{F}}+\frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_{\mathrm{F}}}+\cdots \tag{94}$$

Dipol elementer
energi rata-rata

$$\overline{E}=-N\mu B\tanh(\beta\mu B) \tag{95}$$

$$U=-N\mu B\tanh(\beta\mu B) \tag{96}$$

$$\overline{\mu_z}=\mu\tanh(\beta\mu B) \tag{97}$$

$$M=N\overline{\mu_z}=N\mu\tanh(\beta\mu B) \tag{98}$$

Meand Field Theoreme

$$\mathcal{H}=-J\sum_j\sigma_j\sigma_{j+1}-h\sum_j\sigma_j \tag{99}$$

$$\langle\mathcal{H}-\mathcal{H}_0\rangle_0=$$

$$-\frac{\left(\sum_{\{s\}} -J\sum_{<ij>}\sigma_j\,\sigma_{j+1} \cdot \exp\left[\beta H_0 \sum_j \sigma_j\right]\right)}{\sum_{\{s\}}\exp\left[BH_0\sum_j\sigma_j\right]}$$

$$-\frac{\sum_{\{s\}} h\sum_j\sigma_j \exp\left[\beta H_0\sum_j\sigma_j\right]}{\sum_{\{s\}}\exp\left[BH_0\sum_j\sigma_j\right]}$$

$$+\frac{\sum_{\{s\}} H_0\sum_j\sigma_j\exp\left[\beta H_0\sum_j\sigma_j\right]}{\sum_{\{s\}}\exp\left[BH_0\sum_j\sigma_j\right]}$$

$$= -J\sum_{<ij>}\langle\sigma_i\rangle_0\langle\sigma_j\rangle_0 - h\sum_i\langle\sigma_i\rangle_0 + H_0\sum_i\langle\sigma_i\rangle_0$$

$$= -J\sum_i\langle\sigma\rangle_0^2 - h\sum_i\langle\sigma_i\rangle_0 + H_0\sum_i\langle\sigma_i\rangle_0$$

$$= -\frac{JzN}{2}\tanh^2\beta H_0 - Nh\tanh\beta H_0 + NH_0\tanh\beta H_0 \qquad (100)$$

Jadi

$$\Phi = f_0 + (\mathcal{H}-\mathcal{H}_0)_0 = -NkT\ln(2\cosh\beta H_0) - \frac{JzN}{2}\tanh^2\beta H_0 + NH_0\tanh\beta H_0 - Nh\tanh\beta H_0 \qquad (101)$$

$$\frac{\partial\Phi}{\partial H_0} = 0$$

$$\Rightarrow -\frac{NkT(2\sinh\beta H_0)\beta}{2\cosh\beta H_0} - \frac{JzN\beta\tanh\beta H_0}{\cosh^2\beta H_0} + \frac{NH_0\beta}{\cosh^2\beta H_0} + N\tanh\beta H_0 + \frac{Nh\beta}{\cosh^2\beta H_0} = 0$$

$$\Rightarrow -N\tanh\beta H_0 - \frac{Jz\beta N\tanh\beta H_0}{\cosh^2\beta H_0} + \frac{NH_0\beta}{\cosh^2\beta H_0} + N\tanh\beta H_0 + \frac{Nh\beta}{\cosh^2\beta H_0} = 0$$

$$\Rightarrow \frac{-JzBN\tanh\beta H_0}{\cosh^2\beta H_0} + \frac{N\beta H_0}{\cosh^\beta H_0} - \frac{Nh\beta}{\cosh\beta H_0} = 0$$

$$\Rightarrow= -Jz\tanh\beta H_0 + H_0 - h = 0$$

$$\Rightarrow H_0 = Jz\tanh\beta H_0 + h = Jz\langle s\rangle_0 + h \qquad (102)$$

Dari defenisi $\langle s\rangle_0 = \tanh\beta H_0$, maka

$$\langle s\rangle_0 = \tanh\beta(Jz\langle s\rangle_0 + h) \qquad (103)$$

Susceptibilitas per spin:

$$\chi = \frac{\partial\langle\sigma\rangle_0}{\partial h} \qquad (104)$$

$$\frac{\partial\langle\sigma\rangle_0}{\partial h} = \frac{\partial}{\partial h}\tanh[\beta(Jq\langle\sigma\rangle_0 + h)]$$

$$= \operatorname{sech}^2(\beta(Jq\langle\sigma\rangle_0+h))\,\beta\left(Jq\frac{\partial\langle\sigma\rangle_0}{\partial h} + \frac{\partial}{\partial h}h\right)$$

$$= \left(1-\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]\right)\beta\left(Jq\frac{\partial\langle\sigma\rangle_0}{\partial h}+1\right)$$

maka

$$\frac{\partial\langle\sigma\rangle_0}{\partial h}\left(1 - Jq\left(1-\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]\right)\right) = \left(1-\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]\right)\cdot\beta \qquad (105)$$

atau

$$\frac{\partial\langle\sigma\rangle_0}{\partial h} = \frac{(1-\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)])\cdot\beta}{\left(1-\beta Jq + \beta Jq\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]\right)}$$

$$= \frac{1-\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]}{\left(\frac{1}{\beta} - Jq + Jq\tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]\right)}$$

$$= \frac{1-\langle\sigma\rangle_0^2}{Jq\left(\left(\frac{1}{\beta Jq} - 1 + \tanh^2[\beta(Jq\langle\sigma\rangle_0+h)]\right)\right)}$$

$$= \frac{1-\langle\sigma\rangle_0^2}{Jq\left(\frac{1}{\beta Jq} - 1 + \langle\sigma\rangle_0^2\right)}$$

$$= \frac{1-\langle\sigma\rangle_0^2}{Jq\left(\frac{kT}{kT_c} - 1 + \langle\sigma\rangle_0^2\right)}$$

$$= \boxed{\frac{1-\langle\sigma\rangle_0^2}{Jq(t+\langle\sigma\rangle_0^2)}} \qquad (106)$$

Ekspansi Virial

$$\frac{Pv}{kT} = \frac{\sum_{l=1}^{\infty}\bar{b}z^l}{\sum_{l=1}^{\infty} l\bar{b}_l z^l\bar{b}}\ ;\ v = \frac{N}{V} \qquad (107)$$

Sementara ekspansi virial sendiri didefenisikan sebagai:

$$\frac{Pv}{kT} = \sum_{l=1}^{\infty} a_l(T)\left(\frac{\lambda^3}{v}\right)^{l-1} \qquad (108)$$

sehingga:

$$\sum_{l=1}^{\infty} a_l\left(\sum_{n=1}^{\infty} n\bar{b}z^n\right)^{l-1} = \frac{\sum_{l=1}^{\infty}\bar{b}z^l}{\sum_{l=1}^{\infty} l\bar{b}_l z^l\bar{b}} \qquad (109)$$

atau:

$$(\bar{b}_1z + 2\bar{b}_2z^2 + 3\bar{b}_3z^3 + \cdots)\left[a_1 + a_2\left(\sum_{n=1}^{\infty}n\bar{b}_nz^n\right) + a_3\left(\sum_{n=1}^{\infty}n\bar{b}_nz^n\right)^2 + \cdots\right] = \bar{b}_1z + \bar{b}_2z^2 + \bar{b}_3z^3 + \cdots$$

$$\Rightarrow (\bar{b}_1z + 2\bar{b}_2z^2 + 3\bar{b}_3z^3 + \cdots)\left[a_1 + a_2(\bar{b}_1z + 2\bar{b}_2z^2 + 3\bar{b}_3z^3 + \cdots) + a_3\left(\bar{b}_1^2z^2 + 2\bar{b}_1z\,2\bar{b}_2z^2 + \cdots\right) + a_4(b_1^3z^3 + \cdots)\right] = \bar{b}_1z + \bar{b}_2z^2 + \cdots$$

Jadi: untuk z^1:

$$\bar{b}_1z\,a_1 = \bar{b}_1z \Rightarrow \bar{b}_1 = a_1 = 1 \qquad (110)$$

untuk z^2:

$$2\bar{b}_2z^2\,a_1 + a_2\bar{b}_1z\,\bar{b}_1z = \bar{b}z^2$$

$$\Rightarrow 2\bar{b}_2z^2 + a_2z^2 = \bar{b}_2z^2$$

$$\Rightarrow a_2z^2 = -\bar{b}_2z^2$$

$$\Rightarrow a_2 = -\bar{b}_2 \qquad (111)$$

untuk z^3:

$$3\bar{b}_3z^3a_1 + 2\bar{b}_2z^2a_2\bar{b}_1z + a_2 2\bar{b}_2z^2\bar{b}_1z + a_3\bar{b}_1^2z^2\bar{b}_1z = \bar{b}_3z^3$$

$$\Rightarrow a_3z^3 = (\bar{b}_3 - 3\bar{b}_3 - 2\bar{b}_2a_2 - a_2 2\bar{b}_2)z^3$$

$$a_3 = -2\bar{b}_3 + 2\bar{b}_2^2 + 2\bar{b}_2^2$$

$$\Rightarrow a_3 = -2\bar{b}_3 + 4\bar{b}_2^2 \qquad (112)$$

untuk z^4:

$$2\bar{b}_2z^2a_2 2\bar{b}_2z + 2\bar{b}_2z^2a_3\bar{b}_1^2z^2 + 4\bar{b}_4z^4 + a_4\bar{b}_1^3z^3\bar{b}_1z + \bar{b}_1za_2 3\bar{b}_3z^3 + 2\bar{b}_1za_3\bar{b}_1z2\bar{b}_2z^2 + 3\bar{b}_3z^3a_2\bar{b}_1z = b_4vz^4$$

$$\Rightarrow 4\bar{b}_2^2a_2 + 2\bar{b}_2a_3 + 3a_2\bar{b}_3 + 4a_3\bar{b}_2 + 6\bar{b}_3a_2 + 4\bar{b}_4 + a_4 = \bar{b}_4$$

$$\Rightarrow -3\bar{b}_4 + 4\bar{b}_2^4 - 2\bar{b}_2(4\bar{b}_2^2 - 2\bar{b}_3) + \bar{b}_3a_2 + 4(4\bar{b}_2^2 - 2\bar{b}_3)\bar{b}_2 = a_4$$

$$\Rightarrow a_4 = -3\bar{b}_4 + 4\bar{b}_2^3 - 8\bar{b}_2^3 + 4\bar{b}_2\bar{b}_3 + 6\bar{b}_2\bar{b}_3 - 16\bar{b}_2^3 + 8\bar{b}_2\bar{b}_3$$

$$\Rightarrow a_4 = -3\bar{b}_4 - 20\bar{b}_2^3 + 8\bar{b}_2\bar{b}_3 \qquad (113)$$

Dengan demikian deret virial dapat dinyatakan menjadi:

$$\frac{Pv}{kT} = \frac{PN}{NkT} = a_1 + a_2\left(\frac{N}{V}\right) + a_3\left(\frac{N}{V}\right)^2 + a_4\left(\frac{N}{V}\right)^3 + \cdots$$

$$= 1 + (-\bar{b}_2)\left(\frac{N}{V}\right) + (4\bar{b}_2^2 - 2\bar{b}_3)\left(\frac{N}{V}\right)^2 + (-20\bar{b}_2^3 + 18\bar{b}_2\bar{b}_3 - 3\bar{b}_4)\left(\frac{N}{V}\right)^3 \qquad (114)$$

Ekspansi Cluster :

$$U(r) = \begin{cases} 0 & r > R \\ \infty & r \le R \end{cases} \qquad (115)$$

$$b_j = \frac{1}{j!\lambda^{3(j-1)}V}[\text{jml gugus } j] \qquad (116)$$

untuk $j=1$ maka:

$$b_2 = \frac{1}{2\lambda^3V}\iint f_{12}d^3r_1\,d^3r_2 \approx \frac{1}{2\lambda^3}\int f_{12}d^3r_{12} = \frac{2\pi}{\lambda^3}\int_0^{\infty}f(r)r^2\,dr = \frac{1}{2\lambda^3}\int_0^{\infty}\left(e^{-U(r)/kT} - 1\right)r^2\,dr \qquad (117)$$

dengan demikian:

$$a_2 = -b_2 = \frac{2\pi}{\lambda^3}\int_0^{\infty}(1 - e^{-U(r)} - 1)r^2\,dr$$

$$= \frac{2\pi}{\lambda^3}\left[\int_0^R(1 - e^{-\infty/kT})r^2\,dr + \int_R^{\infty}(1 - e^{-0/kT})r^2\,dr\right]$$

$$= \frac{2\pi}{\lambda^3}\left[\int_0^R r^2\,dr + 0\right]$$

$$= \frac{2\pi}{\lambda^3}\left(\frac{r^3}{3}\right)\Bigg|_0^R = \frac{2\pi}{\lambda^3}\frac{R^3}{3}$$

$$= \frac{2\pi}{3}\left(\frac{R}{\lambda}\right)^3 \qquad (118)$$

Tinjauan sistem dengan interaksi lemah

$$\boldsymbol{Z} = \int_{\Gamma_{6N}} e^{-E/kT}\frac{d\Gamma_{6N}}{h^{3N}} \qquad (119)$$

untuk semi-klasik:

$$\boldsymbol{Z} = \int_{\Gamma_{6N}} e^{-E/kT}\frac{d\Gamma_{6N}}{h^{3N}N!} \qquad (120)$$

Totak energi asembli:

$$E = \frac{1}{2m}\sum_{j=1}^{N}(p_{xj}^2 + p_{yj}^2 + p_{zj}^2) + \sum_j^N\sum_{l>j}U_{jl} \qquad (121)$$

dengan demikian:

$$\boldsymbol{Z} = \frac{1}{N!h^{3N}}\int_{\Gamma_{6N}}\exp\left[-\sum_{j=1}^{N}\left\{\frac{1}{2m}(p_{xj}^2+p_{yj}^2+p_{zj}^2) + \sum\nolimits_{l>j}U_{jl}\right\}/kT\right]d\Gamma_{6N}$$

$$= \frac{1}{N!h^{3N}}\left\{\int_{-\infty}^{\infty}\exp(-p_{x1}^2/2mkT)\,dp_{x1}\right\}^{3N}\times\int_V\int_V\cdots\int_V\exp\left(-\sum_{j=1}^{N}\sum_{l>j}U_{jl}/kT\right)\times\prod_{j=1}^{N}dx_j\,dy_j\,dz_j$$

$$= \frac{(2\pi mkT)^{3N/2}}{N!h^{3N}}\int_V\int_V\cdots\int_V\exp\left(-\sum_{j=1}^{N}\sum_{l>j}U_{jl}/kT\right)\times\prod_{j=1}^{N}dx_j\,dy_j\,dz_j$$

$$= \frac{(2\pi mkT)^{3N/2}}{N!h^{3N}}I_N \qquad (122)$$

persamaan keadaan diperoleh:

$$F = -kT\ln\boldsymbol{Z} = -kT\left[\ln\left\{\frac{(2\pi mkT)^{3N/2}}{N!h^{3N}}\right\} + \ln I_N\right] \qquad (123)$$

tekanan gas kemudian diperoleh:

$$p = -\left\{\frac{\partial F}{\partial V}\right\}_T = \frac{kT}{I_N}\left\{\frac{\partial I_N}{\partial V}\right\}_T \qquad (124)$$

Jika interaksi molekul cukup lemah, yakni $e^{-U_{jl}/kT} \simeq 1$ maka

$$I_N = \int_V\int_V\cdots\int_V\exp\left(-\sum_{j=1}^{N}\sum_{l>j}U_{jl}/kT\right)\times\prod_{j=1}^{N}dx_j\,dy_j\,dz_j \simeq \int_V dx_j\,dy_j\,dz_j = V^N \qquad (125)$$

atau

$$p = \frac{NkT}{V} \qquad (126)$$

Pers. keadaan dengan gugus meyer

$$e^{-U_{jl}/kT} = 1 + f(r_{jl}) \qquad (127)$$

maka

$$\exp\left(-\sum_{j=1}^{N}\sum_{l>j}U_{jl}/kT\right) = \prod_{l>j,j=1}\{1 + f(r_{jl})\}$$

Jika diasumsikan energi interaksi cukup lemah, maka dua suku pertama dari hasil perkalian di atas (bentuk f_{ab}) yang ditinjau, yakni:

$$\prod_{l>j,j=1}^{j=N}\{1 + f(r_{jl})\} \simeq 1 + \sum_{j=1}^{N}\sum_{l>j}^{N}f(r_{jl}) \qquad (128)$$

dengan demikian:

$$I_N = \int_V\int_V\cdots\int_V\left\{1 + \sum_{j=1}^{N}\sum_{l>j}f(r_{jl})\right\}\cdot\prod_{j=1}^{N}dx_j\,dy_j\,dz_j = V^N + \int_V\int_V\cdots\int_V\sum_{j=1}^{N}\sum_{l>j}f(r_{jl})\prod_{j=1}^{N}dx_j \qquad (129)$$

Jika dianggap energi interaksi dua buah molekul tidak bergantung pada $N-2$ molekul lain dan tidak ada keistimewaan antara pasangan molekul yang ditinjau:

$$I_N = V^N + \frac{N(N-1)}{2}\int_V\int_V\cdots\int_V f(r_{jl})\prod_{j=1}^{N}dx_j\,dy_j\,dz_j$$

$$= V^N + \frac{N(N-1)V^{N-2}}{2}\times\int_V\int_V f(r_{jl})dx_j\,dy_j\,dz_j\,dx_l\,dy_l\,dz_l \qquad (130)$$

Jika dianggap koordinat mula-mula dari molekul j berada pada posisi molekul l dan diasumsikan $f(r_{jl})$ akan menuju nol ketika r_{jl} meningkat. maka:

$$\int_V\int_V f(r_{jl})dx_j\,dy_j\,dz_j\,dx_l\,dy_l\,dz_l \simeq \int_V dx_l\,dy_l\,dz_l\int_0^{\pi}\sin\theta\,d\theta\times\int_0^{2\pi}d\psi\int_0^{\infty}f(r)r^2\,dr = V\int_0^{\infty}f(r)4\pi r^2\,dr \qquad (131)$$

Jika dimisalkan $\int_0^{\infty}f(r)4\pi r^2\,dr = a$ maka

$$I_N = V^N + \frac{N(N-1)}{2}V^{N-1}a \qquad (132)$$

sehingga dengan memasukkan ke dalam pers. 124 diperoleh:

$$p = \frac{kT}{\left\{V^N + \frac{N(N-1)}{2}V^{N-1}a\right\}}\times\left\{NV^{N-1} + \frac{N(N-1)^2}{2}V^{N-2}a\right\} \qquad (133)$$

Jika energi interaksi U_{jl} cukup kecil yakni $e^{-U_{jl}/kT} \simeq 1 - U_{jl}/kT$ maka $f(r_{jl}) \simeq -U_{jl}/kT$ dan konstanta a menjadi $a = -(1/kT)\int_0^{\infty}U(r)4\pi r^2\,dr = a'/kT$ dengan a' positif karena $U(r)$ negatif dalam domain tinjauan. Sehingga $a/V = a'/kTV \simeq a'N/pV^2$ yang menghasilkan

$$p \simeq \frac{NkT}{p}\left(1 - \frac{(N-1)}{2}\frac{Na'}{pV^2}\right) \qquad (134)$$

atau

$$\left(p + \frac{a''}{V^2}\right)V \simeq NkT \qquad (135)$$

dengan $a'' = (N-1)Na'/2$

Ekspansi deret temperatur tinggi

karena perkalian s_is_j hanya bernilai ±1 maka

$$e^{\beta Js_is_j} = \cosh\beta J + s_is_j\sinh\beta J \equiv \cosh\beta J(1 + s_is_jv);\quad v = \tanh\beta J \qquad (136)$$

di mana $v \to 0$ ketika $T \to \infty$. Jadi fungsi partisi dapat dituliskan menjadi:

$$\mathcal{Z} = \sum_{\{s\}}\prod_{\langle ij\rangle}e^{\beta Js_is_j} = (\cosh\beta J)^{\mathcal{B}}\sum_{\{s\}}\prod_{\langle ij\rangle}(1 + s_is_jv) = (\cosh\beta J)^{\mathcal{B}}\sum_{\{s\}}\left(1 + v\sum_{\langle ij\rangle}s_is_j + v^2\sum_{\langle ij\rangle;\langle kl\rangle}s_is_js_ks_l + \cdots\right) \qquad (137)$$

Karena $s_i = \pm1$ maka dalam fungsi partisi akan berlaku

$$\sum_{\{s\}}(s_i^{n_i}s_j^{n_j}s_k^{n_k}\cdots) = 2^N \quad \begin{pmatrix}\text{all } n_i\\ \text{even}\end{pmatrix}$$

$$= 0 \ (\text{lainnya})$$

sehingga pernyataan untuk suku-suku dalam orde v^n dapat dinyatakan ke dalam loop tertutup yang menghubungkan titik-titik latis. Dengan demikian fungsi Partisinya menjadi :

$$\mathcal{Z} = (\cosh\beta J)^{\mathcal{B}}2^N\left\{1 + Nv^4 + 2Nv^6 + \frac{1}{2}N(N+9)v^8 + 2N(N+6)v^{10} + O(v^{12})\right\} \qquad (138)$$

sehingga energi bebas :

$$\mathcal{F} = -NkT\left\{\ln 2 + v^2 + \frac{3}{2}v^4 + \frac{7}{3}v^6 + \frac{19}{4}v^8 + \frac{61}{5}v^{10} + O(v^{12})\right\} \qquad (139)$$

Ekspansi deret temperatur rendah

$$\mathcal{Z} = e^{-E_0/kT}\left(1 + \sum_{n=1}^{\infty}\Delta\mathcal{Z}_N^{(n)}\right) \qquad (140)$$

$\Delta\mathcal{Z}_N^{(n)}$ menyatakan penjumlahan terhadap faktor Boltzmann. Untuk model ising tiap ikatan yang salah yang diasosiasikan dengan spin flip akan mempunyai energi $2J$ relative terhadap ground state sehingga faktor Boltzmann:

$$x = e^{-2J/kT} \qquad (141)$$

jadi:

$$\mathcal{Z} = e^{-E_0/kT}\left\{1 + Nx^4 + 2Nx^6 + \frac{1}{2}N(N+9)x^8 + 2N(N+6)x^{10} + O(x^{12})\right\} \qquad (142)$$

Mean Field Critical Exponent

$$T = T_c(1+t) = \frac{Jz}{k}(1+t) \qquad (143)$$

maka

$$\langle\sigma\rangle_0 = \tanh\{\langle\sigma\rangle_0/(1+t)\} \qquad (144)$$

jika diuraikan dalam darat untuk $\langle\sigma\rangle_0$ dan t kecil, maka:

$$\langle\sigma\rangle_0 = \langle\sigma\rangle_0/(1+t) - \langle\sigma\rangle_0^3/3(1+t)^3 + O(\langle\sigma\rangle_0^5/(1+t)^5) = \langle\sigma\rangle_0(1-t) - \langle\sigma\rangle_0^3/3 + O(\langle\sigma\rangle_0t^2, \langle\sigma\rangle_0^3t, \langle\sigma\rangle_0^5) \qquad (145)$$

jika diaransemen kembali menghasilkan:

$$-t = \langle\sigma\rangle_0^2/3 + O(t^2, \langle\sigma\rangle_0^2t, \langle\sigma\rangle_0^4) \qquad (146)$$

atau

$$\langle\sigma\rangle_0 \sim (-t)^{1/2} \qquad (147)$$

$$\langle\sigma\rangle_0 = \tanh\beta(Jz\langle\sigma\rangle_0 + H) \qquad (148)$$

untuk $T = T_c$ maka

$$\langle\sigma\rangle_0 = \tanh(\langle\sigma\rangle_0 + H/jz) \qquad (149)$$

jika diuraikan untuk $\langle\sigma\rangle_0$ dan H kecil:

$$\langle\sigma\rangle_0 = \langle\sigma\rangle_0 + H/jz - \langle\sigma\rangle_0^3/3 + O(\langle\sigma\rangle_0^2H, \langle\sigma\rangle_0H^2, H^3, \langle\sigma\rangle_0^5) \qquad (150)$$

atau

$$\langle\sigma\rangle_0 \sim H^{1/3} \qquad (151)$$

contoh soal MF

Jika dipole menunjuk ke atas:

$$E_{\uparrow} = -\varepsilon\sum_{\text{tetangga}}s_{\text{tetangga}} = -n\bar{s} \qquad (152)$$

$$E_{\downarrow} = +\varepsilon n\bar{s} \qquad (153)$$

fungsi partisi

$$Z_i = e^{\beta\varepsilon n\bar{s}} + e^{-\beta\varepsilon n\bar{s}} = 2\cosh(\beta\varepsilon n\bar{s}) \qquad (154)$$

nilai ekspektasi spin :

$$\bar{s_i} = \frac{1}{Z_i}\left[(1)e^{\beta\varepsilon n\bar{s}} + (-1)e^{-\beta\varepsilon n\bar{s}}\right] = \frac{2\sinh(\beta n\varepsilon\bar{s})}{2\cosh(\beta\varepsilon n\bar{s})} = \tanh(\beta\varepsilon n\bar{s}) \qquad (155)$$

sehingga jika s_i merupakan ekspektasi spin keseluruhan atau $\bar{s}$, maka diperoleh :

$$\bar{s} = \tanh(\beta\varepsilon n\bar{s}) \qquad (156)$$

Transfer Matrix

$$\mathcal{Z} = \sum_{\{s\}}e^{\begin{pmatrix}BJ(s_0s_1 + s_1S_2 + \cdots \\ +s_{N-1}s_0) + \\ BH(s_0 + s_1 + \cdots \\ +s_{N-1})\end{pmatrix}} \qquad (157)$$

atau

$$\mathcal{Z} = \sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1}\exp\left[\beta\sum_{i=1}^{N}\left\{J\sigma_i\,\sigma_{i+1} + \frac{1}{2}\mu B(\sigma_i + \sigma_{i+1})\right\}\right]$$

$$= \sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1}\langle\sigma_1|\boldsymbol{P}|\sigma_N\rangle\langle\sigma_2|\boldsymbol{P}|\sigma_N\rangle\cdots\langle\sigma_{N-1}|\boldsymbol{P}|\sigma_N\rangle\langle\sigma_N|\boldsymbol{P}|\sigma_1\rangle \qquad (158)$$

dengan $\boldsymbol{P}$ menyatakan operator matriks yang elemen-elemennya dapat dinyatakan dalam:

$$\langle\sigma_i|\boldsymbol{P}|\sigma_{i+1}\rangle = \exp\left[\beta\left\{J\sigma_i\,\sigma_{i+1} - \frac{1}{2}\mu B(\sigma_i + \sigma_{i+1})\right\}\right] \qquad (159)$$

yakni:

$$(\boldsymbol{P}) = \begin{pmatrix} e^{\beta(J+\mu B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} \end{pmatrix} \qquad (160)$$

Karena $\sigma_1 = \sigma_{N+1}$ maka pernyataan untuk fungsi partisi di atas dapat dituliskan kembali sebagai:

$$\mathcal{Z} = \sigma_{\sigma_1=\pm1}\langle\sigma_1|\boldsymbol{P}^N|\sigma_1\rangle = \text{Trace}(\boldsymbol{P}^N) = \lambda_1^N + \lambda_2^N \qquad (161)$$

dengan λ_1 dan λ_2 menyatakan nilai eigen dari matriks $\boldsymbol{P}$ yang dapat diperoleh melalui:

$$\begin{vmatrix} e^{\beta(J+\mu B)} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} - \lambda \end{vmatrix} \qquad (162)$$

atau

$$\lambda^2 - 2\lambda e^{\beta J}\cosh(\beta\mu B) + 2\sinh(2\beta J) = 0 \qquad (163)$$

maka

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = e^{\beta J}\cosh(\beta\mu B) \pm \left[e^{-2\beta J} + e^{2\beta J}\sinh^2(\beta\mu B)\right]^{1/2} \qquad (164)$$

Karena $\lambda_2 < \lambda_1$ maka $(\lambda_2/\lambda_1)^N \to 0$ ketika $N \to \infty$ sehingga nilai eigen yang dominan adalah λ_1 atau:

$$\ln\mathcal{Z} \approx N\ln\lambda_1$$

maka

$$\frac{1}{N}\ln\mathcal{Z} \approx \ln\lambda_1 = \ln\left[e^{\beta J}\cosh(\beta\mu B) + \left\{e^{-2\beta J} + e^{2\beta J}\sinh^2(\beta\mu B)\right\}\right] \qquad (165)$$

maka

$$f = -kT\ln\mathcal{Z} = -kT\ln(e^{\beta J}\cosh(\beta h)) + \sqrt{e^{2\beta J}\sinh^2\beta h + e^{-2\beta J}} \qquad (166)$$