Maths for Economists

A Comprehensive Handbook

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Maths for Economists: A Comprehensive Handbook

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Preface

This book is written as a handbook for economists rather than a note for courses like Econ Math, where materials are selected from various sources including course handouts from SUFE, lecture notes from open scources and many great books. To make it a comprehensive one, I'll try to cover almost all maths needed for studying economics, including logic, algebra, analysis, topology, probability theory and the theory of optimization. And I'll add something more, basically based on the syllabus of the course I've taken. In this way can those(especially from SUFE) who're taking courses like Mathematical analysis refer to it.

Though served as a handbook, the book will be written as intuitive as possible so that all techniques and definitions are expressed in a natural way and readers can better understand them.

The book may be updated in a monthly basis.

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Part I

Foundations

Chapter 1

Logic

To make complex ideas clear, we often separate them into statements and check whether they're right or wrong. Here we use symbolic logic. Symbolic logic is basically about statements which can be claimed to be either true or false and "operations" of statements.

We start with what is a statement. For sure, most of the sentences can be seen as a statement. But to notice, not all sentences are statements. For example, "This sentence is false." is not a statement, that is because when it is true, it is false and vice versa.

In maths, we call statements as propositions and we can categorize them into theorems, axioms, lemmas, corollaries, etc. Axioms are statements that we accept to be true without proof. Theorems are statements that can be proven to be true based on axioms and previously proven theorems. Lemmas are "helper" theorems used to prove larger theorems and corollaries are statements that follow directly from a theorem.

1.1 Negation

We use the symbol ¬ to express "NOT". Suppose A is a statement, then $\neg A$ is also a statement and $\neg A$ means "A is NOT true". We can use a truth table to show the relationship between A and $\neg A$.

Statement	A	$\neg A$
Truth	T	F
Value	F	T

1.2 Conjunction and Disjunction

Conjunction and Disjunction are ways to combine statements together. We use the symbol $\wedge$ to express "AND". Suppose A and B are statements, then $A \wedge B$ is also a statement and $A \wedge B$ is true only when both A and B are true. Similarly, we use the symbol $\vee$ to express "OR". Suppose A and B are statements, then $A \vee B$ is also a statement and $A \vee B$ is false only when both A and B are false.

We can use a truth table to show the relationship between A, B, $A \wedge B$ and $A \vee B$.

Statement	A	B	$A \wedge B$	$A \vee B$
	T	T	T	T
Truth	T	F	F	T
Value	F	T	F	T
	F	F	F	F

1.3 Quantifiers

1.3.1 Property

We first define the concept of property. We say $P(x)$ is a property of x (x is from a particular class) if when x is replaced with a certain object, $P(x)$ becomes a statement. For example, let $P(x)$ be "x is even", then $P(2)$ is true and $P(3)$ is false. The set $\{x : P(x)\}$ consists of all values of x such that $P(x)$ is true.

1.3.2 Forall and Exists

Now we can create another kind of statement using quantifiers. There are two quantifiers: "for all" and "there exists". The expression $\forall x \in X : P(x)$ means "for all components x in the set X, $P(x)$ is true". And the expression $\exists x \in X : P(x)$ means "there exists at least one component x in the set X, such that $P(x)$ is true".

A very important proposition is that

$$\neg[\exists x \in X : P(x)] = \forall x \in X : \neg P(x) \tag{1.1}$$

And to understand all statements, we also need $\neg(\neg A) = A, \neg(A \wedge B) = \neg A \vee \neg B, \neg(A \vee B) = \neg A \wedge \neg B$. This becomes trivial to deal with quantifiers that we only have to interchange $\vee$ and $\wedge$ and change $\forall$ to $\exists$ and vice versa when we negate a statement.

1.4 Implications

1.4.1 Implication

The implication $(A \implies B) := (\neg A) \vee B$ is false if and only if A is true and B is false. The definition is simple, but note that $A \implies B$ not necessarily mean that A causes B to be true. When A and B are false, the implication is also true, which makes it different from the familiar meaning of "imply".

We also say that A is a sufficient condition for B, and B is a necessary condition for A when we write the statement $A \implies B$.

1.4.2 Equivalence

When A is both necessary and sufficient for B, we say that A is equivalent to B, and we write $A \iff B$. Note that $A \iff B$ is equivalent to $(A \implies B) \wedge (B \implies A)$.

1.4.3 Prove by Contrapositive

With equivalence, we know that the equation in (1.1) actually means equivalent and we can make clear of the above relationships. Moreover, using the interchanging technique, we can find the counterpositive statement

$$(A \implies B) \iff (\neg B \implies \neg A)$$

This inspires us to prove a statement, we can turn to its contrapositive if difficulties arise when proving the original statement.

1.4.4 Prove by Contradiction

Consider statements $(A \implies C), (C \implies B)$, then

$$(A \implies B) \iff (A \implies C) \wedge (C \implies B)$$

Suppose B is false and assume A is true, then $(C \implies B)$ and $(A \implies C)$ are false, that is to say, we can find a statement C with the false truth value.

This leads us to the proof by contradiction method. To prove $A \implies B$, we can assume A is true and B is false, then try to find a statement with a false truth value and we prove the original statement.

Chapter 2

Set, Relation and Operation

Since all readers should be familiar with the high-school-level set theory, we only give a brief review of the basic concepts and operations of sets.

We can't define what is a set but we can give it a description. A set is all objects that satisfy a certain property. The following lists some of the basics.

Definition 2.0.1 (Subset)**.** Let A, B be sets, when we say $A \subset B$, we mean $\forall x \in A, x \in B$. Notice

$$A = B \iff (A \subset B) \land (B \subset A)$$

2.1 Operations of Sets

2.1.1 Complement, Intersection and Union

Definition 2.1.1 (Complement, Intersection and Union)**.** Let A, B be sets, we define the following operations:

Complement The (relative) complement of set B in set A is defined as $A \setminus B = \{x : (x \in A) \land (x \notin B)\}$.

Intersection The intersection of sets A and B is defined as $A \cap B = \{x : (x \in A) \land (x \in B)\}$.

Union The union of sets A and B is defined as $A \cup B = \{x : (x \in A) \vee (x \in B)\}$.

By definition, one can easily check:

Proposition 2.1.1. *Let A, B, C be sets, then*

Commutativity $A \cap B = B \cap A$ *and* $A \cup B = B \cup A$.

Associativity $(A \cap B) \cap C = A \cap (B \cap C)$ *and* $(A \cup B) \cup C = A \cup (B \cup C)$.

Distributivity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

de Morgan's Laws $(A \cap B)^c = A^c \cup B^c$ *and* $(A \cup B)^c = A^c \cap B^c$

Subset $A \subset B \iff A \cap B = A \iff A \cup B = B$.

2.1.2 Power Set

Definition 2.1.2 (Power Set)**.** Let A be a set, define:

$$2^A = \{X : X \subset A\}$$

2.1.3 Cartesian Product

We define an ordered pair or a n-tuple $x = (x_1, x_2, \cdots, x_n)$ and the equity means that all components are equal. Denote $x_j := pr_j(x)$ and we call it the jth projection of x.

The Cartesian product is to describe the set of ordered pairs.

Definition 2.1.3 (Cartesian Product)**.** Let A, B be sets, define:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

and by deduction,

$$\prod_{i=1}^{n} A_i = (A_1 \times \cdots \times A_{n-1}) \times A_n$$

Specially, when $A_i = A$ for all i, we denote:

$$A^n = \prod_{i=1}^{n} A$$

To practice logic, we show how to prove the following proposition.

Proposition 2.1.2. *Let A, B be sets, then $A \times B = \emptyset \iff (A = \emptyset) \vee (B = \emptyset)$*

Proof. ($\Rightarrow$) We prove by contradiction. Suppose $A \times B$ is non-empty, which implies there exists $(a, b) \in A \times B$, then by definition of Cartesian product, we have $a \in A$ and $b \in B$, contradicting the assumption.

($\Leftarrow$) This side can be similarly proved by contradiction. $\square$

2.1.4 Family of Sets

We now extend the operation of intersection and union a little bit. Consider we're intersecting a series of sets $A, B, C, \cdots$, we first rename the sets as $\{A_i\}_{i \in I}$, where I is called an index set, and intersect them according to the index i, that is

$$\bigcap_{i \in I} A_i = \{x : \forall i \in I, x \in A_i\}.$$

Similarly, for union, we have

$$\bigcup_{i \in I} A_i = \{x : \exists i \in I, x \in A_i\}.$$

Proposition 2.1.3. *Let $\{A_\alpha : \alpha \in I_A\}$ and $\{B_\beta : \beta \in I_B\}$ be families of subsets of a set X, then*

(1) Associativity:

$$\left(\bigcap_{\alpha \in I_A} A_\alpha\right) \cap \left(\bigcap_{\beta \in I_B} B_\beta\right) = \bigcap_{(\alpha,\beta) \in I_A \times I_B} (A_\alpha \cap B_\beta)$$

and

$$\left(\bigcup_{\alpha\in I_A} A_\alpha\right)\cup\left(\bigcup_{\beta\in I_B} B_\beta\right)=\bigcup_{(\alpha,\beta)\in I_A\times I_B}(A_\alpha\cup B_\beta)$$

(2) Distributivity:

$$A\cap\left(\bigcup_{\alpha\in I_A} A_\alpha\right)=\bigcup_{\alpha\in I_A}(A\cap A_\alpha)$$

and

$$A\cup\left(\bigcap_{\alpha\in I_A} A_\alpha\right)=\bigcap_{\alpha\in I_A}(A\cup A_\alpha)$$

(3) de Morgan's Laws:

$$\left(\bigcap_{\alpha\in I_A} A_\alpha\right)^c=\bigcup_{\alpha\in I_A} A_\alpha^c$$

and

$$\left(\bigcup_{\alpha\in I_A} A_\alpha\right)^c=\bigcap_{\alpha\in I_A} A_\alpha^c$$

Proof. We show how to prove (3). □

2.2 Relation and Operation

This subsection is to formally introduce relationship between elements in a set X. A binary relation R on a set X is a subset of the Cartesian product $X \times X$. For $(x, y) \in R$, we denote it as xRy.

There're several important types of relations.

Relation	Meaning
Reflexive	$\forall x \in X, xRx$
Symmetric	$xRy \implies yRx$
Transitive	$(xRy)\wedge(yRz) \implies (xRz)$
Complete	$\forall x, y \in X, (xRy)\vee(yRx)$

We now turn to some special relations.

2.2.1 Order

A relation $\leq$ on a set X is a partial order if it's reflexive, transitive, and anti-symmetric*. We call the pair $(X, \leq)$ a partially ordered set. If, in addition to reflexiveness, transitivity, and anti-symmetric, completeness also holds for $\leq$, we say it's a total order and $(X, \leq)$ is a total ordered set.

We now turn to the elements in a partially ordered set $(X, \leq)$.

Definition 2.2.1 (Monotonicity)**.** Given a partially ordered set $(X, \leq)$, for $\{x_n\}_{n=1}^{\infty} \subset X$, we say it's **monotonically increasing** if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$; similarly, it's **monotonically decreasing** if $x_{n+1} \leq x_n, \forall n \in \mathbb{N}$.

Definition 2.2.2 (Boundedness)**.** Given a partially ordered set $(X, \leq)$ and $A \subseteq X$, an element x is an **upper bound** of A if $a \leq x, \forall a \in A$. We say A is bounded above. And similarly, an element x is a **lower bound** of A if $x \leq a, \forall a \in A$ and it's bounded below.

Definition 2.2.3 (Supremum and Infimum)**.** Given a partially ordered set $(X, \leq)$ and $A \subseteq X$, if A is bounded above, then we define

$$\sup A = \min\{x \in X : x \text{ is an upper bound of } A\}$$

Similarly, if A is bounded below,

$$\inf A = \max\{x \in X : x \text{ is a lower bound of } A\}$$

Note 1. Note that, if $\sup A$ and $\max A$ exists, $\sup A \in A \iff \sup A = \max A$. Similarly, if $\inf A$ and $\min A$ exists, $\inf A \in A \iff \inf A = \min A$.

A very important example of partially order is the subset relation. We introduce the concept of lim inf and lim sup of a sequence of sets as an example, which is of great importance in measure theory.

Example 2.2.1. Let $\{A_i\}_{i=1}^{\infty}$ be a family of subsets of a set X. Let $\mathcal{A}^{(n)}$ and $\mathcal{A}_{(n)}$

* Anti-symmetry means $(x \leq y) \land (y \leq x) \implies x = y$

denote

$$\mathcal{A}^{(n)} = \bigcap_{i=n}^{\infty} A_i, \quad \mathcal{A}_{(n)} = \bigcup_{i=n}^{\infty} A_i$$

and notice that

$$\mathcal{A}^{(n+1)} \subset \mathcal{A}^{(n)}, \mathcal{A}_{(n)} \subset \mathcal{A}_{(n+1)}, \quad \forall n \in \mathbb{N}$$

Since $\left(\left\{\mathcal{A}^{(n)}\right\}_{n=1}^{\infty}, \subset\right)$ and $\left(\left\{\mathcal{A}_{(n)}\right\}_{n=1}^{\infty}, \subset\right)$ are partially ordered sets, we have

$$\inf \left\{\mathcal{A}^{(n)}\right\}_{n=1}^{\infty} = \bigcup_{n=1}^{\infty} \mathcal{A}^{(n)} = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$
$$\sup \left\{\mathcal{A}_{(n)}\right\}_{n=1}^{\infty} = \bigcap_{n=1}^{\infty} \mathcal{A}_{(n)} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$$

denoted as $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ respectively.

The concept of partial order is important in economics as preference is actually a partial order. We can define certain kinds of preferences with ease.

Definition 2.2.4 (Rational Preference)**.** A preference is rational if and only it's complete and transitive.

2.2.2 Mapping

We've encountered mappings before, where we say mapping is a rule that assigns each element of a set, the domain, to exactly one element of the image set. If treated carefully, we may find the word "rule" vague. We now try to give it a formal definition.

Consider the set $G = \{(x, f(x)) : x \in X\}$. To make $f : X \to Y$ a mapping, given any $x \in X$, there should exist only one $y \in Y$ such that $(x, y) \in G$. This leads to the following definition.

Definition 2.2.5 (Mapping)**.** A mapping $f : X \to Y$ is a binary relation

$$G = \{(x, f(x)) : x \in X \wedge (\exists! y \in Y \text{s.t.} f(x) = y)\}$$

and we denote $f(A) = \operatorname{Im} f|_A$, where $A \subset X$.

We now list some important concepts.

Definition 2.2.6. Let $f : X \to Y$ be a mapping.

Surjective f is surjective $\iff \operatorname{Im} f = Y$.

Injective f is injective $\iff (f(x) = f(y) \implies x = y)$.

Bijective f is bijective $\iff f$ is both injective and surjective.

Inverse Suppose f is bijective, then there exists an inverse function $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{Id}_X$.[†]

Proposition 2.2.1 (Set Valued Mapping)**.** *Suppose we have $f : X \to Y$ and families of subsets of X $\{A_i\}_{i \in I_A}$ and $\{B_j\}_{j \in I_B}$. Then we have*

(1) $f\left(\bigcup_{i \in I_A} A_i\right) = \bigcup_{i \in I_A} f(A_i)$
(2) $f\left(\bigcap_{i \in I_A} A_i\right) \subset \bigcap_{i \in I_A} f(A_i)$
(3) $f^{-1}\left(\bigcup_{j \in I_B} B_j\right) = \bigcup_{j \in I_B} f^{-1}(B_j)$
(4) $f^{-1}\left(\bigcap_{j \in I_B} B_j\right) = \bigcap_{j \in I_B} f^{-1}(B_j)$

2.2.3 Correspondence

For mappings, we assign only one element in the image set to each element in the domain. We now consider assigning multiple elements in the image set to each element in the domain, which is called correspondence.

Definition 2.2.7 (Correspondence)**.** A correspondence $\varphi : X \twoheadrightarrow Y$ is a binary relation

$$G = \{(x, \varphi(x)) : x \in X \wedge (\varphi(x) \subset Y)\}.$$

and for the subset A of X, we call $\varphi^u(A) := \{x \in X : \varphi(x) \subset A\}$ the upper inverse image of A; similarly, we call $\varphi^l(A) := \{x \in X : \varphi(x) \cap A \neq \emptyset\}$ the lower inverse image of A.

† f^{-1} is well-defined as we'll show later that an operation has at most one identity element and then we can construct the inverse function with the bijective property. An important proposition is $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

2.2.4 Operation

An operation is a mapping $\circledast : X \times X \to Y$ and we denote $\circledast(x, y)$ as $x \circledast y$ in convention. We say the subset A of X is closed under the operation $\circledast$ if $A \circledast A \subset A$.

Two important kinds of operation we often meet are associative and commutative operations.

Associative An operation $\circledast$ is associative if $(x \circledast y) \circledast z = x \circledast (y \circledast z)$ for all $x, y, z \in X$.

Commutative An operation $\circledast$ is commutative if $x \circledast y = y \circledast x$ for all $x, y \in X$.

Proposition 2.2.2 (Uniqueness of Identity Element)**.** *We call an element e an identity element if and only if $x \circledast e = e \circledast x = x$ for all $x \in X$. For a certain operation, there exsists at most one identity element.*

Proof. We prove by contradiction. Suppose there're two identity elements e and e'. Then $e = e \circledast e' = e'$, contradicting with $e \neq e'$. □

2.3 Countability

For finite sets, we can easily count the element and say it's countable. But what if the set have infinite many elements? Mathematicians believe that natural numbers can be counted and is the smallest infinite set. In this section we formally treat infinity, before which we introduce the axiom of choice.

2.3.1 The Axiom of Choice

The axiom of choice states that for a family of non-empty sets $\{A_i\}_{i \in I}$, the Cartesian product $\prod_{i \in I} A_i$ is non-empty.

That is to say there exists a choice function f such that $\forall i \in I, f(i) \in A_i$.

2.3.2 Cardinality

The cardinality of a finite set is defined as the number of elements, that is $|A| = \#A$. For infinite sets, the definition is complicated and we turn to compare the cardinality of 2 sets.

Definition 2.3.1 (Equality of Cardinality)**.** Two sets A and B are said to have the same cardinality if there exists a bijection from A to B, denoted as $|A| = |B|$. If there exists a subsection of B and a bijection from A to that subsection, we say $|A| \leq |B|$.

Proposition 2.3.1. *Let X, Y be sets, the following statements are equivalent:*

(1) $|X| \leq |Y|$
(2) there exists an injection from X to Y.
(3) there exists a surjection from Y to X.

Proof. (1) $\implies$ (2) is done by definition.

To show (2) $\implies$ (3), let g be a mapping from $f(X)$ to X such that $g \circ f|_{f(X)} = \mathrm{Id}_{f(X)}$, which is a bijection. For a fixed $x \in X$, define

$$h(y) = \begin{cases} g(y), & y \in f(X) \\ x, & y \notin f(X) \end{cases}$$

which is a surjection from Y to X.

(3) $\implies$ (1) needs the axiom of choice. Let $A_x := \{y \in Y : f(y) = x\}$ and A_x is non-empty as f is surjective. By the axiom of choice, there exists a choice function h such that $h(x) \in A_x$. Notice that

$$x \neq x' \implies (A_x \cap A_{x'}) = \emptyset \implies h(x) \neq h(x')$$

Then h is an injection from X to Y. □

Actually, $(|\cdot|, \leq)$ is a total ordered set. See the following theorems.

Theorem 2.3.2 (Anti-symmetry and Completeness of Cardinality)**.** *Let X, Y be sets.*

Bernstein-Schroeder $(|Y| \leq |X| \wedge |X| \leq |Y|) \implies |X| = |Y|$

Completeness $|X| \leq |Y| \vee |Y| \leq |X|$

Note 2. Note that the completeness is equivalent to the axiom of choice.

2.3.3 Countable Set

Definition 2.3.2 (Countability). A set X is countable if $|X| = |\mathbb{N}|$.

Chapter 3

Group, Ring and Field

Part II

Vector Space and Linear Algebra

Chapter 4

Vector Space and Linear Mapping

Chapter 5

Normed Vector Space

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Chapter 6

Multilinear Form and Determinant

One good way to introduce determinant is through solving equation systems and define it by deduction. Consider solving

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

then we have

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

and denote

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

We can rewrite the solution as

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}, x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

Now we turn to the 3-dimensional case:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Similarly, by elimination method, we have

$$x_1 = \frac{b_1 \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} b_2 & a_{22} \\ b_3 & a_{32} \end{pmatrix}}{a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}$$

$$x_2 = \frac{a_{11} \det\begin{pmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{pmatrix} - b_1 \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & b_2 \\ a_{31} & b_3 \end{pmatrix}}{a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}$$

$$x_3 = \frac{a_{11} \det\begin{pmatrix} a_{22} & b_2 \\ a_{32} & b_3 \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & b_2 \\ a_{31} & b_3 \end{pmatrix} + b_1 \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}{a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}$$

This leads us to denote

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

and rewrite the solution as

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}, x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}, x_3 = \frac{\det\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}$$

In this way can we define determinant in a deductive way and achieve the important Cramer's Law easily. However, this definition arises only as a notation, lacking the geometric understanding of determinants. To do this, we start from the concept of **multilinear form**.

6.1 Multilinear Form

6.1.1 Bilinear Form

A bilinear form on V is a function on $V \times V$ and is linear in each slot while holding the other one fixed.

Definition 6.1.1 (Bilinear Form)**.** Let V be a vector space over field $\mathbb{F}$. A function $\beta : V \times V \to \mathbb{F}$ is called a bilinear form on V if

$$v \mapsto \beta(u, v), \quad u \mapsto \beta(u, v)$$

are both linear mapping on V. We denote $V^{(2)}$ as the set of all bilinear forms in V.

An example of bilinear form is the inner product.

Example 6.1.1 (Inner Product)**.** Let β be the inner product $\langle \cdot, \cdot \rangle$. Recall the linearity of inner product in each slot and we can verify by definition that inner product is indeed a bilinear form.

6.1.1.1 Symmetric and Alternating Bilinear Form

However, though the inner product is symmetric, that is $\beta(u,v)=\beta(v,u)$, symmetry does not always hold for bilinear forms. There is a special kind of bilinear form called **symmetric bilinear form**.

Definition 6.1.2 (Symmetric Bilinear Form)**.** A bilinear form β from $V^{(2)}$ is called symmetric if

$$\beta(u,v)=\beta(v,u), \quad \forall u,v\in V.$$

We denote $V^{(2)}_{sym}$ as the set of all symmetric bilinear forms in $V^{(2)}$. It's trivial to show that $V^{(2)}_{sym}$ is a subspace of $V^{(2)}$.

Another special and important kind of bilinear form is **alternating bilinear form**. We'll see its importance when we reach the final theorem that $V^{(2)}$ can be separated as the direct sum of $V^{(2)}_{sym}$ and $V^{(2)}_{alt}$.

Definition 6.1.3 (Alternating Bilinear Form)**.** A bilinear form β from $V^{(2)}$ is called alternating if

$$\beta(v,v)=0, \quad \forall v\in V.$$

We denote $V^{(2)}_{alt}$ as the set of all alternating bilinear forms in $V^{(2)}$. It's trivial to show that $V^{(2)}_{alt}$ is a subspace of $V^{(2)}$.

Proposition 6.1.1 (Characterization of Alternating Bilinear Form)**.** *A bilinear form α is alternating if and only if*

$$\alpha(u,v)=-\alpha(v,u), \quad \forall u,v\in V.$$

Proof. Notice that, by definition of bilinear form, we have

$$\alpha(u+v,u+v)=\alpha(u,u)+\alpha(u,v)+\alpha(v,u)+\alpha(v,v)$$

Suppose α is alternating, then $\alpha(u+v,u+v)=\alpha(u,v)+\alpha(v,u)=0$, which implies $\alpha(u,v)=-\alpha(v,u)$.

Conversely, suppose $\alpha(u,v)=-\alpha(v,u)$, then $\alpha(v,v)=-\alpha(v,v)$, which implies $\alpha(v,v)=0$ for all v. Therefore, α is alternating. □

Now we turn to the final theorem of alternating and symmetric bilinear forms.

Theorem 6.1.2 (Decomposition of Bilinear Form)**.** *Given a vector space V, we have*

$$V^{(2)} = V^{(2)}_{sym} \oplus V^{(2)}_{alt}.$$

Proof. We first show that $V^{(2)} = V^{(2)}_{sym} + V^{(2)}_{alt}$. Suppose $\beta \in V^{(2)}$, Let

$$\beta_{sym} = \frac{\beta(u,v)+\beta(v,u)}{2}, \quad \beta_{alt} = \frac{\beta(u,v)-\beta(v,u)}{2}$$

It's trivial that $\beta_{sym} \in V^{(2)}_{sym}$ and $\beta_{alt} \in V^{(2)}_{alt}$, and $\beta = \beta_{sym} + \beta_{alt}$.

Then we only have to show that $V^{(2)}_{sym} \cap V^{(2)}_{alt} = \{0\}$. Suppose $\beta \in V^{(2)}_{sym} \cap V^{(2)}_{alt}$. Then β is both symmetric and alternating, so $\beta(u,v) = \beta(v,u)$ and $\beta(u,v) = -\beta(v,u)$, which implies $\beta(u,v) = 0$ for all u, v. Therefore, $V^{(2)}_{sym} \cap V^{(2)}_{alt} = \{0\}$. $\square$

6.1.1.2 Matrix of a Bilinear Form

6.1.2 Multilinear Form

We now turn to define determinants and this subsection is for preparation. The definition of multilinear form is a natural extension of bilinear form.

Definition 6.1.4 (Multilinear Form)**.** Let V be a vector space over field $\mathbb{F}$. A k-multilinear form

$$\beta : \underbrace{V \times V \times \cdots \times V}_{k \text{ times}} \longrightarrow \mathbb{F}$$

satisfies that

$$v_i \mapsto \beta(v_1, \ldots, v_i, \ldots, v_k)$$

are all linear mapping on V for each $i = 1, 2, \ldots, k$. We denote $V^{(k)}$ as the set of all k-multilinear forms in V.

6.1.2.1 Alternating Multilinear Form

For multilinear forms, we only concern about alternating ones.

Definition 6.1.5 (Alternating Multilinear Form)**.** A multilinear form β from $V^{(k)}$ is called alternating if

$$\beta(v_1, \dots, v_k) = 0$$

whenever $v_i = v_j$ for some $i \neq j$. We denote $V_{alt}^{(k)}$ as the set of all alternating multilinear forms in $V^{(k)}$. It's trivial to show that $V_{alt}^{(k)}$ is a subspace of $V^{(k)}$.

By linearity at each slot, it's trivial to show that for a linearly dependent set $\{v_1, \dots, v_k\}$, we have $\alpha(v_1, \dots, v_k) = 0$ for any alternating multilinear form $\alpha \in V_{alt}^{(k)}$. The counterpositive satement implies that if $\alpha(v_1, \dots, v_k) \neq 0$ for some alternating multilinear form $\alpha \in V_{alt}^{(k)}$, then $\{v_1, \dots, v_k\}$ is linearly independent. Actucally, with the theorem $\dim V_{alt}^{(\dim V)} = 1$, which will be shown later, the converse statement also holds.

Theorem 6.1.3. *For a nonzero $\alpha \in V_{alt}^{(k)}$, $\alpha(v_1, \dots, v_k) \neq 0 \iff \{v_1, \dots, v_k\}$ is linearly independent.*

6.1.2.2 Swapping the Entries

Theorem 6.1.4 (Swapping 2 Slots of an Alternating Multilinear Form)**.** *For alternating multilinear form α and distinct $i, j \in \{1, 2, \dots, k\}$, we have*

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

Proof. We show the case of swapping v_1 and v_2. Notice that

$$\begin{aligned}
0 &= \alpha(v_1 + v_2, v_1 + v_2, \dots, v_k) \\
&= \alpha(v_1, v_1, \dots, v_k) + \alpha(v_1, v_2, \dots, v_k) + \alpha(v_2, v_1, \dots, v_k) + \alpha(v_2, v_2, \dots, v_k) \\
&= \alpha(v_1, v_2, \dots, v_k) + \alpha(v_2, v_1, \dots, v_k)
\end{aligned}$$

We have $\alpha(v_1, v_2, \dots, v_k) = -\alpha(v_2, v_1, \dots, v_k)$. $\square$

To generalize the swapping theorem, we consider swapping multiple slots and multiple times. The intuitive idea asks us to count how many times we just swapped and we can use the above theorem. That is to consider, given the original list $(v_1, \dots, v_k)$ and the swapped list $(v_{j_1}, \dots, v_{j_k})$ with exactly the same elements but in different orders, how many swaps are made. We don't know how many swaps are made since there're multiple ways to do it and actually, how the swaps were made wouldn't change the result. So we can reconstruct the process of swapping by doing adjacent swaps, that is only swapping v_i and v_{i+1}. In this manner, we can count the swap by comparing the original order and the final order. We introduce the concept of permutation to understand the full picture following our intuition.

Definition 6.1.6 (Permutation and its Sign)**.** A permutation of the list $(1, \dots, k)$ is an element of $\operatorname{perm} k := \{(j_1, \dots, j_k) \in \mathbb{N}^k : j_u \neq j_v \quad \text{and} \quad 1 \leq j_i \leq k\}$.

The sign of a permutation maps a certain element of perm k to either $+1$ or -1. For a permutation $(j_1, \dots, j_k) \in \operatorname{perm} k$, its sign $\operatorname{sign}(j_1, \dots, j_k)$ is defined as

$$\operatorname{sign}(j_1, \dots, j_k) = (-1)^N$$

where N is the cardinality of $\{(m, n) : 1 \leq m < n \leq k \text{ and } j_m > j_n\}$.

The next result is straightforward.

Proposition 6.1.5 (Generalization of the Swapping)**.** *Given a permutation* $(j_1, \dots, j_k) \in \operatorname{perm} k$ *and an alternating multilinear form* $\alpha \in V_{alt}^{(k)}$*, we have*

$$\alpha(v_{j_1}, \dots, v_{j_k}) = \operatorname{sign}(j_1, \dots, j_k)\alpha(v_1, \dots, v_k)$$

Chapter 7

Eigenvalue and Eigenvector

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Part III

Metric Space and Convergence

Part IV

Topology Space and Functions

Part V

Differentiation in One Variable

Part VI

More in Convergence: Uniform Convergence

Appendix A

Notation

Appendix B

Supplementary Scripts

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