Hierarchical (Multilevel) Modeling

Structure and Inference

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Hierarchical Modeling

Hierarchical Modeling: Core Idea

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Addresses limitations of simpler approaches:

Complete Pooling (Ignore Structure)
Analyze all data together.

× Underestimates errors.

× Hides group differences (Ecological Fallacy).

× Doesn't quantify group variability.

No Pooling (Separate Analyses)
Analyze each group separately.

× Ignores group similarities.

× Inefficient.

× Noisy estimates (esp. small groups).

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- "Borrow strength": Groups inform each other.
- Improves individual group estimates (especially for noisy/small groups).
- Simultaneously estimates population-level effects and the extent of variation between groups.

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  where $\epsilon_{ij} \sim \mathcal{N}(0, \sigma_y^2)$ (Residual variance).

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This structure explicitly models the dependency within groups.

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- **Level 2 (Between-Group):**

  Intercepts α_j and slopes β_j are drawn from a population distribution, often modeled as multivariate normal to capture potential correlation (ρ).

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \sim \mathcal{MVN}\left(\begin{pmatrix} \mu_\alpha \\ \mu_\beta \end{pmatrix}, \mathbf{\Sigma} = \begin{pmatrix} \sigma_\alpha^2 & \rho\sigma_\alpha\sigma_\beta \\ \rho\sigma_\alpha\sigma_\beta & \sigma_\beta^2 \end{pmatrix} \right)$$

  - μ_α, μ_β: Population average intercept and slope.
  - $\sigma_\alpha^2, \sigma_\beta^2$: Variance of intercepts and slopes across groups.
  - ρ: Correlation between intercepts and slopes across groups.

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Variance Components Parameters characterizing variability at different levels:

- Level 1: Residual variance σ_y^2.
- Level 2: Random effect variances $(\sigma_\alpha^2, \sigma_\beta^2)$ and their correlation/covariance $(\boldsymbol{\Sigma})$. Quantify the magnitude of group

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- $\text{Var}(\hat{\alpha}_{j,\text{no pool}})$ depends on group size n_j and within-group variance σ_y^2.
- Group estimates are **shrunk** towards the population mean.

Hierarchical Modeling: Adaptive Shrinkage

The amount of shrinkage is adaptive and data-dependent: **More Shrinkage** (towards $\hat{\mu}_\alpha$) when:

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This adaptive regularization prevents overfitting and leads to better out-of-sample predictions compared to no-pooling or complete-pooling models.

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  - Level 1 Likelihood: $P(\text{Data}|\text{Level 1 Params})$
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- **Computation:** Modern MCMC (e.g., HMC/NUTS in Stan, PyMC) handles complex posteriors effectively.
- **Prior Specification:** Requires care!
  - Fixed Effects / Means (μś): Often weakly informative (e.g., wide Normal).
  - Variance Components (σś): Crucial! Use weakly informative priors concentrated away from zero (e.g., Half-Normal, Half-Cauchy) to avoid issues.
  - Correlations (ρś): LKJ priors are common for correlation matrices.

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Interpretation: Focus on population parameters (μś, fixed βś), magnitude of variation (σś), and potentially shrunken group estimates (α_jś, β_jś), always with uncertainty

Hierarchical Modeling: Advanced Topics

Briefly:

- **Crossed vs. Nested Random Effects:**
  - Nested: Students in classrooms, classrooms in schools.
  - Crossed: Participants respond to multiple stimuli (random effects for participant AND stimulus, not nested).
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- **Non-Centered Parameterization (Reparameterization):**
  - Crucial for MCMC efficiency, especially with small group variances or sparse data.
  - Instead of $\alpha_j \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$...
  - ... use $z_j \sim \mathcal{N}(0, 1)$ and define $\alpha_j = \mu_\alpha + z_j \times \sigma_\alpha$.
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- **Multilevel Models for Longitudinal Data:** Analyzing change over time, often

Hierarchical Modeling: Key References i

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