Item response theory

Joachim Vandekerckhove

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- items may differ in their difficulty (or their correspondence to the construct)

Representations in IRT

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We can quantify the dominance, $\eta_{ip} \in \mathbb{R}$, a respondent has over an item

$$\eta_{ip} = \theta_p - \beta_i$$

Linking functions

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For this, we will use a *linking* function – a function that takes as input $\eta_{ip} \in \mathbb{R}$ and gives as output a value $(0, 1)$

$$\mathbb{R} \xrightarrow{\text{link}} (0, 1)$$

Linking functions

In IRT, we use the *logistic function* (also known as the *inverse logit*):

$$P\left(X_{ip}=1\right)=\pi_{ip}=\frac{1}{1+\exp(-\eta_{ip})}$$

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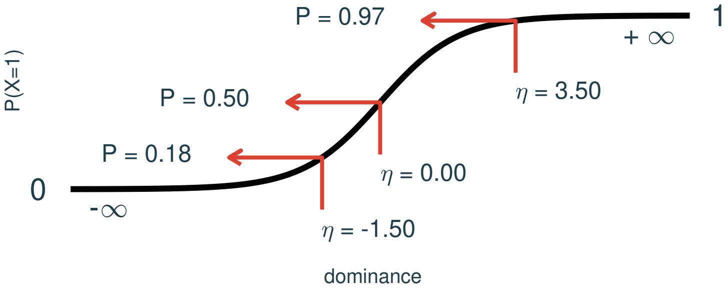

The Rasch model

A much more convenient way of writing that is:

$$
\begin{aligned}
P\left(X_{ip}=1\right) &= \text{ilogit}\left(\eta_{ip}\right) \\
&= \text{ilogit}\left(\theta_p-\beta_i\right) \\
\Leftrightarrow \quad \text{logit}\left(P\left(X_{ip}=1\right)\right) &= \theta_p-\beta_i
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This exact formulation is one of the most common IRT models (there are a few). This is called the *Rasch* model, after Danish psychometrician Georg Rasch (Rasch, 1961)

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$$\begin{aligned} P\left(X_{ip}=1\right) &= \text{ilogit}\left(\eta_{ip}\right) \\ &= \text{ilogit}\left(\theta_p - \beta_i\right) \\ \Leftrightarrow \quad \text{logit}\left(P\left(X_{ip}=1\right)\right) &= \theta_p - \beta_i \end{aligned}$$

This exact formulation is one of the most common IRT models (there are a few). This is called the *Rasch* model, after Danish psychometrician Georg Rasch (Rasch, 1961)

The attraction of this model is that it lets us take the data matrix $\mathbf{X}$ and infer person-specific abilities and item-specific difficulties

Item response data

The data matrix $\mathbf{X}$ is simply the person-by-item matrix of 0s and 1s

person	item 1	item 2	item 3	...
1	1	1	1	...
2	1	1	0	...
3	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
67	1	1	1	...

Guttman scales

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- Only the person's ability and the item difficulty affect the proability of responding correctly
- The response on one item does not affect the response on any other item
- Items can be rank ordered in difficulty
- Participants can be rank ordered in ability

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Consider this segment of the data matrix:

person	I_1	I_2	I_3
1	1	1	1
2	1	1	0
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$\rightarrow$

person	I_2	I_1	I_3
3	1	0	0
2	1	1	0
1	1	1	1
67	1	1	1

If the unidimensionality assumption holds, the data matrix can be rearranged so that all its zeros are in a triangle

Guttman scales

If this rearrangement is possible, then we can combine the item scores into a single sum score, which can then be interpreted as a unidimensional measure of the ability of the respondents.

person	I_2	I_1	I_3	Sum Score
3	1	0	0	1
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A sum score that meets this condition is called a *Guttman scale*

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The most common reasons for violation of these assumptions are:

- There may be randomness in the data (e.g., respondents ‘lapse’ and choose the wrong answer due to distraction), or

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- The underlying construct may really be multidimensional (e.g., a math test that uses difficult phrases is really also a test of English comprehension), or

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- The order of the items causes sequential effects (some examples in Moore, 2002)

Item response theory

The Rasch model, $P(X_{ip}) = \text{ilogit}(\theta_p - \beta_i)$, is a relatively simple IRT model. In psychometric circles, it is often called the *one-parameter logistic model* or simply 1PL

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Some of its relatives are:

- 2PL: $P(X_{ip}) = \text{ilogit}(\alpha_i\,(\theta_p - \beta_i))$
  Here, α_i is a discriminability index that captures how important the item i is for measuring the construct

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- 3PL: $P(X_{ip}) = \gamma_i + (1 - \gamma_i) \times \text{ilogit}(\alpha_i\,(\theta_p - \beta_i))$
  Here, γ_i is a guessing parameter that captures how well a participant could do by guessing the answer
- Explanatory item response models: $P(X_{ip}) = \text{ilogit}\,(\theta_p - \gamma W_i)$
  Here, β_i is replaced by an expression based on some predictor W

References

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