ME4231

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1 Week 1

1.1 Deriving NS equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{1}$$

On the left-hand side, there consists of the time derivative term.

$$\frac{\partial \mathbf{u}}{\partial t} = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{pmatrix} \tag{2}$$

There also consists of the convective term.

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \left(\begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \right) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) u \\ \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) v \\ \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) w \end{pmatrix} \tag{3}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix} \tag{4}$$

On the right-hand side, there consists of the pressure gradient term.

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho} \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ -\frac{1}{\rho}\frac{\partial p}{\partial z} \end{pmatrix} \tag{5}$$

Also, the viscous friction term on RHS.

$$\nu \nabla^2 \mathbf{u} = \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \nu \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix} \tag{6}$$

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} + w \cdot \frac{\partial u}{\partial z} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} + \nu \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + f_x$$

$$\frac{\partial v}{\partial t} + u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} + w \cdot \frac{\partial v}{\partial z} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial y} + \nu \cdot \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + f_y$$

$$\frac{\partial w}{\partial t} + u \cdot \frac{\partial w}{\partial x} + v \cdot \frac{\partial w}{\partial y} + w \cdot \frac{\partial w}{\partial z} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} + \nu \cdot \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + f_z$$

Figure 1: The full form of Naiver Stokes Momentum Equation

This gives the equations listed in figure 1. Conversion of matrices above to PDE is not listed.

1.1.1 Linear Momentum Equations

$$\sum \mathbf{F} = \sum (\dot{m} \cdot \mathbf{v})_{\text{out}} - \sum (\dot{m} \cdot \mathbf{v})_{\text{in}} \tag{7}$$

Assumptions: steady flow.

Net force acting on a system = **rate of change** of linear momentum.

Can separate into 2 components, x and y, if needed.

$$\dot{m} = \rho A v = m v \tag{8}$$

Using equation above, and adding m we obtain

Relating linear momentum $p = mv$ to the NS equations is shown below. Multiply density to all terms

For first term in NS equations,

$$\rho \frac{\partial u}{\partial t} = \frac{m}{V} \frac{\partial u}{\partial t} \tag{9}$$

We obtain forces on a volume.

For second term,

$$\rho u \cdot \frac{\partial u}{\partial x} = \tag{10}$$

continue with derivation of NS equations, slide 27

continue with non-dimensionalized equations

1.2 Mathematics

Prerequisite: Gradient operator ∇, ∇k, where k is a constant, divergence $\nabla \cdot \mathbf{V}$, Laplacian $\nabla^2 = \Delta = \nabla \cdot \nabla$ (Note this is a scalar, treat this like a mathematical operator, similar to $\frac{\partial^2}{\partial x^2}$), curl $\nabla \times \mathbf{V}$.

1.2.1 Vorticity

Vorticity is a vector, $\boldsymbol{\zeta}$. (In notes expressed as $\boldsymbol{\omega}$ but do not wish to confuse with angular velocity).

$$\boldsymbol{\zeta} = \nabla \times \mathbf{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{11}$$

Where $\mathbf{v}$ is a velocity vector.

Recap on Angular Kinematics

$$s = r\theta \tag{12}$$

$$v = r\omega \tag{13}$$

$$a = r\alpha \tag{14}$$

Relate Angular Velocity to Vorticity

Below is a prerequisite on triple product rule of vectors.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\,\mathbf{a} \tag{15}$$

Utilizing the above rule,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{16}$$

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = (\nabla \cdot \mathbf{r})\boldsymbol{\omega} - (\nabla \cdot \boldsymbol{\omega})\mathbf{r} \tag{17}$$

$$= \boldsymbol{\omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) - \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{18}$$

Due to x, y, and z being independent of each other, $\frac{\partial x}{\partial y}$, $\frac{\partial x}{\partial z}$, $\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial z}$, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y} = 0$

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = 3\boldsymbol{\omega} - \boldsymbol{\omega} \tag{19}$$

$$\boldsymbol{\zeta} = \nabla \times \mathbf{v} = 2\boldsymbol{\omega} \tag{20}$$

If flow is irrotational, i.e. no vorticity,

$$\boldsymbol{\zeta} = \boldsymbol{\omega} = \mathbf{0} \tag{21}$$

1.2.2 Potential Flow

$$\nabla \times \nabla k = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial k}{\partial x} \\ \frac{\partial k}{\partial y} \\ \frac{\partial k}{\partial z} \end{pmatrix} \tag{22}$$

where k is a scalar. After cross product is attempted, it is clear that each entry in vector gives 0.

$$\nabla \times \nabla k = \mathbf{0} \tag{23}$$

It is easy to see that if $\nabla \times \mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \nabla k$, gradient of a scalar.

1.2.3 Kinematic Viscosity and Dynamic Viscosity

Kinematic viscosity ν and dynamic viscosity μ is related by the formula below.

$$\nu = \frac{\mu}{\rho} \tag{24}$$

Since density changes with temperature, both ν and μ is temperature dependent.

1.2.4 Shear Stress in a Fluid

Shear stress τ_{xy} is given by the equation below.

$$\tau_{xy} = \mu \cdot \left(\frac{\partial u}{\partial y}\right) \tag{25}$$

Fluid flows in the x direction, from left to right. Fluid gains in elevation in y direction, moves away from boundary plate.

If flow is inviscid, $\mu = 0$, then shear stress $\tau = 0$.

More on this concept in ME2134.

1.2.5 Skin Friction Drag

Skin friction drag D is given by the equation below.

$$D = \int \tau \cdot dA \tag{26}$$

1.2.6 Formulas

Reynolds Number	Mach Number	Speed of Sound	Knudsen Number
$Re = \frac{\rho \cdot u \cdot L}{\mu} = \frac{u \cdot L}{\nu}$	$M = \frac{V}{a}$	$a = \sqrt{\gamma RT}$	$Kn = \frac{\text{Mean Free Path}}{\text{Characteristic Length Scale}} = \frac{\lambda}{d}$