Path integral formalism for polaron impurity

Table of contents

1 The model

The Hamiltonian is

$$\hat{H} = \hat{H}_{\mathrm{imp}} + \hat{H}_{\mathrm{ph}} + \hat{H}_{\mathrm{el}}, \tag{1.1}$$

where $\hat{H}_{\mathrm{imp}} = -\mu a^\dagger a$,

$$\hat{H}_{\mathrm{el}} = \sum_k \lambda_k (\hat{a}^\dagger \hat{c}_k + \hat{c}_k^\dagger \hat{a}) + \sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k, \quad \hat{H}_{\mathrm{ph}} = \hat{a}^\dagger \hat{a} \sum_k g_k(\hat{b}_k + \hat{b}_k^\dagger) + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k. \tag{1.2}$$

The bath spectral functions of phonon and eletron are defined as

$$J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k), \quad \Gamma(\varepsilon) = \sum_k \lambda_k^2 \delta(\varepsilon - \varepsilon_k). \tag{1.3}$$

2 The imaginary time path integral

Consider the partition function and reduced partition function

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]. \tag{2.1}$$

Let us express the phonon in coherent state $\hat{b}|\varphi\rangle = \varphi|\varphi\rangle$, the bath electron in particle number represetation $|m\rangle$ and the impurity electron in number represetation $|n\rangle$, then

$$Z = \sum_{\boldsymbol{m},\boldsymbol{n}} \int \mathscr{D}[\overline{\varphi}\varphi] \langle m_N n_N \varphi_N | e^{-\delta\tau\hat{H}_{\mathrm{imp}}} e^{-\delta\tau\hat{H}_{\mathrm{ph}}} e^{-\delta\tau\hat{H}_{\mathrm{el}}} | m_{N-1} n_{N-1} \varphi_{N-1} \rangle$$

$$\times \cdots \langle m_1 n_1 \varphi_1 | e^{-\delta\tau\hat{H}_{\mathrm{imp}}} e^{-\delta\tau\hat{H}_{\mathrm{ph}}} e^{-\delta\tau\hat{H}_{\mathrm{el}}} | m_0 n_0 \varphi_0 \rangle, \tag{2.2}$$

where

$$\boldsymbol{m} = (m_0, \dots, m_N), \quad \boldsymbol{n} = (n_0, \dots, n_N), \quad \mathscr{D}[\overline{\varphi}\varphi] = \prod_{\alpha=0}^N \frac{\mathrm{d}\overline{\varphi}_\alpha \mathrm{d}\varphi_\alpha}{2\pi i} e^{-\overline{\varphi}_\alpha \varphi_\alpha}. \tag{2.3}$$

The boundary condition is $\langle m_N| = \langle m_0|, \langle n_N| = \langle n_0|, \langle \varphi_N| = \langle \varphi_0|$. We have

$$
\begin{aligned}
Z &= \sum_{m,n} \int \mathcal{D}[\overline{\varphi}\varphi] \langle m_N n_N \varphi_N | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} e^{-\delta\tau \hat{H}_{\mathrm{el}}} e^{-\delta\tau[g_k n_{N-1}(\overline{\varphi}_N + \varphi_{N-1}) + \omega_k \overline{\varphi}_N \varphi_{N-1}]} | m_{N-1} n_{N-1} \varphi_{N-1} \rangle \\
&\quad \times \cdots \langle m_1 n_1 \varphi_1 | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} e^{-\delta\tau \hat{H}_{\mathrm{el}}} e^{-\delta\tau[g_k n_0(\overline{\varphi}_1 + \varphi_0) + \omega_k \overline{\varphi}_1 \varphi_0]} | m_0 n_0 \varphi_0 \rangle, \quad \hat{n} = \hat{a}^\dagger a
\end{aligned}
\tag{2.4}
$$

Following the standard procedures, the phonon bath can be integrated out that

$$
\begin{aligned}
Z &= Z_{\mathrm{ph}} \sum_{m,n} \langle m_N n_N | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} e^{-\delta\tau \hat{H}_{\mathrm{el}}} | m_{N-1} n_{N-1} \rangle \cdots \langle m_1 n_1 | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} e^{-\delta\tau \hat{H}_{\mathrm{el}}} | m_0 n_0 \rangle \\
&\quad \times e^{-\int d\tau' \int d\tau'' \hat{n}(\tau') \Lambda(\tau', \tau'') \hat{n}(\tau'')}.
\end{aligned}
\tag{2.5}
$$

Now we transform the electron represetation into coherent state $\hat{c}_k|c\rangle = c_k|c\rangle$ and $\hat{a}|a\rangle = a|a\rangle$, then

$$
\begin{aligned}
Z &= Z_{\mathrm{ph}} \int \mathcal{D}[\bar{a}a] \int \mathcal{D}[\bar{c}c] \langle a_N c_N | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} e^{-\delta\tau[\lambda_k(\bar{a}_N + a_{N-1}) + \varepsilon_k \bar{a}_N a_{N-1}]} | a_{N-1} c_{N-1} \rangle \\
&\quad \times \cdots \langle a_1 c_1 | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} e^{-\delta\tau[\lambda_k(\bar{a}_N + a_{N-1}) + \varepsilon_k \bar{a}_N a_{N-1}]} | a_0 c_0 \rangle \\
&\quad \times e^{-\int d\tau' \int d\tau'' \bar{a}(\tau') a(\tau') \Lambda(\tau', \tau'') \bar{a}(\tau'') a(\tau'')(\tau'')},
\end{aligned}
\tag{2.6}
$$

the boundary condition is $\langle a_N| = \langle -a_0|, \langle c_N| = \langle -c_0|$. The electron bath can be integrated out via standard procedure, which yields

$$
\begin{aligned}
Z &= Z_{\mathrm{ph}} Z_{\mathrm{el}} \int \mathcal{D}[\bar{a}a] \langle a_N | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} | a_{N-1} \rangle \cdots \langle a_1 | e^{-\delta\tau \hat{H}_{\mathrm{imp}}} | a_0 \rangle \\
&\quad \times e^{-\int d\tau' \int d\tau'' a(\tau') \Delta(\tau', \tau'') a(\tau'')} e^{-\int d\tau' \int d\tau'' \bar{a}(\tau') a(\tau') \Lambda(\tau', \tau'') \bar{a}(\tau'') a(\tau'')} \\
&= Z_{\mathrm{ph}} Z_{\mathrm{el}} \int \mathcal{D}[\bar{a}a] K[\bar{a}a] I_{\mathrm{el}}[\bar{a}a] I_{\mathrm{ph}}[\bar{a}a].
\end{aligned}
\tag{2.7}
$$

Here we have

$$
\Lambda(\tau', \tau'') = \int J(\omega) D_\omega^{(0)}(\tau', \tau'') d\omega, \tag{2.8}
$$

$$
\Delta(\tau', \tau'') = \int \Gamma(\varepsilon) G_\varepsilon^{(0)}(\tau, \tau'') d\varepsilon. \tag{2.9}
$$

3 The real time path integral

Suppose the bath is at thermal equilibrium that $\hat{\rho}_E = e^{-\beta(\omega_k \hat{b}^\dagger \hat{b} + \varepsilon_k \hat{c}^\dagger \hat{c})}$, then following the similar procedure we should have

$$
Z(t) = Z_{\mathrm{ph}} Z_{\mathrm{el}} \int \mathcal{D}[\bar{a}a] K[\bar{a}a] I_{\mathrm{el}}[\bar{a}a] I_{\mathrm{ph}}[\bar{a}a], \tag{3.1}
$$

where (the boundary condition is $\langle a_N^+| = \langle -a_{\bar{N}}|$)

$$
K[\bar{a}a] = \langle a_N^+ | e^{-i\hat{H}_{\mathrm{imp}}\delta t} | a_{N-1}^+ \rangle \cdots \langle a_1^+ | e^{-i\hat{H}_{\mathrm{imp}}\delta t} | a_0^+ \rangle \langle a_0^+ | \hat{\rho}_{\mathrm{imp}}(0) | a_0^- \rangle \langle a_0^- | e^{i\hat{H}_{\mathrm{imp}}\delta t} | a_1^- \rangle \cdots \langle a_{N-1}^- | e^{i\hat{H}_{\mathrm{imp}}\delta t} | a_{\bar{N}} \rangle \tag{3.2}
$$

$$
I_{\mathrm{el}}[\bar{a}a] = e^{-i\int_C dt' \int_C dt'' \bar{a}(t') \Delta(t', t'') a(t'')} \tag{3.3}
$$

$$
I_{\mathrm{ph}}[\bar{a}a] = e^{-i\int_C dt' \int_C dt'' \bar{a}(t') a(t') \Lambda(t', t'') \bar{a}(t'') a(t'')} \tag{3.4}
$$

4 Imaginary time QUAPI

4.1 Phonon part

We have

$$
\Lambda(\tau', \tau'') = \int d\omega J(\omega) D_\omega^{(0)}(\tau, \tau''), \tag{4.1}
$$

where

$$D_\omega^{(0)}(\tau', \tau'') = -\langle T\hat{b}_\omega(\tau')\hat{b}_\omega^\dagger(\tau'')\rangle_0 = \begin{cases} -(1+n_p)e^{-\omega(\tau'-\tau'')} & \tau' > \tau'' \\ -n_p e^{-\omega(\tau'-\tau'')} & \tau' < \tau'' \end{cases}. \tag{4.2}$$

After discretization, we have

$$e^{-\int d\tau' \int d\tau'' \bar{a}(\tau')a(\tau')\Lambda(\tau',\tau'')\bar{a}(\tau'')a(\tau'')} \rightarrow e^{-\sum_{ij}\bar{a}_{i+1}a_i\Delta_{ij}\bar{a}_{j+1}a_j}. \tag{4.3}$$

We have

$$\begin{aligned} \int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau' \int_{k\Delta\tau}^{(k+1)\Delta\tau} d\tau'' e^{-\omega(\tau'-\tau'')} &= \frac{1}{\omega}\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau'[e^{-\omega[\tau'-(k+1)\Delta\tau]} - e^{-\omega[\tau'-k\Delta\tau]}] \\ &= -\frac{1}{\omega^2}[e^{-\omega[(j+1)\Delta\tau-(k+1)\Delta\tau]} - e^{-\omega[j\Delta\tau-(k+1)\Delta\tau]}] \\ &\quad + \frac{1}{\omega^2}[e^{-\omega[(j+1)\Delta\tau-k\Delta\tau]} - e^{-\omega[j\Delta\tau-k\Delta\tau]}] \\ &= -\frac{1}{\omega^2}[2e^{-\omega(j-k)\Delta\tau} - e^{-\omega(j-k)\Delta\tau}(e^{\omega\Delta\tau} + e^{-\omega\Delta\tau})] \\ &= -\frac{2}{\omega^2}e^{-\omega(j-k)\Delta\tau}(1 - \cosh\omega\Delta\tau), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau' \int_{j\Delta\tau}^{\tau'} d\tau'' e^{-\omega(\tau'-\tau'')} &= \frac{1}{\omega}\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau'[1 - e^{-\omega(\tau'-j\Delta\tau)}] \\ &= \frac{\delta\tau}{\omega} + \frac{1}{\omega^2}[e^{-\omega\Delta\tau} - 1] \\ &= -\frac{1}{\omega^2}[(1 - \omega\Delta\tau) - e^{-\omega\Delta\tau}], \quad (\tau' > \tau''), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau'' \int_{j\Delta\tau}^{\tau''} d\tau' e^{-\omega(\tau'-\tau'')} &= -\frac{1}{\omega}\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau''[1 - e^{-\omega(j\Delta\tau-\tau'')}] \\ &= -\frac{\Delta\tau}{\omega} + \frac{1}{\omega^2}[e^{-\omega[j\Delta\tau-(j+1)\Delta\tau]} - 1] \\ &= -\frac{1}{\omega^2}[(1 + \omega\Delta\tau) - e^{\omega\Delta\tau}], \quad (\tau' < \tau''). \end{aligned} \tag{4.6}$$

In summary, we have $[n_p = (e^{\beta\omega} - 1)^{-1}, 1 + n_p = (1 - e^{-\beta\omega})^{-1}]$

$$\Lambda_{jk} = \begin{cases} 2\int d\omega \frac{J(\omega)}{\omega^2}(1 - e^{-\beta\omega})^{-1}e^{-\omega(j-k)\Delta\tau}(1 - \cosh\omega\Delta\tau), & j > k; & (4.7) \\ \int d\omega \frac{J(\omega)}{\omega^2}\{(1 - e^{-\beta\omega})^{-1}[(1-\omega\Delta\tau) - e^{-\omega\Delta\tau}] + (e^{\beta\omega} - 1)^{-1}[(1+\omega\Delta\tau) - e^{\omega\Delta\tau}]\}, & j = k; & (4.8) \\ 2\int d\omega \frac{J(\omega)}{\omega^2}(e^{\beta\omega} - 1)^{-1}e^{-\omega(j-k)\Delta\tau}(1 - \cosh\omega\Delta\tau), & j < k. & (4.9) \end{cases}$$

4.2 Electron part

We have

$$\Delta(\tau', \tau'') = \int d\varepsilon\, \Gamma(\varepsilon) G_\varepsilon^{(0)}(\tau', \tau''), \tag{4.10}$$

where

$$G_\varepsilon^{(0)}(\tau', \tau'') = -\langle T\hat{c}_\varepsilon(\tau')\hat{c}_\varepsilon^\dagger(\tau'')\rangle_0 = -(1 - n_e)e^{-\varepsilon(\tau'-\tau'')}, \quad \tau' > \tau''; \tag{4.11}$$

$$n_e e^{-\varepsilon(\tau'-\tau'')}, \quad \tau' < \tau''. \tag{4.12}$$

After discretization, we have

$$e^{-\int d\tau' \int d\tau'' \bar{a}(\tau')\Delta(\tau',\tau'')a(\tau'')} \rightarrow e^{-\sum_{ij}\bar{a}_i\Delta_{ij}a_j} \tag{4.13}$$

Then similarly we have $[n_e = (e^{\beta\varepsilon} + 1)^{-1}, (1 - n_e) = (1 + e^{-\beta\varepsilon})^{-1}]$

$$2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1 + e^{-\beta\varepsilon})^{-1}e^{-\varepsilon(j-k)\Delta\tau}(1 - \cosh\varepsilon\Delta\tau), \quad j > k; \tag{4.14}$$

$$\Delta_{jk} = \int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}\{(1 + e^{-\beta\varepsilon})^{-1}[(1 - \varepsilon\Delta\tau) - e^{-\varepsilon\Delta\tau}] - (e^{\beta\varepsilon} + 1)^{-1}[(1 + \varepsilon\Delta\tau) - e^{\varepsilon\Delta\tau}]\}, \quad j = k; \tag{4.15}$$

$$-2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon} + 1)^{-1}e^{-\varepsilon(j-k)\Delta\tau}(1 - \cosh\varepsilon\Delta\tau), \quad j < k; \tag{4.16}$$

5   Real time QUAPI

5.1   Phonon part

We have

$$i\Lambda(t', t'') = i\int d\omega J(\omega)D_\omega^{(0)}(t', t''), \tag{5.1}$$

where

$$[iD_\omega^{(0)}(t', t'')]^{++} = \langle T\hat{b}_\omega(t')\hat{b}_\omega^\dagger(t'')\rangle_0 = (1 + n_p)e^{-i\omega(t'-t'')}, \quad t' > t''; \tag{5.2}$$

$$n_p e^{-i\omega(t'-t'')}, \quad t' < t'', \tag{5.3}$$

$$[iD_\omega^{(0)}(t', t'')]^{+-} = \langle \hat{b}_\omega^\dagger(t'')\hat{b}_\omega(t')\rangle_0 = n_p e^{-i\omega(t'-t'')}, \tag{5.4}$$

$$[iD_\omega^{(0)}(t', t'')]^{-+} = \langle \hat{b}_\omega(t')\hat{b}_\omega^\dagger(t'')\rangle_0 = (1 + n_p)e^{-i\omega(t'-t'')}, \tag{5.5}$$

$$[iD_\omega^{(0)}(t', t'')]^{--} = \langle \overline{T}\hat{b}_\omega(t')\hat{b}_\omega^\dagger(t'')\rangle_0 = n_p e^{-i\omega(t'-t'')}; \tag{5.6}$$

$$(1 + n_p)e^{-i\omega(t'-t'')}. \tag{5.7}$$

After discretization, we have $[i\Lambda(t', t'') \to \Lambda_{jk}]$

$$e^{-i\int_C dt'\int_C dt''\bar{a}(t')a(t')\Lambda(t',t'')\bar{a}(t'')a(t'')} \to e^{-\sum_{jk}\sum_{\zeta',\zeta''}\bar{a}_{j+1}^{\zeta'}a_{j+1}^{\zeta'}\Lambda_{jk}^{\zeta'\zeta''}\bar{a}_{k+1}^{\zeta''}a_k^{\zeta''}} \tag{5.8}$$

We have

$$\int_{j\Delta t}^{(j+1)\Delta t} dt'\int_{k\Delta t}^{(k+1)\Delta t} dt'' e^{-i\varepsilon(t'-t'')} = \frac{1}{i\varepsilon}\int_{j\Delta t}^{(j+1)\Delta t} dt'[e^{-i\varepsilon[t'-(k+1)\Delta t]} - e^{-i\varepsilon(t'-k\Delta t)}]$$

$$= \frac{1}{\varepsilon^2}[e^{-i\varepsilon[(j+1)\Delta t-(k+1)\Delta t]} - e^{-i\varepsilon[j\Delta t-(k+1)\Delta t]}]$$

$$- \frac{1}{\varepsilon^2}[e^{-i\varepsilon[(j+1)\Delta t-k\Delta t]} - e^{-i\varepsilon(j\Delta t-k\Delta t)}]$$

$$= \frac{2}{\varepsilon^2}e^{-i\varepsilon(j-k)\Delta t}(1 - \cos\varepsilon\Delta t), \tag{5.9}$$

and

$$\int_{j\Delta t}^{(j+1)\Delta t} dt'\int_{j\Delta t}^{t'} dt'' e^{-i\varepsilon(t'-t'')} = \frac{1}{i\varepsilon}\int_{j\Delta t}^{(j+1)\Delta t} dt'[1 - e^{-i\varepsilon(t'-j\Delta t)}]$$

$$= \frac{\Delta t}{i\varepsilon} - \frac{1}{\varepsilon^2}[e^{-i\varepsilon\Delta t} - 1]$$

$$= \frac{1}{\varepsilon^2}[(1 - i\varepsilon\Delta t) - e^{-i\varepsilon\Delta t}], \tag{5.10}$$

$$\int_{j\Delta t}^{(j+1)\Delta t} dt''\int_{j\Delta t}^{t''} dt' e^{-i\varepsilon(t'-t'')} = -\frac{1}{i\varepsilon}\int_{j\Delta t}^{(j+1)\Delta t} dt''[1 - e^{-i\varepsilon(j\Delta t-t'')}]$$

$$= -\frac{\Delta t}{i\varepsilon} - \frac{1}{\varepsilon^2}[e^{-i\varepsilon[j\Delta t-(j+1)\Delta t]} - 1]$$

$$= \frac{1}{\varepsilon^2}[(1 + i\varepsilon\Delta t) - e^{i\varepsilon\Delta t}]. \tag{5.11}$$

Therefore we have $[n_p = (e^{\beta\omega} + 1)^{-1}, 1 + n_p = (1 + e^{-\beta\omega})^{-1}]$

$$2\int d\omega \frac{J(\omega)}{\omega^2}(1 + e^{-\beta\omega})^{-1}e^{-i\omega(j-k)\Delta t}(1 - \cos\varepsilon\Delta t), \quad j > k \tag{5.12}$$

$$\Lambda_{jk}^{++} = \int d\omega \frac{J(\omega)}{\omega^2}\{(1 + e^{-\beta\omega})^{-1}[(1 - i\omega\Delta t) - e^{-i\omega\Delta t}] + (e^{\beta\omega} + 1)^{-1}[(1 + i\omega\Delta t) - e^{i\omega\Delta t}]\}, \quad j = k \tag{5.13}$$

$$2\int d\omega \frac{J(\omega)}{\omega^2}(e^{\beta\omega} + 1)^{-1}e^{-i\omega(j-k)\Delta t}(1 - \cos\omega\Delta t), \quad i < j. \tag{5.14}$$

$$\Lambda_{jk}^{+-} = -2\int d\omega \frac{J(\omega)}{\omega^2}(e^{\beta\omega} + 1)^{-1}e^{-i\omega(j-k)\Delta t}(1 - \cos\omega\Delta t), \quad j \neq k \tag{5.15}$$

$$-\int d\omega \frac{J(\omega)}{\omega^2}(e^{\beta\omega} + 1)^{-1}\{[(1 - i\omega\Delta t) - e^{-i\omega\Delta t}] + [(1 + i\omega\Delta t) - e^{i\omega\Delta t}]\}, \quad j = k \tag{5.16}$$

$$\Lambda_{jk}^{-+} = -2\int d\omega \frac{J(\omega)}{\omega^2}(1 + e^{-\beta\omega})^{-1}e^{-i\omega(j-k)\Delta t}(1 - \cos\omega\Delta t), \quad j \neq k \tag{5.17}$$

$$-\int d\omega \frac{J(\omega)}{\omega^2}(1 + e^{-\beta\omega})^{-1}\{[(1 - i\omega\Delta t) - e^{-i\omega\Delta t}] + [(1 + i\omega\Delta t) - e^{i\omega\Delta t}]\}, \quad j = k \tag{5.18}$$

$$2\int d\omega \frac{J(\omega)}{\omega^2}(e^{\beta\omega} + 1)^{-1}e^{-i\omega(j-k)\Delta t}(1 - \cos\omega\Delta t), \quad j > k \tag{5.19}$$

$$\Lambda_{jk}^{--} = \int d\omega \frac{J(\omega)}{\omega^2}\{(e^{\beta\omega} + 1)^{-1}[(1 - i\omega\Delta t) - e^{-i\omega\Delta t}] + (1 + e^{-\beta\omega})^{-1}[(1 + i\omega\Delta t) - e^{i\omega\Delta t}]\}, \quad j = k \tag{5.20}$$

$$2\int d\omega \frac{J(\omega)}{\omega^2}(1 + e^{-\beta\omega})^{-1}e^{-i\omega(i-j)\Delta t}(1 - \cos\omega\Delta t), \quad j < k \tag{5.21}$$

Here we have already considered the signs of dt' and dt''.

5.2 Electron part

We have

$$i\Delta(t', t'') = i\mathscr{P}_{t't''}\int d\varepsilon\,\Gamma(\varepsilon)G_\varepsilon^{(0)}(t', t''), \tag{5.22}$$

where ($\mathscr{P}_{t't''} = 1$ when t', t'' on some branch, otherwise $\mathscr{P}_{t't''} = -1$)

$$[iG_\varepsilon^{(0)}(t', t'')]^{++} = \langle T\hat{c}_\varepsilon(t')\hat{c}_\varepsilon^\dagger(t'')\rangle_0 = (1 - n_e)e^{-i\varepsilon(t'-t'')}, \quad t' > t'' \tag{5.23}$$

$$-n_e e^{-i\varepsilon(t'-t'')}, \quad t' < t'' \tag{5.24}$$

$$[iG_\varepsilon^{(0)}(t', t'')]^{+-} = \langle \hat{c}_\varepsilon(t')\hat{c}_\varepsilon^\dagger(t'')\rangle_0 = (1 - n_e)e^{-i\varepsilon(t'-t'')} \tag{5.25}$$

$$[iG_\varepsilon^{(0)}(t', t'')]^{-+} = -\langle \hat{c}_\varepsilon^\dagger(t'')\hat{c}_\varepsilon(t')\rangle = -n_e e^{-i\varepsilon(t'-t'')} \tag{5.26}$$

$$[iG_\varepsilon^{(0)}(t', t'')]^{--} = \langle \overline{T}\hat{c}_\varepsilon(t')\hat{c}_\varepsilon^\dagger\rangle = -n_e e^{-i\varepsilon(t'-t'')}, \quad t' > t'' \tag{5.27}$$

$$= (1 - n_e)e^{-i\varepsilon(t'-t'')}, \quad t' < t''. \tag{5.28}$$

After discretization, we have $[i\Delta(t', t'') \to \Delta_{jk}]$

$$e^{-i\int_C dt'\int_C dt''\bar{a}(t')\Delta(t',t'')a(t'')} \to e^{-\sum_{jk}\sum_{\zeta',\zeta''}\bar{a}_j\Delta_{jk}^{\zeta'\zeta''}a_k}. \tag{5.29}$$

Therefore we have $[n_e = (e^{\beta\varepsilon} + 1)^{-1}, 1 + n_e = (1 + e^{-\beta\varepsilon})^{-1}]$

$$2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1 + e^{-\beta\varepsilon})^{-1}e^{-i\varepsilon(j-k)\Delta t}(1 - \cos\varepsilon\Delta t), \quad j > k \tag{5.30}$$

$$\Delta_{jk}^{++} = \int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}\{(1 + e^{-\beta\varepsilon})^{-1}[(1 - i\varepsilon\Delta t) - e^{-i\varepsilon\Delta t}] - (e^{\beta\varepsilon} + 1)^{-1}[(1 + i\varepsilon\Delta t) - e^{i\varepsilon\Delta t}]\}, \quad j = k \tag{5.31}$$

$$-2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon} + 1)^{-1}e^{-i\varepsilon(j-k)\Delta t}(1 - \cos\varepsilon\Delta t), \quad j < k \tag{5.32}$$

$$\Delta_{jk}^{+-} = -2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon} + 1)^{-1}e^{-i\varepsilon(j-k)\Delta t}(1 - \cos\varepsilon\Delta t), \quad j \neq k \tag{5.33}$$

$$= -\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon} + 1)^{-1}\{[(1 - i\varepsilon\Delta t) - e^{-i\varepsilon\Delta t}] + [(1 + i\varepsilon\Delta t) - e^{i\varepsilon\Delta t}]\}, \quad j = k \tag{5.34}$$

$$\Delta_{jk}^{-+} \;=\; 2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\varepsilon})^{-1}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t), \quad j\neq k \tag{5.35}$$

$$\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\varepsilon})^{-1}\{[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}]+[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}]\}, \quad j=k \tag{5.36}$$

$$-2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\omega}+1)^{-1}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t), \quad j<k \tag{5.37}$$

$$\Delta_{jk}^{--} \;=\; -\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}\{(e^{\beta\omega}+1)^{-1}[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}]-(1+e^{-\beta\omega})^{-1}[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}]\}, \quad j=k \tag{5.38}$$

$$2\int d\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\omega})^{-1}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t), \quad j>k \tag{5.39}$$

Here the sign of $\mathscr{P}_{t't''}$ cancels the signs of dt' and dt''.