Path integral formalism for polaron impurity

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1 The model

The Hamiltonian is

$$\hat{H} = \hat{H}_{\text{imp}} + \hat{H}_{\text{ph}} + \hat{H}_{\text{el}}, \tag{1.1}$$

where $\hat{H}_{\text{imp}} = -\mu a^{\dagger} a$,

$$\hat{H}_{\text{el}} = \sum_k \lambda_k (\hat{a}^{\dagger}\hat{c}_k + \hat{c}_k^{\dagger}\hat{a}) + \sum_k \varepsilon_k \hat{c}_k^{\dagger}\hat{c}_k, \quad \hat{H}_{\text{ph}} = \hat{a}^{\dagger}\hat{a}\sum_k g_k(\hat{b}_k + \hat{b}_k^{\dagger}) + \sum_k \omega_k \hat{b}_k^{\dagger}\hat{b}_k. \tag{1.2}$$

The bath spectral functions of phonon and eletron are defined as

$$J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k), \quad \Gamma(\varepsilon) = \sum_k \lambda_k^2 \delta(\varepsilon - \varepsilon_k). \tag{1.3}$$

2 The imaginary time path integral

Consider the partition function and reduced partition function

$$Z = \mathrm{Tr}[e^{-\beta\hat{H}}]. \tag{2.1}$$

Let us express the phonon in coherent state $\hat{b}|\varphi\rangle = \varphi|\varphi\rangle$, the bath electron in particle number represetation $|m\rangle$ and the impurity electron in number represetation $|n\rangle$, then

$$\begin{aligned} Z \;=\; & \sum_{\boldsymbol{m},\boldsymbol{n}} \int \mathscr{D}[\overline{\varphi}\varphi] \langle m_N n_N \varphi_N | e^{-\delta\tau\hat{H}_{\text{imp}}} e^{-\delta\tau\hat{H}_{\text{ph}}} e^{-\delta\tau\hat{H}_{\text{el}}} | m_{N-1} n_{N-1} \varphi_{N-1}\rangle \\ & \times \cdots \langle m_1 n_1 \varphi_1 | e^{-\delta\tau\hat{H}_{\text{imp}}} e^{-\delta\tau\hat{H}_{\text{ph}}} e^{-\delta\tau\hat{H}_{\text{el}}} | m_0 n_0 \varphi_0\rangle, \end{aligned} \tag{2.2}$$

where

$$\boldsymbol{m} = (m_0, \ldots, m_N), \quad \boldsymbol{n} = (n_0, \ldots, n_N), \quad \mathscr{D}[\overline{\varphi}\varphi] = \prod_{\alpha=0}^{N} \frac{\mathrm{d}\overline{\varphi}_\alpha \mathrm{d}\varphi_\alpha}{2\pi i} e^{-\overline{\varphi}_\alpha\varphi_\alpha}. \tag{2.3}$$

The boundary condition is $\langle m_N| = \langle m_0|, \langle n_N| = \langle n_0|, \langle \varphi_N| = \langle \varphi_0|$. We have

$$\begin{aligned} Z &= \sum_{m,n} \int \mathscr{D}[\bar{\varphi}\varphi] \langle m_N n_N \varphi_N| e^{-\delta\tau \hat{H}_{\text{imp}}} e^{-\delta\tau \hat{H}_{\text{el}}} e^{-\delta\tau[g_k n_{N-1}(\bar{\varphi}_N + \varphi_{N-1}) + \omega_k \bar{\varphi}_N \varphi_{N-1}]} |m_{N-1} n_{N-1} \varphi_{N-1}\rangle \\ &\quad \times \cdots \langle m_1 n_1 \varphi_1| e^{-\delta\tau \hat{H}_{\text{imp}}} e^{-\delta\tau \hat{H}_{\text{el}}} e^{-\delta\tau[g_k n_0(\bar{\varphi}_1 + \varphi_0) + \omega_k \bar{\varphi}_1 \varphi_0]} |m_0 n_0 \varphi_0\rangle, \quad \hat{n} = \hat{a}^\dagger a \end{aligned} \tag{2.4}$$

Following the standard procedures, the phonon bath can be integrated out that

$$\begin{aligned} Z &= Z_{\text{ph}} \sum_{m,n} \langle m_N n_N| e^{-\delta\tau \hat{H}_{\text{imp}}} e^{-\delta\tau \hat{H}_{\text{el}}} |m_{N-1} n_{N-1}\rangle \cdots \langle m_1 n_1| e^{-\delta\tau \hat{H}_{\text{imp}}} e^{-\delta\tau \hat{H}_{\text{el}}} |m_0 n_0\rangle \\ &\quad \times e^{-\int d\tau' \int d\tau'' \hat{n}(\tau') \Lambda(\tau', \tau'') \hat{n}(\tau'')}. \end{aligned} \tag{2.5}$$

Now we transform the electron represetation into coherent state $\hat{c}_k|c\rangle = c_k|c\rangle$ and $\hat{a}|a\rangle = a|a\rangle$, then

$$\begin{aligned} Z &= Z_{\text{ph}} \int \mathscr{D}[\bar{a}a] \int \mathscr{D}[\bar{c}c] \langle a_N c_N| e^{-\delta\tau \hat{H}_{\text{imp}}} e^{-\delta\tau[\lambda_k(\bar{a}_N + a_{N-1}) + \varepsilon_k \bar{a}_N a_{N-1}]} |a_{N-1} c_{N-1}\rangle \\ &\quad \times \cdots \langle a_1 c_1| e^{-\delta\tau \hat{H}_{\text{imp}}} e^{-\delta\tau[\lambda_k(\bar{a}_N + a_{N-1}) + \varepsilon_k \bar{a}_N a_{N-1}]} |a_0 c_0\rangle \\ &\quad \times e^{-\int d\tau' \int d\tau'' \bar{a}(\tau') a(\tau') \Lambda(\tau', \tau'') \bar{a}(\tau'') a(\tau'')(\tau'')}, \end{aligned} \tag{2.6}$$

the boundary condition is $\langle a_N| = \langle -a_0|, \langle c_N| = \langle -c_0|$. The electron bath can be integrated out via standard procedure, which yields

$$\begin{aligned} Z &= Z_{\text{ph}} Z_{\text{el}} \int \mathscr{D}[\bar{a}a] \langle a_N| e^{-\delta\tau \hat{H}_{\text{imp}}} |a_{N-1}\rangle \cdots \langle a_1| e^{-\delta\tau \hat{H}_{\text{imp}}} |a_0\rangle \\ &\quad \times e^{-\int d\tau' \int d\tau'' a(\tau') \Delta(\tau', \tau'') a(\tau'')} e^{-\int d\tau' \int d\tau'' \bar{a}(\tau') a(\tau') \Lambda(\tau', \tau'') \bar{a}(\tau'') a(\tau'')} \\ &= Z_{\text{ph}} Z_{\text{el}} \int \mathscr{D}[\bar{a}a] K[\bar{a}a] I_{\text{el}}[\bar{a}a] I_{\text{ph}}[\bar{a}a]. \end{aligned} \tag{2.7}$$

Here we have

$$\Lambda(\tau', \tau'') = \int J(\omega) D_\omega^{(0)}(\tau', \tau'') d\omega, \tag{2.8}$$

$$\Delta(\tau', \tau'') = \int \Gamma(\varepsilon) G_\varepsilon^{(0)}(\tau, \tau'') d\varepsilon. \tag{2.9}$$

3 The real time path integral

Suppose the bath is at thermal equilibrium that $\hat{\rho}_E = e^{-\beta(\omega_k \hat{b}^\dagger \hat{b} + \varepsilon_k \hat{c}^\dagger \hat{c})}$, then following the similar procedure we should have

$$Z(t) = Z_{\text{ph}} Z_{\text{el}} \int \mathscr{D}[\bar{a}a] K[\bar{a}a] I_{\text{el}}[\bar{a}a] I_{\text{ph}}[\bar{a}a], \tag{3.1}$$

where (the boundary condition is $\langle a_N^+| = \langle -a_N^-|$)

$$K[\bar{a}a] = \langle a_N^+| e^{-i\hat{H}_{\text{imp}}\delta t} |a_{N-1}^+\rangle \cdots \langle a_1^+| e^{-i\hat{H}_{\text{imp}}\delta t} |a_0^+\rangle \langle a_0^+| \hat{\rho}_{\text{imp}}(0) |a_0^-\rangle \langle a_0^-| e^{i\hat{H}_{\text{imp}}\delta t} |a_1^-\rangle \cdots \langle a_{N-1}^-| e^{i\hat{H}_{\text{imp}}\delta t} |a_N^-\rangle \tag{3.2}$$

$$I_{\text{el}}[\bar{a}a] = e^{-i\int_C dt' \int_C dt'' \bar{a}(t') \Delta(t', t'') a(t'')} \tag{3.3}$$

$$I_{\text{ph}}[\bar{a}a] = e^{-i\int_C dt' \int_C dt'' \bar{a}(t') a(t') \Lambda(t', t'') \bar{a}(t'') a(t'')} \tag{3.4}$$

4 Imaginary time QUAPI

4.1 Phonon part

We have

$$\Lambda(\tau', \tau'') = \int d\omega J(\omega) D_\omega^{(0)}(\tau, \tau''), \tag{4.1}$$

where

$$D_\omega^{(0)}(\tau',\tau'') \;=\; -\langle T\hat{b}_\omega(\tau')\hat{b}_\omega^\dagger(\tau'')\rangle_0 = \begin{cases} -(1+n_p)e^{-\omega(\tau'-\tau'')} & \tau'>\tau'' \\ -n_p e^{-\omega(\tau'-\tau'')} & \tau'<\tau'' \end{cases}. \tag{4.2}$$

After discretization, we have

$$e^{-\int d\tau'\int d\tau''\bar{a}(\tau')a(\tau')\Lambda(\tau',\tau'')\bar{a}(\tau'')a(\tau'')} \;\to\; e^{-\sum_{ij}\bar{a}_{i+1}a_i\Delta_{ij}\bar{a}_{j+1}a_j}. \tag{4.3}$$

We have

$$\begin{aligned}
\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau' \int_{k\Delta\tau}^{(k+1)\Delta\tau} d\tau'' e^{-\omega(\tau'-\tau'')} &= \frac{1}{\omega}\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau'[e^{-\omega[\tau'-(k+1)\Delta\tau]}-e^{-\omega[\tau'-k\Delta\tau]}] \\
&= -\frac{1}{\omega^2}[e^{-\omega[(j+1)\Delta\tau-(k+1)\Delta\tau]}-e^{-\omega[j\Delta\tau-(k+1)\Delta\tau]}] \\
&\quad +\frac{1}{\omega^2}[e^{-\omega[(j+1)\Delta\tau-k\Delta\tau]}-e^{-\omega[j\Delta\tau-k\Delta\tau]}] \\
&= -\frac{1}{\omega^2}[2e^{-\omega(j-k)\Delta\tau}-e^{-\omega(j-k)\Delta\tau}(e^{\omega\Delta\tau}+e^{-\omega\Delta\tau})] \\
&= -\frac{2}{\omega^2}e^{-\omega(j-k)\Delta\tau}(1-\cosh\omega\Delta\tau),
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau' \int_{j\Delta\tau}^{\tau'} d\tau'' e^{-\omega(\tau'-\tau'')} &= \frac{1}{\omega}\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau'[1-e^{-\omega(\tau'-j\Delta\tau)}] \\
&= \frac{\delta\tau}{\omega}+\frac{1}{\omega^2}[e^{-\omega\Delta\tau}-1] \\
&= -\frac{1}{\omega^2}[(1-\omega\Delta\tau)-e^{-\omega\Delta\tau}], \quad (\tau'>\tau''),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau'' \int_{j\Delta\tau}^{\tau''} d\tau' e^{-\omega(\tau'-\tau'')} &= -\frac{1}{\omega}\int_{j\Delta\tau}^{(j+1)\Delta\tau} d\tau''[1-e^{-\omega(j\Delta\tau-\tau'')}] \\
&= -\frac{\Delta\tau}{\omega}+\frac{1}{\omega^2}[e^{-\omega[j\Delta\tau-(j+1)\Delta\tau]}-1] \\
&= -\frac{1}{\omega^2}[(1+\omega\Delta\tau)-e^{\omega\Delta\tau}], \quad (\tau'<\tau'').
\end{aligned} \tag{4.6}$$

In summary, we have $[n_p=(e^{\beta\omega}-1)^{-1}, 1+n_p=(1-e^{-\beta\omega})^{-1}]$

$$\Lambda_{jk} \;=\; \begin{cases} 2\int d\omega\frac{J(\omega)}{\omega^2}(1-e^{-\beta\omega})^{-1}e^{-\omega(j-k)\Delta\tau}(1-\cosh\omega\Delta\tau), \quad j>k; & (4.7) \\ \int d\omega\frac{J(\omega)}{\omega^2}\{(1-e^{-\beta\omega})^{-1}[(1-\omega\Delta\tau)-e^{-\omega\Delta\tau}]+(e^{\beta\omega}-1)^{-1}[(1+\omega\Delta\tau)-e^{\omega\Delta\tau}]\}, \quad j=k; & (4.8) \\ 2\int d\omega\frac{J(\omega)}{\omega^2}(e^{\beta\omega}-1)^{-1}e^{-\omega(j-k)\Delta\tau}(1-\cosh\omega\Delta\tau), \quad j<k. & (4.9) \end{cases}$$

4.2 Electron part

We have

$$\Delta(\tau',\tau'') \;=\; \int d\varepsilon\Gamma(\varepsilon)G_\varepsilon^{(0)}(\tau',\tau''), \tag{4.10}$$

where

$$G_\varepsilon^{(0)}(\tau',\tau'') = -\langle T\hat{c}_\varepsilon(\tau')\hat{c}_\varepsilon^\dagger(\tau'')\rangle_0 \;=\; -(1-n_e)e^{-\varepsilon(\tau'-\tau'')}, \quad \tau'>\tau''; \tag{4.11}$$

$$n_e e^{-\varepsilon(\tau'-\tau'')}, \quad \tau'<\tau''. \tag{4.12}$$

After discretization, we have

$$e^{-\int d\tau'\int d\tau''\bar{a}(\tau')\Delta(\tau',\tau'')a(\tau'')} \;\to\; e^{-\sum_{ij}\bar{a}_i\Delta_{ij}a_j} \tag{4.13}$$

Then similarly we have $[n_e=(e^{\beta\varepsilon}+1)^{-1},(1-n_e)=(1+e^{-\beta\varepsilon})^{-1}]$

$$\Delta_{jk} = \begin{cases} 2\int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\varepsilon})^{-1}e^{-\varepsilon(j-k)\Delta\tau}(1-\cosh\varepsilon\Delta\tau), \quad j>k; & (4.14)\\ \int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}\{(1+e^{-\beta\varepsilon})^{-1}[(1-\varepsilon\Delta\tau)-e^{-\varepsilon\Delta\tau}]-(e^{\beta\varepsilon}+1)^{-1}[(1+\varepsilon\Delta\tau)-e^{\varepsilon\Delta\tau}]\}, \quad j=k; & (4.15)\\ -2\int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon}+1)^{-1}e^{-\varepsilon(j-k)\Delta\tau}(1-\cosh\varepsilon\Delta\tau), \quad j<k; & (4.16)\end{cases}$$

5 Real time QUAPI

5.1 Phonon part

We have

$$i\Lambda(t',t'') = i\int d\omega J(\omega)D_\omega^{(0)}(t',t''), \tag{5.1}$$

where

$$[iD_\omega^{(0)}(t',t'')]^{++}=\langle T\hat{b}_\omega(t')\hat{b}_\omega^\dagger(t'')\rangle_0 = \begin{cases}(1+n_p)e^{-i\omega(t'-t'')}, \quad t'>t''; & (5.2)\\ n_pe^{-i\omega(t'-t'')}, \quad t'<t'', & (5.3)\end{cases}$$

$$[iD_\omega^{(0)}(t',t'')]^{+-}=\langle\hat{b}_\omega^\dagger(t'')\hat{b}_\omega(t')\rangle_0 = n_pe^{-i\omega(t'-t'')}, \tag{5.4}$$

$$[iD_\omega^{(0)}(t',t'')]^{-+}=\langle\hat{b}_\omega(t')\hat{b}_\omega^\dagger(t'')\rangle_0 = (1+n_p)e^{-i\omega(t'-t'')}, \tag{5.5}$$

$$[iD_\omega^{(0)}(t',t'')]^{--}=\langle\overline{T}\hat{b}_\omega(t')\hat{b}_\omega^\dagger(t'')\rangle_0 = \begin{cases} n_pe^{-i\omega(t'-t'')}; & (5.6)\\ (1+n_p)e^{-i\omega(t'-t'')}. & (5.7)\end{cases}$$

After discretization, we have $[i\Lambda(t',t'')\rightarrow\Lambda_{jk}]$

$$e^{-i\int_C dt'\int_C dt''\bar{a}(t')a(t')\Lambda(t',t'')\bar{a}(t'')a(t'')} \rightarrow e^{-\sum_{jk}\sum_{\zeta',\zeta''}\bar{a}_{j+1}^{\zeta'}a_{j+1}^{\zeta'}\Lambda_{jk}^{\zeta'\zeta''}\bar{a}_{k+1}^{\zeta''}a_k^{\zeta''}} \tag{5.8}$$

We have

$$\begin{aligned}\int_{j\Delta t}^{(j+1)\Delta t}dt'\int_{k\Delta t}^{(k+1)\Delta t}dt''e^{-i\varepsilon(t'-t'')} &= \frac{1}{i\varepsilon}\int_{j\Delta t}^{(j+1)\Delta t}dt'[e^{-i\varepsilon[t'-(k+1)\Delta t]}-e^{-i\varepsilon(t'-k\Delta t)}]\\ &= \frac{1}{\varepsilon^2}[e^{-i\varepsilon[(j+1)\Delta t-(k+1)\Delta t]}-e^{-i\varepsilon[j\Delta t-(k+1)\Delta t]}]\\ &\quad -\frac{1}{\varepsilon^2}[e^{-i\varepsilon[(j+1)\Delta t-k\Delta t]}-e^{-i\varepsilon(j\Delta t-k\Delta t)}]\\ &= \frac{2}{\varepsilon^2}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t),\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\int_{j\Delta t}^{(j+1)\Delta t}dt'\int_{j\Delta t}^{t'}dt''e^{-i\varepsilon(t'-t'')} &= \frac{1}{i\varepsilon}\int_{j\Delta t}^{(j+1)\Delta t}dt'[1-e^{-i\varepsilon(t'-j\Delta t)}]\\ &= \frac{\Delta t}{i\varepsilon}-\frac{1}{\varepsilon^2}[e^{-i\varepsilon\Delta t}-1]\\ &= \frac{1}{\varepsilon^2}[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}],\end{aligned} \tag{5.10}$$

$$\begin{aligned}\int_{j\Delta t}^{(j+1)\Delta t}dt''\int_{j\Delta t}^{t''}dt'e^{-i\varepsilon(t'-t'')} &= -\frac{1}{i\varepsilon}\int_{j\Delta t}^{(j+1)\Delta t}dt''[1-e^{-i\varepsilon(j\Delta t-t'')}]\\ &= -\frac{\Delta t}{i\varepsilon}-\frac{1}{\varepsilon^2}[e^{-i\varepsilon[j\Delta t-(j+1)\Delta t]}-1]\\ &= \frac{1}{\varepsilon^2}[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}].\end{aligned} \tag{5.11}$$

Therefore we have $[n_p=(e^{\beta\omega}+1)^{-1}, 1+n_p=(1+e^{-\beta\omega})^{-1}]$

$$2\int d\omega\frac{J(\omega)}{\omega^2}(1+e^{-\beta\omega})^{-1}e^{-i\omega(j-k)\Delta t}(1-\cos\varepsilon\Delta t),\quad j>k \tag{5.12}$$

$$\Lambda_{jk}^{++} = \int d\omega\frac{J(\omega)}{\omega^2}\{(1+e^{-\beta\omega})^{-1}[(1-i\omega\Delta t)-e^{-i\omega\Delta t}]+(e^{\beta\omega}+1)^{-1}[(1+i\omega\Delta t)-e^{i\omega\Delta t}]\},\quad j=k \tag{5.13}$$

$$2\int d\omega\frac{J(\omega)}{\omega^2}(e^{\beta\omega}+1)^{-1}e^{-i\omega(j-k)\Delta t}(1-\cos\omega\Delta t),\quad i<j. \tag{5.14}$$

$$\Lambda_{jk}^{+-} = -2\int d\omega\frac{J(\omega)}{\omega^2}(e^{\beta\omega}+1)^{-1}e^{-i\omega(j-k)\Delta t}(1-\cos\omega\Delta t),\quad j\neq k \tag{5.15}$$

$$-\int d\omega\frac{J(\omega)}{\omega^2}(e^{\beta\omega}+1)^{-1}\{[(1-i\omega\Delta t)-e^{-i\omega\Delta t}]+[(1+i\omega\Delta t)-e^{i\omega\Delta t}]\},\quad j=k \tag{5.16}$$

$$\Lambda_{jk}^{-+} = -2\int d\omega\frac{J(\omega)}{\omega^2}(1+e^{-\beta\omega})^{-1}e^{-i\omega(j-k)\Delta t}(1-\cos\omega\Delta t),\quad j\neq k \tag{5.17}$$

$$-\int d\omega\frac{J(\omega)}{\omega^2}(1+e^{-\beta\omega})^{-1}\{[(1-i\omega\Delta t)-e^{-i\omega\Delta t}]+[(1+i\omega\Delta t)-e^{i\omega\Delta t}]\},\quad j=k \tag{5.18}$$

$$2\int d\omega\frac{J(\omega)}{\omega^2}(e^{\beta\omega}+1)^{-1}e^{-i\omega(j-k)\Delta t}(1-\cos\omega\Delta t),\quad j>k \tag{5.19}$$

$$\Lambda_{jk}^{--} = \int d\omega\frac{J(\omega)}{\omega^2}\{(e^{\beta\omega}+1)^{-1}[(1-i\omega\Delta t)-e^{-i\omega\Delta t}]+(1+e^{-\beta\omega})^{-1}[(1+i\omega\Delta t)-e^{i\omega\Delta t}]\},\quad j=k \tag{5.20}$$

$$2\int d\omega\frac{J(\omega)}{\omega^2}(1+e^{-\beta\omega})^{-1}e^{-i\omega(i-j)\Delta t}(1-\cos\omega\Delta t),\quad j<k \tag{5.21}$$

Here we have already considered the signs of dt' and dt''.

5.2 Electron part

We have

$$i\Delta(t',t'') = i\mathscr{P}_{t't''}\int d\varepsilon\,\Gamma(\varepsilon)G_\varepsilon^{(0)}(t',t''), \tag{5.22}$$

where ($\mathscr{P}_{t't''}=1$ when t',t'' on some branch, otherwise $\mathscr{P}_{t't''}=-1$)

$$[iG_\varepsilon^{(0)}(t',t'')]^{++}=\langle T\hat{c}_\varepsilon(t')\hat{c}_\varepsilon^\dagger(t'')\rangle_0 = (1-n_e)e^{-i\varepsilon(t'-t'')},\quad t'>t'' \tag{5.23}$$

$$-n_e e^{-i\varepsilon(t'-t'')},\quad t'<t'' \tag{5.24}$$

$$[iG_\varepsilon^{(0)}(t',t'')]^{+-}=\langle\hat{c}_\varepsilon(t')\hat{c}_\varepsilon^\dagger(t'')\rangle_0 = (1-n_e)e^{-i\varepsilon(t'-t'')} \tag{5.25}$$

$$[iG_\varepsilon^{(0)}(t',t'')]^{-+} = -\langle\hat{c}_\varepsilon^\dagger(t'')\hat{c}_\varepsilon(t')\rangle=-n_e e^{-i\varepsilon(t'-t'')} \tag{5.26}$$

$$[iG_\varepsilon^{(0)}(t',t'')]^{--}=\langle\overline{T}\hat{c}_\varepsilon(t')\hat{c}_\varepsilon^\dagger\rangle = -n_e e^{-i\varepsilon(t'-t'')},\quad t'>t'' \tag{5.27}$$

$$= (1-n_e)e^{-i\varepsilon(t'-t'')},\quad t'<t''. \tag{5.28}$$

After discretization, we have $[i\Delta(t',t'')\to\Delta_{jk}]$

$$e^{-i\int_C dt'\int_C dt''\bar{a}(t')\Delta(t',t'')a(t'')} \to e^{-\sum_{jk}\sum_{\zeta',\zeta''}\bar{a}_j\Delta_{jk}^{\zeta'\zeta''}a_k}. \tag{5.29}$$

Therefore we have $[n_e=(e^{\beta\varepsilon}+1)^{-1}, 1+n_e=(1+e^{-\beta\varepsilon})^{-1}]$

$$2\int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\varepsilon})^{-1}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t),\quad j>k \tag{5.30}$$

$$\Delta_{jk}^{++} = \int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}\{(1+e^{-\beta\varepsilon})^{-1}[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}]-(e^{\beta\varepsilon}+1)^{-1}[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}]\},\quad j=k \tag{5.31}$$

$$-2\int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon}+1)^{-1}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t),\quad j<k \tag{5.32}$$

$$\Delta_{jk}^{+-} = -2\int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon}+1)^{-1}e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t),\quad j\neq k \tag{5.33}$$

$$= -\int d\varepsilon\frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\varepsilon}+1)^{-1}\{[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}]+[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}]\},\quad j=k \tag{5.34}$$

$$\Delta_{jk}^{-+} = 2\int \mathrm{d}\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\varepsilon})^{-1} e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t), \quad j \neq k \tag{5.35}$$

$$\int \mathrm{d}\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\varepsilon})^{-1}\{[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}]+[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}]\}, \quad j=k \tag{5.36}$$

$$-2\int \mathrm{d}\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(e^{\beta\omega}+1)^{-1} e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t), \quad j<k \tag{5.37}$$

$$\Delta_{jk}^{--} = -\int \mathrm{d}\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}\{(e^{\beta\omega}+1)^{-1}[(1-i\varepsilon\Delta t)-e^{-i\varepsilon\Delta t}]-(1+e^{-\beta\omega})^{-1}[(1+i\varepsilon\Delta t)-e^{i\varepsilon\Delta t}]\}, \quad j=k \tag{5.38}$$

$$2\int \mathrm{d}\varepsilon \frac{\Gamma(\varepsilon)}{\varepsilon^2}(1+e^{-\beta\omega})^{-1} e^{-i\varepsilon(j-k)\Delta t}(1-\cos\varepsilon\Delta t), \quad j>k \tag{5.39}$$

Here the sign of $\mathscr{P}_{t't''}$ cancels the signs of $\mathrm{d}t'$ and $\mathrm{d}t''$.

6 Anderson Polaron Impurity Model

When two spin flavors are considered, the phonon Hamiltonian becomes

$$\hat{H}_{\mathrm{ph}} = \hat{n}\sum_k g_k(\hat{b}_k + \hat{b}_k^\dagger) + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k, \tag{6.1}$$

where

$$\hat{n} = \hat{a}_\uparrow^\dagger \hat{a}_\uparrow + \hat{a}_\downarrow^\dagger \hat{a}_\downarrow. \tag{6.2}$$

Then the imaginary-time phonon influence functional becomes

$$I[\bar{\boldsymbol{a}}\boldsymbol{a}] = e^{-\int_0^\beta \mathrm{d}\tau' \int_0^\beta \mathrm{d}\tau'' [\bar{a}_\uparrow(\tau')a_\uparrow(\tau')+\bar{a}_\downarrow(\tau')a_\downarrow(\tau')]\Lambda(\tau',\tau'')[\bar{a}_\uparrow(\tau'')a_\uparrow(\tau'')+\bar{a}_\downarrow(\tau'')a_\downarrow(\tau'')]} \tag{6.3}$$

$$= e^{-\sum_{\sigma'\sigma''}\int_0^\beta \mathrm{d}\tau' \int_0^\beta \mathrm{d}\tau'' \bar{a}_{\sigma'}(\tau')a_{\sigma'}(\tau')\Lambda(\tau',\tau'')\bar{a}_{\sigma''}(\tau'')a_{\sigma''}(\tau'')} \tag{6.4}$$

After the discretization, we should have

$$e^{-\sum_{\sigma'\sigma''}\int_0^\beta \mathrm{d}\tau' \int_0^\beta \mathrm{d}\tau'' \bar{a}_{\sigma'}(\tau')a_{\sigma'}(\tau')\Lambda(\tau',\tau'')\bar{a}_{\sigma''}(\tau'')a_{\sigma''}(\tau'')} \to e^{-\sum_{\sigma'\sigma''}\sum_{jk}\bar{a}_{\sigma'j+1}a_{\sigma'j}\Lambda_{jk}\bar{a}_{\sigma''k+1}a_{\sigma''k}}. \tag{6.5}$$

On Keldysh contour, the expression changes in a similar way.