中山大學
SUN YAT-SEN UNIVERSITY

J1-879 M: Optimization in Energy Networks

Special Topics in Systems and Control
Lecture 2

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Today's Lecture

- Unconstraint Optimization
- Equality Constrained Optimization

Learning Objective

After today's lecture, you should be able to:

- explain the characteristics of a convex/concave function and use them to determine if a function is convex or concave;
- explain why convexity is an important aspect in optimization;
- give the conditions for a local maximum and minimum for an unconstrained optimization problem and apply them to a multi-dimensional function to determine the extrema of this function;
- formulate and apply the first order optimality conditions (KKT conditions) for an equality constrained optimization problem;
- explain the meaning of the Langrange Multipliers and give the Lagrange function of an equality constrained optimization problem;
- explain the graphical meaning of the KKT conditions for an equality constrained optimization problem;

Mathematical Formulation

Goal: to find values for the variables which are optimal with respect to a certain objective and given constraints.

Variables: x (state variables)

u (control variables)

Objective: $f(x,u)$

Equality Constraints: $g(x,u)=0$

Inequality Constraints: $h(x,u)\leq 0$

For optimization to take place, the total number of variables must be greater than the number of equality constraints.

Unconstrained Optimization

Find a solution to: $\min\limits_{x} f(x)$

=> no constraints

First order or necessary condition:

$$\nabla f\left(x^{*}\right) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix}_{x=x^{*}} = 0$$

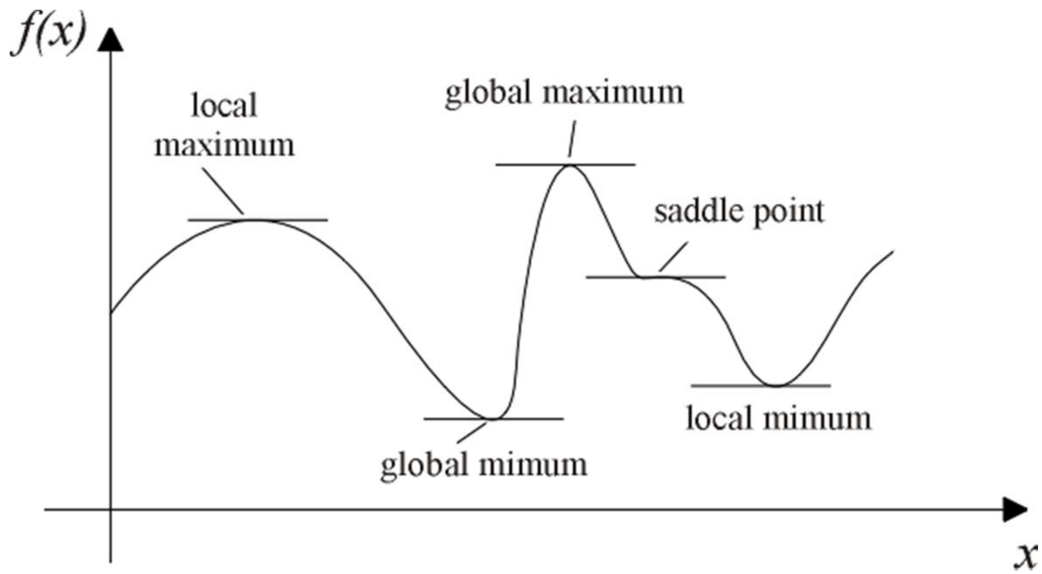
=> gradient, i.e. derivatives with respect to variables must be zero

Unconstrained Optimization

1-dimensional, i.e. only one variable x

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = 0$$

but this holds for global and local minima and maxima as well as saddle points

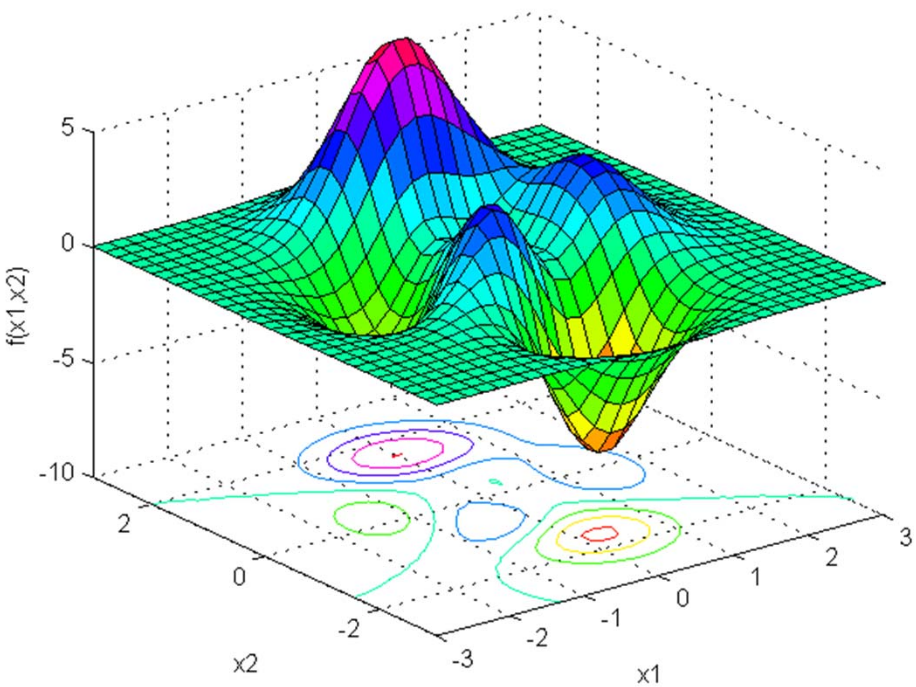


=> sufficient condition for a local minimum is $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x^*} > 0$

Unconstrained Optimization

2-dimensional

$$\nabla f\left(x^*\right) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}_{x=x^*} = 0$$



gradient corresponds to a vector and at any point represents the direction in which the function will increase/decrease the most

Unconstrained Optimization

Second order or sufficient condition for **local minima**:

$$\nabla^2 f\left(x^*\right) = \begin{bmatrix} \dfrac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}_{x=x^*}$$

Hessian matrix (matrix of order derivatives) must be positive definite, i.e. all eigenvalues > 0

There are no general conditions for a global optimum; such conditions exist only for certain classes of objective functions such as convex functions

Convexity

A function is convex if

$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$ for all x, y and $\alpha + \beta = 1, \alpha, \beta \ge 0$

=> $\nabla^2 f(x) \ge 0$ for all x, i.e. Hessian matrix is positive definite

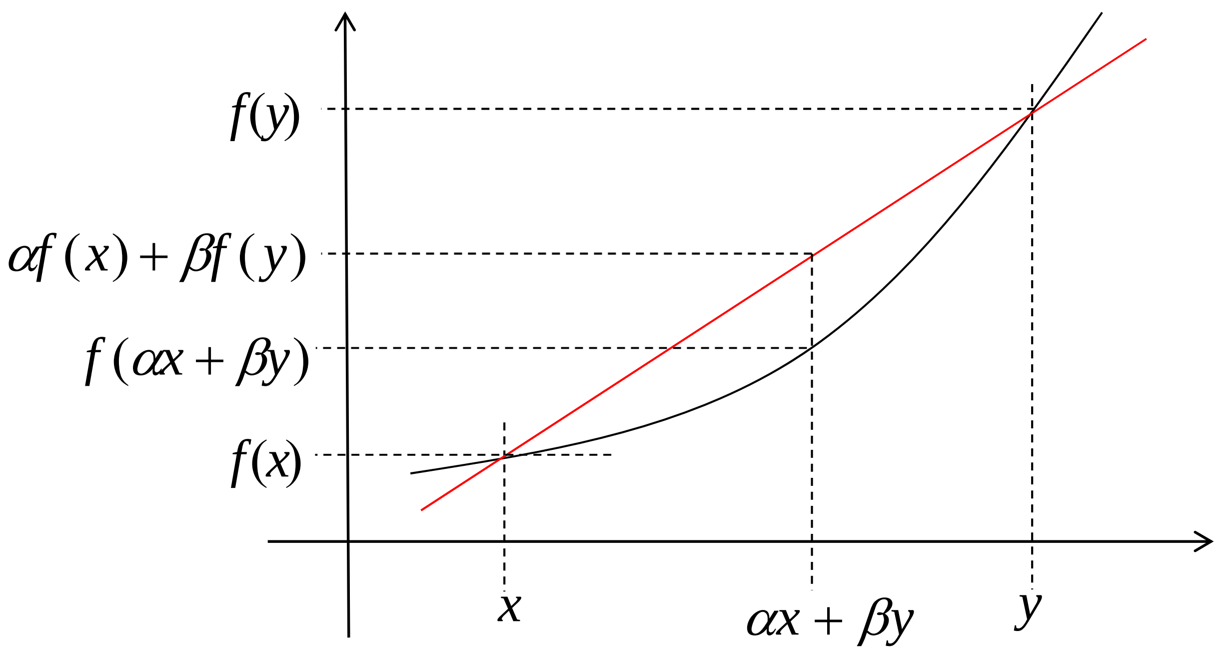
A function is concave if

$f(\alpha x + \beta y) \ge \alpha f(x) + \beta f(y)$ for all x, y and $\alpha + \beta = 1, \alpha, \beta \ge 0$

=> $\nabla^2 f(x) \le 0$ for all x, i.e. Hessian matrix is negative definite

Convexity

One-dimensional example:



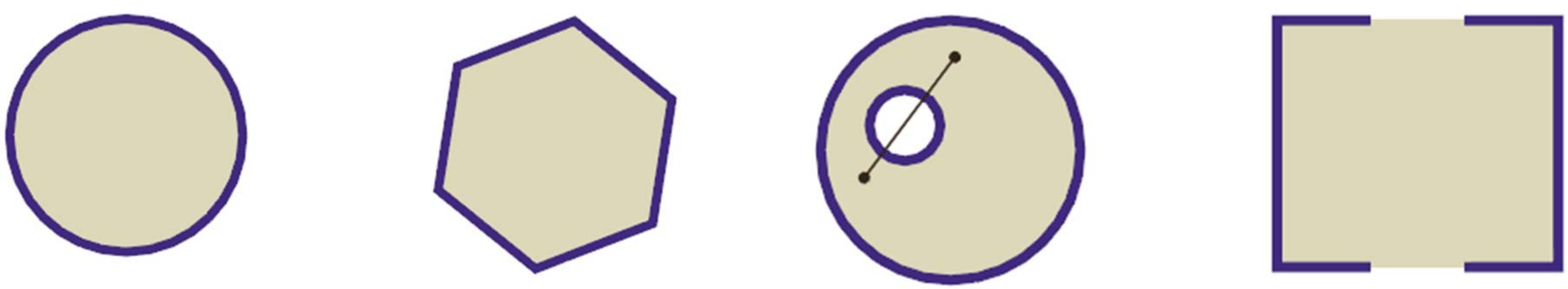
Concave

Convexity

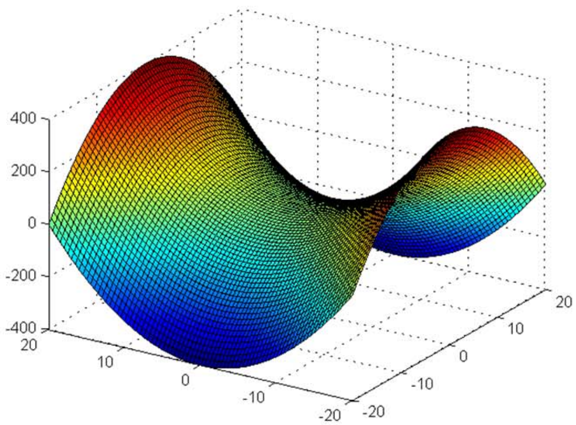
Which of these functions are convex? Which ones are concave?

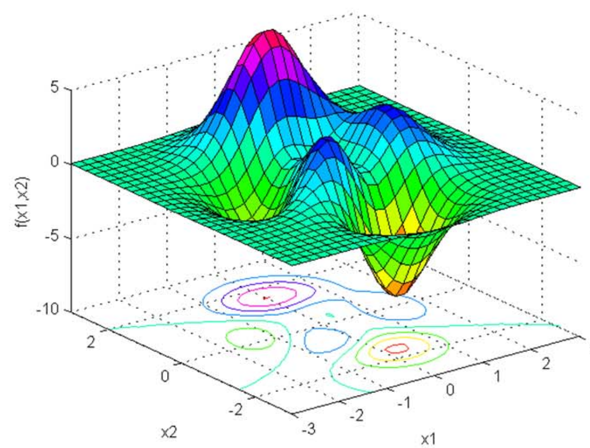
$f(x) = x^2$ $\qquad f(x) = \sin x$ $\qquad f(x) = x^3$

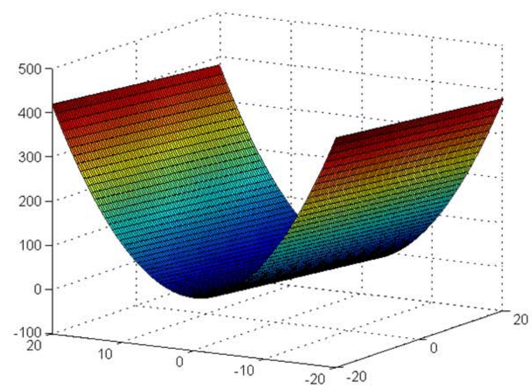
$f(x_1, x_2) = -x_1^2 - x_2^2$ $\qquad f(x_1, x_2) = x_1^2 + x_2^2$



Convexity


400
200
0
-200
-400
20
10
0
-10
-20
20
10
0
-10
-20


5
0
-5
-10
f(x1,x2)
2
0
-2
x2
-3
-2
-1
0
1
2
3
x1


500
400
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-10
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-20
0
20

Convexity

Examples:

$$a. \quad f(x) = x_1^2 + x_2^2$$

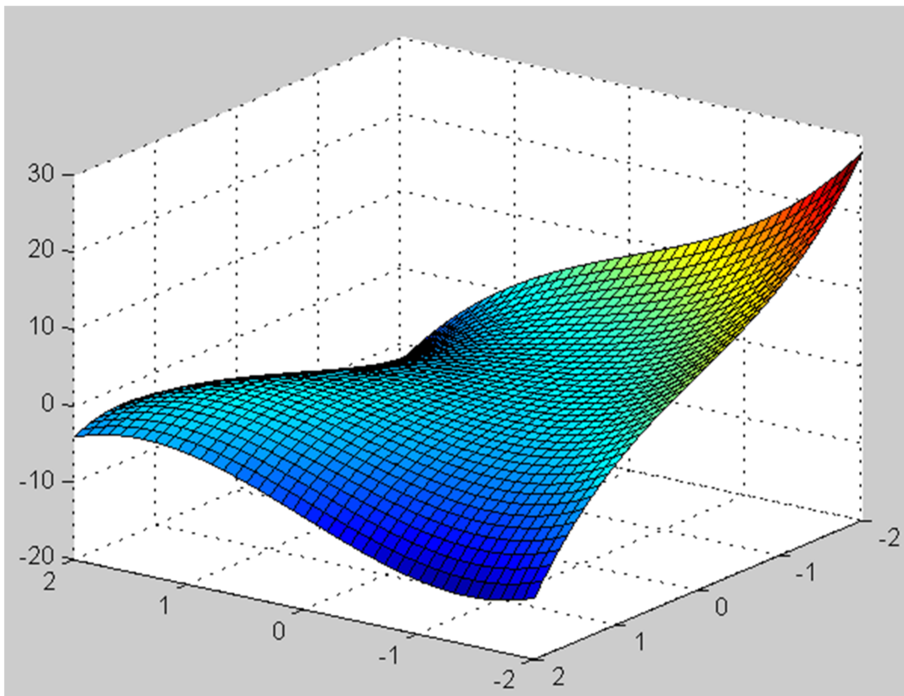
$$b. \quad f(x) = x_1^2 - x_2^2$$

$$c. \quad f(x) = 3x_1x_2 - x_1^3 - x_2^3$$

Convexity

Examples:

c)

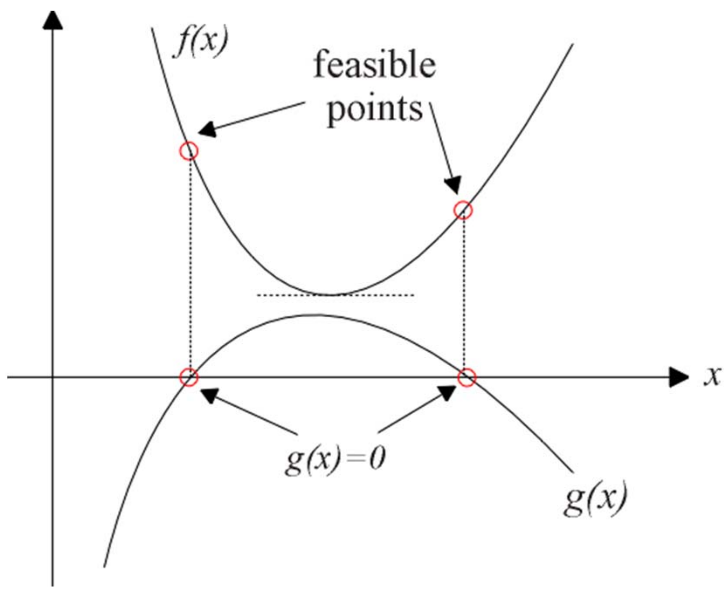


Equality Constrained Optimization

One dimensional example

$$\min_{x} f(x)$$

$$s.t. \quad g(x) = 0$$



$$\frac{\partial f(x)}{\partial x} \neq 0 \quad \text{in feasible points}$$

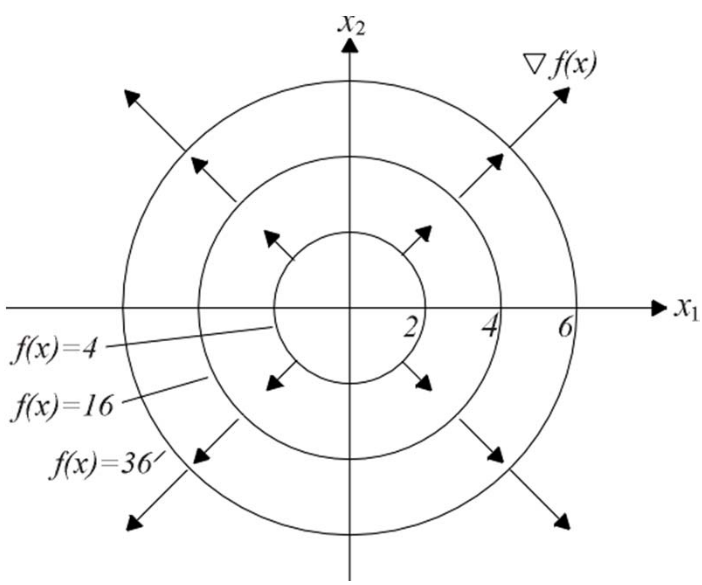
Equality Constrained Optimization

2-dimensional example:

$$\min_x \; x_1^2 + x_2^2$$

$$s.t. \; -x_1 - x_2 + 4 = 0$$

Objective Function:

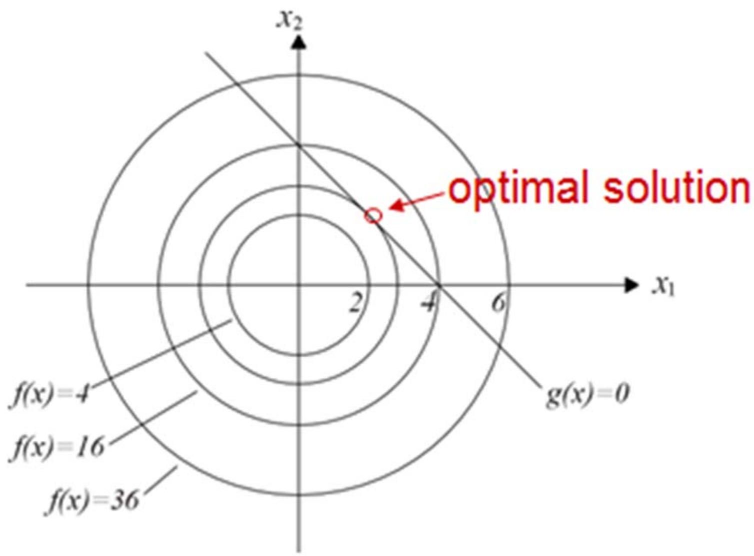


optimal solution without constraint at $\nabla f(x) = 0$

$$\Rightarrow \; x_1 = x_2 = 0$$

Equality Constrained Optimization

Objective Function and Constraint

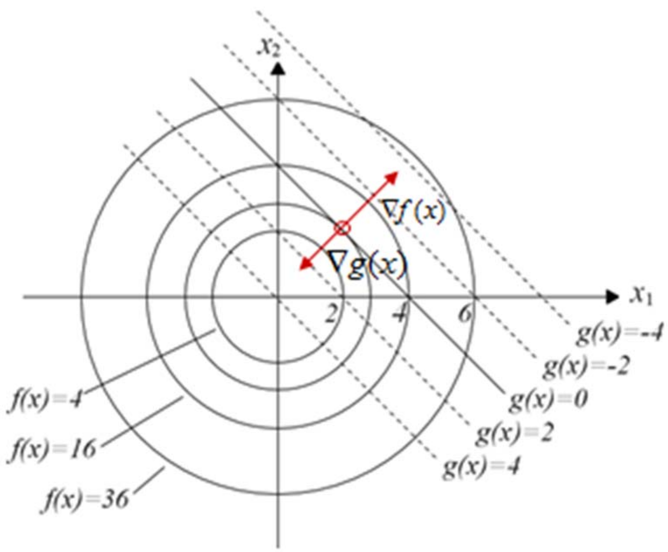


all feasible points lie on $g(x) = 0$

What is special at the optimal point?

Equality Constrained Optimization

Gradients



in the optimal point:

$$\nabla f(x) + \lambda^T \cdot \nabla g(x) = 0$$

$$g(x) = 0$$

=> gradients of the objective function and the constraints are parallel

Equality Constrained Optimization

$$\min_x f(x)$$
$$s.t. \quad g(x) = 0$$

Lagrange Function

$$L = f(x) + \lambda^T \cdot g(x)$$

Minimize Lagrange Function

$$\frac{\partial L}{\partial x} = \frac{\partial f(x)}{\partial x} + \lambda^T \cdot \frac{\partial g(x)}{\partial x} = 0$$
$$\frac{\partial L}{\partial \lambda} = g(x) = 0$$

Equation system with n +m variables and n + m equations

n: number of state variables
m: number of equality constraints

$\Rightarrow$ Solve equation system to obtain solution to optimization problem

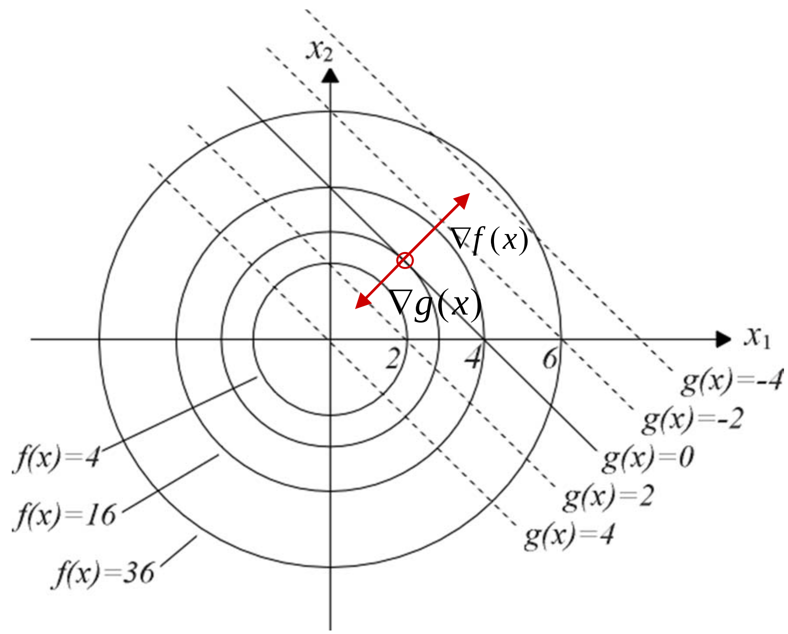
Karush-Kuhn-Tucker (KKT) first order or necessary conditions for equality constrained optimization

Equality Constrained Optimization

Meaning of Lagrange Multiplier

$$\frac{\Delta f(x)}{\Delta x} + \lambda \cdot \frac{\Delta g(x)}{\Delta x} = 0$$

$$\Rightarrow \lambda = -\frac{\Delta f(x)}{\Delta g(x)}$$



Lagrange multipliers correspond to the negative sensitivities of the optimal objective function value with respect to a small change in the equality constraint.

Equality Constrained Optimization

Example:

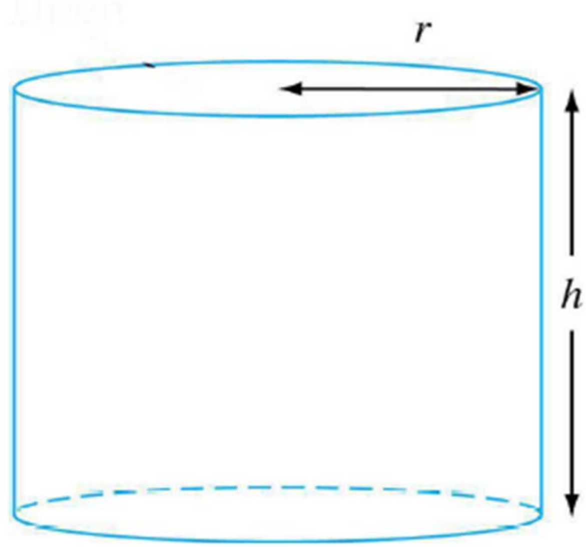
$$\min_{x} \; x_1^2 + x_2^2$$

$$s.t. \; -x_1 - x_2 + 4 = 0$$

Find the solution to this problem using KKT conditions.

Equality Constrained Optimization

Example: Minimize the outer area of a cylinder subject to a fixed volume.



Summary

- Unconstrained Optimization $\min_x f(x)$

  => optimality conditions

  $$\nabla f\left(x^*\right)=0, \qquad \nabla^2 f\left(x^*\right)$$

  Local minimum if positive definite

  Local maximum if negative definite

- Convexity
- Equality Constrained Optimization

  $$\min_x f(x)$$
  $$s.t. \quad g(x)=0$$

  => optimality conditions

  $$\nabla f(x)+\lambda^T \cdot \nabla g(x)=0$$
  $$g(x)=0$$