1

INTRODUCTION TO VECTORS

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $\boldsymbol{v}+\boldsymbol{w}$. We multiply by numbers c and d to get $c\boldsymbol{v}$ and $d\boldsymbol{w}$. Combining those two operations (adding $c\boldsymbol{v}$ to $d\boldsymbol{w}$) gives the *linear combination* $c\boldsymbol{v}+d\boldsymbol{w}$.

Linear combinations are all-important in this subject! Sometimes we want one particular combination, a specific choice of c and d that produces a desired $c\boldsymbol{v}+d\boldsymbol{w}$. Other times we want to visualize *all the combinations* (coming from all c and d). The vectors $c\boldsymbol{v}$ lie along a line. The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ normally fill a two-dimensional plane. (I have to say "two-dimensional" because linear algebra allows higher-dimensional planes.) From four vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ in four-dimensional space, their combinations are likely to fill the whole space.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n-dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n-dimensional space), and the first steps are the operations in Sections 1.1 and 1.2:

1.1 ***Vector addition*** $\boldsymbol{v}+\boldsymbol{w}$ ***and linear combinations*** $c\boldsymbol{v}+d\boldsymbol{w}$.

1.2 ***The dot product*** $v \cdot w$ ***and the length*** $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$.

VECTORS AND LINEAR COMBINATIONS ■ 1.1

"You can't add apples and oranges." In a strange way, this is the reason for vectors! If we keep the number of apples separate from the number of oranges, we have a pair of numbers. That pair is a ***two-dimensional vector*** $\boldsymbol{v}$, with "components" v_1 and v_2:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{array}{l} v_1 = \text{number of apples} \\ v_2 = \text{number of oranges.} \end{array}$$

We write $\boldsymbol{v}$ as a ***column vector***. The main point so far is to have a single letter $\boldsymbol{v}$ (in ***boldface italic***) for this pair of numbers v_1 and v_2 (in *lightface italic*).

Even if we don't add v_1 to v_2, we do *add vectors*. The first components of $\boldsymbol{v}$ and $\boldsymbol{w}$ stay separate from the second components:

VECTOR ADDITION $\quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ add to $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$.

You see the reason. We want to add apples to apples. Subtraction of vectors follows the same idea: *The components of $\boldsymbol{v} - \boldsymbol{w}$ are $v_1 - w_1$ and* ____ .

The other basic operation is *scalar multiplication*. Vectors can be multiplied by 2 or by -1 or by any number c. There are two ways to double a vector. One way is to add $\boldsymbol{v} + \boldsymbol{v}$. The other way (the usual way) is to multiply each component by 2:

SCALAR MULTIPLICATION $\quad 2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$ and $-\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$.

The components of $c\boldsymbol{v}$ are cv_1 and cv_2. The number c is called a "scalar".

Notice that the sum of $-\boldsymbol{v}$ and $\boldsymbol{v}$ is the zero vector. This is $\mathbf{0}$, which is not the same as the number zero! The vector $\mathbf{0}$ has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$—***adding vectors and multiplying by scalars***.

The order of addition makes no difference: $\boldsymbol{v} + \boldsymbol{w}$ equals $\boldsymbol{w} + \boldsymbol{v}$. Check that by algebra: The first component is $v_1 + w_1$ which equals $w_1 + v_1$. Check also by an example:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \boldsymbol{w} + \boldsymbol{v}.$$

Linear Combinations

By combining these operations, we now form "*linear combinations*" of $\boldsymbol{v}$ and $\boldsymbol{w}$. Multiply $\boldsymbol{v}$ by c and multiply $\boldsymbol{w}$ by d; then add $c\boldsymbol{v} + d\boldsymbol{w}$.

DEFINITION *The sum of $c\boldsymbol{v}$ and $d\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$.*

Four special linear combinations are: sum, difference, zero, and a scalar multiple $c\boldsymbol{v}$:

$$\begin{aligned} 1\boldsymbol{v} + 1\boldsymbol{w} &= \text{sum of vectors in Figure 1.1} \\ 1\boldsymbol{v} - 1\boldsymbol{w} &= \text{difference of vectors in Figure 1.1} \\ 0\boldsymbol{v} + 0\boldsymbol{w} &= \textbf{\textit{zero vector}} \\ c\boldsymbol{v} + 0\boldsymbol{w} &= \text{vector } c\boldsymbol{v} \text{ in the direction of } \boldsymbol{v} \end{aligned}$$

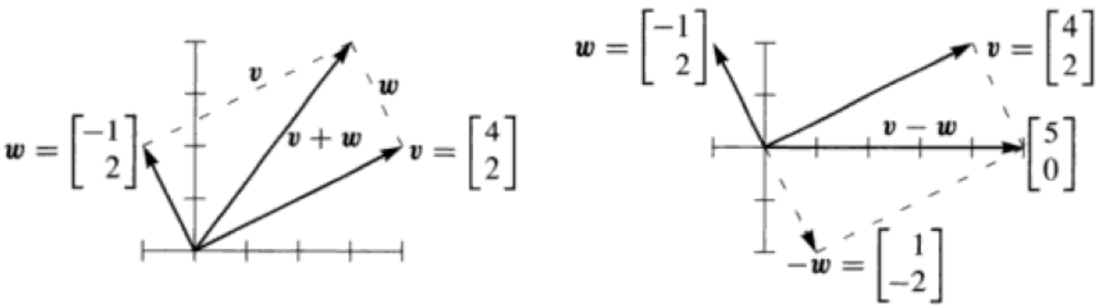
The zero vector is always a possible combination (when the coefficients are zero). Every time we see a "space" of vectors, that zero vector will be included. It is this big view, taking all the combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$, that makes the subject work.

The figures show how you can visualize vectors. For algebra, we just need the components (like 4 and 2). In the plane, that vector $\boldsymbol{v}$ is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up. It ends at the point whose x, y coordinates are $4, 2$. This point is another representation of the vector—so we have three ways to describe $\boldsymbol{v}$, by an *arrow* or a *point* or a *pair of numbers*.

Using arrows, you can see how to visualize the sum $\boldsymbol{v} + \boldsymbol{w}$:

Vector addition (head to tail) ***At the end of*** $\boldsymbol{v}$, ***place the start of*** $\boldsymbol{w}$.

We travel along $\boldsymbol{v}$ and then along $\boldsymbol{w}$. Or we take the shortcut along $\boldsymbol{v} + \boldsymbol{w}$. We could also go along $\boldsymbol{w}$ and then $\boldsymbol{v}$. In other words, $\boldsymbol{w} + \boldsymbol{v}$ gives the same answer as $\boldsymbol{v} + \boldsymbol{w}$. These are different ways along the parallelogram (in this example it is a rectangle). The endpoint in Figure 1.1 is the diagonal $\boldsymbol{v} + \boldsymbol{w}$ which is also $\boldsymbol{w} + \boldsymbol{v}$.



$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Figure 1.1 Vector addition $\boldsymbol{v} + \boldsymbol{w}$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w}$.

The zero vector has $v_1 = 0$ and $v_2 = 0$. It is too short to draw a decent arrow, but you know that $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$. For $2\boldsymbol{v}$ we double the length of the arrow. We reverse its direction for $-\boldsymbol{v}$. This reversing gives the subtraction on the right side of Figure 1.1.

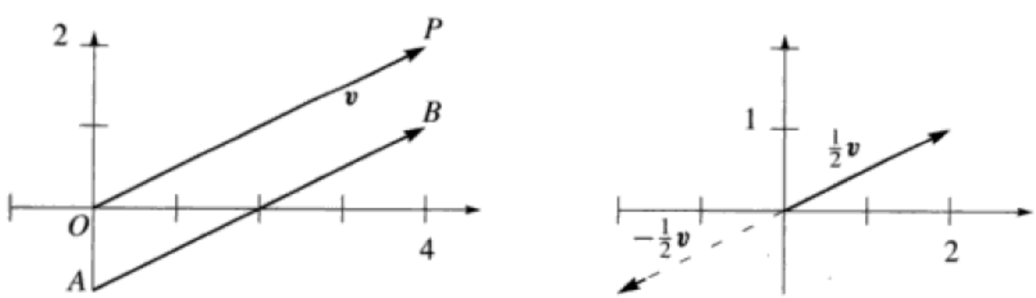


Figure 1.2 The arrow usually starts at the origin $(0, 0)$; $c\boldsymbol{v}$ is always parallel to $\boldsymbol{v}$.

Vectors in Three Dimensions

A vector with two components corresponds to a point in the xy plane. The components of $\boldsymbol{v}$ are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$. Now we allow vectors to have three components (v_1, v_2, v_3). The xy plane is replaced by three-dimensional space.

Here are typical vectors (still column vectors but with three components):

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

The vector $\boldsymbol{v}$ corresponds to an arrow in 3-space. Usually the arrow starts at the origin, where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point with coordinates v_1, v_2, v_3. There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends***.

$$\textit{From now on} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \textit{is also written as} \quad \boldsymbol{v} = (1, 2, 2).$$

The reason for the row form (in parentheses) is to save space. But $\boldsymbol{v} = (1, 2, 2)$ is not a row vector! It is in actuality a column vector, just temporarily lying down. The row vector $[\,1 \;\; 2 \;\; 2\,]$ is absolutely different, even though it has the same three components. It is the "transpose" of the column $\boldsymbol{v}$.

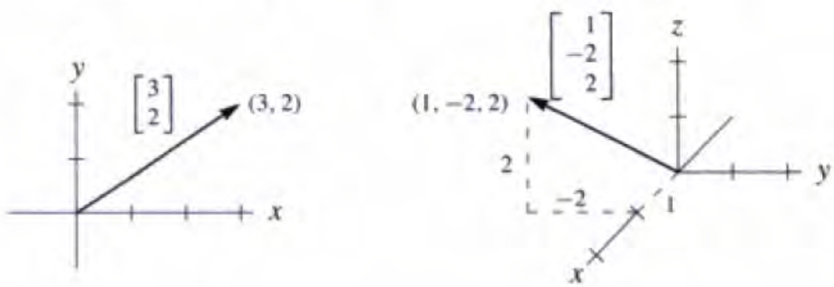


Figure 1.3 Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

In three dimensions, $\boldsymbol{v} + \boldsymbol{w}$ is still done a component at a time. The sum has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$. You see how to add vectors in 4 or 5 or n dimensions. When $\boldsymbol{w}$ starts at the end of $\boldsymbol{v}$, the third side is $\boldsymbol{v} + \boldsymbol{w}$. The other way around the parallelogram is $\boldsymbol{w} + \boldsymbol{v}$. Question: Do the four sides all lie in the same plane? *Yes*. And the sum $\boldsymbol{v} + \boldsymbol{w} - \boldsymbol{v} - \boldsymbol{w}$ goes completely around to produce _____.

A typical linear combination of three vectors in three dimensions is $\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w}$:

$$\textbf{Linear combination} \quad \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}.$$

The Important Questions

For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$. For two vectors, the combinations are $c\boldsymbol{u}+d\boldsymbol{v}$. For three vectors, the combinations are $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$. Will you take the big step from *one* linear combination to *all* linear combinations? Every c and d and e are allowed. Suppose the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in three-dimensional space:

1 What is the picture of all combinations $c\boldsymbol{u}$?

2 What is the picture of all combinations $c\boldsymbol{u} + d\boldsymbol{v}$?

3 What is the picture of all combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

The answers depend on the particular vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. If they were all zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

1 The combinations $c\boldsymbol{u}$ fill a ***line***.

2 The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane***.

3 The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

The line is infinitely long, in the direction of $\boldsymbol{u}$ (forward and backward, going through the zero vector). It is the plane of all $c\boldsymbol{u} + d\boldsymbol{v}$ (combining two lines) that I especially ask you to think about.

Adding all $c\boldsymbol{u}$ *on one line to all* $d\boldsymbol{v}$ *on the other line fills in the plane in Figure* 1.4.

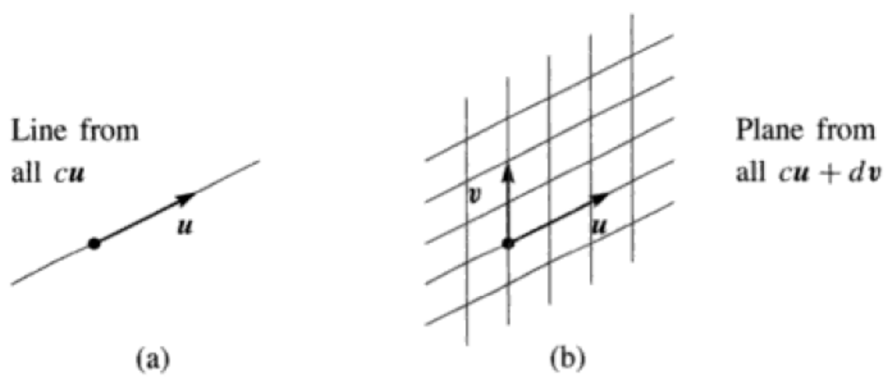


Figure 1.4 (a) The line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

When we include a third vector $\boldsymbol{w}$, the multiples $e\boldsymbol{w}$ give a third line. Suppose that line is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$. Then combining all $e\boldsymbol{w}$ with all $c\boldsymbol{u} + d\boldsymbol{v}$ fills up the whole three-dimensional space.

This is the typical situation! Line, then plane, then space. But other possibilities exist. When $\boldsymbol{w}$ happens to be $c\boldsymbol{u}+d\boldsymbol{v}$, the third vector is in the plane of the first two. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ will not go outside that $\boldsymbol{uv}$ plane. We do not get the full three-dimensional space. Please think about the special cases in Problem 1.

■ REVIEW OF THE KEY IDEAS ■

1. A vector $\boldsymbol{v}$ in two-dimensional space has two components v_1 and v_2.

2. $\boldsymbol{v}+\boldsymbol{w}=(v_1+w_1,\ v_2+w_2)$ and $c\boldsymbol{v}=(cv_1, cv_2)$ are executed a component at a time.

3. A linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ is $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$.

4. Take all linear combinations of $\boldsymbol{u}$, or $\boldsymbol{u}$ and $\boldsymbol{v}$, or $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\boldsymbol{w}$. In three dimensions, those combinations typically fill a line, a plane, and the whole space.

■ WORKED EXAMPLES ■

1.1 A Describe all the linear combinations of $\boldsymbol{v}=(1,1,0)$ and $\boldsymbol{w}=(0,1,1)$. Find a vector that is *not* a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Solution These are vectors in three-dimensional space $\mathbf{R}^3$. Their combinations $c\boldsymbol{v}+d\boldsymbol{w}$ fill a *plane* in $\mathbf{R}^3$. The vectors in that plane allow any c and d:

$$c\boldsymbol{v}+d\boldsymbol{w}=c\begin{bmatrix}1\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}c\\c+d\\d\end{bmatrix}.$$

Four particular vectors in that plane are $(0,0,0)$ and $(2,3,1)$ and $(5,7,2)$ and $(\sqrt{2},0,-\sqrt{2})$. The second component is always the sum of the first and third components. The vector $(1,1,1)$ is *not* in the plane.

Another description of this plane through $(0,0,0)$ is to know a vector *perpendicular* to the plane. In this case $\boldsymbol{n}=(1,-1,1)$ is perpendicular, as Section 1.2 will confirm by testing dot products: $\boldsymbol{v}\cdot\boldsymbol{n}=0$ and $\boldsymbol{w}\cdot\boldsymbol{n}=0$.

1.1 B For $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$, describe all the points $c\boldsymbol{v}$ and all the combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with any d and **(1)** *whole numbers* c **(2)** *nonnegative* $c\geq 0$.

Solution

(1) The vectors $c\boldsymbol{v} = (c, 0)$ with whole numbers c are equally spaced points along the x axis (the direction of $\boldsymbol{v}$). They include $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$. Adding all vectors $d\boldsymbol{w} = (0, d)$ puts a full line in the y direction through those points. We have infinitely many *parallel lines* from $c\boldsymbol{v} + d\boldsymbol{w} =$ (*whole number, any number*). These are vertical lines in the xy plane, through equally spaced points on the x axis.

(2) The vectors $c\boldsymbol{v}$ with $c \geq 0$ fill a "half-line". It is the *positive* x axis, starting at $(0, 0)$ where $c = 0$. It includes $(\pi, 0)$ but not $(-\pi, 0)$. Adding all vectors $d\boldsymbol{w}$ puts a full line in the y direction crossing every point on that half-line. Now we have a *half-plane*. It is the right half of the xy plane, where $x \geq 0$.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1 Describe geometrically (as a line, plane, ...) all linear combinations of

$$\text{(a)}\quad \begin{bmatrix}1\\1\\1\end{bmatrix} \text{ and } \begin{bmatrix}4\\4\\4\end{bmatrix} \qquad \text{(b)}\quad \begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad \text{(c)}\quad \begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\1\\1\end{bmatrix} \text{ and } \begin{bmatrix}1\\1\\1\end{bmatrix}$$

2 Draw the vectors $\boldsymbol{v} = \left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]$ and $\boldsymbol{w} = \left[\begin{smallmatrix}-2\\2\end{smallmatrix}\right]$ and $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ in a single xy plane.

3 If $\boldsymbol{v} + \boldsymbol{w} = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$ and $\boldsymbol{v} - \boldsymbol{w} = \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$, compute and draw $\boldsymbol{v}$ and $\boldsymbol{w}$.

4 From $\boldsymbol{v} = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ and $\boldsymbol{w} = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - 3\boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

5 Compute $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$ and $2\boldsymbol{u} + 2\boldsymbol{v} + \boldsymbol{w}$ when

$$\boldsymbol{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix}-3\\1\\-2\end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix}2\\-3\\-1\end{bmatrix}.$$

6 Every combination of $\boldsymbol{v} = (1, -2, 1)$ and $\boldsymbol{w} = (0, 1, -1)$ has components that add to ______. Find c and d so that $c\boldsymbol{v} + d\boldsymbol{w} = (4, 2, -6)$.

7 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix}3\\1\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

8 The parallelogram in Figure 1.1 has diagonal $\boldsymbol{v}+\boldsymbol{w}$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.

9 If three corners of a parallelogram are $(1,1)$, $(4,2)$, and $(1,3)$, what are all the possible fourth corners? Draw two of them.

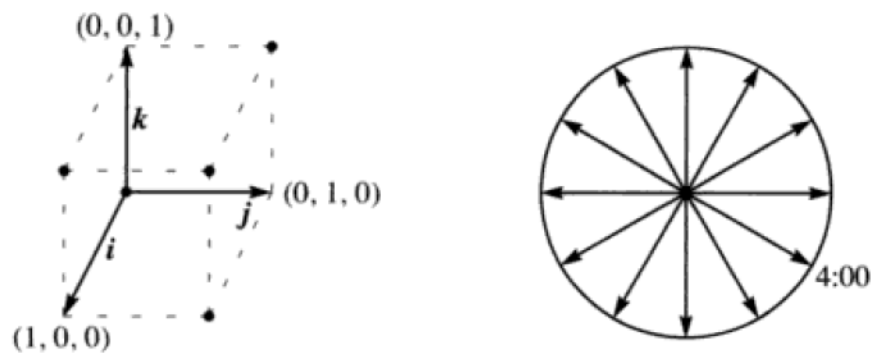


Figure 1.5 Unit cube from $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$; twelve clock vectors.

Problems 10–14 are about special vectors on cubes and clocks.

10 Copy the cube and draw the vector sum of $\boldsymbol{i}=(1,0,0)$ and $\boldsymbol{j}=(0,1,0)$ and $\boldsymbol{k}=(0,0,1)$. The addition $\boldsymbol{i}+\boldsymbol{j}$ yields the diagonal of _____.

11 Four corners of the cube are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces are _____.

12 How many corners does a cube have in 4 dimensions? How many faces? How many edges? A typical corner is $(0,0,1,0)$.

13 (a) What is the sum $\boldsymbol{V}$ of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the vector to 4:00 is removed, find the sum of the eleven remaining vectors.

(c) What is the unit vector to 1:00?

14 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0,0)$ at the center. The vector to 12:00 is doubled to $2\boldsymbol{j}=(0,2)$. Add the new twelve vectors.

Problems 15–19 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (Figure 1.6)

15 The figure shows $\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v}+\frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v}+\frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v}+\boldsymbol{w}$.

16 Mark the point $-\boldsymbol{v}+2\boldsymbol{w}$ and any other combination $c\boldsymbol{v}+d\boldsymbol{w}$ with $c+d=1$. Draw the line of all combinations that have $c+d=1$.

17 Locate $\frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{2}{3}\boldsymbol{v} + \frac{2}{3}\boldsymbol{w}$. The combinations $c\boldsymbol{v} + c\boldsymbol{w}$ fill out what line? Restricted by $c \geq 0$ those combinations with $c = d$ fill out what half line?

18 Restricted by $0 \leq c \leq 1$ and $0 \leq d \leq 1$, shade in all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

19 Restricted only by $c \geq 0$ and $d \geq 0$ draw the "cone" of all combinations $c\boldsymbol{v}+d\boldsymbol{w}$.

Problems 20–27 deal with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in three-dimensional space (see Figure 1.6).

20 Locate $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ in the dashed triangle. Challenge problem: Under what restrictions on c, d, e, will the combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ fill in the dashed triangle?

21 The three sides of the dashed triangle are $\boldsymbol{v} - \boldsymbol{u}$ and $\boldsymbol{w} - \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{w}$. Their sum is _____. Draw the head-to-tail addition around a plane triangle of $(3, 1)$ plus $(-1, 1)$ plus $(-2, -2)$.

22 Shade in the pyramid of combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ with $c \geq 0$, $d \geq 0$, $e \geq 0$ and $c + d + e \leq 1$. Mark the vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ as inside or outside this pyramid.

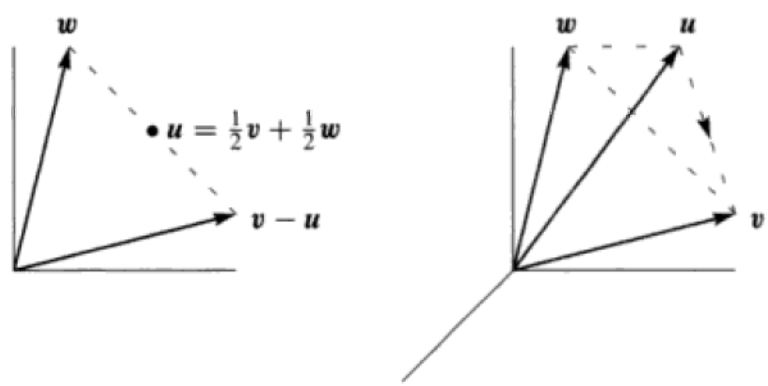


Figure 1.6 Problems **15–19** in a plane Problems **20–27** in 3-dimensional space

23 If you look at *all* combinations of those $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$, is there any vector that can't be produced from $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

24 Which vectors are combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$, and *also* combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$?

25 Draw vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a line. Draw vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a plane.

26 What combination of the vectors $\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix}$ produces $\begin{bmatrix} \mathbf{14} \\ \mathbf{8} \end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

27 *Review Question.* In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$?

28 If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that (a, c) is a multiple of (b, d)*. This is surprisingly important; call it a challenge question. You could use numbers first to see how a, b, c, d are related. The question will lead to:

If $A = \left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ has dependent rows then it has dependent columns.

And eventually: If $AB = \left[\begin{smallmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right]$ then $BA = \left[\begin{smallmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right]$. That looks so simple...

LENGTHS AND DOT PRODUCTS ■ 1.2

The first section mentioned multiplication of vectors, but it backed off. Now we go forward to define the *"dot product"* of $\boldsymbol{v}$ and $\boldsymbol{w}$. This multiplication involves the separate products v_1w_1 and v_2w_2, but it doesn't stop there. Those two numbers are added to produce the single number $\boldsymbol{v} \cdot \boldsymbol{w}$.

DEFINITION The ***dot product*** or ***inner product*** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2. \tag{1}$$

Example 1 The vectors $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ have a *zero* dot product:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular*. The angle between them is 90°. When we drew them in Figure 1.1, we saw a rectangle (not just any parallelogram). The clearest example of perpendicular vectors is $\boldsymbol{i} = (1, 0)$ along the x axis and $\boldsymbol{j} = (0, 1)$ up the y axis. Again the dot product is $\boldsymbol{i} \cdot \boldsymbol{j} = 0 + 0 = 0$. Those vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ form a right angle.

The dot product of $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (2, 1)$ is 4. *Please check this.* Soon that will reveal the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ (not 90°).

Example 2 Put a weight of 4 at the point $x = -1$ and a weight of 2 at the point $x = 2$. The x axis will balance on the center point $x = 0$ (like a see-saw). The weights balance because the dot product is $(4)(-1) + (2)(2) = 0$.

This example is typical of engineering and science. The vector of weights is $(w_1, w_2) = (4, 2)$. The vector of distances from the center is $(v_1, v_2) = (-1, 2)$. The weights times the distances, w_1v_1 and w_2v_2, give the "moments". The equation for the see-saw to balance is $w_1v_1 + w_2v_2 = 0$.

The dot product $\boldsymbol{w} \cdot \boldsymbol{v}$ equals $\boldsymbol{v} \cdot \boldsymbol{w}$. The order of $\boldsymbol{v}$ and $\boldsymbol{w}$ makes no difference.

Example 3 Dot products enter in economics and business. We have three products to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the "price vector" $\boldsymbol{p}$.

The quantities we buy or sell are (q_1, q_2, q_3)—positive when we sell, negative when we buy. Selling q_1 units of the first product at the price p_1 brings in $q_1 p_1$. The total income is the dot product $\boldsymbol{q} \cdot \boldsymbol{p}$:

$$\textbf{\textit{Income}} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3.$$

A zero dot product means that "the books balance." Total sales equal total purchases if $\boldsymbol{q} \cdot \boldsymbol{p} = 0$. Then $\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$ (in three-dimensional space). With three products, ***the vectors are three-dimensional***. A supermarket goes quickly into high dimensions.

Small note: Spreadsheets have become essential in management. They compute linear combinations and dot products. What you see on the screen is a matrix.

Main point To compute the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$, multiply each v_i times w_i. Then add.

Lengths and Unit Vectors

An important case is the dot product of a vector *with itself*. In this case $\boldsymbol{v} = \boldsymbol{w}$. When the vector is $\boldsymbol{v} = (1, 2, 3)$, the dot product with itself is $\boldsymbol{v} \cdot \boldsymbol{v} = 14$:

$$\boldsymbol{v} \cdot \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14.$$

The answer is not zero because $\boldsymbol{v}$ is not perpendicular to itself. Instead of a $90°$ angle between vectors we have $0°$. The dot product $\boldsymbol{v} \cdot \boldsymbol{v}$ gives the *length of $\boldsymbol{v}$ squared*.

DEFINITION The ***length*** (or ***norm***) of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$:

$$\text{length} = \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}.$$

In two dimensions the length is $\sqrt{v_1^2 + v_2^2}$. In three dimensions it is $\sqrt{v_1^2 + v_2^2 + v_3^2}$. By the calculation above, the length of $\boldsymbol{v} = (1, 2, 3)$ is $\|\boldsymbol{v}\| = \sqrt{14}$.

We can explain this definition. $\|\boldsymbol{v}\|$ is just the ordinary length of the arrow that represents the vector. In two dimensions, the arrow is in a plane. If the components are 1 and 2, the arrow is the third side of a right triangle (Figure 1.7). The formula $a^2 + b^2 = c^2$, which connects the three sides, is $1^2 + 2^2 = \|\boldsymbol{v}\|^2$.

For the length of $\boldsymbol{v} = (1, 2, 3)$, we used the right triangle formula twice. The vector $(1, 2, 0)$ in the base has length $\sqrt{5}$. This base vector is perpendicular to $(0, 0, 3)$ that goes straight up. So the diagonal of the box has length $\|\boldsymbol{v}\| = \sqrt{5 + 9} = \sqrt{14}$.

The length of a four-dimensional vector would be $\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$. Thus $(1, 1, 1, 1)$ has length $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. The diagonal in n dimensions has length $\sqrt{n}$.

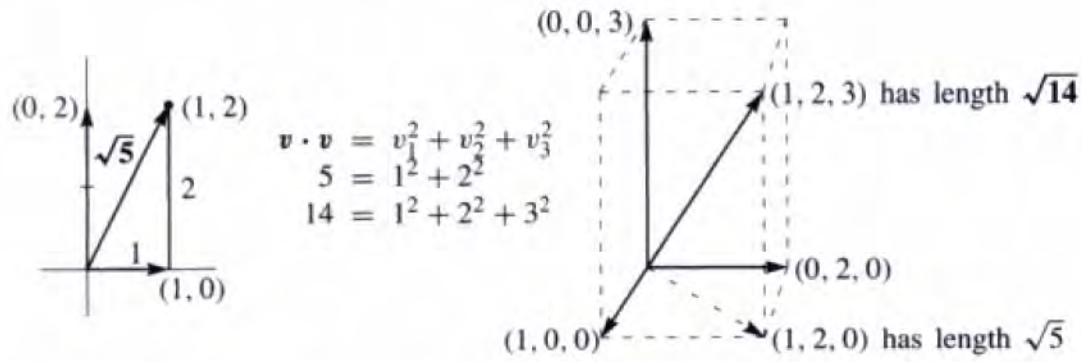


Figure 1.7 The length $\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ of two-dimensional and three-dimensional vectors.

The word "unit" is always indicating that some measurement equals "one." The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we define the idea of a "unit vector."

DEFINITION A *unit vector* ***u*** ***is a vector whose length equals one.*** Then $\boldsymbol{u} \cdot \boldsymbol{u} = 1$.

An example in four dimensions is $\boldsymbol{u} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $\boldsymbol{u} \cdot \boldsymbol{u}$ is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. We divided $\boldsymbol{v} = (1, 1, 1, 1)$ by its length $\|\boldsymbol{v}\| = 2$ to get this unit vector.

Example 4 The standard unit vectors along the x and y axes are written $\boldsymbol{i}$ and $\boldsymbol{j}$. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $(\cos\theta, \sin\theta)$:

$$\textbf{Unit vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

When $\theta = 0$, the horizontal vector $\boldsymbol{u}$ is $\boldsymbol{i}$. When $\theta = 90°$ (or $\frac{\pi}{2}$ radians), the vertical vector is $\boldsymbol{j}$. At any angle, the components $\cos\theta$ and $\sin\theta$ produce $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ because $\cos^2\theta + \sin^2\theta = 1$. These vectors reach out to the unit circle in Figure 1.8. Thus $\cos\theta$ and $\sin\theta$ are simply the coordinates of that point at angle θ on the unit circle.

In three dimensions, the unit vectors along the axes are $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. Their components are $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$. Notice how every three-dimensional vector is a linear combination of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. The vector $\boldsymbol{v} = (2, 2, 1)$ is equal to $2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$. Its length is $\sqrt{2^2 + 2^2 + 1^2}$. This is the square root of 9, so $\|\boldsymbol{v}\| = 3$.

Since $(2, 2, 1)$ has length 3, the vector $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has length 1. Check that $\boldsymbol{u} \cdot \boldsymbol{u} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$. To create a unit vector, just divide $\boldsymbol{v}$ by its length $\|\boldsymbol{v}\|$.

1A Unit vectors ***Divide any nonzero vector v by its length.*** Then $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ is a unit vector in the same direction as $\boldsymbol{v}$.

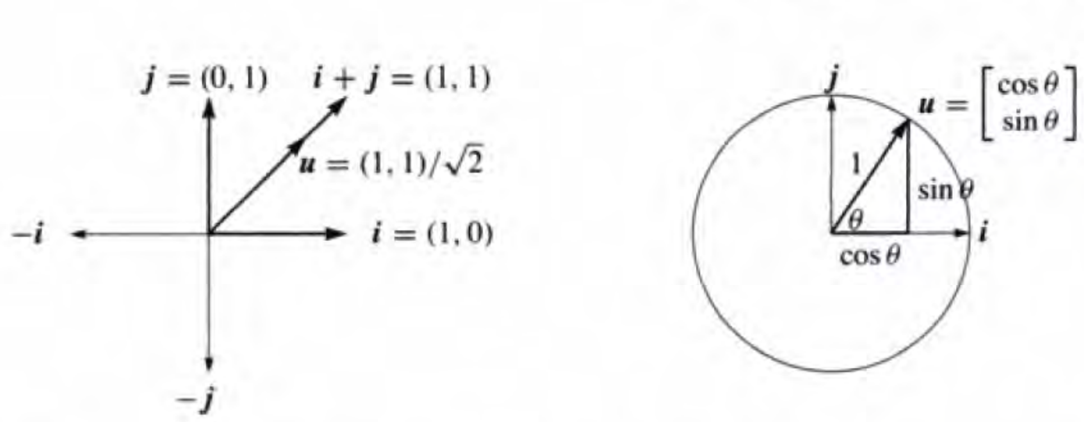


Figure 1.8 The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ at angle 45° (left) and the unit vector $(\cos\theta, \sin\theta)$ at angle θ.

The Angle Between Two Vectors

We stated that perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is 90°. To explain this, we have to connect angles to dot products. Then we show how $\boldsymbol{v} \cdot \boldsymbol{w}$ finds the angle between any two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

1B Right angles *The dot product is $v \cdot w = 0$ when v is perpendicular to w.*

Proof When $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, they form two sides of a right triangle. The third side is $\boldsymbol{v} - \boldsymbol{w}$ (the hypotenuse going across in Figure 1.7). The *Pythagoras Law* for the sides of a right triangle is $a^2 + b^2 = c^2$:

$$\textbf{Perpendicular vectors} \quad \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2 \tag{2}$$

Writing out the formulas for those lengths in two dimensions, this equation is

$$\left(v_1^2 + v_2^2\right) + \left(w_1^2 + w_2^2\right) = (v_1 - w_1)^2 + (v_2 - w_2)^2. \tag{3}$$

The right side begins with $v_1^2 - 2v_1w_1 + w_1^2$. Then v_1^2 and w_1^2 are on both sides of the equation and they cancel, leaving $-2v_1w_1$. Similarly v_2^2 and w_2^2 cancel, leaving $-2v_2w_2$. (In three dimensions there would also be $-2v_3w_3$.) The last step is to divide by -2:

$$0 = -2v_1w_1 - 2v_2w_2 \quad \text{which leads to} \quad v_1w_1 + v_2w_2 = 0. \tag{4}$$

Conclusion Right angles produce $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. We have proved **Theorem 1B**. The dot product is zero when the angle is $\theta = 90°$. Then $\cos\theta = 0$. The zero vector $\boldsymbol{v} = \boldsymbol{0}$ is perpendicular to every vector $\boldsymbol{w}$ because $\boldsymbol{0} \cdot \boldsymbol{w}$ is always zero.

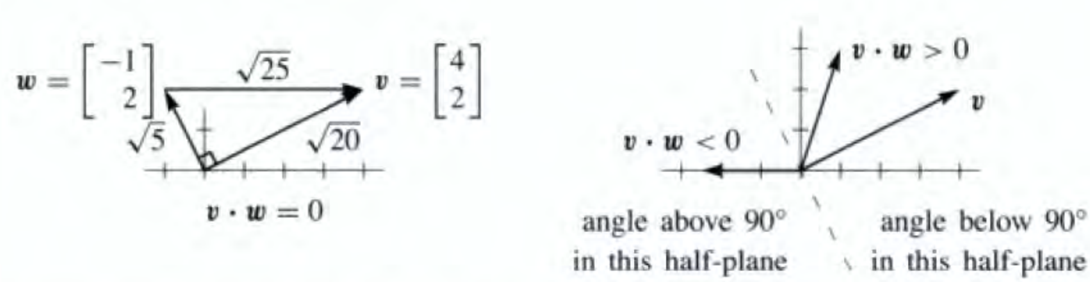


Figure 1.9 Perpendicular vectors have $v \cdot w = 0$. The angle is below 90° when $v \cdot w > 0$.

Now suppose $v \cdot w$ is not zero. It may be positive, it may be negative. The sign of $v \cdot w$ immediately tells whether we are below or above a right angle. The angle is less than 90° when $v \cdot w$ is positive. The angle is above 90° when $v \cdot w$ is negative. Figure 1.9 shows a typical vector $v = (3, 1)$. The angle with $w = (1, 3)$ is less than 90°.

The borderline is where vectors are perpendicular to v. On that dividing line between plus and minus, where we find $w = (1, -3)$, the dot product is zero.

The next page takes one more step, to find the exact angle θ. This is not necessary for linear algebra—you could stop here! Once we have matrices and linear equations, we won't come back to θ. But while we are on the subject of angles, this is the place for the formula.

Start with unit vectors u and U. The sign of $u \cdot U$ tells whether $\theta < 90°$ or $\theta > 90°$. Because the vectors have length 1, we learn more than that. ***The dot product $u \cdot U$ is the cosine of θ.*** This is true in any number of dimensions.

1C If u and U are unit vectors then $\boxed{u \cdot U = \cos\theta}$. Certainly $\boxed{|u \cdot U| \le 1}$.

Remember that $\cos\theta$ is never greater than 1. It is never less than -1. ***The dot product of unit vectors is between -1 and 1.***

Figure 1.10 shows this clearly when the vectors are $u = (\cos\theta, \sin\theta)$ and $i = (1, 0)$. The dot product is $u \cdot i = \cos\theta$. That is the cosine of the angle between them.

After rotation through any angle α, these are still unit vectors. Call the vectors $u = (\cos\beta, \sin\beta)$ and $U = (\cos\alpha, \sin\alpha)$. Their dot product is $\cos\alpha\cos\beta + \sin\alpha\sin\beta$. From trigonometry this is the same as $\cos(\beta - \alpha)$. Since $\beta - \alpha$ equals θ (no change in the angle between them) we have reached the formula $u \cdot U = \cos\theta$.

Problem 26 proves $|u \cdot U| \le 1$ directly, without mentioning angles. The inequality and the cosine formula $u \cdot U = \cos\theta$ are always true for unit vectors.

What if v and w are not unit vectors? Divide by their lengths to get $u = v/\|v\|$ and $U = w/\|w\|$. Then the dot product of those unit vectors u and U gives $\cos\theta$.

Figure 1.10 The dot product of unit vectors is the cosine of the angle θ.

Whatever the angle, this dot product of $\boldsymbol{v}/\|\boldsymbol{v}\|$ with $\boldsymbol{w}/\|\boldsymbol{w}\|$ never exceeds one. That is the ***"Schwarz inequality"*** for dot products—or more correctly the Cauchy-Schwarz-Buniakowsky inequality. It was found in France and Germany and Russia (and maybe elsewhere—it is the most important inequality in mathematics). With the division by $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ from rescaling to unit vectors, we have $\cos\theta$:

1D (a) **COSINE FORMULA** If $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors then $\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \cos\theta$.

(b) **SCHWARZ INEQUALITY** If $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors then $|\boldsymbol{v}\cdot\boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

Example 5 Find $\cos\theta$ for $\boldsymbol{v} = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$ and $\boldsymbol{w} = \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ in Figure 1.9b.

Solution The dot product is $\boldsymbol{v}\cdot\boldsymbol{w} = 6$. Both $\boldsymbol{v}$ and $\boldsymbol{w}$ have length $\sqrt{10}$. The cosine is

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5}.$$

The angle is below 90° because $\boldsymbol{v}\cdot\boldsymbol{w} = 6$ is positive. By the Schwarz inequality, $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\| = 10$ is larger than $\boldsymbol{v}\cdot\boldsymbol{w} = 6$.

Example 6 The dot product of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (b, a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality says that $2ab \le a^2 + b^2$.

Reason The difference between $a^2 + b^2$ and $2ab$ can never be negative:

$$a^2 + b^2 - 2ab = (a - b)^2 \ge 0.$$

This is more famous if we write $x = a^2$ and $y = b^2$. Then the "geometric mean" $\sqrt{xy}$ is not larger than the "arithmetic mean," which is the average $\frac{1}{2}(x + y)$:

$$ab \le \frac{a^2 + b^2}{2} \quad \text{becomes} \quad \sqrt{xy} \le \frac{x + y}{2}.$$

Notes on Computing

Write the components of $\boldsymbol{v}$ as $v(1), \ldots, v(N)$ and similarly for $\boldsymbol{w}$. In FORTRAN, the sum $\boldsymbol{v}+\boldsymbol{w}$ requires a loop to add components separately. The dot product also loops to add the separate $v(i)w(i)$:

```
      DO 10 I = 1,N                DO 10 I = 1,N
10 VPLUSW(I) = v(I)+w(I)    10 VDOTW = VDOTW + V(I) * W(I)
```

MATLAB *works directly with whole vectors, not their components.* No loop is needed. When $\boldsymbol{v}$ and $\boldsymbol{w}$ have been defined, $\boldsymbol{v}+\boldsymbol{w}$ is immediately understood. It is printed unless the line ends in a semicolon. Input $\boldsymbol{v}$ and $\boldsymbol{w}$ as rows—the prime $'$ at the end transposes them to columns. The combination $2\boldsymbol{v}+3\boldsymbol{w}$ uses $*$ for multiplication.

$$\boldsymbol{v}=[2 \quad 3 \quad 4]' \;\; ; \;\; \boldsymbol{w}=[1 \quad 1 \quad 1]' \;\; ; \;\; \boldsymbol{u}=2*\boldsymbol{v}+3*\boldsymbol{w}$$

The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is usually seen as ***a row times a column (with no dot)***:

$$\text{Instead of} \quad \begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} \quad \text{we more often see} \quad [1 \;\, 2]\begin{bmatrix}3\\4\end{bmatrix} \quad \text{or} \quad v' * w$$

The length of $\boldsymbol{v}$ is already known to MATLAB as norm $(\boldsymbol{v})$. We could define it ourselves as sqrt $(\boldsymbol{v}'*\boldsymbol{v})$, using the square root function—also known. The cosine we have to define ourselves! Then the angle (in radians) comes from the *arc cosine* (acos) function:

$$\begin{aligned}\text{cosine} &= \boldsymbol{v}' * \boldsymbol{w}/(\text{norm}\,(\boldsymbol{v}) * \text{norm}\,(\boldsymbol{w}));\\ \text{angle} &= \text{acos}\,(\text{cosine})\end{aligned}$$

An M-file would create a new function **cosine** $(\boldsymbol{v}, \boldsymbol{w})$ for future use. (Quite a few M-files have been created especially for this book. They are listed at the end.)

■ REVIEW OF THE KEY IDEAS ■

1. The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ multiplies each component v_i by w_i and adds the $v_i w_i$.

2. The length $\|\boldsymbol{v}\|$ is the square root of $\boldsymbol{v}\cdot\boldsymbol{v}$.

3. The vector $\boldsymbol{v}/\|\boldsymbol{v}\|$ is a ***unit vector***. Its length is 1.

4. The dot product is $\boldsymbol{v}\cdot\boldsymbol{w}=0$ when $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular.

5. The cosine of θ (the angle between any nonzero $\boldsymbol{v}$ and $\boldsymbol{w}$) never exceeds 1:

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} \qquad \textbf{\textit{Schwarz inequality}} \quad |\boldsymbol{v}\cdot\boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|.$$

■ WORKED EXAMPLES ■

1.2 A For the vectors $\boldsymbol{v} = (3, 4)$ and $\boldsymbol{w} = (4, 3)$ test the Schwarz inequality on $\boldsymbol{v} \cdot \boldsymbol{w}$ and the triangle inequality on $\|\boldsymbol{v} + \boldsymbol{w}\|$. Find $\cos\theta$ for the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. When will we have *equality* $|\boldsymbol{v} \cdot \boldsymbol{w}| = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ and $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$?

Solution The dot product is $\boldsymbol{v} \cdot \boldsymbol{w} = (3)(4) + (4)(3) = 24$. The length of $\boldsymbol{v}$ is $\|\boldsymbol{v}\| = \sqrt{9 + 16} = 5$ and also $\|\boldsymbol{w}\| = 5$. The sum $\boldsymbol{v} + \boldsymbol{w} = (7, 7)$ has length $\|\boldsymbol{v} + \boldsymbol{w}\| = 7\sqrt{2} \approx 9.9$.

Schwarz inequality	$\lvert\boldsymbol{v} \cdot \boldsymbol{w}\rvert \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ is $24 < 25$.
Triangle inequality	$\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ is $7\sqrt{2} < 10$.
Cosine of angle	$\cos\theta = \frac{24}{25}$ (Thin angle!)

If one vector is a multiple of the other as in $\boldsymbol{w} = -2\boldsymbol{v}$, then the angle is $0°$ or $180°$ and $|\cos\theta| = 1$ and $|\boldsymbol{v} \cdot \boldsymbol{w}|$ *equals* $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$. If the angle is $0°$, as in $\boldsymbol{w} = 2\boldsymbol{v}$, then $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$. The triangle is flat.

1.2 B Find a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = (3, 4)$. Find a unit vector $\boldsymbol{U}$ perpendicular to $\boldsymbol{u}$. How many possibilities for $\boldsymbol{U}$?

Solution For a unit vector $\boldsymbol{u}$, divide $\boldsymbol{v}$ by its length $\|\boldsymbol{v}\| = 5$. For a perpendicular vector $\boldsymbol{V}$ we can choose $(-4, 3)$ since the dot product $\boldsymbol{v} \cdot \boldsymbol{V}$ is $(3)(-4) + (4)(3) = 0$. For a *unit* vector $\boldsymbol{U}$, divide $\boldsymbol{V}$ by its length $\|\boldsymbol{V}\|$:

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \boldsymbol{U} = \frac{\boldsymbol{V}}{\|\boldsymbol{V}\|} = \frac{(-4,3)}{5} = \left(-\frac{4}{5}, \frac{3}{5}\right)$$

The only other perpendicular unit vector would be $-\boldsymbol{U} = (\frac{4}{5}, -\frac{3}{5})$.

Problem Set 1.2

1 Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{v} \cdot \boldsymbol{w}$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:

$$\boldsymbol{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

2 Compute the lengths $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ and $\|\boldsymbol{w}\|$ of those vectors. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \le \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

3 Find unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$ in Problem 1, and the cosine of the angle θ. Choose vectors that make $0°$, $90°$, and $180°$ angles with $\boldsymbol{w}$.

4 Find unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in the directions of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (2, 1, 2)$. Find unit vectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ that are perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

5 For any *unit* vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, find the dot products (actual numbers) of

(a) $\boldsymbol{v}$ and $-\boldsymbol{v}$ (b) $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ (c) $\boldsymbol{v}-2\boldsymbol{w}$ and $\boldsymbol{v}+2\boldsymbol{w}$

6 Find the angle θ (from its cosine) between

(a) $\boldsymbol{v}=\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}1\\0\end{bmatrix}$ (b) $\boldsymbol{v}=\begin{bmatrix}2\\2\\-1\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}2\\-1\\2\end{bmatrix}$

(c) $\boldsymbol{v}=\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}$ (d) $\boldsymbol{v}=\begin{bmatrix}3\\1\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}-1\\-2\end{bmatrix}$.

7 (a) Describe every vector $\boldsymbol{w}=(w_1,w_2)$ that is perpendicular to $\boldsymbol{v}=(2,-1)$.

(b) The vectors that are perpendicular to $\boldsymbol{V}=(1,1,1)$ lie on a _____.

(c) The vectors that are perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a _____.

8 True or false (give a reason if true or a counterexample if false):

(a) If $\boldsymbol{u}$ is perpendicular (in three dimensions) to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

(b) If $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}+2\boldsymbol{w}$.

(c) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular unit vectors then $\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{2}$.

9 The slopes of the arrows from $(0,0)$ to (v_1,v_2) and (w_1,w_2) are v_2/v_1 and w_2/w_1. If the product v_2w_2/v_1w_1 of those slopes is -1, show that $\boldsymbol{v}\cdot\boldsymbol{w}=0$ and the vectors are perpendicular.

10 Draw arrows from $(0,0)$ to the points $\boldsymbol{v}=(1,2)$ and $\boldsymbol{w}=(-2,1)$. Multiply their slopes. That answer is a signal that $\boldsymbol{v}\cdot\boldsymbol{w}=0$ and the arrows are _____.

11 If $\boldsymbol{v}\cdot\boldsymbol{w}$ is negative, what does this say about the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$? Draw a 2-dimensional vector $\boldsymbol{v}$ (an arrow), and show where to find all $\boldsymbol{w}$'s with $\boldsymbol{v}\cdot\boldsymbol{w}<0$.

12 With $\boldsymbol{v}=(1,1)$ and $\boldsymbol{w}=(1,5)$ choose a number c so that $\boldsymbol{w}-c\boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$. Then find the formula that gives this number c for any nonzero $\boldsymbol{v}$ and $\boldsymbol{w}$.

13 Find two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that are perpendicular to $(1,0,1)$ and to each other.

14 Find three vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ that are perpendicular to $(1,1,1,1)$ and to each other.

15 The geometric mean of $x=2$ and $y=8$ is $\sqrt{xy}=4$. The arithmetic mean is larger: $\frac{1}{2}(x+y)=$ _____. This came in Example 6 from the Schwarz inequality for $\boldsymbol{v}=(\sqrt{2},\sqrt{8})$ and $\boldsymbol{w}=(\sqrt{8},\sqrt{2})$. Find $\cos\theta$ for this $\boldsymbol{v}$ and $\boldsymbol{w}$.

16 How long is the vector $\boldsymbol{v}=(1,1,\ldots,1)$ in 9 dimensions? Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$ and a vector $\boldsymbol{w}$ that is perpendicular to $\boldsymbol{v}$.

17 What are the cosines of the angles α,β,θ between the vector $(1,0,-1)$ and the unit vectors $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ along the axes? Check the formula $\cos^2\alpha+\cos^2\beta+\cos^2\theta=1$.

Problems 18–31 lead to the main facts about lengths and angles in triangles.

18 The parallelogram with sides $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ is a rectangle. Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for ***right triangles only***:

$$\textbf{(length of } \boldsymbol{v})^2 + \textbf{(length of } \boldsymbol{w})^2 = \textbf{(length of } \boldsymbol{v} + \boldsymbol{w})^2.$$

19 In this 90° case, $a^2 + b^2 = c^2$ also works for $\boldsymbol{v} - \boldsymbol{w}$:

$$(\text{length of } \boldsymbol{v})^2 + (\text{length of } \boldsymbol{w})^2 = (\text{length of } \boldsymbol{v} - \boldsymbol{w})^2.$$

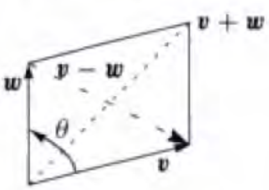
Give an example of $\boldsymbol{v}$ and $\boldsymbol{w}$ (not at right angles) for which this equation fails.

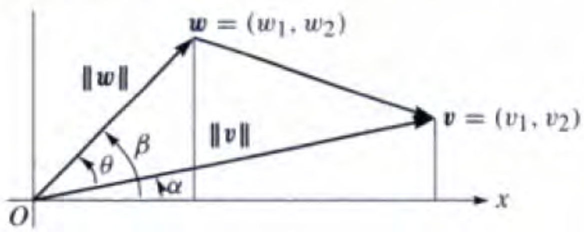
20 (Rules for dot products) These equations are simple but useful:
(1) $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ **(2)** $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ **(3)** $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$
Use **(1)** and **(2)** with $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ to prove $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$.

21 The ***triangle inequality*** says: (length of $\boldsymbol{v} + \boldsymbol{w}$) $\le$ (length of $\boldsymbol{v}$) + (length of $\boldsymbol{w}$). Problem 20 found $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$. Use the Schwarz inequality $\boldsymbol{v} \cdot \boldsymbol{w} \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ to turn this into the triangle inequality:

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \le (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2 \quad \text{or} \quad \boldsymbol{\|v + w\| \le \|v\| + \|w\|}.$$

22 A right triangle in three dimensions still obeys $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} + \boldsymbol{w}\|^2$. Show how this leads in Problem 20 to $v_1w_1 + v_2w_2 + v_3w_3 = 0$.





23 The figure shows that $\cos\alpha = v_1/\|\boldsymbol{v}\|$ and $\sin\alpha = v_2/\|\boldsymbol{v}\|$. Similarly $\cos\beta$ is _____ and $\sin\beta$ is _____. The angle θ is $\beta - \alpha$. Substitute into the formula $\cos\beta\cos\alpha + \sin\beta\sin\alpha$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

24 With $\boldsymbol{v}$ and $\boldsymbol{w}$ at angle θ, the "Law of Cosines" comes from $(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w})$:

$$\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2.$$

If $\theta < 90^\circ$ show that $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ (the third side).

25 The Schwarz inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ by algebra instead of trigonometry:

(a) Multiply out both sides of $(v_1w_1 + v_2w_2)^2 \le (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Show that the difference between those sides equals $(v_1w_2 - v_2w_1)^2$. This cannot be negative since it is a square—so the inequality is true.

26 One-line proof of the Schwarz inequality $|\boldsymbol{u} \cdot \boldsymbol{U}| \leq 1$ for unit vectors:

$$|\boldsymbol{u} \cdot \boldsymbol{U}| \leq |u_1|\,|U_1| + |u_2|\,|U_2| \leq \frac{u_1^2 + U_1^2}{2} + \frac{u_2^2 + U_2^2}{2} = \frac{1+1}{2} = 1.$$

Put $(u_1, u_2) = (.6, .8)$ and $(U_1, U_2) = (.8, .6)$ in that whole line and find $\cos\theta$.

27 Why is $|\cos\theta|$ never greater than 1 in the first place?

28 Pick any numbers that add to $x+y+z=0$. Find the angle between your vector $\boldsymbol{v} = (x, y, z)$ and the vector $\boldsymbol{w} = (z, x, y)$. Challenge question: Explain why $\boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ is always $-\frac{1}{2}$.

29 (*Recommended*) If $||\boldsymbol{v}|| = 5$ and $||\boldsymbol{w}|| = 3$, what are the smallest and largest values of $||\boldsymbol{v} - \boldsymbol{w}||$? What are the smallest and largest values of $\boldsymbol{v} \cdot \boldsymbol{w}$?

30 If $\boldsymbol{v} = (1, 2)$ draw all vectors $\boldsymbol{w} = (x, y)$ in the xy plane with $\boldsymbol{v} \cdot \boldsymbol{w} = 5$. Which is the shortest $\boldsymbol{w}$?

31 Can three vectors in the xy plane have $\boldsymbol{u} \cdot \boldsymbol{v} < 0$ and $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ and $\boldsymbol{u} \cdot \boldsymbol{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would be impossible...).

2

SOLVING LINEAR EQUATIONS

VECTORS AND LINEAR EQUATIONS ■ 2.1

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers—we never see x times y. Our first example of a linear system is certainly not big. It has two equations in two unknowns. But you will see how far it leads:

$$\begin{aligned} x \;-\; 2y &= 1 \\ 3x \;+\; 2y &= 11 \end{aligned} \tag{1}$$

We begin *a row at a time*. The first equation $x - 2y = 1$ produces a straight line in the xy plane. The point $x = 1, y = 0$ is on the line because it solves that equation. The point $x = 3, y = 1$ is also on the line because $3 - 2 = 1$. If we choose $x = 101$ we find $y = 50$. The slope of this particular line is $\frac{1}{2}$ (y increases by 50 when x changes by 100). But slopes are important in calculus and this is linear algebra!

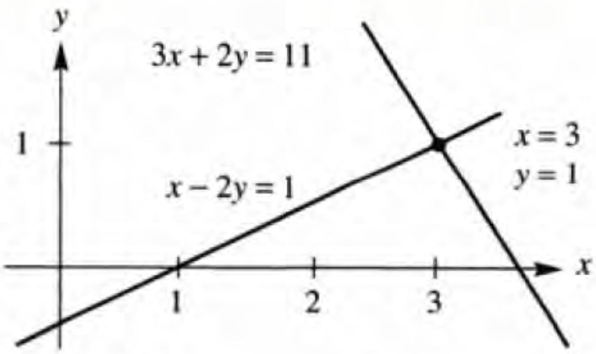


Figure 2.1 *Row picture*: The point (3, 1) where the lines meet is the solution.

Figure 2.1 shows that line $x - 2y = 1$. The second line in this "row picture" comes from the second equation $3x + 2y = 11$. You can't miss the intersection point where the two lines meet. *The point* $x = 3$, $y = 1$ *lies on both lines.* That point solves both equations at once. This is the solution to our system of linear equations.

R *The row picture shows two lines meeting at a single point.*

Turn now to the column picture. I want to recognize the linear system as a "vector equation". Instead of numbers we need to see *vectors*. If you separate the original system into its columns instead of its rows, you get

$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \boldsymbol{b}. \tag{2}$$

This has two column vectors on the left side. The problem is *to find the combination of those vectors that equals the vector on the right.* We are multiplying the first column by x and the second column by y, and adding. With the right choices $x = 3$ and $y = 1$, this produces $3(\textbf{\textit{column 1}}) + 1(\textbf{\textit{column 2}}) = \boldsymbol{b}$.

C *The column picture combines the column vectors on the left side to produce the vector b on the right side.*

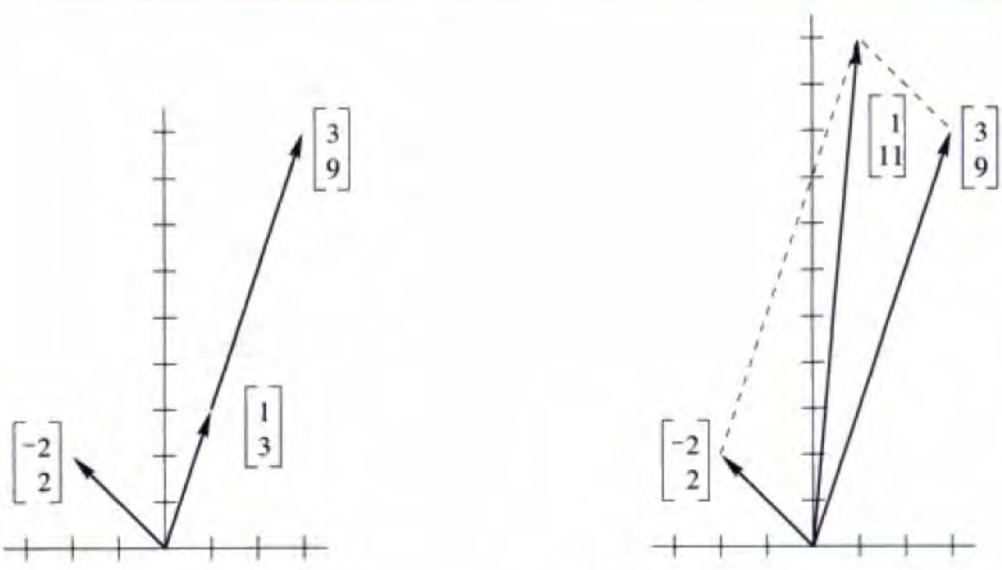


Figure 2.2 *Column picture*: A combination of columns produces the right side (1,11).

Figure 2.2 is the "column picture" of two equations in two unknowns. The first part shows the two separate columns, and that first column multiplied by 3. This multiplication by a *scalar* (a number) is one of the two basic operations in linear algebra:

$$\textbf{Scalar multiplication} \qquad 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

If the components of a vector $\boldsymbol{v}$ are v_1 and v_2, then $c\boldsymbol{v}$ has components cv_1 and cv_2.

The other basic operation is *vector addition*. We add the first components and the second components separately. The vector sum is $(1, 11)$ as desired:

$$\textbf{Vector addition} \qquad \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

The graph in Figure 2.2 shows a parallelogram. The sum $(1, 11)$ is along the diagonal:

$$\textbf{\textit{The sides are}} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \textbf{\textit{ and }} \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \quad \textbf{\textit{The diagonal sum is}} \begin{bmatrix} 3-2 \\ 9+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

We have multiplied the original columns by $x = 3$ and $y = 1$. That combination produces the vector $\boldsymbol{b} = (1, 11)$ on the right side of the linear equations.

To repeat: The left side of the vector equation is a ***linear combination*** of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That step is crucially important, because it contains both of the basic operations:

$$\textbf{Linear combination} \qquad 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Of course the solution $x = 3, y = 1$ is the same as in the row picture. I don't know which picture you prefer! I suspect that the two intersecting lines are more familiar at first. You may like the row picture better, but only for one day. My own preference is to combine column vectors. It is a lot easier to see a combination of four vectors in four-dimensional space, than to visualize how four hyperplanes might possibly meet at a point. (*Even one hyperplane is hard enough. . .*)

The ***coefficient matrix*** on the left side of the equations is the 2 by 2 matrix A:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}.$$

This is very typical of linear algebra, to look at a matrix by rows and by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $A\boldsymbol{x} = \boldsymbol{b}$:

$$\textbf{Matrix equation} \qquad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

The row picture deals with the two rows of A. The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into the solution vector $\boldsymbol{x}$. Then

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is} \quad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Three Equations in Three Unknowns

The three unknowns are x, y, z. The linear equations $Ax = b$ are

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{array} \tag{3}$$

We look for numbers x, y, z that solve all three equations at once. Those desired numbers might or might not exist. For this system, they do exist. When the number of unknowns matches the number of equations, there is *usually* one solution. Before solving the problem, we visualize it both ways:

R ***The row picture shows three planes meeting at a single point.***

C ***The column picture combines three columns to produce the vector*** **(6, 4, 2).**

In the row picture, each equation is a *plane* in three-dimensional space. The first plane comes from the first equation $x + 2y + 3z = 6$. That plane crosses the x and y and z axes at the points $(6, 0, 0)$ and $(0, 3, 0)$ and $(0, 0, 2)$. Those three points solve the equation and they determine the whole plane.

The vector $(x, y, z) = (0, 0, 0)$ does not solve $x + 2y + 3z = 6$. Therefore the plane in Figure 2.3 does not contain the origin.

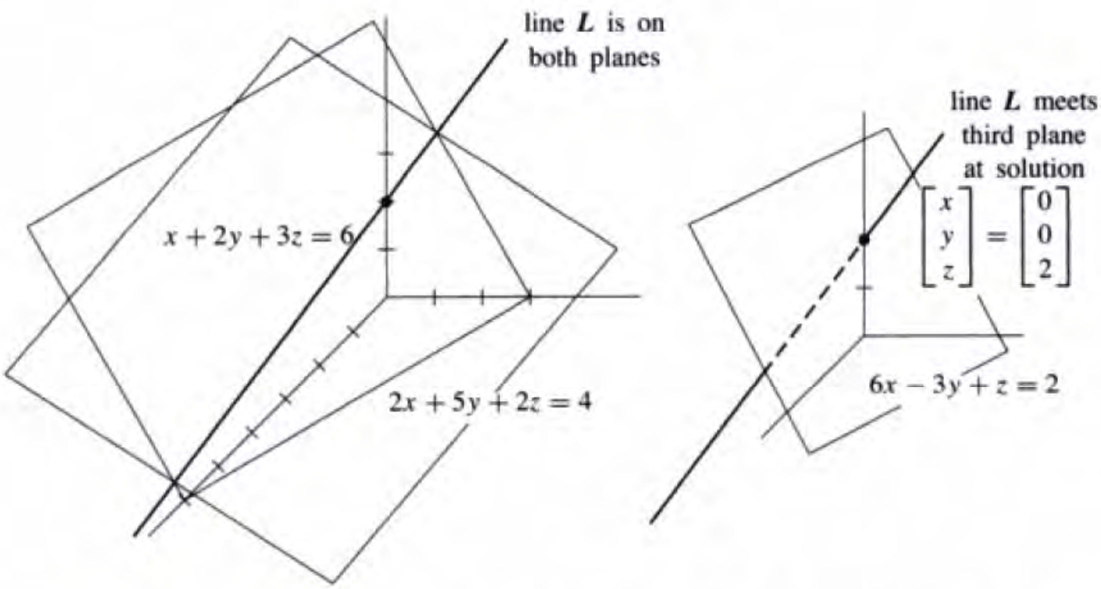


Figure 2.3 Row picture of three equations: Three planes meet at a point.

The plane $x + 2y + 3z = 0$ does pass through the origin, and it is parallel to $x + 2y + 3z = 6$. When the right side increases to 6, the plane moves away from the origin.

The second plane is given by the second equation $2x + 5y + 2z = 4$. *It intersects the first plane in a line* $\boldsymbol{L}$. The usual result of two equations in three unknowns is a line $\boldsymbol{L}$ of solutions.

The third equation gives a third plane. It cuts the line $\boldsymbol{L}$ at a single point. That point lies on all three planes and it solves all three equations. It is harder to draw this triple intersection point than to imagine it. The three planes meet at the solution (which we haven't found yet). The column form shows immediately why $z = 2$!

The column picture starts with the vector form of the equations:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}. \tag{4}$$

The unknown numbers x, y, z are the coefficients in this linear combination. We want to multiply the three column vectors by the correct numbers x, y, z to produce $\boldsymbol{b} = (6, 4, 2)$.

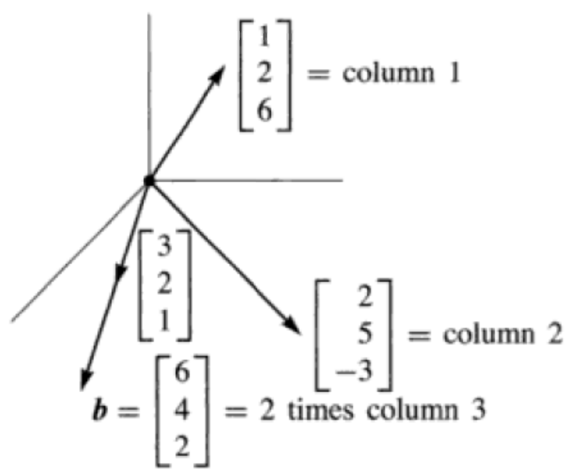


Figure 2.4 *Column picture*: $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = \boldsymbol{b}$.

Figure 2.4 shows this column picture. Linear combinations of those columns can produce any vector $\boldsymbol{b}$! The combination that produces $\boldsymbol{b} = (6, 4, 2)$ is just 2 times the third column. *The coefficients we need are* $x = 0$, $y = 0$, *and* $z = 2$. This is also the intersection point of the three planes in the row picture. It solves the system:

$$\textbf{Correct combination}\quad 0\begin{bmatrix}1\\2\\6\end{bmatrix}+0\begin{bmatrix}2\\5\\-3\end{bmatrix}+2\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}.$$

The Matrix Form of the Equations

We have three rows in the row picture and three columns in the column picture (plus the right side). The three rows and three columns contain nine numbers. *These nine numbers fill a* 3 *by* 3 *matrix*. The "coefficient matrix" has the rows and columns that have so far been kept separate:

$$\textbf{\textit{The coefficient matrix is}} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}.$$

The capital letter A stands for all nine coefficients (in this square array). The letter $\boldsymbol{b}$ denotes the column vector with components $6, 4, 2$. The unknown $\boldsymbol{x}$ is also a column vector, with components x, y, z. (We use boldface because it is a vector, $\boldsymbol{x}$ because it is unknown.) By rows the equations were (3), by columns they were (4), and now by matrices they are (5). The shorthand is $A\boldsymbol{x} = \boldsymbol{b}$:

$$\textbf{\textit{Matrix equation}} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}. \qquad (5)$$

We multiply the matrix A times the unknown vector $\boldsymbol{x}$ to get the right side $\boldsymbol{b}$.

Basic question: What does it mean to "multiply A times $\boldsymbol{x}$"? We can multiply by rows or by columns. Either way, $A\boldsymbol{x} = \boldsymbol{b}$ must be a correct representation of the three equations. You do the same nine multiplications either way.

Multiplication by rows $\quad A\boldsymbol{x}$ comes from *dot products*, each row times the column $\boldsymbol{x}$:

$$Ax = \begin{bmatrix} (\textit{ row 1}) \cdot x \\ (\textit{ row 2}) \cdot x \\ (\textit{ row 3}) \cdot x \end{bmatrix}. \qquad (6)$$

Multiplication by columns $\quad A\boldsymbol{x}$ is a ***combination of column vectors***:

$$\boldsymbol{Ax = x \textbf{ (column 1) } + y \textbf{ (column 2) } + z \textbf{ (column 3)}}. \qquad (7)$$

When we substitute the solution $\boldsymbol{x} = (0, 0, 2)$, the multiplication $A\boldsymbol{x}$ produces $\boldsymbol{b}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \textbf{\textit{2 times column 3}} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The first dot product in row multiplication is $(1, 2, 3) \cdot (0, 0, 2) = 6$. The other dot products are 4 and 2. Multiplication by columns is simply 2 times column 3.

This book sees Ax ***as a combination of the columns of*** A.

Example 1 Here are 3 by 3 matrices A and I, with three ones and six zeros:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad Ix = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

If you are a row person, the product of every row $(1, 0, 0)$ with $(4, 5, 6)$ is 4. If you are a column person, the linear combination is 4 times the first column $(1, 1, 1)$. In that matrix A, the second and third columns are zero vectors.

The example with Ix deserves a careful look, because the matrix I is special. It has ones on the "main diagonal". Off that diagonal, all the entries are zeros. *Whatever vector this matrix multiplies, that vector is not changed.* This is like multiplication by 1, but for matrices and vectors. The exceptional matrix in this example is the 3 by 3 ***identity matrix***:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{always yields the multiplication} \quad Ix = x\,.$$

Matrix Notation

The first row of a 2 by 2 matrix contains a_{11} and a_{12}. The second row contains a_{21} and a_{22}. The first index gives the row number, so that a_{ij} is an entry in row i. The second index j gives the column number. But those subscripts are not convenient on a keyboard! Instead of a_{ij} it is easier to type $A(i, j)$. ***The entry*** $a_{57} = A(5, 7)$ ***would be in row 5, column 7.***

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}.$$

For an m by n matrix, the row index i goes from 1 to m. The column index j stops at n. There are mn entries in the matrix. A square matrix (order n) has n^2 entries.

Multiplication in MATLAB

I want to express A and x and their product Ax using MATLAB commands. This is a first step in learning that language. I begin by defining the matrix A and the vector x. This vector is a 3 by 1 matrix, with three rows and one column. Enter matrices a row at a time, and use a semicolon to signal the end of a row:

$$A = [1 \ \ 2 \ \ 3; \quad 2 \ \ 5 \ \ 2; \quad 6 \ -3 \ \ 1]$$
$$x = [0; \, 0; \, 2]$$

Here are three ways to multiply Ax in MATLAB. In reality, $A * x$ is the way to do it. MATLAB is a high level language, and it works with matrices:

$$\textit{Matrix multiplication} \quad b = A * x$$

We can also pick out the first row of A (as a smaller matrix!). The notation for that 1 by 3 submatrix is $A(1,:)$. **Here the colon symbol keeps all columns of row 1**:

$$\textit{Row at a time} \quad b = [\,A(1,:) * x\,;\ A(2,:) * x\,;\ A(3,:) * x\,]$$

Those are dot products, row times column, 1 by 3 matrix times 3 by 1 matrix.

The other way to multiply uses the columns of A. The first column is the 3 by 1 submatrix $A(:,1)$. Now the colon symbol : is keeping all rows of column 1. This column multiplies $x(1)$ and the other columns multiply $x(2)$ and $x(3)$:

$$\textit{Column at a time} \quad b = A(:,1) * x(1) + A(:,2) * x(2) + A(:,3) * x(3)$$

I think that matrices are stored by columns. Then multiplying a column at a time will be a little faster. So $A * x$ is actually executed by columns.

You can see the same choice in a FORTRAN-type structure, which operates on single entries of A and x. This lower level language needs an outer and inner "DO loop". When the outer loop uses the row number I, multiplication is a row at a time. The inner loop $J = 1, 3$ goes along each row I.

When the outer loop uses J, multiplication is a column at a time. I will do that in MATLAB , which needs two more lines "end" "end" to close "for I" and "for J":

FORTRAN by rows	MATLAB **by columns**
DO 10 $I = 1, 3$	for $J = 1:3$
DO 10 $J = 1, 3$	for $I = 1:3$
10 $B(I) = B(I) + A(I,J) * X(J)$	$b(I) = b(I) + A(I,J) * x(J)$

Notice that MATLAB is sensitive to upper case versus lower case (capital letters and small letters). If the matrix is A then its entries are $A(I,J)$ not $a(I,J)$.

I think you will prefer the higher level $A * x$. FORTRAN won't appear again in this book. *Maple* and *Mathematica* and graphing calculators also operate at the higher level. Multiplication is $A.\,x$ in *Mathematica*. It is **multiply**(A, x); or **evalm**$(A\&{*}x)$; in *Maple*. Those languages allow symbolic entries $a, b, x, \ldots$ and not only real numbers. Like MATLAB's Symbolic Toolbox, they give the symbolic answer.

■ REVIEW OF THE KEY IDEAS ■

1. The basic operations on vectors are multiplication cv and vector addition $v + w$.

2. Together those operations give linear combinations $cv + dw$.

3. Matrix-vector multiplication $A\boldsymbol{x}$ can be executed by rows (dot products). But it should be understood as a combination of the columns of A!

4. Column picture: $A\boldsymbol{x} = \boldsymbol{b}$ asks for a combination of columns to produce $\boldsymbol{b}$.

5. Row picture: Each equation in $A\boldsymbol{x} = \boldsymbol{b}$ gives a line ($n = 2$) or a plane ($n = 3$) or a "hyperplane" ($n > 3$). They intersect at the solution or solutions.

■ WORKED EXAMPLES ■

2.1 A Describe the column picture of these three equations. Solve by careful inspection of the columns (instead of elimination):

$$\begin{aligned} x + 3y + 2z &= -3 \\ 2x + 2y + 2z &= -2 \\ 3x + 5y + 4z &= -5 \end{aligned} \qquad \text{which is } A\boldsymbol{x} = \boldsymbol{b}: \qquad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}.$$

Solution The column picture asks for a linear combination that produces $\boldsymbol{b}$ from the three columns of A. In this example $\boldsymbol{b}$ is *minus the second column*. So the solution is $x = 0, y = -1, z = 0$. To show that $(0, -1, 0)$ is the *only* solution we have to know that "A is invertible" and "the columns are independent" and "the determinant isn't zero". Those words are not yet defined but the test comes from elimination: We need (and we find!) a full set of three nonzero pivots.

If the right side changes to $\boldsymbol{b} = (4, 4, 8) =$ sum of the first two columns, then the right combination has $x = 1, y = 1, z = 0$. The solution becomes $\boldsymbol{x} = (1, 1, 0)$.

2.1 B This system has *no solution*, because the three planes in the row picture don't pass through a point. No combination of the three columns produces $\boldsymbol{b}$:

$$\begin{aligned} x + 3y + 5z &= 4 \\ x + 2y - 3z &= 5 \\ 2x + 5y + 2z &= 8 \end{aligned} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & -3 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} = \boldsymbol{b}$$

(1) Multiply the equations by $1, 1, -1$ and add to show that these planes don't meet at a point. Are any two of the planes parallel? What are the equations of planes parallel to $x + 3y + 5z = 4$?

(2) Take the dot product of each column (and also $\boldsymbol{b}$) with $\boldsymbol{y} = (1, 1, -1)$. How do those dot products show that the system has no solution?

(3) Find three right side vectors $\boldsymbol{b}^*$ and $\boldsymbol{b}^{**}$ and $\boldsymbol{b}^{***}$ that *do* allow solutions.

Solution

(1) Multiplying the equations by $1, 1, -1$ and adding gives

$$\begin{array}{rl} x + 3y + 5z = 4 & \\ x + 2y - 3z = 5 & \\ -[2x + 5y + 2z = 8] & \\ \hline 0x + 0y + 0z = 1 & \textit{No Solution} \end{array}$$

The planes don't meet at any point, but no two planes are parallel. For a plane parallel to $x+3y+5z=4$, just change the "4". The parallel plane $x+3y+5z=0$ goes through the origin $(0, 0, 0)$. And the equation multiplied by any nonzero constant still gives the same plane, as in $2x+6y+10z=8$.

(2) The dot product of each column with $\boldsymbol{y} = (1, 1, -1)$ is *zero*. On the right side, $\boldsymbol{y} \cdot \boldsymbol{b} = (1, 1, -1) \cdot (4, 5, 8) = 1$ is *not zero*. So a solution is impossible. (If a combination of columns could produce $\boldsymbol{b}$, take dot products with $\boldsymbol{y}$. Then a combination of zeros would produce 1.)

(3) There is a solution when $\boldsymbol{b}$ is a combination of the columns. These three examples $\boldsymbol{b}^*, \boldsymbol{b}^{**}, \boldsymbol{b}^{***}$ have solutions $\boldsymbol{x}^* = (1, 0, 0)$ and $\boldsymbol{x}^{**} = (1, 1, 1)$ and $\boldsymbol{x}^{***} = (0, 0, 0)$:

$$\boldsymbol{b}^* = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \text{first column} \quad \boldsymbol{b}^{**} = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} = \text{sum of columns} \quad \boldsymbol{b}^{***} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem Set 2.1

Problems 1–9 are about the row and column pictures of $A\boldsymbol{x} = \boldsymbol{b}$.

1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $\boldsymbol{x} = (x, y, z) = (2, 3, 4)$:

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= 3 \\ 0x + 0y + 1z &= 4 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

2 Draw the vectors in the column picture of Problem 1. Two times column 1 plus three times column 2 plus four times column 3 equals the right side $\boldsymbol{b}$.

3 If the equations in Problem 1 are multiplied by $2, 3, 4$ they become $\widehat{A}\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$:

$$\begin{aligned} 2x + 0y + 0z &= 4 \\ 0x + 3y + 0z &= 9 \\ 0x + 0y + 4z &= 16 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$$

Why is the row picture the same? Is the solution $\widehat{\boldsymbol{x}}$ the same as $\boldsymbol{x}$? What is changed in the column picture—the columns or the right combination to give $\widehat{\boldsymbol{b}}$?

4 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

5 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

6 The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line **L**). Find three solutions on **L**.

7 Move the third plane in Problem 6 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't _____ that line.

8 In Problem 6 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is _____. Find two combinations of the columns that give $\boldsymbol{b} = (2, 3, 5)$. This is only possible for $\boldsymbol{b} = (4, 6, c)$ if $c =$ _____.

9 Normally 4 "planes" in 4-dimensional space meet at a _____. Normally 4 column vectors in 4-dimensional space can combine to produce $\boldsymbol{b}$. What combination of $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$ produces $\boldsymbol{b} = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?

Problems 10–15 are about multiplying matrices and vectors.

10 Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

11 Compute each $A\boldsymbol{x}$ in Problem 10 as a combination of the columns:

$$\text{10(a) becomes} \quad A\boldsymbol{x} = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}.$$

How many separate multiplications for $A\boldsymbol{x}$, when the matrix is "3 by 3"?

12 Find the two components of $A\boldsymbol{x}$ by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

13 Multiply A times $\boldsymbol{x}$ to find three components of $A\boldsymbol{x}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

14 (a) A matrix with m rows and n columns multiplies a vector with _____ components to produce a vector with _____ components.

(b) The planes from the m equations $A\boldsymbol{x} = \boldsymbol{b}$ are in _____-dimensional space. The combination of the columns of A is in _____-dimensional space.

15 Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector $\boldsymbol{x} = (x, y, z, t)$ to produce $\boldsymbol{b}$. The solutions $\boldsymbol{x}$ fill a plane or "hyperplane" in 4-dimensional space. *The plane is 3-dimensional with no 4D volume.*

Problems 16–23 ask for matrices that act in special ways on vectors.

16 (a) What is the 2 by 2 identity matrix? I times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} y \\ x \end{bmatrix}$.

17 (a) What 2 by 2 matrix R rotates every vector by 90°? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} y \\ -x \end{bmatrix}$.

(b) What 2 by 2 matrix rotates every vector by 180°?

18 Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

19 What 2 by 2 matrix E subtracts the first component from the second component? What 3 by 3 matrix does the same?

$$E\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

20 What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z+x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z-x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1}, the two results are (_____) and (_____).

21 What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (_____) and (_____).

22 What 2 by 2 matrix R rotates every vector through 45°? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R.

23 Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication $A\boldsymbol{x}$. The matrix A has one row. The solutions to $A\boldsymbol{x} = \boldsymbol{0}$ lie on a _____ perpendicular to the vector _____. The columns of A are only in _____-dimensional space.

24 In MATLAB notation, write the commands that define this matrix A and the column vectors $\boldsymbol{x}$ and $\boldsymbol{b}$. What command would test whether or not $A\boldsymbol{x} = \boldsymbol{b}$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

25 The MATLAB commands A = eye(3) and v = [3 : 5]′ produce the 3 by 3 identity matrix and the column vector $(3, 4, 5)$. What are the outputs from A ∗ v and v′ ∗ v? (Computer not needed!) If you ask for v ∗ A, what happens?

26 If you multiply the 4 by 4 all-ones matrix A = ones(4,4) and the column v = ones(4,1), what is A ∗ v? (Computer not needed.) If you multiply B = eye(4) + ones(4,4) times w = zeros(4,1) + 2 ∗ ones(4,1), what is B ∗ w?

Questions 27–29 are a review of the row and column pictures.

27 Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.

28 For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a _____.

29 For four linear equations in two unknowns x and y, the row picture shows four _____. The column picture is in _____-dimensional space. The equations have no solution unless the vector on the right side is a combination of _____.

30 Start with the vector $\boldsymbol{u}_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" A below. The next three vectors are $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$:

$$\boldsymbol{u}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = \_\_\_\_\_ \qquad \boldsymbol{u}_3 = A\boldsymbol{u}_2 = \_\_\_\_\_.$$

What property do you notice for all four vectors $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$?

31 With a computer, continue from $\boldsymbol{u}_0 = (1, 0)$ to $\boldsymbol{u}_7$, and from $\boldsymbol{v}_0 = (0, 1)$ to $\boldsymbol{v}_7$. What do you notice about $\boldsymbol{u}_7$ and $\boldsymbol{v}_7$? Here are two MATLAB codes, one with while and one with for. They plot $\boldsymbol{u}_0$ to $\boldsymbol{u}_7$—you can use other languages:

```
u = [1 ; 0]; A = [.8 .3 ; .2 .7];
x = u; k = [0 : 7];
while size(x,2) <= 7
   u = A*u; x = [x u];
end
plot(k, x)
```

```
u = [1 ; 0]; A = [.8 .3 ; .2 .7];
x = u; k = [0 : 7];
for j=1 : 7
   u = A*u; x = [x u];
end
plot(k, x)
```

32 The $\boldsymbol{u}$'s and $\boldsymbol{v}$'s in Problem 31 are approaching a steady state vector $\boldsymbol{s}$. Guess that vector and check that $A\boldsymbol{s} = \boldsymbol{s}$. If you start with $\boldsymbol{s}$, you stay with $\boldsymbol{s}$.

33 This MATLAB code allows you to input $\boldsymbol{x}_0$ with a mouse click, by ginput. With $t = 1$, A rotates vectors by *theta*. The plot will show $A\boldsymbol{x}_0, A^2\boldsymbol{x}_0, \ldots$ going around a circle ($t > 1$ will spiral out and $t < 1$ will spiral in). You can change *theta* and the stop at j=10. We plan to put this code on **web.mit.edu/18.06/www**:

```
theta = 15*pi/180; t = 1.0;
A = t*[cos(theta) -sin(theta) ; sin(theta) cos(theta)];
disp('Click to select starting point')
[x1 , x2] = ginput(1); x = [x1 ; x2];
for j=1:10
   x = [x A*x( : , end)];
end
plot(x(1,:), x(2,:), 'o')
hold off
```


34 Invent a 3 by 3 **magic matrix** M_3 with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be 8, 3, 4. What is M_3 times $(1, 1, 1)$? What is M_4 times $(1, 1, 1, 1)$ if this magic matrix has entries $1, \ldots, 16$?

THE IDEA OF ELIMINATION ■ 2.2

This chapter explains a systematic way to solve linear equations. The method is called ***"elimination"***, and you can see it immediately in our 2 by 2 example. Before elimination, x and y appear in both equations. After elimination, the first unknown x has disappeared from the second equation:

$$\textbf{Before}\quad \begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \qquad \textbf{After}\quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned} \qquad \begin{aligned} &(\textit{multiply by 3 and subtract}) \\ &(x \textit{ has been eliminated}) \end{aligned}$$

The last equation $8y = 8$ instantly gives $y = 1$. Substituting for y in the first equation leaves $x - 2 = 1$. Therefore $x = 3$ and the solution $(x, y) = (3, 1)$ is complete.

Elimination produces an ***upper triangular system***—this is the goal. The nonzero coefficients $1, -2, 8$ form a triangle. The last equation $8y = 8$ reveals $y = 1$, and we go up the triangle to x. This quick process is called ***back substitution***. It is used for upper triangular systems of any size, after forward elimination is complete.

Important point: The original equations have the same solution $x = 3$ and $y = 1$. Figure 2.5 repeats this original system as a pair of lines, intersecting at the solution point $(3, 1)$. After elimination, the lines still meet at the same point! One line is horizontal because its equation $8y = 8$ does not contain x.

How did we get from the first pair of lines to the second pair? We subtracted 3 times the first equation from the second equation. The step that eliminates x from equation 2 is the fundamental operation in this chapter. We use it so often that we look at it closely:

To eliminate x: Subtract a multiple of equation 1 from equation 2.

Three times $x - 2y = 1$ gives $3x - 6y = 3$. When this is subtracted from $3x + 2y = 11$, the right side becomes 8. The main point is that $3x$ cancels $3x$. What remains on the left side is $2y - (-6y)$ or $8y$, and x is eliminated.

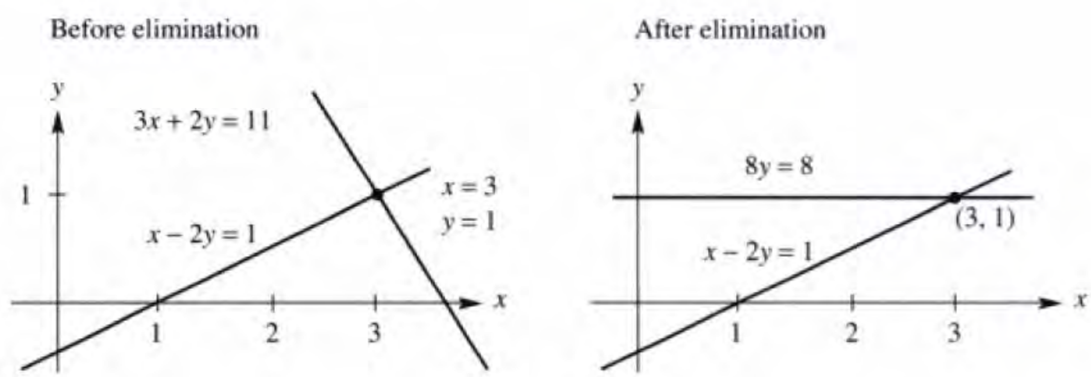


Figure 2.5 Two lines meet at the solution. So does the new line $8y = 8$.

Ask yourself how that multiplier $\ell = 3$ was found. The first equation contains x. ***The first pivot is*** 1 (the coefficient of x). The second equation contains $3x$, so the first equation was multiplied by 3. Then subtraction $3x - 3x$ produced the zero.

You will see the multiplier rule if we change the first equation to $4x - 8y = 4$. (Same straight line but the first pivot becomes 4.) The correct multiplier is now $\ell = \frac{3}{4}$. *To find the multiplier, divide the coefficient "3" to be eliminated by the pivot "4":*

$$\begin{array}{lll} \mathbf{4}x - 8y = 4 & \textbf{Multiply equation 1 by } \tfrac{3}{4} & 4x - 8y = 4 \\ \mathbf{3}x + 2y = 11 & \textbf{Subtract from equation 2} & \quad\;\; 8y = 8. \end{array}$$

The final system is triangular and the last equation still gives $y = 1$. Back substitution produces $4x - 8 = 4$ and $4x = 12$ and $x = 3$. We changed the numbers but not the lines or the solution. ***Divide by the pivot to find that multiplier*** $\boldsymbol{\ell = \frac{3}{4}}$:

Pivot	=	*first nonzero in the row that does the elimination*
Multiplier	=	*(entry to eliminate) divided by (pivot)* $= \frac{3}{4}$.

The new second equation starts with the second pivot, which is 8. We would use it to eliminate y from the third equation if there were one. *To solve n equations we want n pivots.* ***The pivots are on the diagonal of the triangle after elimination.***

You could have solved those equations for x and y without reading this book. It is an extremely humble problem, but we stay with it a little longer. Even for a 2 by 2 system, elimination might break down and we have to see how. By understanding the possible breakdown (when we can't find a full set of pivots), you will understand the whole process of elimination.

Breakdown of Elimination

Normally, elimination produces the pivots that take us to the solution. But failure is possible. At some point, the method might ask us to *divide by zero*. We can't do it. The process has to stop. There might be a way to adjust and continue—or failure may be unavoidable. Example 1 fails with no solution. Example 2 fails with too many solutions. Example 3 succeeds by exchanging the equations.

Example 1 ***Permanent failure with no solution.*** Elimination makes this clear:

$$\begin{array}{rll} x - 2y = 1 & \text{Subtract 3 times} & x - 2y = 1 \\ 3x - 6y = 11 & \text{eqn. 1 from eqn. 2} & \quad\;\; \mathbf{0}y = 8. \end{array}$$

The last equation is $0y = 8$. There is *no* solution. Normally we divide the right side 8 by the second pivot, but *this system has no second pivot.* ***(Zero is never allowed as a pivot!)*** The row and column pictures of this 2 by 2 system show that failure was unavoidable. If there is no solution, elimination must certainly have trouble.

The row picture in Figure 2.6 shows parallel lines—which never meet. A solution must lie on both lines. With no meeting point, the equations have no solution.

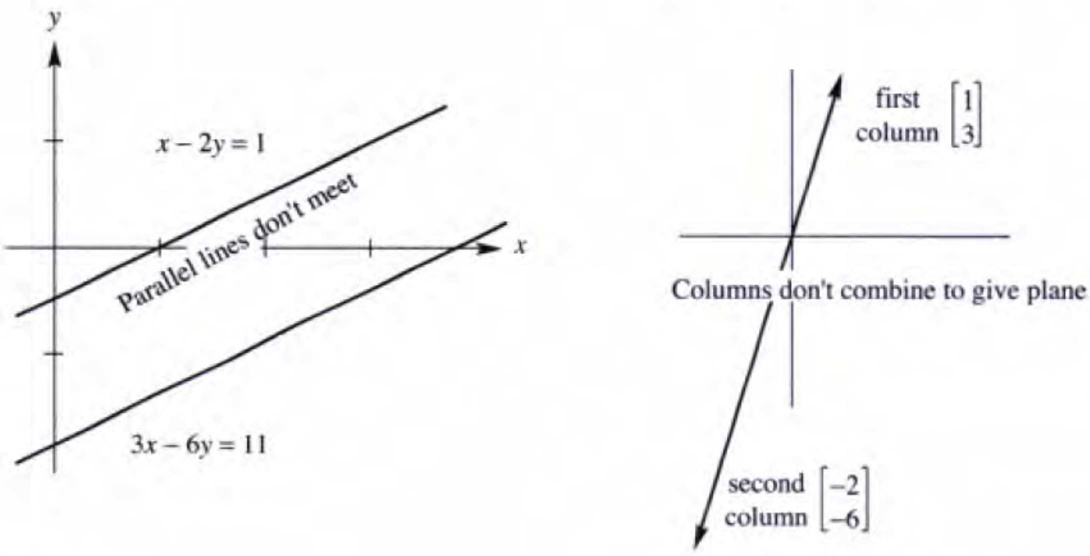


Figure 2.6 Row picture and column picture for Example 1: ***no solution***.

The column picture shows the two columns $(1, 3)$ and $(-2, -6)$ in the same direction. *All combinations of the columns lie along a line.* But the column from the right side is in a different direction $(1, 11)$. No combination of the columns can produce this right side—therefore no solution.

When we change the right side to $(1, 3)$, failure shows as a whole line of solutions. Instead of no solution there are infinitely many:

Example 2 ***Permanent failure with infinitely many solutions:***

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 3 \end{aligned} \qquad \begin{array}{c}\text{Subtract 3 times}\\ \text{eqn. 1 from eqn. 2}\end{array} \qquad \boxed{\begin{aligned} x - 2y &= 1 \\ \mathbf{0}y &= \mathbf{0}. \end{aligned}}$$

Every y satisfies $0y = 0$. There is really only one equation $x - 2y = 1$. The unknown y is ***"free"***. After y is freely chosen, x is determined as $x = 1 + 2y$.

In the row picture, the parallel lines have become the same line. Every point on that line satisfies both equations. We have a whole line of solutions.

In the column picture, the right side $(1, 3)$ is now the same as the first column. So we can choose $x = 1$ and $y = 0$. We can also choose $x = 0$ and $y = -\frac{1}{2}$; the second column times $-\frac{1}{2}$ equals the right side. There are infinitely many other solutions. Every (x, y) that solves the row problem also solves the column problem.

Elimination can go wrong in a third way—but this time it can be fixed. *Suppose the first pivot position contains zero.* We refuse to allow zero as a pivot. When the first equation has no term involving x, we can exchange it with an equation below:

Example 3 ***Temporary failure but a row exchange produces two pivots***:

$$\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned} \qquad \begin{array}{c}\text{Exchange the}\\ \text{two equations}\end{array} \qquad \boxed{\begin{aligned} 3x - 2y &= 5 \\ 2y &= 4. \end{aligned}}$$

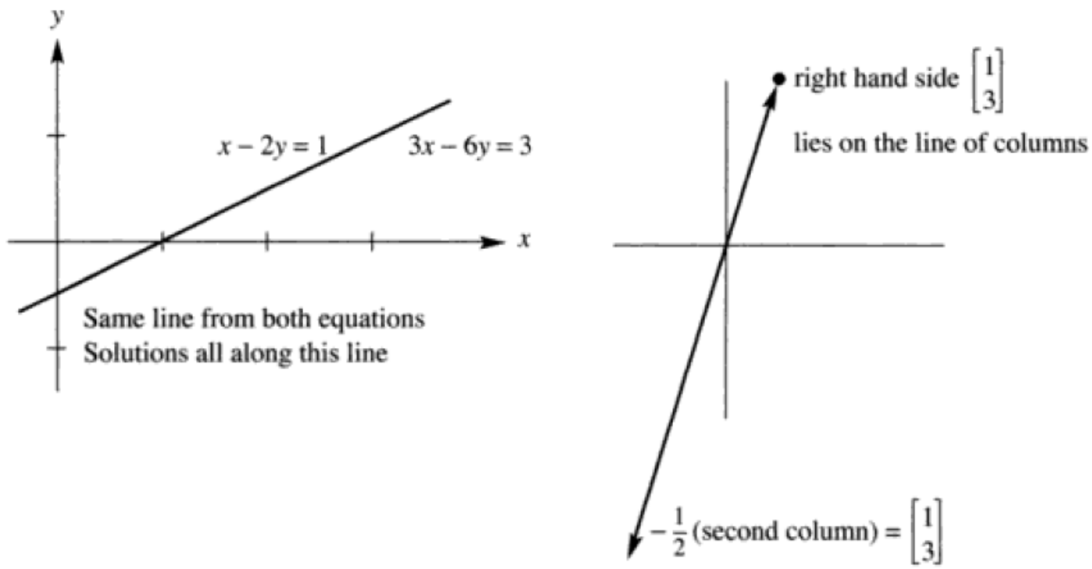


Figure 2.7 Row and column pictures for Example 2: *infinitely many solutions.*

The new system is already triangular. This small example is ready for back substitution. The last equation gives $y = 2$, and then the first equation gives $x = 3$. The row picture is normal (two intersecting lines). The column picture is also normal (column vectors not in the same direction). The pivots 3 and 2 are normal—but an exchange was required to put the rows in a good order.

Examples 1 and 2 are ***singular***—there is no second pivot. Example 3 is ***nonsingular***—there is a full set of pivots and exactly one solution. Singular equations have no solution or infinitely many solutions. Pivots must be nonzero because we have to divide by them.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems. Three by three is enough to see the pattern. For now the matrices are square—an equal number of rows and columns. Here is a 3 by 3 system, specially constructed so that all steps lead to whole numbers and not fractions:

$$\begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \tag{1}$$

What are the steps? The first pivot is the boldface 2 (upper left). Below that pivot we want to create zeros. The first multiplier is the ratio $4/2 = 2$. Multiply the pivot equation by $\ell_{21} = 2$ and subtract. Subtraction removes the $4x$ from the second equation:

Step 1 Subtract 2 times equation 1 from equation 2.

We also eliminate $-2x$ from equation 3—still using the first pivot. The quick way is to add equation 1 to equation 3. Then $2x$ cancels $-2x$. We do exactly that, but the rule in this book is to *subtract rather than add.* The systematic pattern has multiplier $\ell_{31} = -2/2 = -1$. Subtracting -1 times an equation is the same as adding:

Step 2 Subtract -1 times equation 1 from equation 3.

The two new equations involve only y and z. The second pivot (boldface) is 1:

$$\begin{aligned} \mathbf{1}y + 1z &= 4 \\ 1y + 5z &= 12 \end{aligned}$$

We have reached a 2 *by* 2 *system.* The final step eliminates y to make it 1 by 1:

Step 3 Subtract equation 2_{new} from 3_{new}. The multiplier is 1. Then $4z = 8$.

The original system $A\boldsymbol{x} = \boldsymbol{b}$ has been converted into a triangular system $U\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad \text{has become} \quad \begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ \mathbf{1}y + 1z &= 4 \\ \mathbf{4}z &= 8. \end{aligned} \tag{2}$$

The goal is achieved—forward elimination is complete. ***Notice the pivots* 2,1,4 *along the diagonal*.** Those pivots 1 and 4 were hidden in the original system! Elimination brought them out. This triangle is ready for back substitution, which is quick:

$$(4z = 8 \quad \text{gives} \quad z = 2) \quad (y + z = 4 \quad \text{gives} \quad y = 2) \quad (\text{equation 1 gives} \quad x = -1)$$

The solution is $(x, y, z) = (-1, 2, 2)$. The row picture has three planes from three equations. All the planes go through this solution. The original planes are sloping, but the last plane $4z = 8$ after elimination is horizontal.

The column picture shows a combination of column vectors producing the right side $\boldsymbol{b}$. The coefficients in that combination $A\boldsymbol{x}$ are $-1, 2, 2$ (the solution):

$$(-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} \text{ equals } \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \tag{3}$$

The numbers x, y, z multiply columns 1, 2, 3 in the original system $A\boldsymbol{x} = \boldsymbol{b}$ and also in the triangular system $U\boldsymbol{x} = \boldsymbol{c}$.

For a 4 by 4 problem, or an n by n problem, elimination proceeds the same way. Here is the whole idea of forward elimination, column by column:

Column 1. ***Use the first equation to create zeros below the first pivot.***

Column 2. ***Use the new equation 2 to create zeros below the second pivot.***

Columns 3 to *n*. ***Keep going to find the other pivots and the triangular*** U.

After column 2 we have $\begin{bmatrix} \boldsymbol{x} & x & x & x \\ 0 & \boldsymbol{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$. We want $\begin{bmatrix} \boldsymbol{x} & x & x & x \\ & \boldsymbol{x} & x & x \\ & & \boldsymbol{x} & x \\ & & & \boldsymbol{x} \end{bmatrix}$. (4)

The result of forward elimination is an upper triangular system. It is nonsingular if there is a full set of n pivots (never zero!). *Question*: Which x could be changed to boldface $\boldsymbol{x}$ because the pivot is known? Here is a final example to show the original $A\boldsymbol{x} = \boldsymbol{b}$, the triangular system $U\boldsymbol{x} = \boldsymbol{c}$, and the solution from back substitution:

$$\begin{aligned} x + \ y + \ z &= 6 \\ x + 2y + 2z &= 9 \\ x + 2y + 3z &= 10 \end{aligned} \qquad \begin{aligned} x + y + z &= 6 \\ y + z &= 3 \\ z &= 1 \end{aligned} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

All multipliers are 1. All pivots are 1. All planes meet at the solution $(3, 2, 1)$. The columns combine with coefficients $3, 2, 1$ to give $\boldsymbol{b} = (6, 9, 10)$ and $\boldsymbol{c} = (6, 3, 1)$.

■ REVIEW OF THE KEY IDEAS ■

1. A linear system becomes upper triangular after elimination.
2. The upper triangular system is solved by back substitution (starting at the bottom).
3. Elimination subtracts ℓ_{ij} times equation j from equation i, to make the (i, j) entry zero.
4. The multiplier is $\ell_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$. Pivots can not be zero!
5. A zero in the pivot position can be repaired if there is a nonzero below it.
6. When breakdown is permanent, the system has no solution or infinitely many.

■ WORKED EXAMPLES ■

2.2 A When elimination is applied to this matrix A, what are the first and second pivots? What is the multiplier ℓ_{21} in the first step (ℓ_{21} times row 1 is subtracted from row 2)? What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3? Why is the multiplier $\ell_{31} = 0$, subtracting 0 times row 1 from row 3?

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

Solution The first pivot is 3. The multiplier ℓ_{21} is $\frac{6}{3} = 2$. When 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7. If we reduce the entry "9" to "2", that drop of 7 in the (2, 2) position would force a row exchange. (The second row would start with 6, 2 which is an exact multiple of 3, 1 in the first row. Zero will appear in the second pivot position.) The multiplier ℓ_{31} is zero because $a_{31} = 0$. A zero at the start of a row needs no elimination.

2.2 B Use elimination to reach upper triangular matrices U. Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{aligned} x + y + z &= 7 \\ x + y - z &= 5 \\ x - y + z &= 3 \end{aligned} \qquad\qquad \begin{aligned} x + y + z &= 7 \\ x + y - z &= 5 \\ -x - y + z &= 3 \end{aligned}$$

Solution For the first system, subtract equation 1 from equations 2 and 3 (the multipliers are $\ell_{21} = 1$ and $\ell_{31} = 1$). The 2, 2 entry becomes zero, so exchange equations:

$$\begin{aligned} x + y + z &= 7 \\ \mathbf{0}y - 2z &= -2 \\ -2y + 0z &= -4 \end{aligned} \qquad \text{exchanges into} \qquad \begin{aligned} x + y + z &= 7 \\ \mathbf{-2}y + 0z &= -4 \\ -2z &= -2 \end{aligned}$$

Then back substitution gives $z = 1$ and $y = 2$ and $x = 4$. The pivots are $1, -2, -2$.

For the second system, subtract equation 1 from equation 2 as before. Add equation 1 to equation 3. This leaves zero in the 2, 2 entry *and below*:

$$\begin{aligned} x + y + z &= 7 \\ \mathbf{0}y - 2z &= -2 \\ \mathbf{0}y + 2z &= 10 \end{aligned} \qquad \begin{array}{l} \text{There is } \textit{no pivot} \text{ in column 2.} \\ \text{A further elimination step gives } 0z = 8 \\ \text{The three planes don't meet!} \end{array}$$

Plane 1 meets plane 2 in a line. Plane 1 meets plane 3 in a parallel line. *No solution.*

If we change the "3" in the original third equation to "-5" then elimination would leave $2z = 2$ instead of $2z = 10$. Now $z = 1$ would be consistent—we have moved the third plane. Substituting $z = 1$ in the first equation leaves $x + y = 6$. There are infinitely many solutions! *The three planes now meet along a whole line.*

Problem Set 2.2

Problems 1–10 are about elimination on 2 by 2 systems.

1 What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

2 Solve the triangular system of Problem 1 by back substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution?

3 What multiple of equation 1 should be *subtracted* from equation 2?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution?

4 What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y? The second pivot is missing when $ad = bc$.

5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

7 For which numbers a does elimination break down (1) permanently (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

8 For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

9 What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

10 In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ ____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c =$ ____ .

Problems 11–20 study elimination on 3 by 3 systems (and possible failure).

11 Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back substitution for z, y, x.

12 Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \qquad &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ____ times row ____ from row ____ .

13 Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

14 Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \qquad &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

15 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

16 If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned} \qquad\qquad \begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

17 Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\boldsymbol{b} = (1, 10, 100)$ and how many with $\boldsymbol{b} = (0, 0, 0)$?

18 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

19 (Recommended) It is impossible for a system of linear equations to have exactly two solutions. *Explain why.*

(a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?

(b) If 25 planes meet at two points, where else do they meet?

20 Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a _____ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

Problems 21–23 move up to 4 by 4 and n by n.

21 Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \qquad\qquad &= 0 \\ x + 2y + z \qquad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5. \end{aligned}$$

22 This system has the same pivots and right side as Problem 21. How is the solution different (if it is)?

$$\begin{aligned} 2x - y \qquad\qquad &= 0 \\ -x + 2y - z \qquad &= 0 \\ - y + 2z - t &= 0 \\ - z + 2t &= 5. \end{aligned}$$

23 If you extend Problems 21–22 following the $1, 2, 1$ pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot?

24 If elimination leads to these equations, find three possible original matrices A:

$$\begin{aligned} x + y + z &= 0 \\ y + z &= 0 \\ 3z &= 0. \end{aligned}$$

25 For which two numbers a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

26 For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}.$$

27 Look for a matrix that has row sums 4 and 8, and column sums 2 and s:

$$\text{Matrix } = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a + b = 4 & a + c = 2 \\ c + d = 8 & b + d = s \end{matrix}$$

The four equations are solvable only if $s =$ _____. Then find two different matrices that have the correct row and column sums. *Extra credit*: Write down the 4 by 4 system $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} = (a, b, c, d)$ and make A triangular by elimination.

28 Elimination in the usual order gives what pivot matrix and what solution to this "lower triangular" system? We are really solving by *forward substitution*:

$$\begin{aligned} 3x \qquad\quad &= 3 \\ 6x + 2y \qquad &= 8 \\ 9x - 2y + z &= 9. \end{aligned}$$

29 Create a MATLAB command `A(2, : ) = ...` for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix A is already known.

30 Find experimentally the average first and second and third pivot sizes (use the absolute value) in MATLAB's $A = \textbf{rand}(3, 3)$. The average of $\textbf{abs}(A(1, 1))$ should be 0.5 but I don't know the others.

ELIMINATION USING MATRICES ■ 2.3

We now combine two ideas—elimination and matrices. The goal is to express all the steps of elimination (and the final result) in the clearest possible way. In a 3 by 3 example, elimination could be described in words. For larger systems, a long list of steps would be hopeless. You will see how to subtract a multiple of one row from another row—*using matrices*.

The matrix form of a linear system is $A\boldsymbol{x} = \boldsymbol{b}$. Here are $\boldsymbol{b}, \boldsymbol{x}$, and A:

1 The vector of right sides is $\boldsymbol{b}$.
2 The vector of unknowns is $\boldsymbol{x}$. (The unknowns change to $x_1, x_2, x_3, \ldots$ because we run out of letters before we run out of numbers.)
3 The coefficient matrix is A. In this chapter A is square.

The example in the previous section has the beautifully short form $A\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \quad \text{is the same as} \quad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \quad (1)$$

The nine numbers on the left go into the matrix A. That matrix not only sits beside $\boldsymbol{x}$, it *multiplies* $\boldsymbol{x}$. The rule for "A times $\boldsymbol{x}$" is exactly chosen to yield the three equations.

Review of A times $\boldsymbol{x}$. A matrix times a vector gives a vector. The matrix is square when the number of equations (three) matches the number of unknowns (three). Our matrix is 3 by 3. A general square matrix is n by n. Then the vector $\boldsymbol{x}$ is in n-dimensional space. This example is in 3-dimensional space:

$$\textbf{\textit{The unknown is}} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \textbf{\textit{and the solution is}} \quad \boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Key point: $A\boldsymbol{x} = \boldsymbol{b}$ represents the row form and also the column form of the equations. We can multiply by taking a column of A at a time:

$$A\boldsymbol{x} = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \quad (2)$$

This rule is used so often that we express it once more for emphasis.

2A ***The product $A\boldsymbol{x}$ is a combination of the columns of A.*** Components of $\boldsymbol{x}$ multiply columns: $A\boldsymbol{x} = x_1$ times (column 1) $+ \cdots + x_n$ times (column n).

One point to repeat about matrix notation: The entry in row 1, column 1 (the top left corner) is called a_{11}. The entry in row 1, column 3 is a_{13}. The entry in row 3, column 1 is a_{31}. (Row number comes before column number.) The word "entry" for a matrix corresponds to the word "component" for a vector. General rule: ***The entry in row i, column j of the matrix A is a_{ij}.***

Example 1 This matrix has $a_{ij} = 2i + j$. Then $a_{11} = 3$. Also $a_{12} = 4$ and $a_{21} = 5$. Here is Ax with numbers and letters:

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

The first component of Ax is $6 + 4 = 10$. That is the product of the row $[3 \quad 4]$ with the column $(2, 1)$. ***A row times a column gives a dot product!***

The ith component of Ax involves row i, which is $[\, a_{i1} \quad a_{i2} \quad \cdots \quad a_{in} \,]$. The short formula for its dot product with x uses "sigma notation":

2B The ith component of Ax is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$. This is $\displaystyle\sum_{j=1}^{n} a_{ij}x_j$.

The sigma symbol $\sum$ is an instruction to add. Start with $j = 1$ and stop with $j = n$. Start the sum with $a_{i1}x_1$ and stop with $a_{in}x_n$.[1]

The Matrix Form of One Elimination Step

$Ax = b$ is a convenient form for the original equation. What about the elimination steps? The first step in this example subtracts 2 times the first equation from the second equation. On the right side, 2 times the first component of b is subtracted from the second component:

$$b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \quad \text{changes to} \quad b_{\text{new}} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$

We want to do that subtraction with a matrix! The same result $b_{\text{new}} = Eb$ is achieved when we multiply an "elimination matrix" E times b. It subtracts $2b_1$ from b_2:

$$\textit{The elimination matrix is} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplication by E subtracts 2 times row 1 from row 2. Rows 1 and 3 stay the same:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

Notice how $b_1 = 2$ and $b_3 = 10$ stay the same. The first and third rows of E are the first and third rows of the identity matrix I. The new second component is the number 4 that appeared after the elimination step. This is $b_2 - 2b_1$.

[1] Einstein shortened this even more by omitting the $\sum$. The repeated j in $a_{ij}x_j$ automatically meant addition. He also wrote the sum as $a_i^j x_j$. Not being Einstein, we include the $\sum$.

It is easy to describe the "elementary matrices" or "elimination matrices" like E. Start with the identity matrix I. *Change one of its zeros to the multiplier* $-\ell$:

2C The ***identity matrix*** has 1's on the diagonal and otherwise 0's. Then $Ib = b$. The ***elementary matrix or elimination matrix*** E_{ij} that subtracts a multiple ℓ of row j from row i has the extra nonzero entry $-\ell$ in the i, j position.

Example 2

$$\textbf{Identity} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{Elimination} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\boldsymbol{\ell} & 0 & 1 \end{bmatrix}.$$

When you multiply I times $\boldsymbol{b}$, you get $\boldsymbol{b}$. But E_{31} subtracts ℓ times the first component from the third component. With $\ell = 4$ we get $9 - 4 = 5$:

$$I\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \quad \text{and} \quad E\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{-4} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

What about the left side of $A\boldsymbol{x} = \boldsymbol{b}$? The multiplier $\ell = 4$ was chosen to produce a zero, by subtracting 4 times the pivot. ***E_{31} creates a zero in the* (3, 1) *position.***

The notation fits this purpose. Start with A. Apply E's to produce zeros below the pivots (the first E is E_{21}). End with a triangular U. We now look in detail at those steps.

First a small point. The vector $\boldsymbol{x}$ stays the same. The solution is not changed by elimination. (That may be more than a small point.) It is the coefficient matrix that is changed! When we start with $A\boldsymbol{x} = \boldsymbol{b}$ and multiply by E, the result is $EA\boldsymbol{x} = E\boldsymbol{b}$. The new matrix EA is the result of multiplying E times A.

Matrix Multiplication

The big question is: ***How do we multiply two matrices?*** When the first matrix is E (an elimination matrix), there is already an important clue. We know A, and we know what it becomes after the elimination step. To keep everything right, we hope and expect that EA is

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix} \quad (\textit{with the zero}).$$

This step does not change rows 1 and 3 of A. Those rows are unchanged in EA—only row 2 is different. *Twice the first row has been subtracted from the second row.* Matrix multiplication agrees with elimination—and the new system of equations is $EA\boldsymbol{x} = E\boldsymbol{b}$.

$EA\boldsymbol{x}$ is simple but it involves a subtle idea. Multiplying both sides of the original equation gives $E(A\boldsymbol{x}) = E\boldsymbol{b}$. With our proposed multiplication of matrices, this is also

$(EA)\boldsymbol{x} = E\boldsymbol{b}$. The first was E times $A\boldsymbol{x}$, the second is EA times $\boldsymbol{x}$. They are the same! The parentheses are not needed. We just write $EA\boldsymbol{x} = E\boldsymbol{b}$.

When multiplying ABC, you can do BC first or you can do AB first. This is the point of an "associative law" like $3 \times (4 \times 5) = (3 \times 4) \times 5$. We multiply 3 times 20, or we multiply 12 times 5. Both answers are 60. That law seems so obvious that it is hard to imagine it could be false. But the "commutative law" $3 \times 4 = 4 \times 3$ looks even more obvious. For matrices, EA is different from AE.

2D ASSOCIATIVE LAW $A(BC) = (AB)C$

NOT COMMUTATIVE LAW Often $AB \neq BA$.

There is another requirement on matrix multiplication. Suppose B has only one column (this column is $\boldsymbol{b}$). The matrix-matrix law for EB should be consistent with the old matrix-vector law for $E\boldsymbol{b}$. Even more, we should be able to *multiply matrices a column at a time*:

If B has several columns $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$, then EB has columns $E\boldsymbol{b}_1, E\boldsymbol{b}_2, E\boldsymbol{b}_3$.

This holds true for the matrix multiplication above (where the matrix is A instead of B). If you multiply column 1 of A by E, you get column 1 of EA:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and } E(\text{column } j \text{ of } A) = \text{column } j \text{ of } EA.$$

This requirement deals with columns, while elimination deals with rows. The next section describes each individual entry of the product. The beauty of matrix multiplication is that all three approaches (rows, columns, whole matrices) come out right.

The Matrix P_{ij} for a Row Exchange

To subtract row j from row i we use E_{ij}. To exchange or "permute" those rows we use another matrix P_{ij}. Row exchanges are needed when zero is in the pivot position. Lower down that pivot column may be a nonzero. By exchanging the two rows, we have a pivot (never zero!) and elimination goes forward.

What matrix P_{23} exchanges row 2 with row 3? We can find it by exchanging rows of the identity matrix I:

$$\textbf{Permutation matrix} \qquad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is a ***row exchange matrix***. Multiplying by P_{23} exchanges components 2 and 3 of any column vector. Therefore it also exchanges rows 2 and 3 of any matrix:

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix}=\begin{bmatrix}1\\5\\3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}2&4&1\\0&0&3\\0&6&5\end{bmatrix}=\begin{bmatrix}2&4&1\\0&6&5\\0&0&3\end{bmatrix}.$$

On the right, P_{23} is doing what it was created for. With zero in the second pivot position and "6" below it, the exchange puts 6 into the pivot.

Matrices *act*. They don't just sit there. We will soon meet other permutation matrices, which can change the order of several rows. Rows 1, 2, 3 can be moved to 3, 1, 2. Our P_{23} is one particular permutation matrix—it exchanges rows 2 and 3.

2E Row Exchange Matrix P_{ij} is the identity matrix with rows i and j reversed. When P_{ij} multiplies a matrix A, it exchanges rows i and j of A.

$$\textit{\textbf{To exchange equations 1 and 3 multiply by}} \quad P_{13}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Usually row exchanges are not required. The odds are good that elimination uses only the E_{ij}. But the P_{ij} are ready if needed, to move a pivot up to the diagonal.

The Augmented Matrix

This book eventually goes far beyond elimination. Matrices have all kinds of practical applications, in which they are multiplied. Our best starting point was a square E times a square A, because we met this in elimination—and we know what answer to expect for EA. The next step is to allow a *rectangular matrix*. It still comes from our original equations, but now it includes the right side $\boldsymbol{b}$.

Key idea: Elimination does the same row operations to A and to $\boldsymbol{b}$. ***We can include b as an extra column and follow it through elimination***. The matrix A is enlarged or "augmented" by the extra column $\boldsymbol{b}$:

$$\textit{Augmented matrix} \quad \begin{bmatrix}A & b\end{bmatrix}=\begin{bmatrix}2&4&-2&\mathbf{2}\\4&9&-3&\mathbf{8}\\-2&-3&7&\mathbf{10}\end{bmatrix}.$$

Elimination acts on whole rows of this matrix. The left side and right side are both multiplied by E, to subtract 2 times equation 1 from equation 2. With $\begin{bmatrix}A & \boldsymbol{b}\end{bmatrix}$ those steps happen together:

$$\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&4&-2&\mathbf{2}\\4&9&-3&\mathbf{8}\\-2&-3&7&\mathbf{10}\end{bmatrix}=\begin{bmatrix}2&4&-2&\mathbf{2}\\0&1&1&\mathbf{4}\\-2&-3&7&\mathbf{10}\end{bmatrix}.$$

The new second row contains 0, 1, 1, 4. The new second equation is $x_2+x_3=4$. Matrix multiplication works by rows and at the same time by columns:

R (by rows): Each row of E acts on $[\,A \;\; \boldsymbol{b}\,]$ to give a row of $[\,EA \;\; E\boldsymbol{b}\,]$.

C (by columns): E acts on each column of $[\,A \;\; \boldsymbol{b}\,]$ to give a column of $[\,EA \;\; E\boldsymbol{b}\,]$.

Notice again that word "acts." This is essential. Matrices do something! The matrix A acts on $\boldsymbol{x}$ to produce $\boldsymbol{b}$. The matrix E operates on A to give EA. The whole process of elimination is a sequence of row operations, alias matrix multiplications. A goes to $E_{21}A$ which goes to $E_{31}E_{21}A$. Finally $E_{32}E_{31}E_{21}A$ is a triangular matrix.

The right side is included in the augmented matrix. The end result is a triangular system of equations. We stop for exercises on multiplication by E, before writing down the rules for all matrix multiplications (including block multiplication).

▪ REVIEW OF THE KEY IDEAS ▪

1. $A\boldsymbol{x} = x_1$ times column $1 + \cdots + x_n$ times column n. And $(A\boldsymbol{x})_i = \sum_{j=1}^{n} a_{ij}x_j$.
2. Identity matrix $= I$, elimination matrix $= E_{ij}$, exchange matrix $= P_{ij}$.
3. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ by E_{21} subtracts a multiple ℓ_{21} of equation 1 from equation 2. The number $-\ell_{21}$ is the $(2, 1)$ entry of the elimination matrix E_{21}.
4. For the augmented matrix $\left[\,A \;\; \boldsymbol{b}\,\right]$, that elimination step gives $\left[\,E_{21}A \;\; E_{21}\boldsymbol{b}\,\right]$.
5. When A multiplies any matrix B, it multiplies each column of B separately.

▪ WORKED EXAMPLES ▪

2.3 A What 3 by 3 matrix E_{21} subtracts 4 times row 1 from row 2? What matrix P_{32} exchanges row 2 and row 3? If you multiply A on the *right* instead of the left, describe the results AE_{21} and AP_{32}.

Solution By doing those operations on the identity matrix I, we find

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying by E_{21} on the right side will subtract 4 times **column 2** from **column 1**. Multiplying by P_{32} on the right will exchange **columns 2** and **3**.

2.3 B Write down the augmented matrix $[A \;\; \boldsymbol{b}]$ with an extra column:

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

Apply E_{21} and then P_{32} to reach a triangular system. Solve by back substitution. What combined matrix $P_{32}\,E_{21}$ will do both steps at once?

Solution The augmented matrix and the result of using E_{21} are

$$[A \quad \boldsymbol{b}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad E_{21}[A \quad \boldsymbol{b}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

P_{32} exchanges equation 2 and 3. Back substitution produces (x, y, z):

$$P_{32}\, E_{21}[A \quad \boldsymbol{b}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

For the matrix $P_{32}\, E_{21}$ that does both steps at once, apply P_{32} to E_{21}!

$$P_{32}\, E_{21} = \text{exchange the rows of } E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 1 & 0 \end{bmatrix}.$$

2.3 C Multiply these matrices in two ways: first, rows of A times columns of B to find each entry of AB, and second, columns of A times rows of B to produce two matrices that add to AB. How many separate ordinary multiplications are needed?

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 5 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = (3 \text{ by } 2)(2 \text{ by } 2)$$

Solution Rows of A times columns of B are dot products of vectors:

$$(\text{row } 1) \cdot (\text{column } 1) = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 10 \qquad \text{is the } (1,1) \text{ entry of } AB$$

$$(\text{row } 2) \cdot (\text{column } 1) = \begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 7 \qquad \text{is the } (2,1) \text{ entry of } AB$$

The first columns of AB are $(10, 7, 4)$ and $(16, 9, 8)$. We need 6 dot products, 2 multiplications each, 12 in all $(3 \cdot 2 \cdot 2)$. The same AB comes from *columns of* A *times rows of* B:

$$AB = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{16} \\ \mathbf{7} & \mathbf{9} \\ \mathbf{4} & \mathbf{8} \end{bmatrix}.$$

Problem Set 2.3

Problems 1–15 are about elimination matrices.

1 Write down the 3 by 3 matrices that produce these elimination steps:

(a) E_{21} subtracts 5 times row 1 from row 2.

(b) E_{32} subtracts -7 times row 2 from row 3.

(c) P exchanges rows 1 and 2, then rows 2 and 3.

2 In Problem 1, applying E_{21} and then E_{32} to the column $\boldsymbol{b} = (1, 0, 0)$ gives $E_{32}E_{21}\boldsymbol{b} =$ ____. Applying E_{32} before E_{21} gives $E_{21}E_{32}\boldsymbol{b} =$ ____. When E_{32} comes first, row ____ feels no effect from row ____.

3 Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

4 Include $\boldsymbol{b} = (1, 0, 0)$ as a fourth column in Problem 3 to produce $[\,A \ \ \boldsymbol{b}\,]$. Carry out the elimination steps on this augmented matrix to solve $A\boldsymbol{x} = \boldsymbol{b}$.

5 Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ____. If you change a_{33} to ____, there is no third pivot.

6 If every column of A is a multiple of $(1, 1, 1)$, then $A\boldsymbol{x}$ is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

7 Suppose E_{31} subtracts 7 times row 1 from row 3. To reverse that step you should ____ 7 times row ____ to row ____. This "inverse matrix" is $R_{31} =$ ____.

8 Suppose E_{31} subtracts 7 times row 1 from row 3. What matrix R_{31} is changed into I? Then $E_{31}R_{31} = I$ where Problem 7 has $R_{31}E_{31} = I$. Both are true!

9 (a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?

(b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

10 (a) What 3 by 3 matrix E_{13} will add row 3 to row 1?

(b) What matrix adds row 1 to row 3 and *at the same time* row 3 to row 1?

(c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

11 Create a matrix that has $a_{11} = a_{22} = a_{33} = 1$ but elimination produces two negative pivots without row exchanges. (The first pivot is 1.)

12 Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

13 Explain these facts. If the third column of B is all zero, the third column of EB is all zero (for any E). If the third *row* of B is all zero, the third row of EB might *not* be zero.

14 This 4 by 4 matrix will need elimination matrices E_{21} and E_{32} and E_{43}. What are those matrices?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

15 Write down the 3 by 3 matrix that has $a_{ij} = 2i - 3j$. This matrix has $a_{32} = 0$, but elimination still needs E_{32} to produce a zero in the 3, 2 position. Which previous step destroys the original zero and what is E_{32}?

Problems 16–23 are about creating and multiplying matrices.

16 Write these ancient problems in a 2 by 2 matrix form $A\boldsymbol{x} = \boldsymbol{b}$ and solve them:

(a) X is twice as old as Y and their ages add to 33.

(b) $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c.

17 The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

18 Multiply these matrices in the orders EF and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Also compute $E^2 = EE$ and $F^3 = FFF$.

19 Multiply these row exchange matrices in the orders PQ and QP and P^2:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find four matrices whose squares are $M^2 = I$.

20 (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times ____.

(b) Suppose all rows of B are $[1\ \ 2\ \ 4]$. Show by example that all rows of EB are *not* $[1\ \ 2\ \ 4]$. It is true that those rows are ____.

21 If E adds row 1 to row 2 and F adds row 2 to row 1, does EF equal FE?

22 The entries of A and $\boldsymbol{x}$ are a_{ij} and x_j. So the first component of $A\boldsymbol{x}$ is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for

(a) the third component of $A\boldsymbol{x}$

(b) the $(2, 1)$ entry of $E_{21}A$

(c) the $(2, 1)$ entry of $E_{21}(E_{21}A)$

(d) the first component of $EA\boldsymbol{x}$.

23 The elimination matrix $E = \left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$ subtracts 2 times row 1 of A from row 2 of A. The result is EA. What is the effect of $E(EA)$? In the opposite order AE, we are subtracting 2 times ____ of A from ____. (Do examples.)

Problems 24–29 include the column $\boldsymbol{b}$ in the augmented matrix $[\,A\ \ \boldsymbol{b}\,]$.

24 Apply elimination to the 2 by 3 augmented matrix $[\,A\ \ \boldsymbol{b}\,]$. What is the triangular system $U\boldsymbol{x} = \boldsymbol{c}$? What is the solution $\boldsymbol{x}$?

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

25 Apply elimination to the 3 by 4 augmented matrix $[\,A\ \ \boldsymbol{b}\,]$. How do you know this system has no solution? Change the last number 6 so there *is* a solution.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

26 The equations $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{x}^* = \boldsymbol{b}^*$ have the same matrix A. What double augmented matrix should you use in elimination to solve both equations at once?

Solve both of these equations by working on a 2 by 4 matrix:

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

27 Choose the numbers a, b, c, d in this augmented matrix so that there is (a) no solution (b) infinitely many solutions.

$$[\,A\ \ \boldsymbol{b}\,] = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

Which of the numbers a, b, c, or d have no effect on the solvability?

28 If $AB = I$ and $BC = I$ use the associative law to prove $A = C$.

29 Choose two matrices $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det M = ad - bc = 1$ and with a, b, c, d positive integers. Prove that every such matrix M either has

$$\text{EITHER} \quad \text{row } 1 \leq \text{row } 2 \quad \text{OR} \quad \text{row } 2 \leq \text{row } 1.$$

Subtraction makes $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} M$ or $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} M$ nonnegative but smaller than M. If you continue and reach I, write your M's as products of the inverses $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

30 Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Challenge question: Which M (from several E's) reduces Pascal all the way to I?

RULES FOR MATRIX OPERATIONS ■ 2.4

I will start with basic facts. A matrix is a rectangular array of numbers or "entries." When A has m rows and n columns, it is an "m by n" matrix. Matrices can be added if their shapes are the same. They can be multiplied by any constant c. Here are examples of $A + B$ and $2A$, for 3 by 2 matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix} \quad \text{and} \quad 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}.$$

Matrices are added exactly as vectors are—one entry at a time. We could even regard a column vector as a matrix with only one column (so $n = 1$). The matrix $-A$ comes from multiplication by $c = -1$ (reversing all the signs). Adding A to $-A$ leaves the *zero matrix*, with all entries zero.

The 3 by 2 zero matrix is different from the 2 by 3 zero matrix. Even zero has a shape (several shapes) for matrices. All this is only common sense.

The entry in row i and column j is called a_{ij} or $A(i, j)$. The n entries along the first row are $a_{11}, a_{12}, \ldots, a_{1n}$. The lower left entry in the matrix is a_{m1} and the lower right is a_{mn}. The row number i goes from 1 to m. The column number j goes from 1 to n.

Matrix addition is easy. The serious question is ***matrix multiplication.*** When can we multiply A times B, and what is the product AB? We cannot multiply when A and B are 3 by 2. They don't pass the following test:

To multiply AB: If A has n columns, B must have n rows.

If A has two columns, B must have two rows. When A is 3 by 2, the matrix B can be 2 by 1 (a vector) or 2 by 2 (square) or 2 by 20. Every column of B is ready to be multiplied by A. Then AB is 3 by 1 (a vector) or 3 by 2 or 3 by 20.

Suppose A is m by n and B is n by p. We can multiply. The product AB is m by p.

$$\begin{bmatrix} \textbf{m rows} \\ n \textit{ columns} \end{bmatrix} \begin{bmatrix} n \textit{ rows} \\ \textbf{p columns} \end{bmatrix} = \begin{bmatrix} \textbf{m rows} \\ \textbf{p columns} \end{bmatrix}.$$

A row times a column is an extreme case. Then 1 by n multiplies n by 1. The result is 1 by 1. That single number is the "dot product."

In every case AB is filled with dot products. For the top corner, the $(1, 1)$ entry of AB is (row 1 of A) $\cdot$ (column 1 of B). To multiply matrices, take all these dot products: ***(each row of*** A***)*** $\cdot$ ***(each column of*** B***)***.

2F ***The entry in row*** i ***and column*** j ***of*** AB ***is*** (row i of A) $\cdot$ (column j of B).

Figure 2.8 picks out the second row ($i = 2$) of a 4 by 5 matrix A. It picks out the third column ($j = 3$) of a 5 by 6 matrix B. Their dot product goes into row 2 and column 3 of AB. The matrix AB has *as many rows as* A (4 rows), and *as many columns as* B.

$$\begin{bmatrix} * \\ a_{i1}\ a_{i2}\ \cdots\ a_{i5} \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

A is 4 by 5 $\qquad$ B is 5 by 6 $\qquad$ AB is 4 by 6

Figure 2.8 Here $i = 2$ and $j = 3$. Then $(AB)_{23}$ is (row 2) $\cdot$ (column 3) $= \Sigma a_{2k} b_{k3}$.

Example 1 Square matrices can be multiplied if and only if they have the same size:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}.$$

The first dot product is $1 \cdot 2 + 1 \cdot 3 = 5$. Three more dot products give 6, 1, and 0. Each dot product requires two multiplications—thus eight in all.

If A and B are n by n, so is AB. It contains n^2 dot products, row of A times column of B. Each dot product needs n multiplications, so ***the computation of*** AB ***uses*** n^3 ***separate multiplications***. For $n = 100$ we multiply a million times. For $n = 2$ we have $n^3 = 8$.

Mathematicians thought until recently that AB absolutely needed $2^3 = 8$ multiplications. Then somebody found a way to do it with 7 (and extra additions). By breaking n by n matrices into 2 by 2 blocks, this idea also reduced the count for large matrices. Instead of n^3 it went below $n^{2.8}$, and the exponent keeps falling.[1] The best

[1] Maybe the exponent won't stop falling before 2. No number in between looks special.

at this moment is $n^{2.376}$. But the algorithm is so awkward that scientific computing is done the regular way: n^2 dot products in AB, and n multiplications for each one.

Example 2 Suppose A is a row vector (1 by 3) and B is a column vector (3 by 1). Then AB is 1 by 1 (only one entry, the dot product). On the other hand B times A (***a column times a row***) is a full 3 by 3 matrix. This multiplication is allowed!

$$\textbf{\textit{Column times row:}} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

A row times a column is an *"inner"* product—that is another name for dot product. A column times a row is an *"outer"* product. These are extreme cases of matrix multiplication, with very thin matrices. They follow the rule for shapes in multiplication: (n by 1) times (1 by n). The product of column times row is n by n.

Example 3 will show how to multiply AB using columns times rows.

Rows and Columns of AB

In the big picture, A multiplies each column of B. The result is a column of AB. In that column, we are combining the columns of A. ***Each column of AB is a combination of the columns of A.*** That is the column picture of matrix multiplication:

$$\textbf{\textit{Column of}}\ AB \text{ is } (\textit{matrix } A)\ \textbf{\textit{times}}\ (\textit{column of } B).$$

The row picture is reversed. Each row of A multiplies the whole matrix B. The result is a row of AB. It is a combination of the rows of B:

$$\begin{bmatrix} \text{row } i \text{ of } A \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} \text{row } i \text{ of } AB \end{bmatrix}.$$

We see row operations in elimination (E times A). We see columns in A times $\boldsymbol{x}$. The "row-column picture" has the dot products of rows with columns. Believe it or not, ***there is also a "column-row picture."*** Not everybody knows that columns $1, \dots, n$ of A multiply rows $1, \dots, n$ of B and add up to the same answer AB.

The Laws for Matrix Operations

May I put on record six laws that matrices do obey, while emphasizing an equation they don't obey? The matrices can be square or rectangular, and the laws involving $A+B$ are all simple and all obeyed. Here are three addition laws:

$$\begin{aligned} A + B &= B + A && \text{(commutative law)} \\ c(A+B) &= cA + cB && \text{(distributive law)} \\ A + (B + C) &= (A + B) + C && \text{(associative law).} \end{aligned}$$

Three more laws hold for multiplication, but $AB = BA$ is not one of them:

$\boldsymbol{AB \neq BA}$	(the commutative "law" is *usually broken*)
$C(A+B) = CA + CB$	(distributive law from the left)
$(A+B)C = AC + BC$	(distributive law from the right)
$\boldsymbol{A(BC) = (AB)C}$	(associative law for ABC) (***parentheses not needed***).

When A and B are not square, AB is a different size from BA. These matrices can't be equal—even if both multiplications are allowed. For square matrices, almost any example shows that AB is different from BA:

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is true that $AI = IA$. All square matrices commute with I and also with cI. Only these matrices cI commute with all other matrices.

The law $A(B+C) = AB + AC$ is proved a column at a time. Start with $A(\boldsymbol{b} + \boldsymbol{c}) = A\boldsymbol{b} + A\boldsymbol{c}$ for the first column. That is the key to everything—***linearity***. Say no more.

The law $A(BC) = (AB)C$ ***means that you can multiply*** BC ***first or*** AB ***first.*** The direct proof is sort of awkward (Problem 16) but this law is extremely useful. We highlighted it above; it is the key to the way we multiply matrices.

Look at the special case when $A = B = C =$ square matrix. Then (A *times* A^2) $=$ (A^2 *times* A). The product in either order is A^3. The matrix powers A^p follow the same rules as numbers:

$$\boldsymbol{A^p = AAA \cdots A \; (p \text{ factors}) \qquad (A^p)(A^q) = A^{p+q} \qquad (A^p)^q = A^{pq}.}$$

Those are the ordinary laws for exponents. A^3 times A^4 is A^7 (seven factors). A^3 to the fourth power is A^{12} (twelve A's). When p and q are zero or negative these rules still hold, provided A has a "-1 power"—which is the *inverse matrix* A^{-1}. Then $A^0 = I$ is the identity matrix (no factors).

For a number, a^{-1} is $1/a$. For a matrix, the inverse is written A^{-1}. (It is *never* I/A, except this is allowed in MATLAB.) Every number has an inverse except $a = 0$. To decide when A has an inverse is a central problem in linear algebra. Section 2.5 will start on the answer. This section is a Bill of Rights for matrices, to say when A and B can be multiplied and how.

Block Matrices and Block Multiplication

We have to say one more thing about matrices. They can be cut into ***blocks*** (which are smaller matrices). This often happens naturally. Here is a 4 by 6 matrix broken into blocks of size 2 by 2—and each block is just I:

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

If B is also 4 by 6 and its block sizes match the block sizes in A, you can add $A+B$ *a block at a time*.

We have seen block matrices before. The right side vector $\boldsymbol{b}$ was placed next to A in the "augmented matrix." Then $[\,A \ \ \boldsymbol{b}\,]$ has two blocks of different sizes. Multiplying by an elimination matrix gave $[\,EA \ \ E\boldsymbol{b}\,]$. No problem to multiply blocks times blocks, when their shapes permit:

2G Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}. \tag{1}$$

This equation is the same as if the blocks were numbers (which are 1 by 1 blocks). We are careful to keep A's in front of B's, because BA can be different. The cuts between rows of A give cuts between rows of AB. Any column cuts in B are also column cuts in AB.

Main point When matrices split into blocks, it is often simpler to see how they act. The block matrix of I's above is much clearer than the original 4 by 6 matrix A.

Example 3 **(Important special case)** Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up ***columns times rows***:

$$AB = \begin{bmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \boldsymbol{b}_1 & - \\ & \vdots & \\ - & \boldsymbol{b}_n & - \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}. \tag{2}$$

This is another way to multiply matrices! Compare it with the usual rows times columns. Row 1 of A times column 1 of B gave the $(1, 1)$ entry in AB. Now *column* 1 of A

times *row* 1 of B gives a full matrix—not just a single number. Look at this example:

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}. \tag{3}$$

We stop there so you can see columns multiplying rows. If a 2 by 1 matrix (a column) multiplies a 1 by 2 matrix (a row), the result is 2 by 2. That is what we found. Dot products are "inner products," these are "outer products."

When you add the two matrices at the end of equation (3), you get the correct answer AB. In the top left corner the answer is $3 + 4 = 7$. This agrees with the row-column dot product of $(1, 4)$ with $(3, 1)$.

Summary The usual way, rows times columns, gives four dot products (8 multiplications). The new way, columns times rows, gives two full matrices (8 multiplications). The eight multiplications, and also the four additions, are all the same. You just execute them in a different order.

Example 4 (Elimination by blocks) Suppose the first column of A contains $1, 3, 4$. To change 3 and 4 to 0 and 0, multiply the pivot row by 3 and 4 and subtract. Those row operations are really multiplications by elimination matrices E_{21} and E_{31}:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

The "block idea" is to do both eliminations with one matrix E. That matrix clears out the whole first column of A below the pivot $a = 2$:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad \text{multiplies} \quad \begin{bmatrix} \mathbf{1} & x & x \\ \mathbf{3} & x & x \\ \mathbf{4} & x & x \end{bmatrix} \quad \text{to give} \quad EA = \begin{bmatrix} \mathbf{1} & x & x \\ \mathbf{0} & x & x \\ \mathbf{0} & x & x \end{bmatrix}.$$

Block multiplication gives a formula for EA. The matrix A has four blocks a, $\boldsymbol{b}$, $\boldsymbol{c}$, D: the pivot, the rest of row 1, the rest of column 1, and the rest of the matrix. Watch how E multiplies A by blocks:

$$EA = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline -\boldsymbol{c}/a & I \end{array}\right]\left[\begin{array}{c|c} a & \boldsymbol{b} \\ \hline \boldsymbol{c} & D \end{array}\right] = \left[\begin{array}{c|c} a & \boldsymbol{b} \\ \hline \mathbf{0} & D - \boldsymbol{c}\boldsymbol{b}/a \end{array}\right]. \tag{4}$$

Elimination multiplies the first row $[\,a \;\; \boldsymbol{b}\,]$ by $\boldsymbol{c}/a$. It subtracts from $\boldsymbol{c}$ to get zeros in the first column. It subtracts from D to get $D - \boldsymbol{c}\boldsymbol{b}/a$. This is ordinary elimination, a column at a time—written in blocks.

■ REVIEW OF THE KEY IDEAS ■

1. The (i, j) entry of AB is (row i of A) $\cdot$ (column j of B).
2. An m by n matrix times an n by p matrix uses mnp separate multiplications.
3. A times BC equals AB times C (surprisingly important).
4. AB is also the sum of these matrices: (column j of A) times (row j of B).
5. Block multiplication is allowed when the block shapes match correctly.

■ WORKED EXAMPLES ■

2.4 A Put yourself in the position of the author! I want to show you matrix multiplications that are *special*, but mostly I am stuck with small matrices. There is one terrific family of **Pascal matrices**, and they come in all sizes, and above all they have real meaning. I think 4 by 4 is a good size to show some of their amazing patterns.

Here is the lower triangular Pascal matrix L. Its entries come from "*Pascal's triangle*". I will multiply L times the **ones** vector, and the **powers** vector:

$$\begin{array}{c}\textbf{Pascal}\\ \textbf{matrix}\end{array} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{4} \\ \mathbf{8} \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \boldsymbol{x} \\ \boldsymbol{x^2} \\ \boldsymbol{x^3} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1+\boldsymbol{x}} \\ \mathbf{(1+\boldsymbol{x})^2} \\ \mathbf{(1+\boldsymbol{x})^3} \end{bmatrix}.$$

Each row of L leads to the next row: *Add an entry to the one on its left to get the entry below*. In symbols $\ell_{i\,j} + \ell_{i\,j-1} = \ell_{i+1\,j}$. The numbers after $1, 3, 3, 1$ would be $1, 4, 6, 4, 1$. Pascal lived in the 1600's, long before matrices, but his triangle fits perfectly into L.

Multiplying by **ones** is the same as adding up each row, to get powers of 2. In fact **powers** $=$ **ones** when $x = 1$. By writing out the last rows of L times **powers**, you see the entries of L as the "binomial coefficients" that are so essential to gamblers:

$$\mathbf{1} + \mathbf{2}x + \mathbf{1}x^2 = (1 + x)^2 \qquad \mathbf{1} + \mathbf{3}x + \mathbf{3}x^2 + \mathbf{1}x^3 = (1 + x)^3$$

The number "3" counts the ways to get Heads once and Tails twice in three coin flips: HTT and THT and TTH. The other "3" counts the ways to get Heads twice: HHT and HTH and THH. Those are examples of "i choose j" $=$ the number of ways to get j heads in i coin flips. That number is exactly ℓ_{ij}, if we start counting rows and columns of L at $i = 0$ and $j = 0$ (and remember $0! = 1$):

$$\ell_{ij} = \binom{i}{j} = \boldsymbol{i \text{ choose } j} = \frac{i!}{j!\,(i-j)!} \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

There are six ways to choose two aces out of four aces. We will see Pascal's triangle and these matrices again. Here are the questions I want to ask now:

1. What is $H = L^2$? This is the "hypercube matrix".

2. Multiply H times **ones** and **powers**.

3. The last row of H is $8, 12, 6, 1$. A cube has 8 corners, 12 edges, 6 faces, 1 box. *What would the next row of H tell about a hypercube in* **4D**?

Solution Multiply L times L to get the hypercube matrix $H = L^2$:

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & & & \\ \mathbf{2} & \mathbf{1} & & \\ \mathbf{4} & \mathbf{4} & \mathbf{1} & \\ \mathbf{8} & \mathbf{12} & \mathbf{6} & \mathbf{1} \end{bmatrix} = H.$$

Now multiply H times the vectors of **ones** and **powers**:

$$\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{9} \\ \mathbf{27} \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2+x} \\ \mathbf{(2+x)^2} \\ \mathbf{(2+x)^3} \end{bmatrix}$$

If $x = 1$ we get the powers of 3. If $x = 0$ we get powers of 2 (where do $1, 2, 4, 8$ appear in H?). Where L changed x to $1+x$, applying L again changes $1+x$ to $2+x$.

How do the rows of H count corners and edges and faces of a cube? A square in **2D** has 4 corners, 4 edges, 1 face. Add one dimension at a time:

Connect two squares to get a **3D** *cube. Connect two cubes to get a* **4D** *hypercube.*

The cube has 8 corners and 12 edges: 4 edges in each square and 4 between the squares. The cube has 6 faces: 1 in each square and 4 faces between the squares. This row $8, 12, 6, 1$ of H will lead to the next row (one more dimension) by $\mathbf{2}h_{i\,j} + h_{i\,j-1} = h_{i+1\,j}$.

Can you see this in four dimensions? The hypercube has 16 corners, no problem. It has 12 edges from one cube, 12 from the other cube, 8 that connect corners between those cubes: total $2 \times 12 + 8 = 32$ edges. It has 6 faces from each separate cube and 12 more from connecting pairs of edges: total $2 \times 6 + 12 = 24$ faces. It has one box from each cube and 6 more from connecting pairs of faces: total $2 \times 1 + 6 = 8$ boxes. And sure enough, the next row of H is $16, 32, 24, 8, 1$.

2.4 B For these matrices, when does $AB = BA$? When does $BC = CB$? When does A times BC equal AB times C? Give the conditions on their entries p, q, r, z:

$$A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

If $p, q, r, 1, z$ are 4 by 4 blocks instead of numbers, do the answers change?

Solution First of all, A times BC *always* equals AB times C. We don't need parentheses in $A(BC) = (AB)C = ABC$. But we do need to keep the matrices in this order A, B, C. Compare AB with BA:

$$AB = \begin{bmatrix} p & p \\ q & q+r \end{bmatrix} \qquad BA = \begin{bmatrix} p+q & r \\ q & r \end{bmatrix}.$$

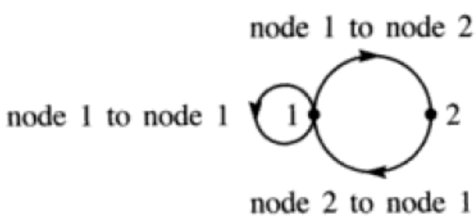
We only have $AB = BA$ if $q = 0$ and $p = r$. Now compare BC with CB:

$$BC = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \qquad CB = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}.$$

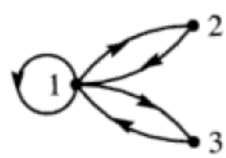
B and C happen to commute. One explanation is that the diagonal part of B is I, which commutes with all 2 by 2 matrices. The off-diagonal part of B looks exactly like C (except for a scalar factor z) and every matrix commutes with itself.

When p, q, r, z are 4 by 4 blocks and 1 changes to the 4 by 4 identity matrix, all these products remain correct. So the answers are the same. (If the I's in B were changed to blocks t, t, t, then BC would have the block tz and CB would have the block zt. Those would normally be different—the order is important in block multiplication.)

2.4 C A **directed graph** starts with n nodes. There are n^2 possible edges—each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n **adjacency matrix** has $a_{ij} = 1$ when an edge leaves node i and enters node j; if no edge then $a_{ij} = 0$. Here are two directed graphs and their adjacency matrices:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The $\boldsymbol{i, j}$ ***entry of*** $\mathbf{A^2}$ ***is*** $\boldsymbol{a_{i1}a_{1j} + \cdots + a_{in}a_{nj}}$. Why does that sum count the *two-step paths* from i to any node to j? The i, j entry of A^k counts k-step paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \text{counts the paths} \\ \text{with two edges} \end{matrix} \quad \begin{bmatrix} \text{1 to 2 to 1, 1 to 1 to 1} & \text{1 to 1 to 2} \\ \text{2 to 1 to 1} & \text{2 to 1 to 2} \end{bmatrix}$$

List all of the 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the ***diameter*** of the graph—the number of edges needed to connect the most distant pair of nodes. What is the diameter of the second graph?

Solution The number $a_{ik}a_{kj}$ will be "1" if there is an edge from node i to k and an edge from k to j. This is a 2-step path. The number $a_{ik}a_{kj}$ will be "0" if either of those edges (i to k, k to j) is missing. So the sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j. Matrix multiplication is just right for this count.

The 3-step paths are counted by A^3; we look at paths to node 2:

$$A^3 = \begin{bmatrix} 3 & \mathbf{2} \\ 2 & \mathbf{1} \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with three steps} \end{array} \quad \begin{bmatrix} \cdots & \text{1 to 1 to 1 to 2, 1 to 2 to 1 to 2} \\ \cdots & \text{2 to 1 to 1 to 2} \end{bmatrix}$$

These A^k contain the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ coming in Section 6.2. Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ (as in $13 = 8 + 5$) shows up in $(A)(A^k) = A^{k+1}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} = A^{k+1}.$$

There are 13 six-step paths from node 1 to node 1, but I can't find them all.

A^k *also counts words.* A path like 1 to 1 to 2 to 1 corresponds to the number 1121 or the word **aaba**. The number 2 (the letter **b**) is not allowed to repeat because the graph has no edge from node 2 to node 2. The i, j entry of A^k counts the allowed numbers (or words) of length $k+1$ that start with the ith letter and end with the jth.

The second graph also has diameter 2; A^2 *has no zeros.*

Problem Set 2.4

Problems 1–17 are about the laws of matrix multiplication.

1 A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B+C).$$

2 What rows or columns or matrices do you multiply to find

(a) the third column of AB?

(b) the first row of AB?

(c) the entry in row 3, column 4 of AB?

(d) the entry in row 1, column 1 of CDE?

3 Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

4 In Problem 3, multiply A times BC. Then multiply AB times C.

5 Compute A^2 and A^3. Make a prediction for A^5 and A^n:

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

6 Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1&2\\0&0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1&0\\3&0\end{bmatrix}.$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ ____ $+B^2$.

7 True or false. Give a specific example when false:

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

(d) $(AB)^2=A^2B^2$.

8 How is each row of DA and EA related to the rows of A, when

$$D=\begin{bmatrix}3&0\\0&5\end{bmatrix} \quad \text{and} \quad E=\begin{bmatrix}0&1\\0&1\end{bmatrix} \quad \text{and} \quad A=\begin{bmatrix}a&b\\c&d\end{bmatrix}?$$

How is each column of AD and AE related to the columns of A?

9 Row 1 of A is added to row 2. This gives EA below. Then column 1 of EA is added to column 2 to produce $(EA)F$:

$$EA=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\a+c&b+d\end{bmatrix}$$

$$\text{and} \quad (EA)F=(EA)\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}a&a+b\\a+c&a+c+b+d\end{bmatrix}.$$

(a) Do those steps in the opposite order. First add column 1 of A to column 2 by AF, then add row 1 of AF to row 2 by $E(AF)$.

(b) Compare with $(EA)F$. What law is obeyed by matrix multiplication?

10 Row 1 of A is again added to row 2 to produce EA. Then F adds row 2 of EA to row 1. The result is $F(EA)$:

$$F(EA)=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}a&b\\a+c&b+d\end{bmatrix}=\begin{bmatrix}2a+c&2b+d\\a+c&b+d\end{bmatrix}.$$

(a) Do those steps in the opposite order: first add row 2 to row 1 by FA, then add row 1 of FA to row 2.

(b) What law is or is not obeyed by matrix multiplication?

11 (3 by 3 matrices) Choose the only B so that for every matrix A

(a) $BA=4A$

(b) $BA=4B$

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged

(d) All rows of BA are the same as row 1 of A.

12 Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I. The only matrices that commute with B and C and all other 2 by 2 matrices are A = multiple of I.

13 Which of the following matrices are guaranteed to equal $(A-B)^2$: $A^2 - B^2$, $(B-A)^2$, $A^2 - 2AB + B^2$, $A(A-B) - B(A-B)$, $A^2 - AB - BA + B^2$?

14 True or false:

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

15 If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector $\boldsymbol{x}$ with n components?

(b) A multiplies an n by p matrix B?

(c) A multiplies itself to produce A^2? Here $m = n$.

16 To prove that $(AB)C = A(BC)$, use the column vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ of B. First suppose that C has only one column $\boldsymbol{c}$ with entries $c_1, \ldots, c_n$:

AB has columns $A\boldsymbol{b}_1, \ldots, A\boldsymbol{b}_n$ and $B\boldsymbol{c}$ has one column $c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n$.

Then $(AB)\boldsymbol{c} = c_1 A\boldsymbol{b}_1 + \cdots + c_n A\boldsymbol{b}_n$ equals $A(c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n) = A(B\boldsymbol{c})$.

Linearity gives equality of those two sums, and $(AB)\boldsymbol{c} = A(B\boldsymbol{c})$. The same is true for all other _____ of C. Therefore $(AB)C = A(BC)$.

17 For $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, compute these answers *and nothing more*:

(a) column 2 of AB

(b) row 2 of AB

(c) row 2 of $AA = A^2$

(d) row 2 of $AAA = A^3$.

Problems 18–20 use a_{ij} for the entry in row i, column j of A.

18 Write down the 3 by 3 matrix A whose entries are

(a) a_{ij} = minimum of i and j

(b) $a_{ij} = (-1)^{i+j}$

(c) $a_{ij} = i/j$.

19 What words would you use to describe each of these classes of matrices? Give a 3 by 3 example in each class. Which matrix belongs to all four classes?

(a) $a_{ij} = 0$ if $i \neq j$

(b) $a_{ij} = 0$ if $i < j$

(c) $a_{ij} = a_{ji}$

(d) $a_{ij} = a_{1j}$.

20 The entries of A are a_{ij}. Assuming that zeros don't appear, what is

(a) the first pivot?

(b) the multiplier ℓ_{31} of row 1 to be subtracted from row 3?

(c) the new entry that replaces a_{32} after that subtraction?

(d) the second pivot?

Problems 21–25 involve powers of A.

21 Compute A^2, A^3, A^4 and also $A\boldsymbol{v}$, $A^2\boldsymbol{v}$, $A^3\boldsymbol{v}$, $A^4\boldsymbol{v}$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

22 Find all the powers $A^2, A^3, \ldots$ and $AB, (AB)^2, \ldots$ for

$$A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

23 By trial and error find real nonzero 2 by 2 matrices such that

$$A^2 = -I \qquad BC = 0 \qquad DE = -ED \text{ (not allowing } DE = 0).$$

24 (a) Find a nonzero matrix A for which $A^2 = 0$.

(b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.

25 By experiment with $n = 2$ and $n = 3$ predict A^n for

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Problems 26–34 use column-row multiplication and block multiplication.

26 Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\qquad} = \underline{\qquad}.$$

27 The product of upper triangular matrices is always upper triangular:

$$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix}.$$

Row times column is dot product (Row 2 of A) $\cdot$ (column 1 of B) $= 0$. Which other dot products give zeros?

Column times row is full matrix Draw x's and 0's in (column 2 of A) times (row 2 of B) and in (column 3 of A) times (row 3 of B).

28 Draw the cuts in A (2 by 3) and B (3 by 4) and AB to show how each of the four multiplication rules is really a block multiplication:

(1) Matrix A times columns of B.

(2) Rows of A times matrix B.

(3) Rows of A times columns of B.

(4) Columns of A times rows of B.

29 Draw cuts in A and $\boldsymbol{x}$ to multiply $A\boldsymbol{x}$ a column at a time: x_1(column 1) $+ \cdots$.

30 Which matrices E_{21} and E_{31} produce zeros in the (2, 1) and (3, 1) positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA.

31 Block multiplication says in the text that column 1 is eliminated by

$$EA = \begin{bmatrix} 1 & \boldsymbol{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{0} & D - \boldsymbol{cb}/a \end{bmatrix}.$$

In Problem 30, what are $\boldsymbol{c}$ and D and what is $D - \boldsymbol{cb}/a$?

32 With $i^2 = -1$, the product of $(A+iB)$ and $(\boldsymbol{x}+i\boldsymbol{y})$ is $A\boldsymbol{x}+iB\boldsymbol{x}+iA\boldsymbol{y}-B\boldsymbol{y}$. Use blocks to separate the real part without i from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

33 Suppose you solve $A\boldsymbol{x} = \boldsymbol{b}$ for three special right sides $\boldsymbol{b}$:

$$A\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are the columns of a matrix X, what is A times X?

34 If the three solutions in Question 33 are $\boldsymbol{x}_1 = (1, 1, 1)$ and $\boldsymbol{x}_2 = (0, 1, 1)$ and $\boldsymbol{x}_3 = (0, 0, 1)$, solve $A\boldsymbol{x} = \boldsymbol{b}$ when $\boldsymbol{b} = (3, 5, 8)$. Challenge problem: What is A?

35 *Elimination for a 2 by 2 block matrix*: When you multiply the first block row by CA^{-1} and subtract from the second row, what is the "*Schur complement*" S that appears?

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

36 Find all matrices $A = \left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ that satisfy $A\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]A$.

37 Suppose a "circle graph" has 5 nodes connected (in both directions) by edges around a circle. What is its adjacency matrix from Worked Example **2.4 C**? What are A^2 and A^3 and the diameter of this graph?

38 If 5 edges in Question 37 go in one direction only, from nodes 1, 2, 3, 4, 5 to 2, 3, 4, 5, 1, what are A and A^2 and the diameter of this one-way circle?

39 If you multiply a *northwest matrix* A and a *southeast matrix* B, what type of matrices are AB and BA? "Northwest" and "southeast" mean zeros below and above the antidiagonal going from $(1, n)$ to $(n, 1)$.

INVERSE MATRICES ■ 2.5

Suppose A is a square matrix. We look for an "***inverse matrix***" A^{-1} of the same size, such that A^{-1} *times* A *equals* I. Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing. But A^{-1} might not exist.

What a matrix mostly does is to multiply a vector $\boldsymbol{x}$. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ by A^{-1} gives $A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The left side is just $\boldsymbol{x}$! The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. An ordinary number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix A^{-1} is called "A inverse."

DEFINITION The matrix A is ***invertible*** if there exists a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \tag{1}$$

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible? We don't mean that we immediately calculate A^{-1}. In most problems we never compute it! Here are six "notes" about A^{-1}.

Note 1 ***The inverse exists if and only if elimination produces n pivots*** (row exchanges allowed). Elimination solves $A\boldsymbol{x} = \boldsymbol{b}$ without explicitly using A^{-1}.

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \tag{2}$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

Note 3 If A is invertible, the one and only solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$:

$$\textbf{\textit{Multiply}} \quad A\boldsymbol{x} = \boldsymbol{b} \quad \textbf{\textit{by}} \quad A^{-1}. \quad \textbf{\textit{Then}} \quad \boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

Note 4 (Important) ***Suppose there is a nonzero vector $\boldsymbol{x}$ such that*** $A\boldsymbol{x} = \boldsymbol{0}$. *Then A cannot have an inverse.* No matrix can bring $\boldsymbol{0}$ back to $\boldsymbol{x}$.

If A is invertible, then $A\boldsymbol{x} = \boldsymbol{0}$ can only have the zero solution $\boldsymbol{x} = \boldsymbol{0}$.

Note 5 A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\textbf{2 by 2 Inverse:} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad - bc$ is the *determinant* of A. A matrix is invertible if its determinant is not zero (Chapter 5). The test for n pivots is usually decided before the determinant appears.

Note 6 A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

Example 1 The 2 by 2 matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right]$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $A\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{x} = (2, -1)$. It fails to have two pivots as required by Note 1. Elimination turns the second row of A into a zero row.

The Inverse of a Product AB

For two nonzero numbers a and b, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For two matrices A and B, the situation is similar. It is hard to say much about the invertibility of $A + B$. But the *product* AB has an inverse, whenever the factors A and B are separately invertible (and the same size). The important point is that A^{-1} and B^{-1} come in ***reverse order***:

2H If A and B are invertible then so is AB. The inverse of a product AB is

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{4}$$

To see why the order is reversed, multiply AB times $B^{-1}A^{-1}$. The inside step is $BB^{-1} = I$:

$$(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I. This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____. The same idea applies to three or more matrices:

$$\textbf{Reverse order} \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \tag{5}$$

Example 2 ***Inverse of an Elimination Matrix***. If E subtracts 5 times row 1 from row 2, then E^{-1} *adds* 5 times row 1 to row 2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiply EE^{-1} to get the identity matrix I. Also multiply $E^{-1}E$ to get I. We are adding and subtracting the same 5 times row 1. Whether we add and then subtract (this is EE^{-1}) or subtract and then add (this is $E^{-1}E$), we are back at the start.

For square matrices, an inverse on one side is automatically an inverse on the other side. If $AB = I$ then automatically $BA = I$. In that case B is A^{-1}. This is very useful to know but we are not ready to prove it.

Example 3 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 2 to find FE. Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the orders FE and $E^{-1}F^{-1}$!

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \tag{6}$$

The result is strange but correct. The product FE contains "20" but its inverse doesn't. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the *new* row 2 (changed by row 1) from row 3. ***In this order FE, row 3 feels an effect from row 1.***

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn't change again. ***In this order, row 3 feels no effect from row 1.***

> For elimination with normal order FE, ***the product of inverses $E^{-1}F^{-1}$ is quick. The multipliers fall into place below the diagonal of 1's.***

This special property of $E^{-1}F^{-1}$ and $E^{-1}F^{-1}G^{-1}$ will be useful in the next section. We will explain it again, more completely. In this section our job is A^{-1}, and we expect some serious work to compute it. Here is a way to organize that computation.

Calculating A^{-1} by Gauss-Jordan Elimination

I hinted that A^{-1} might not be explicitly needed. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is solved by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. But it is not necessary or efficient to compute A^{-1} and multiply it times $\boldsymbol{b}$. *Elimination goes directly to* $\boldsymbol{x}$. Elimination is also the way to calculate A^{-1}, as we now show. The Gauss-Jordan idea is to solve $AA^{-1} = I$, *finding each column of* A^{-1}.

A multiplies the first column of A^{-1} (call that $\boldsymbol{x}_1$) to give the first column of I (call that $\boldsymbol{e}_1$). This is our equation $A\boldsymbol{x}_1 = \boldsymbol{e}_1 = (1, 0, 0)$. Each of the columns $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ of A^{-1} is multiplied by A to produce a column of I:

$$AA^{-1} = A\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = I. \tag{7}$$

To invert a 3 by 3 matrix A, we have to solve three systems of equations: $A\boldsymbol{x}_1 = \boldsymbol{e}_1$ and $A\boldsymbol{x}_2 = \boldsymbol{e}_2 = (0, 1, 0)$ and $A\boldsymbol{x}_3 = \boldsymbol{e}_3 = (0, 0, 1)$. This already shows why computing A^{-1} is expensive. We must solve n equations for its n columns. To solve $Ax = \boldsymbol{b}$ without A^{-1}, we deal only with *one* column.

In defense of A^{-1}, we want to say that its cost is not n times the cost of one system $Ax = \boldsymbol{b}$. Surprisingly, the cost for n columns is only multiplied by 3. This saving is because the n equations $A\boldsymbol{x}_i = \boldsymbol{e}_i$ all involve the same matrix A. Working with the right sides is relatively cheap, because elimination only has to be done once on A. The complete A^{-1} needs n^3 elimination steps, where a single $\boldsymbol{x}$ needs $n^3/3$. The next section calculates these costs.

The ***Gauss-Jordan method*** computes A^{-1} by solving all n equations together. Usually the "augmented matrix" has one extra column $\boldsymbol{b}$, from the right side of the equations. Now we have three right sides $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ (when A is 3 by 3). They are the columns of I, so the augmented matrix is really the block matrix $[\,A \quad I\,]$. Here is a worked-out example when A has 2's on the main diagonal and -1's next to the 2's:

$$\begin{bmatrix} A & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & 0 & 0 & 1 \end{bmatrix} \quad \textbf{Start Gauss-Jordan}$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{\frac{3}{2}} & \mathbf{-1} & \mathbf{\frac{1}{2}} & \mathbf{1} & \mathbf{0} \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \quad (\tfrac{1}{2}\ \textbf{\textit{row}}\ \mathbf{1} + \textbf{\textit{row}}\ \mathbf{2})$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{\frac{4}{3}} & \mathbf{\frac{1}{3}} & \mathbf{\frac{2}{3}} & \mathbf{1} \end{bmatrix} \quad (\tfrac{2}{3}\ \textbf{\textit{row}}\ \mathbf{2} + \textbf{\textit{row}}\ \mathbf{3})$$

We are now halfway. The matrix in the first three columns is U (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. The contribution of Jordan is *to continue with elimination*! He goes all the way to the ***"reduced echelon form"***. Rows are added to rows above them, to produce ***zeros above the pivots***:

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{\frac{3}{2}} & \mathbf{0} & \mathbf{\frac{3}{4}} & \mathbf{\frac{3}{2}} & \mathbf{\frac{3}{4}} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (\tfrac{3}{4}\ \textbf{\textit{row}}\ \mathbf{3} + \textbf{\textit{row}}\ \mathbf{2})$$

$$\to \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{\frac{3}{2}} & \mathbf{1} & \mathbf{\frac{1}{2}} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (\tfrac{2}{3}\ \textbf{\textit{row}}\ \mathbf{2} + \textbf{\textit{row}}\ \mathbf{1})$$

The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1. We have reached I in the first half of the matrix, because A is invertible. ***The three columns of A^{-1} are in the second half of*** $[\, I \ A^{-1}\,]$:

$$\begin{matrix}(\text{divide by } 2)\\ (\text{divide by } \frac{3}{2})\\ (\text{divide by } \frac{4}{3})\end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = [\, I \ \ x_1 \ \ x_2 \ \ x_3 \,].$$

Starting from the 3 by 6 matrix $[\, A \ \ I\,]$, we ended with $[\, I \ \ A^{-1}\,]$. Here is the whole Gauss-Jordan process on one line:

$$\textit{Multiply} \ \ \begin{bmatrix} A & I \end{bmatrix} \ \ \textit{by} \ \ A^{-1} \ \ \textit{to get} \ \ [I \ A^{-1}].$$

The elimination steps gradually create the inverse matrix. For large matrices, we probably don't want A^{-1} at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular A^{-1} because it is an important example. We introduce the words *symmetric*, *tridiagonal*, and *determinant* (Chapter 5):

1. A is ***symmetric*** across its main diagonal. So is A^{-1}.

2. A is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. That is another reason we don't often compute A^{-1}.

3. The product of pivots is $2(\frac{3}{2})(\frac{4}{3}) = 4$. This number 4 is the ***determinant*** of A.

$$\boldsymbol{A^{-1}} \textbf{\textit{ involves division by the determinant}} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \tag{8}$$

Example 4 Find A^{-1} by Gauss-Jordan elimination starting from $A = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 7 \end{smallmatrix}\right]$. There are two row operations and then a division to put 1's in the pivots:

$$[\, A \ \ I\,] = \begin{bmatrix} \mathbf{2} & \mathbf{3} & 1 & 0 \\ \mathbf{4} & \mathbf{7} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

$$\to \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{7} & \mathbf{-3} \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & -2 & 1 \end{bmatrix} = [\, I \ \ A^{-1}\,].$$

The reduced echelon form of $[\, A \ \ I\,]$ is $[\, I \ \ A^{-1}\,]$. This A^{-1} involves division by the determinant $2 \cdot 7 - 3 \cdot 4 = 2$. The code for $X = \text{inverse}(A)$ has three important lines!

```
I = eye (n, n);            % Define the identity matrix
R = rref ([A I]);          % Eliminate on the augmented matrix
X = R(:, n + 1 : n + n)    % Pick A^-1 from the last n columns of R
```

A must be invertible, or elimination will not reduce it (in the left half of R) to I.

Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: A^{-1} ***exists exactly when A has a full set of n pivots.*** (Row exchanges allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With n pivots, elimination solves all the equations $Ax_i = e_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ and A^{-1} is at least a ***right-inverse***.

2. Elimination is really a sequence of multiplications by E's and P's and D^{-1}:

$$(D^{-1} \cdots E \cdots P \cdots E)A = I. \tag{9}$$

D^{-1} divides by the pivots. The matrices E produce zeros below and above the pivots. P will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a ***left-inverse***. With n pivots we reach $A^{-1}A = I$.

The right-inverse equals the left-inverse. That was Note 2 in this section. So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will now show that A must have n pivots if $AC = I$. Then we deduce that C is also a left-inverse. Here is one route to those conclusions:

1. If A doesn't have n pivots, elimination will lead to a *zero row*.
2. Those elimination steps are taken by an invertible M. *So a row of MA is zero.*
3. If $AC = I$ then $MAC = M$. The zero row of MA, times C, gives a zero row of M.
4. The invertible matrix M can't have a zero row! A *must* have n pivots if $AC = I$.
5. Then equation (9) displays the left inverse in $BA = I$, and Note 2 proves $B = C$.

That argument took five steps, but the outcome is short and important.

2I A complete test for invertibility of a square matrix A comes from elimination. A^{-1} ***exists (and Gauss-Jordan finds it) exactly when A has n pivots.*** The full argument shows more:

$$\text{If} \quad AC = I \quad \text{then} \quad CA = I \quad \text{and} \quad C = A^{-1} \quad !$$

Example 5 If L is lower triangular with 1's on the diagonal, so is L^{-1}.

Use the Gauss-Jordan method to construct L^{-1}. Start by subtracting multiples of pivot rows from rows *below*. Normally this gets us halfway to the inverse, but for L it gets us all the way. L^{-1} appears on the right when I appears on the left:

$$\begin{bmatrix} L & I \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{1} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \\ \to \\ \to \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \\ \text{(3 times row 1 from row 2)} \\ \text{(4 times row 1 from row 3)} \end{matrix}$$

$$\begin{matrix} \\ \\ \to \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{-3} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{11} & \mathbf{-5} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} I & \mathbf{L^{-1}} \end{bmatrix}.$$

When L goes to I by elimination, I goes to L^{-1}. In other words, the product of elimination matrices $E_{32}E_{31}E_{21}$ is L^{-1}. All pivots are 1's (a full set). L^{-1} is lower triangular. The strange entry "11" in L^{-1} does not appear in $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$.

■ REVIEW OF THE KEY IDEAS ■

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.

2. A is invertible if and only if it has n pivots (row exchanges allowed).

3. If $A\boldsymbol{x} = \mathbf{0}$ for a nonzero vector $\boldsymbol{x}$, then A has no inverse.

4. The inverse of AB is the reverse product $B^{-1}A^{-1}$.

5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the n columns of A^{-1}. The augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ is row-reduced to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

■ WORKED EXAMPLES ■

2.5 A Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $A\boldsymbol{x} = \mathbf{0}$) for the other three, in that order. The matrices A, B, C, D, E, F are

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

$$B^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36}\begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. D is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. F is not invertible because a combination of the columns (the second column minus the first column) is zero—in other words $F\boldsymbol{x} = \mathbf{0}$ has the solution $\boldsymbol{x} = (-1, 1, 0)$.

Of course all three reasons for noninvertibility would apply to each of A, D, F.

2.5 B Apply the Gauss-Jordan method to find the inverse of this triangular "Pascal matrix" A = abs(pascal(4,1)). You see **Pascal's triangle**—adding each entry to the entry on its left gives the entry below. The entries are "binomial coefficients":

$$\textbf{Triangular Pascal matrix} \quad A = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \end{bmatrix}.$$

Solution Gauss-Jordan starts with $[A \quad I]$ and produces zeros by subtracting row 1:

$$[A \quad I] = \left[\begin{array}{cccc|cccc} \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{-1} & \mathbf{3} & \mathbf{-3} & \mathbf{1} \end{array}\right] = [I \quad \mathbf{A^{-1}}].$$

All the pivots were 1! So we didn't need to divide rows by pivots to get I. The inverse matrix A^{-1} looks like A itself, except odd-numbered diagonals are multiplied by -1.

Please notice that 4 by 4 matrix A^{-1}, we will see Pascal matrices again. The same pattern continues to n by n Pascal matrices—the inverse has "alternating diagonals".

Problem Set 2.5

1 Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

2 For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

3 Solve for the columns of $A^{-1} = \left[\begin{smallmatrix} \mathbf{x} & \mathbf{t} \\ \mathbf{y} & \mathbf{z} \end{smallmatrix}\right]$:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4 Show that $\left[\begin{smallmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{6} \end{smallmatrix}\right]$ has no inverse by trying to solve for the column (x, y):

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & t \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{must include} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

5 Find an upper triangular U (not diagonal) with $U^2 = I$ and $U = U^{-1}$.

6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \left[\begin{smallmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{smallmatrix}\right]$, find two matrices $B \neq C$ such that $AB = AC$.

7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\boldsymbol{x} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{x} = \boldsymbol{b}$?

(c) What happens to row 3 in elimination?

8 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

9 Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible and how would you find B^{-1} from A^{-1}?

10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

14 If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

$$\text{Notice the order. The inverse of} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} & A & \end{bmatrix} \quad \text{is} \quad \underline{\qquad}.$$

15 Prove that a matrix with a column of zeros cannot have an inverse.

16 Multiply $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ times $\left[\begin{smallmatrix} \mathbf{d} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{a} \end{smallmatrix}\right]$. What is the inverse of each matrix if $ad \neq bc$?

17 (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

18 If B is the inverse of A^2, show that AB is the inverse of A.

19 Find the numbers a and b that give the inverse of 5 * eye(4) − ones(4,4):

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 * eye(5) − ones(5,5)?

20 Show that A = 4 * eye(4) − ones(4,4) is *not* invertible: Multiply A * ones(4,1).

21 There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1}.

22 Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

23 Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below the pivots to reduce $[\,A \;\; I\,]$ to $[\,I \;\; A^{-1}\,]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

24 Use Gauss-Jordan elimination on $[\,A \quad I\,]$ to solve $AA^{-1} = I$:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

25 Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A \quad I\,]$ and $[\,B \quad I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 6 \end{smallmatrix}\right]$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1}.

27 Invert these matrices A by the Gauss-Jordan method starting with $[\,A \quad I\,]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

28 Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) A matrix with 1's down the main diagonal is invertible.

(c) If A is invertible then A^{-1} is invertible.

(d) If A is invertible then A^2 is invertible.

30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A \quad I\,]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

33 Use the 4 by 4 inverse in Question 32 to solve $A\boldsymbol{x} = (1, 1, 1, 1)$.

34 Suppose P and Q have the same rows as I but in any order. Show that $P - Q$ is singular by solving $(P - Q)\boldsymbol{x} = \mathbf{0}$.

35 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

36 If an invertible matrix A commutes with C (this means $AC = CA$) show that A^{-1} commutes with C. If also B commutes with C, show that AB commutes with C. Translation: If $AC = CA$ and $BC = CB$ then $(AB)C = C(AB)$.

37 Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

38 In the worked example **2.5 B**, the triangular Pascal matrix A has an inverse with "alternating diagonals". Check that this A^{-1} is DAD, where the diagonal matrix D has alternating entries $1, -1, 1, -1$. Then $ADAD = I$, so what is the inverse of $AD =$ pascal (4,1)?

39 The Hilbert matrices have $H_{ij} = 1/(i + j - 1)$. Ask MATLAB for the exact 6 by 6 inverse invhilb(6). Then ask for inv(hilb(6)). How can these be different, when the computer never makes mistakes?

40 Use inv(S) to invert MATLAB's 4 by 4 symmetric matrix S = pascal(4). Create Pascal's lower triangular A = abs(pascal(4,1)) and test inv(S) = inv(A') * inv(A).

41 If $A =$ ones(4,4) and $\boldsymbol{b} =$ rand(4,1), how does MATLAB tell you that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution? If $\boldsymbol{b} =$ ones(4,1), which solution to $A\boldsymbol{x} = \boldsymbol{b}$ is found by $A\backslash\boldsymbol{b}$?

42 If $AC = I$ and $AC^* = I$ (all square matrices) use **2I** to prove that $C = C^*$.

43 Direct multiplication gives $MM^{-1} = I$, and I would recommend doing *#3*. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted from A:

1	$M = I - \boldsymbol{uv}$	and	$M^{-1} = I + \boldsymbol{uv}/(1 - \boldsymbol{vu})$
2	$M = A - \boldsymbol{uv}$	and	$M^{-1} = A^{-1} + A^{-1}\boldsymbol{uv}A^{-1}/(1 - \boldsymbol{v}A^{-1}\boldsymbol{u})$
3	$M = I - UV$	and	$M^{-1} = I_n + U(I_m - VU)^{-1}V$
4	$M = A - UW^{-1}V$	and	$M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$

The Woodbury-Morrison formula **4** is the "matrix inversion lemma" in engineering. The four identities come from the 1, 1 block when inverting these matrices ($\boldsymbol{v}$ is 1 by n, $\boldsymbol{u}$ is n by 1, V is m by n, U is n by m, $m \le n$):

$$\begin{bmatrix} I & \boldsymbol{u} \\ \boldsymbol{v} & 1 \end{bmatrix} \qquad \begin{bmatrix} A & \boldsymbol{u} \\ \boldsymbol{v} & 1 \end{bmatrix} \qquad \begin{bmatrix} I_n & U \\ V & I_m \end{bmatrix} \qquad \begin{bmatrix} A & U \\ V & W \end{bmatrix}$$

ELIMINATION = FACTORIZATION: $A = LU$ ■ 2.6

Students often say that mathematics courses are too theoretical. Well, not this section. It is almost purely practical. The goal is to describe Gaussian elimination in the most useful way. Many key ideas of linear algebra, when you look at them closely, are really *factorizations* of a matrix. The original matrix A becomes the product of two or three special matrices. The first factorization—also the most important in practice—comes now from elimination. ***The factors are triangular matrices. The factorization that comes from elimination is $A = LU$.***

We already know U, the upper triangular matrix with the pivots on its diagonal. The elimination steps take A to U. We will show how reversing those steps (taking U back to A) is achieved by a lower triangular L. ***The entries of L are exactly the multipliers*** ℓ_{ij}—which multiplied row j when it was subtracted from row i.

Start with a 2 by 2 example. The matrix A contains 2, 1, 6, 8. The number to eliminate is 6. ***Subtract 3 times row*** 1 ***from row*** 2. That step is E_{21} in the forward direction. The return step from U to A is $L = E_{21}^{-1}$ (an addition using +3):

$$\textbf{\textit{Forward from } } A \textbf{\textit{ to }} U: \quad E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$\textbf{\textit{Back from }} U \textbf{\textit{ to }} A: \quad E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A.$$

The second line is our factorization. Instead of $E_{21}^{-1}U = A$ we write $LU = A$. Move now to larger matrices with many E's. ***Then L will include all their inverses.***

Each step from A to U multiplies by a matrix E_{ij} to produce zero in the (i, j) position. To keep this clear, we stay with the most frequent case—***when no row exchanges are involved.*** If A is 3 by 3, we multiply by E_{21} and E_{31} and E_{32}. The multipliers ℓ_{ij} produce zeros in the (2, 1) and (3, 1) and (3, 2) positions—all below the diagonal. Elimination ends with the upper triangular U.

Now move those E's onto the other side, *where their inverses multiply* U:

$$(E_{32}E_{31}E_{21})A = U \quad \textbf{becomes} \quad A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U \quad \textbf{which is} \quad A = LU. \quad (1)$$

The inverses go in opposite order, as they must. That product of three inverses is L. ***We have reached $A = LU$.*** Now we stop to understand it.

Explanation and Examples

First point: Every inverse matrix E_{ij}^{-1} is *lower triangular*. Its off-diagonal entry is ℓ_{ij}, to undo the subtraction with $-\ell_{ij}$. The main diagonals of E and E^{-1} contain 1's. Our example above had $\ell_{21} = 3$ and $E = \left[\begin{smallmatrix} 1 & 0 \\ -3 & 1 \end{smallmatrix}\right]$ and $E^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]$.

Second point: Equation (1) shows a lower triangular matrix (the product of E_{ij}) multiplying A. It also shows a lower triangular matrix (the product of E_{ij}^{-1}) multiplying U to bring back A. ***This product of inverses is L.***

One reason for working with the inverses is that we want to factor A, not U. The "inverse form" gives $A = LU$. The second reason is that we get something extra, almost more than we deserve. This is the third point, showing that L is exactly right.

Third point: Each multiplier ℓ_{ij} goes directly into its i, j position—*unchanged*—in the product of inverses which is L. Usually matrix multiplication will mix up all the numbers. Here that doesn't happen. The order is right for the inverse matrices, to keep the ℓ's unchanged. The reason is given below in equation (3).

Since each E^{-1} has 1's down its diagonal, the final good point is that L does too.

2J ($A = LU$) ***This is elimination without row exchanges.*** The upper triangular U has the pivots on its diagonal. The lower triangular L has all 1's on its diagonal. ***The multipliers ℓ_{ij} are below the diagonal of L.***

Example 1 The matrix A has $1, 2, 1$ on its diagonals. Elimination subtracts $\frac{1}{2}$ times row 1 from row 2. The last step subtracts $\frac{2}{3}$ times row 2 from row 3. The lower triangular L has $\ell_{21} = \frac{1}{2}$ and $\ell_{32} = \frac{2}{3}$. Multiplying LU produces A:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU.$$

The $(3, 1)$ multiplier is zero because the $(3, 1)$ entry in A is zero. No operation needed.

Example 2 Change the top left entry from 2 to 1. The pivots all become 1. The multipliers are all 1. That pattern continues when A is 4 by 4:

$$A = \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}.$$

These LU examples are showing something extra, which is very important in practice. Assume no row exchanges. When can we predict *zeros* in L and U?

When a row of A starts with zeros, so does that row of L.

When a column of A starts with zeros, so does that column of U.

If a row starts with zero, we don't need an elimination step. L has a zero, which saves computer time. Similarly, zeros at the *start* of a column survive into U. But please realize: Zeros in the *middle* of a matrix are likely to be filled in, while elimination sweeps forward. We now explain why L has the multipliers ℓ_{ij} in position, with no mix-up.

The key reason why A equals LU: Ask yourself about the pivot rows that are subtracted from lower rows. Are they the original rows of A? *No*, elimination probably changed them. Are they rows of U? *Yes*, the pivot rows never change again. When computing the third row of U, we subtract multiples of earlier rows of U (*not rows of* A!):

$$\text{Row 3 of } U = (\text{Row 3 of } A) - \ell_{31}(\text{Row 1 of } U) - \ell_{32}(\text{Row 2 of } U). \tag{2}$$

Rewrite this equation to see that the row $[\,\ell_{31} \quad \ell_{32} \quad 1\,]$ is multiplying U:

$$(\text{Row 3 of } A) = \ell_{31}(\text{Row 1 of } U) + \ell_{32}(\text{Row 2 of } U) + 1(\text{Row 3 of } U). \tag{3}$$

This is exactly row 3 ***of*** $A = LU$. All rows look like this, whatever the size of A. With no row exchanges, we have $A = LU$.

Remark The LU factorization is "unsymmetric" because U has the pivots on its diagonal where L has 1's. This is easy to change. ***Divide U by a diagonal matrix D that contains the pivots.*** That leaves a new matrix with 1's on the diagonal:

$$\text{Split } U \text{ into } \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}.$$

It is convenient (but a little confusing) to keep the same letter U for this new upper triangular matrix. It has 1's on the diagonal (like L). Instead of the normal LU, the new form has D in the middle: ***Lower triangular L times diagonal D times upper triangular U.***

The triangular factorization can be written $A = LU$ or $A = LDU$.

Whenever you see LDU, it is understood that U has 1's on the diagonal. *Each row is divided by its first nonzero entry—the pivot.* Then L and U are treated evenly in LDU:

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} \quad \text{splits further into} \quad \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & \\ & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

The pivots 2 and 5 went into D. Dividing the rows by 2 and 5 left the rows $[\,1 \quad 4\,]$ and $[\,0 \quad 1\,]$ in the new U. The multiplier 3 is still in L.

My own lectures sometimes stop at this point. The next paragraphs show how elimination codes are organized, and how long they take. If MATLAB (or any software) is available, I strongly recommend the last problems 32 to 35. You can measure the computing time by just counting the seconds!

One Square System = Two Triangular Systems

The matrix L contains our memory of Gaussian elimination. It holds the numbers that multiplied the pivot rows, before subtracting them from lower rows. When do we need this record and how do we use it?

We need L as soon as there is a *right side* $\boldsymbol{b}$. The factors L and U were completely decided by the left side (the matrix A). On the right side of $A\boldsymbol{x} = \boldsymbol{b}$, we use ***Solve***:

1 ***Factor*** (into L and U, by forward elimination on A)

2 ***Solve*** (forward elimination on $\boldsymbol{b}$ using L, then back substitution using U).

Earlier, we worked on $\boldsymbol{b}$ while we were working on A. No problem with that—just augment A by an extra column $\boldsymbol{b}$. But most computer codes keep the two sides separate. The memory of forward elimination is held in L and U, at no extra cost in storage. Then we process $\boldsymbol{b}$ whenever we want to. The User's Guide to LINPACK remarks that "This situation is so common and the savings are so important that no provision has been made for solving a single system with just one subroutine."

How does ***Solve*** work on $\boldsymbol{b}$? First, apply forward elimination to the right side (the multipliers are stored in L, use them now). This changes $\boldsymbol{b}$ to a new right side $\boldsymbol{c}$—*we are really solving* $L\boldsymbol{c} = \boldsymbol{b}$. Then back substitution solves $U\boldsymbol{x} = \boldsymbol{c}$ as always. The original system $A\boldsymbol{x} = \boldsymbol{b}$ is factored into ***two triangular systems***:

$$\textbf{\textit{Solve}} \quad L\boldsymbol{c} = \boldsymbol{b} \quad \textbf{\textit{and then solve}} \quad U\boldsymbol{x} = \boldsymbol{c}\,. \tag{5}$$

To see that $\boldsymbol{x}$ is correct, multiply $U\boldsymbol{x} = \boldsymbol{c}$ by L. Then $LU\boldsymbol{x} = L\boldsymbol{c}$ is just $A\boldsymbol{x} = \boldsymbol{b}$.

To emphasize: There is *nothing new* about those steps. This is exactly what we have done all along. We were really solving the triangular system $L\boldsymbol{c} = \boldsymbol{b}$ as elimination went forward. Then back substitution produced $\boldsymbol{x}$. An example shows it all.

Example 3 Forward elimination on $A\boldsymbol{x} = \boldsymbol{b}$ ends at $U\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{aligned} u + 2v &= 5 \\ 4u + 9v &= 21 \end{aligned} \qquad \text{becomes} \qquad \begin{aligned} u + 2v &= 5 \\ v &= 1\,. \end{aligned}$$

The multiplier was 4, which is saved in L. The right side used it to find $\boldsymbol{c}$:

$$L\boldsymbol{c} = \boldsymbol{b} \text{ The lower triangular system } \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \text{ gives } \boldsymbol{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

$$U\boldsymbol{x} = \boldsymbol{c} \text{ The upper triangular system } \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

It is satisfying that L and U can take the n^2 storage locations that originally held A. The ℓ's go below the diagonal. The whole discussion is only looking to see what elimination actually did.

The Cost of Elimination

A very practical question is cost—or computing time. Can we solve 1000 equations on a PC? What if $n = 10,000$? Large systems come up all the time in scientific computing, where a three-dimensional problem can easily lead to a million unknowns. We can let the calculation run overnight, but we can't leave it for 100 years.

The first stage of elimination, on column 1, produces zeros below the first pivot. To find each new entry below the pivot row requires one multiplication and one subtraction. *We will count this first stage as n^2 multiplications and n^2 subtractions.* It is actually less, $n^2 - n$, because row 1 does not change.

The next stage clears out the second column below the second pivot. The working matrix is now of size $n - 1$. Estimate this stage by $(n-1)^2$ multiplications and subtractions. The matrices are getting smaller as elimination goes forward. The rough count to reach U is the sum of squares $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$.

There is an exact formula $\frac{1}{3}n(n+\frac{1}{2})(n+1)$ for this sum of squares. When n is large, the $\frac{1}{2}$ and the 1 are not important. *The number that matters is $\frac{1}{3}n^3$.* The sum of squares is like the integral of x^2! The integral from 0 to n is $\frac{1}{3}n^3$:

Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.

What about the right side $\boldsymbol{b}$? Going forward, we subtract multiples of b_1 from the lower components $b_2, \ldots, b_n$. This is $n - 1$ steps. The second stage takes only $n - 2$ steps, because b_1 is not involved. The last stage of forward elimination takes one step.

Now start back substitution. Computing x_n uses one step (divide by the last pivot). The next unknown uses two steps. When we reach x_1 it will require n steps ($n - 1$ substitutions of the other unknowns, then division by the first pivot). The total count on the right side, from $\boldsymbol{b}$ to $\boldsymbol{c}$ to $\boldsymbol{x}$—*forward to the bottom and back to the top*—is exactly n^2:

$$[(n-1) + (n-2) + \cdots + 1] \; + \; [1 + 2 + \cdots + (n-1) + n] = n^2. \qquad (6)$$

To see that sum, pair off $(n-1)$ with 1 and $(n-2)$ with 2. The pairings leave n terms, each equal to n. That makes n^2. The right side costs a lot less than the left side!

Each right side needs n^2 multiplications and n^2 subtractions.

Here are the MATLAB codes to factor A into LU and to solve $A\boldsymbol{x} = \boldsymbol{b}$. The program **slu** stops right away if a number smaller than the tolerance "*tol*" appears in a pivot position. Later the program **plu** will look down the column for a pivot, to execute a row exchange and continue solving. These Teaching Codes are on **web.mit.edu/18.06/www**.

```
function [L, U] = slu(A)
%     Square LU factorization with no row exchanges!
[n, n] = size(A);   tol = 1.e - 6;
for k = 1 : n
  if abs(A(k, k)) < tol
  end      % Cannot proceed without a row exchange: stop
  L(k, k) = 1;
  for i = k + 1 : n   % Multipliers for column k are put into L
    L(i, k) = A(i, k)/A(k, k);
    for j = k + 1 : n   % Elimination beyond row k and column k
      A(i, j) = A(i, j) - L(i, k) * A(k, j);   % Matrix still called A
    end
  end
  for j = k : n
    U(k, j) = A(k, j);      % row k is settled, now name it U
  end
end
```



```
function x = slv(A, b)
%     Solve Ax = b using L and U from slu(A). No row exchanges!
[L, U] = slu(A);
for k = 1 : n
  for j = 1 : k - 1
    s = s + L(k, j) * c(j);
  end
  c(k) = b(k) - s;   % Forward elimination to solve Lc = b
end
for k = n : -1 : 1   % Going backwards from x(n) to x(1)
  for j = k + 1 : n   % Back substitution
    t = t + U(k, j) * x(j);
  end
  x(k) = (c(k) - t)/U(k, k);   % Divide by pivot
end
x = x';   % Transpose to column vector
```

How long does it take to solve $Ax = b$? For a random matrix of order $n = 1000$, we tried the MATLAB command tic; $A\backslash b$; toc. The time on my PC was 3 seconds. For $n = 2000$ the time was 20 seconds, which is approaching the n^3 rule. The time is multiplied by about 8 when n is multiplied by 2.

According to this n^3 rule, matrices that are 10 times as large (order 10,000) will take thousands of seconds. Matrices of order 100,000 will take millions of seconds.

This is too expensive without a supercomputer, but remember that these matrices are full. Most matrices in practice are sparse (many zero entries). In that case $A = LU$ is much faster. For tridiagonal matrices of order 10,000, storing only the nonzeros, solving $A\boldsymbol{x} = \boldsymbol{b}$ is a breeze.

■ REVIEW OF THE KEY IDEAS ■

1. Gaussian elimination (with no row exchanges) factors A into L times U.
2. The lower triangular L contains the numbers that multiply pivot rows, going from A to U. The product LU adds those rows back to recover A.
3. On the right side we solve $L\boldsymbol{c} = \boldsymbol{b}$ (forward) and $U\boldsymbol{x} = \boldsymbol{c}$ (backwards).
4. There are $\frac{1}{3}(n^3 - n)$ multiplications and subtractions on the left side.
5. There are n^2 multiplications and subtractions on the right side.

■ WORKED EXAMPLES ■

2.6 A The lower triangular Pascal matrix P_L was in the worked example **2.5 B**. (It contains the "*Pascal triangle*" and Gauss-Jordan found its inverse.) This problem connects P_L to the *symmetric* Pascal matrix P_S and the upper triangular P_U. The symmetric P_S has Pascal's triangle tilted, so each entry is the sum of the entry above and the entry to the left. The n by n symmetric P_S is pascal(n) in MATLAB.

Problem: *Establish the amazing lower-upper factorization* $P_S = P_L P_U$:

$$\text{pascal(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_L P_U.$$

Then predict and check the next row and column for 5 by 5 Pascal matrices.

Solution You could multiply $P_L P_U$ to get P_S. Better to start with the symmetric P_S and reach the upper triangular P_U by elimination:

$$P_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_U.$$

The multipliers ℓ_{ij} that entered these steps go perfectly into P_L. Then $P_S = P_L P_U$ is a particularly neat example of $A = LU$. *Notice that every pivot is* 1! The pivots are on the diagonal of P_U. The next section will show how symmetry produces a special relationship between the triangular L and U. You see P_U as the "transpose" of P_L.

You might expect the MATLAB command lu(pascal(4)) to produce these factors P_L and P_U. That doesn't happen because the **lu** subroutine chooses the largest available pivot in each column (it will exchange rows so the second pivot is 3). But a different command chol factors without row exchanges. Then $[L, U] = $ chol(pascal(4)) produces the triangular Pascal matrices as L and U. Try it.

In the 5 by 5 case the new fifth rows do maintain $P_S = P_L P_U$:

Next Row $\quad 1 \quad 5 \quad 15 \quad 35 \quad 70$ for P_S $\qquad 1 \quad 4 \quad 6 \quad 4 \quad 1$ for P_L

I will only check that this fifth row of P_L times the (same) fifth column of P_U gives $1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$ in the fifth row of P_S. The full proof of $P_S = P_L P_U$ is quite fascinating—this factorization can be reached in at least four different ways. I am going to put these proofs on the course web page **web.mit.edu/18.06/www**, which is also available through MIT's *OpenCourseWare* at **ocw.mit.edu**.

These Pascal matrices P_S, P_L, P_U have so many remarkable properties—we will see them again. You could locate them using the Index at the end of the book.

2.6 B The problem is: *Solve* $P_S \boldsymbol{x} = \boldsymbol{b} = (1, 0, 0, 0)$. This special right side means that $\boldsymbol{x}$ will be the first column of P_S^{-1}. That is Gauss-Jordan, matching the columns of $P_S P_S^{-1} = I$. We already know the triangular P_L and P_U from **2.6 A**, so we solve

$$P_L \boldsymbol{c} = \boldsymbol{b} \text{ (forward substitution)} \qquad P_U \boldsymbol{x} = \boldsymbol{c} \text{ (back substitution).}$$

Use MATLAB to find the full inverse matrix P_S^{-1}.

Solution The lower triangular system $P_L \boldsymbol{c} = \boldsymbol{b}$ is solved *top to bottom*:

$$\begin{array}{llcl} c_1 & = 1 & & c_1 = +1 \\ c_1 + c_2 & = 0 & \text{gives} & c_2 = -1 \\ c_1 + 2c_2 + c_3 & = 0 & & c_3 = +1 \\ c_1 + 3c_2 + 3c_3 + c_4 & = 0 & & c_4 = -1 \end{array}$$

Forward elimination is multiplication by P_L^{-1}. It produces the upper triangular system $P_U \boldsymbol{x} = \boldsymbol{c}$. The solution $\boldsymbol{x}$ comes as always by back substitution, *bottom to top*:

$$\begin{array}{rcl} x_1 + x_2 + x_3 + x_4 = 1 & & x_1 = +4 \\ x_2 + 2x_3 + 3x_4 = -1 & \text{gives} & x_2 = -6 \\ x_3 + 3x_4 = 1 & & x_3 = +4 \\ x_4 = -1 & & x_4 = -1 \end{array}$$

The complete inverse matrix P_S^{-1} has that $\boldsymbol{x}$ in its first column:

$$\text{inv(pascal(4))} = \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Problem Set 2.6

Problems 1–14 compute the factorization $A = LU$ (and also $A = LDU$).

1 (Important) Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b}$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{c}$:

$$\begin{aligned} x + \ y &= 5 \\ x + 2y &= 7 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x + \ y &= 5 \\ y &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 1 & 2 & \mathbf{7} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 0 & 1 & \mathbf{2} \end{bmatrix}$$

That step subtracted $\ell_{21} =$ ____ times row 1 from row 2. The reverse step *adds* ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L =$ ____. Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to get ____ $=$ ____. In letters, L multiplies $U\boldsymbol{x} = \boldsymbol{c}$ to give ____.

2 (Move to 3 by 3) Forward elimination changes $A\boldsymbol{x} = \boldsymbol{b}$ to a triangular $U\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{aligned} x + \ y + \ z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned} \qquad \begin{aligned} x + \ y + \ z &= 5 \\ y + 2z &= 2 \\ 2y + 5z &= 6 \end{aligned} \qquad \begin{aligned} x + \ y + \ z &= 5 \\ y + 2z &= 2 \\ z &= 2 \end{aligned}$$

The equation $z = 2$ in $U\boldsymbol{x} = \boldsymbol{c}$ comes from the original $x + 3y + 6z = 11$ in $A\boldsymbol{x} = \boldsymbol{b}$ by subtracting $\ell_{31} =$ ____ times equation 1 and $\ell_{32} =$ ____ times the *final* equation 2. Reverse that to recover $[\,1 \ \ 3 \ \ 6 \ \ 11\,]$ in A and $\boldsymbol{b}$ from the final $[\,1 \ \ 1 \ \ 1 \ \ 5\,]$ and $[\,0 \ \ 1 \ \ 2 \ \ 2\,]$ and $[\,0 \ \ 0 \ \ 1 \ \ 2\,]$ in U and $\boldsymbol{c}$:

$$\text{Row 3 of } \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 + 1 \text{ Row } 3) \text{ of } \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $\boldsymbol{b} = L\boldsymbol{c}$.

3 Write down the 2 by 2 triangular systems $L\boldsymbol{c} = \boldsymbol{b}$ and $U\boldsymbol{x} = \boldsymbol{c}$ from Problem 1. Check that $\boldsymbol{c} = (5, 2)$ solves the first one. Find $\boldsymbol{x}$ that solves the second one.

4 What are the 3 by 3 triangular systems $L\boldsymbol{c} = \boldsymbol{b}$ and $U\boldsymbol{x} = \boldsymbol{c}$ from Problem 2? Check that $\boldsymbol{c} = (5, 2, 2)$ solves the first one. Which $\boldsymbol{x}$ solves the second one?

5 What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

6 What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

7 What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

8 Suppose A is already lower triangular with 1's on the diagonal. Then $U = I$!

$$A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

The elimination matrices E_{21}, E_{31}, E_{32} contain $-a$ then $-b$ then $-c$.

(a) Multiply $E_{32}E_{31}E_{21}$ to find the single matrix E that produces $EA = I$.

(b) Multiply $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ to bring back L (nicer than E).

9 When zero appears in a pivot position, $A = LU$ *is not possible*! (We are requiring nonzero pivots in U.) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}.$$

This difficulty is fixed by a row exchange. That needs a "permutation" P.

10 Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

11 What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

12 A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

13 (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

14 Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

Problems 15-16 use L and U (without needing A) to solve $A\boldsymbol{x} = \boldsymbol{b}$.

15 Solve the triangular system $L\boldsymbol{c} = \boldsymbol{b}$ to find $\boldsymbol{c}$. Then solve $U\boldsymbol{x} = \boldsymbol{c}$ to find $\boldsymbol{x}$:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety find $A = LU$ and solve $A\boldsymbol{x} = \boldsymbol{b}$ as usual. Circle $\boldsymbol{c}$ when you see it.

16 Solve $L\boldsymbol{c} = \boldsymbol{b}$ to find $\boldsymbol{c}$. Then solve $U\boldsymbol{x} = \boldsymbol{c}$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

17 (a) When you apply the usual elimination steps to L, what matrix do you reach?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}.$$

(b) When you apply the same steps to I, what matrix do you get?

(c) When you apply the same steps to LU, what matrix do you get?

18 If $A = LDU$ and also $A = L_1D_1U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$. *"The factors are unique."*

Derive the equation $L_1^{-1}LD = D_1U_1U^{-1}$. Are the two sides triangular or diagonal? Deduce $L = L_1$ and $U = U_1$ (they all have diagonal 1's). Then $D = D_1$.

19 *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ and $A = LDL^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

20 When T is tridiagonal, its L and U factors have only two nonzero diagonals. How would you take advantage of the zeros in T in a computer code for Gaussian elimination? Find L and U.

$$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

21 If A and B have nonzeros in the positions marked by x, which zeros (marked by 0) are still zero in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

22 After elimination has produced zeros below the first pivot, put x's to show which blank entries are known in the final L and U:

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} & & \\ 0 & & \\ 0 & & \end{bmatrix}.$$

23 Suppose you eliminate upwards (almost unheard of). Use the last row to produce zeros in the last column (the pivot is 1). Then use the second row to produce zero above the second pivot. Find the factors in $A = UL$(!):

$$A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

24 Collins uses elimination in both directions, meeting at the center. Substitution goes out from the center. After eliminating both 2's in A, one from above and one from below, what 4 by 4 matrix is left? Solve $A\boldsymbol{x} = \boldsymbol{b}$ his way.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 8 \\ 8 \\ 2 \end{bmatrix}.$$

25 (Important) If A has pivots $2, 7, 6$ with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix B (without row 3 and column 3)? Explain why.

26 Starting from a 3 by 3 matrix A with pivots $2, 7, 6$, add a fourth row and column to produce M. What are the first three pivots for M, and why? What fourth row and column are sure to produce 9 as the fourth pivot?

27 Use chol(pascal(5)) to find the triangular Pascal factors as in Worked Example **2.6 A**. Show how row exchanges in $[L, U]$ = lu(pascal(5)) spoil Pascal's pattern!

28 (Careful review) For which numbers c is $A = LU$ impossible—with three pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

29 Change the program **slu**(A) into **sldu**(A), so that it produces L, D, and U. Put L, D, U into the n^2 storage locations that held the original A. The extra storage used for L is not required.

30 Explain in words why $x(k)$ is $(c(k) - t)/U(k,k)$ at the end of **slv**(A, b).

31 Write a program that multiplies a two-diagonal L times a two-diagonal U. Don't loop from 1 to n when you know there are zeros! L times U should undo **slu**.

32 I just learned MATLAB's tic–toc command, which measures computing time. Previously I counted seconds until the answer appeared, which required very large problems—now A = rand(1000) and $\boldsymbol{b}$ = rand(1000,1) is large enough. How much faster is tic; $A\backslash\boldsymbol{b}$; toc for elimination than tic; inv$(A) * \boldsymbol{b}$; toc which computes A^{-1}?

33 Compare tic; inv(A); toc for A = rand(500) and A = rand(1000). The n^3 operation count says that doubling n should multiply computing time by 8.

34 I = eye(1000); A = rand(1000); B = triu(A); produces a random *triangular* matrix B. Compare the times for inv(B) and $B \setminus I$. Backslash is engineered to use the zeros in B, while inv uses the zeros in I when reducing $[\,B \;\; I\,]$ by Gauss-Jordan. (Compare also with inv(A) and $A \setminus I$ for the full matrix A.)

35 Estimate the time difference for each new right side $\boldsymbol{b}$ when $n = 800$. Create A = rand(800) and $\boldsymbol{b}$ = rand(800,1) and B = rand(800,9). Compare tic; $A\backslash\boldsymbol{b}$; toc and tic; $A\backslash B$; toc (which solves for 9 right sides).

36 Show that L^{-1} has entries j/i on and below its main diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}.$$

I think this pattern continues for $L = \text{eye}(5) - \text{diag}(1{:}5)\backslash\text{diag}(1{:}4, -1)$ and $\text{inv}(L)$.

TRANSPOSES AND PERMUTATIONS ■ 2.7

We need one more matrix, and fortunately it is much simpler than the inverse. It is the ***"transpose"*** of A, which is denoted by A^{T}. *The columns of* A^{T} *are the rows of* A.

When A is an m by n matrix, the transpose is n by m:

$$\text{If} \quad A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 0 & 0 & 4 \end{bmatrix} \quad \text{then} \quad A^{\mathrm{T}} = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{2} & 0 \\ \mathbf{3} & 4 \end{bmatrix}.$$

You can write the rows of A into the columns of A^{T}. Or you can write the columns of A into the rows of A^{T}. The matrix "flips over" its main diagonal. The entry in row i, column j of A^{T} comes from row j, column i of the original A:

$$\mathbf{(A^{T})_{ij} = A_{ji}.}$$

The transpose of a lower triangular matrix is upper triangular. (But the inverse is still lower triangular.) The transpose of A^{T} is A.

Note MATLAB's symbol for the transpose of A is A'. Typing [1 2 3] gives a row vector and the column vector is $\boldsymbol{v} = [\,1 \;\; 2 \;\; 3\,]'$. To enter a matrix M with second column $\boldsymbol{w} = [\ 4\ 5\ 6\ ]'$ you could define $M = [\ \boldsymbol{v}\ \boldsymbol{w}\ ]$. Quicker to enter by rows and then transpose the whole matrix: $M = [\,1 \;\; 2 \;\; 3;\;\; 4 \;\; 5 \;\; 6\,]'$.

The rules for transposes are very direct. We can transpose $A+B$ to get $(A+B)^{\mathrm{T}}$. Or we can transpose A and B separately, and then add $A^{\mathrm{T}} + B^{\mathrm{T}}$—same result. The serious questions are about the transpose of a product AB and an inverse A^{-1}:

$$\text{The transpose of} \quad A+B \quad \text{is} \quad A^{\mathrm{T}} + B^{\mathrm{T}}. \tag{1}$$

$$\textbf{The transpose of} \quad \mathbf{AB} \quad \textbf{is} \quad \mathbf{(AB)^{T} = B^{T}A^{T}.} \tag{2}$$

$$\text{The transpose of} \quad A^{-1} \quad \text{is} \quad (A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}. \tag{3}$$

Notice especially how $B^{\mathrm{T}}A^{\mathrm{T}}$ comes in reverse order. For inverses, this reverse order was quick to check: $B^{-1}A^{-1}$ times AB produces I. To understand $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, start with $(A\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$:

$$\boldsymbol{A\,x \text{ combines the columns of } A \text{ while } x^{\mathrm{T}}A^{\mathrm{T}} \text{ combines the rows of } A^{\mathrm{T}}.}$$

It is the same combination of the same vectors! In A they are columns, in A^{T} they are rows. So the transpose of the column $A\boldsymbol{x}$ is the row $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$. That fits our formula $(A\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$. Now we can prove the formula for $(AB)^{\mathrm{T}}$.

When $B = [\boldsymbol{x}_1\ \ \boldsymbol{x}_2]$ has two columns, apply the same idea to each column. The columns of AB are $A\boldsymbol{x}_1$ and $A\boldsymbol{x}_2$. Their transposes are the rows of $B^{\mathrm{T}}A^{\mathrm{T}}$:

$$\text{Transposing}\quad AB = \begin{bmatrix} & & \\ A\boldsymbol{x}_1 & A\boldsymbol{x}_2 & \cdots \\ & & \end{bmatrix} \quad\text{gives}\quad \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}}A^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}}A^{\mathrm{T}} \\ \vdots \end{bmatrix} \quad\text{which is}\quad B^{\mathrm{T}}A^{\mathrm{T}}. \tag{4}$$

The right answer $B^{\mathrm{T}}A^{\mathrm{T}}$ comes out a row at a time. There is also a "***transparent proof***" by looking through the page at the end of the problem set. Here are numbers!

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 9 & 1 \end{bmatrix} \quad\text{and}\quad B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 0 & 1 \end{bmatrix}.$$

The reverse order rule extends to three or more factors: $(ABC)^{\mathrm{T}}$ equals $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}$.

$$\textbf{\textit{If}}\ \ \boldsymbol{A = LDU}\ \ \textbf{\textit{then}}\ \ \boldsymbol{A^{\mathrm{T}} = U^{\mathrm{T}}D^{\mathrm{T}}L^{\mathrm{T}}}. \quad \textbf{\textit{The pivot matrix has}}\ \ \boldsymbol{D = D^{\mathrm{T}}}.$$

Now apply this product rule to both sides of $A^{-1}A = I$. On one side, I^{T} is I. We confirm the rule that $(A^{-1})^{\mathrm{T}}$ is the inverse of A^{T}:

$$A^{-1}A = I \quad\text{is transposed to}\quad A^{\mathrm{T}}(A^{-1})^{\mathrm{T}} = I. \tag{5}$$

Similarly $AA^{-1} = I$ leads to $(A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = I$. We can invert the transpose or transpose the inverse. Notice especially: A^{T} ***is invertible exactly when*** A ***is invertible***.

Example 1 The inverse of $A = \left[\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right]$ is $A^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ -6 & 1 \end{smallmatrix}\right]$. The transpose is $A^{\mathrm{T}} = \left[\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right]$.

$$(A^{-1})^{\mathrm{T}} \quad \textbf{\textit{and}} \quad (A^{\mathrm{T}})^{-1} \quad \textbf{\textit{are both equal to}} \quad \left[\begin{smallmatrix} 1 & -6 \\ 0 & 1 \end{smallmatrix}\right].$$

Before leaving these rules, we call attention to dot products. The following statement looks extremely simple, but it actually contains the deep purpose for the transpose. For any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} \quad\text{equals}\quad \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} \quad\text{equals}\quad \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}). \tag{6}$$

When A moves from one side of a dot product to the other side, it becomes A^{T}.

Here are two quick applications to electrical engineering and mechanical engineering (with more in Chapter 8). The same A and A^{T} appear in both applications.

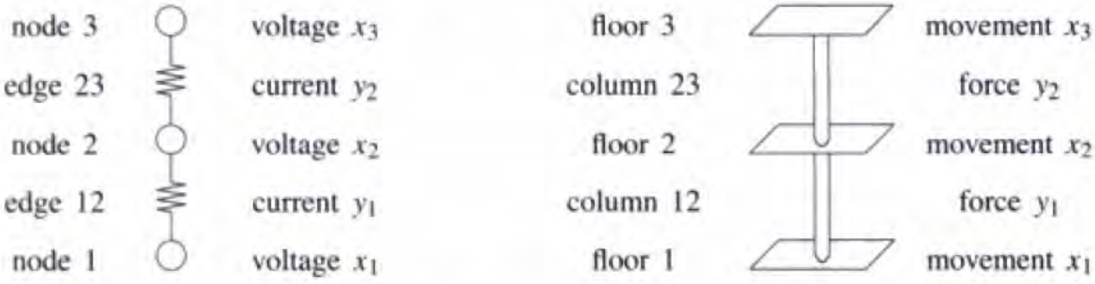


Figure 2.9 A line of resistors and a structure, both governed by A and A^{T}.

Electrical Networks The vector $\boldsymbol{x} = (x_1, x_2, x_3)$ gives voltages at 3 nodes, and $A\boldsymbol{x}$ gives the voltage differences across 2 edges. The "difference matrix" A is 2 by 3:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix} = \text{voltage differences.}$$

The vector $\boldsymbol{y} = (y_1, y_2)$ gives currents on those edges (node 1 to 2, and node 2 to 3). Look how $A^{\mathrm{T}}\boldsymbol{y}$ finds the total currents leaving each node in Kirchhoff's Current Law:

$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} \text{current leaving node 1} \\ \textbf{out minus in} \text{ at node 2} \\ \text{current leaving node 3} \end{bmatrix}.$$

Section 8.2 studies networks in detail. Here we look at the energy $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y}$ lost as heat:

Energy **(voltages $\boldsymbol{x}$) $\cdot$ (inputs $A^{\mathrm{T}}\boldsymbol{y}$)** = *Heat loss* **(voltage drops $A\boldsymbol{x}$) $\cdot$ (currents $\boldsymbol{y}$)**.

Forces on a Structure The vector $\boldsymbol{x} = (x_1, x_2, x_3)$ gives the movement of each floor under the weight of the floors above. The matrix A takes differences of the x's to give the strains $A\boldsymbol{x}$, the movements between floors:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} \text{movement between 1 and 2} \\ \text{movement between 2 and 3} \end{bmatrix}.$$

The vector $\boldsymbol{y} = (y_1, y_2)$ gives the stresses (internal forces from the columns that resist the movement and save the structure). Then $A^{\mathrm{T}}\boldsymbol{y}$ gives the forces that balance the weight:

$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - y_1 \\ -y_2 \end{bmatrix} \quad \text{balances} \quad \begin{bmatrix} \text{weight of floor 1} \\ \text{weight of floor 2} \\ \text{weight of floor 3} \end{bmatrix}$$

In resistors, the relation of $\boldsymbol{y}$ to $A\boldsymbol{x}$ is Ohm's Law (current proportional to voltage difference). For elastic structures this is Hooke's Law (stress proportional to strain). The catastrophe on September 11 came when the fires in the World Trade Center weakened the steel columns. Hooke's Law eventually failed. The internal forces couldn't balance the weight of the tower. After the first columns buckled, the columns below couldn't take the extra weight.

For a linearly elastic structure, the work balance equation is $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$:

Internal work (**strain** $A\boldsymbol{x}$) $\cdot$ (**stress** $\boldsymbol{y}$) $=$ *External work* (**movement** $\boldsymbol{x}$) $\cdot$ (**force** $A^{\mathrm{T}}\boldsymbol{y}$).

Symmetric Matrices

For a *symmetric* matrix—these are the most important matrices—transposing A to A^{T} produces no change. Then $A^{\mathrm{T}} = A$. The matrix is symmetric across the main diagonal. A symmetric matrix is necessarily square. Its (j, i) and (i, j) entries are equal.

DEFINITION A ***symmetric matrix*** has $A^{\mathrm{T}} = A$. This means that $a_{ji} = a_{ij}$.

Example 2 $$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^{\mathrm{T}} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = D^{\mathrm{T}}.$$

A is symmetric because of the 2's on opposite sides of the diagonal. The rows agree with the columns. In D those 2's are zeros. Every diagonal matrix is symmetric.

The inverse of a symmetric matrix is also symmetric. (We have to add: "*If A is invertible.*") The transpose of A^{-1} is $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = A^{-1}$, so A^{-1} is symmetric:

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Now we show that ***multiplying any matrix R by R^{T} gives a symmetric matrix.***

Symmetric Products $R^{\mathrm{T}}R$ and RR^{T} and LDL^{T}

Choose any matrix R, probably rectangular. Multiply R^{T} times R. Then the product $R^{\mathrm{T}}R$ is automatically a square symmetric matrix:

$$\textit{The transpose of} \quad R^{\mathrm{T}}R \quad \textit{is} \quad R^{\mathrm{T}}(R^{\mathrm{T}})^{\mathrm{T}} \quad \textit{which is} \quad R^{\mathrm{T}}R. \tag{7}$$

That is a quick proof of symmetry for $R^{\mathrm{T}}R$. We could also look at the (i, j) entry of $R^{\mathrm{T}}R$. It is the dot product of row i of R^{T} (column i of R) with column j of R. The (j, i) entry is the same dot product, column j with column i. So $R^{\mathrm{T}}R$ is symmetric.

The matrix RR^{T} is also symmetric. (The shapes of R and R^{T} allow multiplication.) But RR^{T} is a different matrix from $R^{\mathrm{T}}R$. In our experience, most scientific problems that start with a rectangular matrix R end up with $R^{\mathrm{T}}R$ or RR^{T} or both.

Example 3 $R = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $R^{\mathrm{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ produce $R^{\mathrm{T}}R = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $RR^{\mathrm{T}} = [\,5\,]$.

The product $R^{\mathrm{T}}R$ is n by n. In the opposite order, RR^{T} is m by m. Even if $m = n$, it is not very likely that $R^{\mathrm{T}}R = RR^{\mathrm{T}}$. Equality can happen, but it is abnormal.

When elimination is applied to a symmetric matrix, $A^{\mathrm{T}} = A$ is an advantage. The smaller matrices stay symmetric as elimination proceeds, and we can work with half the matrix! It is true that the upper triangular U cannot be symmetric. ***The symmetry is in** LDU*. Remember how the diagonal matrix D of pivots can be divided out, to leave 1's on the diagonal of both L and U:

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad (LU \text{ misses the symmetry})$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} (LDU \text{ captures the symmetry}) \\ \text{Now } U \textbf{\textit{ is the transpose of }} L. \end{matrix}$$

When A is symmetric, the usual form $A = LDU$ becomes $A = LDL^{\mathrm{T}}$. The final U (with 1's on the diagonal) is the transpose of L (also with 1's on the diagonal). The diagonal D—the matrix of pivots—is symmetric by itself.

2K If $A = A^{\mathrm{T}}$ can be factored into LDU with no row exchanges, then $U = L^{\mathrm{T}}$. *The symmetric factorization of a symmetric matrix is* $A = LDL^{\mathrm{T}}$.

Notice that the transpose of LDL^{T} is automatically $(L^{\mathrm{T}})^{\mathrm{T}}D^{\mathrm{T}}L^{\mathrm{T}}$ which is LDL^{T} again. The work of elimination is cut in half, from $n^3/3$ multiplications to $n^3/6$. The storage is also cut essentially in half. We only keep L and D, not U.

Permutation Matrices

The transpose plays a special role for a *permutation matrix*. This matrix P has a single "1" in every row and every column. Then P^{T} is also a permutation matrix—maybe the same or maybe different. Any product P_1P_2 is again a permutation matrix. We now create every P from the identity matrix, by reordering the rows of I.

The simplest permutation matrix is $P = I$ (*no exchanges*). The next simplest are the row exchanges P_{ij}. Those are constructed by exchanging two rows i and j of I. Other permutations reorder more rows. By doing all possible row exchanges to I, we get all possible permutation matrices:

DEFINITION *A permutation matrix P has the rows of I in any order.*

Example 4 There are six 3 by 3 permutation matrices. Here they are without the zeros:

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There are $n!$ permutation matrices of order n. The symbol $n!$ means "n factorial," the product of the numbers $(1)(2)\cdots(n)$. Thus $3! = (1)(2)(3)$ which is 6. There will be 24 permutation matrices of order $n = 4$. And 120 permutations of order 5.

There are only two permutation matrices of order 2, namely $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Important: P^{-1} is also a permutation matrix. Among the six 3 by 3 P's displayed above, the four matrices on the left are their own inverses. The two matrices on the right are inverses of each other. In all cases, a single row exchange is its own inverse. If we repeat the exchange we are back to I. But for $P_{32}P_{21}$, the inverses go in opposite order (of course). The inverse is $P_{21}P_{32}$.

More important: ***P^{-1} is always the same as P^{T}.*** The two matrices on the right are transposes—and inverses—of each other. When we multiply PP^{T}, the "1" in the first row of P hits the "1" in the first column of P^{T} (since the first row of P is the first column of P^{T}). It misses the ones in all the other columns. So $PP^{\mathrm{T}} = I$.

Another proof of $P^{\mathrm{T}} = P^{-1}$ looks at P as a product of row exchanges. A row exchange is its own transpose and its own inverse. P^{T} and P^{-1} both come from the product of row exchanges in the opposite order. So P^{T} and P^{-1} are the same.
Symmetric matrices led to $A = LDL^{\mathrm{T}}$. Now permutations lead to $PA = LU$.

The LU Factorization with Row Exchanges

We sure hope you remember $A = LU$. It started with $A = (E_{21}^{-1}\cdots E_{ij}^{-1}\cdots)U$. Every elimination step was carried out by an E_{ij} and it was inverted by E_{ij}^{-1}. Those inverses were compressed into one matrix L, bringing U back to A. The lower triangular L has 1's on the diagonal, and the result is $A = LU$.

This is a great factorization, but it doesn't always work! Sometimes row exchanges are needed to produce pivots. Then $A = (E^{-1}\cdots P^{-1}\cdots E^{-1}\cdots P^{-1}\cdots)U$. Every row exchange is carried out by a P_{ij} and inverted by that P_{ij}. We now compress those row exchanges into a *single permutation matrix* P. This gives a factorization for every invertible matrix A—which we naturally want.

The main question is where to collect the P_{ij}'s. There are two good possibilities—do all the exchanges before elimination, or do them after the E_{ij}'s. The first way gives $PA = LU$. The second way has a permutation matrix P_1 in the middle.

1. The row exchanges can be done *in advance*. Their product P puts the rows of A in the right order, so that no exchanges are needed for PA. ***Then $PA = LU$.***

2. If we hold row exchanges until *after elimination*, the pivot rows are in a strange order. P_1 puts them in the correct triangular order in U_1. ***Then*** $\boldsymbol{A = L_1 P_1 U_1}$.

$PA = LU$ is constantly used in almost all computing (and always in MATLAB). ***We will concentrate on this form*** $PA = LU$. The factorization $A = L_1 P_1 U_1$ might be more elegant. If we mention both, it is because the difference is not well known. Probably you will not spend a long time on either one. Please don't. The most important case has $P = I$, when A equals LU with no exchanges.

For this matrix A, exchange rows 1 and 2 to put the first pivot in its usual place. Then go through elimination on PA:

$$\underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \rightarrow \underset{PA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \rightarrow \underset{\ell_{31} = 2}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}} \rightarrow \underset{\ell_{32} = 3}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}.$$

The matrix PA is in good order, and it factors as usual into LU:

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU. \tag{8}$$

We started with A and ended with U. *The only requirement is invertibility of* A.

2L If A is invertible, a permutation P will put its rows in the right order to factor $\boxed{PA = LU}$. There must be a full set of pivots, after row exchanges.

In the MATLAB code, $A([r\ k], :) = A([k\ r], :)$ exchanges row k with row r below it (where the kth pivot has been found). Then we update L and P and the sign of P:

$$\begin{aligned} &A([r\ k], :) = A([k\ r], :); \\ &L([r\ k], 1:k-1) = L([k\ r], 1:k-1); \\ &P([r\ k], :) = P([k\ r], :); \\ &\text{sign} = -\text{sign} \end{aligned}$$

The "**sign**" of P tells whether the number of row exchanges is even (sign $= +1$) or odd (sign $= -1$). At the start, P is I and sign $= +1$. When there is a row exchange, the sign is reversed. The final value of sign is the **determinant of** P and it does not depend on the order of the row exchanges.

For PA we get back to the familiar LU. This is the usual factorization. In reality, MATLAB might not use the first available pivot. Mathematically we can accept a small pivot—anything but zero. It is better if the computer looks down the column for the largest pivot. (Section 9.1 explains why this "***partial pivoting***" reduces the roundoff error.) P may contain row exchanges that are not algebraically necessary. Still $PA = LU$.

Our advice is to understand permutations but let MATLAB do the computing. Calculations of $A = LU$ are enough to do by hand, without P. The Teaching Code **splu** (A) factors $PA = LU$ and **splv** (A, b) solves $Ax = \boldsymbol{b}$ for any invertible A. The program **splu** stops if no pivot can be found in column k. That fact is printed.

■ REVIEW OF THE KEY IDEAS ■

1. The transpose puts the rows of A into the columns of A^{T}. Then $(A^{\mathrm{T}})_{ij} = A_{ji}$.
2. The transpose of AB is $B^{\mathrm{T}}A^{\mathrm{T}}$. The transpose of A^{-1} is the inverse of A^{T}.
3. The dot product $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y}$ equals the dot product $\boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$.
4. When A is symmetric ($A^{\mathrm{T}} = A$), its LDU factorization is symmetric: $A = LDL^{\mathrm{T}}$.
5. A permutation matrix P has a 1 in each row and column, and $P^{\mathrm{T}} = P^{-1}$.
6. If A is invertible then a permutation P will reorder its rows for $PA = LU$.

■ WORKED EXAMPLES ■

2.7 A Applying the permutation P to the rows of A destroys its symmetry:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \qquad PA = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

What permutation matrix Q applied to the columns of PA will recover symmetry in PAQ? The numbers $1, 2, 3$ must come back to the main diagonal (not necessarily in order). How is Q related to P, when symmetry is saved by PAQ?

Solution To recover symmetry and put "2" on the diagonal, column 2 of PA must move to column 1. Column 3 of PA (containing "3") must move to column 2. Then the "1" moves to the $3, 3$ position. The matrix that permutes columns is Q:

$$PA = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad PAQ = \begin{bmatrix} 2 & 6 & 4 \\ 6 & 3 & 5 \\ 4 & 5 & 1 \end{bmatrix} \text{ is symmetric.}$$

The matrix Q is P^{T}. This choice always recovers symmetry, because PAP^{T} is guaranteed to be symmetric. (Its transpose is again PAP^{T}.) *The matrix Q is also P^{-1}, because the inverse of every permutation matrix is its transpose.*

If we look only at the main diagonal D of A, we are finding that PDP^{T} is guaranteed diagonal. When P moves row 1 down to row 3, P^{T} on the right will move column 1 to column 3. The $(1,1)$ entry moves down to $(3,1)$ and over to $(3,3)$.

2.7 B Find the symmetric factorization $A = LDL^{\mathrm{T}}$ for the matrix A above. Is A invertible? Find also the $PQ = LU$ factorization for Q, which needs row exchanges.

Solution To factor A into LDL^{T} we eliminate below the pivots:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & -14 & -22 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} = U.$$

The multipliers were $\ell_{21} = 4$ and $\ell_{31} = 5$ and $\ell_{32} = 1$. The pivots $1, -14, -8$ go into D. When we divide the rows of U by those pivots, L^{T} should appear:

$$A = LDL^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -14 & \\ & & -8 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix A is invertible because *it has three pivots*. Its inverse is $(L^{\mathrm{T}})^{-1}D^{-1}L^{-1}$ and it is also symmetric. The numbers 14 and 8 will turn up in the denominators of A^{-1}. The "determinant" of A is the product of the pivots $(1)(-14)(-8) = 112$.

The matrix Q is certainly invertible. But elimination needs two row exchanges:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[1 \leftrightarrow 2]{\text{rows}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[2 \leftrightarrow 3]{\text{rows}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Then $L = I$ and $U = I$ are the LU factors. We only need the permutation P that put the rows of Q into their right order in I. Well, P must be Q^{-1}. It is the same P as above! We could find it as a product of the two row exchanges, $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$:

$$P = P_{23}P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{reorders } Q \text{ into } PQ = I.$$

Problem Set 2.7

Questions 1–7 are about the rules for transpose matrices.

1 Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

2 Verify that $(AB)^{\mathrm{T}}$ equals $B^{\mathrm{T}}A^{\mathrm{T}}$ but those are different from $A^{\mathrm{T}}B^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

In case $AB = BA$ (not generally true!) how do you prove that $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$?

3 (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order?*

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is _____ triangular.

4 Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matrix).

5 (a) The row vector $\boldsymbol{x}^{\mathrm{T}}$ times A times the column $\boldsymbol{y}$ produces what number?

$$\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\qquad}.$$

(b) This is the row $\boldsymbol{x}^{\mathrm{T}}A =$ _____ times the column $\boldsymbol{y} = (0, 1, 0)$.

(c) This is the row $\boldsymbol{x}^{\mathrm{T}} = [0 \;\; 1]$ times the column $A\boldsymbol{y} =$ _____.

6 When you transpose a block matrix $M = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ the result is $M^{\mathrm{T}} =$ _____. Test it. Under what conditions on A, B, C, D is the block matrix symmetric?

7 True or false:

(a) The block matrix $\left[\begin{smallmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{smallmatrix}\right]$ is automatically symmetric.

(b) If A and B are symmetric then their product AB is symmetric.

(c) If A is not symmetric then A^{-1} is not symmetric.

(d) When A, B, C are symmetric, the transpose of ABC is CBA.

Questions 8–15 are about permutation matrices.

8 Why are there $n!$ permutation matrices of order n?

9 If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

10 There are 12 "*even*" permutations of $(1, 2, 3, 4)$, with an *even number of exchanges*. Two of them are $(1, 2, 3, 4)$ with no exchanges and $(4, 3, 2, 1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, use the numbers 4, 3, 2, 1 to give the position of the 1 in each row.

11 (Try this question) Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? ***Multiplying A on the right by P_2 exchanges the _____ of A.***

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

12 Explain why the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$ equals the dot product of $P\boldsymbol{x}$ and $P\boldsymbol{y}$. Then from $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ deduce that $P^{\mathrm{T}}P = I$ for any permutation. With $\boldsymbol{x} = (1, 2, 3)$ and $\boldsymbol{y} = (1, 4, 2)$ choose P to show that $P\boldsymbol{x} \cdot \boldsymbol{y}$ is not always equal to $\boldsymbol{x} \cdot P\boldsymbol{y}$.

13 Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

14 If you take powers of a permutation matrix, why is some P^k eventually equal to I?

Find a 5 by 5 permutation P so that the smallest power to equal I is P^6. (This is a challenge question. Combine a 2 by 2 block with a 3 by 3 block.)

15 Row exchange matrices are symmetric: $P^{\mathrm{T}} = P$. Then $P^{\mathrm{T}}P = I$ becomes $P^2 = I$. Some other permutation matrices are also symmetric.

(a) If P sends row 1 to row 4, then P^{T} sends row _____ to row _____. When $P^{\mathrm{T}} = P$ the row exchanges come in pairs with no overlap.

(b) Find a 4 by 4 example with $P^{\mathrm{T}} = P$ that moves all four rows.

Questions 16–21 are about symmetric matrices and their factorizations.

16 If $A = A^{\mathrm{T}}$ and $B = B^{\mathrm{T}}$, which of these matrices are certainly symmetric?

(a) $A^2 - B^2$

(b) $(A + B)(A - B)$

(c) ABA

(d) $ABAB$.

17 Find 2 by 2 symmetric matrices $A = A^{\mathrm{T}}$ with these properties:

(a) A is not invertible.

(b) A is invertible but cannot be factored into LU (row exchanges needed).

(c) A can be factored into LDL^{T} but not into LL^{T} (because of negative D).

18 (a) How many entries of A can be chosen independently, if $A = A^{\mathrm{T}}$ is 5 by 5?

(b) How do L and D (still 5 by 5) give the same number of choices?

(c) How many entries can be chosen if A is *skew-symmetric*? ($A^{\mathrm{T}} = -A$).

19 Suppose R is rectangular (m by n) and A is symmetric (m by m).

(a) Transpose $R^{\mathrm{T}}AR$ to show its symmetry. What shape is this matrix?

(b) Show why $R^{\mathrm{T}}R$ has no negative numbers on its diagonal.

20 Factor these symmetric matrices into $A = LDL^{\mathrm{T}}$. The pivot matrix D is diagonal:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

21 After elimination clears out column 1 below the first pivot, find the symmetric 2 by 2 matrix that appears in the lower right corner:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Questions 22–30 are about the factorizations $PA = LU$ and $A = L_1P_1U_1$.

22 Find the $PA = LU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

23 Find a 3 by 3 permutation matrix (call it A) that needs two row exchanges to reach the end of elimination. For this matrix, what are its factors P, L, and U?

24 Factor the following matrix into $PA = LU$. Factor it also into $A = L_1P_1U_1$ (hold the exchange of row 3 until 3 times row 1 is subtracted from row 2):

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}.$$

25 Write out P after each step of the MATLAB code splu, when

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix}.$$

26 Write out P and L after each step of the code splu when

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 5 & 4 \end{bmatrix}.$$

27 Extend the MATLAB code splu to a code spldu which factors PA into LDU.

28 What is the matrix L_1 in $A = L_1P_1U_1$?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = P_1U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

29 Prove that the identity matrix cannot be the product of three row exchanges (or five). It can be the product of two exchanges (or four).

30 (a) Choose E_{21} to remove the 3 below the first pivot. Then multiply $E_{21}AE_{21}^{\mathrm{T}}$ to remove both 3's:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 11 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{is going toward} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Choose E_{32} to remove the 4 below the second pivot. Then A is reduced to D by $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}} = D$. Invert the E's to find L in $A = LDL^{\mathrm{T}}$.

The next questions are about applications of the identity $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$.

31 Wires go between Boston, Chicago, and Seattle. Those cities are at voltages x_B, x_C, x_S. With unit resistances between cities, the currents between cities are in $\boldsymbol{y}$:

$$\boldsymbol{y} = A\boldsymbol{x} \quad \text{is} \quad \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}.$$

(a) Find the total currents $A^{\mathrm{T}}\boldsymbol{y}$ out of the three cities.

(b) Verify that $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y}$ agrees with $\boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$—six terms in both.

32 Producing x_1 trucks and x_2 planes needs $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. If the unit costs y_1, y_2, y_3 are \$700 per ton, \$3 per pound, and \$3000 per month, what are the values of one truck and one plane? Those are the components of $A^{\mathrm{T}}\boldsymbol{y}$.

33 $A\boldsymbol{x}$ gives the amounts of steel, rubber, and labor to produce $\boldsymbol{x}$ in Problem 32. Find A. Then $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the ______ of inputs while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the value of ______.

34 The matrix P that multiplies (x, y, z) to give (z, x, y) is also a rotation matrix. Find P and P^3. The rotation axis $\boldsymbol{a} = (1, 1, 1)$ doesn't move, it equals $P\boldsymbol{a}$. What is the angle of rotation from $\boldsymbol{v} = (2, 3, -5)$ to $P\boldsymbol{v} = (-5, 2, 3)$?

35 Write $A = \left[\begin{smallmatrix} 1 & 2 \\ 4 & 9 \end{smallmatrix}\right]$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

36 Here is a new factorization of A into *triangular times symmetric*:

$$\text{Start from } A = LDU. \text{ Then } A = L(U^{\mathrm{T}})^{-1} \text{ times } U^{\mathrm{T}}DU.$$

Why is $L(U^{\mathrm{T}})^{-1}$ triangular? Its diagonal is all 1's. Why is $U^{\mathrm{T}}DU$ symmetric?

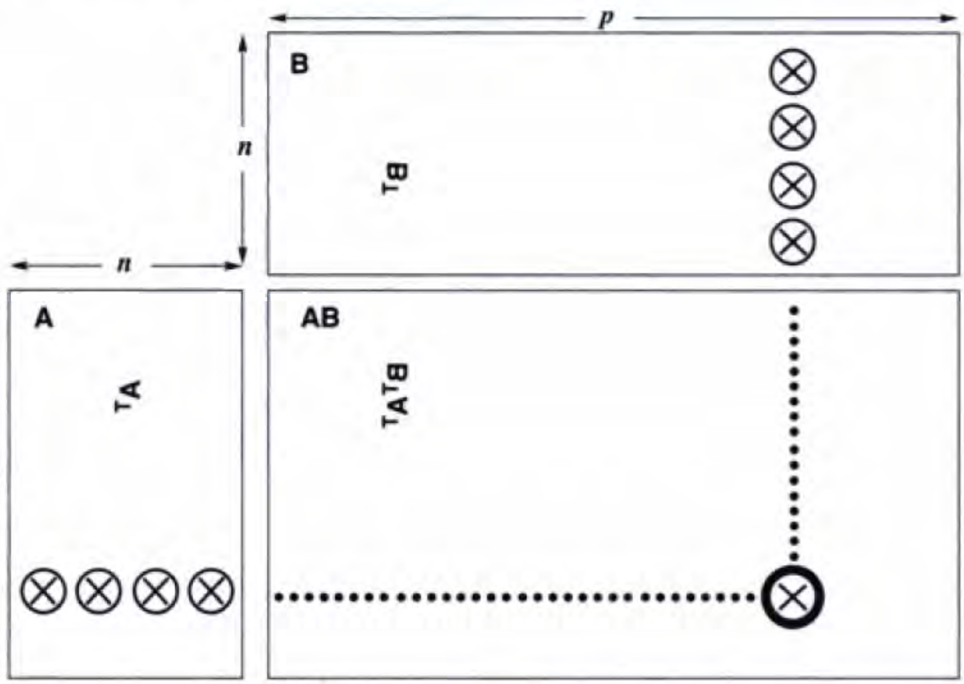
37 A *group* of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups? Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^{\mathrm{T}} = Q^{-1}$. Invent two more matrix groups.

38 If every row of a 4 by 4 matrix contains the numbers 0, 1, 2, 3 in some order, can the matrix be symmetric?

39 Prove that no reordering of rows and reordering of columns can transpose a typical matrix.

40 A square ***northwest matrix*** B is zero in the southeast corner, below the antidiagonal that connects $(1, n)$ to $(n, 1)$. Will B^{T} and B^2 be northwest matrices? Will B^{-1} be northwest or southeast? What is the shape of BC = ***northwest times southeast***? OK to combine permutations with the usual L and U (southwest and northeast).

41 If P has 1's on the antidiagonal from $(1, n)$ to $(n, 1)$, describe PAP.



* **Transparent proof that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.** Matrices can be transposed by looking through the page from the other side. Hold up to the light and practice with B. Its column with four entries $\otimes$ becomes a row, when you look from the back and the symbol B^{T} is upright.

The three matrices are in position for matrix multiplication: the row of A times the column of B gives the entry in AB. Looking from the reverse side, the row of B^{T} times the column of A^{T} gives the correct entry in $B^{\mathrm{T}}A^{\mathrm{T}} = (AB)^{\mathrm{T}}$.