# Hidden Markov Models I


Machine Learning 10-601B

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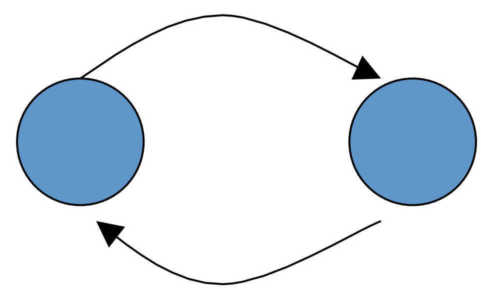
Many of these slides are derived from Tom Mitchell, Ziv Bar-Joseph. Thanks!

# What's wrong with Bayesian networks

- Bayesian networks are very useful for modeling joint distributions
- But they have their limitations:
    - Cannot account for temporal / sequence models
    - DAG's (no self or any other loops)

**This is not a valid Bayesian network!**



# Hidden Markov models

- Model a set of observation with a set of hidden states

  - Robot movement

  Observations: range sensor, visual sensor

  Hidden states: location (on a map)

# Hidden Markov models

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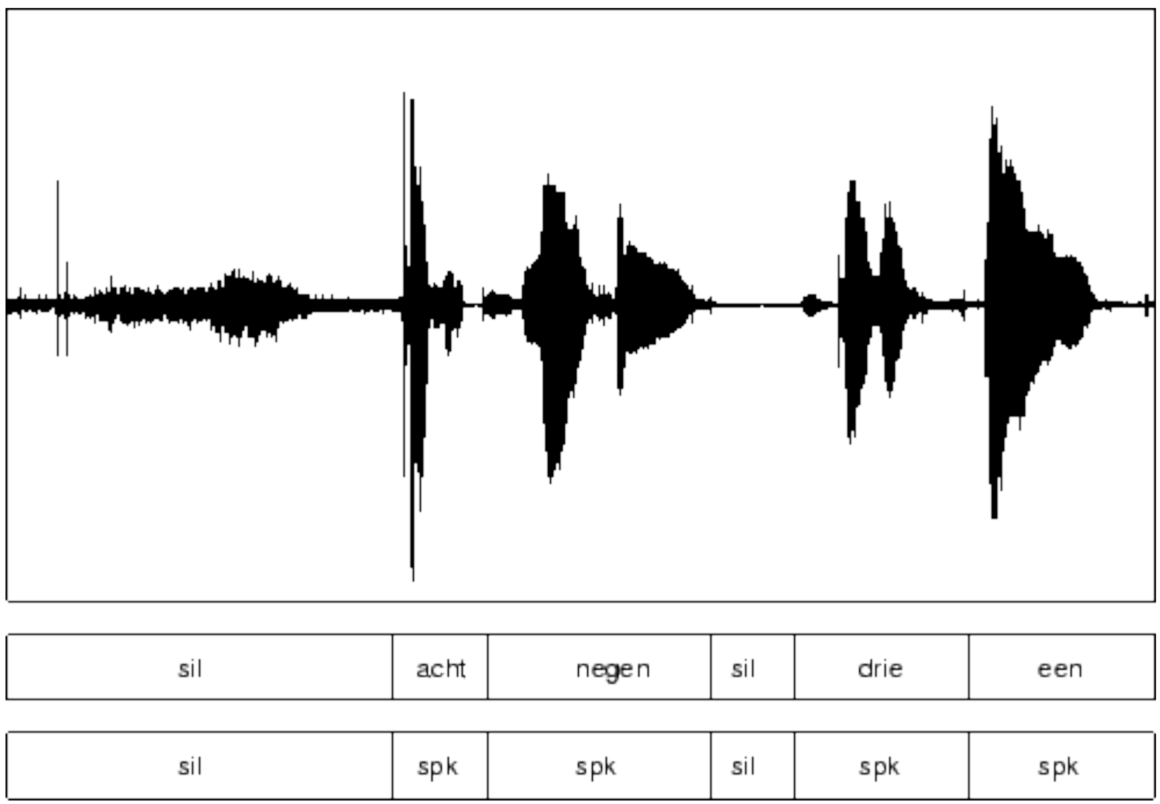
  Hidden states: location (on a map)

1. Hidden states generate observations
2. Hidden states transition to other hidden states

# Examples: Speech processing

Speech processing

- Observations: sound signals
- Hidden states: parts of speech, words

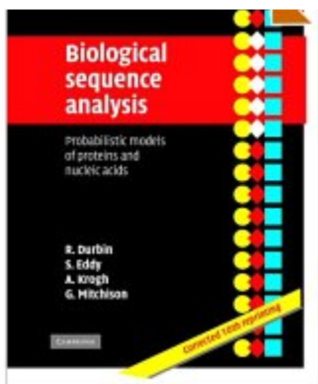


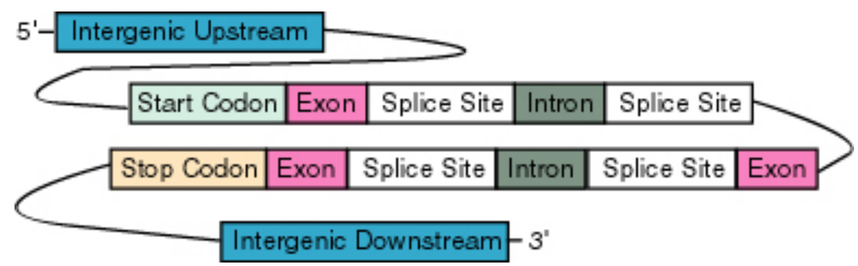
# Example: Biological data

Biology

Observations: DNA base pairs

Hidden states: Genes



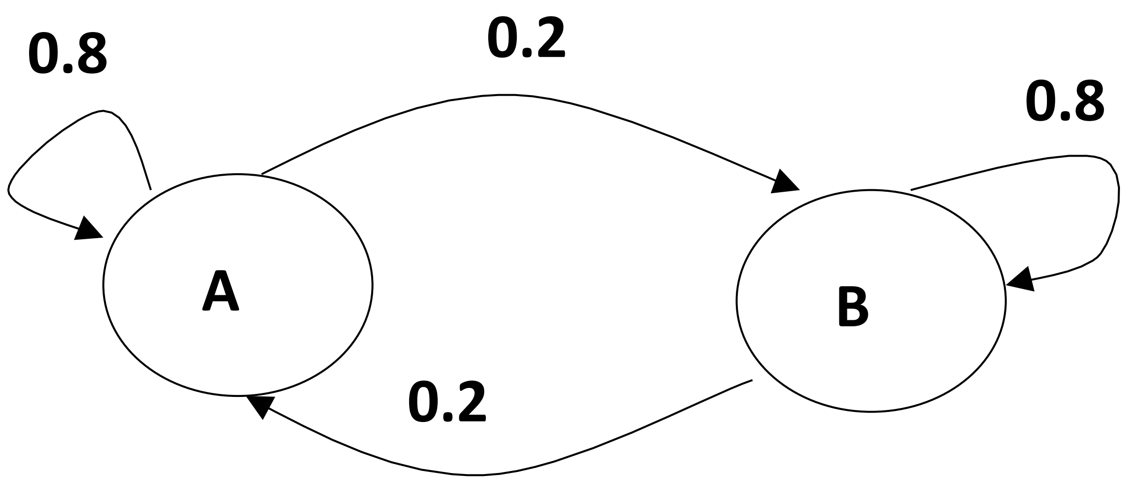




ATGAAGCTACTGTCTTCTATCGAACAAGCATGCGATATTTGCCG
ACTTAAAAAGCTCAAG
TGCTCCAAAGAAAAACCGAAGTGCGCCAAGTGTCTGAAGAA
CAACTGGGAGTGTCGCTAC
TCTCCCAAAACCAAAAGGTCTCCGCTGACTAGGGCACATCTG
ACAGAAGTGGAATCAAGG
CTAGAAAGACTGGAACAGCTATTTCTACTGATTTTTCCTCGAG
AAGACCTTGACATGATT

# Example: Gambling on dice outcome

- Two dice A and B, both skewed (output model).
- Can either stay with the same die or switch to the second die (transition model).

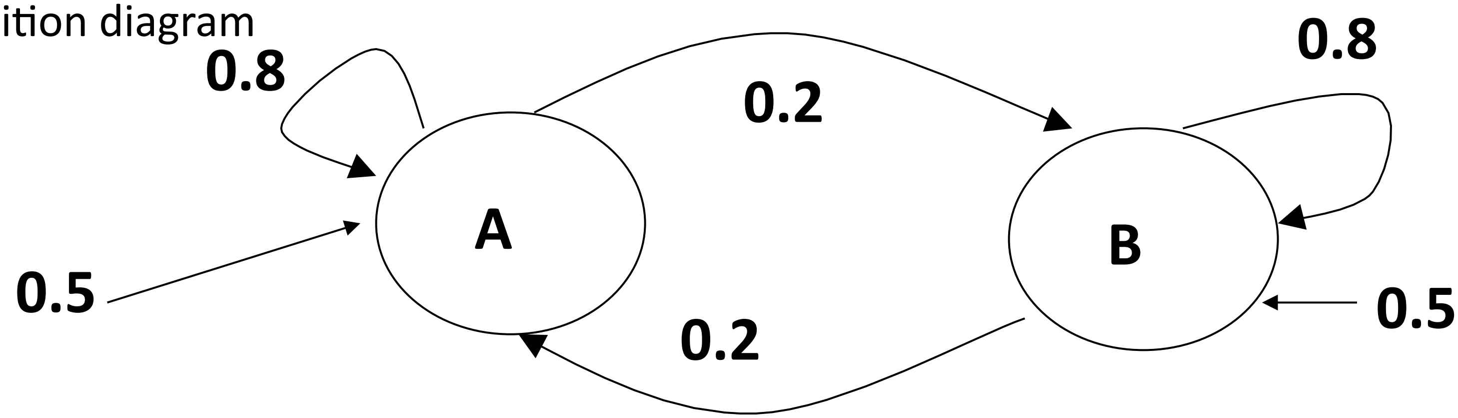
State transition diagram



# A Hidden Markov Model

- A set of states $S = \{s_1 \ldots s_n\}$
  - In each time point we are in exactly one of these states
- A set of possible outputs $\Sigma=\{\sigma_1, \ldots, \sigma_m\}$
  - In each time point we emit a symbol $\sigma_j \in \Sigma$

State transition diagram



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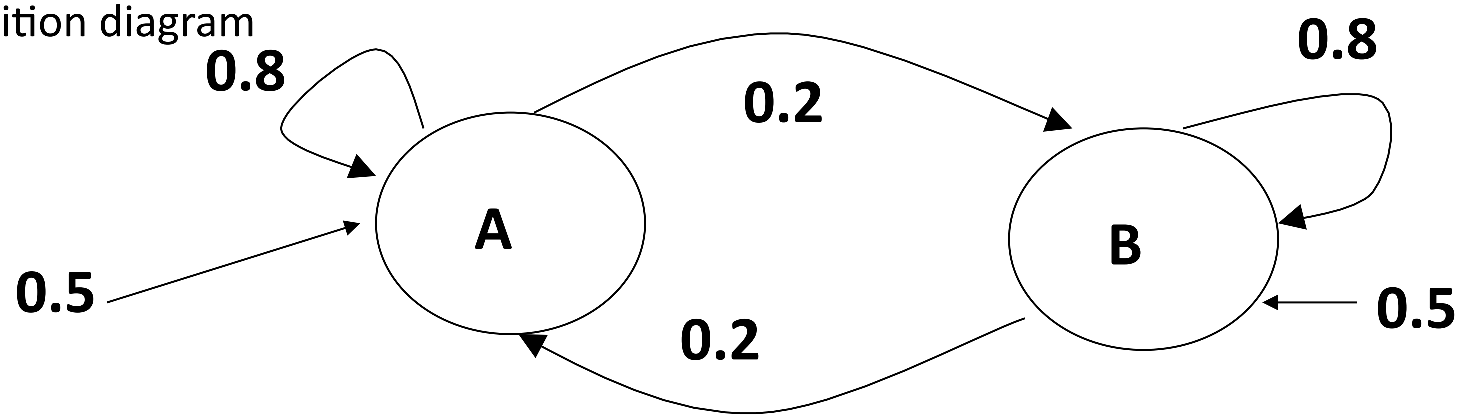
States: A A A B B B B B B A A

Observations: 1 2 2 1 1 2 1 1 1 2 2

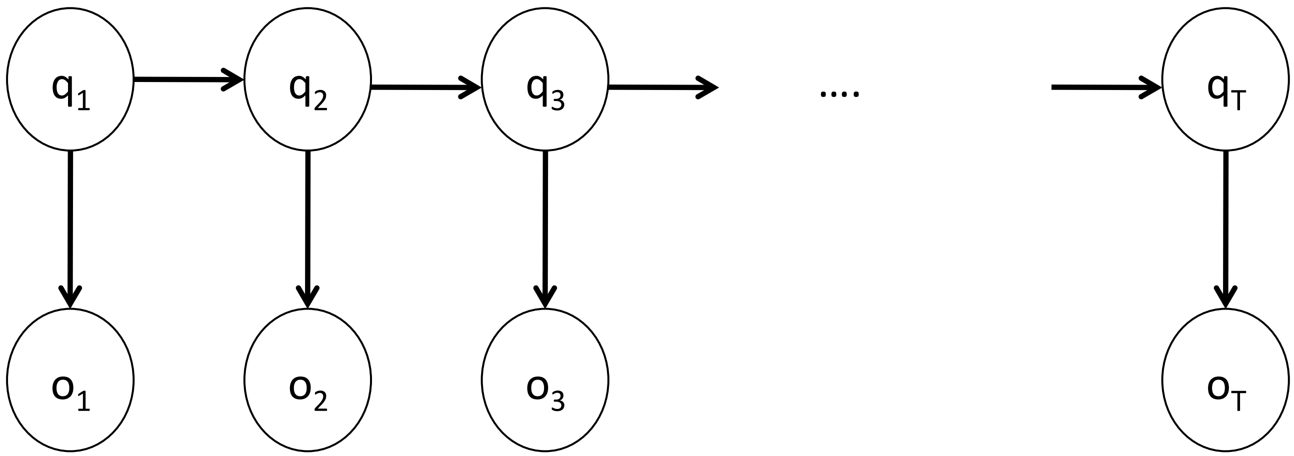
State transition diagram



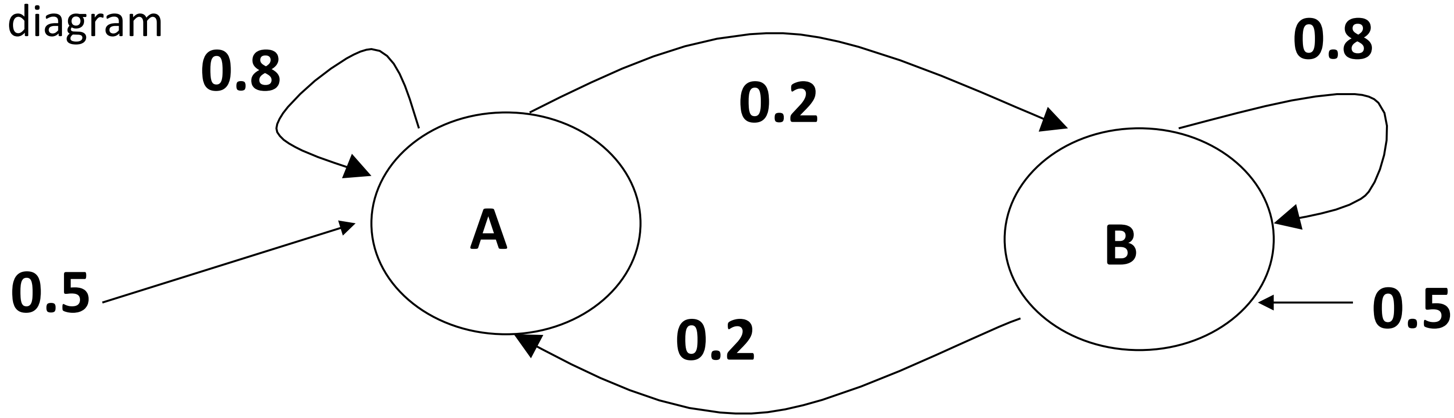
# A Hidden Markov Model

- Probabilistic graphical models

States: A A A B B B B B B A A
Observations: 1 2 2 1 1 2 1 1 1 2 2




State transition diagram



# A Hidden Markov Model

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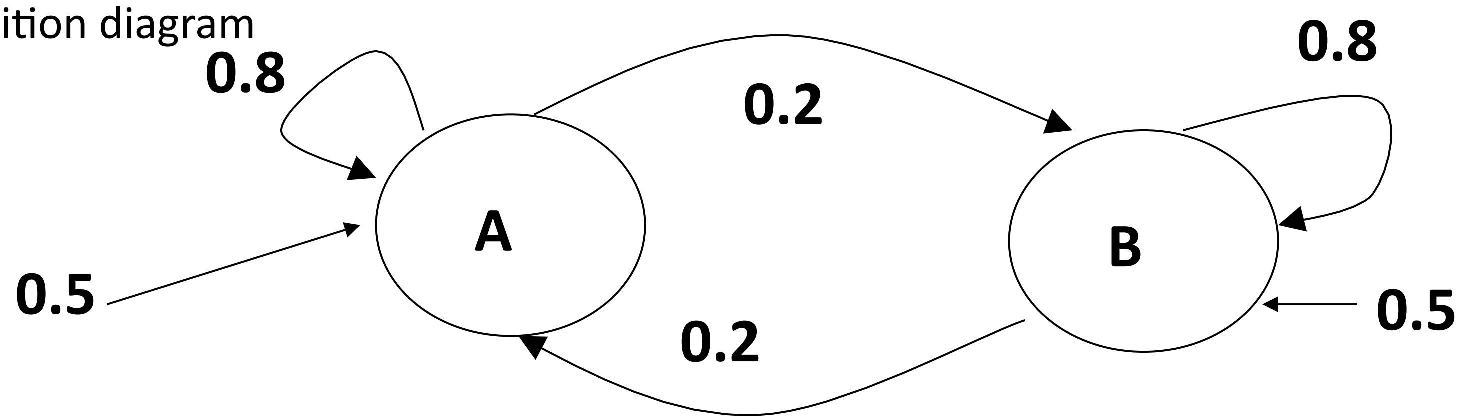
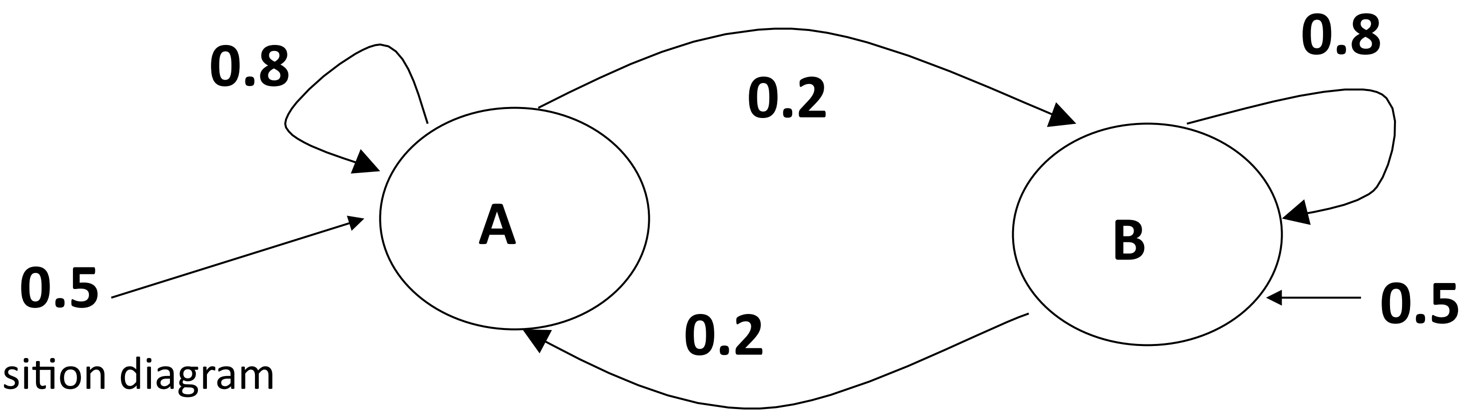
- Random variables
  - States at each time point $Q = \{q_1, \dots, q_T\}$
    - Each $q_t$ can take on values from $\{s_1 \dots s_n\}$
  - Outputs at each time point $O = \{o_1, \dots, o_T\}$
    - Each $o_t$ can take on values from $\Sigma$

State transition diagram



# A Hidden Markov Model

- Parameters of the model
  - $\Pi_i = \{\pi_1, \ldots, \pi_n\}$: initial state probabilities $P(q_1=s_i)$
    - the probability that we *start* at state $s_{i,}$ $i=1,\ldots,n$
  - A transition probability model, $P(q_t = s_i \mid q_{t-1} = s_j)$
    - nxn matrix of transition probabilities
  - An emission probability model, $p(o_t = \sigma_j \mid q_t = s_i)$
    - nxm matrix of emission probabilities




State transition diagram

# A Hidden Markov Model

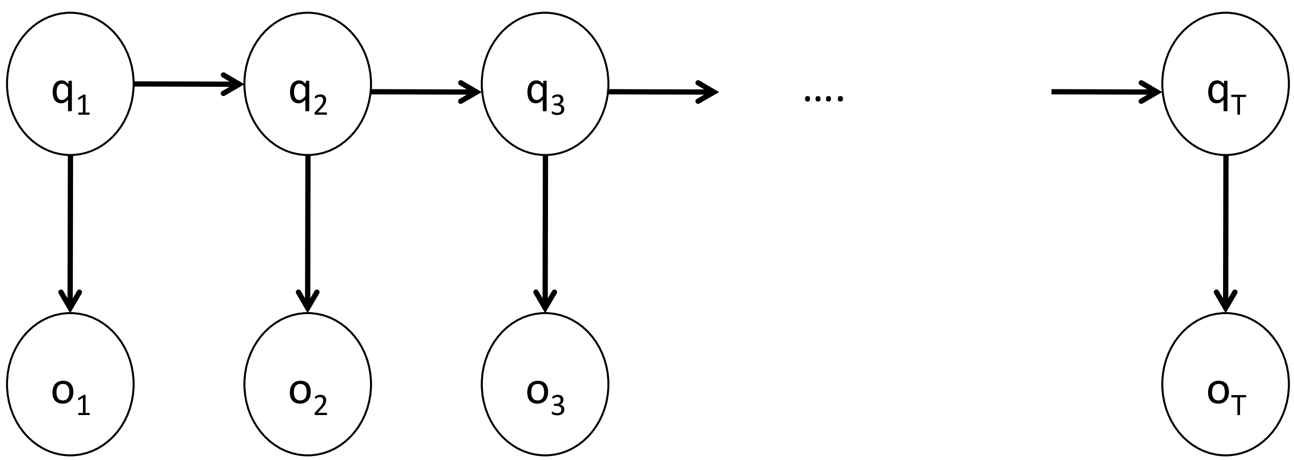
- The joint probability of (Q,O) is defined as

$$P(Q,O) = p(q_1)\prod_{t=1}^{T} p(q_t \mid q_{t-1})\, p(o_t \mid q_t)$$

Initial probability

transition probability

emission probability



# A Hidden Markov Model

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Initial probability

transition probability

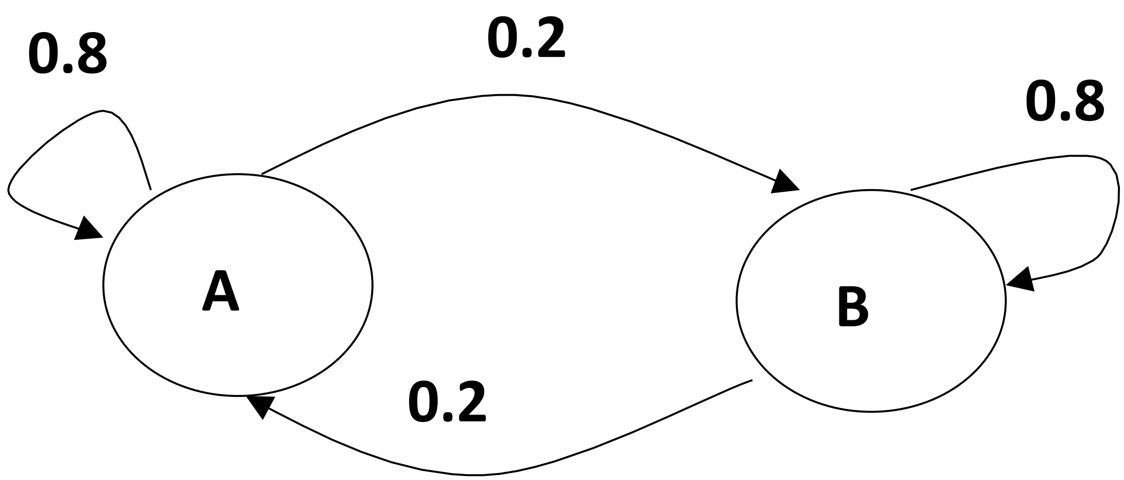
emission probability

An important aspect of this definitions is the Markov property:
$q_{t+1}$ is conditionally independent of $q_{t-1}$ (and any earlier time points) given $q_t$

More formally $P(q_{t+1} = s_i \mid q_t = s_j) = P(q_{t+1} = s_i \mid q_t = s_j, q_{t-1} = s_j)$

# What can we ask when using a HMM?

- A few examples:
  - "Which die is currently being used?"
  - "What is the probability of a 6 in the next role?"
  - "What is the probability of 6 in any of the next 3 roles?"



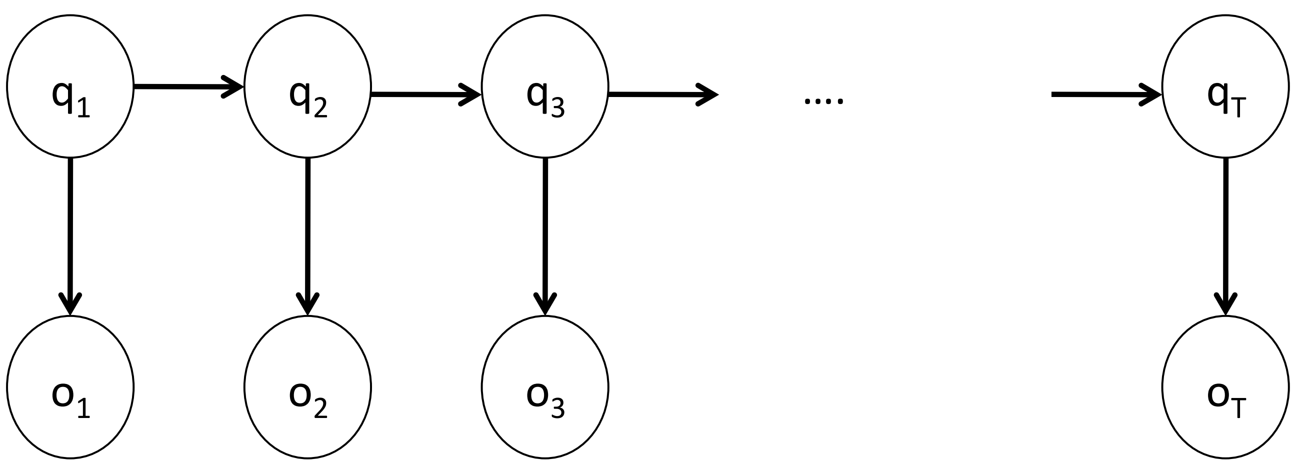
# Inference in HMMs

- Computing $P(Q)$ and $P(q_t = s_i)$

  - If we cannot look at observations
- Computing $P(Q \mid O)$ and $P(q_t = s_i \mid O)$

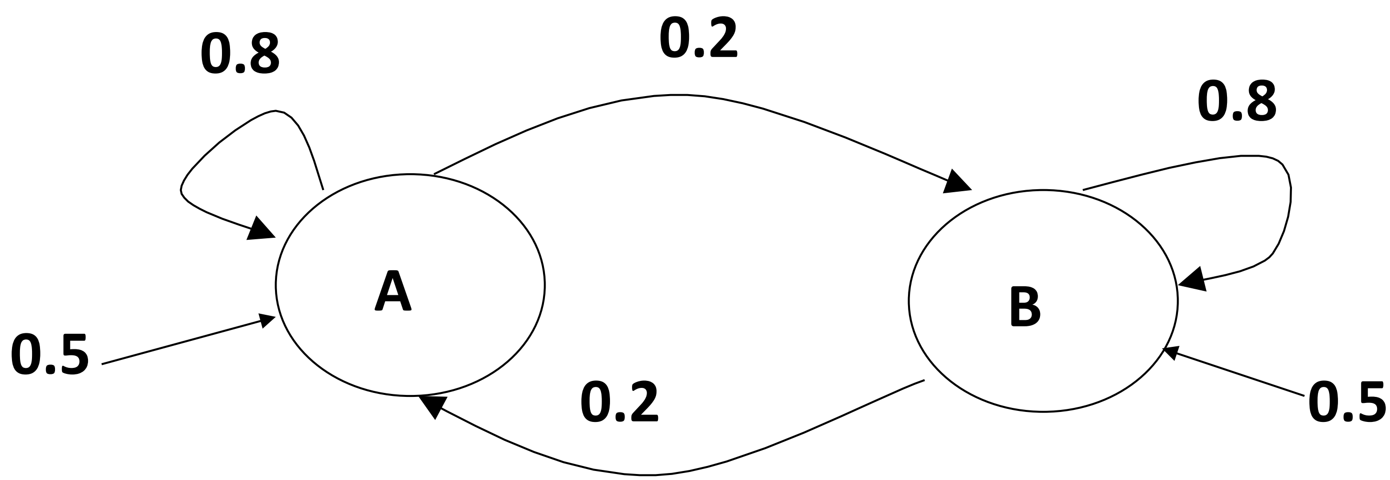
  - When we have observation and care about the last state only
- Computing $\operatorname{argmax}_Q P(Q \mid O)$

  - When we care about the entire path



# Which die is currently being used?

- We played *t* rounds so far
- We want to determine $P(q_t = A)$
- Let's assume for now that we cannot observe any outputs (we are blind folded)
- How can we compute this?



# $P(q_t = A)$?

- Simple answer: Consider “all” paths that end in A. For each such path Q, let’s determine P(Q)

$Q = q_1, \ldots q_{t-1}, A$

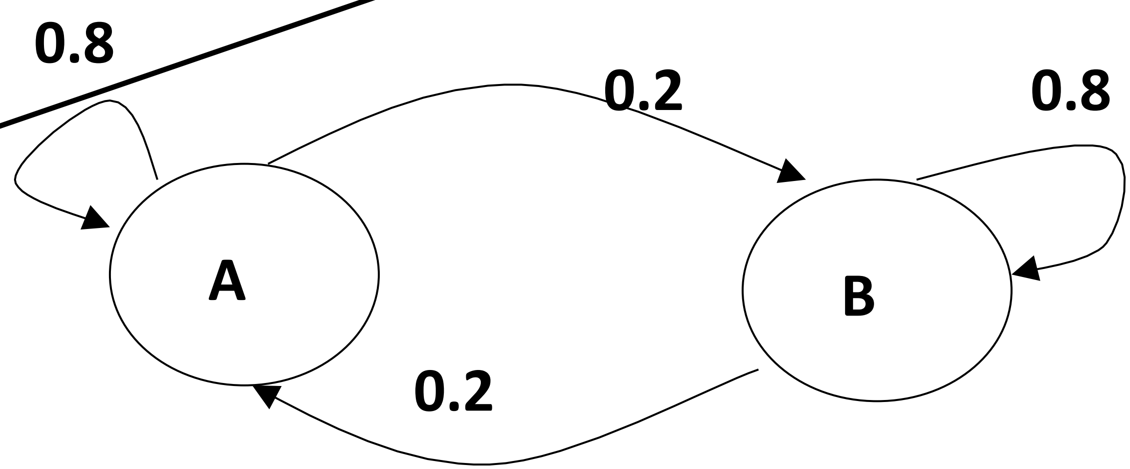
$$P(Q) = P(q_1, \ldots q_{t-1}, A) = P(A \mid q_1, \ldots q_{t-1})\, P(q_1, \ldots q_{t-1})$$

$$= P(A \mid q_{t-1})\, P(q_1, \ldots q_{t-1})$$

$$= P(A \mid q_{t-1}) \ldots P(q_2 \mid q_1)\, P(q_1)$$

Markov property!

Initial probability



# $P(q_t = A)$?

- Simple answer:

1. Let's determine P(Q) where Q is any path that ends in A

$Q = q_1, \ldots q_{t-1}, A$

$P(Q) = P(q_1, \ldots q_{t-1}, A)$

$= P(A \mid q_1, \ldots q_{t-1})\, P(q_1, \ldots q_{t-1})$

$= P(A \mid q_{t-1})\, P(q_1, \ldots q_{t-1})$

$= P(A \mid q_{t-1}) \ldots P(q_2 \mid q_1)\, P(q_1)$

2. $P(q_t = A) = \Sigma P(Q)$

where the sum is over all sets of t states
that end in A

# $P(q_t = A)$?

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2. $P(q_t = A) = \Sigma P(Q)$

where the sum is over all sets of t states that end in A

Q: How many sets Q are there?

A: A lot! ($2^{t-1}$)

Not a feasible solution

# $P(q_t = A)$, the smart way

- Let's define $p_t(i)$ as the probability of being in state *i* at time t: $p_t(i) = p(q_t = s_i)$
- We can determine $p_t(i)$ by induction

1. $p_1(i) = \Pi_i$
2. $p_t(i) = ?$

# $P(q_t = A)$, the smart way

- Let's define $p_t(i)$ = probability state i at time t = $p(q_t = s_i)$
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This type of computation is called dynamic programming

Complexity: $O(n^2{*}t)$

Number of states in our HMM

| Time / state | t1 | t2 | t3 |
|---|---|---|---|
| s1 | .3 | | |
| s2 | .7 | | |

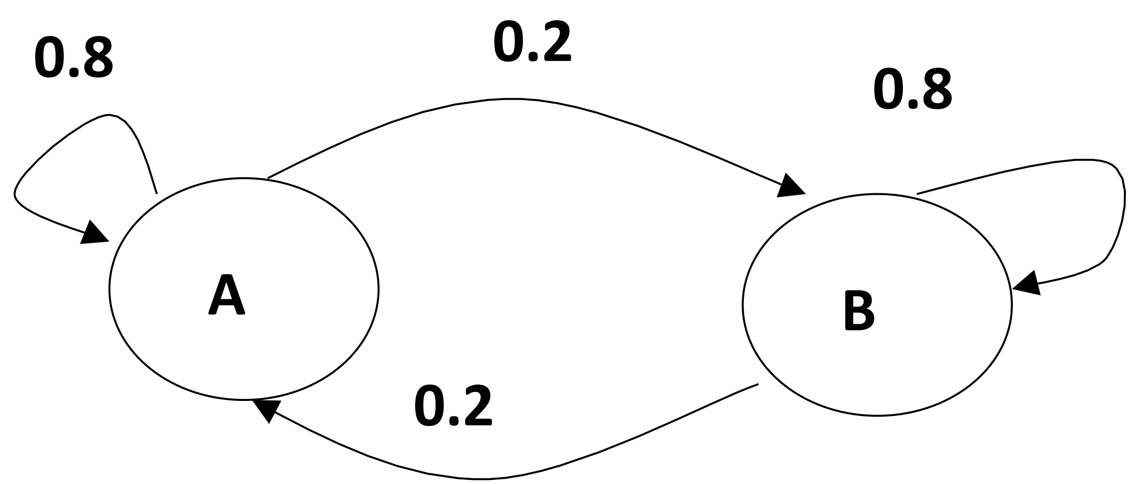
# Inference in HMMs

- Computing $P(Q)$ and $P(q_t = s_i)$ √
- Computing $P(Q \mid O)$ and $P(q_t = s_i \mid O)$
- Computing $\mathrm{argmax}_Q P(Q)$

# But what if we observe outputs?

- So far, we assumed that we could not observe the outputs
- In reality, we almost always can.

| v | P(v \|A) | P(v \|B) |
|---|---|---|
| 1 | .3 | .1 |
| 2 | .2 | .1 |
| 3 | .2 | .1 |
| 4 | .1 | .2 |
| 5 | .1 | .2 |
| 6 | .1 | .3 |



# But what if we observe outputs?

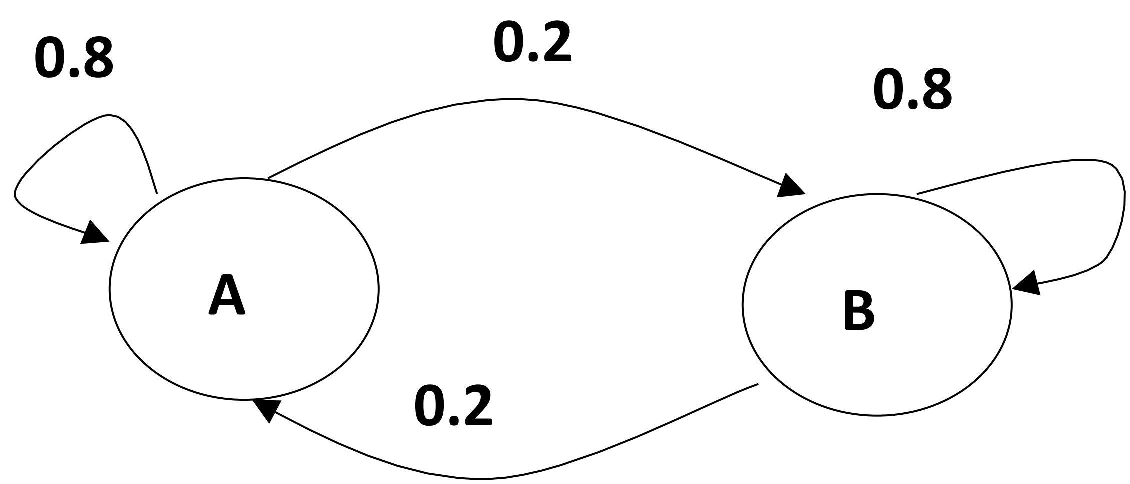
- So far, we assumed that we could not observe the outputs
- In reality, we almost always can.

Does observing the sequence

5, 6, 4, 5, 6, 6

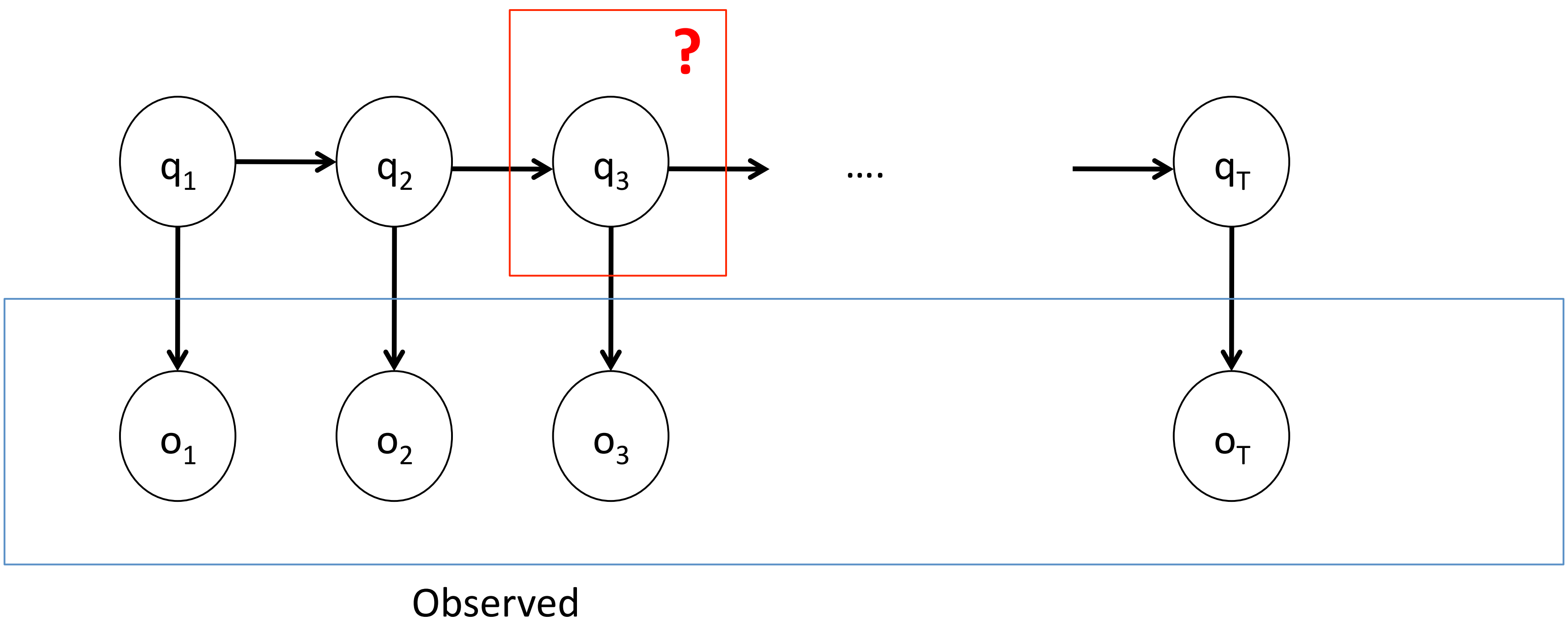
Change our belief about the state?

| v | P(v \|A) | P(v \|B) |
|---|---|---|
| 1 | .3 | .1 |
| 2 | .2 | .1 |
| 3 | .2 | .1 |
| 4 | .1 | .2 |
| 5 | .1 | .2 |
| 6 | .1 | .3 |



# When outputs are observed

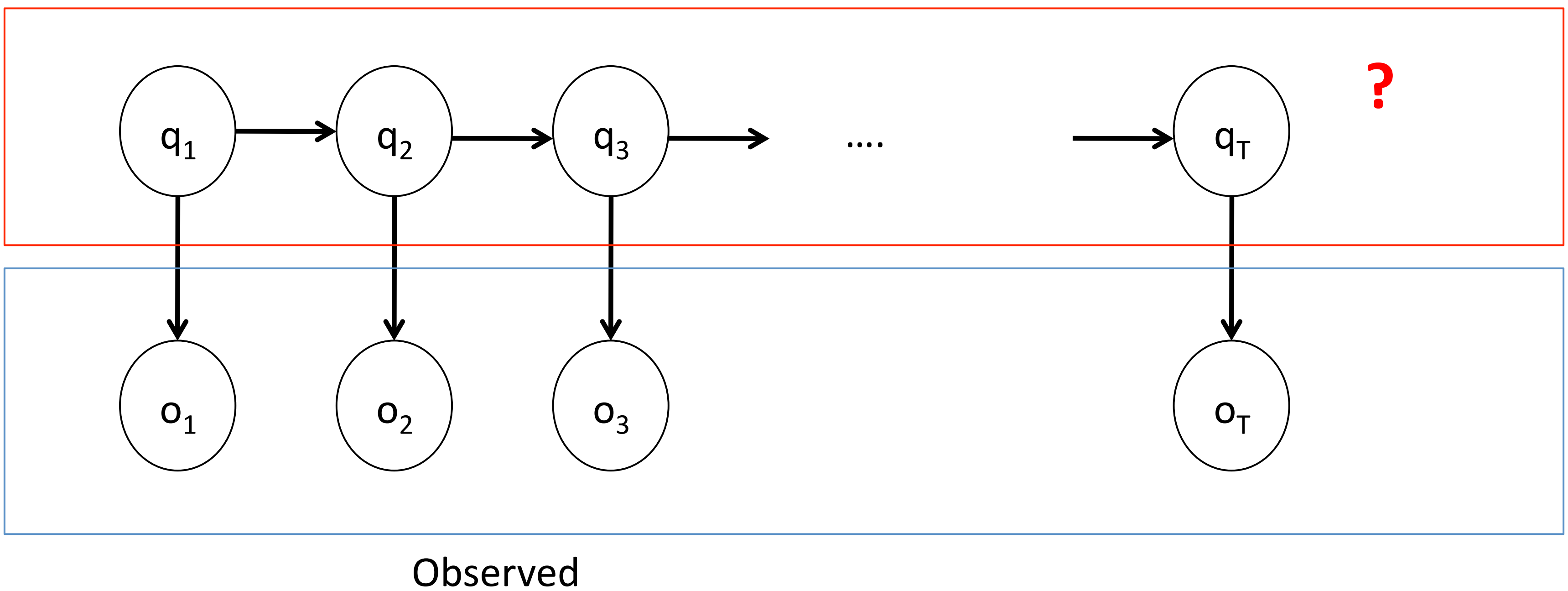
- We want to compute $P(q_t = A \mid O_1 \ldots O_t)$



# $P(q_t = A|O)$ when outputs are observed

- We want to compute $P(q_t = A \mid O_1 \ldots O_t)$
- Let's start with a simpler question. Given a sequence of states Q, what is $P(Q \mid O_1 \ldots O_t) = P(Q \mid O)$?

- It is pretty simple to move from P(Q) to $P(q_t = A)$



# $P(q_t = A|O)$ when outputs are observed

- We want to compute $P(q_t = A \mid O_1 \dots O_t)$
- Let's start with a simpler question. Given a sequence of states Q, what is $P(Q \mid O_1 \dots O_t) = P(Q \mid O)$?

  - It is pretty simple to move from P(Q) to $P(q_t = A)$

  - In some cases P(Q) is the more important question

    - Speech processing

    - NLP

# P(Q | O)

- We can use Bayes rule:

$$P(Q|O) = \frac{P(O \mid Q)P(Q)}{P(O)}$$

Easy, $P(O \mid Q) = P(o_1 \mid q_1)\, P(o_2 \mid q_2) \ldots P(o_t \mid q_t)$

# P(Q | O)

- We can use Bayes rule:

$$P(Q|O) = \frac{P(O \mid Q)P(Q)}{P(O)}$$

Easy, $P(Q) = P(q_1)\, P(q_2 \mid q_1) \ldots P(q_t \mid q_{t-1})$

# P(Q | O)

- We can use Bayes rule:

$$P(Q|O) = \frac{P(O \mid Q)P(Q)}{P(O)}$$

Hard!

# P(O)

- What is the probability of seeing a set of observations:
  - An important question in it own rights, for example classification using two HMMs $H_1$ and $H_2$
    - Compute $P(O|H_1)$ and $P(O|H_1)$, classify to the model with higher probability

# P(O)

- Define $\alpha_t(i) = P(o_1, o_2 \ldots, o_t \wedge \ q_t = s_i)$
- $\alpha_t(i)$ is the probability that we:
    1. Observe $o_1, o_2 \ldots, o_t$
    2. End up at state i

How do we compute $\alpha_t(i)$?

# When outputs are observed

- We want to compute $P(q_t = A \mid O_1 \ldots O_t)$
- For ease of writing we will use the following notations (commonly used in the literature)
- $a_{j,i} = P(q_t = s_i \mid q_{t-1} = s_j)$
- $b_i(o_t) = P(o_t \mid s_i)$

Transition probability

Emission probability

# Computing $\alpha_t(i)$

- $\alpha_1(i) = P(o_1 \wedge q_1 = i) = P(o_1 \mid q_1 = s_i)\Pi_i$

We must be at a state in time t

chain rule

Markov property

# Computing $\alpha_t(i)$

- $\alpha_1(i) = P(o_1 \wedge q_1 = i) = P(o_1 \mid q_1 = s_i)\Pi_i$

We must be at a state in time t

$$\alpha_{t+1}(i) = P(O_1 \ldots O_{t+1} \wedge q_{t+1} = s_i) =$$

chain rule

$$\sum_j P(O_1 \ldots O_t \wedge q_t = s_j \wedge O_{t+1} \wedge q_{t+1} = s_i) =$$

$$\sum_j P(O_{t+1} \wedge q_{t+1} = s_i \mid O_1 \ldots O_t \wedge q_t = s_j) P(O_1 \ldots O_t \wedge q_t = s_j) =$$

Markov property

$$\sum_j P(O_{t+1} \wedge q_{t+1} = s_i \mid O_1 \ldots O_t \wedge q_t = s_j) \alpha_t(j) =$$

$$\sum_j P(O_{t+1} \mid q_{t+1} = s_i) P(q_{t+1} = s_i \mid q_t = s_j) \alpha_t(j) =$$

$$\sum_j b_i(O_{t+1}) a_{j.i} \alpha_t(j)$$

# Example: Computing $\alpha_3(B)$

- We observed 2,3,6

$\alpha_1(A) = P(2 \wedge q_1 = A) = P(2 \mid q_1 = A)\Pi_A = .2*.7 = .14$, $\alpha_1(B) = .1*.3 = .03$

$\alpha_2(A) = \Sigma_{j=A,,B} b_A(3) a_{j,A}\ \alpha_1(j) = .2*.8*.14 + .2*.2*.03 = 0.0236$, $\alpha_2(B) = 0.0052$

$\alpha_3(B) = \Sigma_{j=A,,B} b_B(6) a_{j,B}\ \alpha_2(j) = .3*.2*.0236 + .3*.8*.0052 = 0.00264$

$\Pi_A$=0.7
$\Pi_b$=0.3

| v | P(v \|A) | P(v \|B) |
|---|---|---|
| 1 | .3 | .1 |
| 2 | .2 | .1 |
| 3 | .2 | .1 |
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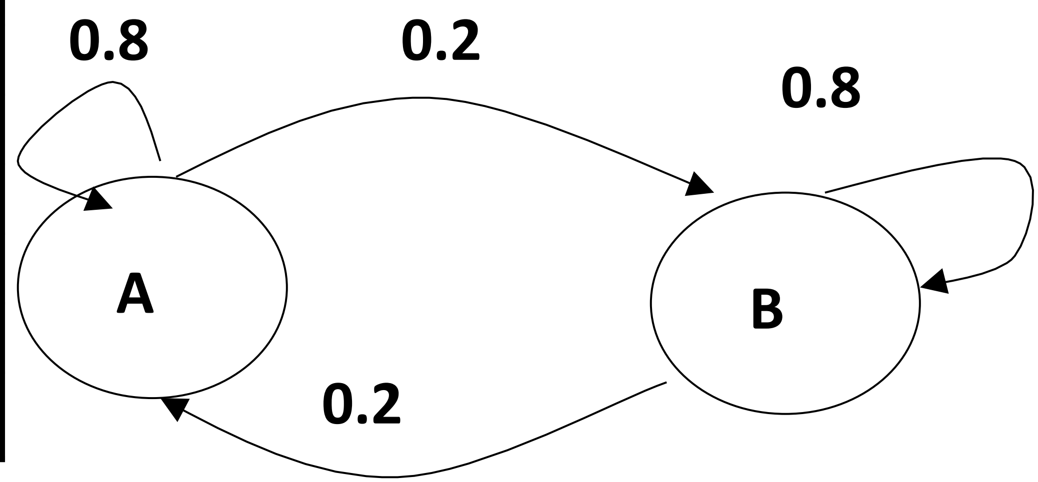

# Where we are

- We want to compute P(Q | O)
- For this, we only need to compute P(O)
- We know how to compute $\alpha_t(i)$

From now its easy

$\alpha_t(i) = P(o_1, o_2 \ldots, o_t \wedge q_t = s_i)$

so

$P(O) = P(o_1, o_2 \ldots, o_t) = \Sigma_i P(o_1, o_2 \ldots, o_t \wedge q_t = s_i) = \Sigma_i \alpha_t(i)$

note that

$$p(q_t = s_i \mid o_1, o_2 \ldots, o_t) = \frac{\alpha_t(i)}{\sum_j \alpha_t(j)}$$

P(A | B) = P(A ∧ B) / P(B)

# Complexity

- How long does it take to compute P(Q | O)?
  - P(Q): O(n)
  - P(O|Q): O(n)
  - P(O): O($n^2$t)

# Inference in HMMs

- Computing $P(Q)$ and $P(q_t = s_i)$ √
- Computing $P(Q \mid O)$ and $P(q_t = s_i \mid O)$ √
- Computing $\text{argmax}_Q P(Q)$

# Most probable path

- We are almost done …
- One final question remains

  How do we find the most probable path, that is Q* such that

$$P(Q^* \mid O) = \text{argmax}_Q P(Q|O)?$$

- This is an important path

  - The words in speech processing

  - The set of genes in the genome

  - etc.

# Example

- What is the most probable set of states leading to the sequence:

  1,2,2,5,6,5,1,2,3 ?

$\Pi_A$=0.7
$\Pi_b$=0.3

| v | P(v \|A) | P(v \|B) |
|---|---|---|
| 1 | .3 | .1 |
| 2 | .2 | .1 |
| 3 | .2 | .1 |
| 4 | .1 | .2 |
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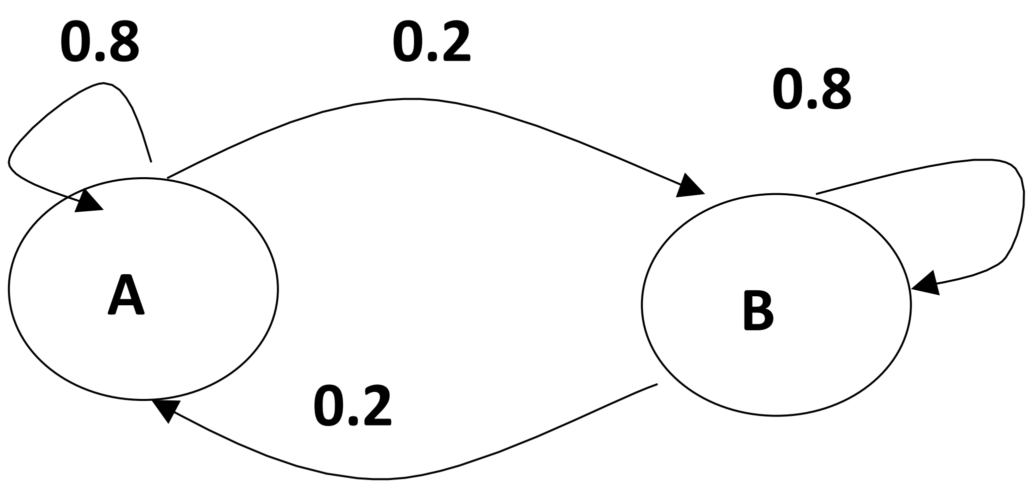

# Most probable path

$$\arg\max_Q P(Q \mid O) = \arg\max_Q \frac{P(O \mid Q)P(Q)}{P(O)}$$
$$= \arg\max_Q P(O \mid Q)P(Q)$$

We will use the following definition:

$$\delta_t(i) = \max_{q_1 \ldots q_{t-1}} p(q_1 \ldots q_{t-1} \wedge q_t = s_i \wedge O_1 \ldots O_t)$$

In other words we are interested in the most likely path from 1 to t that:

1. Ends in $S_i$
2. Produces outputs $O_1 \ldots O_t$

# Computing $\delta_t(i)$

$$\delta_t(i) = \max_{q_1 \ldots q_{t-1}} p(q_1 \ldots q_{t-1} \wedge q_t = s_i \wedge O_1 \ldots O_t)$$

Initialization at t=1

$$\begin{aligned}\delta_1(i) &= p(q_1 = s_i \wedge O_1) \\ &= p(q_1 = s_i) p(O_1 \mid q_1 = s_i) \\ &= \pi_i b_i(O_1)\end{aligned}$$

Q: Given $\delta_t(i)$, how can we compute $\delta_{t+1}(i)$?

A: To get from $\delta_t(i)$ to $\delta_{t+1}(i)$ we need to

1. Add an emission for time t+1 ($O_{t+1}$)
2. Transition to state $s_i$

$$\begin{aligned}\delta_{t+1}(i) &= \max_{q_1 \ldots q_t} p(q_1 \ldots q_t \wedge q_{t+1} = s_i \wedge O_1 \ldots O_{t+1}) \\ &= \max_j \delta_t(j) p(q_{t+1} = s_i \mid q_t = s_j) p(O_{t+1} \mid q_{t+1} = s_i) \\ &= \max_j \delta_t(j) a_{j,i} b_i(O_{t+1})\end{aligned}$$

# The Viterbi algorithm

$$\delta_{t+1}(i) = \max_{q_1 \ldots q_t} p(q_1 \ldots q_t \wedge q_{t+1} = s_i \wedge O_1 \ldots O_{t+1})$$

$$= \max_j \delta_t(j) p(q_{t+1} = s_i \mid q_t = s_j) p(O_{t+1} \mid q_{t+1} = s_i)$$

$$= \max_j \delta_t(j) a_{j,i} b_i(O_{t+1})$$

- Once again we use dynamic programming for solving $\delta_t(i)$
- Once we have $\delta_t(i)$, we can solve for our P(Q*|O) by:

P(Q* | O) = $\text{argmax}_Q$P(Q|O) =

path defined by $\text{argmax}_j$ $\delta_t(j)$,

# Inference in HMMs

- Computing $P(Q)$ and $P(q_t = s_i)$ √
- Computing $P(Q \mid O)$ and $P(q_t = s_i \mid O)$ √
- Computing $\mathrm{argmax}_Q P(Q)$ √

# What you should know

- Why HMMs? Which applications are suitable?
- Inference in HMMs
  - No observations
  - Probability of next state w. observations
  - Maximum scoring path (Viterbi)