Pre-Class Assignments: Basic Math I

Biomedical Imaging & Analysis (ECE J1-791) - Fall 2014

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Instructions

Please show your solutions to each problem in full, writing them neatly. For computer programs, please remember to turn in your code through the course's blackboard session, as well as any plots / figures that are requested. If you have collaborated with another student on solving this homework assignment please state so (e.g. "I helped John with question 1" or "Mary helped with question 3").

This assignment is due via Blackboard (MS Word Documents of typed out assignments or scanned PDFs of hand-written ones) on **Tuesday, 2 Sept 2014.**

I. LINEAR ALGEBRA

1. **(15 points) Apply Gauss elimination to solve the following linear system of equations after first writing out the Augmented matrix for the system and computing the Echelon form of the system:**

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 3 \\ 2x_1 - 2x_2 - x_3 + 2x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + 2x_4 = 2 \\ x_1 - x_2 - 2x_3 + x_4 = 0 \end{cases}$$

- Also find the determinant of the 4x4 matrix made of the left-hand-side of the above system of equations (show your work in steps!).
- Is the determinant of the system the same as the determinant of the Echelon form of the left-hand-side of the system of equations obtained after Gauss Elimination..?

2. **(30 points) Are the following sets of vectors linearly independent? Show why or why not. Recall, *p* vectors with *n* components are linearly independent if the matrix with these vectors as row vectors has rank *p*. They are linearly dependent if that rank is less than *p*.**

(1) $\begin{bmatrix}3 & -2 & 0 & 4\end{bmatrix}$, $\begin{bmatrix}5 & 0 & 0 & 1\end{bmatrix}$, $\begin{bmatrix}-6 & 1 & 0 & 1\end{bmatrix}$, $\begin{bmatrix}2 & 0 & 0 & 3\end{bmatrix}$

(2) $\begin{bmatrix}1 & 1 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 1 & 1\end{bmatrix}$.

a) If the above vectors are linearly independent are they mutually "orthogonal"..? Why..?
b) Compute a set of three mutually orthogonal vectors to the space defined by the linearly independent basis defined by the three vectors in part (2). **HINT**: Gram-Schmidt Orthogonalization (see Reading Material attached with this assignment).

3. **(35 points)** Consider the signal, s[k], with k = 1…N, and the signal model f[k] which is the Fourier series expansion of s[k], such that:

$$f[k] = (1/N)\ \Sigma_{n=0}^{N-1}\ c[n]\ \Phi_n[k] \qquad (1)$$

where, c[k] are coefficients and $\Phi_n[k]$ are terms of the Fourier series with N total terms.

a) Write down the formulation of this fitting problem expressed in the matrix form, assuming that the Fourier terms, $\Phi_n[k]$ form a matrix, A.
b) Provide a means of solving this problem for the coefficients, c, of the Fourier series expansion in the least squares sense and prove that:

$$\mathbf{c} = (\mathbf{A}^T\,\mathbf{A})^{-1}\,\mathbf{A}^T\,\mathbf{f} \qquad (2)$$

where, **c** is a vector, **A** is a symmetric matrix and **f** is a vector with *k* elements.

c) In this specific case of **A** representing Fourier terms, **given properties of being "orthonormal", how can you further simplify equation (2) ..?**

II. BASIC VECTOR CALCULUS & MATRIX MANIPULATION

4. (20 points) Consider the 2x2 matrix, A = [1, 3; 3, 1]

- Calculate the eigen-values and eigen-vectors (**v1** and **v2**, say) for **A.**
  - o What is the vector norm of the eigen-vectors of **A ..?**
    Hint: what are |v1| and |v2|..?

5. (10 points) Show that:

$$f(\bar{x}) = x_1^2 + x_1 x_2 + x_2^2 = x_1^2 + 0.5 \cdot x_1 x_2 + 0.5 \cdot x_2 x_1 + x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$where, (\bar{x}) = (x_1, x_2)$$

HINT: Start with the matrix on the right-hand-side and work back to get the left-hand-side of the equation. Show your steps and remember to pay attention to the correct order of multiplying a series of matrices !