Boosting

Machine Learning 10-601B

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Many of these slides are derived fromTom Mitchell, Ziv-Bar Joseph. Thanks!

Simple Learners

- Simple (a.k.a. weak) learners are good
  - e.g., naïve Bayes, logistic regression, decision stumps (or shallow decision trees)
  - don't usually overfit

- Simple (a.k.a. weak) learners are bad
  - can't solve hard learning problems

- Can we make weak learners always good???
  - No!!!
  - But often yes…

Voting (Ensemble Methods)

- Instead of learning a single (weak) classifier, learn **many weak classifiers** that are **good at different parts of the input space**

- **Output class:** (Weighted) vote of each classifier
  - Classifiers that are most "sure" will vote with more conviction
  - Classifiers will be most "sure" about a particular part of the space
  - On average, do better than single classifier!

- **But how do you ???**
  - force classifiers to learn about different parts of the input space?
  - weight the votes of different classifiers?

Boosting [Schapire, 1989]

- Idea: given a weak learner, run it multiple times on (reweighted) training data, then let the learned classifiers vote

- On each iteration *t*:
  - weight each training example by how incorrectly it was classified
  - Learn a hypothesis – h_t
  - A strength for this hypothesis – α_t

- Final classifier:

  \- A linear combination of the votes of the different classifiers weighted by their strength

$$H(X) = \text{sign}(\Sigma \alpha_t h_t(X))$$

- **Practically useful**
- **Theoretically interesting**

Learning from weighted data

- **Sometimes not all data points are equal**
  - Some data points are more equal than others
- **Consider a weighted dataset**
  - D(i) – weight of *i* th training example ($\mathbf{x}^i, y^i$)
  - Interpretations:
    - *i* th training example counts as D(i) examples
    - If I were to “resample” data, I would get more samples of “heavier” data points
- **Now, in all calculations, whenever used, *i* th training example counts as D(i) “examples”**
  - e.g., MLE for Naïve Bayes, redefine *Count(Y=y)* to be weighted count

Learning From Weighted Data

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  - D(i) – weight of i th training example (xi,yi)
  - Interpretatons:
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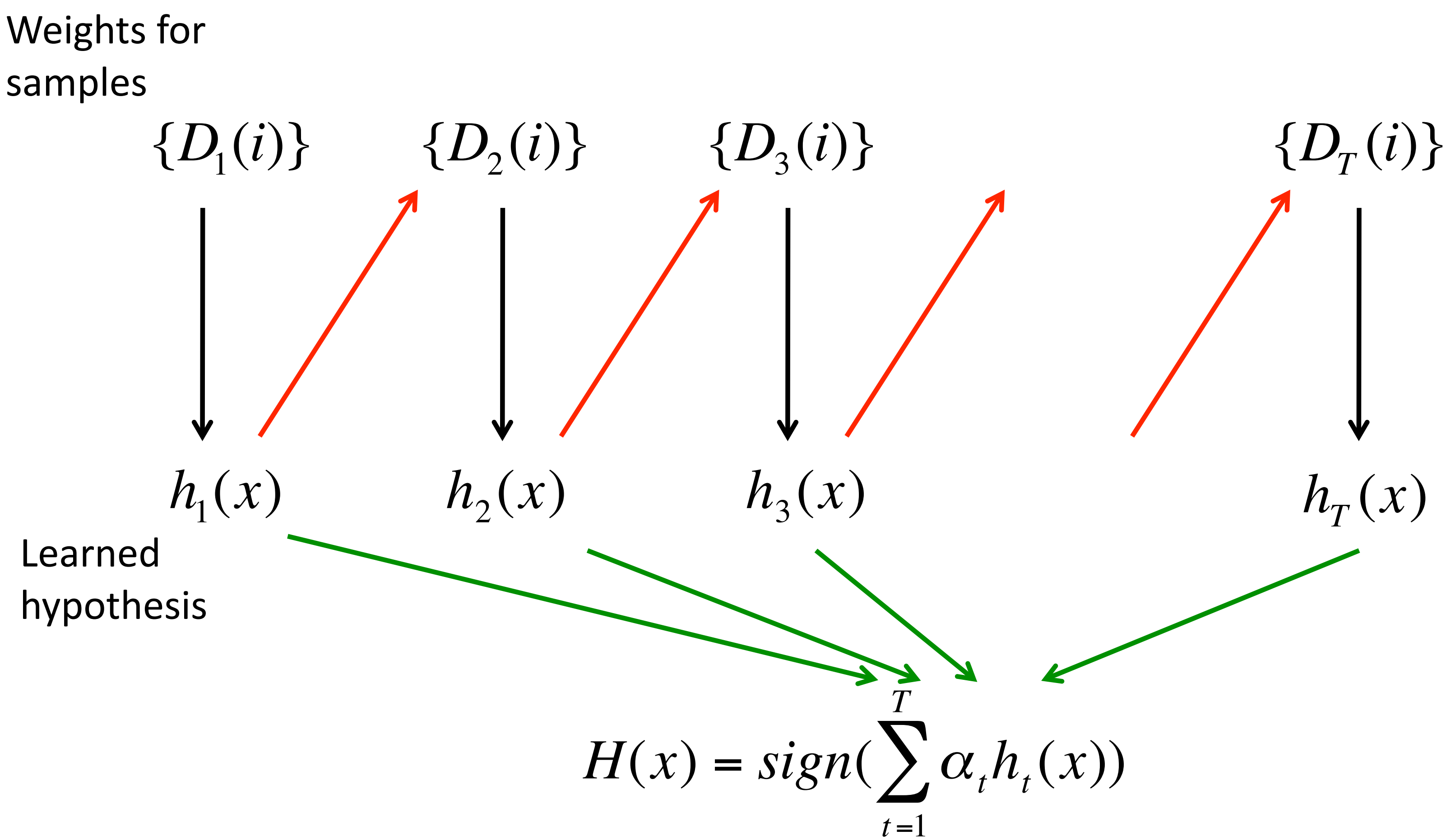
- Now, in all calculations, whenever used, *i*th training example counts as D(i) "examples"
  - e.g., in MLE redefine Count(Y=y) to be weighted count

Unweighted data	Weights D(i)
$\text{Count}(Y=y) = \sum_{i=1}^{m} I(Y^i=y)$	$\text{Count}(Y=y) = \sum_{i=1}^{m} D(i) I(Y^i=y)$

Boosting


Weights for samples
$\{D_1(i)\}$
$\{D_2(i)\}$
$\{D_3(i)\}$
$\{D_T(i)\}$
$h_1(x)$
$h_2(x)$
$h_3(x)$
$h_T(x)$
Learned hypothesis
$H(x) = sign(\sum_{t=1}^{T} \alpha_t h_t(x))$

Given: $(x_1, y_1), \ldots, (x_m, y_m)$ where $x_i \in X$, $y_i \in Y = \{-1, +1\}$

Initialize $D_1(i) = 1/m$. Initially equal weights

For $t = 1, \ldots, T$:

- Train weak learner using distribution D_t. Naïve Bayes, decision stump
- Get weak classifier $h_t : X \to \mathbb{R}$.
- Choose $\alpha_t \in \mathbb{R}$. Positive value
- Update:

Why?

$$\boxed{D_{t+1}(i) = \frac{D_t(i) \exp(-\alpha_t y_i h_t(x_i))}{Z_t}}$$

where Z_t is a normalization factor

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

Weights for all samples should sum to 1

$\sum_i D_{t+1}(i)=1$

Output the final classifier:

$$H(x) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(x)\right).$$

Given: $(x_1, y_1), \ldots, (x_m, y_m)$ where $x_i \in X$, $y_i \in Y = \{-1, +1\}$

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- Update: **Why?**

$$\boxed{D_{t+1}(i) = \frac{D_t(i) \exp(-\alpha_t y_i h_t(x_i))}{Z_t}}$$

$$\boxed{D_{t+1}(i) = \frac{D_t(i)}{Z_t} \begin{cases} e^{-\alpha_t} & \text{if } y_i = h_t(x_i) \\ e^{\alpha_t} & \text{if } y_i \neq h_t(x_i) \end{cases}}$$

Increase weight
if wrong on sample i

Output the final classifier:

$$H(x) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(x)\right).$$

What αt to choose for hypothesis h_t?

[Schapire, 1989]

- Weight update rule:

$$D_{t+1}(i) = \frac{D_t(i)\exp(-\alpha_t y_i h_t(x_i))}{Z_t}$$

$$\boxed{\alpha_t = \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)}$$

[Freund & Schapire '97]

$$\boxed{\epsilon_t = \sum_{i=1}^{m} D_t(i)\delta(h_t(x_i) \neq y_i)}$$

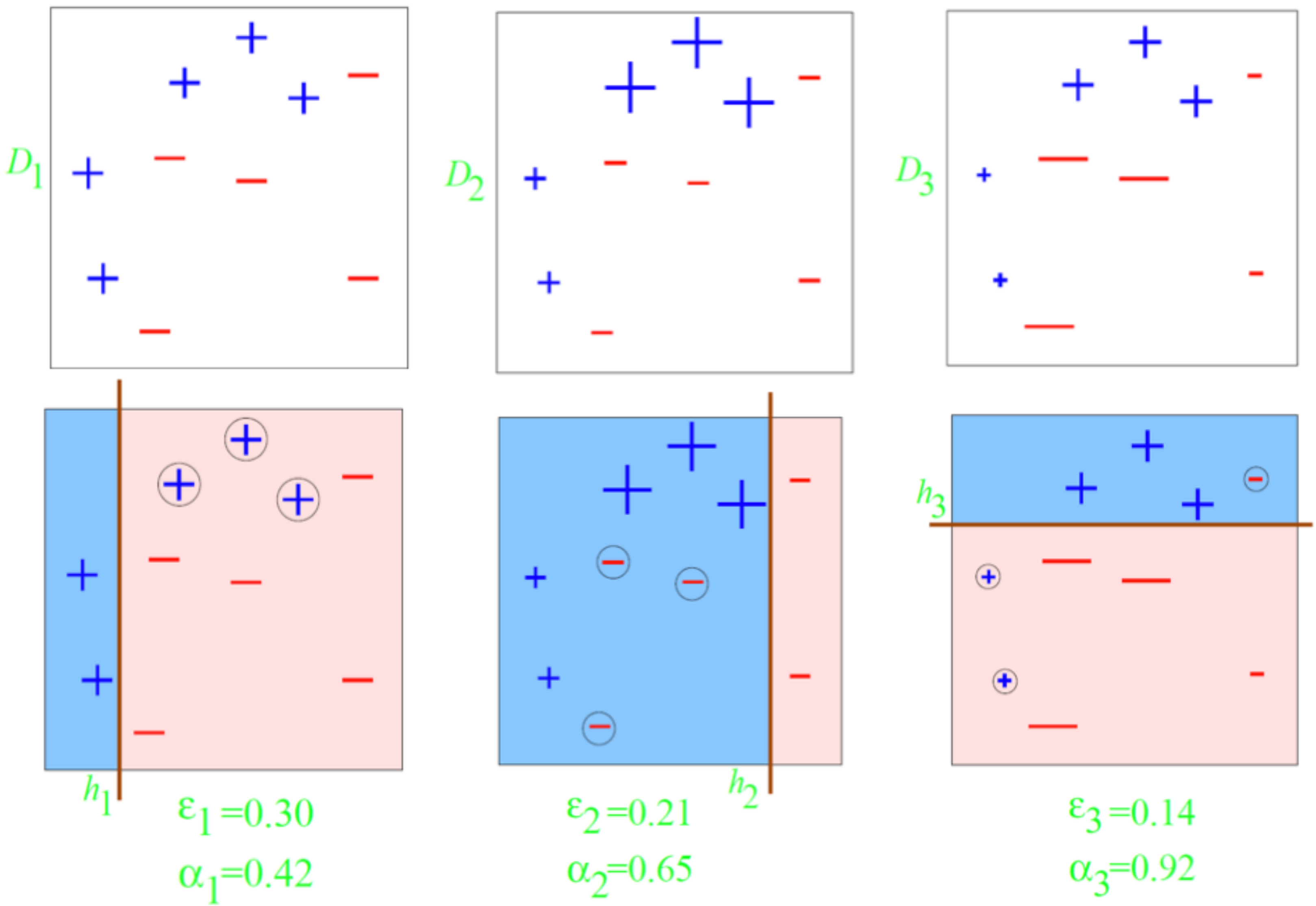
Weighted training error

$\varepsilon_t = 0$ if h_t perfectly classifies all weighted data pts — $\alpha_t = \infty$

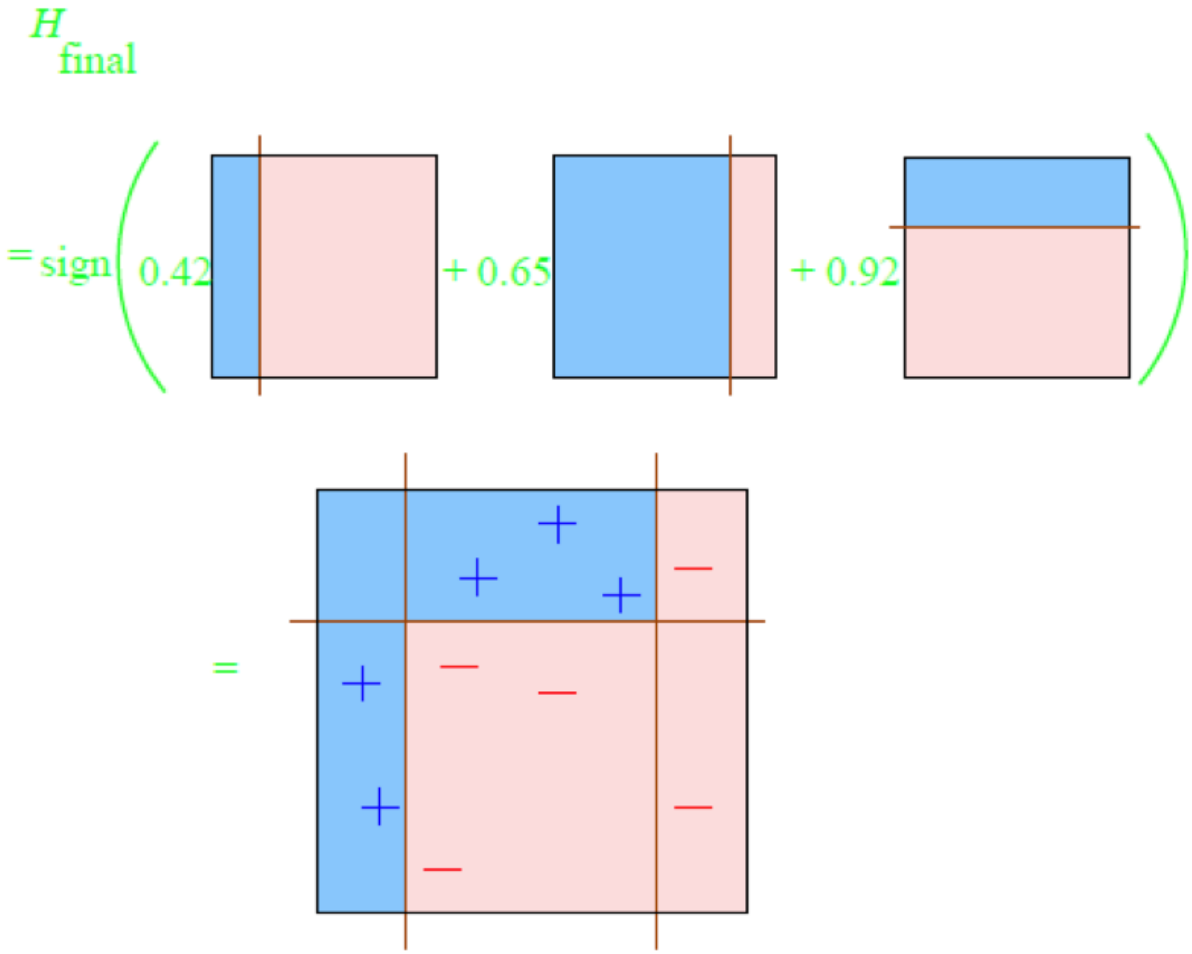
$\varepsilon_t = 1$ if h_t perfectly wrong => $-h_t$ perfectly right — $\alpha_t = -\infty$

$\varepsilon_t = 0.5$ — $\alpha_t = 0$

Boosting Example (Decision Stump)



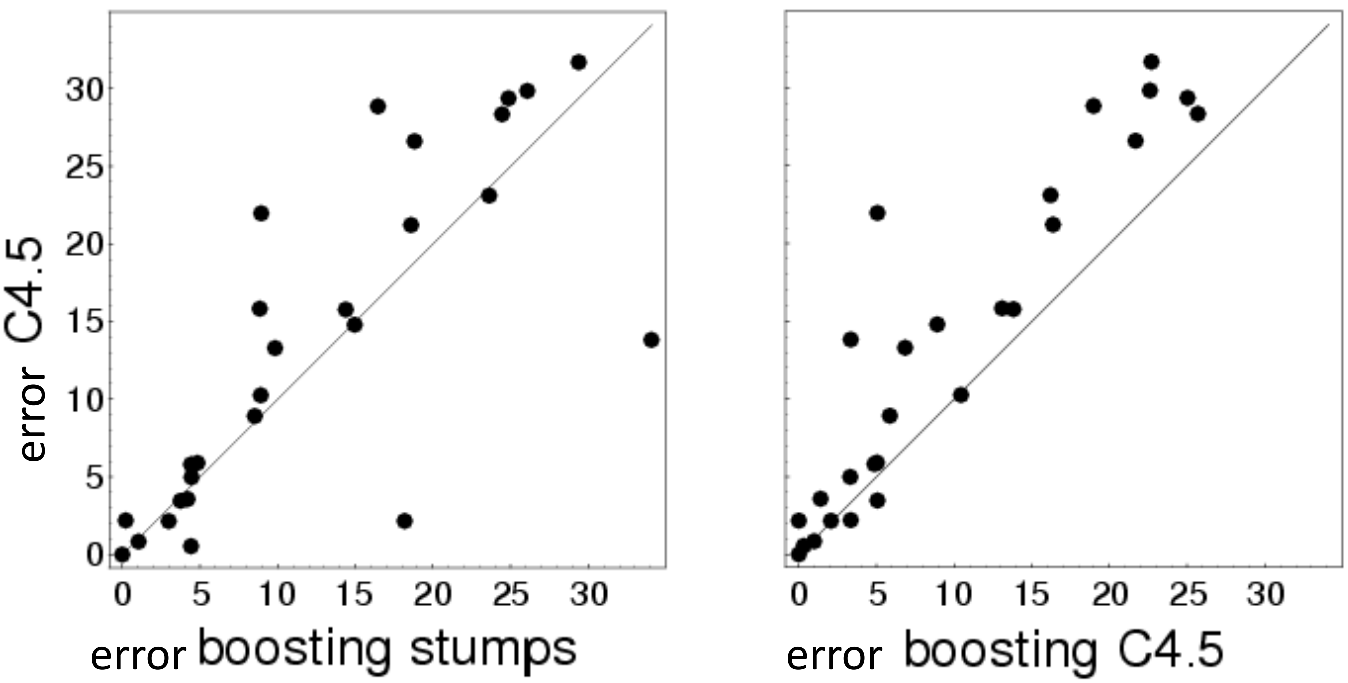
Boosting Example



Boosting: Experimental Results

[Freund & Schapire, 1996]

Comparison of C4.5, Boosting C4.5, Boosting decision stumps (depth 1 trees), 27 benchmark datasets



Analyzing Training Error

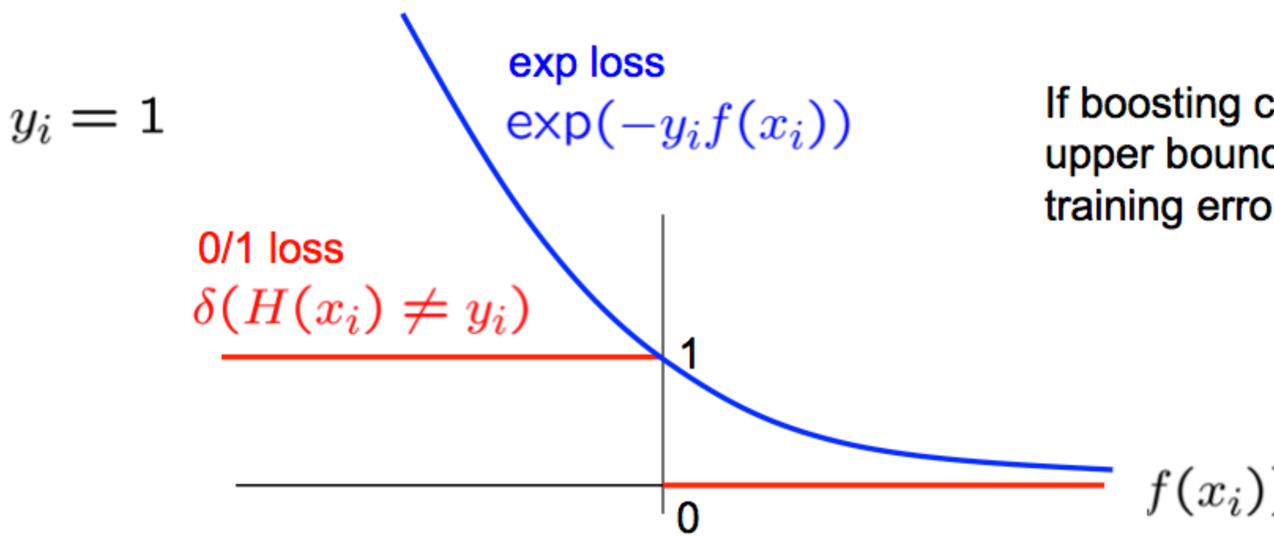
- Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i))$$

Convex upper bound

where

$$f(x)=\sum \alpha_t h_t(x);\, H(x)=sign(f(x))$$



If boosting can make upper bound → 0, then training error → 0

Analyzing Training Error

- Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \underline{\frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i)) = \prod_t Z_t}$$

where $f(x) = \sum_t \alpha_t h_t(x); H(x) = sign(f(x))$

$$\boxed{Z_t = \sum_{i=1}^{m} D_t(i)\exp(-\alpha_t y_i h_t(x_i))}$$

Analyzing Training Error

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$$\boxed{Z_t = \sum_{i=1}^{m} D_t(i)\exp(-\alpha_t y_i h_t(x_i))}$$

Proof: **Using Weight Update Rule**

$$D_{t+1}(i) = \frac{D_t(i)\exp(-\alpha_t y_i h_t(x_i))}{Z_t}$$

$$D_1(i) = 1/m.$$

$$D_2(i) = \frac{1}{m}\frac{e^{-\alpha_1 y_i h_1(x_i)}}{Z_1}$$

$$D_3(i) = \frac{1}{m}\frac{e^{-\alpha_1 y_i h_1(x_i)}e^{-\alpha_2 y_i h_2(x_i)}}{Z_1 Z_2}$$

$$\vdots$$

$$D_{T+1}(i) = \frac{1}{m}\frac{\exp(-y_i f(x_i))}{\prod_t Z_t}$$

Wts of all pts add to 1

$$\sum_{i=1}^{m} D_{T+1}(i) = 1$$

Analyzing Training Error

- Training error of final classifier is bounded by:

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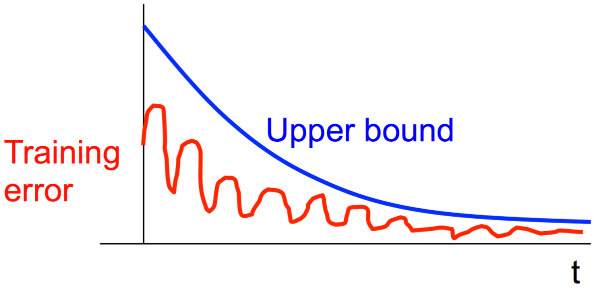
where

$$f(x) = \sum_t \alpha_t h_t(x); H(x) = sign(f(x))$$

If $Z_t < 1$, training error decreases exponentially (even though weak learners may not be good ε_t ~0.5)



Analyzing Training Error

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where

$$f(x)=\sum_t \alpha_t h_t(x);\, H(x)=sign(f(x))$$

If we minimize $\prod_t Z_t$, we minimize our training error

We can tighten this bound greedily, by choosing α_t and h_t on each iteration to minimize Z_t.

$$Z_t=\sum_{i=1}^{m} D_t(i)\exp(-\alpha_t y_i h_t(x_i))$$

What α_t to choose for hypothesis h_t?

[Schapire, 1989]

We can minimize this bound by choosing α_t on each iteration to minimize Z_t.

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

For boolean target function, this is accomplished by [Freund & Schapire '97]:

$$\boxed{\alpha_t = \frac{1}{2} \ln \left(\frac{1-\epsilon_t}{\epsilon_t} \right)}$$

Proof:
$$\begin{aligned} Z_t &= \sum_{i: y_i \neq h_t(x_i)} D_t(i) e^{\alpha_t} + \sum_{i: y_i = h_t(x_i)} D_t(i) e^{-\alpha_t} \\ &= \epsilon_t e^{\alpha_t} + (1-\epsilon_t) e^{-\alpha_t} \end{aligned}$$

$$\frac{\partial Z_t}{\alpha_t} = \epsilon_t e^{\alpha_t} - (1-\epsilon_t) e^{-\alpha_t} = 0 \qquad \Rightarrow e^{2\alpha_t} = \frac{1-\epsilon_t}{\epsilon_t}$$

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$$= 2\sqrt{\epsilon_t (1 - \epsilon_t)} = \sqrt{1 - (1 - 2\epsilon_t)^2}$$

Strong, weak classifiers

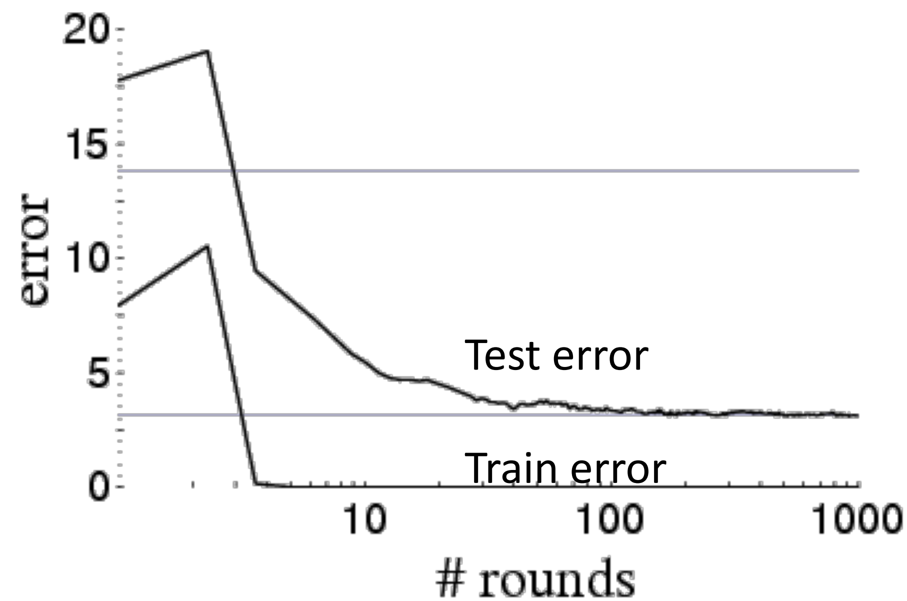
- Training error of the final classifier is bounded by

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i)\leq\prod_t Z_t\leq\exp\left(-2\sum_{t=1}^{T}(1/2-\epsilon_t)^2\right)$$

- If each classifier is (at least slightly) better than random ($\varepsilon_t < 0.5$), AdaBoost will achieve zero training error exponen$ally fast (in number of rounds T) !!

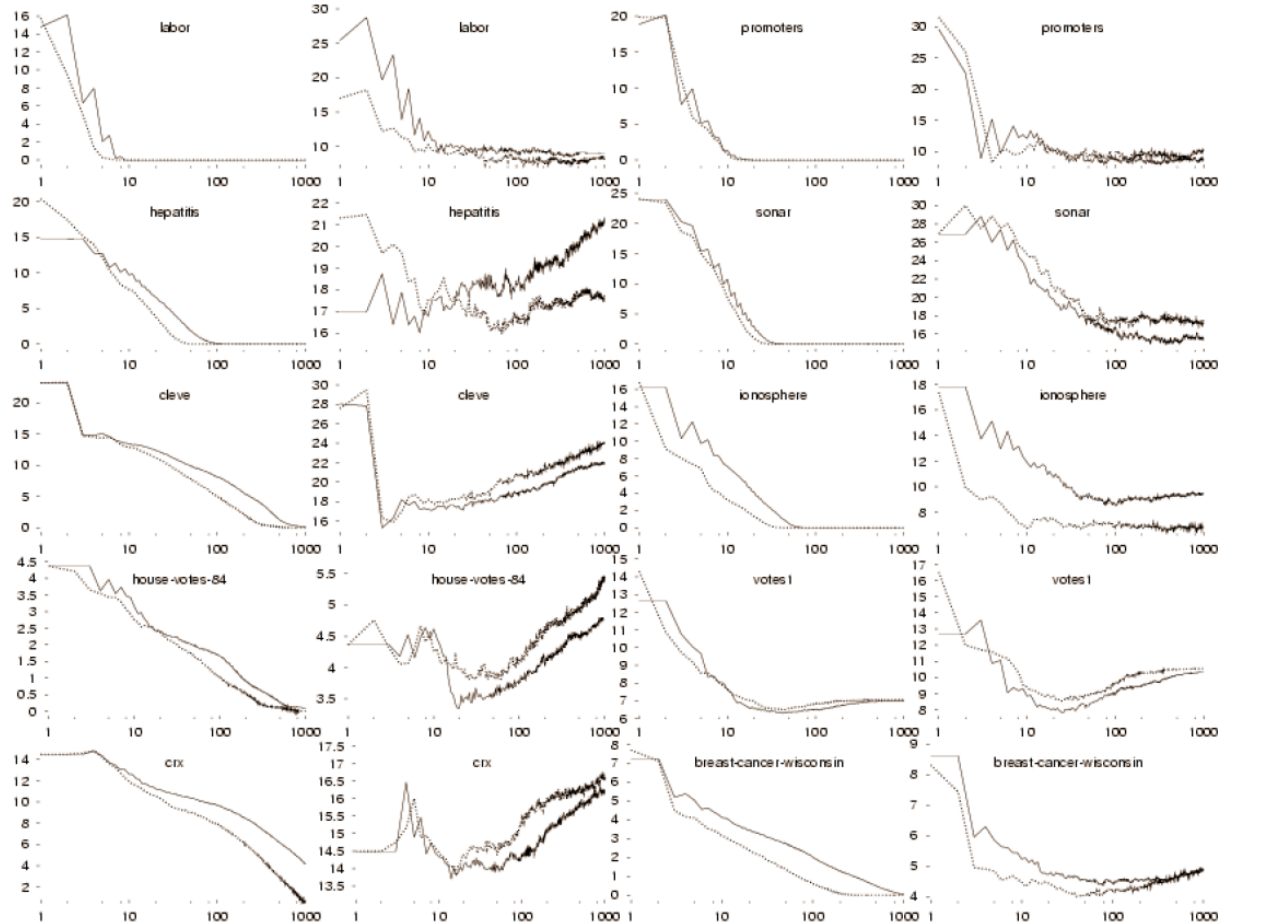
Boosting results – Digit recognition

[Schapire, 1989]



- Boosting often
  - Robust to overfitting
  - Test set error decreases even after training error is zero

AdaBoost and AdaBoost.MH on Train (left) and Test (right) data from Irvine repository. [Schapire and Singer, ML 1999]



Boosting and Logistic Regression

Logistic regression assumes:

$$P(Y = 1|X) = \frac{1}{1 + \exp(f(x))}$$

And tries to maximize data likelihood:

$$P(\mathcal{D}|H) = \prod_{i=1}^{m} \frac{1}{1 + \exp(-y_i f(x_i))}$$

Equivalent to minimizing log loss

$$\sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i)))$$

Boosting and Logistic Regression

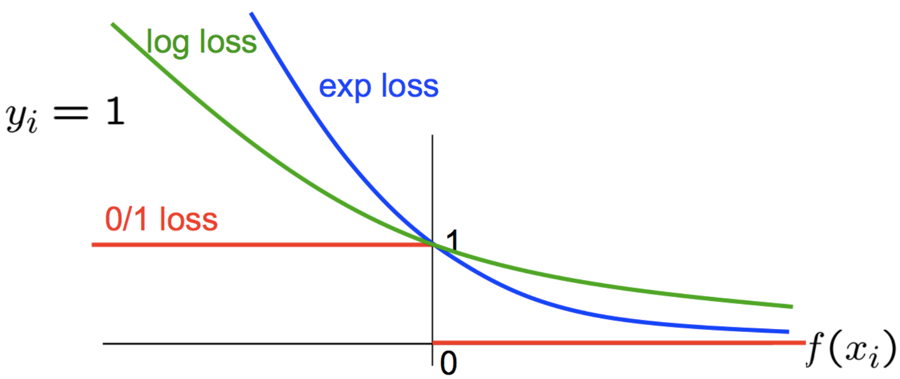
Logistic regression equivalent to minimizing log loss

$$\sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i)))$$

Boosting minimizes similar loss function!!

$$\frac{1}{m} \sum_{i} \exp(-y_i f(x_i))$$

Both smooth approximations of 0/1 loss!



Logistic regression and Boosting

Logistic regression:

- Minimize loss fn

$$\sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i)))$$

- Define

$$f(x) = \sum_j w_j x_j$$

where x_j predefined

Boosting:

- Minimize loss fn

$$\sum_{i=1}^{m} \exp(-y_i f(x_i))$$

- Define

$$f(x) = \sum_t \alpha_t h_t(x)$$

where $h_t(x_i)$ defined dynamically to fit data

(not a linear classifier)

- Weights α_t learned incrementally over t

Bagging

- Related approach to combining classifiers:

  1. Run independent weak learners on bootstrap replicates (sample with replacement) of the training set

  2. Average/vote over weak hypotheses

Bagging	Boosting
Resamples data points	Reweights data points (modifies their distribution)
Weight of each classifier is the same	Weight is dependent on classifier's accuracy

Effect of Outliers

- **Good:** Can identify outliers since focuses on examples that are hard to categorize
- **Bad:** Too many outliers can degrade classification performance dramatically increase time to convergence

What you need to know about Boosting

- Combine weak classifiers to obtain very strong classifier
  - Weak classifier – slightly better than random on training data
  - Resulting very strong classifier – can eventually provide zero training error
- AdaBoost algorithm
- Boosting vs Logistic Regression
  - Similar loss functions
  - Single optimization (LR) vs Incrementally improving classification (B)
- Most popular application of Boosting:
  - Boosted decision stumps!
  - Very simple to implement, very effective classifier