Markov Decision Process and Reinforcement Learning

Machine Learning 10-601B

Many of these slides are derived from Tom Mitchell, William Cohen, Eric Xing. Thanks!

Types of Learning

- Supervised Learning
  - A situation in which sample (input, output) pairs of the function to be learned can be perceived or are given
  - You can think about it as if there is a kind teacher

    - Training data: (X,Y). (features, label)

    - Predict Y, minimizing some loss.

    - Regression, Classification.

- Unsupervised Learning

    - Training data: X. (features only)

    - Find “similar” points in high-dim X-space.

    - Clustering.

Example of Supervised Learning

- Predict the price of a stock in 6 months from now, based on economic data. (Regression)
- Predict whether a patient, hospitalized due to a heart attack, will have a second heart attack. The prediction is to be based on demographic, diet and clinical measurements for that patient. (Logistic Regression)
- Identify the numbers in a handwritten ZIP code, from a digitized image (pixels). (Classification)

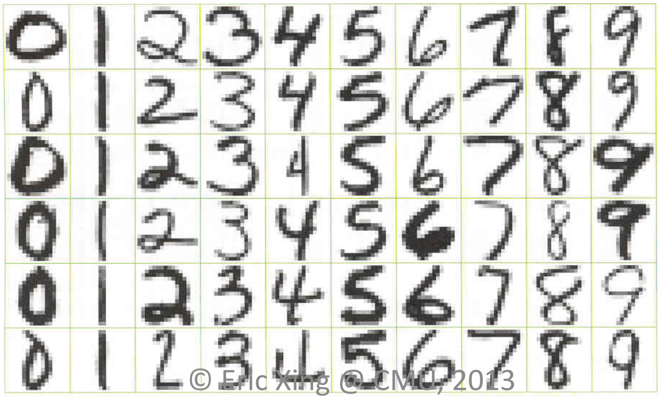


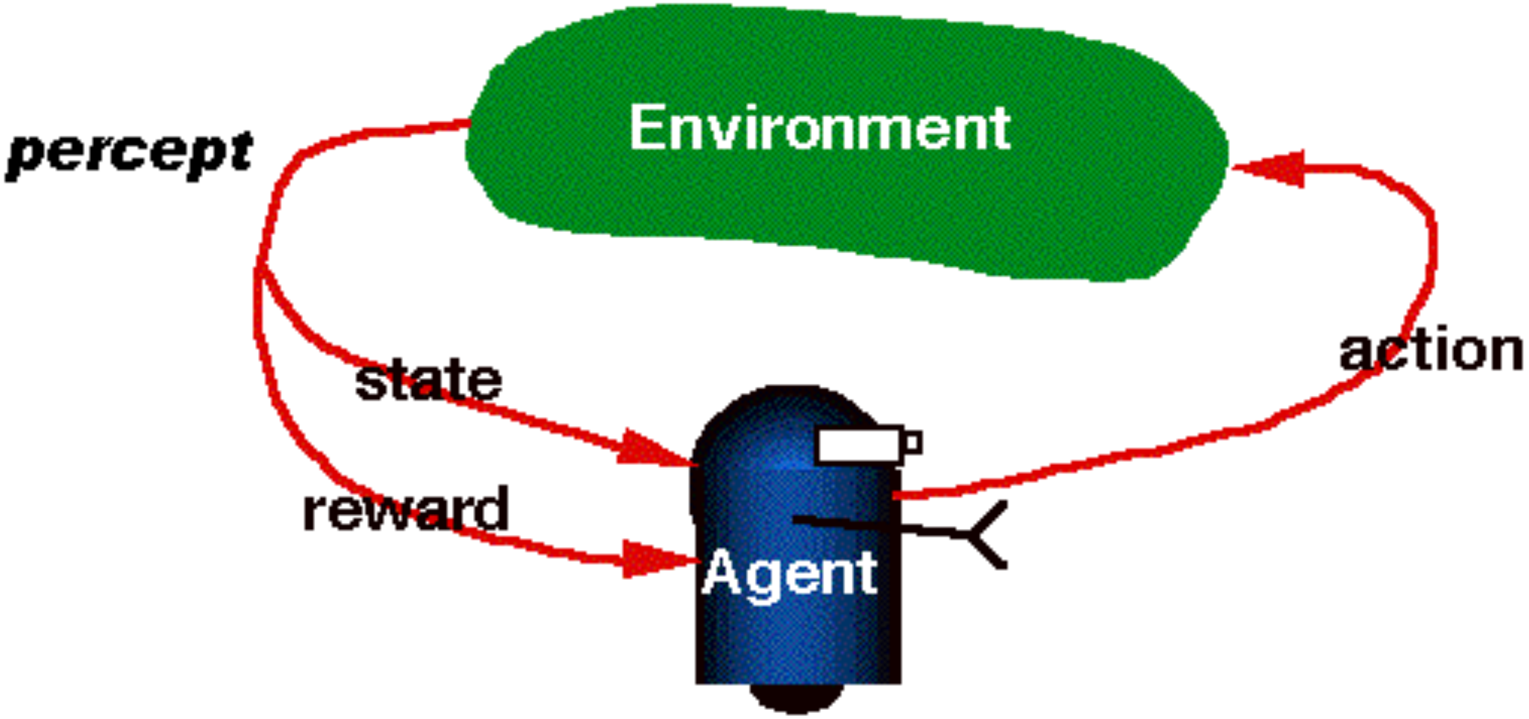
FIGURE 1.2. *Examples of handwritten digits from U.S. postal envelopes.*

What is Learning?

- Learning takes place as a result of interaction between an agent and the world. The idea behind learning is that
  - Percepts received by an agent should be used not only for understanding/interpreting/prediction, as in the machine learning tasks we have addressed so far,

    but also for **acting**, and furthermore for improving the agent's ability to behave optimally in the future to achieve the goal.

RL is learning from interaction



Examples of Reinforcement Learning

- How should a robot behave so as to optimize its "performance"? (Robotics)

- How to automate the motion of a helicopter? (Control Theory)

- How to make a good chess-playing program? (Artificial Intelligence)

Supervised Learning vs Reinforcement Learning

- Supervised learning

  $$f : X \rightarrow Y$$

  - X: inputs
  - Y: outputs
  - The predicted outputs can be evaluated immediately by the teacher
  - f evaluated with loss function

- Reinforcement learning for learning a policy

  $$\pi \ : \ S \mapsto A$$

  - S: states
  - A: actions
  - Take action A to affect state S
  - The predicted action A cannot be evaluated directly but can be given positive/negative rewards
  - Policy evaluated with value functions

Reinforcement Learning

- Reinforcement Learning
  - in the case of the agent acting on its environment, it receives some evaluation of its action (reinforcement), but is not told about which action is the correct one to achieve its goal
  - Imagine learning to go back to GHC from this room …

- Training data: (S, A, R). (State-Action-Reward)

- Develop an optimal policy (sequence of decision rules) for the learner so as to maximize its **long-term reward**.

- Robotics, Board game playing programs.

Elements of RL

- policy $\pi \; : \; S \mapsto A$
  - A map from state space to action space.
  - May be stochastic.
- reward function *R(S)*
  - It maps each state (or, state-action pair) to a real number, called reward.
- value function
  - Value of a state (or, state-action pair) is the total expected reward, starting from that state (or, state-action pair).

History of Reinforcement Learning

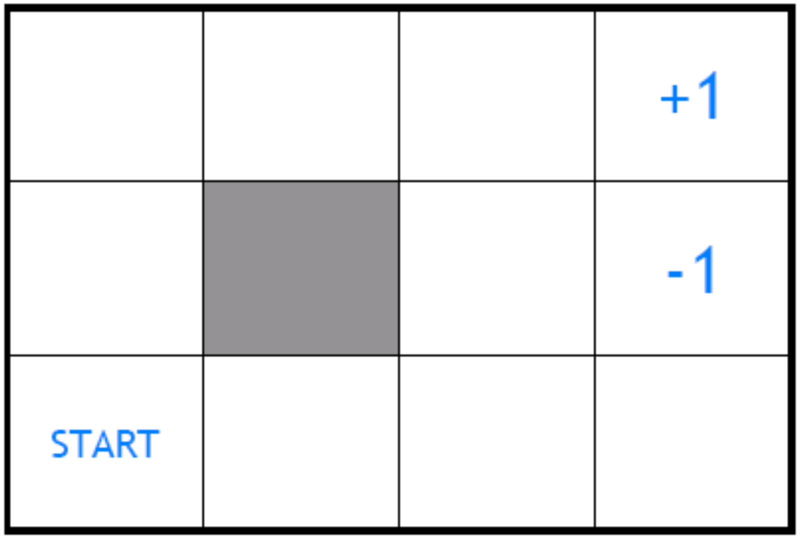
- Roots in the psychology of animal learning (Thorndike,1911).

- Another independent thread was the problem of optimal control, and its solution using dynamic programming (Bellman, 1957).

- Idea of temporal difference learning (on-line method), e.g., playing board games (Samuel, 1959).

- A major breakthrough was the discovery of Q-learning (Watkins, 1989).

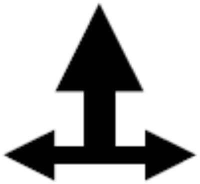
Robot in a room



actions: UP, DOWN, LEFT, RIGHT

UP

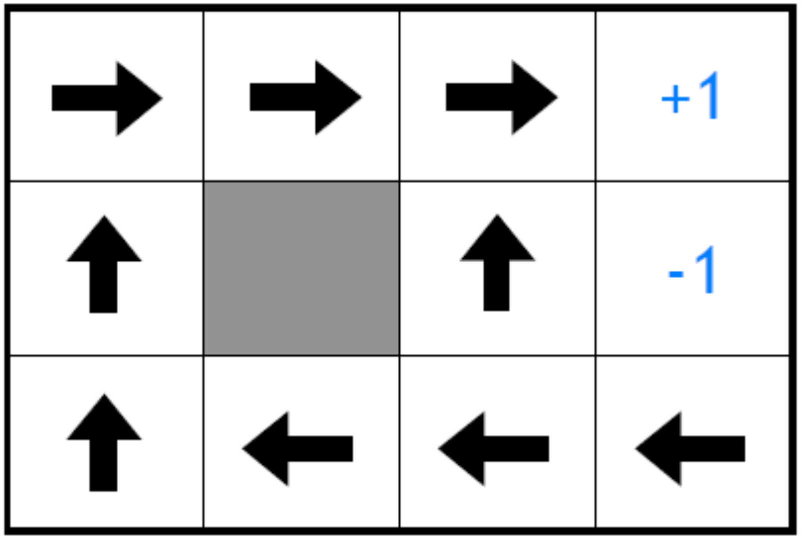
80%	move UP
10%	move LEFT
10%	move RIGHT



- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step

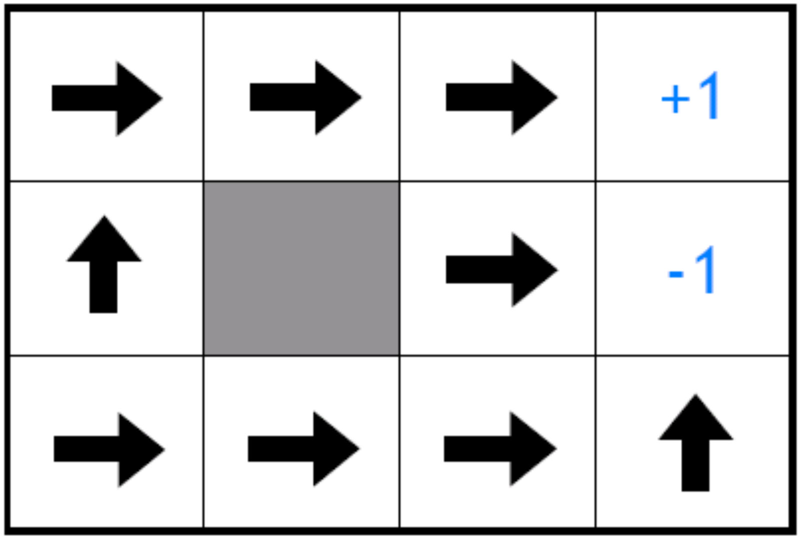
- what’s the strategy to achieve max reward?
- what if the actions were NOT deterministic?

Policy



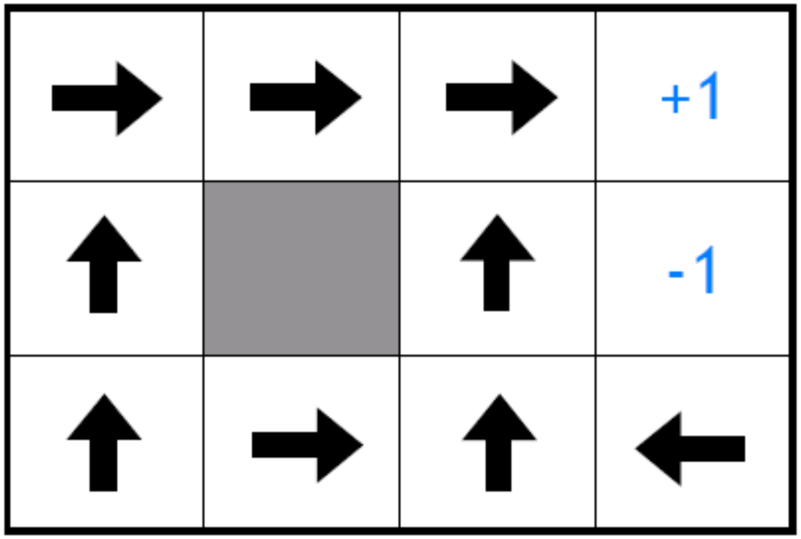
→	→	→	+1
↑	▨	↑	-1
↑	←	←	←

Reward for each step -2



→	→	→	+1
↑		→	-1
→	→	→	↑

Reward for each step: -0.1



Reward for each step: -0.04

→	→	→	+1
↑	▨	↑	-1
↑	←	←	←

The Precise Goal

- To find a policy $\pi : S \mapsto A$ that maximizes the Value function.

- There are different approaches to achieve this goal in various situations.

- Value iteration and Policy iteration are two more classic approaches to this problem. But essentially both are dynamic programming.

- Q-learning is a more recent approaches to this problem. Essentially it is a temporal-difference method.

Markov Decision Processes

A Markov decision process is a tuple $(S, A, \{P_{sa}\}, \gamma, R)$ where:

- S is a set of **states**. (For example, in autonomous helicopter flight, S might be the set of all possible positions and orientations of the helicopter.)

- A is a set of **actions**. (For example, the set of all possible directions in which you can push the helicopter's control sticks.)

- P_{sa} are the state transition probabilities. For each state $s \in S$ and action $a \in A$, P_{sa} is a distribution over the state space. We'll say more about this later, but briely, P_{sa} gives the distribution over what states we will transition to if we take action a in state s.

- $\gamma \in [0, 1)$ is called the **discount factor**.

- $R : S \times A \mapsto \mathbb{R}$ is the **reward function**. (Rewards are sometimes also written as a function of a state S only, in which case we would have $R : S \mapsto \mathbb{R}$).

The dynamics of an MDP

- We start in some state s_0, and get to choose some action $a_0 \in A$
- As a result of our choice, the state of the MDP randomly transitions to some successor state s_1, drawn according to $s_1 \sim P_{s0a0}$
- Then, we get to pick another action a_1
- …

$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} s_3 \xrightarrow{a_3} \dots$$

The dynamics of an MDP, (Cont'd)

- Upon visiting the sequence of states s_0, s_1, …, with actions a_0, a_1, …, our total payoff is given by

$$R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \ldots$$

- Or, when we are writing rewards as a function of the states only, this becomes

$$R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

  - For most of our development, we will use the simpler state-rewards R(s), though the generalization to state-action rewards R(s; a) offers no special diffculties.

- Our goal in reinforcement learning is to choose actions over time so as to maximize the expected value of the total payoff:

$$\mathrm{E}\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots\right]$$

Policy

- A policy is any function $\pi : S \mapsto A$ ..apping from the states to the actions.

- We say that we are executing some policy if, whenever we are in state s, we take action $a = \pi(s)$.

- We also define the value function for a policy π according to

$$V^{\pi}(s) = \mathrm{E}\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots \mid s_0 = s, \pi\right]$$

  - $V^{\pi}(s)$ is simply the expected sum of discounted rewards upon starting in state s, and taking actions according to π.

Value Function

- Given a fixed policy π, its value function V^π satisfies the **Bellman equations**:

$$V^\pi(s) = R(s) + \gamma \sum_{s' \in \mathcal{S}} P_{s\pi(s)}(s') V^\pi(s')$$

Immediate reward

expected sum of future discounted rewards

  - Bellman's equations can be used to efficiently solve for V^π (see later)

The Grid world

M = 0.8 in direction you want to go

0.2 in perpendicular < 0.1 left
0.1 right

Policy: mapping from states to actions

An optimal policy for the stochastic environment:

3	→	→	→	+1
2	↑	▨	↑	-1
1	↑	←	←	←
	1	2	3	4

utilities of states:

3	0.812	0.868	0.912	+1
2	0.762	▨	0.660	-1
1	0.705	0.655	0.611	0.388
	1	2	3	4

Environment < Observable (accessible): percept identifies the state
Partially observable

Markov property: Transition probabilities depend on state only, not on the path to the state.

Markov decision problem (MDP).

Partially observable MDP (POMDP): percepts does not have enough info to identify transition probabilities.

Optimal value function

- We define the optimal value function according to

$$V^*(s) = \max_{\pi} V^{\pi}(s) \qquad (1)$$

  - In other words, this is the best possible expected sum of discounted rewards that can be attained using any policy

- There is a version of Bellman's equations for the optimal value function:

$$V^*(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s') \qquad (2)$$

  - Why?

Optimal policy

- We also define the optimal policy: $\pi^* : S \mapsto A$ as follows:

$$\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in \mathcal{S}} P_{sa}(s') V^*(s') \qquad (3)$$

- Fact:
  - $$V^*(s) = V^{\pi^*}(s) \geq V^{\pi}(s)$$
  - Policy π^* has the interesting property that it is the optimal policy for all states s.
    - It is not the case that if we were starting in some state s then there'd be some optimal policy for that state, and if we were starting in some other state s_0 then there'd be some other policy that's optimal policy for s_0.
    - The same policy π^* attains the maximum above for all states s. This means that we can use the same policy no matter what the initial state of our MDP is.

The Basic Setting for Learning

- Training data: n finite horizon trajectories, of the form

  $$\{s_0, a_0, r_0, \ldots, s_T, a_T, r_T, s_{T+1}\}.$$

- Deterministic or stochastic policy: A sequence of decision rules

  $$\{\pi_0, \pi_1, \ldots, \pi_T\}.$$

- Each π maps from the observable history (states and actions) to the action space at that time point.

Algorithm 1: Value iteration

- Consider only MDPs with finite state and action spaces $(|S| < \infty,\ |A| < \infty)$

- The value iteration algorithm:

```
1. For each state $s$, initialize $V(s) := 0$.

2. Repeat until convergence {

       • For every state, update
         $V(s) := R(s) + \max_{a \in A} \gamma \sum_{s' \in \mathcal{S}} P_{sa}(s') V^{\cancel{*}}(s')$.

   }
```

- It can be shown that value iteration will cause V to converge to $V*$. Having found $V*$, we can find the optimal policy as follows:

$$\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in \mathcal{S}} P_{sa}(s') V^*(s')$$

Algorithm 2: Policy iteration

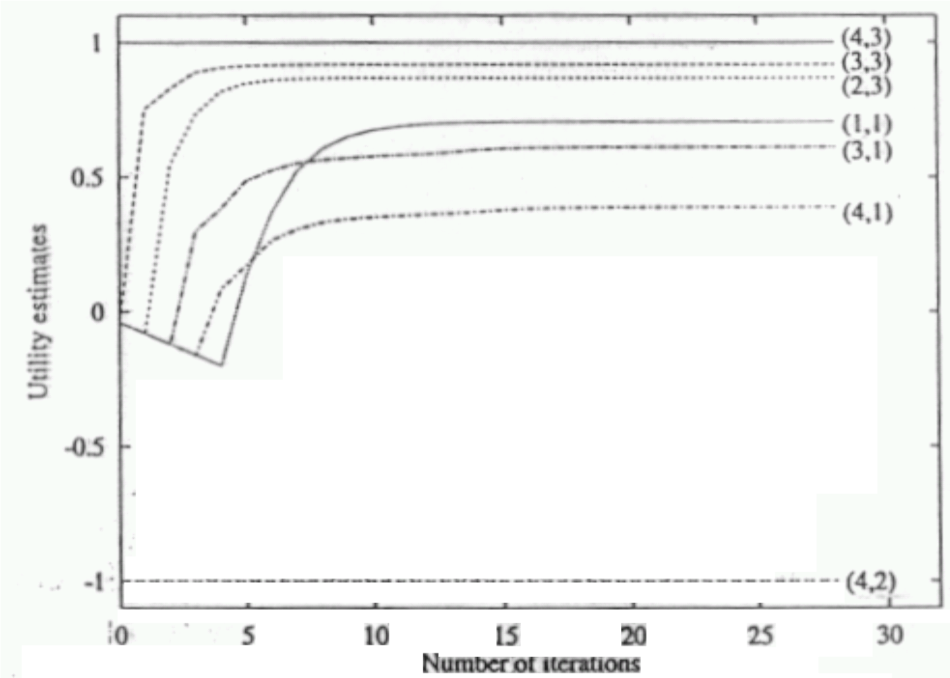
- The policy iteration algorithm:

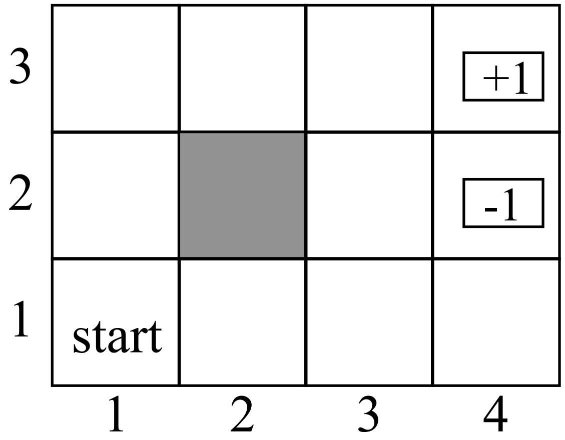
```
1. Initialize $\pi$ randomly.
2. Repeat until convergence {
       • Let $V := V^{\pi}$
       • For each state $s$, let $\pi(s) := \max_{a \in A} \sum_{s' \in \mathcal{S}} P_{sa}(s') V^{*}(s')$.
   }
```

  - The inner-loop repeatedly computes the value function for the current policy, and then updates the policy using the current value function.
  - Greedy update
  - After at most a finite number of iterations of this algorithm, V will converge to $V*$, and π will converge to $\pi*$.

Convergence

- The utility values for selected states at each iteration step in the application of VALUE-ITERATION to the 4x3 world in our example





Thrm: As t→∞, value iteration converges to exact U even if updates are done asynchronously & *i* is picked randomly at every step.

Convergence

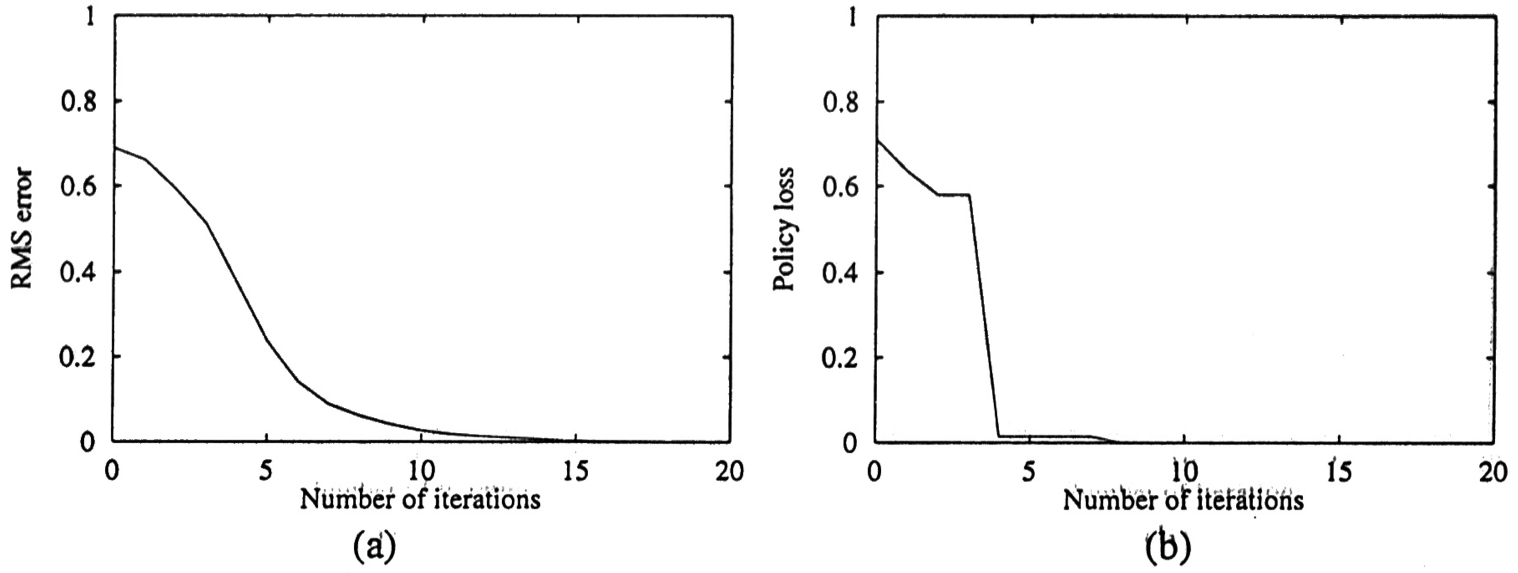


Figure 17.6 (a) The RMS (root mean square) error of the utility estimates compared to the correct values, as a function of iteration number during value iteration. (b) The expected policy loss compared to the optimal policy.

When to stop value iteration?

Q learning

- Define Q-value function

$$V(s) = \max_a Q(s, a)$$

- Rule to choose the action to take

$$a = \arg\max_a Q(s, a)$$

Algorithm 3: Q learning

```
For each pair (s, a), initialize Q(s,a)
Observe the current state s
Loop forever
{
    Select an action a (optionally with ε-exploration) and execute it
        $a = \arg\max_{a} Q(s,a)$
    Receive immediate reward r and observe the new state s’
    Update Q(s,a)
        $Q(s,a) \leftarrow Q(s,a) + \alpha[r_{t+1} + \gamma \max_{a'} Q(s',a') - Q(s,a)]$
    s=s’
}
```

Exploration

- Tradeoff between exploitation (control) and exploration (identification)

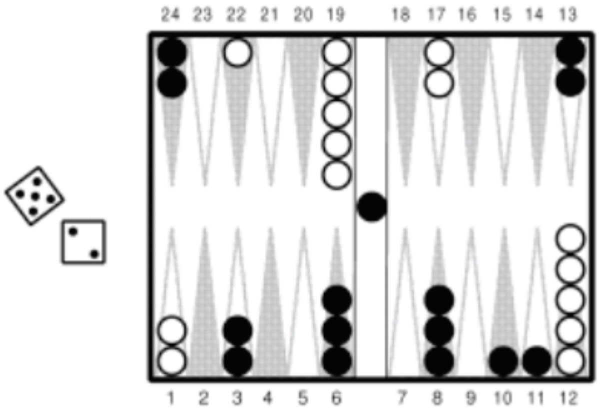
- Extremes: greedy vs. random acting

  (n-armed bandit models)

Q-learning converges to optimal Q-values if

- Every state is visited infinitely often (due to exploration),

A Success Story



- TD Gammon (Tesauro, G., 1992)

  - A Backgammon playing program.

  - Application of temporal difference learning.

  - The basic learner is a neural network.

  - It trained itself to the world class level by playing against itself and learning from the outcome. So smart!!

What is special about RL?

- RL is learning how to map states to actions, so as to maximize a numerical reward over time.
- Unlike other forms of learning, it is a multistage decision-making process (often Markovian).
- An RL agent must learn by trial-and-error. (Not entirely supervised, but interactive)
- Actions may affect not only the immediate reward but also subsequent rewards (Delayed effect).

Summary

- Both value iteration and policy iteration are standard algorithms for solving MDPs, and there isn't currently universal agreement over which algorithm is better.
- For small MDPs, value iteration is often very fast and converges with very few iterations. However, for MDPs with large state spaces, solving for V explicitly would involve solving a large system of linear equations, and could be difficult.
- In these problems, policy iteration may be preferred. In practice value iteration seems to be used more often than policy iteration.
- Q-learning is model-free, and explore the temporal difference

Types of Learning

- Supervised Learning

  - Training data: (X,Y). (features, label)
  - Predict Y, minimizing some loss.
  - Regression, Classification.

- Unsupervised Learning

  - Training data: X. (features only)
  - Find "similar" points in high-dim X-space.
  - Clustering.

- Reinforcement Learning

  - Training data: (S, A, R). (State-Action-Reward)

  - Develop an optimal policy (sequence of decision rules) for the learner so as to maximize its long-term reward.
  - Robotics, Board game playing programs