Median graphs and CAT(0)-cube complex

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Abstract

In this talk, we will introduce median graphs and hyperplanes in median graphs. We will also see some concrete examples that illustrate these definitions. After the first two parts, we will talk about the equivalence between median graphs and CAT(0)-cube complex.

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1 Median graphs

A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$. We assume the graph G to be undirected and simple (no multi-edges). The elements of V are the *vertices*, and the elements of E are the *edges*. The vertex set of the graph G is referred to as $V(G)$, its edge set as $E(G)$. Two vertices u and v are adjacent if there is an edge uv connecting them.

We then introduce some typical examples of graphs:

Definition 1.1 (Complete graph)**.** A graph G is complete, if all the vertices of G are pairwise adjacent.

A complete graph of order n is denoted as K_n.

Definition 1.2 (Path)**.** A *path* is a non-empty graph $P = (V, E)$ of the form

$$\begin{aligned} V(P) &= \{x_0, x_1, \cdots, x_k\} \\ E(P) &= \{x_0x_1, x_1x_2, \cdots x_{k-1}x_k\} \end{aligned}$$

where all x_i are distinct.

The *length* of a path is its number of edges.

Definition 1.3 (Cycle)**.** In the definition above, if $k \geq 3$ and x_0 coincides with x_k, the graph is called a k-cycle, denoted by C_k.

Definition 1.4 (Forest and tree)**.** An acyclic graph G that does not contain any cycles is called a *forest*. A connected forest is called a *tree*.

It follows that a forest is a graph whose components are trees.

The definitions above are more or less standard, interested readers may read [2].

In this talk, we are mainly concerned with a specific type of graphs:

Definition 1.5 (Median graph)**.** A median graph is a graph G, in which every three vertices u, v, and w have a unique *median*: a vertex $m(u, v, w)$ that belongs to the shortest paths between each pair of vertices (u, v), (u, w), and (v, w).

To introduce an important property of median graphs, we need to define the *Cartesian product of graphs*:

Definition 1.6 (Cartesian product of graphs)**.** The Cartesian product $G \square H$ of graphs G and H is a graph, such that

1. Its vertex set is the Cartesian product of $V(G) \times V(H)$,
2. Two vertices (u, u') and (v, v') are adjacent in the product if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.

We will now give some examples of median graphs. Specifically, we will try to prove that trees are median graphs.

Proposition 1.7. *The* 4*-cycle* C_4 *is a median graph.*

Proof. The proof is left as an exercise. □

Proposition 1.8. *Any tree graph is a median graph.*

Proof. We may choose any three arbitrary vertices a, b, and c in a tree graph G, which means that there exist unique (shortest) paths with (unordered) pairs of starting and end points (a, b), (a, c), and (b, c). We denote them by $P(a, b)$, $P(a, c)$, and $P(b, c)$, respectively.

Without loss of generality, we may choose any two paths, say $P(a, b)$ and $P(b, c)$. Because the tree is connected and acyclic, there exists a unique path from a to c that includes the path from a to b, via some vertex m, from which the path goes to c. It implies that m is the median vertex that also belongs to the third path $P(a, c)$. □

Fun fact: Some trees have interesting names, such as *cherry* $K_{1,2}$ and *claw* $K_{1,3}$.

Given what we already have and prove, how would you try to prove the following:

Exercise 1.9 (Grid graphs are median graphs)**.** The vertices of the *grid graph* are the points in the plane with integer coordinates, and they form edges whenever two points are at distance 1.

Try to show that grid graphs are median graphs.

With examples of median graphs in our hand, it is also beneficial to equip ourselves with non-examples, so that we could better understand median graphs:

Non-example 1.10. The triangle K_3 and the complete bipartite graph $K_{2,3}$ are not median graphs.

In fact, with the above non-examples, we have the following (folklore) theorem that can help us detect any other non-examples.

Theorem 1.11. *Any median graph does not contain subgraph isomorphic to* K_3 *or* $K_{2,3}$*.*

Exercise 1.12. We have seen that the triangle K_3 (or C_3) is not median, the 4-cycle C_4 is median. Are the k-cycles median, for $k \geq 5$?

At the end of this section, I would like to include the following two interesting facts related to median graphs:

Theorem 1.13 ([3]). *It is computational equivalent to test whether a graph is median and whether it is triangle-free.*

Theorem 1.14 ([4]). *The only regular median graphs are the hypercubes.*

Interested readers may refer to [1] for more information on median graphs.

References

[1] Hans-Jurgen Bandelt and Victor Chepoi. Metric graph theory and geometry: a survey.

[2] Béla Bollobás. *Modern graph theory*, volume 184. Springer Science & Business Media, 1998.

[3] Wilfried Imrich, Sandi Klavzar, and Henry Martyn Mulder. Median graphs and triangle-free graphs. *SIAM Journal on Discrete Mathematics*, 12(1):111–118, 1999.

[4] Martyn Mulder. n-cubes and median graphs. *Journal of Graph Theory*, 4(1):107–110, 1980.