

# Notes for MATH 3210: Foundation of Analysis I

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## Contents

<b>1</b>	<b>Ring and Field</b>	<b>1</b>
1.1	Ring . . . . .	2
1.1.1	Addition Axiom . . . . .	2
1.1.2	Multiplication Axiom . . . . .	2
1.2	Field . . . . .	2
1.3	Construction of Integers . . . . .	2
1.4	Construction of Fractions (Rational Numbers) . . . . .	4
1.5	Ordered Set . . . . .	5
1.5.1	Partial Order . . . . .	5
1.5.2	Total Order . . . . .	5
1.5.3	Least-upper-bound Property . . . . .	6
<b>2</b>	<b>Basic Topology</b>	<b>7</b>
2.1	Metric Space $M$ . . . . .	7
2.2	Open Sets . . . . .	7
2.2.1	Metric Spaces $X$ . . . . .	7
2.2.2	Topological Spaces $(X, U)$ . . . . .	8
2.3	Compact Sets . . . . .	8

## 1 Ring and Field

Notations:

- $\mathbb{N}$ : The set of natural numbers;
- $\mathbb{Z}$ : The set of integers (ring, not field, has no inverse);
- $\mathbb{Q}$ : The set of rational numbers;
- $\mathbb{R}$ : The set of real numbers (ring and field).

## 1.1 Ring

The set  $A$  has two binary operations, addition and multiplication.

For any  $a, b \in A$ ,

$$\begin{aligned} A * A &\rightarrow A \\ a, b &\rightarrow a + b \end{aligned}$$

### 1.1.1 Addition Axiom

1.  $a + b = b + a$  (commutative)
2.  $(a + b) + c = a + (b + c)$  (associative)
3. There is an element  $0$  such that  $a + 0 = a$  (additive identity)
4. There exists  $-a$  (additive inverse) such that  $a + (-a) = 0$

### 1.1.2 Multiplication Axiom

1.  $a \cdot b = b \cdot a$
2.  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element  $1$  such that  $a \cdot 1 = a$  and  $0 \neq 1$
4.  $a \cdot (b + c) = a \cdot b + a \cdot c$  (distributive)

## 1.2 Field

$A$  is a field if  $A$  is a *ring* for any  $a \in A$  and  $a \neq 0$ , there exists  $a^{-1}$  (inverse of  $a$ ) such that  $a \cdot a^{-1} = 1$ .

$A$  has no zero divisors if  $x \neq 0, y \neq 0 \rightarrow x \cdot y \neq 0$

**Example:**  $x, y \in A, x \neq 0, y \neq 0, xy = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

## 1.3 Construction of Integers

We need to construct  $\mathbb{Z}$  from  $\mathbb{N}$  with **localization**.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Set of equivalence class:  $\mathbb{N} \times \mathbb{N} / \sim$

Equivalence relation:  $(a, b) \sim (a', b')$  if  $a + b' = b + a'$ , since  $a - b = a' - b'$

$$\begin{aligned}
(a, b) &\sim (a', b') \sim (a'', b'') \\
a' + b'' &= b' + a'' \\
(a + b'') + b' &= (a + b') + b'' = b + a' + b'' = b + b' + a'' = (a'' + b) + b' \\
\therefore a + b'' &= a'' + b \\
\therefore (a, b) &\sim (a'', b'')
\end{aligned}$$

We then should define addition:

$$[(a, b)] + [(a', b')] = [(a + a', b + b')] \quad (1)$$

$$[(a_1, b_1)] + [(a', b')] = [(a_1 + a', b_1 + b')] \quad (2)$$

Equation 1 and equation 2 are equal since independent of choice of equivalence classes:

$$\begin{aligned}
a + a' + b_1 + b' &= a_1 + a' + b + b' \\
a + b_1 &= a_1 + b
\end{aligned}$$

Additive identity:  $0 = [(0, 0)]$

For example:  $[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)] = 0$

We then need to define multiplication:

$$[(a, b)] \cdot [(a', b')] = [(aa' + bb', ab' + a'b)]$$

$$\begin{aligned}
&(a - b)(a' - b') \\
&= aa' - ab' - a'b + bb' \\
&= (aa' + bb') - (ab' + a'b)
\end{aligned}$$

Multiplicative identity:  $1 = [(1, 0)]$

For example:  $[(a, b)] \cdot [(1, 0)] = [(a, b)]$

$\therefore \mathbb{N} \times \mathbb{N} / \sim$  is a ring, that is,  $\mathbb{Z}$ .

$$\begin{aligned}
\mathbb{N} &\longrightarrow \mathbb{Z} \\
m &\longrightarrow [(m, 0)] \\
-m &\longrightarrow [(0, m)] = -[(m, 0)]
\end{aligned}$$

For example:

$$\begin{aligned}
[(a, b)] &= [(a, 0)] + [(0, b)] \\
&= [(a, 0)] - [(b, 0)]
\end{aligned}$$

## 1.4 Construction of Fractions (Rational Numbers)

We need to show that  $K$  is a field (with addition and multiplication). Assume  $A$  is a commutative ring with no zero divisors, and  $A^* = A - \{0\}$ .

$$A \times A^* = \{(a, b) \mid a, b \in A, b \neq 0\}$$

Set of equivalence classes:  $K = (A \times A^*) / \sim$  (elements of  $K$  are fractions)

We first need to define addition:

$$\begin{aligned}(a, b) &\sim (a', b') \\ \frac{a}{b} &= \frac{a'}{b'} \\ ab' &= a'b\end{aligned}$$

$[(a, b)]$ : The equivalence class of  $(a, b)$

$$\begin{aligned}[(a, b)] + [(a', b')] &= [(ab' + a'b, bb')] = \frac{ab' + a'b}{bb'} \\ [(0, 1)] &= 0 \in K \quad (\text{additive identity})\end{aligned}$$

The following is to prove that it is independent of the choice of equivalence class:

$$\begin{aligned}[(a_1, b_1)] + [(a', b')] &= [(a_1b' + a'b_1, b_1b')] \\ \therefore (ab' + a'b) \cdot b_1b' &= (a_1b' + a'b_1) \cdot bb' \\ \therefore left &= right\end{aligned}$$

We then need to define multiplication:

$$\begin{aligned}[(a, b)][(a', b')] &= [(aa', bb')] = \frac{aa'}{bb'} \\ [(1, 1)] &= 1 \in K \quad (\text{multiplicative identity})\end{aligned}$$

**Examples:**

$$\begin{aligned}[(x, y)] + [(0, 1)] &= [(x, y)] \\ [(x, y)] \cdot [(1, 1)] &= [(x, y)] \\ [(x, y)] &\neq [(0, 1)]\end{aligned}$$

So we need to show that  $K$  is a field:

$$\begin{aligned}
[(x, y)] \cdot [(y, x)] &= [(xy, xy)] = [(1, 1)] \quad (\text{Non-zeros have inverse}) \\
\therefore ab' &= a'b \\
(a, b) &\sim (a', b') \\
\therefore [(xy, xy)] &= [(1, 1)]
\end{aligned}$$

**Therefore**  $K$  is a field of fractions of  $A$ .

We need to show the following map is injective:

If  $A = \mathbb{Z}$  and  $K = \mathbb{Q}$ :

$$\begin{aligned}
A &\longrightarrow K \quad (\text{injective}) \\
x &\longrightarrow [(x, 1)] \\
x + y &\longrightarrow [(x + y, 1)] = [(x, 1)] + [(y, 1)] \\
x \cdot y &\longrightarrow [(xy, 1)] = [(x, 1)] \cdot [(y, 1)] \\
\text{Comm. ring without } 0 &\longrightarrow \text{Field containing the ring}
\end{aligned}$$

Assume  $[(a, 1)] = [(b, 1)]$  where  $a, b \in A$ :

$$\begin{aligned}
a \cdot 1 &= b \cdot 1 \Rightarrow a = b \\
\therefore \text{The map is injective.}
\end{aligned}$$

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, a \in R$   
Ring of polynomials:  $P(x) \times Q(x) \Rightarrow R(x)$  (with no zero divisors)

## 1.5 Ordered Set

### 1.5.1 Partial Order

$(S, \leq)$  is a (partially) ordered set (poset),  $\forall x, y, z \in S$ :

1.  $x \leq x$
2.  $x \leq y, y \leq x \Rightarrow x = y$
3.  $x \leq y, y \leq z \Rightarrow x \leq z$

### 1.5.2 Total Order

Total/Linear order: Either  $x \leq y$  or  $y \leq x$ .

Ordered field is a field  $K$  with  $\leq$  total order.

**Definition:**

1. If  $x \leq y$  then  $x + z \leq y + z$  and
2. if  $x \geq 0, y \geq 0$  then  $x \times y \geq 0$

**First example:**

$$\mathbb{Q} = \{(a, b)\}, (a, b) = \frac{a}{b}$$

Since  $a \geq 0, b > 0$

$\therefore \mathbb{Q}$  is an ordered set.

**Second example:**

$$x \geq 0 \Leftrightarrow -x \leq 0$$

$$x \geq 0 \Rightarrow x^2 \geq 0$$

$$x \leq 0 \Rightarrow (-x)^2 = x^2 \geq 0$$

$$1^2 = 1 > 0$$

$$2 > 1$$

$$3 > 2$$

$$\vdots$$

$$n > n - 1$$

### 1.5.3 Least-upper-bound Property

$\mathbb{Q}$ : Ordered field of rational numbers (with gaps)

Least-upper-bound property:  $K$  has least-upper-bound property if any bounded subset  $E \subset K$  has least upper bound.

$K$  is a field with order relation  $\leq$ ,  $E \subset K$  and  $E$  is *bounded above* if  $\exists \alpha \in K$  such that  $p \leq \alpha, p \in E$ .



$\beta$  is the *least-upper-bound* of  $E$  if

1.  $\beta$  is an upper bound,
2. for any other upper bound  $\alpha$ , we have  $\beta \leq \alpha$ .

We call  $\beta = \sup E$ .

We then need to prove that  $\mathbb{Q}$  does not have least-upper-bound property, that is, there are gaps in  $\mathbb{Q}$ .

*Proof.*

$$\begin{aligned} E = \{q \in \mathbb{Q} : q^2 \leq 2\} &\Rightarrow \text{no sup } E \\ q \in E &\Leftrightarrow -q \in E \quad \text{and} \quad q \geq 0 \end{aligned}$$

Let  $p \in \mathbb{Q}$  such that  $p \geq 0, p^2 > 2$ .  $\therefore p^2 > 2 \geq q^2 \Rightarrow p^2 \geq q^2$

□

## 2 Basic Topology

Metric (distance function) of  $x_0$  and  $x$ :  $d(x_0, x) = |x - x_0|$

### 2.1 Metric Space $M$

$$d: M \times M \rightarrow \mathbb{R}$$

1.  $d(x_0, x_1) \geq 0$  where  $x_0, x_1 \in M$
2.  $d(x_0, x_1) = 0 \iff x_0 = x_1$
3.  $d(x_0, x_1) = d(x_1, x_0)$  (irrespective of order)
4.  $d(x, z) \leq d(x, y) + d(y, z)$  (triangular inequity)

Euclidean metric on  $\mathbb{R}^2$ :  $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

Discrete metric:  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Open ball of radius  $\epsilon$ :  $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$

### 2.2 Open Sets

#### 2.2.1 Metric Spaces $X$

$U \subset X$  is open if for any  $x \in U$  there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subset U$ .

Open interval is open set; closed interval is not open set.

#### 2.2.2 Topological Spaces $(X, U)$

Let  $U$  be a family of all sets,  $X$  be a set.  $U$  is a **topology** on  $X$  if

1.  $\emptyset$  (Empty set) is always open;  $X$  is open.  $\Leftrightarrow \emptyset$  and  $X$  itself belong to  $U$ .
2.  $F$  is a collection of open sets, then  $\bigcup_{U \in F} U$  is open.  $\Leftrightarrow$  Any union of members of  $U$  still belongs to  $U$ . (Union)
3.  $F$  is a *finite* collection of open sets, then  $\bigcap_{U \in F} U$  is open.  $\Leftrightarrow$  The intersection of any finite number of members of  $U$  belongs to  $U$ . (Intersection)

Finite case:  $x \in \bigcap_{U \in F} U$ ,  $x \in U$ , hence  $B(x, \epsilon_U) \subset U$

$$\delta = \min \epsilon_U > 0 \quad \text{where } U \in F$$

$$B(x, \delta) \subset B(x, \epsilon_U) \subset U$$

$$\therefore B(x, \delta) \subset \bigcap_{U \in F} U$$

Infinite case: For example, the intersection of all intervals of  $(-\frac{1}{n}, \frac{1}{n})$ , where  $n$  is a positive number, is the set  $\{0\}$  which is not open in the real line.

## 2.3 Compact Sets

In metric space  $X$ , compact sets are closed.

Compact  $\Leftrightarrow$  closed and bounded (only for Euclidean metric,  $\mathbb{R}^n$ )