

# Notes for MATH 3210: Foundation of Analysis I

Jing Guo

September 24, 2017

## Contents

<b>1</b>	<b>Ring and Field</b>	<b>1</b>
1.1	Ring . . . . .	2
1.1.1	Addition Axiom . . . . .	2
1.1.2	Multiplication Axiom . . . . .	2
1.2	Field . . . . .	2
1.3	Construction of Integers . . . . .	3
1.4	Construction of Fractions (Rational Numbers) . . . . .	4
1.5	Ordered Set . . . . .	5
1.5.1	Partial Order . . . . .	5
1.5.2	Total Order . . . . .	6
1.5.3	Least-upper-bound Property . . . . .	6
1.6	Real Numbers . . . . .	8
1.6.1	Construction of Real Numbers . . . . .	8
1.6.2	Archimedean Property . . . . .	8
1.6.3	Density of Real Numbers . . . . .	9
1.6.4	Property of Real Numbers . . . . .	9
1.7	Complex Number . . . . .	11
1.7.1	Construction of Complex Numbers . . . . .	12
1.7.2	Automorphism of Complex Numbers . . . . .	13
1.8	Inner Product Space . . . . .	13
<b>2</b>	<b>Basic Topology</b>	<b>13</b>
2.1	Metric Space $M$ . . . . .	14
2.2	Open Sets . . . . .	14
2.2.1	Metric Spaces $X$ . . . . .	14
2.2.2	Topological Spaces $(X, U)$ . . . . .	14
2.3	Compact Sets . . . . .	14

## 1 Ring and Field

Notations:

- $\mathbb{N}$ : The set of natural numbers;
- $\mathbb{Z}$ : The set of integers (ring, not field, has no inverse);
- $\mathbb{Q}$ : The set of rational numbers;
- $\mathbb{R}$ : The set of real numbers (ring and field).

## 1.1 Ring

The set  $A$  has two binary operations, addition and multiplication.  
For any  $a, b \in A$ ,

$$\begin{aligned} A * A &\rightarrow A \\ a, b &\rightarrow a + b \end{aligned}$$

### 1.1.1 Addition Axiom

1.  $a + b = b + a$  (commutative)
2.  $(a + b) + c = a + (b + c)$  (associative)
3. There is an element  $0$  such that  $a + 0 = a$  (additive identity)
4. There exists  $-a$  (additive inverse) such that  $a + (-a) = 0$

### 1.1.2 Multiplication Axiom

1.  $a \cdot b = b \cdot a$
2.  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element  $1$  such that  $a \cdot 1 = a$  and  $0 \neq 1$
4.  $a \cdot (b + c) = a \cdot b + a \cdot c$  (distributive)

## 1.2 Field

$A$  is a field if  $A$  is a *ring* for any  $a \in A$  and  $a \neq 0$ , there exists  $a^{-1}$  (inverse of  $a$ ) such that  $a \cdot a^{-1} = 1$ .

$A$  has no zero divisors if  $x \neq 0, y \neq 0 \rightarrow x \cdot y \neq 0$

**Example:**  $x, y \in A, x \neq 0, y \neq 0, xy = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

### 1.3 Construction of Integers

We need to construct  $\mathbb{Z}$  from  $\mathbb{N}$  with **localization**.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Set of equivalence class:  $\mathbb{N} \times \mathbb{N} / \sim$

Equivalence relation:  $(a, b) \sim (a', b')$  if  $a + b' = b + a'$ , since  $a - b = a' - b'$

$$\begin{aligned} (a, b) &\sim (a', b') \sim (a'', b'') \\ a' + b'' &= b' + a'' \\ (a + b'') + b' &= (a + b') + b'' = b + a' + b'' = b + b' + a'' = (a'' + b) + b' \\ \therefore a + b'' &= a'' + b \\ \therefore (a, b) &\sim (a'', b'') \end{aligned}$$

We then should define addition:

$$[(a, b)] + [(a', b')] = [(a + a', b + b')] \quad (1)$$

$$[(a_1, b_1)] + [(a', b')] = [(a_1 + a', b_1 + b')] \quad (2)$$

Equation 1 and equation 2 are equal since independent of choice of equivalence classes:

$$\begin{aligned} a + a' + b_1 + b' &= a_1 + a' + b + b' \\ a + b_1 &= a_1 + b \end{aligned}$$

Additive identity:  $0 = [(0, 0)]$

For example:  $[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)] = 0$

We then need to define multiplication:

$$[(a, b)] \cdot [(a', b')] = [(aa' + bb', ab' + a'b)]$$

$$\begin{aligned} &(a - b)(a' - b') \\ &= aa' - ab' - a'b + bb' \\ &= (aa' + bb') - (ab' + a'b) \end{aligned}$$

Multiplicative identity:  $1 = [(1, 0)]$

For example:  $[(a, b)] \cdot [(1, 0)] = [(a, b)]$

$\therefore \mathbb{N} \times \mathbb{N} / \sim$  is a ring, that is,  $\mathbb{Z}$ .

$$\begin{aligned}
\mathbb{N} &\longrightarrow \mathbb{Z} \\
m &\longrightarrow [(m, 0)] \\
-m &\longrightarrow [(0, m)] = -[(m, 0)]
\end{aligned}$$

For example:

$$\begin{aligned}
[(a, b)] &= [(a, 0)] + [(0, b)] \\
&= [(a, 0)] - [(b, 0)]
\end{aligned}$$

## 1.4 Construction of Fractions (Rational Numbers)

We need to show that  $K$  is a field (with addition and multiplication). Assume  $A$  is a commutative ring with no zero divisors, and  $A^* = A - \{0\}$ .

$$A \times A^* = \{(a, b) \mid a, b \in A, b \neq 0\}$$

Set of equivalence classes:  $K = (A \times A^*) / \sim$  (elements of  $K$  are fractions)

We first need to define addition:

$$\begin{aligned}
(a, b) &\sim (a', b') \\
\frac{a}{b} &= \frac{a'}{b'} \\
ab' &= a'b
\end{aligned}$$

$[(a, b)]$ : The equivalence class of  $(a, b)$

$$\begin{aligned}
[(a, b)] + [(a', b')] &= [(ab' + a'b, bb')] = \frac{ab' + a'b}{bb'} \\
[(0, 1)] &= 0 \in K \quad (\text{additive identity})
\end{aligned}$$

The following is to prove that it is independent of the choice of equivalence class:

$$\begin{aligned}
[(a_1, b_1)] + [(a', b')] &= [(a_1b' + a'b_1, b_1b')] \\
\therefore (ab' + a'b) \cdot b_1b' &= (a_1b' + a'b_1) \cdot bb' \\
\therefore \text{left} &= \text{right}
\end{aligned}$$

We then need to define multiplication:

$$\begin{aligned}
[(a, b)][(a', b')] &= [(aa', bb')] = \frac{aa'}{bb'} \\
[(1, 1)] &= 1 \in K \quad (\text{multiplicative identity})
\end{aligned}$$

**Examples:**

$$\begin{aligned} [(x, y)] + [(0, 1)] &= [(x, y)] \\ [(x, y)] \cdot [(1, 1)] &= [(x, y)] \\ [(x, y)] &\neq [(0, 1)] \end{aligned}$$

So we need to show that  $K$  is a field:

$$\begin{aligned} [(x, y)] \cdot [(y, x)] &= [(xy, xy)] = [(1, 1)] \quad (\text{Non-zeros have inverse}) \\ \therefore ab' &= a'b \\ (a, b) &\sim (a', b') \\ \therefore [(xy, xy)] &= [(1, 1)] \end{aligned}$$

**Therefore**  $K$  is a field of fractions of  $A$ .

We need to show the following map is injective:

If  $A = \mathbb{Z}$  and  $K = \mathbb{Q}$ :

$$\begin{aligned} A &\longrightarrow K \quad (\text{injective}) \\ x &\longrightarrow [(x, 1)] \\ x + y &\longrightarrow [(x + y, 1)] = [(x, 1)] + [(y, 1)] \\ x \cdot y &\longrightarrow [(xy, 1)] = [(x, 1)] \cdot [(y, 1)] \\ \text{Comm. ring without } 0 &\longrightarrow \text{Field containing the ring} \end{aligned}$$

Assume  $[(a, 1)] = [(b, 1)]$  where  $a, b \in A$ :

$$\begin{aligned} a \cdot 1 &= b \cdot 1 \Rightarrow a = b \\ \therefore \text{The map is injective.} \end{aligned}$$

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n, a \in R$   
 Ring of polynomials:  $P(x) \times Q(x) \Rightarrow R(x)$  (with no zero divisors)

## 1.5 Ordered Set

### 1.5.1 Partial Order

$(S, \leq)$  is a (partially) ordered set (poset),  $\forall x, y, z \in S$ :

1.  $x \leq x$
2.  $x \leq y, y \leq x \Rightarrow x = y$
3.  $x \leq y, y \leq z \Rightarrow x \leq z$

### 1.5.2 Total Order

Total/Linear order: Either  $x \leq y$  or  $y \leq x$ .

Ordered field is a field  $K$  with  $\leq$  total order.

**Definition:**

1. If  $x \leq y$  then  $x + z \leq y + z$  and
2. if  $x \geq 0, y \geq 0$  then  $x \times y \geq 0$

There is no total order in complex numbers.

**First example:**

$$\mathbb{Q} = \{(a, b)\}, (a, b) = \frac{a}{b}$$

$$\text{Since } a \geq 0, b > 0$$

$\therefore \mathbb{Q}$  is an ordered set.

**Second example:**

$$x \geq 0 \Leftrightarrow -x \leq 0$$

$$x \geq 0 \Rightarrow x^2 \geq 0$$

$$x \leq 0 \Rightarrow (-x)^2 = x^2 \geq 0$$

$$1^2 = 1 > 0$$

$$2 > 1$$

$$3 > 2$$

$$\vdots$$

$$n > n - 1$$

### 1.5.3 Least-upper-bound Property

$\mathbb{Q}$ : Ordered field of rational numbers (with gaps)

Least-upper-bound property:  $K$  has least-upper-bound property if any bounded subset  $E \subset K$  has least upper bound.

$K$  is a field with order relation  $\leq$ ,  $E \subset K$  and  $E$  is *bounded above* if  $\exists \alpha \in K$  such that  $p \leq \alpha, p \in E$ .



$\beta$  is the *least-upper-bound* of  $E$  if

1.  $\beta$  is an upper bound,
2. for any other upper bound  $\alpha$ , we have  $\beta \leq \alpha$ .

We call  $\beta = \sup E$ .

We then need to prove that  $\mathbb{Q}$  does not have least-upper-bound property, that is, there are gaps in  $\mathbb{Q}$ .

*Proof.*

$$\begin{aligned} E = \{q \in \mathbb{Q} : q^2 \leq 2\} &\Rightarrow \text{no } \sup E \\ q \in E &\Leftrightarrow -q \in E \quad \text{and} \quad q \geq 0 \end{aligned}$$

Let  $p \in \mathbb{Q}$  such that  $p \geq 0, p^2 > 2$ .

$$\begin{aligned} \therefore p^2 > 2 &\geq q^2 \Rightarrow p^2 \geq q^2 \\ p^2 - q^2 &= (p+q)(p-q) \geq 0 \quad (\text{property of ordered field}) \\ \therefore p &\geq 0, q > 0 \\ \therefore p+q &\geq 0+q = q \geq 0 \\ \therefore p &\geq q \\ p \in \mathbb{Q}, p &\geq 0, p^2 > 2 \text{ are upper bound of } E. \end{aligned}$$

Remark: Let  $p$  be upper bound of  $E$ ,  $\forall q \in E, q \leq p, p > 0$ .  
Define  $p'$ :

$$\begin{aligned} p' &= p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} = 2\frac{p + 1}{p + 2} \\ \therefore p'^2 - 2 &= 2\frac{p^2 - 2}{(p + 2)^2} \end{aligned}$$

In the above equation,  $p^2 - 2 \leq 0$  and  $(p + 2)^2 \geq 0$ .

Assume  $p \in E \Rightarrow p^2 \leq 2 \Rightarrow p^2 - 2 \leq 0$ .

$\therefore p'^2 - 2 \leq 0 \Rightarrow p' \in E$  and  $p' \geq p$ .

$\therefore$  Either  $p' = p$  or  $p' > p$ .

But neither of them is possible since  $\sqrt{2}$  is irrational:

$$\begin{aligned} p = p' &\Rightarrow p^2 = 0, p \in \mathbb{Q} \text{ but } p \text{ is irrational.} \\ \therefore p &\notin E. \text{ All upper bounds of } E \text{ are } \{p \in \mathbb{Q}, p^2 > 2\} \end{aligned}$$

$r$ : an upper bound, and  $r^2 \geq 2$

$$r' = r - \frac{r^2 - 2}{r^2 + 2} \quad \therefore r' < r$$

$$r'^2 - 2 = 2 \frac{r^2}{(r + 2)^2} > 0 \quad \therefore r'^2 > 2$$

$\therefore r'$  is an upper bound of  $E$ .

So we have no upper bound property in  $E$ . □

## 1.6 Real Numbers

### 1.6.1 Construction of Real Numbers

**Theorem.** *There exists a total ordered field  $\mathbb{R}$  which has least upper bound property. So every subset  $E \subset \mathbb{R}$ , are bounded above, has a supremum.*

*Such field is **unique**.*

$$K \xrightarrow{\varphi} K'$$

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$$

if bijective, then isomorphism; otherwise, morphism.

History of construction of real numbers: Dedekind used cuts and Cantor used Cauchy sequence, which works for more general occasions, in 1872.

What is *cuts*?

$E \subset \mathbb{Q}, E \neq \mathbb{Q} \rightarrow E$  is bounded above

$q \in \mathbb{Q}, E_q = \{p \in \mathbb{Q} : p < q\}$

If  $\alpha \in E, \beta < \alpha \Rightarrow \beta \in \alpha$

$\gamma \in E, \delta \in E, \delta > \gamma$

### 1.6.2 Archimedean Property

**Theorem** (Archimedean Property).

$$x, y \in \mathbb{R}, x > 0, y > 0$$

$$\exists n \in \mathbb{Z}_+, n \cdot x > y$$

*Proof.* Assume  $\{nx \mid n \in \mathbb{Z}\}$  is bounded above, i.e. has least upper bound property.



$$\begin{aligned} \exists \alpha &= \sup\{nx \mid n \in \mathbb{Z}\} \\ n \cdot x &\leq \alpha, \forall n \text{ and } \alpha - x < \alpha \\ \therefore \alpha &= \sup\{nx \mid n \in \mathbb{Z}\} \\ \therefore \alpha - x &\text{ is NOT an upper bound.} \\ \exists m &\text{ such that } \alpha - x < n \cdot x \\ \therefore \alpha &< (n+1)x \\ \therefore &\text{ Contridication.} \\ \therefore \{nx \mid n \in \mathbb{R}\} &\text{ is not bounded above.} \end{aligned}$$

□

### 1.6.3 Density of Real Numbers

*Proof.* Assume  $x, y \in \mathbb{R}$  and  $x < y$ .

There exists  $q \in \mathbb{Q}$  such that  $x < q < y$ . Because  $y - x > 0$  and  $1 > 0$ , so there exists  $n$  such that  $n(y - x) > 1$ .

$$\begin{aligned} nx &< m_1, \quad m_1 \in \mathbb{Z} \\ -nx &< m_2, \quad m_2 \in \mathbb{Z} \\ \therefore -m_2 &< nx < m_1 \end{aligned}$$

Conclusion:  $\exists m \in \mathbb{Z}, m - 1 \leq nx < m$ , so  $nx < m \leq nx + 1 < ny$ .

$$\begin{aligned} nx &< m < ny \\ m &\leq mx + 1 \\ \therefore x &< \frac{m}{n} < y \end{aligned}$$

□

### 1.6.4 Property of Real Numbers

We have  $x > 0, y > 0, n \geq 2$ , and  $y^n = x$ , such  $y$  is unique.

We first need to prove **uniqueness**:

*Proof.* We have  $y_1, y_2 > 0, y_1^n = x, y_2^n = x \Rightarrow y_1 = y_2$

We assume  $y_1 \neq y_2$  and  $0 < y_1 < y_2$ . If  $a > 0$ , then  $ay_1 > ay_2$ .

We claim  $y_1^n < y_2^n$ , then we apply mathematical induction:

When  $n = 1$ ,  $y_1 < y_2$ .

$$\begin{aligned}
y_1 y_1^n &< y_1^n y_2 < y_2 y_2^n \\
y_1^{n+1} &< y_2^{n+1} \\
\therefore y_1^n &< y_2^n
\end{aligned}$$

But they should both be equal to  $x$ , therefore we have a contradiction.  $\square$

We then need to prove **existence**:

$$E = \{t \in \mathbb{R} \mid t^n < x, x > 0, x \in \mathbb{R}\}$$

*Proof.* We first need to show that set  $E$  is not empty.

We construct  $t = \frac{x}{x+1} \Rightarrow 0 < t < 1$  and  $t^n < t^{n-1} < \dots < t < 1$

$$\begin{aligned}
t &= \frac{x}{x+1} \\
x &= t + tx \\
t &= x - tx < x \\
\therefore t^n &< t < x
\end{aligned}$$

$\square$

We then need to show that  $E$  is bounded above:

*Proof.* Suppose  $S \geq x + 1 \Rightarrow S > 1$ , so  $S^m > S^{m-1} > \dots > S$ , therefore  $S^m > x + 1 > S$ .

It follows that if  $t \in E$ , then  $t < x + 1 \Rightarrow x + 1$  is an upper bound of  $E$ .

$\exists y = \sup E, y > 0$   $\square$

We claim  $y^n = x$ , since  $y^n < x$  and  $y^n > x$  are both contradictions.

For the first case of contradictions:

$$\frac{x - y^n}{n(y+1)^{n-1}} > 0$$

We have  $0 < n < 1$  and  $h < \frac{x - y^n}{n(y+1)^{n-1}}$ .

$$(y + h)^n - y^n$$

$$\begin{aligned}
(y+h)^{n-k-1} * k &< (y+h)^{n-k-1} * (y+h)^k \\
&= (y+h)^{n-1} \\
&< n * h * (y+h)^{n-1} < x - y^n \\
\therefore (y+h)^n - y^n &< x - y^n \\
(y+h)^n &< x \\
y+h &\in E \text{ and } \sup E = y < y+h
\end{aligned}$$

Contradiction.

For the first case of contradictions:

$$y^n > x$$

We have  $k = \frac{y^n - x}{ny^{n-1}} > 0$ .

$$\begin{aligned}
\therefore 0 < k &< \frac{y^n}{ny^{n-1}} = \frac{y}{n} < y \\
\therefore 0 < k &< y \text{ and } 0 < y - k \leq t
\end{aligned}$$

$$\begin{aligned}
y^n - t^n &\leq y^n - (y-k)^n \\
&< y^{n-1} < kny^{n-1} = y^n - x \\
\therefore t^n &> x \Rightarrow t \notin E \\
\therefore y - k &\notin E \text{ is an upper bound of } E
\end{aligned}$$

## 1.7 Complex Number

Properties:

1.  $|z| \geq 0$
2.  $|z| = 0 \iff z = 0$
3.  $|z| = |-z|$
4. Triangular inequity:  $|z_1 + z_2| \leq |z_1| + |z_2|$

*Proof of Triangular Inequity.*

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1}z_2) \\
&\leq z_1\overline{z_1} + z_1\overline{z_2} + 2|z_1\overline{z_2}| \\
&\leq (|z_1| + |z_2|)^2
\end{aligned}$$

□

### 1.7.1 Construction of Complex Numbers

$\mathbb{R}^2 = \{(a, b) | a, b \in \mathbb{R}\}$  forms abelian group

$$(a, b) + (a', b') = (a + a', b + b') \quad \therefore (0, 0) = 0$$

$$-(a, b) = (-a, -b)$$

$$\text{Multiplication: } (a, b) \cdot (a', b') = (aa' - bb', a'b + ab')$$

$$\text{Identity for mult.: } (1, 0) = \mathbf{1} \text{ and } (a, b) \cdot (1, 0) = (a, b)$$

$\therefore \mathbb{R}^2$  is a ring and  $(a, b) \neq (0, 0)$ .

For any such elements, we can form

$$(a, b)^{-1} = \left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$$

$$(a, b) \cdot (a, b)^{-1} = (1, 0) = \mathbf{1}$$

$\therefore \mathbb{R}^2$  is a field of complex numbers, that is,  $\mathbb{C}$ .

$$\therefore (0, 1)^2 = (-1, 0) = -\mathbf{1}$$

$$(0, -1)^2 = -\mathbf{1}$$

$$\therefore i = (0, 1)$$

$$(a, b) = a \cdot (1, 0) + b \cdot (0, 1) = a + ib$$

### 1.7.2 Automorphism of Complex Numbers

$$\begin{aligned} z &= (a, b) \text{ and } \bar{z} = (a, -b) \text{ (complex conjugate)} \\ \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 * z_2} &= \bar{z}_1 * \bar{z}_2 \\ z * \bar{z} &= (a, b) * (a, -b) = (a^2 + b^2, 0) = (a^2, b^2) \end{aligned}$$

We can extend from  $\mathbb{R}$  to  $\mathbb{C}$ , so we have  $a \in \mathbb{R}$  and  $(a, 0) \in \mathbb{C}$ , therefore  $\mathbb{R}$  is a subfield of  $\mathbb{C}$ .

Absolute value of complex number  $z$ :

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = |z| > 0$$

## 1.8 Inner Product Space

$V$  : vector space over  $\mathbb{R}$

$$\begin{aligned} V \times V &\rightarrow \mathbb{R} \\ (u, v) &\rightarrow (v \mid u) \quad \text{inner product} \end{aligned}$$

Some properties:

We have  $\alpha, \beta \in \mathbb{R}$ :

1.  $(u \mid v) = (v \mid u)$
2.  $(u \mid u) \geq 0$  and  $(u \mid u) = 0 \iff u = 0$
3. Norm:  $\|u\| = (u \mid u)^{\frac{1}{2}}$
4.  $(\alpha v + \beta w \mid u) = \alpha(v \mid u) + \beta(w \mid u)$
5.  $(v \mid \alpha u + \beta w) = \alpha(v \mid u) + \beta(v \mid w)$

Examples:

Euclidean inner product: In  $\mathbb{R}^n$ ,  $x = \{x_1, x_2, \dots, x_n\}$ ,  $y = \{y_1, y_2, \dots, y_n\}$ ,  
 $(x \mid y) = \sum_{i=1}^n x_i y_i$

## 2 Basic Topology

Metric (distance function) of  $x_0$  and  $x$ :  $d(x_0, x) = |x - x_0|$

## 2.1 Metric Space $M$

$$d: M \times M \rightarrow \mathbb{R}$$

1.  $d(x_0, x_1) \geq 0$  where  $x_0, x_1 \in M$
2.  $d(x_0, x_1) = 0 \iff x_0 = x_1$
3.  $d(x_0, x_1) = d(x_1, x_0)$  (irrespective of order)
4.  $d(x, z) \leq d(x, y) + d(y, z)$  (triangular inequity)

Euclidean metric on  $\mathbb{R}^2$ :  $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

Discrete metric:  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Open ball of radius  $\epsilon$ :  $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$

## 2.2 Open Sets

### 2.2.1 Metric Spaces $X$

$U \subset X$  is open if for any  $x \in U$  there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subset U$ .

Open interval is open set; closed interval is not open set.

### 2.2.2 Topological Spaces $(X, U)$

Let  $U$  be a family of all sets,  $X$  be a set.  $U$  is a **topology** on  $X$  if

1.  $\emptyset$  (Empty set) is always open;  $X$  is open.  $\Leftrightarrow \emptyset$  and  $X$  itself belong to  $U$ .
2.  $F$  is a collection of open sets, then  $\bigcup_{U \in F} U$  is open.  $\Leftrightarrow$  Any union of members of  $U$  still belongs to  $U$ . (Union)
3.  $F$  is a *finite* collection of open sets, then  $\bigcap_{U \in F} U$  is open.  $\Leftrightarrow$  The intersection of any finite number of members of  $U$  belongs to  $U$ . (Intersection)

Finite case:  $x \in \bigcap_{U \in F} U$ ,  $x \in U$ , hence  $B(x, \epsilon_U) \subset U$

$$\delta = \min \epsilon_U > 0 \quad \text{where } U \in F$$

$$B(x, \delta) \subset B(x, \epsilon_U) \subset U$$

$$\therefore B(x, \delta) \subset \bigcap_{U \in F} U$$

Infinite case: For example, the intersection of all intervals of  $(-\frac{1}{n}, \frac{1}{n})$ , where  $n$  is a positive number, is the set  $\{0\}$  which is not open in the real line.

## 2.3 Compact Sets

In metric space  $X$ , compact sets are closed.

Compact  $\Leftrightarrow$  closed and bounded (only for Euclidean metric,  $\mathbb{R}^n$ )