Notes for MATH 3210: Foundation of Analysis I

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Contents

1 Ring and Field

Notations:

- $\mathbb{N}$: The set of natural numbers;
- $\mathbb{Z}$: The set of integers (ring, not field, has no inverse);
- $\mathbb{Q}$: The set of rational numbers;
- $\mathbb{R}$: The set of real numbers (ring and field).

1.1 Ring

The set A has two binary operations, addition and multiplication.

For any $a, b \in A$,

$$A * A \to A$$
$$a, b \to a + b$$

1.1.1 Addition Axiom

1. $a + b = b + a$ (commutative)
2. $(a + b) + c = a + (b + c)$ (associative)
3. There is an element 0 such that $a + 0 = a$ (additive identity)
4. There exists $-a$ (additive inverse) such that $a + (-a) = 0$

1.1.2 Multiplication Axiom

1. $a \cdot b = b \cdot a$
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element 1 such that $a \cdot 1 = a$ and $0 \neq 1$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive)

1.2 Field

A is a field if A is a *ring* for any $a \in A$ and $a \neq 0$, there exists a^{-1} (inverse of a) such that $a \cdot a^{-1} = 1$.

Example: $x, y \in A$, $x \neq 0$, $y \neq 0$, $xy = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A has no zero divisors if $x \neq 0, y \neq 0 \to x \cdot y \neq 0$

1.3 Construction of Integers

We need to construct $\mathbb{Z}$ from $\mathbb{N}$ with **localization**.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Set of equivalence class: $\mathbb{N} \times \mathbb{N} / \sim$

Equivalence relation: $(a, b) \sim (a', b')$ if $a + b' = b + a'$, since $a - b = a' - b'$

$$
\begin{gathered}
(a,b) \sim (a',b') \sim (a'',b'') \\
a' + b'' = b' + a'' \\
\boldsymbol{(a+b'')+b'} = (a+b')+b'' = b+a'+b'' = b+b'+a'' = \boldsymbol{(a''+b)+b'} \\
\therefore a+b'' = a''+b \\
\therefore (a,b) \sim (a'',b'')
\end{gathered}
$$

We then should define addition:

$$[(a,b)] + [(a',b')] = [(a+a', b+b')] \tag{1}$$

$$[(a_1,b_1)] + [(a',b')] = [(a_1+a', b_1+b')] \tag{2}$$

Equation 1 and equation 2 are equal since independent of choice of equivalence classes:

$$
\begin{gathered}
a+a'+b_1+b' = a_1+a'+b+b' \\
a+b_1 = a_1+b
\end{gathered}
$$

Additive identity: $0 = [(0,0)]$
For example: $[(a,b)] + [(b,a)] = [(a+b, a+b)] = [(0,0)] = 0$
We then need to define multiplication:

$$[(a,b)] \cdot [(a',b')] = [(aa'+bb', ab'+a'b)]$$

$$
\begin{aligned}
& (a-b)(a'-b') \\
= \quad & aa' - ab' - a'b + bb' \\
= \quad & (aa'+bb') - (ab'+a'b)
\end{aligned}
$$

Multiplicative identity: $1 = [(1,0)]$
For example: $[(a,b)] \cdot [(1,0)] = [(a,b)]$
$\therefore \mathbb{N} \times \mathbb{N}/\sim$ is a ring, that is, $\mathbb{Z}$.

$$
\begin{aligned}
\mathbb{N} &\longrightarrow \mathbb{Z} \\
m &\longrightarrow [(m,0)] \\
-m &\longrightarrow [(0,m)] = -[(m,0)]
\end{aligned}
$$

For example:

$$
\begin{aligned}
[(a,b)] &= [(a,0)] + [(0,b)] \\
&= [(a,0)] - [(b,0)]
\end{aligned}
$$

1.4 Construction of Fractions (Rational Numbers)

We need to show that K is a field (with addition and multiplication). Assume A is a commutative ring with no zero divisors, and $A^* = A - \{0\}$.

$A \times A^* = \{(a, b) \mid a, b \in A, b \neq 0\}$

Set of equivalence classes: $K = (A \times A^*)/\sim$ (elements of K are fractions)

We first need to define addition:

$$\begin{gathered}(a, b) \sim (a', b') \\ \frac{a}{b} = \frac{a'}{b'} \\ ab' = a'b\end{gathered}$$

$[(a, b)]$: The equivalence class of (a, b)

$$\begin{gathered}[(a, b)] + [(a', b')] = [(ab' + a'b, bb')] = \frac{ab' + a'b}{bb'} \\ [(0, 1)] = 0 \in K \quad \text{(additive identity)}\end{gathered}$$

The following is to prove that it is independent of the choice of equivalence class:

$$\begin{gathered}[(a_1, b_1)] + [(a', b')] = [(a_1 b' + a' b_1, b_1 b')] \\ \therefore (ab' + a'b) \cdot b_1 b' = (a_1 b' + a' b_1) \cdot bb' \\ \therefore left = right\end{gathered}$$

We then need to define multiplication:

$$\begin{gathered}[(a, b)][(a', b')] = [(aa', bb')] = \frac{aa'}{bb'} \\ [(1, 1)] = 1 \in K \quad \text{(multiplicative identity)}\end{gathered}$$

Examples:

$$\begin{aligned}[(x, y)] + [(0, 1)] &= [(x, y)] \\ [(x, y)] \cdot [(1, 1)] &= [(x, y)] \\ [(x, y)] &\neq [(0, 1)]\end{aligned}$$

So we need to show that K is a field:

$$
\begin{aligned}
[(x,y)]\cdot[(y,x)] = [(xy,xy)] = [(1,1)] \quad &\text{(Non-zeros have inverse)}\\
\because ab' = a'b&\\
(a,b) \sim (a',b')&\\
\therefore [(xy,xy)] = [(1,1)]&
\end{aligned}
$$

Therefore K is a field of fractions of A.
We need to show the following map is injective:
If $A = \mathbb{Z}$ and $K = \mathbb{Q}$:

$$
\begin{aligned}
A &\longrightarrow K \quad \text{(injective)}\\
x &\longrightarrow [(x,1)]\\
x+y &\longrightarrow [(x+y,1)] = [(x,1)] + [(y,1)]\\
x\cdot y &\longrightarrow [(xy,1)] = [(x,1)]\cdot[(y,1)]\\
\text{Comm. ring without } 0 &\longrightarrow \text{Field containing the ring}
\end{aligned}
$$

Assume $[(a,1) = [(b,1)]]$ where $a, b \in A$:

$$
\begin{aligned}
a\cdot 1 = b\cdot 1 &\Rightarrow a = b\\
\therefore &\text{ The map is injective.}
\end{aligned}
$$

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, a \in R$
Ring of polynomials: $P(x) \times Q(x) \Rightarrow R(x)$ (with no zero divisors)

1.5 Ordered Set

1.5.1 Partial Order

$(S, \leq)$ is a (partially) ordered set (poset), $\forall x, y, z \in S$:

1. $x \leq$ x
2. $x \leq y, y \leq x \Rightarrow x = y$
3. $x \leq y, y \leq z \Rightarrow x \leq z$

1.5.2 Total Order

Total/Linear order: Either $x \leq y$ or $y \leq x$.
Ordered field is a field K with $\leq$ total order.
Definition:

1. If $x \leq y$ then $x + z \leq y + z$ and
2. if $x \geq 0, y \geq 0$ then $x \times y \geq 0$

First example:

$$\mathbb{Q} = \{(a,b)\}, (a,b) = \frac{a}{b}$$
$$\text{Since} \quad a \geq 0, b > 0$$

$\therefore \mathbb{Q}$ is an ordered set.
Second example:

$$x \geq 0 \Leftrightarrow -x \leq 0$$
$$x \geq 0 \Rightarrow x^2 \geq 0$$
$$x \leq 0 \Rightarrow (-x)^2 = x^2 \geq 0$$

$$\begin{aligned} 1^2 = 1 &> 0 \\ 2 &> 1 \\ 3 &> 2 \\ &\vdots \\ n &> n-1 \end{aligned}$$

2 Basic Topology

Metric (distance function) of x_0 and x: $d(x_0, x) = |x - x_0|$

2.1 Metric Space M

$$d\colon M \times M \to \mathbb{R}$$

1. $d(x_0, x_1) \geq 0$ where $x_0, x_1 \in M$
2. $d(x_0, x_1) = 0 \iff x_0 = x_1$
3. $d(x_0, x_1) = d(x_1, x_0)$ (irrespective of order)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequity)

Euclidean metric on $\mathbb{R}^2$: $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
Discrete metric: $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
Open ball of radius ϵ: $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$

2.2 Open Sets

2.2.1 Metric Spaces X

$U \subset X$ is open if for any $x \in U$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$.

Open interval is open set; closed interval is not open set.

2.2.2 Topological Spaces (X, U)

Let U be a family of all sets, X be a set. U is a **topology** on X if

1. $\emptyset$ (Empty set) is always open; X is open. $\Leftrightarrow$ $\emptyset$ and X itself belong to U.
2. F is a collection of open sets, then $\bigcup_{U \in F} U$ is open. $\Leftrightarrow$ Any union of members of U still belongs to U. (Union)
3. F is a *finite* collection of open sets, then $\bigcap_{U \in F} U$ is open. $\Leftrightarrow$ The intersection of any finite number of members of U belongs to U. (Intersection)

Finite case: $x \in \bigcap_{U \in F} U$, $x \in U$, hence $B(x, \epsilon_U) \subset U$

$$\delta = \min \epsilon_U > 0 \quad \textit{where} \quad U \in F$$
$$B(x, \delta) \subset B(x, \epsilon_U) \subset U$$
$$\therefore B(x, \delta) \subset \bigcap_{U \in F} U$$

Infinite case: For example, the intersection of all intervals of $(-\frac{1}{n}, \frac{1}{n})$, where n is a positive number, is the set $\{0\}$ which is not open in the real line.

2.3 Compact Sets

In metric space X, compact sets are <u>closed</u>.

Compact $\Leftrightarrow$ closed and bounded (only for Euclidean metric, $\mathbb{R}^n$)