

Notes for MATH 3210: Foundation of Analysis I

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Contents

1	Ring and Field	1
1.1	Ring	1
1.1.1	Addition Axiom	2
1.1.2	Multiplication Axiom	2
1.2	Field	2
1.3	Construction of Integers	2
1.4	Construction of Fractions (Rational Numbers)	3
2	Basic Topology	5
2.1	Metric Space M	5
2.2	Open Sets	5
2.2.1	Metric Spaces X	5
2.2.2	Topological Spaces (X, U)	5
2.3	Compact Sets	6

1 Ring and Field

Notations:

- \mathbb{N} : The set of natural numbers;
- \mathbb{Z} : The set of integers (ring, not field, has no inverse);
- \mathbb{Q} : The set of rational numbers;
- \mathbb{R} : The set of real numbers (ring and field).

1.1 Ring

The set A has two binary operations, addition and multiplication.

For any $a, b \in A$,

$$\begin{aligned} A * A &\rightarrow A \\ a, b &\rightarrow a + b \end{aligned}$$

1.1.1 Addition Axiom

1. $a + b = b + a$ (commutative)
2. $(a + b) + c = a + (b + c)$ (associative)
3. There is an element 0 such that $a + 0 = a$ (additive identity)
4. There exists $-a$ (additive inverse) such that $a + (-a) = 0$

1.1.2 Multiplication Axiom

1. $a \cdot b = b \cdot a$
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element 1 such that $a \cdot 1 = a$ and $0 \neq 1$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive)

1.2 Field

A is a field if A is a *ring* for any $a \in A$ and $a \neq 0$, there exists a^{-1} (inverse of a) such that $a \cdot a^{-1} = 1$.

Example: $x, y \in A$, $x \neq 0$, $y \neq 0$, $xy = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A has no zero divisors if $x \neq 0, y \neq 0 \rightarrow x \cdot y \neq 0$

1.3 Construction of Integers

We need to construct \mathbb{Z} from \mathbb{N} with **localization**.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Set of equivalence class: $\mathbb{N} \times \mathbb{N} / \sim$

Equivalence relation: $(a, b) \sim (a', b')$ if $a + b' = b + a'$, since $a - b = a' - b'$

$$(a, b) \sim (a', b') \sim (a'', b'')$$

$$a' + b'' = b' + a''$$

$$(a + b'') + b' = (a + b') + b'' = b + a' + b'' = b + b' + a'' = (a'' + b) + b'$$

$$\therefore a + b'' = a'' + b$$

$$\therefore (a, b) \sim (a'', b'')$$

We then should define addition:

$$[(a, b)] + [(a', b')] = [(a + a', b + b')] \quad (1)$$

$$[(a_1, b_1)] + [(a', b')] = [(a_1 + a', b_1 + b')] \quad (2)$$

Equation 1 and equation 2 are equal since independent of choice of equivalence classes:

$$\begin{aligned} a + a' + b_1 + b' &= a_1 + a' + b + b' \\ a + b_1 &= a_1 + b \end{aligned}$$

Additive identity: $0 = [(0, 0)]$

For example: $[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)] = 0$

We then need to define multiplication:

$$[(a, b)] \cdot [(a', b')] = [(aa' + bb', ab' + a'b)]$$

$$\begin{aligned} &(a - b)(a' - b') \\ &= aa' - ab' - a'b + bb' \\ &= (aa' + bb') - (ab' + a'b) \end{aligned}$$

Multiplicative identity: $1 = [(1, 0)]$

For example: $[(a, b)] \cdot [(1, 0)] = [(a, b)]$

$\therefore \mathbb{N} \times \mathbb{N} / \sim$ is a ring, that is, \mathbb{Z} .

$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{Z} \\ m &\longrightarrow [(m, 0)] \\ -m &\longrightarrow [(0, m)] = -[(m, 0)] \end{aligned}$$

For example:

$$\begin{aligned} [(a, b)] &= [(a, 0)] + [(0, b)] \\ &= [(a, 0)] - [(b, 0)] \end{aligned}$$

1.4 Construction of Fractions (Rational Numbers)

We need to show that K is a field (with addition and multiplication). Assume A is a commutative ring with no zero divisors, and $A^* = A - \{0\}$.

$A \times A^* = \{(a, b) \mid a, b \in A, b \neq 0\}$

Set of equivalence classes: $K = (A \times A^*) / \sim$ (elements of K are fractions)

We first need to define addition:

$$\begin{aligned}
(a, b) &\sim (a', b') \\
\frac{a}{b} &= \frac{a'}{b'} \\
ab' &= a'b
\end{aligned}$$

$[(a, b)]$: The equivalence class of (a, b)

$$\begin{aligned}
[(a, b)] + [(a', b')] &= [(ab' + a'b, bb')] = \frac{ab' + a'b}{bb'} \\
[(0, 1)] &= 0 \in K \quad (\text{additive identity})
\end{aligned}$$

The following is to prove that it is independent of the choice of equivalence class:

$$\begin{aligned}
[(a_1, b_1)] + [(a', b')] &= [(a_1b' + a'b_1, b_1b')] \\
\therefore (ab' + a'b) \cdot b_1b' &= (a_1b' + a'b_1) \cdot bb' \\
\therefore \text{left} &= \text{right}
\end{aligned}$$

We then need to define multiplication:

$$\begin{aligned}
[(a, b)][(a', b')] &= [(aa', bb')] = \frac{aa'}{bb'} \\
[(1, 1)] &= 1 \in K \quad (\text{multiplicative identity})
\end{aligned}$$

Examples:

$$\begin{aligned}
[(x, y)] + [(0, 1)] &= [(x, y)] \\
[(x, y)] \cdot [(1, 1)] &= [(x, y)] \\
[(x, y)] &\neq [(0, 1)]
\end{aligned}$$

So we need to show that K is a field:

$$\begin{aligned}
[(x, y)] \cdot [(y, x)] &= [(xy, xy)] = [(1, 1)] \quad (\text{Non-zeros have inverse}) \\
\therefore ab' &= a'b \\
(a, b) &\sim (a', b') \\
\therefore [(xy, xy)] &= [(1, 1)]
\end{aligned}$$

Therefore K is a field of fractions of A .

We need to show the following map is injective:

If $A = \mathbb{Z}$ and $K = \mathbb{Q}$:

$$A \longrightarrow K \quad (\text{injective})$$

$$x \longrightarrow [(x, 1)]$$

$$x + y \longrightarrow [(x + y, 1)] = [(x, 1)] + [(y, 1)]$$

$$x \cdot y \longrightarrow [(xy, 1)] = [(x, 1)] \cdot [(y, 1)]$$

Comm. ring without 0 \longrightarrow Field containing the ring

Assume $[(a, 1)] = [(b, 1)]$ where $a, b \in A$:

$$a \cdot 1 = b \cdot 1 \Rightarrow a = b$$

\therefore The map is injective.

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, a \in R$$

Ring of polynomials: $P(x) \times Q(x) \Rightarrow R(x)$ (with no zero divisors)

2 Basic Topology

Metric (distance function) of x_0 and x : $d(x_0, x) = |x - x_0|$

2.1 Metric Space M

$$d: M \times M \rightarrow \mathbb{R}$$

1. $d(x_0, x_1) \geq 0$ where $x_0, x_1 \in M$
2. $d(x_0, x_1) = 0 \iff x_0 = x_1$
3. $d(x_0, x_1) = d(x_1, x_0)$ (irrespective of order)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequity)

Euclidean metric on \mathbb{R}^2 : $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

Discrete metric: $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Open ball of radius ϵ : $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$

2.2 Open Sets

2.2.1 Metric Spaces X

$U \subset X$ is open if for any $x \in U$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$.

Open interval is open set; closed interval is not open set.

2.2.2 Topological Spaces (X, U)

Let U be a family of all sets, X be a set. U is a **topology** on X if

1. \emptyset (Empty set) is always open; X is open. $\Leftrightarrow \emptyset$ and X itself belong to U .
2. F is a collection of open sets, then $\bigcup_{U \in F} U$ is open. \Leftrightarrow Any union of members of U still belongs to U . (Union)
3. F is a *finite* collection of open sets, then $\bigcap_{U \in F} U$ is open. \Leftrightarrow The intersection of any finite number of members of U belongs to U . (Intersection)

Finite case: $x \in \bigcap_{U \in F} U$, $x \in U$, hence $B(x, \epsilon_U) \subset U$

$$\begin{aligned}\delta &= \min \epsilon_U > 0 \quad \text{where } U \in F \\ B(x, \delta) &\subset B(x, \epsilon_U) \subset U \\ \therefore B(x, \delta) &\subset \bigcap_{U \in F} U\end{aligned}$$

Infinite case: For example, the intersection of all intervals of $(-\frac{1}{n}, \frac{1}{n})$, where n is a positive number, is the set $\{0\}$ which is not open in the real line.

2.3 Compact Sets

In metric space X , compact sets are closed.

Compact \Leftrightarrow closed and bounded (only for Euclidean metric, \mathbb{R}^n)