Notes for MATH 3210: Foundation of Analysis I

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1 Ring and Field

Notations:

- $\mathbb{N}$: The set of natural numbers;
- $\mathbb{Z}$: The set of integers (ring, not field, has no inverse);
- $\mathbb{Q}$: The set of rational numbers;
- $\mathbb{R}$: The set of real numbers (ring and field).

1.1 Ring

The set A has two binary operations, addition and multiplication.

For any $a, b \in A$,

$$A * A \to A$$
$$a, b \to a + b$$

1.1.1 Addition Axiom

1. $a + b = b + a$ (commutative)
2. $(a + b) + c = a + (b + c)$ (associative)
3. There is an element 0 such that $a + 0 = a$ (additive identity)
4. There exists $-a$ (additive inverse) such that $a + (-a) = 0$

1.1.2 Multiplication Axiom

1. $a \cdot b = b \cdot a$
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element 1 such that $a \cdot 1 = a$ and $0 \neq 1$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive)

1.2 Field

A is a field if A is a *ring* for any $a \in A$ and $a \neq 0$, there exists a^{-1} (inverse of a) such that $a \cdot a^{-1} = 1$.

A has no zero divisors if $x \neq 0, y \neq 0 \to x \cdot y \neq 0$

Example: $x, y \in A$, $x \neq 0$, $y \neq 0$, $xy = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

1.3 Construction of Integers

We need to construct $\mathbb{Z}$ from $\mathbb{N}$ with **localization**.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Set of equivalence class: $\mathbb{N} \times \mathbb{N} / \sim$

Equivalence relation: $(a, b) \sim (a', b')$ if $a + b' = b + a'$, since $a - b = a' - b'$

$$
\begin{gathered}
(a,b) \sim (a',b') \sim (a'',b'') \\
a' + b'' = b' + a'' \\
\boldsymbol{(a+b'')+b'} = (a+b') + b'' = b + a' + b'' = b + b' + a'' = \boldsymbol{(a''+b)+b'} \\
\therefore a + b'' = a'' + b \\
\therefore (a,b) \sim (a'',b'')
\end{gathered}
$$

We then should define addition:

$$[(a,b)] + [(a',b')] = [(a+a', b+b')] \tag{1}$$

$$[(a_1,b_1)] + [(a',b')] = [(a_1+a', b_1+b')] \tag{2}$$

Equation 1 and equation 2 are equal since independent of choice of equivalence classes:

$$
\begin{gathered}
a + a' + b_1 + b' = a_1 + a' + b + b' \\
a + b_1 = a_1 + b
\end{gathered}
$$

Additive identity: $0 = [(0,0)]$
For example: $[(a,b)] + [(b,a)] = [(a+b, a+b)] = [(0,0)] = 0$
We then need to define multiplication:

$$[(a,b)] \cdot [(a',b')] = [(aa'+bb', ab'+a'b)]$$

$$
\begin{aligned}
& (a-b)(a'-b') \\
= \quad & aa' - ab' - a'b + bb' \\
= \quad & (aa'+bb') - (ab'+a'b)
\end{aligned}
$$

Multiplicative identity: $1 = [(1,0)]$
For example: $[(a,b)] \cdot [(1,0)] = [(a,b)]$
$\therefore \mathbb{N} \times \mathbb{N} / \sim$ is a ring, that is, $\mathbb{Z}$.

$$
\begin{aligned}
\mathbb{N} &\longrightarrow \mathbb{Z} \\
m &\longrightarrow [(m,0)] \\
-m &\longrightarrow [(0,m)] = -[(m,0)]
\end{aligned}
$$

For example:

$$
\begin{aligned}
[(a,b)] &= [(a,0)] + [(0,b)] \\
&= [(a,0)] - [(b,0)]
\end{aligned}
$$

1.4 Construction of Fractions (Rational Numbers)

We need to show that K is a field (with addition and multiplication). Assume A is a commutative ring with no zero divisors, and $A^* = A - \{0\}$.

$A \times A^* = \{(a, b) \mid a, b \in A, b \neq 0\}$

Set of equivalence classes: $K = (A \times A^*)/\sim$ (elements of K are fractions)

We first need to define addition:

$$(a, b) \sim (a', b')$$
$$\frac{a}{b} = \frac{a'}{b'}$$
$$ab' = a'b$$

$[(a, b)]$: The equivalence class of (a, b)

$$[(a, b)] + [(a', b')] = [(ab' + a'b, bb')] = \frac{ab' + a'b}{bb'}$$
$$[(0, 1)] = 0 \in K \quad \text{(additive identity)}$$

The following is to prove that it is independent of the choice of equivalence class:

$$[(a_1, b_1)] + [(a', b')] = [(a_1b' + a'b_1, b_1b')]$$
$$\therefore (ab' + a'b) \cdot b_1b' = (a_1b' + a'b_1) \cdot bb'$$
$$\therefore left = right$$

We then need to define multiplication:

$$[(a, b)][(a', b')] = [(aa', bb')] = \frac{aa'}{bb'}$$
$$[(1, 1)] = 1 \in K \quad \text{(multiplicative identity)}$$

Examples:

$$[(x, y)] + [(0, 1)] = [(x, y)]$$
$$[(x, y)] \cdot [(1, 1)] = [(x, y)]$$
$$[(x, y)] \neq [(0, 1)]$$

So we need to show that K is a field:

$$
\begin{aligned}
[(x,y)]\cdot[(y,x)] = [(xy,xy)] = [(1,1)] \quad &\text{(Non-zeros have inverse)}\\
\because ab' = a'b\\
(a,b) \sim (a',b')\\
\therefore [(xy,xy)] = [(1,1)]
\end{aligned}
$$

Therefore K is a field of fractions of A.
We need to show the following map is injective:
If $A = \mathbb{Z}$ and $K = \mathbb{Q}$:

$$
\begin{aligned}
A &\longrightarrow K \quad \text{(injective)}\\
x &\longrightarrow [(x,1)]\\
x+y &\longrightarrow [(x+y,1)] = [(x,1)] + [(y,1)]\\
x\cdot y &\longrightarrow [(xy,1)] = [(x,1)]\cdot[(y,1)]\\
\text{Comm. ring without 0} &\longrightarrow \text{Field containing the ring}
\end{aligned}
$$

Assume $[(a,1) = [(b,1)]]$ where $a, b \in A$:

$$
\begin{aligned}
a\cdot 1 = b\cdot 1 \Rightarrow a = b\\
\therefore \text{The map is injective.}
\end{aligned}
$$

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, a \in R$
Ring of polynomials: $P(x) \times Q(x) \Rightarrow R(x)$ (with no zero divisors)

1.5 Ordered Set

1.5.1 Partial Order

$(S, \leq)$ is a (partially) ordered set (poset), $\forall x, y, z \in S$:

1. $x \leq$ x
2. $x \leq y, y \leq x \Rightarrow x = y$
3. $x \leq y, y \leq z \Rightarrow x \leq z$

1.5.2 Total Order

Total/Linear order: Either $x \leq y$ or $y \leq x$.
Ordered field is a field K with $\leq$ total order.
Definition:

1. If $x \leq y$ then $x + z \leq y + z$ and
2. if $x \geq 0, y \geq 0$ then $x \times y \geq 0$

There is no total order in complex numbers.
First example:

$$\mathbb{Q} = \{(a,b)\}, (a,b) = \frac{a}{b}$$
$$\text{Since} \quad a \geq 0, b > 0$$

$\therefore \mathbb{Q}$ is an ordered set.
Second example:

$$\begin{aligned} & x \geq 0 \Leftrightarrow -x \leq 0 \\ & x \geq 0 \Rightarrow x^2 \geq 0 \\ & x \leq 0 \Rightarrow (-x)^2 = x^2 \geq 0 \end{aligned}$$

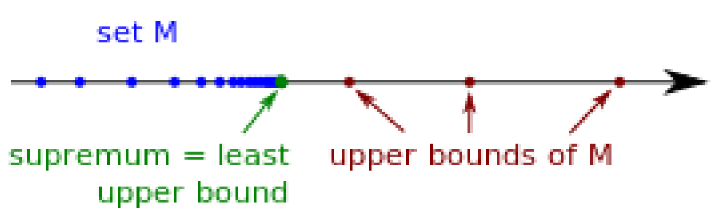
$$\begin{aligned} 1^2 = 1 &> 0 \\ 2 &> 1 \\ 3 &> 2 \\ &\vdots \\ n &> n-1 \end{aligned}$$

1.5.3 Least-upper-bound Property

$\mathbb{Q}$: Ordered field of rational numbers (with gaps)

Least-upper-bound property: K has least-upper-bound property if any bounded subset $E \subset K$ has least upper bound.

K is a field with order relation $\leq$, $E \subset K$ and E is *bounded above* if $\exists \alpha \in K$ such that $p \leq \alpha, p \in E$.



β is the *least-upper-bound* of E if

1. β is an upper bound,
2. for any other upper bound α, we have $\beta \leq \alpha$.

We call $\beta = \sup E$.

We then need to prove that $\mathbb{Q}$ does not have least-upper-bound property, that is, there are gaps in $\mathbb{Q}$.

Proof.

$$E = \{q \in \mathbb{Q} : q^2 \leq 2\} \quad \Rightarrow \quad \text{no } \sup E$$
$$q \in E \Leftrightarrow -q \in E \quad \text{and} \quad q \geq 0$$

Let $p \in \mathbb{Q}$ such that $p \geq 0, p^2 > 2$.

$$\begin{aligned}
&\therefore p^2 > 2 \geq q^2 \Rightarrow p^2 \geq q^2 \\
&\ p^2 - q^2 = (p+q)(p-q) \geq 0 \quad \text{(property of ordered field)} \\
&\because p \geq 0, q > 0 \\
&\therefore p + q \geq 0 + q = q \geq 0 \\
&\therefore p \geq q \\
&\ p \in \mathbb{Q}, p \geq 0, p^2 > 2 \text{ are upper bound of } E.
\end{aligned}$$

Remark: Let p be upper bound of E, $\forall q \in E, q \leq p, p > 0$.
Define p':

$$p' = p - \frac{p^2 - 2}{p+2} = \frac{2p+2}{p+2} = 2\frac{p+1}{p+2}$$

$$\therefore p'^2 - 2 = 2\frac{p^2 - 2}{(p+2)^2}$$

In the above equation, $p^2 - 2 \leq 0$ and $(p+2)^2 \geq 0$.
Assume $p \in E \Rightarrow p^2 \leq 2 \Rightarrow p^2 - 2 \leq 0$.
$\therefore p'^2 - 2 \leq 0 \Rightarrow p' \in E$ and $p' \geq p$.
$\therefore$ Either $p' = p$ or $p' > p$.
But neither of them is possible since $\sqrt{2}$ is irrational:

$$\begin{aligned}
&p = p' \Rightarrow p^2 = 0, p \in \mathbb{Q} \text{ but } p \text{ is irrational.} \\
&\ \therefore p \notin E. \text{ All upper bounds of } E \text{ are } \{p \in \mathbb{Q}, p^2 > 2\}
\end{aligned}$$

r: an upper bound, and $r^2 \geq 2$

$$\begin{aligned}
r' = r - \frac{r^2 - 2}{r^2 + 2} \qquad & \therefore r' < r \\
r'^2 - 2 = 2\frac{r^2}{(r+2)^2} > 0 \quad & \therefore r'^2 > 2
\end{aligned}$$

$\therefore r'$ is an upper bound of E.
So we have no upper bound property in E. □

1.6 Construction of Real Numbers

Theorem. *There exists a total ordered field $\mathbb{R}$ which has least upper bound property. So every subset $E \subset \mathbb{R}$, are bounded above, has a supremum.*

*Such field is **unique**.*

$$K \xrightarrow{\varphi} K'$$
$$\varphi(x+y) = \varphi(x) + \varphi(y)$$
$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$$
if bijective, then isomorphism; otherwise, morphism.

History of construction of real numbers: Dedekind used cuts and Cantor used Cauchy sequence, which works for more general occasions, in 1872.

What is *cuts*?

$$\begin{aligned}
&E \subset \mathbb{Q}, E \neq \mathbb{Q} \to E \text{ is bounded above} \\
&q \in \mathbb{Q}, E_q = \{p \in \mathbb{Q} : p < q\} \\
&\text{If } \alpha \in E, \beta < \alpha \Rightarrow \beta \in \alpha \\
&\gamma \in E, \delta \in E, \delta > \gamma
\end{aligned}$$

1.6.1 Archimedean Property

Theorem (Archimedean Property)**.**

$$x, y \in \mathbb{R}, x > 0, y > 0$$
$$\exists n \in \mathbb{Z}_+, n \cdot x > y$$

Proof. Assume $\{nx \mid n \in \mathbb{Z}\}$ is bounded above, i.e. has least upper bound property.

$$\begin{aligned}
&\exists \alpha = \sup\{nx \mid n \in \mathbb{Z}\} \\
&n \cdot x \leq \alpha, \forall n \text{ and } \alpha - x < \alpha \\
\because\ &\alpha = \sup\{nx \mid n \in \mathbb{Z}\} \\
\therefore\ &\alpha - x \text{ is NOT an upper bound.} \\
&\exists m \text{ such that } \alpha - x < n \cdot x \\
\therefore\ &\alpha < (n+1)x \\
\therefore\ &\text{Contridication.} \\
\therefore\ &\{nx \mid n \in \mathbb{R}\} \text{ is not bounded above.}
\end{aligned}$$

□

1.6.2 Density of Real Numbers

Proof. Assume $x, y \in \mathbb{R}$ and $x < y$.

There exists $q \in \mathbb{Q}$ such that $x < q < y$. Because $y - x > 0$ and $1 > 0$, so there exists n such that $n(y - x) > 1$.

$$
\begin{aligned}
nx &< m_1, \quad m_1 \in \mathbb{Z} \\
-nx &< m_2, \quad m_2 \in \mathbb{Z} \\
\therefore -m_2 < &nx < m_1
\end{aligned}
$$

Conclusion: $\exists m \in \mathbb{Z}, m - 1 \leq nx < m$, so $nx < m \leq nx + 1 < ny$.

$$
\begin{aligned}
nx < m &< ny \\
m &\leq mx + 1 \\
\therefore x < \frac{m}{n} &< y
\end{aligned}
$$

□

1.7 Construction of Complex Numbers

2 Basic Topology

Metric (distance function) of x_0 and x: $d(x_0, x) = |x - x_0|$

2.1 Metric Space M

$$d\colon M \times M \to \mathbb{R}$$

1. $d(x_0, x_1) \geq 0$ where $x_0, x_1 \in M$
2. $d(x_0, x_1) = 0 \iff x_0 = x_1$
3. $d(x_0, x_1) = d(x_1, x_0)$ (irrespective of order)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequity)

Euclidean metric on $\mathbb{R}^2$: $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

Discrete metric: $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Open ball of radius ϵ: $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$

2.2 Open Sets

2.2.1 Metric Spaces X

$U \subset X$ is open if for any $x \in U$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$.

Open interval is open set; closed interval is not open set.

2.2.2 Topological Spaces (X, U)

Let U be a family of all sets, X be a set. U is a **topology** on X if

1. $\emptyset$ (Empty set) is always open; X is open. $\Leftrightarrow$ $\emptyset$ and X itself belong to U.
2. F is a collection of open sets, then $\bigcup_{U \in F} U$ is open. $\Leftrightarrow$ Any union of members of U still belongs to U. (Union)
3. F is a *finite* collection of open sets, then $\bigcap_{U \in F} U$ is open. $\Leftrightarrow$ The intersection of any finite number of members of U belongs to U. (Intersection)

Finite case: $x \in \bigcap_{U \in F} U$, $x \in U$, hence $B(x, \epsilon_U) \subset U$

$$\delta = \min \epsilon_U > 0 \quad where \quad U \in F$$
$$B(x, \delta) \subset B(x, \epsilon_U) \subset U$$
$$\therefore B(x, \delta) \subset \bigcap_{U \in F} U$$

Infinite case: For example, the intersection of all intervals of $(-\frac{1}{n}, \frac{1}{n})$, where n is a positive number, is the set $\{0\}$ which is not open in the real line.

2.3 Compact Sets

In metric space X, compact sets are <u>closed</u>.

Compact $\Leftrightarrow$ closed and bounded (only for Euclidean metric, $\mathbb{R}^n$)