Notes for MATH 3210: Foundation of Analysis I

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Contents

1 Ring and Field

Notations:

- $\mathbb{Z}$: The set of integers (ring, not field, has no inverse);
- $\mathbb{Q}$: The set of rational numbers;
- $\mathbb{R}$: The set of real numbers (ring and field).

1.1 Ring

The set A has two binary operations, addition and multiplication.
For any $a, b \in A$,

$$\begin{aligned} A * A &\to A \\ a, b &\to a + b \end{aligned}$$

1.1.1 Addition Axiom

1. $a + b = b + a$ (commutative)
2. $(a + b) + c = a + (b + c)$ (associative)
3. There is an element 0 such that $a + 0 = a$ (additive identity)
4. There exists $-a$ (additive inverse) such that $a + (-a) = 0$

1.1.2 Multiplication Axiom

1. $a \cdot b = b \cdot a$
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element 1 such that $a \cdot 1 = a$ and $0 \neq 1$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive)

1.2 Field

A is a field if A is a *ring* for any $a \in A$ and $a \neq 0$, there exists a^{-1} (inverse of a) such that $a \cdot a^{-1} = 1$.

Example: $x, y \in A$, $x \neq 0$, $y \neq 0$, $xy = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A has no zero divisors if $x \neq 0, y \neq 0 \to x \cdot y \neq 0$

1.3 Construction of Fractions (Rational Numbers)

We need to show that K is a field (with addition and multiplication). Assume A is a commutative ring with no zero divisors, and $A^* = A - \{0\}$.

$A \times A^* = \{(a, b) \mid a, b \in A, b \neq 0\}$

Set of equivalence classes: $K = (A \times A^*)/\sim$ (elements of K are fractions)

We first need to define addition:

$$(a, b) \sim (a', b')$$
$$\frac{a}{b} = \frac{a'}{b'}$$
$$ab' = a'b$$

$[(a, b)]$: The equivalence class of (a, b)

$$[(a,b)]+[(a',b')]=[(ab'+a'b,bb')]=\frac{ab'+a'b}{bb'}$$
$$[(0,1)]=0\in K(\text{additive})$$

The following is to prove that it is independent of the choice of equivalence class:

$$[(a_1,b_1)]+[(a',b')]=[(a_1b'+a'b_1,b_1b')]$$
$$\therefore (ab'+a'b)\cdot b_1b'=(a_1b'+a'b_1)\cdot bb'$$
$$\therefore left=right$$

We then need to define multiplication:

$$[(a,b)][(a',b')]=[(aa',bb')]=\frac{aa'}{bb'}$$
$$[(1,1)]=1\in K(\text{multiplicative identity})$$

Examples:

$$[(x,y)]+[(0,1)]=[(x,y)]$$
$$[(x,y)]\cdot[(1,1)]=[(x,y)]$$
$$[(x,y)]\neq[(0,1)]$$

So we need to show that K is a field:

$$[(x,y)]\cdot[(y,x)]=[(xy,xy)]=[(1,1)] \quad \text{(Non-zeros have inverse)}$$
$$\because ab'=a'b$$
$$(a,b)\sim(a',b')$$
$$\therefore[(xy,xy)]=[(1,1)]$$

Therefore K is a field of fractions of A.
We need to show the following map is injective:
If $A=\mathbb{Z}$ and $K=\mathbb{Q}$:

$$A\longrightarrow K \quad \text{(injective)}$$
$$x\longrightarrow[(x,1)]$$
$$x+y\longrightarrow[(x+y,1)]=[(x,1)]+[(y,1)]$$
$$x\cdot y\longrightarrow[(xy,1)]=[(x,1)]\cdot[(y,1)]$$
$$\text{Comm. ring without } 0\longrightarrow\text{Field containing the ring}$$

Assume $[(a,1) = [(b,1)]]$ where $a, b \in A$:

$$a \cdot 1 = b \cdot 1 \Rightarrow a = b$$
$$\therefore \text{The map is injective.}$$

2 Basic Topology

Metric (distance function) of x_0 and x: $d(x_0, x) = |x - x_0|$

2.1 Metric Space M

$$d \colon M \times M \to \mathbb{R}$$

1. $d(x_0, x_1) \geq 0$ where $x_0, x_1 \in M$
2. $d(x_0, x_1) = 0 \iff x_0 = x_1$
3. $d(x_0, x_1) = d(x_1, x_0)$ (irrespective of order)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequity)

Euclidean metric on $\mathbb{R}^2$: $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
Discrete metric: $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
Open ball of radius ϵ: $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$

2.2 Open Sets

2.2.1 Metric Spaces X

$U \subset X$ is open if for any $x \in U$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$.
Open interval is open set; closed interval is not open set.

2.2.2 Topological Spaces (X, U)

Let U be a family of all sets, X be a set. U is a **topology** on X if

1. $\emptyset$ (Empty set) is always open; X is open. $\Leftrightarrow$ $\emptyset$ and X itself belong to U.
2. F is a collection of open sets, then $\bigcup_{U \in F} U$ is open. $\Leftrightarrow$ Any union of members of U still belongs to U. (Union)
3. F is a *finite* collection of open sets, then $\bigcap_{U \in F} U$ is open. $\Leftrightarrow$ The intersection of any finite number of members of U belongs to U. (Intersection)

Finite case: $x \in \bigcap_{U \in F} U$, $x \in U$, hence $B(x, \epsilon_U) \subset U$

$$
\begin{gathered}
\delta = \min \epsilon_U > 0 \quad \textit{where} \quad U \in F \\
B(x, \delta) \subset B(x, \epsilon_U) \subset U \\
\therefore B(x, \delta) \subset \bigcap_{U \in F} U
\end{gathered}
$$

Infinite case: For example, the intersection of all intervals of $(-\frac{1}{n}, \frac{1}{n})$, where n is a positive number, is the set $\{0\}$ which is not open in the real line.

2.3 Compact Sets

In metric space X, compact sets are closed.

Compact $\Leftrightarrow$ closed and bounded (only for Euclidean metric, $\mathbb{R}^n$)