Notes for MATH 3210: Foundation of Analysis I

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Contents

1 Ring and Field

Notations:

- $\mathbb{N}$: The set of natural numbers;
- $\mathbb{Z}$: The set of integers (ring, not field, has no inverse);
- $\mathbb{Q}$: The set of rational numbers;
- $\mathbb{R}$: The set of real numbers (ring and field).

1.1 Ring

The set A has two binary operations, addition and multiplication.

For any $a, b \in A$,

$$
\begin{aligned}
A * A &\to A \\
a, b &\to a + b
\end{aligned}
$$

1.1.1 Addition Axiom

1. $a + b = b + a$ (commutative)
2. $(a + b) + c = a + (b + c)$ (associative)
3. There is an element 0 such that $a + 0 = a$ (additive identity)
4. There exists $-a$ (additive inverse) such that $a + (-a) = 0$

1.1.2 Multiplication Axiom

1. $a \cdot b = b \cdot a$
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. There is an element 1 such that $a \cdot 1 = a$ and $0 \neq 1$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive)

1.2 Field

A is a field if A is a *ring* for any $a \in A$ and $a \neq 0$, there exists a^{-1} (inverse of a) such that $a \cdot a^{-1} = 1$.

A has no zero divisors if $x \neq 0, y \neq 0 \to x \cdot y \neq 0$

Example: $x, y \in A$, $x \neq 0$, $y \neq 0$, $xy = 0$

$$
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

1.3 Construction of Integers

We need to construct $\mathbb{Z}$ from $\mathbb{N}$ with **localization**.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Set of equivalence class: $\mathbb{N} \times \mathbb{N} / \sim$
Equivalence relation: $(a, b) \sim (a', b')$ if $a + b' = b + a'$, since $a - b = a' - b'$

$$(a, b) \sim (a', b') \sim (a'', b'')$$
$$a' + b'' = b' + a''$$
$$\boldsymbol{(a + b'') + b'} = (a + b') + b'' = b + a' + b'' = b + b' + a'' = \boldsymbol{(a'' + b) + b'}$$
$$\therefore a + b'' = a'' + b$$
$$\therefore (a, b) \sim (a'', b'')$$

We then should define addition:

$$[(a, b)] + [(a', b')] = [(a + a', b + b')] \tag{1}$$

$$[(a_1, b_1)] + [(a', b')] = [(a_1 + a', b_1 + b')] \tag{2}$$

Equation 1 and equation 2 are equal since independent of choice of equivalence classes:

$$a + a' + b_1 + b' = a_1 + a' + b + b'$$
$$a + b_1 = a_1 + b$$

Additive identity: $0 = [(0, 0)]$
For example: $[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)] = 0$
We then need to define multiplication:

$$[(a, b)] \cdot [(a', b')] = [(aa' + bb', ab' + a'b)]$$

$$\begin{aligned} & (a - b)(a' - b') \\ = \quad & aa' - ab' - a'b + bb' \\ = \quad & (aa' + bb') - (ab' + a'b) \end{aligned}$$

Multiplicative identity: $1 = [(1, 0)]$
For example: $[(a, b)] \cdot [(1, 0)] = [(a, b)]$
$\therefore \mathbb{N} \times \mathbb{N} / \sim$ is a ring, that is, $\mathbb{Z}$.

$$
\begin{aligned}
\mathbb{N} &\longrightarrow \mathbb{Z} \\
m &\longrightarrow [(m,0)] \\
-m &\longrightarrow [(0,m)] = -[(m,0)]
\end{aligned}
$$

For example:

$$
\begin{aligned}
[(a,b)] &= [(a,0)] + [(0,b)] \\
&= [(a,0)] - [(b,0)]
\end{aligned}
$$

1.4 Construction of Fractions (Rational Numbers)

We need to show that K is a field (with addition and multiplication). Assume A is a commutative ring with no zero divisors, and $A^* = A - \{0\}$.

$A \times A^* = \{(a,b) \mid a, b \in A, b \neq 0\}$

Set of equivalence classes: $K = (A \times A^*)/\sim$ (elements of K are fractions)

We first need to define addition:

$$
\begin{gathered}
(a,b) \sim (a',b') \\
\frac{a}{b} = \frac{a'}{b'} \\
ab' = a'b
\end{gathered}
$$

$[(a,b)]$: The equivalence class of (a,b)

$$
\begin{aligned}
[(a,b)] + [(a',b')] = [(ab' + a'b, bb')] &= \frac{ab' + a'b}{bb'} \\
[(0,1)] = 0 \in K &\quad \text{(additive identity)}
\end{aligned}
$$

The following is to prove that it is independent of the choice of equivalence class:

$$
\begin{gathered}
[(a_1,b_1)] + [(a',b')] = [(a_1 b' + a' b_1, b_1 b')] \\
\therefore (ab' + a'b) \cdot b_1 b' = (a_1 b' + a' b_1) \cdot bb' \\
\therefore left = right
\end{gathered}
$$

We then need to define multiplication:

$$
\begin{aligned}
[(a,b)][(a',b')] = [(aa', bb')] &= \frac{aa'}{bb'} \\
[(1,1)] = 1 \in K &\quad \text{(multiplicative identity)}
\end{aligned}
$$

Examples:

$$
\begin{aligned}
[(x,y)] + [(0,1)] &= [(x,y)] \\
[(x,y)] \cdot [(1,1)] &= [(x,y)] \\
[(x,y)] &\neq [(0,1)]
\end{aligned}
$$

So we need to show that K is a field:

$$
\begin{aligned}
[(x,y)] \cdot [(y,x)] = [(xy,xy)] = [(1,1)] \quad &\text{(Non-zeros have inverse)} \\
\because ab' = a'b \\
(a,b) \sim (a',b') \\
\therefore [(xy,xy)] = [(1,1)]
\end{aligned}
$$

Therefore K is a field of fractions of A.
We need to show the following map is injective:
If $A = \mathbb{Z}$ and $K = \mathbb{Q}$:

$$
\begin{aligned}
A &\longrightarrow K \quad \text{(injective)} \\
x &\longrightarrow [(x,1)] \\
x + y &\longrightarrow [(x+y,1)] = [(x,1)] + [(y,1)] \\
x \cdot y &\longrightarrow [(xy,1)] = [(x,1)] \cdot [(y,1)] \\
\text{Comm. ring without } 0 &\longrightarrow \text{Field containing the ring}
\end{aligned}
$$

Assume $[(a,1) = [(b,1)]]$ where $a, b \in A$:

$$
\begin{aligned}
a \cdot 1 = b \cdot 1 \Rightarrow a = b \\
\therefore \text{The map is injective.}
\end{aligned}
$$

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, a \in R$
Ring of polynomials: $P(x) \times Q(x) \Rightarrow R(x)$ (with no zero divisors)

1.5 Ordered Set

1.5.1 Partial Order

$(S, \leq)$ is a (partially) ordered set (poset), $\forall x, y, z \in S$:

1. $x \leq$ x
2. $x \leq y, y \leq x \Rightarrow x = y$
3. $x \leq y, y \leq z \Rightarrow x \leq z$

1.5.2 Total Order

Total/Linear order: Either $x \leq y$ or $y \leq x$.

Ordered field is a field K with $\leq$ total order.

Definition:

1. If $x \leq y$ then $x + z \leq y + z$ and
2. if $x \geq 0, y \geq 0$ then $x \times y \geq 0$

There is no total order in complex numbers.

First example:

$$\mathbb{Q} = \{(a, b)\}, (a, b) = \frac{a}{b}$$
$$\text{Since} \quad a \geq 0, b > 0$$

$\therefore \mathbb{Q}$ is an ordered set.

Second example:

$$\begin{aligned} &x \geq 0 \Leftrightarrow -x \leq 0 \\ &x \geq 0 \Rightarrow x^2 \geq 0 \\ &x \leq 0 \Rightarrow (-x)^2 = x^2 \geq 0 \end{aligned}$$

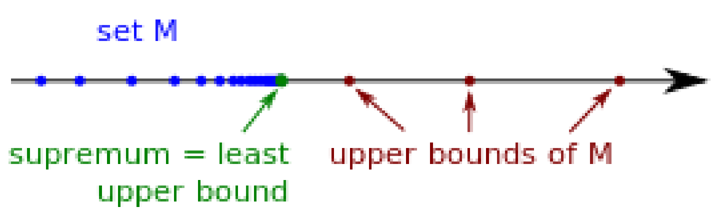
$$\begin{aligned} 1^2 = 1 &> 0 \\ 2 &> 1 \\ 3 &> 2 \\ &\vdots \\ n &> n-1 \end{aligned}$$

1.5.3 Least-upper-bound Property

$\mathbb{Q}$: Ordered field of rational numbers (with gaps)

Least-upper-bound property: K has least-upper-bound property if any bounded subset $E \subset K$ has least upper bound.

K is a field with order relation $\leq$, $E \subset K$ and E is *bounded above* if $\exists \alpha \in K$ such that $p \leq \alpha, p \in E$.



β is the *least-upper-bound* of E if

1. β is an upper bound,
2. for any other upper bound α, we have $\beta \leq \alpha$.

We call $\beta = \sup E$.

We then need to prove that $\mathbb{Q}$ does not have least-upper-bound property, that is, there are gaps in $\mathbb{Q}$.

Proof.

$$
\begin{aligned}
E = \{q \in \mathbb{Q} : q^2 \leq 2\} \quad &\Rightarrow \quad \text{no } \sup E \\
q \in E \Leftrightarrow -q \in E \quad &\text{and} \quad q \geq 0
\end{aligned}
$$

Let $p \in \mathbb{Q}$ such that $p \geq 0, p^2 > 2$.

$$
\begin{aligned}
&\therefore p^2 > 2 \geq q^2 \Rightarrow p^2 \geq q^2 \\
&\quad p^2 - q^2 = (p+q)(p-q) \geq 0 \quad \text{(property of ordered field)} \\
&\because p \geq 0, q > 0 \\
&\therefore p + q \geq 0 + q = q \geq 0 \\
&\therefore p \geq q \\
&\quad p \in \mathbb{Q}, p \geq 0, p^2 > 2 \text{ are upper bound of } E.
\end{aligned}
$$

Remark: Let p be upper bound of E, $\forall q \in E, q \leq p, p > 0$.
Define p':

$$
p' = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} = 2\frac{p + 1}{p + 2}
$$

$$
\therefore p'^2 - 2 = 2\frac{p^2 - 2}{(p+2)^2}
$$

In the above equation, $p^2 - 2 \leq 0$ and $(p+2)^2 \geq 0$.
Assume $p \in E \Rightarrow p^2 \leq 2 \Rightarrow p^2 - 2 \leq 0$.
$\therefore p'^2 - 2 \leq 0 \Rightarrow p' \in E$ and $p' \geq p$.
$\therefore$ Either $p' = p$ or $p' > p$.
But neither of them is possible since $\sqrt{2}$ is irrational:

$$
\begin{aligned}
&p = p' \Rightarrow p^2 = 0, p \in \mathbb{Q} \text{ but } p \text{ is irrational.} \\
&\therefore p \notin E. \text{ All upper bounds of } E \text{ are } \{p \in \mathbb{Q}, p^2 > 2\}
\end{aligned}
$$

r: an upper bound, and $r^2 \geq 2$

$$
\begin{aligned}
r' &= r - \frac{r^2-2}{r^2+2} \qquad && \therefore r' < r \\
r'^2 - 2 &= 2\frac{r^2}{(r+2)^2} > 0 \qquad && \therefore r'^2 > 2
\end{aligned}
$$

$\therefore r'$ is an upper bound of E.
So we have no upper bound property in E. $\square$

1.6 Real Numbers

1.6.1 Construction of Real Numbers

Theorem. *There exists a total ordered field $\mathbb{R}$ which has least upper bound property. So every subset $E \subset \mathbb{R}$, are bounded above, has a supremum.*
*Such field is **unique**.*

$$
\begin{gathered}
K \xrightarrow{\varphi} K' \\
\varphi(x+y) = \varphi(x) + \varphi(y) \\
\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \\
\text{if bijective, then isomorphism; otherwise, morphism.}
\end{gathered}
$$

History of construction of real numbers: Dedekind used cuts and Cantor used Cauchy sequence, which works for more general occasions, in 1872.
What is *cuts*?

$$
\begin{aligned}
&E \subset \mathbb{Q}, E \neq \mathbb{Q} \to E \text{ is bounded above} \\
&q \in \mathbb{Q}, E_q = \{p \in \mathbb{Q} : p < q\} \\
&\text{If } \alpha \in E, \beta < \alpha \Rightarrow \beta \in \alpha \\
&\gamma \in E, \delta \in E, \delta > \gamma
\end{aligned}
$$

1.6.2 Archimedean Property

Theorem (Archimedean Property)**.**

$$
\begin{gathered}
x, y \in \mathbb{R}, x > 0, y > 0 \\
\exists n \in \mathbb{Z}_+, n \cdot x > y
\end{gathered}
$$

Proof. Assume $\{nx \mid n \in \mathbb{Z}\}$ is bounded above, i.e. has least upper bound property.

$$
\begin{aligned}
& \exists \alpha = \sup\{nx \mid n \in \mathbb{Z}\} \\
& n \cdot x \leq \alpha, \forall n \text{ and } \alpha - x < \alpha \\
\because\ & \alpha = \sup\{nx \mid n \in \mathbb{Z}\} \\
\therefore\ & \alpha - x \text{ is NOT an upper bound.} \\
& \exists m \text{ such that } \alpha - x < n \cdot x \\
\therefore\ & \alpha < (n+1)x \\
\therefore\ & \text{Contridication.} \\
\therefore\ & \{nx \mid n \in \mathbb{R}\} \text{ is not bounded above.}
\end{aligned}
$$

□

1.6.3 Density of Real Numbers

Proof. Assume $x, y \in \mathbb{R}$ and $x < y$.

There exists $q \in \mathbb{Q}$ such that $x < q < y$. Because $y - x > 0$ and $1 > 0$, so there exists n such that $n(y - x) > 1$.

$$
\begin{aligned}
nx &< m_1, \quad m_1 \in \mathbb{Z} \\
-nx &< m_2, \quad m_2 \in \mathbb{Z} \\
\therefore -m_2 &< nx < m_1
\end{aligned}
$$

Conclusion: $\exists m \in \mathbb{Z}, m - 1 \leq nx < m$, so $nx < m \leq nx + 1 < ny$.

$$
\begin{aligned}
nx &< m < ny \\
m &\leq mx + 1 \\
\therefore x &< \frac{m}{n} < y
\end{aligned}
$$

□

1.6.4 Property of Real Numbers

We have $x > 0, y > 0, n \geq 2$, and $y^n = x$, such y is unique.

We first need to prove **uniqueness**:

Proof. We have $y_1, y_2 > 0, y_1^n = x, y_2^n = x \Rightarrow y_1 = y_2$

We assume $y_1 \neq y_2$ and $0 < y_1 < y_2$. If $a > 0$, then $ay_1 > ay_2$.

We claim $y_1^n < y_2^n$, then we apply mathematical induction:

When $n = 1$, $y_1 < y_2$.

$$y_1 y_1^n < y_1^n y_2 < y_2 y_2^n$$
$$y_1^{n+1} < y_2^{n+1}$$
$$\therefore y_1^n < y_2^n$$

But they should both be equal to x, therefore we have a contradiction. $\square$

We then need to prove **existence**:

$$E = \{t \in \mathbb{R} \mid t^n < x, x > 0, x \in \mathbb{R}\}$$

Proof. We first need to show that set E is not empty.
We construct $t = \frac{x}{x+1} \Rightarrow 0 < t < 1$ and $t^n < t^{n-1} < \cdots < t < 1$

$$t = \frac{x}{x+1}$$
$$x = t + tx$$
$$t = x - tx < x$$
$$\therefore t^n < t < x$$

$\square$

We then need to show that E is bounded above:

Proof. Suppose $S \geq x + 1 \Rightarrow S > 1$, so $S^m > S^{m-1} > \cdots > S$, therefore $S^m > x + 1 > S$.
It follows that if $t \in E$, then $t < x + 1 \Rightarrow x + 1$ is an upper bound of E.
$\exists y = \sup E, y > 0$ $\square$

We claim $y^n = x$, since $y^n < x$ and $y^n > x$ are both contradictions.
For the first case of contradictions:

$$\frac{x - y^n}{n(y+1)^{n-1}} > 0$$

We have $0 < n < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.

$$(y + h)^n - y^n$$

$$
\begin{aligned}
(y+h)^{n-k-1} * k < (y+h)^{n-k-1} * (y+h)^k \\
&= (y+h)^{n-1} \\
&< n * h * (y+h)^{n-1} < x - y^n \\
&\therefore (y+h)^n - y^n < x - y^n \\
&(y+h)^n < x \\
&y + h \in E \text{ and } \sup E = y < y + h
\end{aligned}
$$

Contradiction.
For the first case of contradictions:

$$y^n > x$$

We have $k = \frac{y^n - x}{ny^{n-1}} > 0$.

$$
\begin{aligned}
&\because 0 < k < \frac{y^n}{ny^{n-1}} = \frac{y}{n} < y \\
&\therefore 0 < k < y \text{ and } 0 < y - k \leq t
\end{aligned}
$$

$$
\begin{aligned}
y^n - t^n &\leq y^n - (y-k)^n \\
&< y^{n-1} < kny^{n-1} = y^n - x \\
&\therefore t^n > x \Rightarrow t \notin E \\
\therefore y - k \notin E &\text{ is an upper bound of } E
\end{aligned}
$$

1.7 Complex Number

Properties:

1. $|z| \geq 0$
2. $|z| = 0 \iff z = 0$
3. $|z| = |-z|$
4. Triangular inequity: $|z_1 + z_2| \leq |z_1| + |z2|$

Proof of Triangular Inequity.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1}z_2) \\
&\leq z_1\overline{z_1} + z_1\overline{z_2} + 2\,|z_1\overline{z_2}| \\
&\leq (|z_1| + |z_2|)^2
\end{aligned}$$

□

1.7.1 Construction of Complex Numbers

$$\begin{aligned}
\mathbb{R}^2 = \{(a,b)|a,b \in \mathbb{R}\} &\text{ forms abelian group} \\
(a,b) + (a',b') = (a+a', b+b') &\quad \therefore (0,0) = 0 \\
-(a,b) = (-a,-b)& \\
\text{Multiplication: } (a,b)\cdot(a',b') = (aa' - bb', a'b + ab')& \\
\text{Identity for mult.: } (1,0) = \mathbb{1} \text{ and } (a,b)\cdot(1,0) = (a,b)&
\end{aligned}$$

$\therefore \mathbb{R}^2$ is a ring and $(a,b) \neq (0,0)$.
For any such elements, we can form

$$\begin{aligned}
(a,b)^{-1} &= (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}) \\
(a,b)\cdot(a,b)^{-1} &= (1,0) = \mathbb{1}
\end{aligned}$$

$\therefore \mathbb{R}^2$ is a field of complex numbers, that is, $\mathbb{C}$.

$$\begin{aligned}
&\because (0,1)^2 = (-1,0) = -\mathbb{1} \\
&\quad (0,-1)^2 = -\mathbb{1} \\
&\therefore i = (0,1) \\
&\quad (a,b) = a\cdot(1,0) + b\cdot(0,1) = a + ib
\end{aligned}$$

1.7.2 Automorphism of Complex Numbers

$$z = (a, b) \text{ and } \overline{z} = (a, -b) \text{ (complex conjugate)}$$
$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
$$\overline{z_1 * z_2} = \overline{z_1} * \overline{z_2}$$
$$z * \overline{z} = (a, b) * (a, -b) = (a^2 + b^2, 0) = (a^2, b^2)$$

We can extend from $\mathbb{R}$ to $\mathbb{C}$, so we have $a \in \mathbb{R}$ and $(a, 0) \in \mathbb{C}$, therefore $\mathbb{R}$ is a subfield of $\mathbb{C}$.

Absolute value of complex number z:

$$\sqrt{z\overline{z}} = \sqrt{a^2 + b^2} = |z| > 0$$

1.8 Inner Product Space

V : vector space over $\mathbb{R}$

$$V \times V \to \mathbb{R}$$
$$(u, v) \to (v \mid u) \quad \text{inner product}$$

Some properties:

We have $\alpha, \beta \in \mathbb{R}$:

1. $(u \mid v) = (v \mid u)$
2. $(u \mid u) \geq 0$ and $(u \mid u) = 0 \iff u = 0$
3. Norm: $\|u\| = (u \mid u)^{\frac{1}{2}}$
4. $(\alpha v + \beta w \mid u) = \alpha(v \mid u) + \beta(w \mid u)$
5. $(v \mid \alpha u + \beta w) = \alpha(v \mid u) + \beta(v \mid w)$

Euclidean inner product: In $\mathbb{R}^n, x = \{x_1, x_2, \cdots, x_n\}, y = \{y_1, y_2, \cdots, y_n\}$, $(x \mid y) = \sum_{i=1}^{n} x_i y_i$

1.9 Cauchy-Schwarz Inequality

Theorem (Cauchy-Schwarz Inequality)**.**

$$\forall u, v \in V, |(u \mid v)| \leq \|u\| \cdot \|v\|$$

Proof. We can assume that $u, v \neq 0$ and $\forall t \in \mathbb{R}$, $(tu - v \mid tv - u) \geq 0$

$$
\begin{aligned}
& t^2(u \mid u) - t(u \mid v) - t(v \mid u) + (v \mid v) \geq 0 \\
\because\, & b^2 - 4ac \leq 0 \\
& 4(u \mid v)^2 \leq 4\|u\|^2\|v\|^2 \\
& |(u \mid v)| \leq \|u\|\|v\|
\end{aligned}
$$

□

Properties of norms:

1. $\|v\| \geq 0$
2. $\|v\| = 0 \iff v = 0$
3. $\|\alpha v\| = |a|\, \|v\|, \alpha \in \mathbb{R}$
4. $\|u + v\| \leq \|u\| + \|v\|$

2 Basic Topology

Metric (distance function) of x_0 and x: $d(x_0, x) = |x - x_0|$

2.1 Metric Space M

$$d\colon M \times M \to \mathbb{R}_+$$

1. $d(x_0, x_1) \geq 0$ where $x_0, x_1 \in M$
2. $d(x_0, x_1) = 0 \iff x_0 = x_1$
3. $d(x_0, x_1) = d(x_1, x_0)$ (irrespective of order)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequity)

Norm in metric space: $d(u, v) = \|u - v\| \geq 0$
$\|u - v\| = \|(u - w) + (w - v)\| \leq \|u - w\| + \|w - v\|$
Euclidean metric in $\mathbb{R}^2$: $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
Euclidean metric in $\mathbb{R}^n$: $d(u, v) = (\sum_{i-1}^{n}(u_i - v_i)^2)^{\frac{1}{2}}$
Discrete metric: $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
Discrete metric is open and closed.
Open ball of radius ϵ: $B_\epsilon(x_0) = B(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$
Open ball in M centered in x of radius ϵ/r, and Rudin called it "neighborhood."

Example: In discrete metric, if $r > 1$, $y \in B_r(x) \Rightarrow d(x, y) < r \Rightarrow B_r(x) = S$; if $r \leq 1$, $B_r(x) = \{x\}$

2.2 Open Sets

$U \subset X$ is open (without boundary) if for any $u \in U$ there exists $r > 0$ such that $B(u, r) \subset U$.

Open interval is open set; closed interval is not open set.

Prove $B_r(x)$ is an open set (https://math.stackexchange.com/questions/104083/an-open-ball-is-an-open-set):

Proof. $y \in B_r(x)$ and $\delta = d(x, y)$, so $\epsilon < r - \delta$. We have $z \in B_\epsilon(y)$, so $d(x, z) < \delta + \epsilon < r$, therefore $z \in B_\epsilon(y) \Rightarrow z \in B_r(x)$. $\square$

Theorem.

$$(\bigcup_{i \in I} U_i)^C = \bigcap_{i \in I} U_i^C$$

Proof. Suppose x in the left and y in the right, so $x \notin \cup U_i$, then $x \notin U_i$, so $x \in U_i^C$ for any i, so x in the right. Conversely, we can prove that y in the left. Therefore it follows that left equals to right. $\square$

Properties of open sets:

1. $\emptyset \in S$ is open;
2. S itself is open;
3. If $U_i, i \in I$ is a family of open sets, then $\bigcup_{i \in I} U_i$ is an open set. (Union of arbitrary (finite and infinite) collection of open sets is open)

   Proof. If $x \in \bigcup_{i \in I} U_i$, $x \in U_i$, so there exists $r > 0$, such that $B_r(x) \in U_i \Rightarrow B_r(x) \subset \bigcup_{i \in I} U_i$. $\square$

4. Intersection of *finite* open sets is open. (Why not infinite: consider $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$, where $0 < \epsilon < 1$, which is not an open set.)

   Proof. If $x \in \bigcap_{i \in I}^{n} U_i$, then $x \in U_i$. So U_i is open and $B_r(x) \subset U_i$. Let $\rho = \min(r_1, r_2, \cdots, r_n) > 0$, $B_\rho(x) \subset B_{r_i}(x) \subset \bigcap_{i \in I}^{n} U_i \Rightarrow B_\rho(x) \subset \bigcap_{i \in I}^{n} U_i$. $\square$

2.3 Closed Sets

$Z \subset S$ is closed if Z^C is open.

Third part of last subsection implies that $\bigcup_{i \in I} Z_i^C$ is open, which means that $\bigcap_{i \in I} Z_i$ is closed. Similarly, fourth part of last subsection implies that $\bigcup_{i=1}^{n} Z_i$ is closed.

Example (Chaos/Indiscrete topology): If $S \neq \emptyset$,then $u = \{\emptyset, S\}$. $(S, \{\emptyset, S\})$ is called a discrete space.

2.4 Compact Sets

In metric space X, compact sets are closed.

Compact $\Leftrightarrow$ closed and bounded (only for Euclidean metric, $\mathbb{R}^n$)

2.5 Topological Spaces (X, U)

Let U be a family of all sets, X be a set. U is a **topology** on X if

1. $\emptyset$ (Empty set) is always open; X is open. $\Leftrightarrow$ $\emptyset$ and X itself belong to U.
2. F is a collection of open sets, then $\bigcup_{U \in F} U$ is open. $\Leftrightarrow$ Any union of members of U still belongs to U. (Union)
3. F is a *finite* collection of open sets, then $\bigcap_{U \in F} U$ is open. $\Leftrightarrow$ The intersection of any finite number of members of U belongs to U. (Intersection)

Finite case: $x \in \bigcap_{U \in F} U$, $x \in U$, hence $B(x, \epsilon_U) \subset U$

$$\delta = \min \epsilon_U > 0 \quad where \quad U \in F$$
$$B(x, \delta) \subset B(x, \epsilon_U) \subset U$$
$$\therefore B(x, \delta) \subset \bigcap_{U \in F} U$$

Infinite case: For example, the intersection of all intervals of $(-\frac{1}{n}, \frac{1}{n})$, where n is a positive number, is the set $\{0\}$ which is not open in the real line.

Remark: Some topological spaces are not metric.

Discrete topology: (S, d) is discrete metric, and $B_{\frac{1}{2}}(x) = \{x\}$ since all subsets of S are open.

$U \subset S$, $\bar{U}$ is the smallest closed set that contains U, and it is the closure of U. U^o is the largest open set contained in U, and it is the interior of U.

Since $T^o \subset T \subset \bar{T}$,boundary of T:

$$\partial T = \bar{T} - T^o$$

Example: We have $\mathbb{Q} \subset \mathbb{R}$, so $\mathbb{Q}^o(\emptyset) \subset \mathbb{Q} \subset \bar{\mathbb{Q}}(\mathbb{R})$, so the boundary is $\mathbb{R}$, which is much larger than $\mathbb{Q}$.

Example: In discrete topology, $T^o = T = \bar{T}$, so $\partial T = \emptyset$.

2.6 Continuous Map

We have two topological spaces, (S, U) and (T, V). A function $f : S \to T$ is continuous if $\forall v \in V, f^{-1}(V) \in U$, where $f^{-1}(V) = \{u \in S \mid f(u) \in V\}$.

Identity map: $id : S \to S$ is continuous.

Composition: $(g \circ f)(s) = g(f(s))$; if f and g are continuous, then $f \circ g$ is continuous.

Example:

1. S (discrete topology) $\rightarrow T$ all functions from S to T with f is continuous.
2. $S \rightarrow T$ (indiscrete topology), $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(T) = S$ are both open.