Computation of Tail Probability Distributions via double exponential transform and Sinc method

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Abstract. In this article, we use the double exponential method, combine with Sinc interpolation formula to calculate five common probability distributions - Gaussian, inverse Gaussian, t-student, Gamma and Fischer distributions. Using this algorithm, we are able to compute integrals to high degree of pre-determined accuracy.

Keywords.

Double exponential method. Sinc interpolation formula. Numerical integration.

1 Introduction

The study of probability distributions is of great interest and challenge to diverse scientific community. Many random phenomenon can easily model by probability distributions, for example, in mathematical finance, Black - Scholes equation is use to model the price of European-style options [1]. In many application, the probability distributions become complicate and it is impossible to evaluate the integral analytically. While Numerical method provide a less satisfactory answer, it is necessary to study and develop it for scientific applications. Indeed, Numerical method has been an active and interdisciplinary area of research. With advent of supercomputer, numerical methods become more accessible to implement yet highly sophisticated.

In this article, we develop a method to evaluate the tail integral of probability distributions, in particular the positive end of the tail integral. The problem of calculating the tail integral arise commonly in mathematical finance - Tail risk is a form of portfolio risk that have a small probability of occurring , and occur at both ends of a normal distribution curve. Understanding of the risk would assist investors to hedge against risk and maximize long term return, more informations of tail risk can be find in Articles [2,3]. Because of this, it is necessary to evaluate the tail integral to a high degree of accuracy. Straightforward numerical integration techniques such as quadrature formulas does not provide sufficient accuracy, so more sophisticate methods need to be considered.

Previous article by Gaudreau, Slevinsky and Safouhi [4] develop a quiet accurate integration technique which apply to five probability distributions - Normal(Gaussian) distribution, student's t-distribution, F distribution, Gamma distribution and Inverse Gaussian distribution. The method rely on the fact that these probability distributions satisfy the first order linear homogeneous differential equation, so it is justify to apply $G_n^{(1)}$ transform to the PDFs. More information of the $G_n^{(1)}$ transform refer to article [5–7]. The $G_n^{(1)}$ transform require calculation of higher dimensional determinants for large n, which can be cumbersome and present a challenge for its implementation. They used the recursive algorithm they introduced earlier [8], combined with Slevinsky - Safouhi formula 1 (SSF1) formula [9], which make the computation of higher order derivative significantly easier. The numerical result was then compare to the values obtain from Maple. The method was quiet accurate as the result obtained match highly with Maple.

One deficiency of the method is the lack of control of accuracy. The numerical error of approximation would oscillate around the exact value rather than converge to it, as shown in FIG. 4.1. of [4]. The amount of accuracy become uncertain when the exact solution might not exist in practical application. Second, there is no general reliable test which stops the computation once it reach desire accuracy because in some case after certain order of the transformation, the relative error grows larger and diverge. Additional stopping criterion was needed for some cases. Last, The method require sophisticate knowledge of extrapolation technique which can be inaccessible to wider scientific audience.

In this article, we introduce Sinc quadratic rule combine with double exponential method, the purpose is to provide an alternative method and address the weakness mentioned above. The double exponential transformation was first proposed by Takahasi and Mori [10] in 1974. The method was extended and applied to Fourier type integral by Ooura and Mori [11]. Recently, it was discovered that the double transformation method is also useful for a variety of so-called Sinc numerical methods [12, 13]. The trapezoidal rule and Sinc quadrature rule are intimately related and can be derive from one another. Similar method like single exponential transformation combined with Sinc formula was introduced by Haber [14]. The benefit of using double exponential transformation is that the error of the formula behaves approximately as $\exp(\frac{-c_1 N}{\log(c_2 N)})$, while the single exponential transformation formula behave approximately as $\sqrt{N}\exp(-\sqrt{c_1 N})$. If we compare the dependence on N we see that the error of double exponential formula converges to zero much faster than single exponential formula. This corresponds to significant saving in computation time and resource. It has theoretically shown double exponential transform method is optimal with respect to other transformation - using functional analysis approach, Sugihara was able to prove the double exponential function decay fastest other than the trivial function [15].

In practical implementation, our method involve infinite summation of function. We use the stopping criterion which enable to approximate infinite summation using finite number of terms. The numerical result compare to values from article [4], the result replicate for up to 16 correct digits.

We organize the articles as follow: in section 2, we will brief introduce the probability distributions(PDF) and its various application; in section 3, we then define the double transform method and Sinc interpolation formula. In section 4, we present the application of method to five PDF and the numerical result. In section 5, we present several numerical discussion.

2 Definitions of probability distribution

In this section, we define the normal distribution, the gamma distribution, the t student distribution, the inverse Gaussian distribution and the F distribution.

The normal distribution (Gaussian distribution) has the probability density function (PDF) given by:

$$f_N(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad \text{for} \qquad -\infty < x < +\infty, \tag{1}$$

where μ denotes the mean of the distribution and σ^2 represents the variance. Normal distribution(Gaussian distribution) is one of the most important distribution in statistics and are often used to represent random variables whose distribution are not known. Central limit theorem states that every independent, identically distributed random variables become normal distribution, when the number of observations is sufficiently large.

The gamma distribution has the PDF:

$$f_g(x) = \frac{x^{a-1}\, e^{-\frac{x}{b}}}{\Gamma(a)\, b^a} \qquad \text{for} \qquad 0 \, < \, x \, < \, +\infty, \tag{2}$$

in which $a > 0$ and $b > 0$ are parameters and Γ is the gamma function. Gamma function is a two-parameter family of continuous probability distribution.The exponential distribution and chi-squared distribution are special cases of the gamma function. The Gamma distribution has been used to model the size of insurance claims [16] and amount of rainfalls [17].

The t student distribution has the PDF:

$$f_t(x) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\,\pi}\,\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{x^2}{v}\right)^{-(\frac{v+1}{2})} \qquad \text{for} \qquad -\infty \, < \, x \, < \, +\infty, \tag{3}$$

where the parameter $v > 0$ stands for the number of degrees of freedom. Its arises when estimating the mean of normal distribution when the sample size is small and the standard deviation is unknown. When the number of degrees of freedom v become large, the t-distribution converges to standard normal distribution.

The inverse Gaussian distribution has the PDF:

$$f_i(x) = \left(\frac{\lambda}{2\,\pi\, x^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda\,(x-\mu)^2}{2\,\mu^2\, x}\right) \qquad \text{for} \qquad 0 \, < \, x \, < \, +\infty, \tag{4}$$

where μ and λ are two parameters. The mean of the inverse Gaussian distribution is μ and the variance is $\sigma^2 = \frac{\mu^3}{\lambda}$. The inverse Gaussian distribution was first used to model Brownian motion [18].

The F Fisher distribution has the PDF:

$$f_F(x) = \frac{\Gamma\left(\frac{a+b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\,\Gamma\left(\frac{b}{2}\right)} \left(\frac{a}{b}\right)^{a/2} \frac{x^{\frac{a-2}{2}}}{\left(1 + \left(\frac{a}{b}\right) x\right)^{\frac{a+b}{2}}} \qquad \text{for} \qquad 0 \, < \, x \, < \, +\infty, \tag{5}$$

which the integers a and b are two parameters. The mean of the F distribution is $\mu = \dfrac{b}{b-2}$ for $b > 2$ and the variance is $\sigma^2 = \dfrac{2\,b^2\,(a+b-2)}{a\,(b-2)^2\,(b-4)}$ for $b > 4$.It is used most notably in the analysis of variance(ANOVA) and F test [19].

3 Double exponential transform and Sinc method

The double exponential method is given as: Suppose $f(x)$ is a function that we would like to integral over some interval $[a, b]$, where $a, b \in \mathbb{R}$. We find a transform function $\phi(t)$ which satisfy

$$\phi(-\infty) = a, \;\; \phi(\infty) = b \tag{6}$$

so that the interval $[a, b]$ is map to $(-\infty, \infty)$. Then by the chain rule,

$$\int_a^b f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(\phi(t))\phi^{'}(t)\mathrm{d}t \tag{7}$$

The weight function $f(\phi(t))\phi^{'}(t)$ is characteristics by its double exponential decay behavior at the end point $|t| \to \infty$. Sinc interpolation formula is define as

$$u(t) = \sum_{k=-\infty}^{\infty} u(kh)\,\mathrm{sinc}(\frac{t}{h} - k) \tag{8}$$

Where h is the mesh size, in this article we use $h = 0.1$. $\mathrm{sinc}(x)$ is define as $\frac{\sin(\pi x)}{\pi x}$. Notice that the sinc function is normalized such that $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\mathrm{d}x = 1$.

Combine the double exponential transformation and Sinc interpolation the integral become

$$\int_a^b f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(\phi(t))\phi^{'}(t)\mathrm{d}t = \sum_{k=-\infty}^{\infty} f(\phi(kh))\phi^{'}(kh) \int_{-\infty}^{\infty} \mathrm{sinc}(\frac{t}{h} - k)\mathrm{d}t = \sum_{k=-\infty}^{\infty} f(\phi(kh))\phi^{'}(kh) \quad (9)$$

the interchange of summation and integration was justified since the double exponential decay property of the weight function. In practical computation, we must truncate the summation at some $k = m, n$ for some $m, n \in \mathbb{Z}$. In this article, We stop the summation whenever $|f(\phi(kh))\phi^{'}(kh)| < \epsilon$, where ϵ is a very small positive number. We find that for $\epsilon = 10^{-15}$, the stopping criterion provides sufficient accuracy for all probability distributions we study. More over, ϵ can be arbitrary smaller to achieve better accuracy.

4 Computing the probability distributions

The objective of this section is to compute integral tails $\int_z^{\infty} f(t)\mathrm{d}t$, where $f(t)$ is a probability distribution function and $z \in \mathbb{R}^+$, using the method described above. First we need to find an applicable transform function so that the transformed integrand satisfy double exponential decay property. Some of the basic transformed function are given in Article by Takahasi and Mori [10]. We will modify the transform function so it would map the interval $[z, \infty)$ to $(-\infty, \infty)$. The five probability distribution function we study have different asymptotic behavior at the end points, so it require to have different transform function $\phi(t)$ for each one.

4.1 Gaussian distribution

The PDF 1 behaves as $\exp(-x^2)$ as $|x| \to \infty$. We use the transform function $\phi(t) = e^{t-e^{-t}} + z$. It is easy to see that $\phi(-\infty) = z$ and $\phi(\infty) = \infty$. Hence our integral become

$$\int_z^{\infty} f(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty} u(kh) \int_{-\infty}^{\infty} \mathrm{sinc}(\frac{t}{h} - k)\mathrm{d}t = \sum_{k=-\infty}^{\infty} u(kh) \approx \sum_{k=m}^{n} u(kh) \quad (10)$$

where m, n correspond to right and left stopping index. And

$$u(t) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \exp\left(-\frac{(\phi(t) - \mu)^2}{2\,\sigma^2}\right) \phi^{'}(t) \quad (11)$$

In table 1 , we list several numerical results for Gaussian distribution. We start the function evaluation at index $k = 0$, the right index is the stopping index on the positive side; Then we set index $k = -1$, similarly the left index is the stopping index on the negative side. In general, the left stopping index is higher than right stopping index, because of non-symmetric of the weight function. The sum is numerical values we obtained through the method present above.

Table 1: Numerical evaluation of the tail integral of the normal distribution.

z	μ	σ	Right Index	Left Index	Sum(15)	Maple Value(15)
1.2	‘0	1.0	22	36	.115 069 670 221 708(00)	.115 069 670 221 708(00)
1.6	‘0	1.0	21	36	.547 992 916 995 579(-01)	.547 992 916 995 579(-01)
2.0	‘0	1.0	21	36	.227 501 319 481 791(-01)	.227 501 319 481 792(-01)
3.0	‘0	1.0	19	35	.134 989 803 163 004(-02)	.134 989 803 163 009(-02)
6.0	‘0	1.0	12	28	.986 587 548 902 742(-09)	.986 587 645 037 698(-09)
45	18	6.0	32	31	.339 767 312 459 529(-05)	.339 767 312 473 006(-05)
54.2	‘2.0	25	51	34	.183 989 173 418 576(-01)	.183 989 173 418 576(-01)
0.30	‘0	1.0	23	36	.382 088 577 811 047(-01)	.382 088 577 811 047(-01)

4.2 The gamma distribution

The PDF 2 has single exponential decay property $\exp(-x)$ as $x \to \infty$ We use the transform function $\phi(t) = e^{t-e^{-t}} + z$, which has double exponential decay property as $t \to -\infty$ and single exponential decay property as $t \to \infty$. Then our integral become

$$\int_z^\infty f(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty} u(kh) \int_{-\infty}^{\infty} \mathrm{sinc}(\frac{t}{h} - k)\mathrm{d}t = \sum_{k=-\infty}^{\infty} u(kh) \tag{12}$$

Where $u(t)$ is given by

$$u(t) = \frac{1}{\Gamma(a)\, b^a} (\phi(t))^{a-1} \exp(-\frac{\phi(t)}{b})\phi^{'}(t) \tag{13}$$

Table 2: Numerical evaluation of the tail integral of the gamma distribution.

z	a	b	Right Index	Left Index	Sum(15)	Maple Value(15)
13	7	2	47	35	.526 523 622 517 999(00)	.526 523 622 517 999(00)
15	7	2	46	35	.378 154 694 323 469(00)	.378 154 694 323 469(00)
20	7	2	46	35	.130 141 420 882 482(00)	.130 141 420 882 482(00)
35	7	2	45	34	.147 001 977 487 619(-02)	.147 001 977 487 619(-02)
40	7	2	44	34	.255 122 495 856 297(-03)	.255 122 495 856 300(-03)
45	7	2	43	33	.407 935 571 774 482(-04)	.407 935 571 774 571(-04)
50	7	2	43	32	.610 629 446 192 140(-05)	.610 629 446 192 790(-05)
60	7	2	41	30	.117 319 420 010 362(-06)	.117 319 420 023 469(-06)
25.5	4.432	2.0230	44	34	.251 747 197 371 770(-02)	.251 747 197 371 771(-02)
45	5.432	4.5432	53	34	.453 930 946 920 783(-01)	.453 930 946 920 784(-01)
14	1.111	9	59	35	.245 873 088 348 520(00)	.245 873 088 348 520(00)

4.3 The t student distribution

The transform function $\phi(t) = \exp(\frac{\pi}{2}\sinh(t)) + z$ maps the interval from $[z, \infty)$ to $(-\infty, \infty)$. let $f_t(x)$ be the student's t-distribution we obtain:

$$\int_z^\infty f_t(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty} u(kh) \int_{-\infty}^{\infty} \mathrm{sinc}(\frac{t}{h} - k)\mathrm{d}t = \sum_{k=-\infty}^{\infty} u(kh) \tag{14}$$

Where

$$u(t) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\,\pi}\,\Gamma(\frac{v}{2})}\left(1+\frac{(\phi(t))^2}{v}\right)^{-\frac{v+1}{2}}\phi^{'}(t) \tag{15}$$

Table 3: Numerical evaluation of the tail integral of the t student distribution.

z	v	Right index	Left index	Sum(15)	Maple value(15)
1.812	10	19	39	.500 376 310 329 236(-01)	.500 376 310 329 236(-01)
2.228	10	19	38	.250 058 859 085 556(-01)	.250 058 859 085 556(-01)
3.169	10	19	38	.500 231 668 219 243(-02)	.500 231 668 219 242(-02)
4.587	10	19	37	.499 918 645 938 128(-03)	.499 918 645 938 171(-03)
6.927	20	14	35	.500 032 563 400 663(-06)	.500 032 563 506 499(-06)
5.449	60	10	35	.499 901 999 380 232(-06)	.499 901 999 489 723(-06)
3.373	120	11	37	.500 752 580 749 910(-03)	.500 752 580 749 990(-03)

4.4 The inverse Gaussian distribution

We use the transform function $\phi(t) = \exp(t - \exp(-t)) + z$:

$$\int_z^\infty f_t(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty} u(kh)\int_{-\infty}^{\infty}\mathrm{sinc}(\frac{t}{h}-k)\mathrm{d}t = \sum_{k=-\infty}^{\infty} u(kh) \tag{16}$$

where $u(t) = \left(\frac{\lambda}{2\,\pi\,\phi(t)^3}\right)^{1/2}\exp\left(-\frac{\lambda\,(\phi(t)-\mu)^2)}{2\,\mu^2\,\phi(t)}\right)\phi^{'}(t)$

Table 4: Numerical evaluation of the tail integral of the inverse Gaussian distribution.

z	μ	λ	Right index	Left index	Sum(15)	Maple Value(15)
1.50	1.00	1.00	42	35	.189 232 007 000 020(0)	.189 232 007 000 020(0)
2.00	1.00	1.00	42	35	.114 524 574 013 994(0)	.114 524 574 013 993(0)
3.00	1.00	1.00	42	35	.468 120 792 572 116(-1)	.468 120 792 572 116(-1)
4.50	1.00	1.00	42	35	.143 011 829 460 931(-1)	.143 011 829 460 931(-1)
16.00	1.00	1.00	40	32	.943 916 863 492 529(-5)	.943 916 863 494 723(-5)
32.00	1.00	1.00	35	27	.122 006 553 368 850(-8)	.122 006 566 375 975(-8)
24.00	2.00	4.00	38	30	.510 429 100 274 007(-6)	.510 429 100 438 016(-6)
33.46	4.54	2.78	61	34	.621 975 008 388 138(-2)	.621 975 008 388 144(-2)
23.00	6.54	6.00	61	34	.333 636 164 607 370(-1)	.333 636 164 607 370(-1)

4.5 The F Fisher distribution

We use the transform $\phi(t) = \exp(\frac{\pi}{2}\sinh(t)) + z$, the integral become

$$\int_z^\infty f(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty} u(kh)\int_{-\infty}^{\infty}\mathrm{sinc}(\frac{t}{h}-k)\mathrm{d}t = \sum_{k=-\infty}^{\infty} u(kh) \tag{17}$$

where $u(t)$ is given by:

$$u(t) = \frac{\Gamma(\frac{a+b}{2})}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}(\frac{a}{b})^{\frac{a}{2}}\frac{(\phi(t))^{\frac{a-2}{2}}}{(1+\frac{a}{b}\,\phi(t))^{\frac{a+b}{2}}}\,\phi^{'}(t) \tag{18}$$

Table 5: Numerical evaluation of the tail integral of the F distribution.

z	a	b	Right index	Left index	Sum(15)	Maple value(15)
4.190	3	4	33	38	.100 029 643 896 895(0)	.100 029 643 896 895(00)
6.590	3	4	33	38	.500 168 891 790 405(-1)	.500 168 891 790 405(-01)
9.980	3	4	33	38	.249 965 339 234 568(-1)	.249 965 339 234 568(-01)
16.70	3	4	33	37	.999 383 733 001 456(-2)	.999 383 733 001 462(-02)
5.750	5	1	46	38	.306 042 577 763 857(0)	.306 042 577 763 857(00)
3.340	1	1	46	38	.318 737 836 141 656(0)	.318 737 836 141 636(00)
23.23	10	5	31	37	.142 310 351 602 084(-2)	.142 310 351 602 084(-02)
12.05	8	3	35	38	.325 796 489 130 336(-1)	.325 796 489 130 337(-01)

5 Numerical discussion

The double exponential and Sinc method simplified the calculation of tail integral of probability distributions. It allows us to express the integral tail as the summation of sample points of transformed function. The algorithm is easy to implement and provide high degree of accuracy , as shown above by the numerical table. Further, The degree of accuracy can be easily adjust by changing the sample points computed. We examine couple distributions with different number of stopping index, to investigate how change of stopping index affect the degree of accuracy (figure 1). it is easy to see the negative linear relation between the stopping index and relative error.

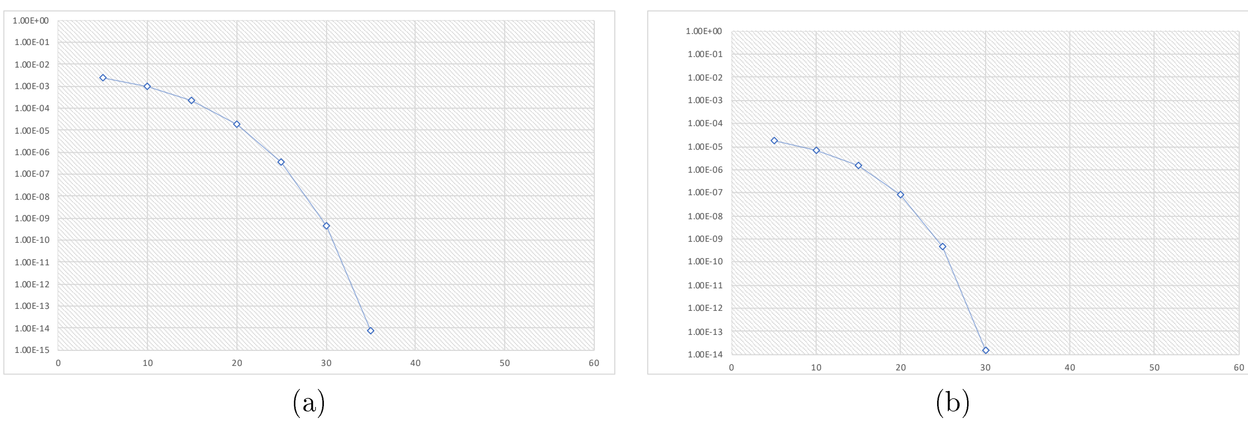

Figure 1: Plot of the base 10 logarithm of the relative error as stopping index n of the approximations for (a) the Gamma distribution with $z = 45$, $\mu = 18$, and $\sigma = 6$ (corresponds to Table 6) and for (b) the t distribution with $x = 4.587$ and $v = 10$ (corresponds to Table 7).

Table 6 shows numerical values for Gaussian distribution with different fixed number of stopping index N. Summation is the approximation we obtain, whereas error is the difference of summation and Maple values. The error is less than 10^{-15} with $N = 35$.

Table 6: Evaluation of Gaussian distribution for $z = 45, \mu = 18, \sigma = 6$ by (1).

N	Summation	Error
5	.157 277 161 863 221(-05)	.182 490(-05)
10	.268 194 627 626 314(-05)	.715 727(-06)
15	.325 234 019 829 763(-05)	.145 333(-06)
20	.338 910 920 125 176(-05)	.856 392(-08)
25	.339 762 839 629 612(-05)	.447 284(-10)
30	.339 767 312 320 389(-05)	.152 617(-14)
35	.339 767 312 473 006(-05)	0
40	.339 767 312 473 006(-05)	0
45	.339 767 312 473 006(-05)	0
50	.339 767 312 473 006(-05)	0

Table 7 shows numerical values for student distribution with different fixed number of stopping index N. The error is less than 10^{-15} with $N = 40$.

Table 7: Error table for the t distribution for $x = 4.587$ and $v = 10$ by (3).

N	Summation	Error
5	.255 561 532 102 862(-3)	.244 357(-03)
10	.404 152 451 294 424(-3)	.957 662(-04)
15	.478 159 434 669 979(-3)	.217 592(-04)
20	.498 047 743 664 632(-3)	.187 092(-05)
25	.499 885 082 172 954(-3)	.335 638(-07)
30	.499 918 601 120 739(-3)	.448 174(-10)
35	.499 918 645 937 391(-3)	.781 059(-15)
40	.499 918 645 938 172(-3)	0
45	.499 918 645 938 172(-3)	0
50	.499 918 645 938 172(-3)	0

6 Conclusion

In this paper we develop double exponential method and sinc interpolation formula to calculate the tail integral of five probability distributions. The key part of the method is to find an appropriate transform function so that transformed integrand has double exponential property. The Sinc interpolation allows us to sum up the sample points of the transformed integrand. Compare to the extrapolation method by [4], the double exponential method present several advantages. As discussed in the introduction, the stopping criterion we use is sufficient for all PDFs studied; the degree of accuracy can easily adjust by changing stopping index depends on its applications; Further this algorithm is relatively easier to implement and more accessible to the general public. One possible drawback of this method is we have to use different transformation function for PDFs, which depends on the interval of integration and asymptotic behavior of integrands.

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