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FAST MARCHING METHODS FOR STATIONARY HAMILTON–JACOBI EQUATIONS WITH AXIS-ALIGNED ANISOTROPY*

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Abstract. The fast marching method (FMM) has proved to be a very efficient algorithm for solving the isotropic Eikonal equation. Because it is a minor modification of Dijkstra's algorithm for finding the shortest path through a discrete graph, FMM is also easy to implement. In this paper we describe a new class of Hamilton–Jacobi (HJ) PDEs with axis-aligned anisotropy which satisfy a causality condition for standard finite-difference schemes on orthogonal grids and can hence be solved using the FMM; the only modification required to the algorithm is in the local update equation for a node. This class of HJ PDEs has applications in anelliptic wave propagation and robotic path planning, and brief examples are included. Since our class of HJ PDEs and grids permit asymmetries, we also examine some methods of improving the efficiency of the local update that do not require symmetric grids and PDEs. Finally, we include explicit update formulas for variations of the Eikonal equation that use the Manhattan, Euclidean, and infinity norms on orthogonal grids of arbitrary dimension and with variable node spacing.

Key words. fast marching method, anisotropic optimal control, Hamilton–Jacobi equation, viscosity solution

AMS subject classifications. 35F30, 49L20, 49L25, 49N90, 65N06, 65N12

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1. Introduction.

The fast marching method (FMM) [29, 23] has become a popular algorithm to use when solving the Dirichlet problem for an isotropic static Hamilton–Jacobi partial differential equation (HJ PDE), also known as the Eikonal equation $\|Du(x)\|_2 = c(x)$. FMM has proven to be particularly efficient in practice because it can approximately solve this problem in a single pass through the nodes of a grid. It is also straightforward to implement, requiring only a small modification of Dijkstra's algorithm [9], which is a popular method for finding the shortest path through a graph.

While the isotropic case is the most common, there are applications which require the solution of anisotropic HJ PDEs. Unfortunately, FMM produces a correct approximation only under certain causality conditions on the values of nodes and their neighbors. This limitation has motivated the development of a more generally applicable version of FMM called ordered upwind methods (OUMs) [21] and also several recent works such as [31, 13, 19] on sweeping methods. However, OUMs are much more complex to implement than FMM, and sweeping methods can be much less efficient for problems with curved characteristics and practical grid sizes [12, 11].

Consequently, we have motivation to seek classes of anisotropic problems to which FMM might still be applied. One such class of problems was identified in [20] and includes the Eikonal equation where an energy norm replaces the standard Euclidean norm. In [3] we identified another such class of problems. Because its characteristics are minimum time paths to the boundary, the Eikonal equation has often been

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proposed for robotic path planning; for example, see [15]. However, for some robots, using the Euclidean norm in this equation is inappropriate. Consider a robot arm, where each joint has its own motor. If each motor can rotate at some maximum speed independent of the action of the other motors, then the action of the whole arm is best bounded in an approprately-scaled infinity norm. The corresponding Eikonal equation should use the dual Manhattan norm and is thus anisotropic. Other scenarios where such problems arise were considered in [3]—such as planning collision-free optimal paths for multiple robots—and experimental evidence suggested that FMM would be successful on these problems.

As a group, the anisotropy in these problems is axis-aligned. In this paper we describe a broader class of such axis-aligned problems (section 2) and demonstrate that FMM can be applied to approximate their solution on axis-aligned orthogonal grids without modification of the algorithm beyond the local update function for a single node (section 3). The examples (section 4) include an anelliptic wave propagation problem and a new multirobot scenario. In Appendix A, we propose some methods by which the local update's efficiency might be improved even if the grid and/or PDE lack symmetry. Lastly, in Appendix B, we provide analytic update formulas for the Eikonal equation with the $p = 1$, 2, and ∞ norms on variably spaced orthogonal grids in any dimension.

Some proofs of theorems and experimental details have been omitted from this paper and may be found in [2].

1.1. The problem. The Dirichlet problem of a static HJ PDE is to find a function u such that

$$H(x, Du(x)) = 0, \qquad x \in \Omega, \tag{1.1a}$$

$$u(x) = g(x), \quad x \in \partial\Omega, \tag{1.1b}$$

where $Du(x)$ is the gradient of u at x, $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, and $\partial\Omega$ is the domain's boundary. In general, it is not possible to find a classical solution to the Dirichlet problem (1.1) where u is differentiable for all x, so we seek instead the viscosity solution [7], a unique weak solution which is continuous and almost everywhere differentiable.

To appreciate the difference between isotropic and anisotropic problems, it is useful to consider a control-theoretic formulation of the Hamiltonian

$$H(x, q) = \max_{a \in \mathcal{A}(x)} (-q \cdot a) - 1, \tag{1.2}$$

where a is an action and $\mathcal{A}(x) \subset \mathbb{R}^d$ is a compact, convex action set containing the origin in its interior. In an isotropic problem $\mathcal{A}(x)$ is a hypersphere centered on the origin for all x, although its radius may depend on x. In such a problem (1.2) reduces to

$$H(x, q) = \|q\|_2 - c(x), \tag{1.3}$$

where $c(x) = 1/r(x)$ and $r(x)$ is the radius of the hyperspherical $\mathcal{A}(x)$. In this case (1.1a) becomes the Eikonal equation. For an anisotropic problem, $\mathcal{A}(x)$ is not always an origin-centered hypersphere. Since not all Hamiltonians H fit the control-theoretic formulation, more generally, for an isotropic problem, the set of q solving $H(x, q) = 0$ is the surface of an origin-centered hypersphere. Several examples of anisotropic problems that do not fit this criterion are included in sections 2.2 and 4.

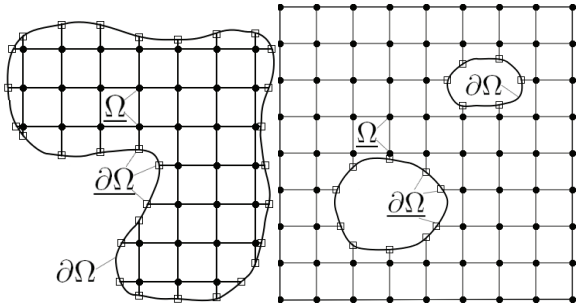


FIG. 1.1. *Orthogonal grids combining discretizations* $\underline{\Omega}$ *and* $\underline{\partial\Omega}$. (a) *boundary conditions are given around the outside of* Ω. (b) *boundary conditions are given on the inside of* Ω.

1.2. The FMM. Since we typically cannot solve for the viscosity solution exactly, we compute an approximate solution $\underline{u}$ on an axis-aligned orthogonal grid with nodes forming both a discretization $\underline{\Omega}$ of Ω, and a discretization $\underline{\partial\Omega}$ of $\partial\Omega$; for example, see Figure 1.1. We take $\underline{\Omega}$ and $\underline{\partial\Omega}$ to be disjoint sets. We allow any axis-aligned orthogonal grid, including those with node spacing that varies between dimensions and within a single dimension; the latter capability makes it easier to more accurately manage an irregular boundary [11]. It is important that the orthogonal grid and the Hamiltonian H are aligned to the same axis. What it means for H to be aligned to an axis is explained in section 2.

Let $\mathcal{N}(\underline{x})$ be the set of neighbors of node $\underline{x} \in \underline{\Omega}$. Whenever we refer to a simplex of a node $\underline{x}$, we mean a simplex specified by the node $\underline{x}$ and d of its neighbors, each in a distinct dimension. Since we are restricted to orthogonal grids, each simplex of $\underline{x}$ corresponds to a particular orthant.

Informally, we refer to $\underline{u}(\underline{x})$ as the value of node $\underline{x}$. In what follows, we may use $\underline{u}$ to refer to either the values of the nodes in the operation or the output of FMM or to the solution of the discretized PDE (3.3). This ambiguity becomes less bothersome when we point out in Proposition 3.2 that the output of FMM is, in fact, the solution to (3.3).

Algorithm 1 outlines a simple dynamic programming algorithm. The algorithm can become either Dijkstra's algorithm or FMM depending on the choice of the `Update` function. Consider, for example, the `Update` function in the context of optimal control, where we are computing the minimal cost over all possible paths. For Dijkstra's algorithm, `Update` computes $\underline{u}(\underline{x}_0)$ as a simple minimization over the neighboring nodes of $\underline{x}_0$ of the path costs to $\underline{x}_0$ via each neighbor. For FMM, the `Update` function computes $\underline{u}(\underline{x}_0)$ as a minimization over the neighboring simplices of $\underline{x}_0$ of the minimum path costs to $\underline{x}_0$ via each simplex.

The `Update` function must satisfy a *causality* property in order for Algorithm 1 to terminate with a correct solution: `Update` must compute a node value $\underline{u}(\underline{x})$ based only on information from neighboring nodes with smaller values, so that $\underline{u}$ is computed in increasing order of $\underline{u}(\underline{x})$ [28, 24]. In Dijkstra's algorithm and FMM for a standard Euclidean norm Eikonal equation on an orthogonal grid, this property is automatic. A major contribution of this paper is to demonstrate that, for a class of static HJ PDEs with axis-aligned anisotropy, an `Update` function that is consistent with the PDE and satisfies the causality property can be defined, and thus FMM can be used.

While the `Update` function in Algorithm 1 is determined by the underlying equation which we seek to solve, it is assumed that its execution time is independent of

```
1 foreach x ∈ Ω do u(x) ← ∞
2 foreach x ∈ ∂Ω do u(x) ← g(x)
3 Q ← Ω ∪ ∂Ω
4 while Q ≠ ∅ do
5     y ← ExtractMin(Q)
6     foreach x_0 ∈ (N(y) ∩ Q) \ ∂Ω do  u(x_0) ← Update(x_0, u)
7 end
```

Algorithm 1: Dynamic Programming Algorithm.

grid resolution, and hence it does not affect the algorithm's asymptotic complexity. The `Update` functions in this paper maintain this property. FMM is usually described as being $\mathcal{O}(n \log n)$, where $n = |\underline{\Omega}|$ is the number of grid points in the discretized domain. This complexity is derived by noting that each node is removed from $\mathcal{Q}$ once by `ExtractMin` and, in the usual binary heap implementation of $\mathcal{Q}$, extraction of the minimum value node costs $\mathcal{O}(\log |\mathcal{Q}|) \leq \mathcal{O}(\log n)$. Note that the heap need only sort nodes with finite values. Because we restrict our modifications of Algorithm 1 to the `Update` function, all of the results here can be used with other versions of FMM; for example, the $\mathcal{O}(n)$ algorithm described in [30], which uses an untidy priority queue for $\mathcal{Q}$ to reduce the cost of `ExtractMin` and hence the whole algorithm. However, for implementation simplicity, we have used the standard binary heap version of $\mathcal{Q}$ in our experiments.

1.3. Related work. The first Dijkstra-like method for a first-order semi-Lagrangian discretization of the isotropic Eikonal PDE on an orthogonal grid was developed in [28]. The Dijkstra-like FMM was later independently developed in [23] for the first-order upwind Eulerian finite-difference discretization of the same Eikonal PDE. FMM was then extended to handle higher-order upwind discretizations on grids and unstructured meshes in $\mathbb{R}^n$ and on manifolds [14, 25, 20]. In [24] it was shown that Dijkstra's method on a uniform orthogonal grid produces the solution for the anisotropic maximum norm Eikonal equation. By solving an isotropic problem on a manifold and then projecting the solution into a subspace, FMM can solve certain anisotropic problems [20]; for example, (1.2) with a constant elliptic $\mathcal{A}(x) = \mathcal{A}$ can be solved by running isotropic FMM on an appropriately tilted planar manifold and then projecting away one dimension. Some anisotropic etching problems have also been solved using FMM [17].

The fact that correct operation of Dijkstra-like algorithms for approximating the Eikonal PDE requires the causality property that $\underline{u}(\underline{x})$ can be written only in terms of smaller values $\underline{u}$ at neighboring nodes was stated in [28], but a reader might incorrectly infer from further comments in that paper that such algorithms would not work for any unstructured grid or anisotropic problem. That FMM is applicable for any consistent, orthogonal, causality satisfying, finite-difference discretization of a general static convex HJ PDE is stated in [24]; however, it is now understood that this criterion applies even more generally, since a Dijkstra-like method can be used to efficiently solve on a graph any nonlinear system of equations for which $\underline{u}(\underline{x})$ is dependent only on smaller values $\underline{u}$ at neighboring nodes. A sufficient criterion (see section 2.1) under which FMM can be used for orthogonal, finite-difference discretizations of static HJ PDEs—now commonly referred to as "Osher's criterion"—is widely attributed to an unpublished work by Osher and Helmsen, but the earliest published description seems

to be [17]. While it is stronger than the causality conditions described earlier, it is useful because it is stated as a condition on the analytic Hamiltonian instead of the equations created by the discretization. In this paper we likewise seek conditions under which FMM is applicable that are closer to the problem's definition than the algorithm's implementation.

OUMs [21, 22] can solve general convex anisotropic problems on unstructured grids with an asymptotic complexity only a constant factor (related to the degree of anisotropy) worse than FMM. FMM fails for these general problems because the neighboring simplex from which the characteristic approaches a node $\underline{x}_0$ may contain another node $\underline{x}$ such that causality does not hold: $\underline{u}(\underline{x}_0) < \underline{u}(\underline{x})$. OUM avoids this difficulty by searching along the active front to find a set of neighboring nodes (which may not be direct neighbors of $\underline{x}_0$) whose values have been accepted, and then constructing a virtual simplex with these nodes from which to update $\underline{u}(\underline{x}_0)$. Although this search along the active front does not degrade the asymptotic complexity, it does significantly increase the computational cost in practice. This effect can be partially mitigated by using nontrivial data structures such as 2^d-trees to speed up the search.

An alternative to these single-pass (or label-setting) algorithms are the sweeping (or label-correcting) algorithms, which are often even simpler to implement than FMM. Sweeping algorithms are also capable of handling anisotropic and even nonconvex problems. The simplest sweeping algorithm is to just iterate through the grid updating each node in a Gauss–Seidel (GS) fashion (so a new value for a node is used immediately in subsequent updates) until $\underline{u}$ converges. GS converges quickly if the node update order is aligned with the characteristics of the solution, so better sweeping algorithms [8, 6, 31, 13, 19] alternate among a collection of static node orderings so that all possible characteristic directions will align with at least one ordering. It is argued in [31] that these methods achieve $\mathcal{O}(n)$ asymptotic complexity (assuming that the node orderings are already determined); however, unlike FMM and OUM, the constant depends on the problem. For practical grid resolutions on problems with curved characteristics, FMM does better despite the difference in asymptotic complexity [12, 11].

There are also a number of sweeping algorithms which use dynamic node orderings; for example [18, 5]. These algorithms attempt to approximate the optimal ordering generated by single-pass methods such as FMM without the overhead associated with managing an accurate queue. These methods have been demonstrated to be comparable to or better than single-pass methods for certain problems and grid resolutions [18, 5]. However, in general, these methods may need to revisit nodes multiple times.

Accurate robotic path planning is only required in cluttered environments where optimal paths—and hence the characteristics of the HJ PDE—are not straight. No alternative algorithm proposed approaches the simple implementation and guaranteed speed of FMM for these types of problems. Consequently, we set out in this paper to characterize another class of anisotropic HJ PDEs for which FMM will work and also to explore their efficient implementation. It should be noted that the update procedures discussed in this paper can be applied to any of the sweeping algorithms without modification.

2. Class of Hamiltonians. FMM can be extended to handle a class of axis-aligned anisotropic problems, defined by a restriction of the Hamiltonian H to that satisfying Properties 1 to 4. We let $q, \tilde{q} \in \mathbb{R}^d$ and make the following definitions.

DEFINITION 2.1. *Write* $q \trianglerighteq \tilde{q}$ *if* $q_j \tilde{q}_j \geq 0$ *and* $|q_j| \geq |\tilde{q}_j|$, *for* $1 \leq j \leq d$.

DEFINITION 2.2. *Write* $q \rhd \tilde{q}$ *if* (i) $q \neq 0$ *and* (ii) $q_j \tilde{q}_j \geq 0$ *and* $|q_j| > |\tilde{q}_j|$ *or* $q_j = \tilde{q}_j = 0$, *for* $1 \leq j \leq d$.

The following properties are satisfied by H.

PROPERTY 1. *H is continuous:* $H \in C(\Omega \times \mathbb{R}^d)$.

PROPERTY 2. *H is coercive:* $H(x, q) \to \infty$ *as* $\|q\| \to \infty$ *for all* $x \in \Omega$.

PROPERTY 3. *H is strictly compatible:* $H(x, 0) < 0$ *for all* $x \in \Omega$.

PROPERTY 4. *H is strictly one-sided monotone: If* $q \rhd \tilde{q}$, *then* $H(x, q) > H(x, \tilde{q})$.

We note that Properties 1, 2, and 3 are similar to some properties on the Hamiltonian in [5]. In this paper, we typically deal only with the `Update` function. For this reason, we usually consider a fixed $x \in \Omega$ and may write $H(q) = H(x, q)$ wherever no ambiguity results. When discussing properties of H, these are in reference to the q parameter. The source of the *axis-aligned* description of the problem class is the strict one-sided monotonicity property of H.

2.1. Connection to Osher's criterion. Although there are earlier statements of the conditions on node values under which a Dijkstra-like algorithm can or cannot be used to solve the problem [28, 23], in this section we outline the connection between the properties described above and Osher's criterion [17] because the latter directly provides a condition on the Hamiltonian rather than on the solution values. In section 3.3, we make the connection between Properties 1 to 4 and the earlier conditions.

Osher's fast marching criterion is defined in [17, 27] as

$$q_j \frac{\partial H(x, q)}{\partial q_j} \geq 0$$

for $1 \leq j \leq d$. The authors state there that as long as this criterion is satisfied, a simple fast marching algorithm based on a one-sided upwind finite-difference discretization can be applied to solve the problem. However, we use Properties 1 to 4 instead of Osher's criterion because Osher's criterion requires H to be differentiable so that $D_q H(x, q)$ exists, but we are interested in potentially nondifferentiable H (e.g., see section 2.2). Note that strict one-sided monotonicity is applicable even when $D_q H(x, q)$ does not exist for all x.

Propositions 2.3, 2.4, and 2.5 explain the relationship between strict one-sided monotonicity of H (Property 4) and Osher's criterion. Proposition 2.3 shows that Property 4 implies one-sided monotonicity (Property 5). Then, Proposition 2.4 shows that Property 5 is the same as Osher's criterion as long as H is differentiable. Finally, Proposition 2.5 demonstrates that Property 5 with the addition of one-sided homogeneity (Property 6) implies Property 4.

PROPERTY 5. *H is one-sided monotone: If* $q \trianglerighteq \tilde{q}$, *then* $H(x, q) \geq H(x, \tilde{q})$.

PROPOSITION 2.3. *Let H be continuous (Property 1). Then strict one-sided monotonicity of H (Property 4) implies one-sided monotonicity of H (Property 5).*

Proof. Let H be strictly one-sided monotone. Let $q, \tilde{q} \in \mathbb{R}^d$ be such that $q \trianglerighteq \tilde{q}$. Let $r \in \{-1, 1\}^d$ be such that

$$r_j = \begin{cases} +1, & \text{if } q_j \geq 0, \\ -1, & \text{otherwise,} \end{cases}$$

and let $\epsilon > 0$. Note that $q + \epsilon r \rhd \tilde{q}$ and thus we have $H(q + \epsilon r) > H(\tilde{q})$. By the continuity of H, we have

$$\lim_{\epsilon \to 0^+} H(q + \epsilon r) \geq H(\tilde{q})$$

and also

$$\lim_{\epsilon \to 0^+} H(q + \epsilon r) = H(q).$$

Therefore, $H(q) \geq H(\tilde{q})$. ∎

PROPOSITION 2.4. *Let H be continuous (Property 1), and let $D_q H(q)$ exist for all $q \in \mathbb{R}^d$. Then the following conditions on H are equivalent:*

(a) $q_j \frac{\partial H(q)}{\partial q_j} \geq 0$ *for all j and $q \in \mathbb{R}^d$.*

(b) *H is one-sided monotone (Property 5).*

Proof. We begin by proving that (a) implies (b). Let $q, \tilde{q} \in \mathbb{R}^d$ be such that $q \trianglerighteq \tilde{q}$. If $q = \tilde{q}$, then $H(q) = H(\tilde{q})$. Otherwise, define the function $\bar{q} : [0,1] \to \mathbb{R}^d$ such that $\bar{q}(t) = \tilde{q} + t(q - \tilde{q})$ to represent the line segment between $\tilde{q}$ and q parameterized by $t \in [0,1]$. Because $q \trianglerighteq \tilde{q}$ we have

$$(q - \tilde{q})_j \bar{q}_j(t) \geq 0$$

for $1 \leq j \leq d$ and for $t \in [0,1]$. Thus, by condition (a), we have

$$(q - \tilde{q})_j \frac{\partial H(\bar{q}_j(t))}{\partial \bar{q}_j(t)} \geq 0 \tag{2.1}$$

for $1 \leq j \leq d$ and for $t \in [0,1]$. We know that

$$\begin{aligned} H(q) &= H(\tilde{q}) + \int_0^1 \frac{d\bar{q}(t)}{dt} \cdot D_q H(\bar{q}(t)) dt \\ &= H(\tilde{q}) + \int_0^1 (q - \tilde{q}) \cdot D_q H(\bar{q}(t)) dt \\ &= H(\tilde{q}) + \int_0^1 \sum_{i=1}^{n} (q - \tilde{q})_j \frac{\partial H(\bar{q}_j(t))}{\partial \bar{q}_j(t)} dt \\ &\geq H(\tilde{q}). \end{aligned}$$

The first equality follows from integrating the change in H along the line segment connecting $\tilde{q}$ and q. The second equality is because the derivative $\frac{d\bar{q}(t)}{dt}$ is simply the vector $q - \tilde{q}$. The third equality breaks up the vector dot product into a sum of scalar products. The inequality results from (2.1) and the fact that an integral of a nonnegative function is nonnegative. Thus, for all $q, \tilde{q}$ such that $q \trianglerighteq \tilde{q}$, including $q = \tilde{q}$, we have $H(q) \geq H(\tilde{q})$.

We now prove that (b) implies (a). Let $q \in \mathbb{R}^d$ and $1 \leq j \leq d$. Define the function $s : \mathbb{R} \to \{-1, +1\}$ such that

$$s(y) = \begin{cases} +1, & \text{if } y \geq 0, \\ -1, & \text{otherwise,} \end{cases}$$

let $\epsilon > 0$, and let e_j be the jth vector in the standard basis. Note that $q + \epsilon s(q_j) e_j \trianglerighteq q$ and thus by (b) we have $H(q + \epsilon s(q_j) e_j) - H(q) \geq 0$. Consequently, by the existence of $D_q H(q)$ for all $q \in \mathbb{R}^d$, we have

$$q_j \frac{\partial H(q)}{\partial q_j} = \lim_{\epsilon \to 0^+} q_j \frac{H(q + \epsilon s(q_j) e_j) - H(q)}{\epsilon s(q_j)} \geq 0. \qquad \blacksquare$$

The following property is used to state Proposition 2.5.

PROPERTY 6. *H is one-sided homogeneous: $H(tq) - H(0) = t(H(q) - H(0))$ for all $t \geq 0$ and $q \in \mathbb{R}^d$.*

PROPOSITION 2.5. *Let H satisfy Properties* 1, 2, *and* 3, *and let H be one-sided monotone (Property* 5*) and one-sided homogeneous (Property* 6*). Then H is strictly one-sided monotone (Property* 4*).*

Proof. Let $q \rhd \tilde{q}$. Then $q \trianglerighteq \tilde{q}$ and $H(q) \geq H(\tilde{q})$ by one-sided monotonicity.

First consider the case $\tilde{q} = 0$. Assume $H(q) = H(\tilde{q}) = H(0)$. By the one-sided homogeneity of H,

$$\lim_{t\to\infty} [H(tq) - H(0)] = \lim_{t\to\infty} [t(H(q) - H(0))] = 0.$$

But by the coercivity of H,

$$\lim_{t\to\infty} [H(tq) - H(0)] = \infty,$$

since $\lim_{t\to\infty} \|tq\| = \infty$ and by compatability $H(0) < 0$. Thus, we have a contradiction, and it must be that $H(q) > H(\tilde{q})$.

Second, consider the case where $\tilde{q} \neq 0$. Let $\mathcal{J} = \{j \mid |q_j| > |\tilde{q}_j|\}$. Note that by Definition 2.2 since $\tilde{q} \neq 0$, we have $\mathcal{J} \neq \emptyset$ and there exist $j \in \mathcal{J}$ such that $\tilde{q}_j \neq 0$. Define a scalar multiple of q:

$$\check{q} = tq = \left(\max_{j\in\mathcal{J}} \frac{|\tilde{q}_j|}{|q_j|} \right) q.$$

Since $|q_j| > |\tilde{q}_j|$, for all $j \in \mathcal{J}$, we have $0 < t < 1$. Furthermore, for $j \in \mathcal{J}$,

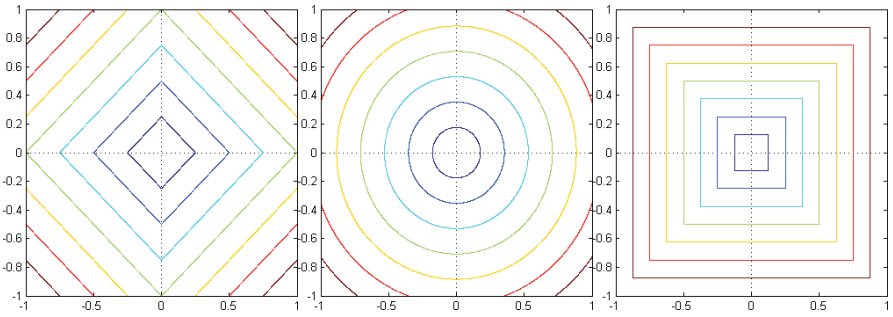
$$|\check{q}_j| = \left(\max_{j\in\mathcal{J}} \frac{|\tilde{q}_j|}{|q_j|} \right) |q_j| \geq |\tilde{q}_j|,$$

while for $j \notin \mathcal{J}$,

$$\check{q}_j = tq_j = 0 = \tilde{q}_j.$$

Consequently, $|\check{q}_j| \geq |\tilde{q}_j|$ for $1 \leq j \leq d$. Also, since $t > 0$, we have $\check{q}_j \tilde{q}_j = tq_j \tilde{q}_j \geq 0$ for $1 \leq j \leq d$. This implies, by one-sided monotonicity of H, that $H(\check{q}) \geq H(\tilde{q})$. Moreover, by one-sided homogeneity of H, $H(\check{q}) - H(0) = H(tq) - H(0) = t(H(q) - H(0))$. It follows that $H(q) - H(0) = (H(\check{q}) - H(0))/t > H(\check{q}) - H(0)$, since $0 < t < 1$ and $H(\check{q}) \geq H(0)$ by one-sided monotonicity. Therefore, $H(q) > H(\check{q}) \geq H(\tilde{q})$. ∎

We impose strict one-sided monotonicity on H because it guarantees a unique solution to a first-order upwind finite-difference discretization of (1.1a), as shown in section 3.1. Simply imposing one-sided monotonicity on H or Osher's condition on differentiable H is not sufficient for a unique solution. However, Proposition 2.5 states that when H satisfies one-sided homogeneity in addition to one-sided monotonicity, then it also satisfies strict one-sided monotonicity, and there is a unique solution to the discretization. Moreover, by Propositions 2.4 and 2.5, when differentiable H satisfies one-sided homogeneity in addition to Osher's criterion, then H also satisfies strict one-sided monotonicity, and there is a unique solution to the discretization. Note that there exist conditions other than one-sided homogeneity, such as strict convexity, that in combination with Osher's criterion, result in strict one-sided monotonicity of H.



(a) $p = 1$. (b) $p = 2$. (c) $p = \infty$.

FIG. 2.1. *Contour plots of* $\|q\|_p$.

2.2. Example H functions. A Hamiltonian H that satisfies Properties 1 to 4 encompasses a fairly broad range of anisotropic problems. We consider examples of H that satisfy Properties 1 to 4. In particular, we look at the case

$$H(x, q) = G(x, q) - c(x), \tag{2.2}$$

where G is a p-norm or some variant and c is a positive cost. We must ensure that G is strictly one-sided monotone, which is not true of all norms.

The p-norm is a useful category of strictly one-sided monotone norms. Let a p-norm, $\|\cdot\|_p$, be defined by

$$\|q\|_p = \left(\sum_{j=1}^{d} |q_j|^p \right)^{1/p},$$

where $p \geq 1$. Commonly used p-norms, illustrated in Figure 2.1, are the Manhattan norm ($p = 1$), the Euclidean norm ($p = 2$), and the maximum norm ($p = \infty$). The following proposition is proved in [2].

PROPOSITION 2.6. *$\|\cdot\|_p$ is strictly one-sided monotone.*

Define a linearly-transformed p-norm $\|\cdot\|_{B,p}$ to be

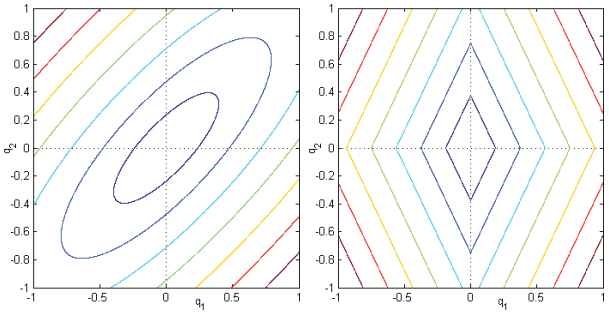
$$\|q\|_{B,p} = \|Bq\|_p,$$

where $p \geq 1$ and B is a nonsingular $d \times d$ matrix. Note that B must be nonsingular so that $\|\cdot\|_{B,p}$ satisfies the properties of a norm such as definiteness and homogeneity. Such a norm is not strictly one-sided monotone in general. Figure 2.2(a) shows a simple example where a vector is rotated by $-\pi/4$ and scaled by 3 in the q_2-axis before the Euclidean norm is taken; i.e.,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -3/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}. \tag{2.3}$$

Let $q = (2, 2)^T$ and $\tilde{q} = (\sqrt{2}, 0)^T$. We have $q \trianglerighteq \tilde{q}$, but

$$\|Bq\|_2 = \left\| \left(2\sqrt{2}, 0\right)^T \right\|_2 = \sqrt{8} < \sqrt{10} = \|(1, -3)^T\|_2 = \|B\tilde{q}\|_2.$$



(a) (b)

FIG. 2.2. *Contour plots of* $\|Bq\|_p$. (a) *is not strictly one-sided monotone:* $p = 2$ *and* B *is defined by* (2.3). (b) *is strictly one-sided monotone:* $p = 1$ *and* B *scales by* 2 *in the* q_1*-axis.*

Consequently, this particular linearly transformed p-norm is not strictly one-sided monotone. However, in this case an inverse transformation B^{-1} of the grid coordinates will result in a strictly one-sided monotone p-norm, while maintaining the grid's orthogonality. More generally, we conjecture that if the Hamiltonian is of the form $H(q) = \tilde{H}(Bq)$, where B is a rotation (which may be followed by scaling) and $\tilde{H}$ satifies Properties 1 to 4, a transformation of the grid coordinates by B^{-1} will result in a transformed H that also satifies Properties 1 to 4, while maintaining the grid's orthogonality. More complex coordinate modifications might be possible, but we have not yet adequately investigated conditions or procedures.

A scaled p-norm (Figure 2.2(b)) is a special case of a linearly transformed p-norm. Such a norm scales the components of its argument before applying a p-norm by restricting B to be a nonsingular diagonal matrix. It is simple to show that a scaled p-norm is strictly one-sided monotone, considering Proposition 2.6.

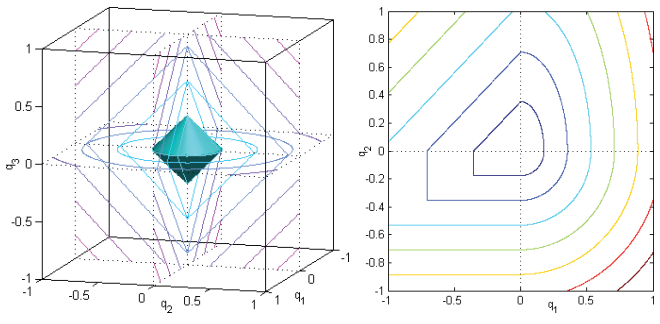
A mixed p-norm is a recursive composition of p-norms, and it is strictly one-sided monotone. The following is an example (Figure 2.3(a)) of a mixed p-norm that takes the Euclidean norm of the first two components and then takes the Manhattan norm of the result and the last component:

$$\begin{aligned} \|q\| &= \| \left(\| \left(q_1, q_2 \right) \|_2, q_3 \right) \|_1 \\ &= \sqrt{(q_1)^2 + (q_2)^2} + |q_3|, \end{aligned} \tag{2.4}$$

where $q = (q_1, q_2, q_3)$. This particular norm was used as a G function in [3] for a simple two-robot coordinated optimal control problem.

Finally, the one-sidedness of Property 4 allows G to be asymmetric, which is not permitted for a norm. An example of such an asymmetric norm-like function is shown in Figure 2.3(b) and is given by

$$G(q) = \begin{cases} \|B_a q\|_\infty, & \text{if } q_1 \le 0 \text{ and } q_2 \le 0, \\ \|B_b q\|_1, & \text{if } q_1 \le 0 \text{ and } q_2 > 0, \\ \|B_c q\|_2, & \text{if } q_1 > 0 \text{ and } q_2 \le 0, \\ \|B_d q\|_2, & \text{if } q_1 > 0 \text{ and } q_2 > 0, \end{cases} \tag{2.5}$$



(a) (b)

FIG. 2.3. *Contour plots of $G(q)$.* (a) *mixed p-norm: G is defined by* (2.4). (b) *asymmetric norm-like function: G is defined by* (2.5).

where

$$B_a = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad B_b = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B_d = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

We solve the anisotropic problem characterized by (2.5) as well as an anelliptic wave propagation problem and a multirobot optimal path planning problem in section 4. Other examples of G functions which satisfy strict one-sided monotonicity are some polygonal norms such as axis aligned hexagonal or octagonal norms; however, we do not further investigate these options here.

3. FMM and the discretized problem. We define a discretized analogue of the Dirichlet problem (1.1). By describing the `Update` function in Algorithm 1, we also formalize the FMM algorithm. Finally, we examine important properties of the `Update` function.

We recall that the nodes in $\underline{\Omega}$ lie on an axis-aligned orthogonal grid. Let $\underline{x}_0 \in \underline{\Omega}$. The neighborhood of $\underline{x}_0$ is shown in Figure 3.1. Let $\underline{x}_j^{\pm}$ be the neighbors of $\underline{x}_0$ in the $\pm e_j$ directions, e_j being the jth vector in the standard basis. The set of neighbors is

$$\mathcal{N}(\underline{x}_0) = \left\{\underline{x}_1^{\pm}, \underline{x}_2^{\pm}, \ldots, \underline{x}_d^{\pm}\right\},$$

and the neighborhood vector is

$$N(\underline{x}_0) = \left(\underline{x}_0, \underline{x}_1^{\pm}, \underline{x}_2^{\pm}, \ldots, \underline{x}_d^{\pm}\right).$$

Let $h_j^{\pm} = \pm\|\underline{x}_0 - \underline{x}_j^{\pm}\|$ be signed distances to the neighbors in the $\pm e_j$ directions. Let

$$\mathcal{S} = \{(s_1, s_2, \ldots, s_d) \mid s_j \in \{-1, +1\}, 1 \le j \le d\}$$

such that $s \in \mathcal{S}$ represents one of the 2^d neighboring simplices of $\underline{x}_0$. Note that we abuse notation by using $s_j \in \{-1, +1\}$ as a superscript indexing $\underline{x}_j^{\pm}$ or $h_j^{\pm}$.

Let $B(\Omega)$ be the set of bounded functions on domain Ω. We define the numerical Hamiltonian $\underline{H} : \Omega^{1+2d} \times B(\Omega) \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\underline{H}(N, \phi, \mu) = \max_{s \in \mathcal{S}} [H(\underline{x}_0, \underline{D}^s(N, \phi, \mu))], \tag{3.1}$$

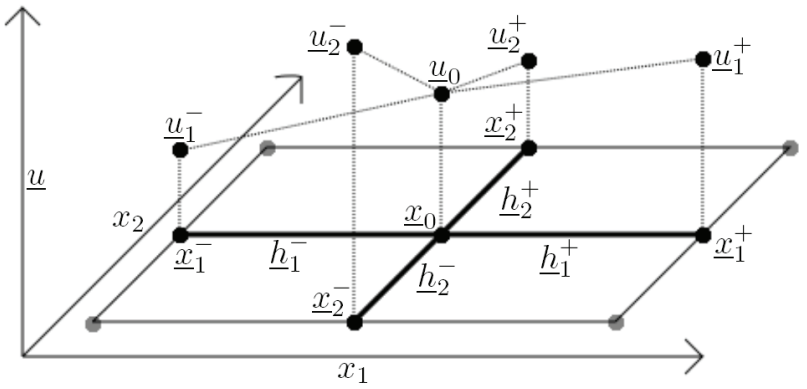


FIG. 3.1. *Neighborhood of* $\underline{x}_0$ *with* $d = 2$.

where H is as defined in section 2 and

$$\underline{D}^s(N, \phi, \mu) = (\underline{D}_1^s(N, \phi, \mu), \underline{D}_2^s(N, \phi, \mu), \ldots, \underline{D}_d^s(N, \phi, \mu))$$

is a first-order, upwind, finite-difference gradient approximation from the simplex represented by s; that is,

$$\underline{D}_j^s(N, \phi, \mu) = \frac{\max(0, \mu - \phi(\underline{x}_j^{s_j}))}{-h_j^{s_j}} \tag{3.2}$$

for $1 \le j \le d$. Although $\underline{H}$ is defined on domain $\Omega^{1+2d} \times B(\Omega) \times \mathbb{R}$, for FMM it will only be used on domain $\underline{\Omega}^{1+2d} \times B(\underline{\Omega}) \times \mathbb{R}$. The broader definition of domain is important for consistency [4]. The restriction of Ω^{1+2d} to $\underline{\Omega}^{1+2d}$ poses no problems to the definition of $\underline{H}$. Furthermore, to evaluate $\underline{H}$, ϕ need only be defined on $\mathcal{N}$, which is true of any function in $B(\underline{\Omega})$.

The discretized Dirichlet problem is to find a function $\underline{u} : (\underline{\Omega} \cup \underline{\partial\Omega}) \to \mathbb{R}$ such that

$$\underline{H}(N(\underline{x}), \underline{u}, \underline{u}(\underline{x})) = 0, \qquad \underline{x} \in \underline{\Omega}, \tag{3.3a}$$

$$\underline{u}(\underline{x}) = g(\underline{x}), \quad \underline{x} \in \underline{\partial\Omega}. \tag{3.3b}$$

DEFINITION 3.1. *Let FMM be Algorithm* 1 *with the* `Update` *function defined as follows. A call to* `Update`$(\underline{x}_0, \underline{u})$ *returns the solution* $\mu = \tilde{\mu}$ *to*

$$\underline{H}(N(\underline{x}_0), \underline{u}, \mu) = 0. \tag{3.4}$$

In this way it determines a node's value $\underline{u}(\underline{x}_0) \leftarrow \tilde{\mu}$ given the values of its neighbors, $\underline{u}_j^{\pm} = \underline{u}(\underline{x}_j^{\pm})$. When we are varying only μ, it will be convenient to write $\underline{H}(\mu) = \underline{H}(N, \phi, \mu)$ and $\underline{D}^s(\mu) = \underline{D}^s(N, \phi, \mu)$. For the lemmas and theorems stated below, we assume H satisfies Properties 1 to 4.

PROPOSITION 3.2. *Let* $\underline{u} : (\underline{\Omega} \cup \underline{\partial\Omega}) \to \mathbb{R}$ *be the grid function after FMM terminates. Then* $\underline{u}$ *is the unique solution of* (3.3).

This proposition states that the grid function $\underline{u}$ that results from running FMM solves the discretized problem (3.3). We use a method similar to those for isotropic FMM in [29, 23] to prove Proposition 3.2 in [2]. The causality of the `Update` function is essential so that FMM can be used to solve (3.3).

A method for proving the convergence of $\underline{u}$ to the solution of (1.1) as the grid spacing goes to zero is presented in [4]. It is shown there that the consistency, monotonicity, and stability of the numerical scheme are sufficient for convergence. We closely follow the technique described in [4] to prove convergence in [2]. Also, uniqueness and monotonicity of the solution to (3.4) are useful for using numerical root finders to implement `Update`. We include proofs of uniqueness, monotonicity, and causality of the `Update` function below. For more details regarding convergence, including the proofs of consistency and stability, see [2].

3.1. Unique update. Let the minimum value of all neighbors of $\underline{x}_0$ be

$$\breve{\underline{u}} = \min_{\underline{x} \in \mathcal{N}(\underline{x}_0)} (\underline{u}(\underline{x})) \,. \tag{3.5}$$

We show there is a unique solution $\mu = \tilde{\mu}$ to (3.4) such that $\tilde{\mu} > \breve{\underline{u}}$. First, we prove two useful lemmas.

LEMMA 3.3. *$\underline{H}(\mu)$ is strictly increasing on $\mu \geq \breve{\underline{u}}$.*

Proof. Let $\mu_a > \mu_b \geq \breve{\underline{u}}$. Let $s \in \mathcal{S}$ and $1 \leq j \leq d$. If $\mu_a > \underline{u}_j^{s_j}$, then $\underline{D}_j^s(\mu_a)\underline{D}_j^s(\mu_b) \geq 0$ and $|\underline{D}_j^s(\mu_a)| > |\underline{D}_j^s(\mu_b)|$. On the other hand, if $\mu_a \leq \underline{u}_j^{s_j}$, then $\underline{D}_j^s(\mu_a) = \underline{D}_j^s(\mu_b) = 0$. Also, there exists at least one $s \in \mathcal{S}$ and $1 \leq j \leq d$ such that $\underline{D}_j^s(\mu_a) \neq 0$, since $\mu_a > \breve{\underline{u}}$. For such s, $H(\underline{D}^s(\mu_a)) > H(\underline{D}^s(\mu_b))$, by strict one-sided monotonicity (Property 4). For all other s, $H(\underline{D}^s(\mu_a)) = H(\underline{D}^s(\mu_b)) = H(0)$. Therefore, by (3.1) $\underline{H}(\mu_a) > \underline{H}(\mu_b)$, so $\underline{H}(\mu)$ is strictly increasing on $\mu \geq \breve{\underline{u}}$. ∎

LEMMA 3.4. *The numerical Hamiltonian $\underline{H}(\mu)$ satisfies the following:*

(a) *$\underline{H}(\mu) = H(0) < 0$ for $\mu \leq \breve{\underline{u}}$.*

(b) *$\underline{H}(\mu) \to \infty$ as $\mu \to \infty$.*

(c) *$\underline{H}(\mu)$ is nondecreasing on all μ.*

Proof. If $\mu \leq \breve{\underline{u}}$, then by (3.2) and (3.5), we have $\underline{D}_j^s(\mu) = 0$ for all $s \in \mathcal{S}$, $1 \leq j \leq d$. By the strict compatibility of H, $H(\underline{D}^s(v_j)) = H(0) < 0$ for all s. By (3.1), we have $\underline{H}(\mu) = H(0) < 0$, for $\mu \leq \breve{\underline{u}}$, proving (a).

Let $s \in \mathcal{S}$ and $1 \leq j \leq d$. As $\mu \to \infty$, we have $\underline{D}_j^s(\mu) \to \infty$ and $\|\underline{D}^s(\mu)\| \to \infty$ for all $s \in \mathcal{S}$, $1 \leq j \leq d$. By the coercivity of H, as $\mu \to \infty$, we have $H(\underline{D}^s(\mu)) \to \infty$ for all $s \in \mathcal{S}$. By (3.1), we have $\underline{H}(\mu) \to \infty$ as $\mu \to \infty$, proving (b).

Because $\underline{H}(\mu)$ is constant on $\mu \leq \breve{\underline{u}}$ and by Lemma 3.3 increasing on $\mu \geq \breve{\underline{u}}$, $\underline{H}(\mu)$ is nondecreasing on all μ, proving (c). ∎

THEOREM 3.5. *There exists a unique solution $\mu = \tilde{\mu}$ to $\underline{H}(\mu) = 0$ such that $\tilde{\mu} > \breve{\underline{u}}$.*

Proof. Each $\underline{D}_j^s(\mu)$ is continuous on μ. Furthermore, by the continuity of H, $H(\underline{D}^s(\mu))$ in continuous on μ for all s. Since max is continuous, $\underline{H}(\mu)$ is continuous. By Lemma 3.4(a/b), $\underline{H}(\mu) < 0$ for $\mu \leq \breve{\underline{u}}$ and $\underline{H}(\mu) \to \infty$ as $\mu \to \infty$. Therefore, by the intermediate value theorem, there exists a solution $\mu = \tilde{\mu}$ to $\underline{H}(\mu) = 0$ such that $\breve{\underline{u}} < \tilde{\mu} < \infty$. Moreover, since $\underline{H}$ is strictly increasing on $\mu \geq \breve{\underline{u}}$ by Lemma 3.3, the solution is unique. ∎

Remark 1. We note that strict one-sided monotonicity (Property 4) of H is used to prove Lemma 3.3, and Lemma 3.3 is then used to show that the solution to $\underline{H}(\mu) = 0$ is unique. We might consider whether or not one-sided monotonicity (Property 5) of H is sufficient for a unique solution. However, Property 5 would not be sufficient to prove Lemma 3.3, and we would find that $\underline{H}(\mu)$ is only nondecreasing on $\mu \geq \breve{\underline{u}}$. A solution to $\underline{H}(\mu) = 0$ would still be guaranteed but not unique in this case. Analogously, for differentiable H, Osher's criterion on H implies a solution that

may not be unique unless H satisfies some additional property, such as one-sided homogeneity (Property 6) or convexity.

3.2. Monotonicity. We show that $\underline{H}$ and the `Update` function are monotone in the neighbor's values. Monotonicity of $\underline{H}$ requires that if none of the neighbor's values decreases, the numerical Hamiltonian $\underline{H}$ should not increase. Additionally, monotonicity of the `Update` function requires that if none of the neighbor's values decreases, the solution to (3.4) should not decrease. Monotonicity is useful both for showing that FMM finds a unique solution and for proving convergence. We note that monotonicity does not require strict one-sided monotonicity of H, but rather one-sided monotonicity of H is sufficient.

THEOREM 3.6. *Let $\underline{v}$ and $\underline{u}$ be grid functions. Let $\underline{v}_j^{\pm} \geq \underline{u}_j^{\pm}$ for $1 \leq j \leq d$. Then for $\mu \in \mathbb{R}$, we have $\underline{H}(N, \underline{v}, \mu) \leq \underline{H}(N, \underline{u}, \mu)$. Furthermore, if $\mu = \mu_v$ is the unique solution to $\underline{H}(N, \underline{v}, \mu) = 0$ and $\mu = \mu_u$ is the unique solution to $\underline{H}(N, \underline{u}, \mu) = 0$, then $\mu_v \geq \mu_u$.*

Proof. Let $\mu \in \mathbb{R}$. We have $\underline{D}^s(N, \underline{u}, \mu) \trianglerighteq \underline{D}^s(N, \underline{v}, \mu)$ for all $s \in \mathcal{S}$. Also, by Proposition 2.3, H satisfies one-sided monotonicity (Property 5). Thus,

$$H(\underline{D}^s(N, \underline{u}, \mu)) \geq H(\underline{D}^s(N, \underline{v}, \mu)) = 0$$

for all $s \in \mathcal{S}$. Consequently, $\underline{H}(N, \underline{u}, \mu) \geq \underline{H}(N, \underline{v}, \mu)$, proving the first claim.

To prove the second claim, we let μ_v and μ_u be as defined above. We note that $\underline{H}(N, \underline{u}, \mu_u) = 0 \geq \underline{H}(N, \underline{v}, \mu_u)$. By Lemma 3.4(c), $\underline{H}(N, \underline{v}, \mu)$ is nondecreasing on all μ, so in order that $\underline{H}(N, \underline{v}, \mu_v) = 0$, it must be that $\mu_v \geq \mu_u$. ∎

3.3. Causality. We note that (3.3) defines a very large system of nonlinear equations, one equation for each node $\underline{x} \in \underline{\Omega}$. FMM can be used to solve this system very efficiently, if the solution $\mu = \tilde{\mu}$ to (3.4) is dependent only on neighbors with smaller values. This property represents a causal relationship between node values. There is an information flow from nodes with smaller values to those with larger values. The causal relationship is meant to mimic that of the PDE (1.1). The solution u of (1.1) is completely defined at x using only values of u from states that are backwards along the characteristic line that passes through x.

FMM exploits the causal property of $\underline{H}$ by computing $\underline{u}(\underline{x})$ in increasing order in a single pass through the nodes. This causal property has been discussed as a requirement for Dijkstra-like single-pass methods in several works [28, 24, 26, 17, 22]. The following theorem states that $\underline{H}$ and the `Update` function are causal. The `Update` function is considered causal if any change to the value of a neighboring node, such that both the new and old values are no smaller than the solution $\mu = \tilde{\mu}$ to $\underline{H}(N, \underline{u}, \mu) = 0$, has no effect on the solution.

THEOREM 3.7. *Let $\underline{v}$ and $\underline{u}$ be grid functions. Let*

$$\tilde{\mathcal{N}}(\underline{x}_0) = \{\underline{x} \in \mathcal{N}(\underline{x}_0) \mid \underline{v}(\underline{x}) \neq \underline{u}(\underline{x})\}.$$

Let

$$\breve{w} = \begin{cases} \min_{\underline{x} \in \tilde{\mathcal{N}}(\underline{x}_0)} \min(\underline{v}(\underline{x}), \underline{u}(\underline{x})), & \text{if } \tilde{\mathcal{N}}(\underline{x}_0) \neq \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $\underline{H}(N, \underline{v}, \mu) = \underline{H}(N, \underline{u}, \mu)$ for $\mu \leq \breve{w}$.

Furthermore, let $\mu = \tilde{\mu}_u$ be the unique solution to $\underline{H}(N, \underline{u}, \mu) = 0$, and let $\mu = \tilde{\mu}_v$ be the unique solution to $\underline{H}(N, \underline{v}, \mu) = 0$. If $\tilde{\mu}_u \leq \breve{w}$ or $\tilde{\mu}_v \leq \breve{w}$, then $\tilde{\mu}_u = \tilde{\mu}_v$.

TABLE 4.1

Errors of approximate solution computed by FMM compared to exact solution of (1.1)*, where* H *is as in* (2.2) *and* $G(Du(x)) = \|Du(x)\|_p$*. The variables* d*,* m*, and* n *are the dimension, the number of nodes in each dimension, and the total number of nodes, respectively. Other variables are the spacing* h *between grid nodes, the* $\mathcal{L}_\infty$*-error* e_∞*, the* $\mathcal{L}_\infty$ *convergence rate* r_∞*, the* $\mathcal{L}_1$*-error* e_1*, and the* $\mathcal{L}_1$ *convergence rate* r_1*.*

				$p=1$				$p=2$			
d	m	n	h	e_∞	r_∞	e_1	r_1	e_∞	r_∞	e_1	r_1
2	11	1.2e2	2.0e-1	2.2e-1		6.3e-2		1.2e-1		6.2e-2	
	21	4.4e2	1.0e-1	1.7e-1	.41	3.7e-2	.77	7.8e-2	.56	4.3e-2	.53
	41	1.7e3	5.0e-2	1.2e-1	.46	2.0e-2	.85	5.0e-2	.65	2.8e-2	.63
	81	6.6e3	2.5e-2	8.8e-2	.48	1.1e-2	.90	3.1e-2	.70	1.7e-2	.69
	161	2.6e4	1.3e-2	6.3e-2	.49	5.7e-3	.94	1.8e-2	.75	1.0e-2	.73
	321	1.0e5	6.3e-3	4.4e-2	.49	2.9e-3	.96	1.1e-2	.78	6.1e-3	.77
	641	4.1e5	3.1e-3	3.1e-2	.50	1.5e-3	.97	6.1e-3	.81	3.5e-3	.79
	1281	1.6e6	1.6e-3	2.2e-2	.50	7.6e-4	.98	3.4e-3	.83	2.0e-3	.82
3	11	1.3e3	2.0e-1	3.5e-1		1.2e-1		2.1e-1		1.2e-1	
	21	9.3e3	1.0e-1	2.6e-1	.43	6.9e-2	.78	1.4e-1	.58	8.4e-2	.57
	41	6.9e4	5.0e-2	1.9e-1	.47	3.9e-2	.85	8.7e-2	.66	5.4e-2	.65
	81	5.3e5	2.5e-2	1.3e-1	.49	2.1e-2	.89	5.3e-2	.72	3.3e-2	.70
	161	4.2e6	1.3e-2	9.5e-2	.50	1.1e-2	.92	3.1e-2	.76	2.0e-2	.74
4	11	1.5e4	2.0e-1	4.4e-1		1.7e-1		2.9e-1		1.8e-1	
	21	1.9e5	1.0e-1	3.2e-1	.45	9.8e-2	.78	1.9e-1	.60	1.2e-1	.58
	41	2.8e6	5.0e-2	2.3e-1	.48	5.5e-2	.83	1.2e-1	.67	7.7e-2	.66

Proof. Let $\mu \le \check{w}$. By (3.2) and the definition of $\check{w}$, we have $D_j^s(N, \underline{v}, \mu) = D_j^s(N, \underline{u}, \mu)$ for all $s \in \mathcal{S}$, $1 \le j \le d$. This implies that $\underline{H}(N, \underline{v}, \mu) = \underline{H}(N, \underline{u}, \mu)$, proving the first claim.

For the second claim, let $\tilde{\mu}_u$ and $\tilde{\mu}_v$ be as defined above. Let $\tilde{\mu}_u \le \check{w}$. Then $\underline{H}(N, \underline{v}, \tilde{\mu}_u) = \underline{H}(N, \underline{u}, \tilde{\mu}_u) = 0$, so $\mu = \tilde{\mu}_u$ is a solution to $\underline{H}(N, \underline{v}, \mu) = 0$. By Theorem 3.5, this solution is unique. By a symmetric argument, if $\tilde{\mu}_v \le \check{w}$, then $\mu = \tilde{\mu}_v$ is the unique solution to $\underline{H}(N, \underline{u}, \mu) = 0$. ∎

4. Experiments. We conduct experiments to show numerical evidence that the result of FMM converges to the viscosity solution of (1.1), to demonstrate types of anisotropic problems that can be solved, and to determine the effectivenesss of the node and simplex elimination techniques described in Appendix A. Throughout this section, the boundary conditions are $g(x) = 0$ for $x \in \partial\Omega$. For all experiments below, excluding that in section 4.4, we discretize $[-1, 1]^d$ such that there are m uniformly spaced nodes in each dimension, and we ensure that there is a node at the origin O.

4.1. Convergence study. We examine the difference between the solution to (3.3) and the solution to (1.1) for two simple Dirichlet problems. In particular, we look at how the absolute error changes as the grid spacing decreases toward zero. For the problems considered, $\Omega = [-1, 1]^d \setminus \{O\}$. We take H to have the form in (2.2), where $G(Du(x)) = \|Du(x)\|_p$ and $p = 1$ or $p = 2$. The boundary conditions are $g(O) = 0$. We use the analytic node value update equations provided in Appendix B.

Since there is a node at O, any error introduced is from the discretization of H and not from the discretization of the boundary condition. The approximation errors are summarized in Table 4.1.

4.2. Asymmetric anisotropic problem. For this anisotropic problem, H is as in (2.2), where G is defined by (2.5) (see Figure 2.3(b)). The domain is given by $\Omega = [-1, 1]^2 \setminus \{O\}$ and $\partial\Omega = \{O\}$. The cost is $c(x) = 1$, except in four rectangular regions shown in black in Figure 4.1, where $c(x) \gg 1$. In the `Update` function, we

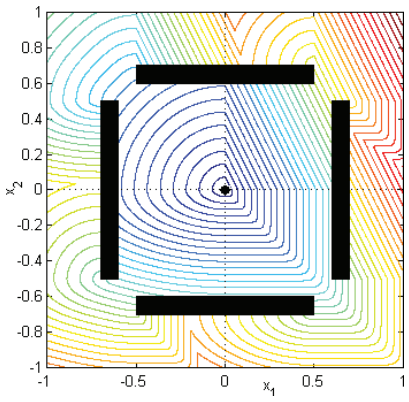


FIG. 4.1. *Contours of $\underline{u}$ computed for the anisotropic problem where Hamiltonian H is as in* (2.2) *and G is as in* (2.5). *The black circle at $O = (0, 0)$ indicates $\underline{\partial\Omega}$, and in the black rectangles $c(x) \gg 1$. In these regions, $\underline{u}$ has purposefully not been computed.*

analytically computed the solution to (3.4) using the equations for updating from a single simplex given in Appendix B. The number of nodes in each dimension is $m = 1281$. We plot the contours of $\underline{u}$ computed by FMM in Figure 4.1. Note the asymmetric contours where the characteristics bend through gaps. The relationship between the shape of the contours of G in Figure 2.3(b) and those of $\underline{u}$ is explained by the duality articulated in Proposition 2.7 of [1].

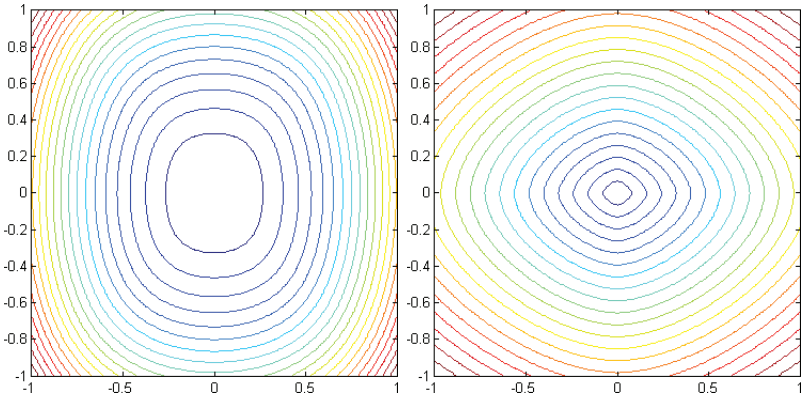
4.3. Anelliptic elastic wave propagation. As is done in [10], we consider elastic wave propagation in VTI media, which are transversely isotropic media with a vertical axis of symmetry. In particular, we wish to find the arrival times of quasi-longitudinal (quasi-P or qP) waves propagating in two dimensions from a point source at the origin O. We solve the anisotropic HJ PDE given by defining the Hamiltonian

(4.1)
$$H(q) = \frac{1}{2}\left(q_1^2 + q_2^2\right)\left\{(a+l)q_1^2 + (c+l)q_2^2 + \sqrt{\left[(a-l)q_1^2 - (c-l)q_2^2\right]^2 + 4(f+l)^2 q_1^2 q_2^2}\right\} - 1.$$

This Hamiltonian is derived from the anisotropic Eikonal equation and the exact qP-wave phase velocity equation in [10]. The parameters $a = 14.47$, $l = 2.28$, $c = 9.57$, and $f = 4.51$ are taken from [10].

The Hamiltonian H and the approximate solution $\underline{u}$ resulting from FMM are shown in Figure 4.2. We have not shown analytically that (4.1) satisfies strict one-sided monotonicity for some range of parameters. However, the level sets of H as shown in Figure 4.2(a) indicate that H is strictly one-sided monotone for the given parameters. Furthermore, the level sets of H indicate that H is convex and a computation of the derivative of H using the symbolic mathematics program Maple shows that H satisfies Osher's criterion for the given parameters. As a result, the analysis in this paper can be applied to the problem, and FMM can be used to compute the solution.

We used a grid of size 201×201. In the `Update` function, we used the interval method to solve (3.4) numerically. We computed the maximum relative error of $\underline{u}$ to be 0.0076 when compared to the travel-time computed with the group-velocity



(a) (b)

FIG. 4.2. *Using FMM for computation of travel times of qP-waves in two-dimensional VTI media.* (a) *Contours of Hamiltonian H as given in* (4.1). (b) *Contours of approximate solution $\underline{u}$ computed by FMM.*

approximation for qP-waves presented in [10]. In turn, the group-velocity approximation is claimed to have a maximum error of about 0.003 when compared to the true solution.

4.4. Two robots. We consider the two-robot coordinated navigation problem illustrated in Figure 4.3. The circular robots are free to move independently in a two-dimensional plane but may not collide with each other or the obstacles (black region). Each may travel at a maximum speed of $1/c(x)$ in any direction. The robots attempt to achieve a joint goal state. This goal should be achieved in minimal time from any initial state in the domain without incurring collisions.

Let the state of the dark-colored robot be $(x_1, x_2) \in \mathbb{R}^2$ and the state of the light-colored robot be $(x_3, x_4) \in \mathbb{R}^2$ so that the combined state of the two robots is $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. We define the control-theoretic action set

$$\mathcal{A}(x) = \{a \mid F(a) = \|(\|(a_1, a_2)\|_2, \|(a_3, a_4)\|_2)\|_\infty \leq 1/c(x)\}.$$

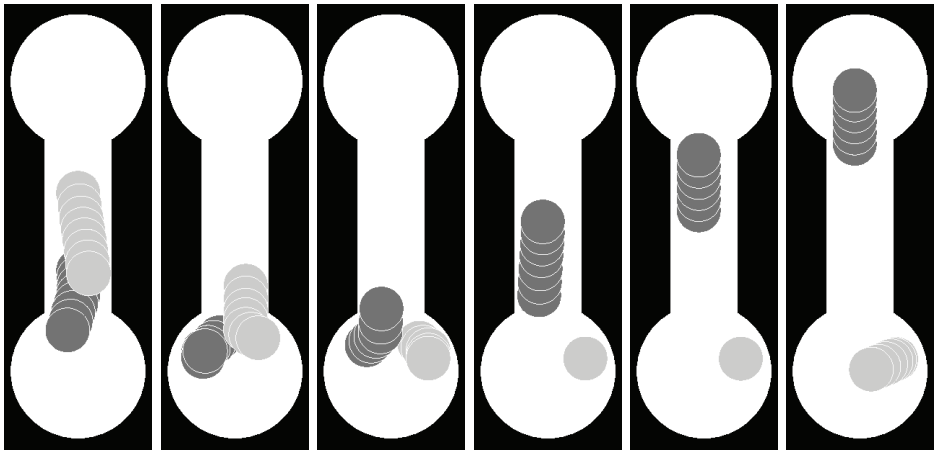
Proposition 2.7 of [1] states that we can use the dual of F to obtain

$$G(x, Du(x)) = \|(\|(\partial_1 u(x), \partial_2 u(x))\|_2, \|(\partial_3 u(x), \partial_4 u(x))\|_2)\|_1, \tag{4.2}$$

where $Du(x) = (\partial_1 u(x), \partial_2 u(x), \partial_3 u(x), \partial_4 u(x))$. Where x is a collision state, we set $c(x) \gg 1$. For all other states x, $c(x) = 1$.

We can compute $\underline{u}$ using FMM since G is a mixed p-norm, and thus H satisfies Properties 1 to 4 (see section 2.2). The domain Ω is discretized using a uniform orthogonal grid of $(81 \times 21)^2$ nodes. The discretization of (4.2) is quartic in $\underline{u}_0$, so it is difficult to solve analytically. However, Theorem 3.5 tells us that we can determine the solution to (3.4) uniquely. As a result, numerical root-finders can easily be used to compute this solution in the `Update` function. Once an approximation of u is generated by FMM, a gradient descent algorithm is used to find optimal paths [3, 2]. The optimal trajectories from a single starting condition are shown in Figure 4.3.

4.5. Efficient implementation. Appendix A describes three different methods for improving the efficiency of the `Update` function: symmetry, causality, and solution



(a) (b) (c) (d) (e) (f)

FIG. 4.3. *Two-robot coordinated optimal navigation problem. The joint goal is for the dark-colored robot to reach the center of the upper bulb and the light-colored robot to reach the center of the lower bulb. Black indicates an obstacle region. The sequence shows the robots achieving their joint goal without collision from a particular initial state. The solution of* (1.1)*, where* H *is given by* (2.2) *and* G *is given by* (4.2) *allows quick determination of the optimal collision-free trajectories for both robots from any initial condition* [3].

elimination. Some of these methods are related to those found in [15, 31]. However, experimental results indicate that the efficiency gains from using these methods are not substantial for an already efficient implementation of FMM. In such an implementation the `Update` function computes only updates from those nodes that have already been extracted from $\mathcal{Q}$ using the `ExtractMin` function. Also, only simplices that include the most-recently extracted node $\underline{y}$ are considered in `Update`. Our experiments show that in many calls to `Update`, only a single simplex fits these criteria, and the fraction of updates for which only a single simplex fits the criteria grows as the grid is refined. For this reason, further techniques for eliminating nodes and simplices, such as those described in Appendix A, are largely ineffective.

However, for coarse grid resolutions and problems where characteristics intersect often, multiple simplices are considered by `Update` frequently enough that symmetry elimination, which is very cheap, significantly improves efficiency. In some cases, a node value update can be ignored altogether if the most-recently extracted node is eliminated by symmetry.

Despite the fact that the node and simplex elimination techniques described in Appendix A are useful only in limited circumstances, we include them for theoretical interest and because they may be applied in other algorithms, such as sweeping methods, that also require the `Update` function.

5. Conclusion. We have described a new class of static HJ PDEs with axis-aligned but potentially asymmetric anisotropy. Assuming Properties 1 to 4 of the Hamiltonian, we showed that uniqueness, monotonicity, and causality hold for a standard finite-difference discretization of these PDEs on an orthogonal grid, and so the FMM can be used to approximate their solution. In the appendix, we also demonstrate several methods for reducing the number of neighboring simplices which must

be considered when computing node updates, including novel methods which work when the PDE and/or grid are asymmetric. In future work, these results might be generalized to unstructured grids.

Appendix A. Efficient Implementation of `Update`. We discuss ways to improve the efficiency of the `Update` function, which calculates the unique solution $\mu = \tilde{\mu}$ to (3.4). We note that these improvements may be used for any type of solution method, including FMM and sweeping methods, as long as (3.4) is being solved. Some efficiency improvements are related to similar ideas specific to the isotropic Eikonal equation found in [15, 31].

Efficiency can be gained by determining which neighbors $\underline{x} \in \mathcal{N}(\underline{x}_0)$ have no influence on the solution and eliminating them from consideration. Let

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d),$$

where $\sigma_j \subseteq \{\pm 1\}$, indicate which $\underline{x} \in \mathcal{N}$ are considered in determining the solution $\mu = \tilde{\mu}$. Let $\mathcal{N}_\sigma$ be the reduced set of neighbor nodes defined by σ. Let $\mathcal{S}_\sigma$ be the set of neighboring simplices that can be formed by the neighbors in $\mathcal{N}_\sigma$. For example, in $d = 4$ dimensions, take

$$\sigma = (\emptyset, \{\pm 1\}, \{-1\}, \{\pm 1\}).$$

We have

$$\mathcal{N}_\sigma = \left\{\underline{x}_2^{\pm}, \underline{x}_3^{-}, \underline{x}_4^{\pm}\right\} \quad \text{and}$$
$$\mathcal{S}_\sigma = \{(0,-1,-1,-1), (0,+1,-1,-1), (0,-1,-1,+1), (0,+1,-1,+1)\}.$$

Let $\underline{H}_\sigma(N, \phi, \mu) = \max_{s \in \mathcal{S}_\sigma}[H(\underline{x}_0, \underline{D}^s(N, \phi, \mu))]$ be the *reduced-neighbor numerical Hamiltonian*, a modification of (3.1) that considers only the neighbors and simplices indicated by σ. For $s \in \mathcal{S}_\sigma$ and $1 \le j \le d$, $s_j = 0$ indicates that $\underline{x}_j^{s_j}$ is not considered in computing the gradient approximation $D^s\underline{u}(\mu)$; that is, $\underline{D}_j^s(N, \phi, \mu) = 0$ if $s_j = 0$, and $\underline{D}_j^s$ satisfies (3.2) otherwise.

To implement `Update`, we first reduce the set of considered neighbors and then solve

$$\underline{H}_\sigma(N(\underline{x}_0), \underline{u}, \mu) = 0 \tag{A.1}$$

for $\mu = \tilde{\mu}$ to determine a node's value $\underline{u}(\underline{x}_0)$. As in section 3, we may write $\underline{H}_\sigma(\mu) = \underline{H}_\sigma(N, \phi, \mu)$ and $\underline{D}^s(\mu) = \underline{D}^s(N, \phi, \mu)$, where no ambiguity results. Note that some properties of (A.1) are retained from (3.4) as long as at least one considered neighbor remains in σ. Let

$$\breve{u}_\sigma = \min_{\underline{x} \in \mathcal{N}_\sigma} \left(\underline{u}(\underline{x})\right).$$

Proposition A.1 (analogue of Lemma 3.3). *$\underline{H}_\sigma(\mu)$ is strictly increasing on $\mu \ge \breve{u}_\sigma$.*

Proposition A.2 (analogue of Lemma 3.4). *The numerical Hamiltonian $\underline{H}_\sigma(\mu)$ satisfies the following:*

(a) *$\underline{H}_\sigma(\mu) = H(0) < 0$ for $\mu \le \breve{u}_\sigma$.*

(b) *$\underline{H}_\sigma(\mu) \to \infty$ as $\mu \to \infty$.*

(c) *$\underline{H}_\sigma(\mu)$ is nondecreasing on all μ.*

Proposition A.3 (analogue of Theorem 3.5). *There exists a unique solution $\mu = \tilde{\mu}$ to $\underline{H}_\sigma(\mu) = 0$ such that $\tilde{\mu} > \breve{u}_\sigma$.*

A.1. Symmetry. We show how the considered neighbors σ can be reduced by keeping only the neighbor with the smaller value of a pair of opposite neighbors in the jth dimension when (3.1) is symmetric in that dimension. This procedure is a generalization of those in [15, 31] to all axis-aligned anisotropic problems on unequally spaced grids. First, we introduce useful notation.

Let $q \in \mathbb{R}^d$. Let $T^i(q)$ be a reflection of q in the hyperplane orthogonal to the ith axis, such that

$$T^i_j(q) = \begin{cases} -q_j, & \text{if } j = i, \\ q_j, & \text{otherwise,} \end{cases}$$

for $1 \le j \le d$. Let Ψ_j indicate symmetry of (3.1) in the jth dimension, as follows:

$$\Psi_j = \begin{cases} 1, & \text{if } |h_j^-| = |h_j^+| \text{ and for all } q \in \mathbb{R}^d,\ H(q) = H(T^j(q)), \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\Psi_j = 1$ if and only if the grid spacing and H are symmetric in the jth dimension. The following theorem is proved in [2].

THEOREM A.4. *Let σ be such that $\sigma_j \subseteq \{\pm 1\}$ for $1 \le j \le d$. Let $\tilde{\sigma}$ be defined by*

$$\tilde{\sigma}_j = \begin{cases} \{-1\}, & \text{if } \sigma_j = \{\pm 1\},\ \Psi_j = 1, \text{ and } \underline{u}_j^- \le \underline{u}_j^+, \\ \{+1\}, & \text{if } \sigma_j = \{\pm 1\},\ \Psi_j = 1, \text{ and } \underline{u}_j^- > \underline{u}_j^+, \\ \sigma_j, & \text{otherwise,} \end{cases}$$

for $1 \le j \le d$. Let $\mu = \mu_\sigma$ be the unique solution to $\underline{H}_\sigma(\mu) = 0$. Let $\mu = \mu_{\tilde{\sigma}}$ be the unique solution to $\underline{H}_{\tilde{\sigma}}(\mu) = 0$. Then $\mu_{\tilde{\sigma}} = \mu_\sigma$.

An implementation of the `Update` function can use the result obtained in Theorem A.4 to eliminate $\underline{x} \in \mathcal{N}$ from consideration in solving (A.1) by exploiting symmetries in (3.1). We call this *symmetry elimination*.

Remark 2. Theorem A.4 can be generalized to an asymmetric version. We let $1 \le j \le d$, and let $s_j, \tilde{s}_j \in \{\pm 1\}$ such that $s_j \ne \tilde{s}_j$. Node $\underline{x}_j^{s_j} \in \mathcal{N}$ may be eliminated from consideration if

- $|h_j^{\tilde{s}_j}| \le |h_j^{s_j}|$;
- for all $q \in \mathbb{R}^d$ such that $s_j q_j \ge 0$, $H(q) \le H(T^j(q))$;
- and $\underline{u}_j^{\tilde{s}_j} \le \underline{u}_j^{s_j}$.

A.2. Causality. The causality of (3.1) can also be exploited to eliminate $\underline{x} \in \mathcal{N}_\sigma$ from consideration. This observation was used in two distinct but equivalent methods for analytically computing the `Update` from a single simplex to solve an isotropic Eikonal equation [15, 31]. We show with the following theorem that the condition $\underline{H}_\sigma(\underline{u}(\underline{x})) \ge 0$ can be checked to determine that a node $\underline{x}$ is noncausal, i.e., that the solution $\mu = \mu_\sigma$ to (A.1) is not dependent on the node $\underline{x}$ and its value $\underline{u}(\underline{x})$.

THEOREM A.5. *Let σ be such that $\sigma_j \subseteq \{\pm 1\}$ for $1 \le j \le d$. Pick any $s \in \mathcal{S}_\sigma$ and $i \in \{1, 2, \ldots, d\}$ such that $s_i \ne 0$ and $\underline{H}_\sigma(\underline{u}_i^{s_i}) \ge 0$. Let $\tilde{\sigma}$ be defined by*

$$\tilde{\sigma}_j = \begin{cases} \sigma_j \setminus \{s_j\}, & \text{if } j = i, \\ \sigma_j, & \text{otherwise.} \end{cases}$$

Let $\mu = \mu_\sigma$ be the unique solution to $\underline{H}_\sigma(\mu) = 0$. Let $\mu = \mu_{\tilde{\sigma}}$ be the unique solution to $\underline{H}_{\tilde{\sigma}}(\mu) = 0$. Then $\mu_{\tilde{\sigma}} = \mu_\sigma$.

Proof. Let σ, s, i, $\tilde{\sigma}$, μ_σ, and $\mu_{\tilde{\sigma}}$ be as defined above. By Proposition A.2(c), $\underline{H}_\sigma(\mu)$ is nondecreasing. Since $\underline{H}_\sigma(\underline{u}_i^{s_i}) \geq 0 = \underline{H}_\sigma(\mu_\sigma)$, it must be that $\mu_\sigma \leq \underline{u}_i^{s_i}$. Note that $\underline{H}_{\tilde{\sigma}}(\mu)$ is identical to $\underline{H}_\sigma(\mu)$ except for $D_i^s \underline{u}(\mu)$, which is set to zero in $\underline{H}_{\tilde{\sigma}}(\mu)$. But for $\mu \leq \underline{u}_i^{s_i}$, we also have $D_i^s \underline{u}(\mu) = 0$ in $\underline{H}_\sigma(\mu)$. Consequently, $\underline{H}_{\tilde{\sigma}}(\mu) = \underline{H}_\sigma(\mu)$ for $\mu \leq \underline{u}_i^{s_i}$. In particular, $\underline{H}_{\tilde{\sigma}}(\mu_\sigma) = \underline{H}_\sigma(\mu_\sigma) = 0$. Therefore, $\mu_{\tilde{\sigma}} = \mu_\sigma$. ∎

Theorem A.5 states that the unique solution μ to (A.1) does not change when a noncausal node is removed from σ. This node removal can be repeated until all noncausal nodes have been removed, and the solution $\mu = \mu_\sigma$ will remain unchanged. We call this *causality elimination*. A binary or linear search through sorted neighbors' values can be used to determine the largest node value that might be causal. Note that causality elimination does not require symmetry in (3.1). However, the test for noncausality requires an evaluation of $\underline{H}_\sigma$, which is more expensive than the comparison of two neighbors' values used for symmetry elimination.

A.3. Solution. After eliminating from consideration nodes in σ using symmetry and causality elimination, we can determine the solution $\mu = \tilde{\mu}$ to (A.1). Let

$$\check{\mu} = \min_{s \in \mathcal{S}_\sigma} (\mu_s), \tag{A.2}$$

where $\mu = \mu_s$ is the unique solution to

$$H(D^s \underline{u}(\mu)) = 0. \tag{A.3}$$

We show with the following proposition that, instead of solving (A.1) directly, we can solve (A.3) for each $s \in \mathcal{S}_\sigma$ and take the minimum such solution $\check{\mu}$. It can be shown that $H(D^s \underline{u}(\mu))$ is continuous and nondecreasing on μ and that (A.3) has a unique solution in an analogous but simpler manner as the proof of Theorem 3.5.

PROPOSITION A.6. *Let $\hat{\mu}$ be the unique solution to* (A.1). *Then $\hat{\mu} = \check{\mu}$.*

Proof. Let μ_s, $\check{\mu}$, and $\hat{\mu}$ be as defined above. For any $s \in \mathcal{S}_\sigma$, we know $\mu_s \geq \check{\mu}$. Since $H(D^s \underline{u}(\mu))$ is nondecreasing on μ, it must be that $H(D^s \underline{u}(\mu)) \leq H(D^s \underline{u}(\mu_s)) = 0$ for all $\mu \leq \mu_s$. In particular, $H(D^s \underline{u}(\check{\mu})) \leq 0$. Furthermore, by the definition of $\check{\mu}$, there exists an $\check{s} \in \mathcal{S}_\sigma$ such that $H(D^{\check{s}} \underline{u}(\check{\mu})) = 0$. Consequently,

$$\underline{H}_\sigma(\check{\mu}) = \max_{s \in \mathcal{S}_\sigma} H(D^s \underline{u}(\check{\mu})) = 0. \tag{A.4}$$

Therefore, $\hat{\mu} = \check{\mu}$ solves (A.1), and it is a unique solution by Proposition A.3. ∎

We further show that we may be able to determine $\check{\mu}$ without solving (A.3) for each $s \in \mathcal{S}_\sigma$. We demonstrate using the following proposition that if we have computed a solution $\mu = \mu_s$ of (A.3) for some $s \in \mathcal{S}_\sigma$, we can easily determine if $\mu_{\tilde{s}} \geq \mu_s$, where $\mu = \mu_{\tilde{s}}$ is the solution to $H(D^{\tilde{s}} \underline{u}(\mu)) = 0$ for some other $\tilde{s} \in \mathcal{S}_\sigma$. Note we do not necessarily need to compute $\mu_{\tilde{s}}$ to rule it out as a minimal solution.

PROPOSITION A.7. *Let $s \in \mathcal{S}_\sigma$ and $\tilde{s} \in \mathcal{S}_\sigma$. Let $\mu = \mu_s$ be the unique solution to $H(D^s \underline{u}(\mu)) = 0$ and $\mu = \mu_{\tilde{s}}$ be the unique solution to $H(D^{\tilde{s}} \underline{u}(\mu)) = 0$. Then $\mu_{\tilde{s}} < \mu_s$ if and only if $H(D^{\tilde{s}} \underline{u}(\mu_s)) > H(D^s \underline{u}(\mu_s))$.*

Proof. Let μ_s and $\mu_{\tilde{s}}$ be as defined above. If $H(D^{\tilde{s}} \underline{u}(\mu_s)) > H(D^s \underline{u}(\mu_s)) = 0$, then the unique solution $\mu = \mu_{\tilde{s}}$ to $H(D^{\tilde{s}} \underline{u}(\mu)) = 0$ must be such that $\mu_{\tilde{s}} < \mu_s$, since $H(D^{\tilde{s}} \underline{u}(\mu))$ is nondecreasing on μ. Similarily, if $H(D^{\tilde{s}} \underline{u}(\mu_s)) \leq H(D^s \underline{u}(\mu_s))$, then the unique solution $\mu = \mu_{\tilde{s}}$ to $H(D^{\tilde{s}} \underline{u}(\mu)) = 0$ must be such that $\mu_{\tilde{s}} \geq \mu_s$. ∎

The result of Proposition A.7 can be used to eliminate simplices $s \in \mathcal{S}_\sigma$ for which solutions to (A.3) are irrelevant to the computation. We call this process *solution elimination*.

Appendix B. Analytic solutions. We provide analytic node value update equations for the cases where H is given by (2.2), where $G(Du(x)) = \|Du(x)\|_p$ and $p = 1$, $p = 2$, or $p = \infty$. In these cases, there is an exact solution to (3.4). For derivations of these equations, see [2]. The equation for $p = 2$ fixes some errors in the appendix of [16]. In [3] we demonstrated that these cases could be treated by FMM and are useful for robotic applications. However, here we generalize the update equations to any dimension and grid spacing.

Let $(v_1, v_2, \ldots, v_m)$ be the values of the neighboring nodes in the simplex $s \in \mathcal{S}_\sigma$ and $(h_1, h_2, \ldots, h_m)$ be the corresponding grid spacings. We are solving for μ. In order to use the analytic updates below, noncausal node values must already have been eliminated using causality elimination, so $\mu > \max_{1 \le j \le m} v_j$. However, in the case of the efficient implementation of FMM discussed in section 4.5, any nodes that would be removed from consideration by causality elimination could not already have been extracted from $\mathcal{Q}$, and so the anaytic updates below can be applied directly.

The update formula for $p = 1$ is

$$\mu = \frac{\sum_j \left(\prod_{l \neq j} h_l \right) v_j + \prod_l h_l c}{\sum_j \prod_{l \neq j} h_l}.$$

The update formula for $p = 2$ is

$$\mu = \frac{\sum_j \left(\prod_{l \neq j} h_l^2 \right) v_j + \prod_l h_l \sqrt{\begin{array}{l} \left(\sum_j \prod_{l \neq j} h_l^2 \right) c^2 \\ - \sum_{j_1} \sum_{j_2 > j_1} \left(\prod_{l \neq j_1, j_2} h_l^2 \right) (v_{j_1} - v_{j_2})^2 \end{array}}}{\sum_j \prod_{l \neq j} h_l^2}.$$

The update formula for $p = \infty$ is

$$\mu = \min_j \left(v_j + h_j c \right).$$

The $p = \infty$ case is identical to the update formula for Dijkstra's algorithm for shortest path on a discrete graph.

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