Proceedings of the
2007 IEEE Intelligent Vehicles Symposium
Istanbul, Turkey, June 13-15, 2007

Trajectory Generation based on Rational Bezier Curves as Clothoids

Nicolás Montés, Marta C. Mora and Josep Tornero, *Member, IEEE*

Abstract- **This paper explains a fast overtaking algorithm for IVHS based on clothoidal trajectories.**

In spite of the fact that clothoidal trajectory generation requires complicated computation of Fresnel integrals, in this paper, an on-line clothoidal path is obtained just by scaling a general parametric curve. This is based on a general offline approximation of the Fresnel integrals into Rational Bezier Curves (RBC) which are later used in the generation of on-line clothoidal paths. In addition, in order to obtain a fast algorithm, a new methodology to approximate the Fresnel integrals into RBC is proposed. This approach guarantees that the resulting curve has the same behavior than the clothoid.

In particular for overtaking maneuvers, a clothoidal path composed of two equal elementary paths is generated. Each one constructed joining two piecewise identical clothoids.

A 3D simulation environment has been used for testing the trajectory generation algorithm described in the paper. As can be seen, the obstacle avoidance module generates free-collision intermediate positions which are reached by clothoidal paths.

I. Introduction

Intelligent Vehicle Highway Systems (IVHS) have been subject to extensive research; however automated lane changing has not been fully accomplished yet. The complexity of the lane changing maneuver, since it incorporates lateral and longitudinal control in the presence of obstacles, is one of the main problems [1].

The simplest solution is to generate a trajectory concatenating line and arc segments [2], [3]. The main disadvantage of this technique is the discontinuity between concatenated paths. This problem can be overcome by the use of clothoids. This curve has a linear relationship between the curvature and the arc length. This fundamental feature guarantees the comfort of the passengers because it maintains a constant variation of the centrifugal acceleration. Therefore, it can be used as a curve transition between lines and arcs [4],[5]. Another possibility is to combine only clothoids [6],[7]. In these papers, an elementary path consisting of two equal clothoids is defined. Then, any pose in the plane can be reached through a bi-elementary path, built using two different paths. For our particular case, that is, lane changing maneuver, the required path can be simplified using two equal elementary paths, as it is proposed in [9] and previously published in [8].

Unfortunately, the clothoid is a transcendental curve defined in terms of Fresnel integrals. Therefore, it cannot be solved analytically. The relevance of the clothoid [6], [7], [10] has encouraged the development of many approximation techniques [11]-[14]. These techniques have been used in CAD/CAM programs for road design. However, they are not fast enough for online path planning. In this sense, we present a general offline approximation of the Fresnel integrals into Rational Bezier Curves (RBC) which are later used in the generation of the on-line clothoidal path just scaling the parametric curves.

Virtual environments are widely used for algorithm validation in mobile robotics. They are particularly useful when real implementation does not guarantee safety. In such situations, it is essential to detect any possible fault that may exist in the algorithm. In spite of the fact that simulation does not contemplate every factor involved in real situations, it is still a good approximation [16]. In order to test the suitability of the clothoidal trajectory generation, a simulation application has been used.

The simulation application, shown in the paper, represents an autonomous vehicle with an obstacle avoidance process based on clothoidal trajectory. The obstacle avoidance algorithm uses the computation of the MTD, [17].

This paper is organized as follows. In section II, a brief review of the properties of clothoid curves is presented. Section III shows the methodology to approximate the Fresnel integrals into RBCs. These curves are used to construct a clothoidal path for lane changing maneuvers, sections IV and V. This methodology is tested in a simulation environment as described in section VI. Conclusions and future work are presented in section VII.

II. Properties Of The Clothoid Curve

The clothoid curve or Cornu spiral is defined as follow.

$$Q(\gamma)=\begin{pmatrix} x(\gamma) \\ y(\gamma) \end{pmatrix} = K\cdot\begin{pmatrix} C(\gamma) \\ S(\gamma) \end{pmatrix} = K\cdot\begin{pmatrix} \int_0^\gamma \cos\frac{\pi\cdot\xi^2}{2}d\xi \\ \int_0^\gamma \sin\frac{\pi\cdot\xi^2}{2}d\xi \end{pmatrix} \quad (1)$$

where K is a positive real number, γ is a non-negative real number. Clothoid curves have the following properties:

This research has been partially sponsored by The Spanish Government, MCyT, Project Ref. DPI2000-0362-P4-05, DPI2001-2689-C03-02, DPI2004-07417-C04-01 and BIA2005-09377-C03-02.

N. Montés is with the Systems Engineering and Control Department, Technical University of Valencia, Valencia, Spain (corresponding author to provide phone: +34 963 877 060; e-mail: nimonsan@upvnet.upv.es).

M.C. Mora is now with the Mechanical Engineering and Construction Department, Universitat Jaume I, Spain (e-mail: mmora@emc.uji.es).

J. Tornero is with the Systems Engineering and Control Department, Technical University of Valencia, Spain (e-mail: jtornero@isa.upv.es).

1. Angle of tangent: $\tau = \pi \cdot \gamma^2/2$
2. Curvature: $k = \pi \cdot \gamma / K$, Radius $R = 1/k$
3. Arc length L: $L = K \cdot \gamma = \sqrt{\pi} \cdot A \cdot \gamma$, where A is the well-known clothoid constant parameter.
4. Homotecial factor $K = \sqrt{\pi} \cdot A$

The most attractive property of the clothoid curve is that:

$$\frac{1}{k} = R = \frac{A^2}{L} \qquad (2)$$

where R is the radius of the curvature. This property guarantees the passengers comfort. In particular, the variation of the centrifugal acceleration, J, is defined by the next equation, [8], [9], [10].

$$A^2 = \frac{V_e^3}{J}$$

where Ve is the vehicle velocity. This property has main relevance in road design [10] and AGVs [6], [7]. As mentioned in the introduction, Fresnel integrals must be solved numerically. The approximation methods can be divided into polynomial and non-polynomial functions. In particular, all existing techniques involving non-polynomial functions [11] are just useful to approximate the Fresnel integrals in a single point. However, CAD/CAM systems or automated guided vehicles trajectory generation require a continuous function. For this purpose, polynomial functions are an ideal solution [1].

Usually, the standard polynomial functions used in CAD/CAM fields are Bezier, Rational Bezier, B-spline and NURBS. In this sense, there are some techniques that approximate the clothoid into these curves [12]-[14]. Although these methods produce accurate approximations of the clothoid, they cannot be used in AGVs because the resulting order of the curves is too high and the calculus is very complicated for online path generation. For this reason, we have developed a new technique to approximate the clothoid into parametric curves. In particular, an approximation of the clothoid to Bezier and Rational Bezier curves (RBC) is presented. In this sense, each Fresnel integral is approximated to one parametric curve in a selected working interval. Therefore, one parametric curve is obtained for each coordinate as explained afterwards.

III. Fresnel Integrals Approach To Bezier And Rational Bezier Curves

A Bezier curve is the most common form to represent planar curves for CAD/CAM applications. This curve has the next formulation:

$$P(u) = \sum_{k=0}^{N} C_k \cdot \frac{N!}{k!(N-k)!} \cdot u^k \cdot (1-u)^{N-k}$$

where:

C_k: Bezier control points
u: Intrinsic parameter [0…1]
N: Order of the Bezier equation

Parameter γ must be normalized in a selected work interval $\gamma_s \le \gamma \le \gamma_e$, where γ_s, γ_e are the limits of the Fresnel integrals. These values can be calculated by the clothoid properties explained in section II. With this, Bezier equation can be rewritten as:

$$P(\gamma) = \sum_{k=0}^{N} C_k \cdot \frac{N!}{k!(N-k)!} \cdot \left(1 - \frac{\gamma_e - \gamma}{\gamma_e - \gamma_i}\right)^k \cdot \left(\frac{\gamma_e - \gamma}{\gamma_e - \gamma_i}\right)^{N-k}$$

It can be expressed as a linear equation:

$$P_\gamma = C_0 \cdot B_\gamma^0 + C_1 \cdot B_\gamma^1 + \ldots + C_N \cdot B_\gamma^N$$

Where B_γ^k is the kth Bernstein basis function, which is:

$$B_\gamma^k = \frac{N!}{k!(N-k)!} \cdot \left(1 - \frac{\gamma_e - \gamma}{\gamma_e - \gamma_i}\right)^k \cdot \left(\frac{\gamma_e - \gamma}{\gamma_e - \gamma_i}\right)^{N-k}$$

This equation can be expressed in the next matrix form:

$$\begin{bmatrix} P_{\gamma_s} \\ . \\ . \\ P_{\gamma_e} \end{bmatrix} = \begin{bmatrix} B_{\gamma_s}^0 & . & . & B_{\gamma_s}^N \\ . & . & & . \\ . & & . & . \\ B_{\gamma_e}^0 & . & . & B_{\gamma_e}^N \end{bmatrix} \cdot \begin{bmatrix} C_k \\ . \\ . \\ C_N \end{bmatrix}$$

$$\mathbf{P} = \mathbf{B} \cdot \mathbf{C}$$

This representation permits the use of LS-techniques in order to compute the Bezier control points. Fresnel points ($C(\gamma)$, $S(\gamma)$), represented by P_γ in the equations , are obtained using a non-polynomial approximation explained in [11], with an accuracy of $2 \cdot 10^{-10}$. The clothoid property 1 (see section II) permits to establish the working interval in terms of the tangent angle $[\tau_i, \tau_e]$. For instance, Fig. 1 shows an approximation of Fresnel points in the interval $[0, \pi/2]$.

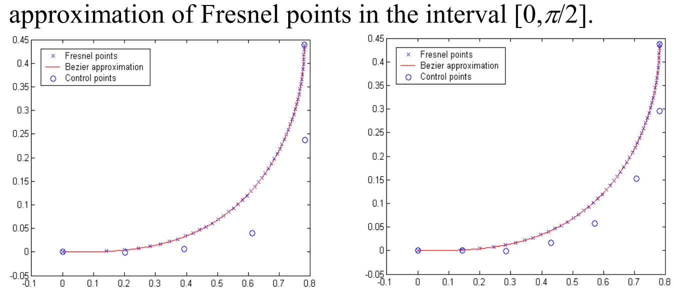

Fig. 1. Bezier curve of 5th (left) and 7th (right) order.

Although this methodology produces an accurate approximation, the resulting Bezier curve does not pass across the start and end points. In addition, it doesn't guarantee G^2 continuity at the start point unless the first three control points must be aligned with the line. Therefore, in order to overcome discontinuities, the control points have to be forced in these locations.

Forcing the location of these three points reduces the degree of freedom, so it is necessary to increase the order of the curve to maintain the accuracy of the approximation. This drawback can be avoided using Rational Bezier Curves (RBC) as a way of introducing new degrees of freedom. RBC requires relocating control points with weights w_k, which allow reallocating starts and ending control points at the desired positions without increase the order of the RBC.

The RBC formulation is:

$$P(\gamma)=\frac{\sum_{k=0}^{N} w_k\cdot C_k\cdot B_\gamma^k}{\sum_{k=0}^{N} w_k\cdot B_\gamma^k}$$

This equation can be expressed in the matrix form **P**=**B**·**w**. To approximate the $C(\gamma)$ Fresnel integral term, it is only necessary to adjust the start and end control points. In this case, the resulting matrixes are:

$$\mathbf{P}=\begin{bmatrix} P_{\gamma_s}\cdot\left[B_{\gamma_s}^0+B_{\gamma_s}^N\right]-P_{\gamma_s}\cdot B_{\gamma_s}^0-P_{\gamma_e}\cdot B_{\gamma_s}^N \\ \cdot \\ \cdot \\ P_{\gamma_f}\cdot\left[B_{\gamma_f}^0+B_{\gamma_f}^N\right]-P_{\gamma_s}\cdot B_{\gamma_f}^0-P_{\gamma_e}\cdot B_{\gamma_f}^N \end{bmatrix}$$

$$\mathrm{B}=\begin{bmatrix} \left(\mathrm{C}_1-\mathrm{P}_{\gamma_s}\right)\cdot \mathrm{B}_{\gamma_s}^1 & \cdot & \cdot & \left(\mathrm{C}_{N-1}-\mathrm{P}_{\tau_f}\right)\cdot \mathrm{B}_{\gamma_s}^{N-1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \left(\mathrm{C}_1-\mathrm{P}_{\gamma_s}\right)\cdot \mathrm{B}_{\gamma_f}^1 & \cdot & \cdot & \left(\mathrm{C}_{N-1}-\mathrm{P}_{\gamma_f}\right)\cdot \mathrm{B}_{\gamma_f}^{N-1} \end{bmatrix}$$

$$\mathbf{w}=\left[w_1 \quad . \quad . \quad w_{N-1}\right]^T$$

To approximate the $S(\gamma)$ Fresnel integral term the first three control points must be aligned with the line. Therefore, hereinbefore matrixes must be changed. In particular, the required changes are $w_1=w_2=1$ and $C_1=C_2=0$. For instance, Fig. 2 shows the RBC approximation in the interval $[0,\pi/2]$.

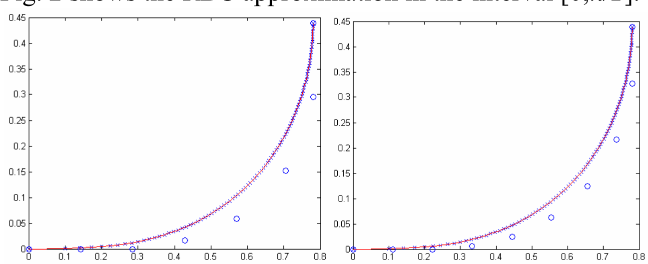

Fig. 2. RBC approximation of 7th (left) and 9th (right) order.

The accuracy of the approximation can be measured as:

$$\sigma^2=\sum_{\gamma_s}^{\gamma_f}\left(P_\gamma-\sum_{k=0}^{N} w_k\cdot C_k\cdot B_\gamma^k\right)^2$$

Next table shows the variance and maximum variance for each Fresnel integral component with different orders of complexity.

TABLE I

RBC WITH DIFFERENT ORDERS OF COMPLEXITY

N	σ^2_C	σ^2_S	Max(σ^2_C)	Max(σ^2_S)
7	$1.3\cdot10^{-11}$	$1.4\cdot10^{-9}$	$1.3\cdot10^{-12}$	$2.7\cdot10^{-10}$
9	$2.1\cdot10^{-13}$	$1.1\cdot10^{-13}$	$2.5\cdot10^{-14}$	$3.9\cdot10^{-14}$
11	$5\cdot10^{-18}$	$4.6\cdot10^{-15}$	$9.1\cdot10^{-19}$	$6.1\cdot10^{-16}$
13	$3\cdot10^{-21}$	$7.9\cdot10^{-20}$	$6.3\cdot10^{-22}$	$9.5\cdot10^{-21}$

Although this approximation is very accurate, the most interesting feature is the behavior of the RBC. In fact, if the approximation is accurate enough, RBC should be a clothoid with homotecial factor K=1 and curvature varying linearly with the arc length (well-known property of the clothoid). In order to test this statement, the Rational Bezier Curves that approximate $C(\gamma)$ and $S(\gamma)$ in the interval $[\tau_i,\tau_e]$ ($P_c(\gamma)$ and $P_s(\gamma)$ respectively) have been resampled in terms of the curvature k using clothoid properties 1 and 2, explained in Section II, obtaining $P_c(k)$ and $P_s(k)$. The work interval has changed to $[k_i, k_e]$. The arc length for every k-value $L(k)$ can be calculated as:

$$L(k)=\sum_{m=k_i}^{k}\sqrt{\left(P_C(m)-P_C(m-1)\right)^2+\left(P_S(m)-P_S(m-1)\right)^2}$$

The linear relationship between the arc length and the curvature is defined by the homotecial factor K. Equation (2) shows this linear relationship using the well-known clothoid constant parameter A. It can be rewritten using equation (2) and the 4th clothoid property:

$$K(k)=\sqrt{\frac{L(k)}{k}\cdot\pi}$$

Fig.3 shows this linear relationship and demonstrates that the RBC has the same behaviour as a clothoid. It is only guaranteed when the approximation error less than $1\cdot10^{-20}$.

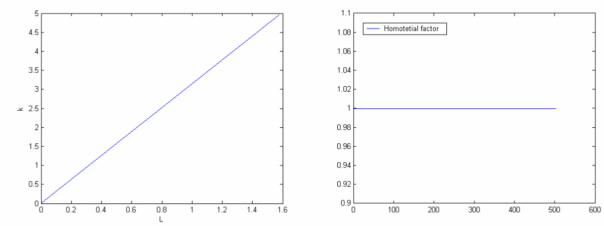

Fig. 3. RBC behaving as a clothoid.

The selected work interval depends on the application. In this case, that is, for lane changing maneuvers, the work interval is $[0,\pi/8]$. The coefficients of the RBC are:

TABLE II

COEFFICIENTS OF THE RBC FOR LANE CHANGING MANOEUVERS

i	C		S	
	C_i	w_i	C_i	w_i
0	0	1	0	1
1	0.0555	$1-76\cdot10^{-11}$	0	1
2	0.1111	$1-52\cdot10^{-10}$	0	1
3	0.1666	$1+95\cdot10^{-10}$	0.0007	$1-17\cdot10^{-4}$
4	0.2222	$1-16\cdot10^{-9}$	0.0031	$1+67\cdot10^{-7}$
5	0.2777	$1+14\cdot10^{-9}$	0.0077	$1+19\cdot10^{-7}$
6	0.3329	$1-12\cdot10^{-9}$	0.015	$1-34\cdot10^{-8}$
7	0.3876	$1+49\cdot10^{-10}$	0.027	$1+24\cdot10^{-9}$
8	0.4410	$1-34\cdot10^{-10}$	0.043	$1-17\cdot10^{-9}$
9	0.4923	1	0.064	1

These coefficients are constant for any computed trajectory just lacking the computation of the homotecial factor, which is coincides with the scaling factor of the RBC.

$$P(\gamma)=\frac{\sum_{k=0}^{N} w_k\cdot\left(K\cdot C_k\right)\cdot B_\gamma^k}{\sum_{k=0}^{N} w_k\cdot B_\gamma^k}$$

As we can observe in the previous equation, it is only necessary to apply the homotecial factor to the control points

C_k to obtain the required clothoid. It has the effect of scaling the obtained curve which is the same as multiplying the Fresnel integral points ($C(\gamma)$, $S(\gamma)$) by the homotecial factor. This allows defining a general RBC that will be particularized to any desired trajectory. It is due to RBC guarantees that the resulting curve is invariant under scaling, rotation and translation. This is one of the properties of the parametric curves, named as *transformation invariance*. It implies that the RBC can be translated, rotated and scaled without loses in the clothoid properties hereinbefore demonstrated. It permits an easy construction of clothoidal paths, as explained in the next sections.

IV. Elementary Path

Because of the interesting properties of the clothoid, it is being applied in road design for many years [10]. These properties have also been considered by researches in automated guided vehicles, as in [6], [7]. In these papers the concept of *elementary path* is introduced. It is built throughout the concatenation of two equal piecewise clothoids. Besides, the concatenation of two different elementary paths joins any two poses in the plane (x,y,θ) [6], [7]. In particular, to link two (x,y)-positions it is only necessary to generate an elementary path because always exists a tangent angle τ that guarantees the symmetry between these two poses. This matter has been extensively treated in our previous work [8].

For the first clothoid of the elementary path, the tangent angle can be computed as:

$$\tau = \frac{\pi}{2} - \mathrm{atan}\left(\frac{x_e}{y_e}\right)$$

where (x_e,y_e) is the end position to reach. In this calculus, initial position (x_i,y_i) is always (0,0). Changing the initial position just implies the translation of the resulting RBC.

The second clothoid of the elementary path is equal and symmetrical to the first one. This involves that the control points of the second clothoid can be obtained by generating the symmetry of the control points of the first clothoid, see Fig.4.

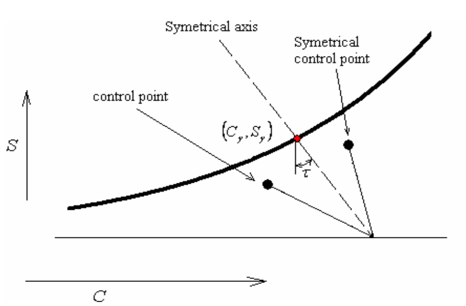

Fig. 4. Control points symmetry: Construction of an elementary path.

Any symmetrical point ($C\tau$,$S\tau$) is obtained resampling the RBC for the required tangent angle. As the tangent angle does not depend on the homotecial factor (see property 1 of the clothoids), it is possible to construct the elementary path without taking it into account.

As mentioned in the previous section, the homotecial factor acts as a scaling factor of the general RBC curve. Thus, the last step will be to scale the control points considering that the last control point (Cc_{2k},Cs_{2k}) has to coincide with the desired final position.

The mentioned homotecial factor K, and the A parameter of the clothoids, is computed from the desired end position (x_e,y_e) and the last control point of the second clothoid (Cc_{2k},Cs_{2k}) as follows :

$$K = \sqrt{\pi} \cdot A = \frac{x_e}{C_{C_{2k}}} = \frac{y_e}{C_{S_{2k}}}$$

For instance, fig. 5 shows an elementary path calculated to join the start position (0,0) with the end position (*20,10*). The resulting homotecial factor is K=21.81.

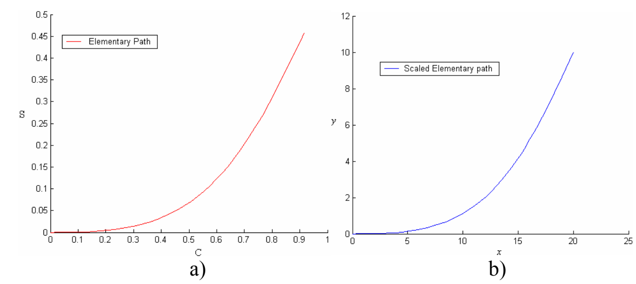

Fig. 5. Elementary path: a) without scaling factor, b) with scaling factor.

This simple technique allows the easy construction of an elementary path. In particular, lane changing maneuvers only require the combination of two elementary paths, as explained in the next section.

V. Four Symmetrical Clothoids

In [7], the concatenation of two different elementary paths is used to achieve any pose in the plane (x,y,θ). A particular case is when the start and end pose have the same inclination. In this situation, the final positions of the first elementary path are located in a straight line joining the start and end positions. In [15] is demonstrated that the minimum length of the clothoidal path is obtained forcing the joint point to be located on the intermediate point of the straight line. In this case, both elementary paths are equal. The required tangent angle to reach the intermediate position has been computed in the hereinbefore section. The control points of the second elementary path must be obtained by symmetry, as shown in Fig 6.

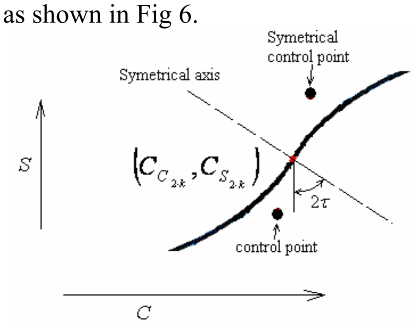

Fig. 6. Control point symmetry in a four clothoidal path.

The homotecial factor K, and then, the A parameter of the clothoids can be calculated as:

$$K = \sqrt{\pi} \cdot A = \frac{x_e}{C_{C_{4k}}} = \frac{y_e}{C_{S_{4k}}}$$

where $(C_{C_{4k}}, C_{S_{4k}})$ is the end control point of the four clothoidal path.

For instance, Fig. 7 shows four symmetrical clothoids to join the start pose (0,0,0) with the end pose (*200,5,0*). The resulting homotecial factor is K=369.56.

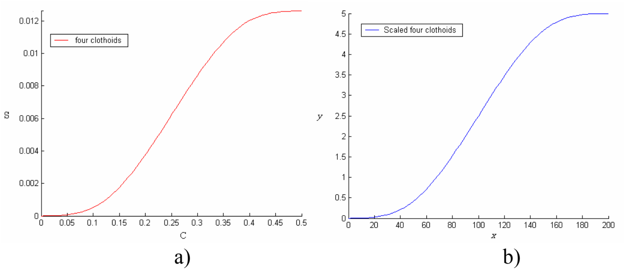

Fig. 7. Four-clothoids path: a) without scaling factor, b) with scaling factor.

A four-clothoids path is ideal for lane changing maneuvers because its smooth curvature guarantees comfort passengers. Besides, its easy and fast construction permits an online path planning. In this sense, a simulation application is explained in the next section.

VI. Non-Structured Simulation Environment

In this section, it is described a virtual environment employed for testing the goodness of the clothoidal trajectory generation explained in previous sections when avoiding obstacles in mobile robotics.

A commercial programming software called Dark Basic Professional® has been used for modeling the environment. It offers powerful specific commands to represent and move 3D objects and handle with control devices such as mouse, joysticks, steering wheels, etc. It also provides an interface between DirectX commands and the programmer.


Fig. 8. Simulation environment showing the AGV and an obstacle.

The simulation environment, already presented in [18], represents a typical four-lane highway where vehicles can travel in two directions, as depicted in Fig. 8.

A motorbike represents the autonomous vehicle and cars are used for simulating obstacles.

The simulation code has been designed to include clothoidal trajectory generation and collision avoidance algorithms as independent modules. The modular structure proposed is achieved by using Dynamic Link Libraries.

The diagram in Fig 9 shows the basic structure of the application. It represents the case of an autonomous vehicle following a predefined trajectory (straight line). When a vehicle occluding the predefined trajectory of the AGV is detected, the obstacle avoidance (OA) algorithm computes an intermediate non-collision position that prevents them from colliding. Once the new position is known the trajectory generation module computes the appropriate clothoidal path that leads the AGV to the non-collision position in the right time interval (established by the OA algorithm). As the environment represents an Intelligent Vehicle Highway System, the generated trajectory obtained will always be a lane changing maneuver, whose line width is a prestablished parameter of the OA algorithm.

The avoidance maneuver can be observed in the picture sequence shown in Fig. 10. It clarifies that the use of clothoids for trajectory generation produces continuous-in-curvature paths that guarantee the comfort of the passengers.

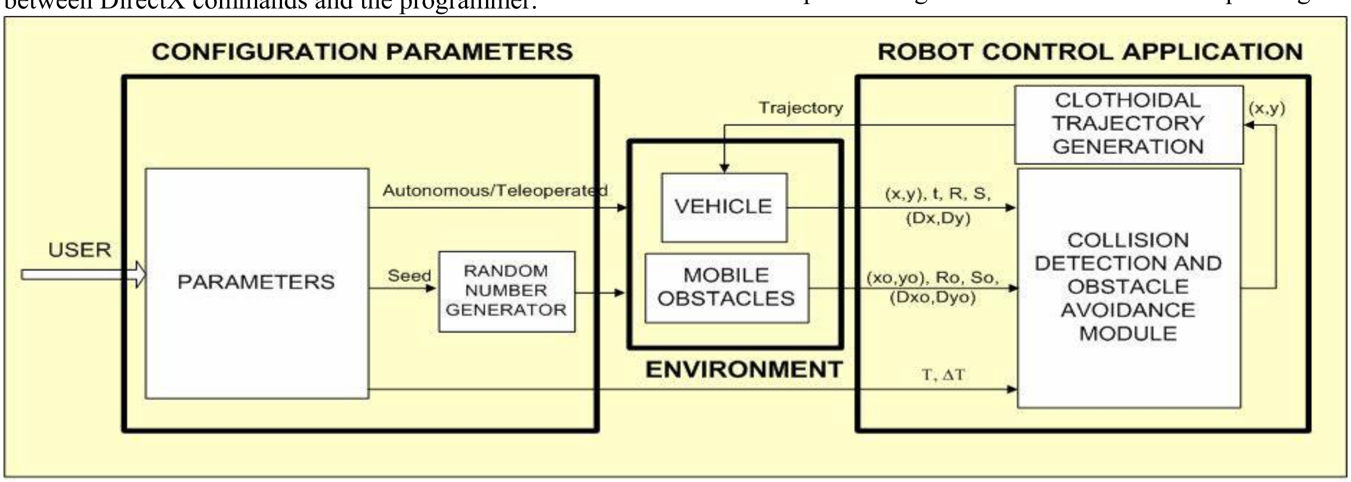

Fig. 9. Simulation application structure.

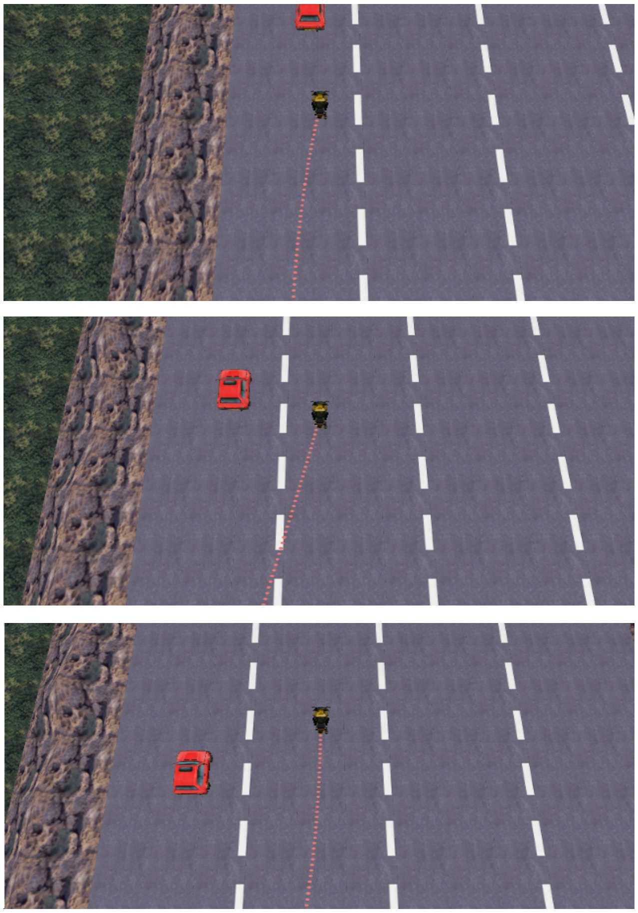

Fig. 10. Lane changing maneuver in obstacle avoidance (top view).

The OA algorithm is based on the computation of the Minimum Translational Distance (MTD) [17]. This avoidance algorithm uses spheres for distance computation as a way of reducing computational time. In particular, bi-spheres are considered for modeling the volume swept by each vehicle. The algorithm provides the MTD of separation (positive value) or penetration (negative value) between two objects, as well as the direction of possible collision.

VII. Conclusion

In this paper, an approximation of clothoids by RBCs has been presented and applied for trajectory generation in mobile robotics and vehicles. This approximation guarantees that the RBC has the same behaviour as a clothoid with lower order than other approaches.

Although clothoidal trajectory generation requires complicated computation of Fresnel integrals, in this paper, an on-line clothoidal path is obtained just by scaling a general parametric curve. Therefore, we have been able to develop a fast algorithm for generating accurate approximations of clothoidal paths. Its simplicity allows the implementation of fast modules for obstacle avoidance applications, 27 ms for a Pentium IV, 2.4 Ghz.

In particular, a four-clothoid path has been taken as ideal solution for lane changing maneuvers due to its smooth curvature that guarantees comfort passengers. In such situations, lane changing maneuvers only require the combination of two equal elementary paths.

In order to test the suitability of the clothoidal trajectory generation, a simulation environment for overtaking manoeuvres of vehicles has been used. When an obstacle is detected, the obstacle avoidance algorithm computes intermediate positions to compute the appropriate clothoidal path that leads to non-collision trajectories. Some videos of this methodology can be found in [19].

References

[1] I. Papadimitriou and M. Tomizuka, “Fast lane changing computations using polynomials”**.** *IEEE Proc. of the American Control Conference*. Vol.1. pp. 48-53. 2003.

[2] P. Jacobs and J. Canny, “Planning smooth paths for mobile robots,” *Proc. IEEE Int. Conf. Robotics and Automation*, vol.1, pp. 2-7, 1989.

[3] J.P. Laumond, J. Jacobs, M. Taix and M.R. Murray, “A motion planner for nonholonomic mobile robots”, *IEEE Trans. Robotics& Automation*, vol.10, issue 5, pp.577-593, 1994.

[4] T. Fraichard, A. Scheuer, and R. Desvigne, “From reeds and sheep´s to continuous curvature paths,” *IEEE Int. Conf. on Advanced Robotics*, pp 585-590, 1999.

[5] A. Scheuer and T. Fraichard, “Collision-free and continuous curvature path planning for car-like robots”, *IEEE Int. Conf. Rob. & Aut.*, 1997.

[6] A. Scheuer and T. Fraichard, “Continuous-curvature path planning for multiple car-like vehicles”, *IEEE Int. Conf. on Intelligent Robots& Systems*, vol.2. pp. 997-1003, 1997.

[7] A. Scheuer and T. Fraichard, “Planning continuous-curvature paths for car-like robots,” *IEEE Int. Conf. on Intelligent Robots and Systems*, vol.3, pp.1304-1311, 1996.

[8] N. Montes and J. Tornero, “Lane changing using s-series clothoidal approximation and dual-rate based on Bezier points for controlling vehicle”, *WSEAS Trans. on Circuits & Systems*, vol.3, issue 10, pp. 2285-2291, 2004.

[9] K. Fotiades and J. Seimenis, “Fast lane changing algorithm for intelligent vehicle highway systems using clothoidal theory and Bezier points,” *IEEE Proc.Intelligent Vehicles Symp.*,pp.73-77, 2005.

[10] I. de Corral Manuel de Villena. Topografía de obras. SPUPC 1999.

[11] K. D. Mielenz, “Computation of Fresnel Integrals II,” *J.of Research of the NIST*, vol. 105, nº4, pp.589, 2000.

[12] L.Z. Wang, K.T. Miura, E. Nakamae, T. Yamamoto and T.J. Wang. “An approximation approach of the clothoid curve defined in the interval [0, π/2] and its offset by free-form curves,” *Computer Aided Design*, vol. 33, nº 14, pp. 1049-1058(10), 2001.

[13] D.S. Meek and D.J. Walton, “An arc spline approximation to a clothoid,”*J. of Comp.l and App. Math.*,vol.170,(1),pp.59-77, 2004.

[14] J. Sánchez-Reyes and J.M. Chacón, “Polinomial approximation to clothoids via s-power series,” *CAD*, vol.35,nº14,pp. 1305-1319, 2003.

[15] Y. Kanayama and B.I. Hartman, “Smooth local path planing for autonomous vehicles,” I*EEE Proc. Int. Conf. on Robotics and Automation*, vol.3, pp.1265-1270, 1989.

[16] R. R. Barton, “Designing simulation experiments,” *in Proc. 33rd Conf. on Winter Simulation*, 2001, pp. 47–52.

[17] E.J. Bernabeu, J. Tornero and M. Tomizuka, “Collision prediction and avoidance amidst moving objects for trajectory planning applications,” *IEEE Press*, pp. 3801-3806, 2001.

[18] M. C. Mora and J. Tornero, “Non-structured simulation environment for collision detection and obstacle avoidance algorithm testing,” *Proc. of the Industrial Simulation Conference*, 2003, pp. 223-227.

[19] http://www.institutoidf.com/instituto/index.php?option=com_content &task=view&id=17&Itemid=28