Chapter 2
Mathematical Programming Problems with Complementarity Constraints

We study mathematical programming problems with complementarity constraints (MPCC) from the topological point of view. The (topological) stability of the MPCC feasible set is addressed. Therefore, we introduce Mangasarian-Fromovitz condition (MFC) and its stronger version (SMFC). Under SMFC, the MPCC feasible set is shown to be a Lipschitz manifold. The links to other well-known constraint qualifications for MPCCs are elaborated. The critical point theory for MPCCs is presented. We also characterize the strong stability of C-stationary points for MPCC, dealing with parametric aspects for MPCCs.

2.1 Applications and examples

We consider the mathematical programming problem with complementarity constraints (MPCC)

$$\text{MPCC:} \quad \min f(x) \text{ s.t. } x \in M[h,g,F_1,F_2] \tag{2.1}$$

with

$$\begin{aligned} M[h,g,F_1,F_2] := \{x \in \mathbb{R}^n \mid\ & F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0, \\ & F_{1,m}(x)F_{2,m}(x) = 0,\ m = 1,\ldots,k, \\ & h_i(x) = 0,\ i \in I,\ g_j(x) \geq 0,\ j \in J\}, \end{aligned}$$

where $h := (h_i,\ i \in I)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|I|})$, $g := (g_j,\ j \in J)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|J|})$, $F_1 := (F_{1,i},\ i = 1,\ldots,k)^T, F_2 := (F_{2,i},\ i = 1,\ldots,k)^T \in C^2(\mathbb{R}^n, \mathbb{R}^k)$, $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $k + |I| \leq n$, $|J| < \infty$. For simplicity, we write M for $M[h,g,F_1,F_2]$ if no confusion is possible.

For $m = 1,\ldots,k$, the constraint

$$F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0,\ F_{1,m}(x)F_{2,m}(x) = 0$$

is called a complementarity constraint. Note that it can be equivalently written as $\min\{F_{1,m}(x), F_{2,m}(x)\} = 0$.

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The MPCC is a special case of the so-called mathematical programming problem with equilibrium constraints (MPEC) (see [88]). In what follows, we show that MPCCs appear quite naturally in bilevel optimization (via Karush-Kuhn-Tucker or Fritz John conditions at the lower level) and when solving nonlinear complementarity problems. Moreover, complementarity constraints arise in the context of variational inequalities. For other applications, we refer the reader to [23, 88, 97].

Bilevel optimization with convexity at the lower level

We model the bilevel optimization problem in the so-called optimistic formulation. To this aim, assume that the follower solves the parametric optimization problem (lower-level problem L)

$$L(x): \quad \min_{y} g(x,y) \quad \text{s.t.} \quad h_j(x,y) \geq 0,\ j \in J$$

and that the leader's optimization problem (upper-level problem U) is

$$U: \quad \min_{(x,y)} f(x,y) \quad \text{s.t.} \quad y \in \text{Argmin}\, L(x).$$

Above we have $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and the real-valued mappings $f, g, h_j, j \in J$ belong to $C^2(\mathbb{R}^n \times \mathbb{R}^m)$, $|J| < \infty$. Argmin $L(x)$ denotes the solution set of the optimization problem $L(x)$. For simplicity, additional (in)equality constraints in defining U are omitted.

We assume convexity at the lower level $L(\cdot)$; that is, for all $x \in \mathbb{R}^n$, let the functions $g(x,\cdot)$, $-h_j(x,\cdot)$, $j \in J$ be convex. For example, the Slater constraint qualification (CQ) hold for $L(\cdot)$. Then, it is well-known that $y \in \text{Argmin}\, L(x)$ if and only if there exist Lagrange multipliers $\mu_j \in \mathbb{R}$, $j \in J$ such that

$$D_y g(x,y) = \sum_{j \in J} \mu_j D_y h_j(x,y),\ \mu_j \geq 0,\ h_j \geq 0,\ \mu_j h_j(x,y) = 0. \tag{2.2}$$

Hence, we can write the corresponding MPCC:

$$U\text{-KKT}: \quad \min_{(y,\mu) \in \mathbb{R}^m \times \mathbb{R}^{|J|}} g(x,y) \quad \text{s.t.}$$

$$D_y g(x,y) = \sum_{j \in J} \mu_j D_y h_j(x,y),\ \mu_j \geq 0, h_j \geq 0,\ \mu_j h_j(x,y) = 0.$$

Here, the complementarity constraints are $\mu_j \geq 0$, $h_j \geq 0$, $\mu_j h_j(x,y) = 0$.

The links between U and U-KKT were elaborated in [16]. It turns out that global solutions of U and U-KKT coincide. But, due to the possible non-uniqueness of Lagrange multipliers in (2.2), local solutions of U and U-KKT may differ.

Note that it is very restrictive to assume the Slater CQ in $L(x)$ for all $x \in \mathbb{R}^n$. Hence, one may try to assume the Slater CQ only at the point of interest $\bar{x}$. However, in that case even global solutions of U and U-KKT may differ (as shown in [16]).

Without assuming the Slater CQ, we arrive at the MPCC-relaxation of U:

$$U\text{-John}: \quad \min_{(y,\delta,\mu) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{|J|}} g(x,y) \quad \text{s.t.}$$

$$\delta D_y g(x,y) = \sum_{j \in J} \mu_j D_y h_j(x,y), \tag{2.3}$$

$$\mu_j D_y h_j(x,y),\ \mu_j \geq 0,\ h_j \geq 0,\ \mu_j h_j(x,y) = 0,\ \delta \geq 0.$$

Here, we use the fact that $y \in \operatorname{Argmin} L(x)$ fulfills the Fritz John condition. In fact, generically one cannot exclude the violation of the LICQ or even MFCQ at the lower level. Thus, the case of vanishing δ in (2.3) cannot be omitted (see Chapter 5 for details).

Solving nonlinear complementarity problems

We consider a nonlinear complementarity problem (NCP) of finding $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ F(x) \geq 0,\ x^T F(x) = 0,$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable. Such problems appear in many applications, such as equilibria models of economics, contact and structural mechanics problems, and obstacle problems (see also [98]).

Setting $H(x) := \min\{x, F(x)\}$ componentwise, we obtain a residual optimization problem

$$\text{RES}: \quad \min_x \vartheta(x) := \frac{1}{2} H(x)^T H(x) \text{ s.t. } x \geq 0.$$

Obviously, if $\bar{x}$ is a solution of an NCP, then $\bar{x}$ is a solution of RES with $\vartheta(\bar{x}) = 0$. Moreover, ϑ is nonnegative and vanishes exactly at solutions of the NCP.

With $y := x - \min\{x, F(x)\}$, it is easy to see that RES can be equivalently written as an MPCC:

$$\text{RES-MPCC}: \quad \min_{(x,y)} \frac{1}{2}(x-y)^T(x-y) \text{ s.t.}$$

$$x \geq 0,\ y \geq 0,\ F(x) - x - y \geq 0,\ y^T(F(x) - x - y) = 0.$$

This problem is used to solve an NCP numerically (see [88]).

Variational inequalities setting

Let $K \subset \mathbb{R}^n$ and $F : K \longrightarrow \mathbb{R}^n$ be given. The variational inequality $VI(K,F)$ is the following problem:

$$VI(K,F): \quad \text{Find } x \in \mathbb{R}^n \text{ such that } (y-x)^T F(x) \geq 0 \text{ for all } y \in K.$$

Clearly, $\bar{x}$ is a solution of $VI(K,F)$ if and only if

$$0 \in F(\bar{x}) + N(\bar{x}, K), \tag{2.4}$$

where $N(\bar{x}, K)$ is a normal cone of K at $\bar{x}$:

$$N(\bar{x}, K) := \left\{ d \in \mathbb{R}^n \,|\, d^T(\bar{x} - y) \leq 0 \text{ for all } y \in K \right\}.$$

Equation (2.4) can be seen as a generalization of the first-order optimality conditions to minimize a differentiable function $f : \mathbb{R}^n \longrightarrow R$ on a convex set K.

Furthermore, if K is a cone, we may link variational inequalities with so-called complementarity problems:

$$CP(K,F): \quad \text{Find } x \in \mathbb{R}^n \text{ such that } x \in K,\, F(x) \in K^\star,\, X^T F(x) = 0,$$

where $K^\star := \left\{ d \in \mathbb{R}^n \,|\, v^T d \geq 0 \text{ for all } v \in K \right\}$ is a dual cone of K.

It can be shown (see, e.g., [22]) that, in the case of K being a cone, solutions of $VI(K,F)$ and $CP(K,F)$ coincide. Moreover, let

$$K := \{ x \in \mathbb{R}^n \,|\, Ax \leq b,\, Cx = d \}$$

with matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{l \times n}$ and vectors $b \in \mathbb{R}^m$, $d \in \mathbb{R}^l$. Then, $\bar{x}$ solves $VI(K,F)$ if and only if there exist $\lambda \in \mathbb{R}^m$, $\mu \in R^l$ such that

$$F(x) + A^T \lambda + C^T \mu = 0,\, C - dx = 0,$$

$$\lambda \geq 0,\, b - Ax \geq 0,\, \lambda^T(b - Ax) = 0.$$

The latter system exhibits complementarity constraints and hence fits in the context of an MPCC.

Note that, setting $K := \mathbb{H}^n$ in $CP(K,F)$, we obtain the usual nonlinear complementarity problem.

2.2 Stability and structure of the feasible set

In this section, we concentrate only on the substantial new case of complementarity constraints. Hence, we omit smooth equality and inequality constraints and consider the mathematical programming problem with complementarity constraints (MPCC):

$$\text{MPCC:} \quad \min f(x) \text{ s.t. } x \in M[F_1, F_2] \tag{2.5}$$

with

$$M[F_1, F_2] := \{ x \in \mathbb{R}^n \,|\, F_1(x) \geq 0, F_2(x) \geq 0,\, F_1(x)^T F_2(x) = 0 \},$$

where $F_1 := (F_{1,i}, i = 1,\dots,k)^T, F_2 := (F_{2,i}, i = 1,\dots,k)^T \in C^1(\mathbb{R}^n,\mathbb{R}^k)$, and $f \in C^1(\mathbb{R}^n,\mathbb{R})$ with $k \le n$.

Note that $M[F_1,F_2]$ can be written as

$$M[F_1,F_2] = \{x \in \mathbb{R}^n \mid \min\{F_{1,i}(x), F_{2,i}(x)\} = 0,\, i = 1,\dots,k\}.$$

Here we deal with the local stability property of the feasible set $M[F_1,F_2]$ w.r.t. C^1-perturbations of the defining functions F_1 and F_2. Under a C^1-neighborhood of a function $g \in C^1(\mathbb{R}^n,\mathbb{R}^l)$, we understand a subset of $C^1(\mathbb{R}^n,\mathbb{R}^l)$ that contains for some $\varepsilon > 0$ the set

$$\left\{ \widetilde{g} \in C^1(\mathbb{R}^n,\mathbb{R}^l) \;\middle|\; \sum_{i=1}^{l} \sup_{x\in\mathbb{R}^n} (|\widetilde{g}_i(x) - g_i(x)| + \|\nabla\widetilde{g}_i(x) - \nabla g_i(x)\|) \le \varepsilon \right\}.$$

Definition 6. The feasible set $M[F_1,F_2]$ from (2.5) is called locally stable at $\bar{x} \in M[F_1,F_2]$ if there exists an $\mathbb{R}^n$-neighborhood V of $\bar{x}$ and a C^1-neighborhood U of (F_1,F_2) in $C^1(\mathbb{R}^n,\mathbb{R}^k) \times C^1(\mathbb{R}^n,\mathbb{R}^k)$ such that for every $(\widetilde{F}_1,\widetilde{F}_2) \in U$ the corresponding feasible set $M[\widetilde{F}_1,\widetilde{F}_2] \cap V$ is homeomorphic with $M[F_1,F_2] \cap V$.

Our main goal is to characterize the local stability property of the feasible set $M[F_1,F_2]$ in terms of the gradients of F_1 and F_2. In the case of standard nonlinear programming, (local) stability of the feasible set was studied in [32, 67] and is characterized by the MFCQ (see Theorem 4).

For stability in the MPCC setting we propose a kind of Mangasarian-Fromovitz condition (MFC) and its stronger version (SMFC). Section 2.2.1 will be devoted to the MFC and SMFC and their relations to other constraint qualifications (linear independence CQ, Mordukhovich's extremal principle, metric regularity, generalized Mangasarian-Fromovitz CQ, and standard subdifferential qualification condition). The conjectured equivalence of the MFC and SMFC is discussed. In Section 2.2.2, we prove that SMFC implies local stability and ensures that the MPCC feasible set is a Lipschitz manifold. Here, the application of nonsmooth versions of implicit function theorems (due to Clarke and Kummer) is crucial. We refer the reader to [19, 70] for details.

2.2.1 Constraint qualifications MFC and SMFC

Definitions of MFC and SMFC

Assume that Assumption A below holds throughout.

Assumption A *For every $\bar{x} \in M[F_1,F_2]$ and $i \in \{1,\dots,k\}$, the set of vectors*

$$\{\nabla F_{j,i}(\bar{x}) \mid F_{j,i}(\bar{x}) = 0, j = 1,2\}$$

is linearly independent.

Furthermore, we define for $\bar{x} \in M[F_1, F_2]$ and $i = 1, \ldots, k$ the (nonempty) convex hull

$$C_i(\bar{x}) := \operatorname{conv}\{\nabla F_{j,i}(\bar{x}) \,|\, F_{j,i}(\bar{x}) = 0\}.$$

Note that $C_i(\bar{x}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x})$ is Clarke's subdifferential of the function $\min\{F_{1,i}(\cdot), F_{2,i}(\cdot)\}$ (see [13]).

Definition 7 (MFC and SMFC). The Mangasarian-Fromovitz condition (MFC) is said to hold at $\bar{x} \in M[F_1, F_2]$ if any k vectors $(w_1, \ldots, w_k) \in C_1(\bar{x}) \times \cdots \times C_k(\bar{x})$ are linearly independent.

The Strong Mangasarian-Fromovitz condition (SMFC) is said to hold at $\bar{x} \in M[F_1, F_2]$ if there exists a k-dimensional linear subspace E of $\mathbb{R}^n$ such that any k vectors $(u_1, \ldots, u_k) \in P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x}))$ are linearly independent, where $P_E : \mathbb{R}^n \longrightarrow E$ denotes the orthogonal projection.

Remark 1. In the presence of additional C^1-equality and -inequality constraints in the description of the MPCC feasible set, the MFC will be enlarged by the standard MFCQ formulation with respect to these constraints.

We give some equivalent reformulations of the MFC and SMFC.

Lemma 1 (MFC and SMFC via Clarke's subdifferentials).

(a) *The MFC at $\bar{x} \in \mathbb{R}^n$ means that Clarke's subdifferentials*

$$\partial \min\{F_{1,i}, F_{2,i}\}(\bar{x}),\ i = 1, \ldots, k \text{ are linearly independent.}$$

(b) *The SMFC holds at $\bar{x} \in \mathbb{R}^n$ if and only if there exists a basis decomposition of $\mathbb{R}^n$ given by a nonsingular $n \times n$ matrix A such that after the linear coordinate transformation $y := Ax$ Clarke's subdifferentials of the functions $h_i(y) := \min\{F_{1,i}(A^{-1}(y)), F_{2,i}(A^{-1}(y))\}$ w.r.t. $z := (y_{n-k+1}, \ldots, y_n)$ are linearly independent, i.e.*

$$\partial_z h_i(\bar{y}),\ i = 1, \ldots, k \text{ are linearly independent,}$$

where $\partial_z h_i(\bar{y}) := \left\{\eta \in \mathbb{R}^k \,|\, \text{there exists } \xi \in \mathbb{R}^{n-k} \text{ with } [\xi, \eta] \in \partial h_i(\bar{y})\right\}.$

Proof. For (a), we only recall that $C_i(\bar{x}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x})$. To prove (b), we first calculate

$$\partial h_i(\bar{y}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x}) \cdot A^{-1} = C_i(\bar{x}) \cdot A^{-1}.$$

Hence, if the SMFC holds at $\bar{x}$, we take as columns of A^{-1} any orthogonal bases of $E^{\perp}$ and E. Conversely, given A, we set the linear subspace E to be spanned by the k last columns of A^{-1}.□

Remark 2 (SMFC as a maximal rank condition). From Lemma 1 (b), we see that the SMFC is the so-called maximal rank condition (in terms of Clarke [13]) w.r.t. some basis decomposition of $\mathbb{R}^n$. It turns out that the concrete choice of such a basis decomposition may affect the validity of the maximal rank condition (see Example 11 for details). This means that the property of maximal rank is not basis-independent. This observation is crucial and motivates the SMFC (see also Section 2.2.2).

Lemma 2 (MFC and SMFC via basis enlargement).

(a) *The MFC holds at $\bar{x} \in \mathbb{R}^n$ if and only if for any $w_i \in C_i(\bar{x}), i = 1, \dots, k$ there exist $\xi_1, \dots, \xi_{n-k} \in \mathbb{R}^n$ such that the vectors $w_1, \dots, w_k, \xi_1, \dots, \xi_{n-k}$ are linearly independent.*

(b) *The SMFC holds at $\bar{x} \in \mathbb{R}^n$ if and only if there exist $\xi_1, \dots, \xi_{n-k} \in \mathbb{R}^n$ such that for any $w_i \in C_i(\bar{x}), i = 1, \dots, k$ the vectors $w_1, \dots, w_k, \xi_1, \dots, \xi_{n-k}$ are linearly independent.*

Proof. The proof of (a) follows immediately from the definition of linear independence. To prove (b), if the SMFC holds, we choose $\xi_1, \dots, \xi_{n-k}$ as basis of $E^{\perp}$. Conversely, we set $E := (\text{span}\{\xi_1, \dots, \xi_{n-k}\})^{\perp}$ in the SMFC. □

Furthermore, we notice that the MFC is a natural constraint qualification for the Clarke stationarity.

Definition 8 (Clarke stationarity; see [22, 105]). A point $\bar{x} \in M[F_1, F_2]$ is called Clarke stationary (C-stationary) if there exist real numbers $\lambda_{j,i}$, $j = 1, 2$, $i = 1, \dots, k$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{k} (\lambda_{1,i} \nabla F_{1,i}(\bar{x}) + \lambda_{2,i} \nabla F_{2,i}(\bar{x})) = 0,$$

$$F_{j,i}(\bar{x}) \lambda_{j,i} = 0 \text{ for every } j = 1, 2,\ i = 1, \dots, k,$$

$$\lambda_{1,i} \lambda_{2,i} \geq 0 \text{ for every } i \in \{1, \dots, k\} \text{ with } F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0.$$

Proposition 2 (MFC and C-stationarity). *If $\bar{x}$ is a local minimizer of the MPCC and MFC holds at $\bar{x}$, then $\bar{x}$ is C-stationary.*

Proof. Due to Lemma 1 in [105], if $\bar{x}$ is a local minimizer of the MPCC, then there exist real numbers λ, $\lambda_{j,i}$, $j = 1, 2$, $i = 1, \dots, k$ (not all vanishing) such that

$$\lambda \nabla f(\bar{x}) + \sum_{i=1}^{k} (\lambda_{1,i} \nabla F_{1,i}(\bar{x}) + \lambda_{2,i} \nabla F_{2,i}(\bar{x})) = 0,$$

$$F_{j,i}(\bar{x}) \lambda_{j,i} = 0 \text{ for every } j = 1, 2,\ i = 1, \dots, k,$$

and

$$\lambda_{1,i} \lambda_{2,i} \geq 0 \text{ for every } i \in \{1, \dots, k\} \text{ with } F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0.$$

Clearly, if $\lambda = 0$, then the MFC is violated at $\bar{x}$. Hence, $\bar{x}$ is C-stationary.□

For more details on C-stationarity and other stationarity concepts, such as W-, A-, M-, and S-stationarity, see [24], [88], [96], [105], [118], and Sections 2.2.1 and 2.4.

Conceptional relations to other CQ

We recall the well-known **LICQ for the MPCC** (e.g., [105, 106]), which is said to hold at $\bar{x} \in M[F_1, F_2]$ if

$$\{\nabla F_{i,j}(\bar{x}) \,|\, F_{i,j}(\bar{x}) = 0,\ i = 1,\dots,k,\ j = 1,2\} \text{ are linearly independent.}$$

The LICQ can be equivalently formulated in terms of the transversal intersection of stratified sets (see [63]). As shown in [106], the LICQ is a generic constraint qualification. However, the LICQ is not necessary for local stability, as one can see from Example 5. In this and all further examples, only the local stability in 0 is of interest.

Example 5 (2D, stable: one point $\longrightarrow$ one point).

The set $M^5 := \{(x,y) \in \mathbb{R}^2 \,|\, \min\{x,y\} = 0, \min\{x-y, 2x-y\} = 0\}$ is a singleton and is locally stable at 0 (see Figure 7). However, the LICQ does not hold at 0.

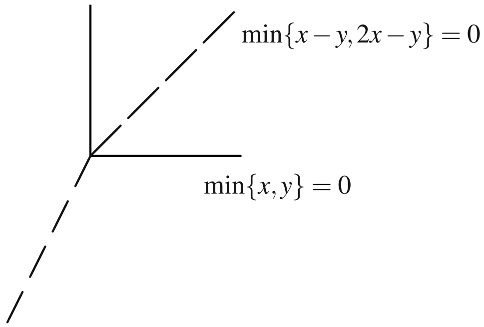

Figure 7 Illustration of Example 5

In this sense, the LICQ appears to be too restrictive. This comes from the fact that the LICQ does not impose the combinatorial structure of the complementarity constraints. Additionally, we notice that the LICQ implies the MFC.

Another condition we intend to discuss comes from the **exact Mordukhovich extremal principle** (see [25, 94]).

Let $\Omega \subset \mathbb{R}^n$ be any arbitrary closed set and $\bar{x} \in \Omega$. The nonempty cone

$$T(\bar{x}, \Omega) := \limsup_{\tau \searrow 0} \frac{\Omega - \bar{x}}{\tau}$$

$$= \left\{ d \in \mathbb{R}^n \,|\, \text{there exist } x_k \longrightarrow \bar{x},\, x_k \in \Omega,\ \tau_k \searrow 0 \text{ such that } \frac{x_k - \bar{x}}{\tau_k} \longrightarrow d \right\}$$

is called the contingent (also Bouligand or tangent) cone to Ω at x.

The Fréchet normal cone is defined via polarization as

$$\hat{N}(\bar{x}, \Omega) := (T(\bar{x}, \Omega))^{\circ}.$$

Finally, the limiting normal cone (also called the Mordukhovich normal cone) is defined by

$$N(\bar{x}, \Omega) := \limsup_{x' \xrightarrow{\Omega} \bar{x}} \hat{N}(x', \Omega)$$

$$= \left\{ \lim_{k\to\infty} w_k \,|\, \text{there exist } x_k \longrightarrow \bar{x},\, x_k \in \Omega,\, w_k \in \hat{N}(x_k,\Omega) \right\}.$$

Definition 9 (local extremal point of set systems; [94]). Let Ω_i, $i = 1,\dots,k$ be nonempty subsets of $\mathbb{R}^n$ and $\bar{x} \in \bigcap_{i=1}^{k} \Omega_i$. We say that $\bar{x}$ is a local extremal point of the set system $\{\Omega_1,\dots,\Omega_k\}$ if there are sequences $\{a_{ij}\} \subset \mathbb{R}^n$, $i = 1,\dots,k$, and a neighborhood V of $\bar{x}$ such that $a_{ij} \longrightarrow 0$ as $j \longrightarrow \infty$ and

$$\bigcap_{i=1}^{k} (\Omega_i - a_{ij}) \cap V = \emptyset \text{ for all large } j \in \mathbb{N}.$$

We recall the finite-dimensional version of the exact Mordukhovich extremal principle.

Theorem 11 (Exact extremal principle in finite dimensions; [94]). *Let Ω_i, $i = 1,\dots,k$ be nonempty closed subsets of $\mathbb{R}^n$ and $\bar{x} \in \bigcap_{i=1}^{k} \Omega_i$ be an extremal point of the set system $\{\Omega_1,\dots,\Omega_k\}$. Then there are $x_i^* \in N(\bar{x},\Omega_i)$, $i = 1,\dots,k$ (not all vanishing) such that $\sum_{i=1}^{k} x_i^* = 0$.*

Actually, Theorem 11 provides a sufficient condition for the property that the intersection of nonempty closed subsets Ω_i, $i = 1,\dots,k$ of $\mathbb{R}^n$ remains locally nonempty with respect to translations. This sufficient condition can be formulated as follows:

$(\triangle)$ For all $x_i^* \in N(\bar{x},\Omega_i)$, $i = 1,\dots,k$, $\sum_{i=1}^{k} x_i^* = 0$ implies $x_i^* = 0$, $i = 1,\dots,k$.

In order to refer to the foregoing discussion in our setting, from now on we set $\Omega_i := M_i$, $i = 1,\dots,k$, where

$$M_i := \{x \in \mathbb{R}^n \,|\, F_{1,i}(x) \geq 0, F_{2,i}(x) \geq 0, F_{1,i}(x)F_{2,i}(x) = 0\}.$$

Proposition 3 (MFC implies $\triangle$). *If the MFC holds at $\bar{x} \in M[F_1,F_2]$, then $\triangle$ also holds at $\bar{x}$.*

Proof. Let $i \in \{1,\dots,k\}$ be fixed. We provide a representation formula for $N(\bar{x},M_i)$. We restrict ourselves to the interesting case that $F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0$. Due to Assumption A, we choose vectors $\xi_1,\dots,\xi_{n-2} \in \mathbb{R}^n$, which form — together with the vectors $\nabla F_{1,i}(\bar{x}), \nabla F_{2,i}(\bar{x})$ — a basis for $\mathbb{R}^n$. Next we put $y = \Phi(x)$ as follows:

$$y_1 := F_{1,i}(x),\; y_2 := F_{2,i}(x),\; y_3 := \xi_1^T(x-\bar{x}),\dots,\; y_n := \xi_{n-2}^T(x-\bar{x}).$$

Note that $\Phi(\bar{x}) = 0$ and $D\Phi(\bar{x})$ is nonsingular. Therefore, Φ maps M_i diffeomorphically to $K := \{y \in \mathbb{R}^n \,|\, y_1 \geq 0, y_2 \geq 0, y_1 y_2 = 0\}$ locally at $\bar{x}$. Setting $L := \{y \in \mathbb{R}^2 \,|\, y_1 \geq 0, y_2 \geq 0, y_1 y_2 = 0\}$, Proposition 6.41 from [104] yields

$$N(0,K) = N(0, L \times \mathbb{R}^{n-2}) = N(0,L) \times N(0, \mathbb{R}^{n-2}).$$

From [25] and [96], we conclude that $N(0,L) = \mathbb{R}^2_- \cup L$. Clearly, $N(0,\mathbb{R}^{n-2}) = \{0_{n-2}\}$. Altogether, we get

$$N(0,K) = \mathbb{R}^2_- \cup L \times \{0_{n-2}\}.$$

Using Exercise 6.7 (change of coordinates) from [104], we get

$$N(\bar{x}, M_i) = \{\beta_1 \nabla F_{1,i}(\bar{x}) + \beta_2 \nabla F_{2,i}(\bar{x}) \mid \text{ either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1\beta_2 = 0\}. \quad (2.6)$$

Analogously, we obtain

$$\widehat{N}(\bar{x}, M_i) = \{\beta_1 \nabla F_{1,i}(\bar{x}) + \beta_2 \nabla F_{2,i}(\bar{x}) \mid \beta_1 \le 0, \beta_2 \le 0\}. \quad (2.7)$$

The representation (2.7) yields that the MFC is equivalent to the following condition:

$$\text{For all } x_i^* \in \pm\widehat{N}(\bar{x}, M_i),\ i = 1,\dots,k,\ \sum_{i=1}^{k} x_i^* = 0 \text{ implies } x_i^* = 0,\ i = 1,\dots,k.$$

Since $N(\bar{x}, M_i) \subset \pm\widehat{N}(\bar{x}, M_i)$, $i = 1,\dots,k$, (see (2.6) and (2.7)), the proposition follows immediately. □

Corollary 1 (MFC via Fréchet normal cones). *MFC is equivalent to the following condition:*

$$\textit{For all } x_i^* \in \pm\widehat{N}(\bar{x}, M_i),\ i = 1,\dots,k,\ \sum_{i=1}^{k} x_i^* = 0 \textit{ implies } x_i^* = 0,\ i = 1,\dots,k,$$

where $M_i = \{x \in \mathbb{R}^n \mid F_{1,i}(x) \ge 0, F_{2,i}(x) \ge 0, F_{1,i}(x)F_{2,i}(x) = 0\}$.

As we show by Example 6, $\triangle$ is not sufficient for $M[F_1,F_2]$ to be locally stable at 0. In this and all further examples in $3D$, we understand under "two-star", "three-star" and "four-star" subsets of $\mathbb{R}^3$ as depicted in Figure 8 up to a homeomorphism.

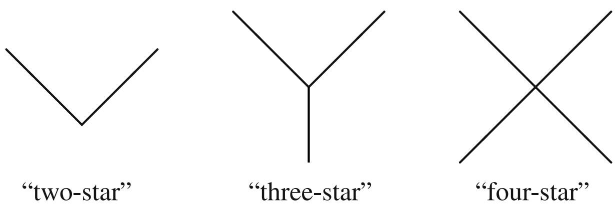

Figure 8

Example 6 (3D, nonstable: "four-star" $\longrightarrow$ 2 "two-stars").

Consider the "four-star" subset

$$M^6 := \{(x,y,z) \in \mathbb{R}^3 \mid \min\{x,y\} = 0, \min\{x+y-\sqrt{2}z, x+y+\sqrt{2}z\} = 0\}$$

(see Figure 9). After an appropriate perturbation the resulting set would have two path-connected components. Therefore, M^6 is not locally stable at 0.

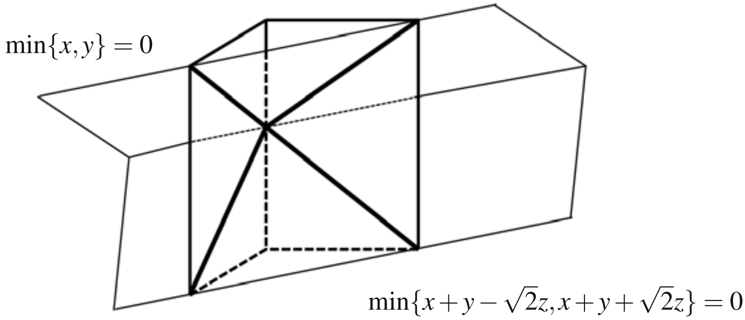

Figure 9 Illustration of Example 6

To show that $\triangle$ holds at 0, we set

$$M_1^6 := \{(x,y,z) \in \mathbb{R}^3 \mid \min\{x,y\} = 0\},$$

$$M_2^6 := \{(x,y,z) \in \mathbb{R}^3 \mid \min\{x+y-\sqrt{2}z, x+y+\sqrt{2}z\} = 0\},$$

and obtain due to (2.6) from the proof of Proposition 3

$$N(0,M_1^6) = \{(\beta_1,\beta_2,0)^T \in \mathbb{R}^3 \mid \text{ either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1\beta_2 = 0\}.$$

$$N(0,M_2^6) = \left\{ \beta_1 \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \;\middle|\; \text{ either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1\beta_2 = 0 \right\}.$$

From the representations of $N(0,M_1^6)$ and $N(0,M_2^6)$, it is easy to see that $\triangle$ (but not MFC) is satisfied at $0 \in M^6$.

The next stability concept we would like to discuss here is **metric regularity**. We recall that a multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ is called metrically regular (with rank $L > 0$) at $(\bar{x},\bar{y}) \in \operatorname{gph} T$ if, for certain neighborhoods U and V of $\bar{x}$ and $\bar{y}$, respectively, it holds that

$$\operatorname{dist}(x, T^{-1}(y)) \le L \operatorname{dist}(y, T(x)) \text{ for all } x \in U, y \in V.$$

Furthermore, a multivalued map $S : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is called pseudo-Lipschitz (with rank $L > 0$) at $(\bar{y},\bar{x}) \in \operatorname{gph} S$ if there are neighborhoods U and V of $\bar{x}$ and $\bar{y}$, respectively, such that, given any points $(y,x) \in (V \times U) \cap \operatorname{gph} S$, it holds that

$$\operatorname{dist}(x, S(y')) \le L\|y' - y\| \text{ for all } y' \in V$$

(see, e.g., [49], [81]).

It holds (see [48]) that T is metrically regular at $(\bar{x},\bar{y}) \in \operatorname{gph} T$ if and only if T^{-1} is pseudo-Lipschitz at $(\bar{y},\bar{x})$.

It is well-known from [100] that the solution map

$$S(y,z) := \{x \in \mathbb{R}^n \,|\, h(x) = y, g(x) \le z\},\ (g,h) \in C^1(\mathbb{R}^n, \mathbb{R}^{k+m}),$$

is pseudo-Lipschitz at $(0,0,\bar{x})$ if and only if the MFCQ is satisfied at $\bar{x} \in S(0,0)$. This means that the local stability of $M_{NLP}[h,g]$ $(= S(0,0))$ at $\bar{x} \in M_{NLP}[h,g]$ is equivalent to the metric regularity of

$$S^{-1}(x) = \{(h(x),z) \,|\, g(x) \le z\}$$

at $(\bar{x},0,0)$.

To apply this idea in our setting, we say that the metric regularity condition (MRC) holds at $\bar{x} \in M[F_1,F_2]$ if and only if

$$G : \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R}^k, \\ \ x \ \ \mapsto \ (\min\{F_{1,i}(x), F_{2,i}(x)\})_{i=1,\dots,k}, \end{cases}$$

is metrically regular at $(\bar{x},0)$.

The next proposition can be derived with the aid of Proposition 3.3 in [52]. For the sake of completeness, we present its proof.

Proposition 4 (MRC is equivalent to $\triangle$). *MRC holds at $\bar{x} \in M[F_1,F_2]$ if and only if $\triangle$ holds at $\bar{x}$.*

Proof. The MRC holds at $\bar{x} \in M[F_1,F_2]$ if and only if the solution map $S(y) := \{x \in \mathbb{R}^n \,|\, G(x) = y\}$, $y \in \mathbb{R}^k$, is pseudo-Lipschitz at $(0,\bar{x})$. Setting

$$F : \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R}^{2k}, \\ \ x \ \ \mapsto \ (F_{1,i}(x), F_{2,i}(x))_{i=1,\dots,k}, \end{cases}$$

and $D_i := \{(a_i,b_i) \in R^2 \,|\, a_i \ge 0,\, b_i \ge 0,\, ab = 0\}$, $i = 1,\dots,k$, we obtain:

$$S(y) = \{x \in \mathbb{R}^n \,|\, F(x) - y \in D_1 \times \cdots \times D_k\},$$

$$S^{-1}(x) = F(x) - D_1 \times \cdots \times D_k.$$

Therefore, the MRC holds at $\bar{x} \in M[F_1,F_2]$ if and only if $F(\cdot) - D_1 \times \cdots \times D_k$ is metrically regular at $(\bar{x},0)$. Since $F \in C^1(\mathbb{R}^n, \mathbb{R}^{2k})$ and $D_1 \times \cdots \times D_k$ is closed, we can apply Example 9.44 from [104]. Due to that example the constraint qualification

$$u \in N(F(\bar{x}), D_1 \times \cdots \times D_k),\ \nabla^T F(\bar{x})u = 0 \Longrightarrow u = 0, \tag{2.8}$$

is equivalent to the metric regularity of $F(\cdot) - D_1 \times \cdots \times D_k$ at $(\bar{x},0)$. Since

$$N(F(\bar{x}), D_1 \times \cdots \times D_k) = N(F_{1,1}(\bar{x}), F_{2,1}(\bar{x}), D_1) \times \cdots \times N(F_{1,k}(\bar{x}), F_{2,k}(\bar{x}), D_k)$$

and

$$N(0,D_i) = \mathbb{R}^2_- \cup D_i,$$

formula (2.6) allows to conclude that the constraint qualification (2.8) is equivalent to $\triangle$.$\square$

We mention some valuable remarks on the previously discussed constraint qualifications (pointed out by an anonymous referee).

Remark 3 (Standard subdifferential qualification condition $\triangle$). $\triangle$ is the standard subdifferential qualification condition for the system Ω_i, $i = 1,\dots,k$ at $\bar{x} \in \bigcap_{i=1}^{k} \Omega_i$ (see [52, 94, 104]). Moreover, $\triangle$ means that the multivalued map $M(z) := \{x \in \mathbb{R}^n \,|\, x + z_i \in \Omega_i,\, i = 1,\dots,k\}, z = (z_1,\dots,z_k) \in \mathbb{R}^{nk}$, is pseudo-Lipschitz (has the Aubin property) at $(0,\dots,0,\bar{x})$. This means that its inverse, $M^{-1}(x) := (\Omega_1 - x) \times \dots \times (\Omega_1 - x), x \in \mathbb{R}^n$, is metrically regular at $(\bar{x},0,\dots,0)$ (e.g., Proposition 3.3 in [52]).

Remark 4 (MFC and GMFCQ). The generalized Mangasarian-Fromovitz constraint qualification (GMFCQ) can be related to the MFC. Indeed, the GMFCQ for the constraint set $M = \{x \in \mathbb{R}^n \,|\, F(x) \in D_1 \times \cdots \times D_k\}$ is exactly (2.8) (see [52]). Thus, it is clear from the proof of Proposition 4 that $\triangle$ is equivalent to the GMFCQ. Hence, the MFC implies the MRC, as well as the GMFCQ. Moreover, Example 6 shows that neither the MRC nor the GMFCQ is sufficient for $M[F_1,F_2]$ being locally stable.

Remark 5 (Constraint qualifications for M-stationarity). It is well-known that under $\triangle$ (or, equivalently, the MRC and GMFCQ) a local minimum for (2.5) is M-stationary. This means that in addition to C-stationarity in Definition 8 it holds that either $\lambda_{1,i}, \lambda_{2,i} < 0$ or $\lambda_{1,i}\lambda_{2,i} = 0$ for every $i \in \{1,\dots,k\}$ with $F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0$.

On equivalence of MFC and SMFC

It is clear that the SMFC implies the MFC. Moreover, these two conditions coincide for $n = k$. The question of whether the SMFC is equivalent to the MFC in general is highly nontrivial.

First, we show that the SMFC implies the MFC at least in the cases where $k = 2$ or LICQ is fulfilled.

This follows mainly from the (linear-algebraic) Lemma 3.

Lemma 3. *Let $C_i := conv\{v_{j,i} \in \mathbb{R}^n \,|\, j = 1,2\}$, $i = 1,\dots,k$, and for every $i \in \{1,\dots,k\}$ let $v_{1,i}, v_{2,i}$ be linearly independent. Let assertions (A) and (B) be given as follows:*

- (A) *Any k vectors $(w_1,\dots,w_k) \in C_1 \times \cdots \times C_k$ are linearly independent.*
- (B) *There exists a k-dimensional linear subspace E of $\mathbb{R}^n$ such that any k vectors $(u_1,\dots,u_k) \in P_E(C_1) \times \cdots \times P_E(C_k)$ are linearly independent, where $P_E : \mathbb{R}^n \longrightarrow E$ denotes the orthogonal projection.*

Then, (A) and (B) are equivalent in the following cases:

1) *The vectors $v_{j,i}$, $j = 1,2$, $i = 1,\dots,k$ are linearly independent.*
2) $k = 2$.

Proof. The nontrivial part is to prove that (A) implies (B) for $n > k$. First, we claim that (B) is equivalent to the following condition (C) (see Lemma 2):

(C) There exist $\xi_1,\dots,\xi_{n-k} \in \mathbb{R}^n$ such that for any $w_i \in C_i, i = 1,\dots,k$ the vectors $w_1,\dots,w_k,\xi_1,\dots,\xi_{n-k}$ are linearly independent.

Indeed, if (B) holds, we choose $\xi_1,\dots,\xi_{n-k}$ as a basis of $E^\perp$ in (C). If (C) holds, we set $E := (\operatorname{span}\{\xi_1,\dots,\xi_{n-k}\})^\perp$ in (B).

Case 1: the vectors $\mathbf{v_{j,i}}$, $\mathbf{j = 1,2}$, $\mathbf{i = 1,\dots,k}$ are linearly independent.

Then, $n \geq 2k$ and $v_{j,i}$, $j = 1,2$, $i = 1,\dots,k$ span a $2k$-dimensional linear subspace of $\mathbb{R}^n$. Hence, w.l.o.g., we may assume that $n = 2k$.

Define a linear coordinate transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as

$$L(v_{1,i}) = e_{2i-1} + e_{2i}, L(v_{2,i}) = e_{2i-1}, i = 1,\dots,k,$$

whereby e_m denotes the m-th standard basis vector for $1 \leq m \leq n$. It holds that $L(C_i) = \{e_{2i-1} + \lambda_i e_{2i} \,|\, \lambda_i \in [0,1]\}$, $i = 1,\dots,k$.

Setting $T := \operatorname{span}\{e_{2i-1}, i = 1,\dots,k\}$, we obviously obtain that

$(\star)$ any k vectors $(v_1,\dots,v_k) \in P_T(L(C_1)) \times \dots \times P_T(L(C_k))$ are linearly independent, where $P_T : \mathbb{R}^n \longrightarrow T$ denotes the orthogonal projection.

As above, $(\star)$ is equivalent to the following condition:

$(\star\star)$ There exist $\gamma_1,\dots,\gamma_{n-k} \in \mathbb{R}^n$ such that for any $v_i \in L(C_i), i = 1,\dots,k$, the vectors $v_1,\dots,v_k,\gamma_1,\dots,\gamma_{n-k}$ are linearly independent.

Setting $\xi_i := L^{-1}(\gamma_i), i = 1,\dots,n-k$, we conclude that (C) is fulfilled due to $(\star\star)$. Thus, B is proved.

Case 2: $k = 2$.

It is clear that the vectors $v_{j,i}$, $j = 1,2$, $i = 1,2$ span at most a *four*-dimensional linear subspace S of $\mathbb{R}^n$ and hence $\dim S \leq 4$. If $\dim S = 4$, then (B) holds as in Case 2. If $\dim S < 4$, we may assume w.l.o.g. that $n = 3$.

For $a \in \{-1,1\}^2$, we set $K_a := \operatorname{cone}\{a_i v_{1,i}, a_i v_{2,i} \,|\, i = 1,2\}$. From the theorem about alternatives (e.g., [102]), we claim that (A) is equivalent to the following condition:

$$\operatorname{int}(K_a^\circ) \neq \emptyset \text{ for all } a \in \{-1,1\}^2.$$

Here, $\operatorname{int}(K_a^\circ)$ denotes the interior of the polar cone of K_a.

Due to this fact, K_a properly lies in a half-space for all $a \in \{-1,1\}^2$. Setting $\{-1,1\}^2 =: \{a^1,-a^1,a^2,-a^2\}$, we can strictly separate K_{a^l} and K_{-a^l} by a plane $\beta_l \ni 0$, $l = 1,2$. Since $0 \in \beta_1 \cap \beta_2$, there exists $\xi \in \beta_1 \cap \beta_2$, $\xi \neq 0$, such that $\xi \notin \bigcup\limits_{a \in \{-1,1\}^2} K_a$ by construction. This means that (C) is fulfilled. Thus, (B) is proved. $\square$

Theorem 12 (MFC implies SMFC for $k = 2$ and under LICQ). *Let $k = 2$ or the LICQ be fulfilled. Then, the SMFC is equivalent to the MFC.*

Proof. It is straightforward to see that the conclusion can be obtained by applying Lemma 3. We have to adjust the proof of Lemma 3 only for the case where only one constraint in $\min\{F_{1,i}(\bar{x}), F_{2,i}(\bar{x})\} = 0$ is active (i.e., $F_{1,i}(\bar{x}) = 0, F_{2,i}(\bar{x}) > 0$, or vice versa). For that we define C_i from Lemma 3 just to be $C_i(\bar{x})$. The respective change in the proof of Lemma 3 is straightforward. In fact, only the so-called biactive set of constraints is crucial (see [88], [118]). □

Remark 6 (MFC implies SMFC for $k = 3$; Rückmann, personal communication). Recently it was proven that the MFC implies the SMFC when $k = 3$. The proof uses a kind of dual description of the SMFC and MFC.

In what follows, we discuss the difficulties by proving that the MFC implies the SMFC for general n and k. These difficulties arise not so much because of linear-algebraic issues but rather because of combinatorial and topological matters of the problem. In fact, using the notation from Lemma 3, we set for $a \in \{-1,1\}^k$

$$K_a := \text{cone}\{a_i v_{1,i}, a_i v_{2,i} \,|\, i = 1,\dots,k\}.$$

Condition (A) means that all cones K_a are pointed; that is,

$$\text{if } x_1 + \cdots + x_p = 0,\, x_s \in K_a,\, s = 1,\dots,p, \text{ then } x_s = 0 \text{ for all } s = 1,\dots,p.$$

Condition (B) means that there exist $n-k$ linearly independent vectors $\xi_1,\dots,\xi_{n-k} \in \mathbb{R}^n$ such that

$$\xi_j \notin \bigcup_{a\in\{-1,1\}^k} K_a \text{ for all } j = 1,\dots,n-k.$$

Thus, for proving "(A) implies (B)", we need to show that (for all k and n)

$$\bigcup_{a\in\{-1,1\}^k} K_a \neq \mathbb{R}^n. \tag{2.9}$$

Here, we deal with a union of pointed cones with the additional property that

$$K_{-a} = -K_a \text{ for all } a \in \{-1,1\}^k.$$

Moreover, 2^k — the number of these cones — grows exponentially in k. It is clear that for proving (2.9) topological properties of $\bigcup_{a\in\{-1,1\}^k} K_a$ (such as, for example, Euler characteristic) are crucial.

We conclude by noting that the conjectured equivalence of the MFC and SMFC is very sophisticated and is a topic of current research.

2.2.2 SMFC implies stability and Lipschitz manifold

We intend to prove that the SMFC implies local stability of the feasible set $M[F_1,F_2]$ (see Theorem 15). The main idea is to show that under the SMFC $M[F_1,F_2]$ appears to be an $(n-k)$-dimensional Lipschitz manifold (the Corollary 2 and Definition 1).

Guiding examples

First, we briefly mention two- and three-dimensional examples with two linear constraints. These examples illustrate which phenomena might occur in general. They mainly highlight the possibilities arising with respect to the stability property of the feasible set $M[F_1,F_2]$ in low dimensions.

Example 7 (2D, nonstable: one point $\longrightarrow$ empty, two points). The set $M^7 := \{(x,y) \in \mathbb{R}^2 \mid \min\{x,y\} = 0, \min\{-x,-y\} = 0\}$ is a singleton (see Figure 10). Note that MFC is not satisfied at 0. After an appropriate perturbation M^7 either becomes empty or contains at least two points.

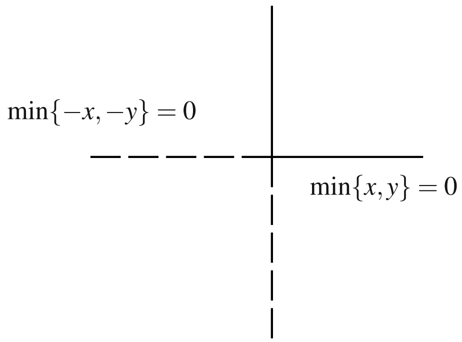

Figure 10 Illustration of Example 7

Example 8 (2D, nonstable: one point $\longrightarrow$ two points). The set $M^8 := \{(x,y) \in \mathbb{R}^2 \mid \min\{x,y\} = 0, \min\{-x+y, x+y\} = 0\}$ is a singleton (see Figure 11). Note that the MFC is not satisfied at 0. After an appropriate perturbation, M^8 contains at least two points.

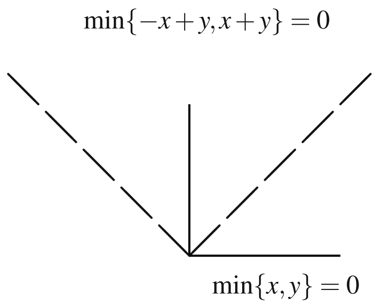

Figure 11 Illustration of Example 8

Example 9 (3D, nonstable: "three-star" $\longrightarrow$ 1 or 2 "two-stars"). The set $M^9 := \{(x,y,z) \in \mathbb{R}^3 \mid \min\{x,y\}=0, \min\{y-z,y+z\}=0\}$ is a "three-star" (see Figure 12). Note that the MFC is not satisfied at 0. After an appropriate perturbation, M^9 either has two path-connected components or is a "two-star".

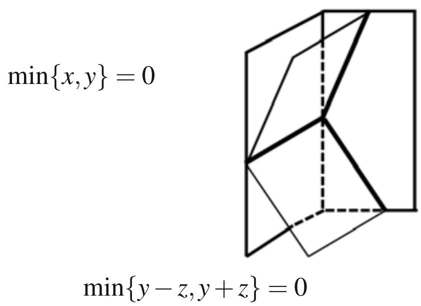

Figure 12 Illustration of Example 9

Example 10 (3D, stable: "two-star" $\longrightarrow$ "two-star"). The set $M^{10} := \{(x,y,z) \in \mathbb{R}^3 \mid \min\{x,y\}=0, \min\{x-y+z,-x+y+z\}=0\}$ is a "two-star"(see Figure 13). Note that the MFC holds at 0. After any sufficiently small perturbation, M^{10} remains to be a "two-star".

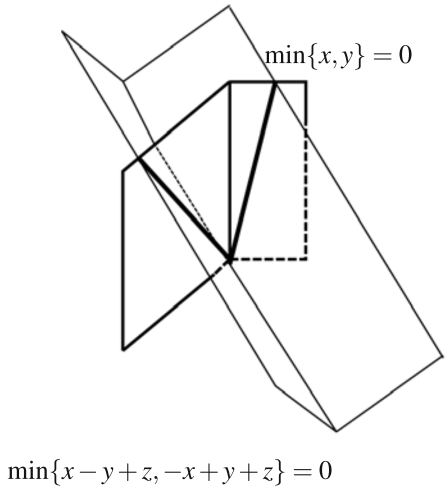

Figure 13 Illustration of Example 10

It is easy to see that in all these examples the MFC holds at 0 if and only if the corresponding feasible set is locally stable. Moreover, these examples emphasize that the locally stable case corresponds to a feasible set being a Lipschitz manifold (see Corollary 2 below).

Main results via Clarke's implicit function theorem

We recall briefly the notion of Clarke's generalized Jacobian and the corresponding inverse and implicit function theorems (see [13] and Section B.1).

For a vector-valued function $G = (g_1, \ldots, g_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ with g_i being Lipschitz near $\bar{x} \in \mathbb{R}^n$, the set

$$\partial G(\bar{x}) := \mathrm{conv}\{\lim DG(x_i) \,|\, x_i \longrightarrow \bar{x}, x_i \notin \Omega_G\}$$

is called Clarke's generalized Jacobian, where $\Omega_G \subset \mathbb{R}^n$ denotes the set of points at which G fails to be differentiable.

Theorem 13 (Clarke's inverse function theorem [13]). *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Lipschitz near $\bar{x}$. If all matrices in $\partial F(\bar{x})$ are nonsingular, then F has the unique Lipschitz inverse function F^{-1} locally around $\bar{x}$.*

Theorem 14 (Clarke's implicit function theorem [13]). *Let $G : \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be Lipschitz near $(\bar{y}, \bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ with $G(\bar{y}, \bar{z}) = 0$. Suppose that*

$$\pi_z \partial G(\bar{y}, \bar{z}) := \{M \in \mathbb{R}^{k \times k} \,|\, \textit{there exists } N \in \mathbb{R}^{k \times n} \textit{ with } [N, M] \in \partial G(\bar{y}, \bar{z})\}$$

is of maximal rank (i.e., contains merely nonsingular matrices). Then there exist an $\mathbb{R}^{n-k}$-neighborhood Y of $\bar{y}$, an $\mathbb{R}^k$-neighborhood Z of $\bar{z}$, and a Lipschitz function

$\zeta : Y \longrightarrow Z$ *such that* $\zeta(\bar{y}) = \bar{z}$ *and for every* $(y,z) \in Y \times Z$ *it holds that*

$$G(y,z) = 0 \textit{ if and only if } z = \zeta(y).$$

However, Example 11 illustrates that Theorem 14 cannot be applied directly in general just for the linear case of a stable $M[F_1,F_2]$.

Example 11 (3D, stable: IFT is not applicable). Consider the set $M^{11} := \{(x,y,z) \in \mathbb{R}^3 \mid \min\{x,y\} = 0, \min\{-y+z,z\} = 0\}$ (see Figure 14). This example shows that although $M[F_1,F_2]$ is a Lipschitz manifold, it cannot be parameterized by means of any splitting of $\mathbb{R}^3$ in the *standard* basis. Therefore, Theorem 14 (and, actually, any implicit function theorem) cannot be applied directly.

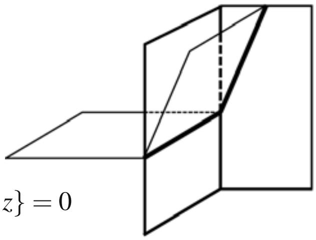

Figure 14 Illustration of Example 11

Indeed, Example 11 suggests first performing a linear coordinate transformation in order to make Theorem 14 applicable. Exactly this idea is incorporated in the SMFC and allows us to prove the following result.

Theorem 15 (Local stability under SMFC). *If the SMFC holds at* $x \in M[F_1,F_2]$*, then the feasible set* $M[F_1,F_2]$ *is locally stable at* $\bar{x}$*.*

Proof. Let $\bar{x} \in M[F_1,F_2]$. Since the SMFC holds at $\bar{x}$, there exists a k-dimensional linear subspace E of $\mathbb{R}^n$ such that any k vectors $(u_1,\ldots,u_k) \in P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x}))$ are linearly independent. Without loss of generality, we may assume that $E = \{0_{n-k}\} \times \mathbb{R}^k$.

Setting $g_i := \min\{F_{1,i},F_{2,i}\}$, $i = 1,\ldots,k$, we define

$$G : \begin{cases} \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow & \mathbb{R}^k, \\ \quad (y,z) \quad\quad \mapsto & (g_1(y,z),\ldots,g_k(y,z)). \end{cases}$$

Let $\bar{x} = (\bar{y},\bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. We obtain from $\partial g_i(\bar{x}) = C_i(\bar{x})$, $i = 1,\ldots,k$, and the choice of E that

$$\pi_z \partial G(\bar{y},\bar{z}) \subset P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x})).$$

Hence, from the SMFC, $\pi_z \partial G(\bar{y},\bar{z})$ is of maximal rank and Theorem 14 can be applied. Then there exist a compact $\mathbb{R}^{n-k}$-neighborhood Y of $\bar{y}$, an $\mathbb{R}^k$-neighborhood

Z of $\bar{z}$ and a Lipschitz function $\zeta : Y \longrightarrow Z$ such that $\zeta(\bar{y}) = \bar{z}$ and for every $(y,z) \in Y \times Z$ it holds that

$$G(y,z) = 0 \text{ if and only if } z = \zeta(y).$$

For $\varepsilon > 0$, we set

$$K_\varepsilon := (\{(y, \zeta(y)) \,|\, y \in Y\} + \bar{B}_{\mathbb{R}^n}(0,\varepsilon)) \cap (Y \times Z),$$

an ε-tube around $M[F_1,F_2] \cap (Y \times Z)$. Due to the compactness of Y, continuity reasonings, and stability of the SMFC within the space of C^1-functions (taking Y smaller if needed), there exists $\varepsilon > 0$ such that:

(•) $K_\varepsilon \subset Y \times Z$ and K_ε is compact.

(••) There exists a C^1-neighborhood U of (F_1,F_2) in $C^1(\mathbb{R}^n,\mathbb{R}^k) \times C^1(\mathbb{R}^n,\mathbb{R}^k)$ such that for every $(\widetilde{F}_1,\widetilde{F}_2) \in U$ it holds that

$$M[\widetilde{F}_1,\widetilde{F}_2] \cap (Y \times Z) \subset K_\varepsilon.$$

We assume U to be a ball of radius $r > 0$ in $C^1(\mathbb{R}^n,\mathbb{R}^k) \times C^1(\mathbb{R}^n,\mathbb{R}^k)$.

(•••) The SMFC is fulfilled at every $x \in M[\widetilde{F}_1,\widetilde{F}_2] \cap (Y \times Z)$ for every $(\widetilde{F}_1,\widetilde{F}_2) \in U$ with the same k-dimensional linear subspace E.

Now $(\widetilde{F}_1,\widetilde{F}_2) \in U$ be arbitrary but fixed. Setting $\widetilde{g}_i := \min\{\widetilde{F}_{1,i},\widetilde{F}_{2,i}\}$, $i = 1,\ldots,k$, we define

$$\widetilde{G} : \begin{cases} \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow & \mathbb{R}^k, \\ \quad (y,z) \quad \mapsto & (\widetilde{g}_1(y,z),\ldots,\widetilde{g}_k(y,z)). \end{cases}$$

Our aim is to show that for every fixed $\widetilde{y} \in Y$ the equation $\widetilde{G}(\widetilde{y},z) = 0$ is uniquely solvable with $(\widetilde{y},z) \in K_\varepsilon$. For that, we set for $(t,y,z) \in [0,1] \times \mathbb{R}^{n-k} \times \mathbb{R}^k$

$$H_{1,i}(t,y,z) := (1-t)F_{1,i}(y,z) + t\widetilde{F}_{1,i}(y,z),$$

$$H_{2,i}(t,y,z) := (1-t)F_{2,i}(y,z) + t\widetilde{F}_{2,i}(y,z),$$

$$g_i(t,y,z) := \min\{H_{1,i}(t,z),H_{2,i}(t,z)\}.$$

Furthermore, we construct a homotopy mapping

$$H : \begin{cases} [0,1] \times \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow & \mathbb{R}^k, \\ \qquad (t,y,z) \qquad \mapsto & (g_1(t,y,z),\ldots,g_k(t,y,z)). \end{cases}$$

We keep in mind that $H(0,y,z) = G(y,z)$ and $H(1,y,z) = \widetilde{G}(y,z)$; moreover,

$$(H_1(t,\cdot,\cdot),H_2(t,\cdot,\cdot)) \in U \text{ for every } t \in [0,1].$$

Next, we fix $\widetilde{y} \in Y$ and consider the equation $H(t,\widetilde{y},z) = 0$ near its solution $(0,\widetilde{y},\zeta(\widetilde{y}))$. Since $(\widetilde{y},\zeta(\widetilde{y})) \in M[F_1,F_2] \cap (Y \times Z)$, we obtain from (•••) that the SMFC holds at $(\widetilde{y},\zeta(\widetilde{y}))$. This means that

$$\pi_z \partial H(0,\widetilde{y},\zeta(\widetilde{y})) = \pi_z \partial G(\widetilde{y},\zeta(\widetilde{y}))$$

is of maximal rank and Theorem 14 can be applied for $H(t,\widetilde{y},z)=0$ near its solution $(0,\widetilde{y},\zeta(\widetilde{y}))$. Thus, we obtain for every $t\in[0,\delta), 0<\delta\leq 1$ a solution $z(t)$ such that $H(t,\widetilde{y},z(t))=0$. Since $(H_1(t,\cdot,\cdot),H_2(t,\cdot,\cdot))\in U$, $(\bullet\bullet)$ yields that $(\widetilde{y},z(t))\in K_\varepsilon$ for every $t\in[0,\delta]$. Here, δ is taken smaller if needed.

These considerations allow us to claim that

$$\begin{aligned}\widetilde{t}:=\sup\{\bar{t}\in[0,1)\mid &\text{ for every } t\in[0,\bar{t}) \text{ there exists at least one}\\ &(\widetilde{y},z(t))\in K_\varepsilon \text{ such that } H(t,\widetilde{y},z(t))=0\}\end{aligned}$$

is well-defined.

Assume that $\widetilde{t}\neq 1$. Then, there is a sequence of solutions $z(t_m), t_m\in[0,\widetilde{t}), t_m\longrightarrow\widetilde{t}$ such that $(\widetilde{y},z(t_m))\in K_\varepsilon$ and $H(t_m,\widetilde{y},z(t_m))=0$. We use the compactness of K_ε from $(\bullet)$ to obtain the existence of $\widetilde{z}$ with $(\widetilde{y},\widetilde{z})\in K_\varepsilon$ and $z_m\longrightarrow\widetilde{z}$. Hence, due to the continuity, we get in the limit $H(\widetilde{t},\widetilde{y},\widetilde{z})=0$. This conclusion allows us to apply Theorem 14 for the equation $H(t,\widetilde{y},z)=0$ near $(\widetilde{t},\widetilde{y},\widetilde{z})$ to extend the solution for $t>\widetilde{t}$. This yields a contradiction with the definition of $\widetilde{t}$.

So, we claim that $\widetilde{t}=1$ and as above we obtain that $\widetilde{G}(\widetilde{y},z)\equiv H(1,\widetilde{y},z)=0$ is solvable with $(\widetilde{y},z)\in K_\varepsilon$.

The unique solvability of $\widetilde{G}(\widetilde{y},z)=0$ for $(\widetilde{y},z)\in K_\varepsilon$ can be proven by contradiction using analogous arguments. One has only to follow different solutions by applying Theorem 14 successively until the unique solution $(0,\widetilde{y},\zeta(\widetilde{y}))$ of $G(\widetilde{y},z)\equiv H(t,\widetilde{y},z)=0$ is reached.

From all of this, it is proven that for every $\widetilde{y}\in Y$ the equation $\widetilde{G}(\widetilde{y},z)=0$ is uniquely solvable with $(\widetilde{y},z(\widetilde{y}))\in K_\varepsilon$. From $(\bullet\bullet)$, one can immediately see that $\widetilde{G}(\widetilde{y},z)=0$ is uniquely solvable, actually, in Z. Therefore, $M[\widetilde{F}_1,\widetilde{F}_2]\cap(Y\times Z)=\{(y,z(y))\,|\,y\in Y\}$. Here, $z:Y\longrightarrow Z$ is Lipschitz due to $(\bullet\bullet\bullet)$ and Theorem 14, which is applicable locally around every $\widetilde{x}\in M[\widetilde{F}_1,\widetilde{F}_2]\cap(Y\times Z)$.

It remains to add that $M[F_1,F_2]\cap(Y\times Z)$ and $M[\widetilde{F}_1,\widetilde{F}_2]\cap(Y\times Z)$, both being Lipschitz graphs on Y, are homeomorphic with $\mathbb{R}^{n-k}$ and thus with each other.□

From the proof of Theorem 15, we deduce the following Corollary 2.

Corollary 2. *If the SMFC holds at every $\bar{x}\in M[F_1,F_2]$, then the feasible set $M[F_1,F_2]$ is an $(n-k)$-dimensional Lipschitz manifold.*

Proof. We use the notation in Theorem 15. From the SMFC at $\bar{x}\in M[F_1,F_2]$, we may assume that after an appropriate linear coordinate transformation it holds that

$$(Y\times Z)\cap M[F_1,F_2]=\{(y,\zeta(y))\,|\,y\in Y\},$$

where $\bar{x}=(\bar{y},\bar{z})\in\mathbb{R}^{n-k}\times\mathbb{R}^k$, Y is an $\mathbb{R}^{n-k}$-neighborhood of $\bar{y}$, Z is an $\mathbb{R}^k$-neighborhood of $\bar{z}$, and $\zeta:Y\longrightarrow Z$ is Lipschitz. Hence, $M[F_1,F_2]$ is locally the graph of a Lipschitz function ζ. $M[F_1,F_2]$ fits Definition 1 and is an $(n-k)$-dimensional Lipschitz manifold.□

On application of Kummer's implicit function theorem

In this section, we link the SMFC with the so-called Thibault limiting sets (or strict graphical derivatives) via Kummer's implicit function theorem (cf. [85] and Section B.1).

For a vector-valued function $G = (g_1, \dots, g_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, the mapping $TG(\bar{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ with

$$TG(\bar{x})(\bar{u}) := \left\{ v \in \mathbb{R}^k \;\middle|\; \begin{array}{l} v = \lim\limits_{k\to\infty} \dfrac{f(x_k + t_k u_k) - f(x_k)}{t_k} \\ \text{for certain } t_k \downarrow 0,\, x_k \longrightarrow \bar{x}, u_k \longrightarrow \bar{u} \end{array} \right\}$$

is called the Thibault derivative at $\bar{x}$ (see [114, 115]) or strict graphical derivative (see [104]).

If, additionally, g_i are Lipschitz near $\bar{x} \in \mathbb{R}^n$, then we may omit the sequence $u_k \longrightarrow \bar{u}$ in the definition of $TG(\bar{x})(\bar{u})$ and we get

$$TG(\bar{x})(\bar{u}) = \left\{ v \in \mathbb{R}^k \;\middle|\; \begin{array}{l} v = \lim\limits_{k\to\infty} \dfrac{f(x_k + t_k \bar{u}) - f(x_k)}{t_k} \\ \text{for certain } t_k \downarrow 0,\, x_k \longrightarrow \bar{x} \end{array} \right\}.$$

Necessary and sufficient conditions for local invertability of Lipschitz functions can be given in terms of Thibault derivatives.

Theorem 16 (Kummer's inverse function theorem [81, 86]). *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Lipschitz near $\bar{x}$. Then the following statements are equivalent:*

(i) *F has the locally unique Lipschitz inverse function F^{-1}.*
(ii) *There exists $c > 0$ such that*

$$\|F(x) - F(x')\| \geq c\|x - x'\| \textit{ for all } x,\, x' \textit{ with } \|\bar{x} - x\| \leq c, \|\bar{x} - x'\| \leq c.$$

(iii) *$TF(\bar{x})$ is injective (i.e., $0 \notin TF(\bar{x})(u)$ for all $u \neq 0$).*

Remark 7. Note that the injectivity of $TF(\bar{x})$ in Theorem 16 is in general weaker than Clarke's requirement that all matrices in $\partial F(\bar{x})$ are nonsingular. In fact, there exists a Lipschitz homeomorphism F of $\mathbb{R}^2$ such that $\partial F(\bar{x})$ contains the zero matrix (see Example BE.3 in [81]).

Remark 8. We point out that (iii) from Theorem 16 implies the existence of the unique Lipschitz inverse of F w.r.t. Lipschitz perturbations of F performed locally. This means that there exists an $\mathbb{R}^n$-neighborhood U of $\bar{x}$ and a neighborhood V of F in the space $C^{0,1}(U, \mathbb{R}^n)$ of Lipschitz functions such that, for all $\widehat{F} \in V$ and $\widehat{x} \in U$, $\widehat{F}$ has the locally unique Lipschitz inverse function $\widehat{F}^{-1}$ around $\widehat{x}$. Note that we equip $F \in C^{0,1}(U, \mathbb{R}^n)$ with the norm

$$|F| := \max \left\{ \sup_{x \in U} \|F(x)\| + \mathrm{Lip}(F, U) \right\},$$

where

$$\mathrm{Lip}(F,U) := \inf\left\{ r > 0 \,|\, \|F(x) - F(x')\| \leq r\|x - x'\| \text{ for all } x, x' \in U \right\}.$$

For details, we refer the reader to Theorem 5.14 and Corollary 4.4 in [81].

Theorem 17 (Kummer's implicit function theorem, [81, 85]). *Let $G : \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be Lipschitz near $(\bar{y},\bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ with $G(\bar{y},\bar{z}) = 0$. Then, the following statements are equivalent:*

(i) *There exist $\mathbb{R}^{n-k}$-neighborhoods Y of $\bar{y}$ and W of 0, an $\mathbb{R}^k$-neighborhood Z of $\bar{z}$, and a Lipschitz function $\zeta : Y \times W \longrightarrow Z$ such that $\zeta(\bar{y},0) = \bar{z}$ and for every $(y,z,w) \in Y \times Z \times W$ it holds that*

$$G(y,z) = w \text{ if and only if } z = \zeta(y,w).$$

(ii) $0 \notin TG(\bar{y},\bar{z})(0,u)$ *for all* $u \neq 0$.

Remark 9. We point out that Theorem 17 gives a necessary and sufficient condition for the existence of implicit functions. Recall that Clarke's IFT (see Theorem 14) gives only a sufficient condition for that fact. Moreover, it is important to note that in Theorem 17 the implicit function ζ depends Lipschitz also on the right-hand-side perturbations w. This issue was used extensively in the proof of Theorem 15.

Now, we turn our attention to the case of min-functions. Let a basis decomposition of $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ be fixed. It turns out that the assumptions of Clarke's and Kummer's implicit function theorems coincide. Moreover, they are also equivalent with the SMFC w.r.t. the subspace $E := \{0_{n-k}\} \times \mathbb{R}^k$ (B. Kummer and O. Stein, personal communication).

Lemma 4. *Setting $g_i := \min\{F_{1,i}, F_{2,i}\}$, $i = 1,\ldots,k$, we define*

$$G : \begin{cases} \mathbb{R}^{n-k} \times \mathbb{R}^k & \longrightarrow & \mathbb{R}^k, \\ (y,z) & \mapsto & (g_1(y,z),\ldots,g_k(y,z)). \end{cases}$$

Then, the following conditions are equivalent for $\bar{x} = (\bar{y},\bar{z})$:

(i) $\pi_z\partial G(\bar{y},\bar{z})$ *is of maximal rank, meaning*

$$\pi_z\partial G(\bar{y},\bar{z}) := \{M \in \mathbb{R}^{k\times k} \,|\, \text{there exists } N \in \mathbb{R}^{k\times n} \text{ with } [N,M] \in \partial G(\bar{y},\bar{z})\}$$

contains merely nonsingular matrices.

(ii) *All matrices in $\partial_z g_1(\bar{x}) \times \partial_z g_2(\bar{x}) \times \ldots \times \partial_z g_k(\bar{x})$ are nonsingular.*

(iii) $0 \notin TG(\bar{y},\bar{z})(0,u)$ *for all* $u \neq 0$.

Proof. **"(i) $\Longrightarrow$ (iii)":** Due to (i), we may apply Clarke's implicit function theorem. Hence, the implicit function $\zeta(y)$ exists. It is not hard to see that ζ depends uniquely and Lipschitz on the w-values of G. Hence, we obtain in fact $\zeta(y,w)$. Applying Kummer's implicit function theorem, we get (iii).

"(ii) $\Longrightarrow$ (i)": In general, it holds (see, e.g., [23]) that

$$\pi_z \partial G(\bar{y},\bar{z}) \subset \partial_z g_1(\bar{x}) \times \partial_z g_2(\bar{x}) \times \ldots \times \partial_z g_k(\bar{x}).$$

This inclusion shows the assertion.

"(iii) $\Longrightarrow$ (ii)": Let $0 \notin TG(\bar{y},\bar{z})(0,u)$ for all $u \neq 0$. For $q \in \mathbb{R}^k$, we set

$$q^+ := (q_1^+,\ldots,q_k^+), \text{ where } q_i^+ := \max\{q,0\}, i = 1,\ldots,k,$$

$$q^- := (q_1^-,\ldots,q_k^-), \text{ where } q_i^- := \min\{q,0\}, i = 1,\ldots,k.$$

We define the mapping $\widehat{G} : \mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R}^{2k}$ as

$$\widehat{G}(y,z,q) = \begin{pmatrix} F_1(y,z) - q^+ \\ -F_2(y,z) - q^- \end{pmatrix}.$$

The zeros of G and $\widehat{G}$ correspond as follows. If $G(x) = 0$, then $\widehat{G}(x,q) = 0$ with $q := F_1 - F_2$. If $\widehat{G}(x,q) = 0$, then $G(x) = 0$.

Setting $\bar{q} = F_1(\bar{y},\bar{z}) - F_2(\bar{y},\bar{z})$, we claim that $0 \notin T\widehat{G}(\bar{x},\bar{q})(0,u,p)$ for all $(u,p) \neq 0$. In fact, from Kummer's IFT, the latter is equivalent to the existence of Lipschitz implicit functions $\zeta(y,w_1,w_2)$ and $q(y,w_1,w_2)$ for the system

$$F_1(y,z) - q^+ = w_1, \; -F_2(y,z) - q^- = w_2. \tag{2.10}$$

The system (2.10) can be equivalently written as

$$F_1(y,z) - w_1 - q^+ = 0, \; -F_2(y,z) - w_2 - q^- = 0.$$

Hence, we need to find the implicit function $\zeta(y,w_1,w_2)$ for

$$\min\{F_{1,i}(y,z) - w_{1,i}, F_{2,i}(y,z) + w_{2,i}\} = 0, \, i = 1,\ldots,k \tag{2.11}$$

and afterwards to set

$$q(y,w_1,w_2) := F_1(y,\zeta(y,w_1,w_2)) - w_1 - F_2(y,\zeta(y,w_1,w_2)) - w_2.$$

Note that (2.11) is a Lipschitz perturbed version of $G(x) = 0$. Remark 8 and (iii) then justify the application of Kummer's IFT for the perturbed system (2.11). Hence, $0 \notin T\widehat{G}(\bar{x},\bar{q})(0,u,p)$ for all $(u,p) \neq 0$.

Now, we compute $T\widehat{G}(\bar{x},\bar{q})(0,u,p)$ using results from [81] on the so-called Kojima functions. For that, we set

$$N(q) := (1,q^+,q^-) \text{ and } M(x) := \begin{pmatrix} F_1(x) & F_2(x) \\ -I_k & 0 \\ 0 & -I_k \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix. It holds that

$$\widehat{F}(x,q) = N(q) \cdot M(x).$$

Applying the product rule (see Theorem 7.5 in [81]), we get:

$$T\widehat{F}(\bar{x},\bar{q})(0,u,p) = N(\bar{q})TM(\bar{x})(0,u) + TN(\bar{q})(p)M(\bar{x}).$$

We compute $TM(\bar{x})$ and $TN(\bar{q})$ (see Lemma 7.3 in [81]) as

$$TM(\bar{x})(0,u) = \begin{pmatrix} D_zF_1(\bar{x})u & D_zF_2(\bar{x})u \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$TN(\bar{q})(p) = \{(0,\lambda,p-\lambda) \,|\, \lambda_i = r_i p_i,, \, r_i \in \mathfrak{R}(\bar{q}),\, i = 1,\dots,k\}, \text{ where}$$
$$\mathfrak{R}(\bar{q}) := \{r \in [0,1]^m \,|\, r_i = 1 \text{ if } \bar{q}_i > 0,\, r_i = 0 \text{ if } \bar{q}_i < 0\}.$$

Combining the above

$$T\widehat{G}(\bar{x},\bar{q})(0,u,p) = \left\{ \begin{pmatrix} D_zF_{1,i}(\bar{x})u - r_i p_i,\, i = 1,\dots,k \\ -D_zF_{2,i}(\bar{x})u - (1-r_i)p_i,\, i = 1,\dots,k \end{pmatrix} \,\middle|\, r \in \mathfrak{R}(\bar{q}) \right\}.$$

Next, let (ii) beassumed to fail. We show that this contradicts the fact that

$$0 \notin T\widehat{G}(\bar{x},\bar{q})(0,u,p) \text{ for all } (u,p) \neq 0.$$

Indeed, if (ii) does not hold, we obtain $u \in \mathbb{R}^k$, $u \neq 0$, and $r \in [0,1]^k$ such that

$$[(1-r_i)D_zF_{1,i}(\bar{x}) + r_i D_zF_{2,i}(\bar{x})]\, u = 0,\; i = 1,\dots,k. \tag{2.12}$$

Note that $r \in \mathfrak{R}(\bar{q})$ due to the definition of Clarke's subdifferentials.

Case $\mathbf{r_i \neq 0}$. Then, we set $p_i := \frac{1}{r_i} D_zF_{1,i}u$ and obtain

$$D_zF_{1,i}(\bar{x})u - r_i p_i = 0.$$

Furthermore, from (2.12) we get,

$$r_i\,[-D_zF_{2,i}(\bar{x})u - (1-r_i)p_i] = r_i \left[-D_zF_{2,i}(\bar{x})u - (1-r_i)\,\frac{1}{r_i} D_zF_{1,i}(\bar{x})u \right]$$
$$= -r_i D_zF_{2,i}(\bar{x})u - (1-r_i)D_zF_{1,i}(\bar{x})u = 0.$$

Hence, $-D_zF_{2,i}(\bar{x})u - (1-r_i)p_i = 0$.

Case $\mathbf{r_i = 0}$. Then, we set $p_i := -D_zF_{2,i}u$ and obtain from (2.12)

$$D_zF_{1,i}(\bar{x})u - r_i p_i = 0.$$

Moreover,

$$-D_zF_{2,i}(\bar{x})u - (1-r_i)p_i = -D_zF_{2,i}(\bar{x})u + D_zF_{2,i}u = 0.$$

Thus, we see that, for $(u,p) \neq 0$ defined as above, it holds that

$$0 \in T\widehat{G}(\bar{x},\bar{q})(0,u,p).\square$$

From Lemma 4, we deduce the following result.

Theorem 18 (SMFC and Kummer's implicit function theorem). *The SMFC holds if and only if Kummer's implicit function theorem is applicable w.r.t. some basis decomposition of* $\mathbb{R}^n$.

Proof. The equivalence of (ii) and (iii) from Lemma 4 immediately inplies the result. In fact, we only need to use the chain rule from [81],

$$T(G \circ A)(x)(u) = TG(Ax)(Au),$$

where A is a nonsingular $(n \times n)$ matrix. See also the characterization of the SMFC in terms of Clarke's subdifferentials in Lemma 1.$\square$

Theorem 18 shows that the remaining difficulty concerning topological stability of the MPCC feasible set lies in the conjectured equivalence between the MFC and SMFC rather than in an application of different implicit function theorems.

2.3 Critical point theory

We study the behavior of the topological properties of lower-level sets

$$M^a := \{x \in M \,|\, f(x) \leq a\}$$

as the level $a \in \mathbb{R}$ varies. It turns out that the concept of C-stationarity is an adequate stationarity concept. In fact, we present two basic theorems from Morse theory (see [63, 93]). First, we show that, for $a < b$, the set M^a is a strong deformation retract of M^b if the (compact) set

$$M_a^b := \{x \in M \,|\, a \leq f(x) \leq b\}$$

does not contain C-stationary points (see Theorem 20(a)). Second, if M_a^b contains exactly one (nondegenerate) C-stationary point, then M^b is shown to be homotopy-equivalent to M^a with a q-cell attached (see Theorem 20(b)). Here, the dimension q is the so-called C-index. It depends on both the restricted Hessian of the Lagrangian and the Lagrange multipliers related to biactive complementarity constraints. The latter fact is the main difference with respect to the well-known case where a feasible set is described only by equality and finitely many inequality constraints (see [63] and Section 1.4).

We would like to refer to some related papers. In [106], the concept of a nondegenerate feasible point for the MPCC is introduced. Some genericity results are obtained. In [99], the concepts of a nondegenerate C-stationary point and its stationary C-index are introduced for quadratic programs with complementarity constraints

(QPCCs). The generic structure of the C-stationary point set for nonparametric and one-parametric QPCCs is discussed, and some homotopy methods for QPCCs are developed. We refer the reader to [69] for details.

Notation and Auxiliary Results

Given $\bar{x} \in M$, we define the following index sets:

$$\begin{aligned} J_0(\bar{x}) &:= \{j \in J \,|\, g_j(\bar{x}) = 0\}, \\ \alpha(\bar{x}) &:= \{m \in \{1,\dots k\} \,|\, F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) > 0\}, \\ \beta(\bar{x}) &:= \{m \in \{1,\dots k\} \,|\, F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) = 0\}, \\ \gamma(\bar{x}) &:= \{m \in \{1,\dots k\} \,|\, F_{1,m}(\bar{x}) > 0, F_{2,m}(\bar{x}) = 0\}. \end{aligned}$$

We call $J_0(\bar{x})$ the active inequality index set and $\beta(\bar{x})$ the biactive index set at $\bar{x}$.

Without loss of generality, we assume that at the particular point of interest $\bar{x} \in M$ it holds that

$$J_0(\bar{x}) = \{1,\dots,|J_0(\bar{x})|\},\ \alpha(\bar{x}) = \{1,\dots,|\alpha(\bar{x})|\},$$
$$\gamma(\bar{x}) = \{|\alpha(\bar{x})| + 1,\dots,|\alpha(\bar{x})| + |\gamma(\bar{x})|\}.$$

We put $s := |I| + |\alpha(\bar{x})| + |\gamma(\bar{x})|$, $q := s + |J_0(\bar{x})|$, $p := n - q - 2|\beta(\bar{x})|$.

Furthermore, we recall the well-known linear independence constraint qualification (LICQ) for the MPCC (e.g. [105]), which is said to hold at $\bar{x} \in M$ if the set of vectors

$$\begin{aligned} &\{D^T h_i(\bar{x}),\ i \in I, D^T F_{1,m_\alpha}(\bar{x}),\ m_\alpha \in \alpha(\bar{x}),\ D^T F_{2,m_\gamma}(\bar{x}),\ m_\gamma \in \gamma(\bar{x}), \\ &D^T g_j(\bar{x}),\ j \in J_0(\bar{x}),\ D^T F_{1,m_\beta}(\bar{x}),\ D^T F_{2,m_\beta}(\bar{x}),\ m_\beta \in \beta(\bar{x})\} \end{aligned}$$

is linearly independent.

Definition 10 (C-stationary point [22, 105]). A point $\bar{x} \in M$ is called Clarke stationary (C-stationary) for the MPCC if there exist real numbers $\bar{\lambda}_i$, $i \in I$, $\bar{\rho}_{m_\alpha}$, $m_\alpha \in \alpha(\bar{x})$, $\bar{\vartheta}_{m_\gamma}$, $m_\gamma \in \gamma(\bar{x})$, $\bar{\mu}_j$, $j \in J_0(\bar{x})$, $\bar{\sigma}_{1,m_\beta}$, $\bar{\sigma}_{2,m_\beta}$, $m_\beta \in \beta(\bar{x})$ (Lagrange multipliers) such that

$$\begin{aligned} Df(\bar{x}) = &\sum_{i\in I} \bar{\lambda}_i Dh_i(\bar{x}) + \sum_{m_\alpha \in \alpha(\bar{x})} \bar{\rho}_{m_\alpha} DF_{1,m_\alpha}(\bar{x}) + \sum_{m_\gamma \in \gamma(\bar{x})} \bar{\vartheta}_{m_\gamma} DF_{2,m_\gamma}(\bar{x}) \\ &+ \sum_{j \in J_0(\bar{x})} \bar{\mu}_j Dg_j(\bar{x}) + \sum_{m_\beta \in \beta(\bar{x})} \Big(\bar{\sigma}_{1,m_\beta} DF_{1,m_\beta}(\bar{x}) + \bar{\sigma}_{2,m_\beta} DF_{2,m_\beta}(\bar{x}) \Big), \end{aligned} \tag{2.13}$$

$$\bar{\mu}_j \geq 0 \text{ for all } j \in J_0(\bar{x}), \tag{2.14}$$

$$\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}). \tag{2.15}$$

In the case where the LICQ holds at $\bar{x} \in M$, the Lagrange multipliers in (2.13) are uniquely determined.

Given a C-stationary point $\bar{x} \in M$ for the MPCC, we set

$$\begin{aligned} M(\bar{x}) := \{x \in \mathbb{R}^n \mid\ & h_i(x) = 0,\, i \in I, F_{1,m_\alpha}(x) = 0,\, m_\alpha \in \alpha(\bar{x}),\\ & F_{2,m_\gamma}(x) = 0,\, m_\gamma \in \gamma(\bar{x}),\, g_j(x) = 0,\, j \in J_0(\bar{x}),\\ & F_{1,m_\beta}(x) = 0,\, F_{2,m_\beta}(x) = 0,\, m_\beta \in \beta(\bar{x})\}. \end{aligned}$$

Obviously, $M(\bar{x}) \subset M$ and, in the case where the LICQ holds at $\bar{x}$, $M(\bar{x})$ is locally a p-dimensional C^2-manifold.

Definition 11 (Nondegenerate C-stationary point [99, 106]). A C-stationary point $\bar{x} \in M$ with Lagrange multipliers as in Definition 10 is called nondegenerate if the following conditions are satisfied:

ND1: LICQ holds at $\bar{x}$.

ND2: $\bar{\mu}_j > 0$ for all $j \in J_0(\bar{x})$.

ND3: $D^2L(\bar{x})\mid_{T_{\bar{x}}M(\bar{x})}$ is nonsingular.

ND4: $\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} > 0$ for all $m_\beta \in \beta(\bar{x})$.

Here, the matrix D^2L stands for the Hessian of the Lagrange function L,

$$\begin{aligned} L(x) := f(x) - \sum_{i\in I} \bar{\lambda}_i h_i(x) - \sum_{m_\alpha \in \alpha(\bar{x})} \bar{\rho}_{m_\alpha} F_{1,m_\alpha}(x) - \sum_{m_\gamma \in \gamma(\bar{x})} \bar{\vartheta}_{m_\gamma} F_{2,m_\gamma}(x)\\ - \sum_{j \in J_0(\bar{x})} \bar{\mu}_j g_j(x) - \sum_{m_\beta \in \beta(\bar{x})} \Big(\bar{\sigma}_{1,m_\beta} F_{1,m_\beta}(x) + \bar{\sigma}_{2,m_\beta} F_{2,m_\beta}(x)\Big), \end{aligned} \tag{2.16}$$

and $T_{\bar{x}}M(\bar{x})$ denotes the tangent space of $M(\bar{x})$ at $\bar{x}$,

$$\begin{aligned} T_{\bar{x}}M(\bar{x}) := \{\xi \in \mathbb{R}^n \mid\ & Dh_i(\bar{x})\,\xi = 0,\, i \in I,\\ & DF_{1,m_\alpha}(\bar{x})\,\xi = 0,\, m_\alpha \in \alpha(\bar{x}),\\ & DF_{2,m_\gamma}(\bar{x})\,\xi = 0,\, m_\gamma \in \gamma(\bar{x}),\\ & Dg_j(\bar{x})\,\xi = 0,\, j \in J_0(\bar{x}),\\ & DF_{1,m_\beta}(\bar{x})\,\xi = 0,\, DF_{2,m_\beta}(\bar{x})\,\xi = 0,\, m_\beta \in \beta(\bar{x})\}. \end{aligned}$$

Condition ND3 means that the matrix $V^T D^2L(\bar{x})V$ is nonsingular, where V is some matrix whose columns form a basis for the tangent space $T_{\bar{x}}M(\bar{x})$.

Definition 12 (C-index [99]). Let $\bar{x} \in M$ be a nondegenerate C-stationary point with Lagrange multipliers as in Definition 11. The number of negative/positive eigenvalues of $D^2L(\bar{x})\mid_{T_{\bar{x}}M(\bar{x})}$ is called the quadratic index (QI)/quadratic coindex (QCI) of $\bar{x}$. The number of pairs $(\bar{\sigma}_{1,m_\beta}, \bar{\sigma}_{2,m_\beta})$, $m_\beta \in \beta(\bar{x})$ with both $\bar{\sigma}_{1,m_\beta}$ and $\bar{\sigma}_{2,m_\beta}$ negative/positive is called the biactive index (BI)/biactive coindex (BCI) of $\bar{x}$. The number $(QI + BI)/(QCI + BCI)$ is called the Clarke index (C-index)/Clarke coindex (C-coindex) of $\bar{x}$.

Note that, in the absence of complementarity constraints, the C-index has only the QI part and coincides with the well-known quadratic index of a nondegenerate Karush-Kuhn-Tucker point in nonlinear programming or, equivalently, with the Morse index (see [63, 83, 93] and Section 1.4).

The following proposition uses the C-index for the characterization of a local minimizer. Its proof is omitted since it can be easily seen (see also [99, 105]).

Proposition 5. *(i) Assume that $\bar{x}$ is a local minimizer for the MPCC and that the LICQ holds at $\bar{x}$. Then, $\bar{x}$ is a C-stationary point for the MPCC.*
(ii) Let $\bar{x}$ be a nondegenerate C-stationary point for the MPCC. Then, $\bar{x}$ is a local minimizer for the MPCC if and only if its C-index is equal to zero.

The next proposition concerning genericity results for the LICQ and for nondegeneracy of C-stationary points mainly follows from [63]. It was shown in [106] and for the special case of the QPCC in [99].

Proposition 6 (Genericity and Stability [99, 106]).

(i) Let $\mathscr{F}$ denote the subset of

$$C^2(\mathbb{R}^n,\mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n,\mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n,\mathbb{R}^k) \times C^2(\mathbb{R}^n,\mathbb{R}^k)$$

consisting of those (h,g,F_1,F_2) for which the LICQ holds at all points $x \in M[h,g,F_1,F_2]$. Then, $\mathscr{F}$ is C^2_s-open and -dense.
(ii) Let $\mathscr{D}$ denote the subset of

$$C^2(\mathbb{R}^n,\mathbb{R}) \times C^2(\mathbb{R}^n,\mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n,\mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n,\mathbb{R}^k) \times C^2(\mathbb{R}^n,\mathbb{R}^k)$$

consisting of those (f,h,g,F_1,F_2) for which each C-stationary point of the MPCC with data functions (f,h,g,F_1,F_2) is nondegenerate. Then, $\mathscr{D}$ is C^2_s-open and -dense.

Morse lemma for the MPCC

For the proof of deformation and cell-attachment results, we locally describe the MPCC feasible set under the LICQ (see Lemma 5). Moreover, an equivariant Morse lemma for the MPCC is derived in order to obtain suitable normal forms for the objective function at C-stationary points (see Theorem 19).

Definition 13. The feasible set M admits a local C^r-coordinate system of $\mathbb{R}^n$ ($r \geq 1$) at $\bar{x}$ by means of a C^r-diffeomorphism $\Phi : U \longrightarrow V$ with open $\mathbb{R}^n$-neighborhoods U and V of $\bar{x}$ and 0, respectively, if it holds that

(i) $\Phi(\bar{x}) = 0$,
(ii) $\Phi(M \cap U) = \left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p\right) \cap V.$

Lemma 5 (see also [106]). *Suppose that the LICQ holds at $\bar{x} \in M$. Then M admits a local C^2-coordinate system of $\mathbb{R}^n$ at $\bar{x}$.*

Proof. Choose vectors $\xi_l \in \mathbb{R}^n$, $l = 1,\dots,p$, which together with the vectors

$$\{D^T h_i(\bar{x}),\, i \in I,,\, D^T F_{1,m_\alpha}(\bar{x}),\, m_\alpha \in \alpha(\bar{x}),\, D^T F_{2,m_\gamma}(\bar{x}),\, m_\gamma \in \gamma(\bar{x}),$$
$$D^T g_j(\bar{x}),\, j \in J_0(\bar{x}),\, D^T F_{1,m_\beta}(\bar{x}),\, D^T F_{2,m_\beta}(\bar{x}),\, m_\beta \in \beta(\bar{x})\},$$

form a basis for $\mathbb{R}^n$. Next, we put

$$\left.\begin{array}{ll}
y_i & := h_i(x),\, i \in I, \\
y_{|I|+m_\alpha} & := F_{1,m_\alpha}(x),\, m_\alpha \in \alpha(\bar{x}), \\
y_{|I|+m_\gamma} & := F_{2,m_\gamma}(x),\, m_\gamma \in \gamma(\bar{x}), \\
y_{s+j} & := g_j(x),\, j \in J_0(\bar{x}), \\
y_{s+|J_0(\bar{x})|+2m_\beta-1} & := F_{1,m_\beta}(x), \\
y_{s+|J_0(\bar{x})|+2m_\beta} & := F_{2,m_\beta}(x),\, m_\beta = 1,\dots,|\beta(\bar{x})|, \\
y_{n-p+l} & := \xi_l^T(x-\bar{x}),\, l = 1,\dots,p,
\end{array}\right\} \tag{2.17}$$

or, for short,

$$y = \Phi(x). \tag{2.18}$$

Note that $\Phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $\Phi(\bar{x}) = 0$, and the Jacobian matrix $D\Phi(\bar{x})$ is nonsingular (by virtue of the LICQ and the choice of ξ_l, $l = 1,\dots,p$). By means of the implicit function theorem, there exist open neighborhoods U of $\bar{x}$ and V of 0 such that $\Phi : U \longrightarrow V$ is a C^2-diffeomorphism. By shrinking U if necessary, we can guarantee that $J_0(x) \subset J_0(\bar{x})$ and $\beta(x) \subset \beta(\bar{x})$ for all $x \in M \cap U$. Thus, property (ii) in Definition 13 follows directly from the definition of Φ. □

Definition 14. We will refer to the C^2-diffeomorphism Φ defined by (2.17) and (2.18) as a standard diffeomorphism.

Remark 10. From the proof of Lemma 5, it follows that the Lagrange multipliers at a nondegenerate C-stationary point are the corresponding partial derivatives of the objective function in new coordinates given by the standard diffeomorphism (see Lemma 2.2.1 of [65]). Moreover, the Hessian with respect to the last p coordinates corresponds to the restriction of the Lagrange function's Hessian on the respective tangent space (see Lemma 2.2.10 of [65]).

Theorem 19 (Morse lemma for MPCC). *Suppose that $\bar{x}$ is a nondegenerate C-stationary point for the MPCC with quadratic index QI, biactive index BI, and C-index = QI + BI. Then, there exists a local C^1-coordinate system $\Psi : U \longrightarrow V$ of $\mathbb{R}^n$ around $\bar{x}$ (according to Definition 13) such that*

$$f \circ \Psi^{-1}(0_s, y_{s+1}, \dots, y_n) =$$

$$f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_{i+s} + \sum_{j=1}^{|\beta(\bar{x})|} \pm\left(y_{2j+q-1} + y_{2j+q}\right) + \sum_{k=1}^{p} \pm y_{k+n-p}^2, \tag{2.19}$$

where $y \in \{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p$. Moreover, in (2.19) there are exactly BI negative linear pairs and QI negative squares.

Proof. Without loss of generality, we may assume $f(\bar{x}) = 0$. Let $\Phi : U \longrightarrow V$ be a standard diffeomorphism according to Definition 14. We put $\bar{f} := f \circ \Phi^{-1}$ on the set $\left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial \mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p\right) \cap V$. From now on, we may assume $s = 0$. In view of Remark 10 we have at the origin

(i) $\dfrac{\partial \bar{f}}{\partial y_i} > 0,\ i \in J_0(\bar{x})$,

(ii) $\dfrac{\partial \bar{f}}{\partial y_{2j+q-1}} \cdot \dfrac{\partial \bar{f}}{\partial y_{2j+q}} > 0,\ j = 1, \ldots |\beta(\bar{x})|$,

(iii) $\dfrac{\partial \bar{f}}{\partial y_{2j+q-1}} < 0$ for exactly BI indices $j \in \{1, \ldots |\beta(\bar{x})|\}$,

(iv) $\dfrac{\partial \bar{f}}{\partial y_{k+n-p}} = 0,\ k = 1, \ldots, p$ and $\left(\dfrac{\partial^2 \bar{f}}{\partial y_{k_1+n-p} \partial y_{k_2+n-p}}\right)_{1 \le k_1, k_2 \le p}$ is a nonsingular matrix with QI negative eigenvalues.

From now on, we denote $\bar{f}$ by f. Under the following coordinate transformations, the set $\mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial \mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ will be transformed in itself (equivariant). As an abbreviation, we put $y = (Y_{n-p}, Y^p)$, where $Y_{n-p} = (y_1, \ldots, y_{n-p})$ and $Y^p = (y_{n-p+1}, \ldots, y_n)$. We write

$$f(Y_{n-p}, Y^p) = f(0, Y^p) + \int_0^1 \frac{d}{dt} f(tY_{n-p}, Y^p) dt = f(0, Y^p) + \sum_{i=1}^{n-p} y_i d_i(y),$$

where $d_i \in C^1$, $i = 1, \ldots, n-p$.

In view of (iv), we may apply the Morse lemma on the C^2-function $f(0, Y^p)$ (see Theorem 2.8.2 of [63]) without affecting the coordinates Y_{n-p}. The corresponding coordinate transformation is of class C^1. Denoting the transformed functions f, d_j again by f, d_j, we obtain

$$f(y) = \sum_{i=1}^{n-p} y_i d_i(y) + \sum_{k=1}^{p} \pm y_{k+n-p}^2.$$

Note that $d_i(0) = \dfrac{\partial f}{\partial y_i}(0)$, $i = 1, \ldots, n-p$. Recalling (i)–(iii), we have

$$y_i |d_i(y)|,\ i = 1, \ldots, n-p, \quad y_j,\ j = n-p+1, \ldots, n \tag{2.20}$$

as new local C^1-coordinates. Denoting the transformed function f again by f and recalling the signs in (i)–(iii), we obtain (2.19). Here, the coordinate transformation Ψ is understood as the composite of all previous ones. □

Theorem 19 allows us to provide two other local representations (normal forms) of the objective function on the MPCC feasible set with respect to Lipschitz and Hölder coordinate systems.

Recall that the set $\partial \mathbb{H}^2$ represents the complementarity relations

$$u \geq 0,\, v \geq 0,\, u \cdot v = 0.$$

Define the mapping $\varphi : \partial\mathbb{H}^2 \longrightarrow \mathbb{R}^1 \times 0_1$ as

$$\varphi(u,0) := (u,0),\; \varphi(0,v) := (-v,0). \tag{2.21}$$

By coordinatewise extension of φ on $\left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|}$ and leaving the other coordinates invariant, (2.21) induces the Lipschitz coordinate transformation Φ,

$$\Phi : \{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p \longrightarrow \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \tag{2.22}$$

On the right-hand side of (2.22), the zeros $\{0_s\}$ and $\{0_1\}$ ($|\beta(\bar{x})|$ times) are deleted. The proof of the following corollary is now straightforward.

Corollary 3 (Normal forms in Lipschitz coordinates). *Let f have the normal form as in (2.19), and let Φ be the Lipschitz coordinate transformation (2.22). Then, we have*

$$f \circ \Phi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_i + \sum_{j=|J_0(\bar{x})|+1}^{J_0(\bar{x})+|\beta(\bar{x})|} \pm |y_j| + \sum_{k=J_0(\bar{x})+|\beta(\bar{x})|+1}^{n-|\beta(\bar{x})|+s} \pm y_{k+n-p}^2. \tag{2.23}$$

In (2.23), there are exactly BI negative absolute value terms and QI negative squares.

On $\mathbb{R}^1$, we introduce the transformation ψ:

$$\psi(y) := \operatorname{sgn}(y)\sqrt{|y|}. \tag{2.24}$$

Note that the function $\pm|y|$ transforms into $\pm y^2$ under ψ^{-1}. By coordinatewise extension of ψ on $\mathbb{R}^{|\beta(\bar{x})|}$ and leaving the other coordinates invariant, (2.24) induces the Hölder coordinate transformation Ψ,

$$\Psi : \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \longrightarrow \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p. \tag{2.25}$$

The proof of the following corollary is again straightforward.

Corollary 4 (Normal forms in Hölder coordinates). *Let f have the normal form as in (2.23), and let Ψ be the Hölder coordinate transformation (2.25). Then, we have*

$$f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_i + \sum_{j=|J_0(\bar{x})|+1}^{n-|\beta(\bar{x})|+s} \pm y_j^2. \tag{2.26}$$

The number of negative squares in (2.26) equals the C-index BI+QI.

Deformation and Cell Attachment

We state and prove the main deformation and cell-attachment theorems for the MPCC. Recall that for $a,b \in \mathbb{R}$, $a < b$ the sets M^a and M_a^b are defined as

$$M^a := \{x \in M \,|\, f(x) \leq a\}$$

and

$$M_a^b := \{x \in M \,|\, a \leq f(x) \leq b\}.$$

Theorem 20. *Let M_a^b be compact, and suppose that the LICQ is satisfied at all points $x \in M_a^b$.*

(a) **(Deformation theorem)** *If M_a^b does not contain any C-stationary point for the MPCC, then M^a is a strong deformation retract of M^b.*

(b) **(Cell-attachment theorem)** *If M_a^b contains exactly one C-stationary point for the MPCC, say $\bar{x}$, and if $a < f(\bar{x}) < b$ and the C-index of $\bar{x}$ is equal to q, then M^b is homotopy-equivalent to M^a with a q-cell attached.*

Proof. (a) Due to the LICQ at all $x \in M_a^b$, there exist real numbers $\lambda_i(x)$, $i \in I$, $\rho_{m_\alpha}(x)$, $m_\alpha \in \alpha(x)$, $\vartheta_{m_\gamma}(x)$, $m_\gamma \in \gamma(x)$, $\mu_j(x)$, $j \in J_0(x)$, $\sigma_{1,m_\beta}(x)$, $\sigma_{2,m_\beta}(x)$, $m_\beta \in \beta(x)$, $\nu_l(x)$, $l = 1,\ldots,p$ such that

$$Df(x) = \sum_{i\in I}\lambda_i(x)Dh_i(x) + \sum_{m_\alpha \in \alpha(x)} \rho_{m_\alpha}(x)DF_{1,m_\alpha}(x)$$

$$+ \sum_{m_\gamma \in \gamma(x)} \vartheta_{m_\gamma}(x)DF_{2,m_\gamma}(x) + \sum_{j \in J_0(x)} \mu_j(x)Dg_j(x)$$

$$+ \sum_{m_\beta \in \beta(x)} \Big(\sigma_{1,m_\beta}(x)DF_{1,m_\beta}(x) + \sigma_{2,m_\beta}(x)DF_{2,m_\beta}(x)\Big) + \sum_{l=1}^{p} \nu_l(x)\,\xi_l,$$

where vectors ξ_l, $l = 1,\ldots,p$ are chosen as in Lemma 5. We set:

$$A := \{x \in M_a^b \,|\, \text{there exists } l \in \{1,\ldots,p\} \text{ with } \nu_l(x) \neq 0\},$$

$$B := \{x \in M_a^b \,|\, \text{there exists } j \in J_0(x) \text{ with } \mu_j(x) < 0\},$$

$$C := \{x \in M_a^b \,|\, \text{there exists } m_\beta \in \beta(x) \text{ with } \sigma_{1,m_\beta}(x) \cdot \sigma_{2,m_\beta}(x) < 0\}.$$

Since each $\bar{x} \in M_a^b$ is not C-stationary for the MPCC, we get $\bar{x} \in A \cup B \cup C$.

The proof consists of a local argument and its globalization. First, we show the **local argument**. For each $\bar{x} \in M_a^b$, there exist an $(\mathbb{R}^n)$-neighborhood $U_{\bar{x}}$ of $\bar{x}$, $t_{\bar{x}} > 0$, and a mapping

$$\Psi^{\bar{x}} : \begin{cases} [0,t_{\bar{x}}) \times \big(M^b \cap U_{\bar{x}}\big) \longrightarrow M \\ \qquad (t,x) \qquad\qquad \mapsto \Psi^{\bar{x}}(t,x) \end{cases} \text{ such that}$$

(i) $\Psi^{\bar{x}}(t, M^b \cap U_{\bar{x}}) \subset M^{b-t}$ for all $t \in [0, t_{\bar{x}})$,
(ii) $\Psi^{\bar{x}}(t_1 + t_2, \cdot) = \Psi^{\bar{x}}(t_1, \Psi^{\bar{x}}(t_2, \cdot))$ for all $t_1, t_2 \in [0, t_{\bar{x}})$ with $t_1 + t_2 \in [0, t_{\bar{x}})$,
(iii) if $\bar{x} \in A \cup B$, then $\Psi^{\bar{x}}(\cdot, \cdot)$ is a C^1-flow corresponding to a C^1-vector field $F^{\bar{x}}$, and
(iv) if $\bar{x} \in C$, then $\Psi^{\bar{x}}(\cdot, \cdot)$ is a Lipschitz flow.

Obviously, the level sets of f are mapped locally onto the level sets of $f \circ \Phi^{-1}$, where Φ is a C^1-diffeomorphism according to Definition 13. Applying the standard diffeomorphism Φ from Definition 14, we consider $f \circ \Phi^{-1}$ (denoted by f again). Thus, we have $\bar{x} = 0$ and f is given on the feasible set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial \mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p$.

Case (a): $\bar{\mathbf{x}} \in \mathbf{A}$
Then, from Remark 10 there exists $l \in \{1, \ldots, p\}$ with $\frac{\partial f}{\partial x_l}(\bar{x}) \neq 0$. Define a local C^1-vector field $F^{\bar{x}}$ as

$$F^{\bar{x}}(x_1, \ldots, x_l, \ldots, x_n) := \left(0, \ldots, -\frac{\partial f}{\partial x_l}(x) \cdot \left(\frac{\partial f}{\partial x_l}(x)\right)^{-2}, \ldots, 0\right)^T.$$

After respective inverse changes of local coordinates, $F^{\bar{x}}$ induces the flow $\Psi^{\bar{x}}$, which fits the local argument (see Theorem 2.7.6 of [63] for details).

Case (b): $\bar{\mathbf{x}} \in \mathbf{B}$
Then, from Remark 10, there exists $j \in J_0(x)$ with $\frac{\partial f}{\partial x_j}(\bar{x}) < 0$. By means of a C^1-coordinate transformation (along the lines of Theorem 3.2.26 of [63]) in the j-th coordinate on $\mathbb{H}$, leaving the other coordinates unchanged, we obtain locally for f

$$f(x_1, \ldots, x_j, \ldots, x_n) = -x_j + f(x_1, \ldots, \bar{x}_j, \ldots, x_n).$$

Define a local C^1-vector field $F^{\bar{x}}$ as

$$F^{\bar{x}}(x_1, \ldots, x_j, \ldots, x_n) := (0, \ldots, 1, \ldots, 0)^T.$$

After respective inverse changes of local coordinates, $F^{\bar{x}}$ induces the flow $\Psi^{\bar{x}}$, which fits the local argument (see Theorem 3.3.25 of [63] for details).

Case (c): $\bar{\mathbf{x}} \in \mathbf{C}$
Then, from Remark 10 there exists $m_\beta \in \beta(x)$ with

$$\frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) \cdot \frac{\partial f}{\partial x_{2,m_\beta}}(\bar{x}) < 0.$$

Without loss of generality, we assume that $\dfrac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) < 0$ and $\dfrac{\partial f}{\partial x_{2,m_\beta}}(\bar{x}) > 0$.

From the proof of Theorem 19, formula (2.20) , we can obtain for f in new C^1-coordinates the representation

$$f(x_1,\dots,x_j,\dots,x_n) = -x_{1,m_\beta} + x_{2,m_\beta} + f(x_1,\dots,\bar{x}_{1,m_\beta},\bar{x}_{2,m_\beta},\dots,x_n).$$

Define the mapping $\Psi^{\bar{x}}$ locally as

$$\Psi^{\bar{x}}(t,x_1,\dots,x_{1,m_\beta},x_{2,m_\beta},\dots,x_n) :=$$

$$(x_1,\dots,x_{1,m_\beta} + \max\{0,t-x_{2,m_\beta}\},\max\{0,x_{2,m_\beta}-t\},\dots,x_n)^T.$$

After respective inverse changes of local coordinates, $\Psi^{\bar{x}}$ fits the local argument.

Note that in all of Cases (a)–(c), $\Psi^{\bar{x}}(t,\cdot)$ leaves the feasible set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ invariant.

Globalization. Consider the open covering $\{U_x \,|\, x \in C\} \cup \{U_{\bar{x}} \,|\, \bar{x} \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ of M_a^b. From continuity arguments, $U_{\bar{x}}$, $\bar{x} \in M_a^b \backslash \{U_x \,|\, x \in C\}$ can be taken smaller, if necessary, to be disjoint with C. Since M_a^b is compact, we get a finite open subcovering $\{U_{x_i} \,|\, x_i \in C\} \cup \{U_{\bar{x}_j} \,|\, \bar{x}_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ of M_a^b. Using a C^∞-partition of unity $\{\phi_j\}$ subordinate to $\{U_{\bar{x}_j} \,|\, \bar{x}_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$, we define with $F^{\bar{x}_j}$ (see Cases (a) and (b)) a C^1-vector field $F := \sum\limits_j \phi_j F^{\bar{x}_j}$. The last induces a flow Ψ on $\{U_{\bar{x}_j} \,|\, \bar{x}_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ (see Theorem 3.3.14 of [63] for details). Note that in each nonempty overlapping region $U_{x_i} \cap U_{x_j}$, $x_i \in C$, $x_j \in M_a^b \backslash \{U_x \,|\, x \in C\}$, the flow Ψ^{x_i} induces exactly the vector field F (see Case (c)). Hence, local trajectories can be glued together on M_a^b, named by Ψ again. Moreover, moving along the local pieces of the trajectories $\Psi(\cdot,x)$, $x \in M_a^b$ reduces the level of f at least by a positive real number

$$\frac{\min\{t_{x_i}, t_{x_j} \,|\, x_i \in C, x_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}}{2}.$$

Thus, we obtain for $x \in M_a^b$ a unique $t_a(x) > 0$ with $\Psi(t_a(x),x) \in M^a$. It is not hard (but technical) to realize that $t_a : x \longrightarrow t_a(x)$ is Lipschitz. Finally, we define $r : [0,1] \times M^b \longrightarrow M^b$ as

$$r(\tau,x) : \begin{cases} x & \text{for } x \in M^a,\ \tau \in [0,1] \\ \Psi(\tau t_a(x),x) & \text{for } x \in M_a^b,\ \tau \in [0,1]. \end{cases}$$

The mapping r provides that M^a is a strong deformation retract of M^b.

(b) By virtue of the deformation theorem and the normal forms (2.19), (2.23) and (2.26), the proof of the cell-attachment part becomes standard. In fact, the deformation theorem allows deformations up to an arbitrarily small neighborhood of the C-stationary point $\bar{x}$. In such a neighborhood, we can work in continuous local coordinates and use the explicit normal form (2.26). In the normal form (2.26), the origin is a nondegenerate KKT point and the cell attachment can be performed as in Theorem 3.3.33 of [63]. □

Remark 11. We emphasize that the linear terms y_i, $i \in J_0(\bar{x})$, in (2.26) do not contribute to the dimension of the cell to be attached. In fact, w.r.t. lower-level sets, the one-dimensional constrained singularity y, $y \geq 0$, plays the same role as the unconstrained singularity y^2. In this sense, the constrained linear terms in (2.26) do not contribute to the number of negative squares.

Remark 12. Another way of looking at the cell-attachment part is via stratified Morse theory (Section 3.7 of [29]). In fact, recall the normal form (2.19). The set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ can be interpreted as the product of the "tangential part" $\{0_s\} \times \mathbb{R}^p$ and the "normal part" $\mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|}$. The main theorem in [29] states that the local "Morse data" is the product of the tangential "Morse data" with the normal "Morse data". The tangential Morse index equals QI and, in view of Remark 11, the normal Morse index equals BI. In the product, the index then becomes the sum QI+BI, which is precisely the C-index (see Figure 15).

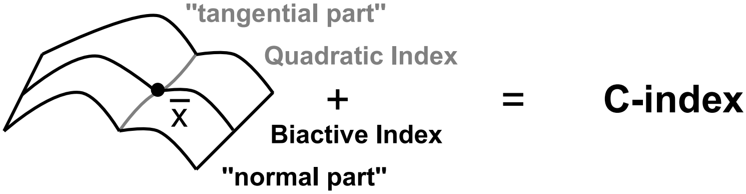

Figure 15 C-index

Remark 13. As pointed out by an anonymous referee, Theorem 20 can be interpreted as follows. The complementarity constraints can be reformulated as Lipschitzian equality constraints of the minimum type. For $u, v \in \mathbb{R}$, we have

$$u \geq 0, v \geq 0, u \cdot v = 0 \Longleftrightarrow \min\{u, v\} = 0.$$

Regarding this issue, Corollary 3 provides a normal form of f in Lipschitzian coordinates. Finally, Theorem 20 shows why the Morse index from the smooth nonlinear programming has to be modified into the Clarke index for the MPCC.

Discussion of different stationarity concepts

We briefly review well-known definitions of various stationarity concepts and connections between them (see [24], [96], [105]).

Definition 15. Let $\bar{x} \in M$.

(i) $\bar{x}$ is called W-stationary if (2.13) and (2.14) hold.
(ii) $\bar{x}$ is called A-stationary if (2.13) and (2.14) hold and

$$\bar{\sigma}_{1,m_\beta} \geq 0 \text{ or } \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}).$$

(iii) $\bar{x}$ is called M-stationary if (2.13) and (2.14) hold and

$$(\bar{\sigma}_{1,m_\beta} > 0 \text{ and } \bar{\sigma}_{1,m_\beta} > 0) \text{ or } \bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} = 0 \text{ for all } m_\beta \in \beta(\bar{x}).$$

(iv) $\bar{x}$ is called S-stationary if (2.13) and (2.14) hold and

$$\bar{\sigma}_{1,m_\beta} \geq 0, \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}).$$

(v) $\bar{x}$ is called B-stationary if $d = 0$ is a local solution of the linearized problem

$$\min f(\bar{x}) + Df(\bar{x})d \text{ s.t.}$$

$$\begin{cases} F_{1,m}(\bar{x}) + DF_{1,m}(\bar{x})d \geq 0, F_{2,m}(\bar{x}) + DF_{2,m}(\bar{x})d \geq 0, \\ (F_{1,m}(\bar{x}) + DF_{1,m}(\bar{x})d) \cdot (F_{2,m}(\bar{x}) + DF_{2,m}(\bar{x})d) = 0, \, m = 1, \ldots, k, \\ h(\bar{x}) + Dh(\bar{x})d = 0, g(\bar{x}) + Dg(\bar{x})d \geq 0. \end{cases}$$

The following diagram (see Figure 16) summarizes the relations between the stationarity concepts mentioned (e.g., [118]):

$$\begin{array}{ccc} & \textbf{S-stationary point} \quad \Longleftrightarrow & \textbf{B-stationary point} \\ & \Downarrow \quad \text{under LICQ} & \\ & \textbf{M-stationary point} & \\ & \Downarrow \qquad \Downarrow & \\ \textbf{C-stationary point} & & \textbf{A-stationary point} \\ & \Downarrow \qquad \Downarrow & \\ & \textbf{W-stationary point} & \end{array}$$

Figure 16 Stationarity concepts in MPCC

Assuming nondegeneracy (as in Definition 11), we see that A-, M-, S-, and B-stationary points describe local minima tighter than C-stationary points. However, they exclude C-stationary points with $BI > 0$. These points are also crucial for the topological structure of the MPCC (see the cell-attachment theorem). For global optimization, points of $C\text{-}index = 1$ play an important role; see also Section 1.2. We emphasize that among the points of $C\text{-}index = 1$ from a topological point of view there is no substantial difference between the points with $BI = 1$, $QI = 0$ and $BI = 0$, $QI = 1$. It is worth mentioning that a linear descent direction might exist in a nondegenerate C-stationary point with positive $C\text{-}index$ (see [87] and [105] and the following discussion). However, at points with $BI = 1$, $QI = 0$, there are exactly two directions of linear decrease. Both of them are important from a global point of view. In turn, W-stationary points contain those with negative and positive Lagrange multipliers corresponding to the same complementarity constraint. Rrom the deformation theorem, such points are irrelevant for the topological structure of the MPCC.

Furthermore, we illustrate the foregoing considerations by Example 12 from [105] (see also [87]).

Example 12.

$$\min (x-1)^2 + (y-1)^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \tag{2.27}$$

It is clear that C-stationary points for (2.27) are $(1,0)$, $(1,0)$, and $(0,0)$. Moreover, $(1,0)$ and $(1,0)$ are local (and global) minimizers with *C-index* 0. The biactive Lagrange multipliers for $(0,0)$ are both -2; hence, its *C-index* is 1. One might think that the C-stationary point $(0,0)$ is irrelevant for numerical purposes since it possesses linear descent directions. However, globally it precisely connects the local minima. Moreover, if we consider the problem (2.27) with smoothed complementarity constraints,

$$\min\ (x-1)^2+(y-1)^2 \text{ s.t. } x \geq 0,\, y \geq 0,\, x\cdot y = \varepsilon, \tag{2.28}$$

where $\varepsilon > 0$ is sufficiently small. Then, it is easily seen that the critical points for (2.28) are

$$(x_1,y_1) = \left(\frac{1+\sqrt{1-4\varepsilon}}{2}, \frac{1-\sqrt{1-4\varepsilon}}{2}\right),$$
$$(x_2,y_2) = \left(\frac{1-\sqrt{1-4\varepsilon}}{2}, \frac{1+\sqrt{1-4\varepsilon}}{2}\right),$$
$$(x_3,y_3) = \left(\sqrt{\varepsilon}, \sqrt{\varepsilon}\right).$$

Obviously, $(x_1,y_1) \longrightarrow (1,0)$, $(x_2,y_2) \longrightarrow (0,1)$, and $(x_3,y_3) \longrightarrow (0,0)$ as $\varepsilon \longrightarrow 0$. Moreover, (x_1,y_1) and (x_2,y_2) are local (and global) minimizers for (2.28) with quadratic index 0, and the quadratic index of (x_3,y_3) is 1 (local maximum). Hence, by the smoothing procedure, the C-stationary point $(0,0)$ with *C-index* 1 corresponds to the critical point (x_3,y_3) with quadratic index 1. In particular, the smoothed version preserves the global topological structure.

We notice that adding positive squares to the objective function in (2.27) provides a higher-dimensional example with the same features.

2.4 Parametric aspects

The aim of this section is the introduction and characterization of the strong stability notion in the MPCC (see Definition 17). In 1980, M. Kojima introduced in [83] the (topological) concept of strong stability of stationary solutions (Karush-Kuhn-Tucker points) for nonlinear programming (see also Robinson [101] and Section 1.4). This concept plays an important role in optimization theory, for example in sensitivity and parametric optimization [64, 84] and structural stability [78]. It turns out that the concept of C-stationarity is an adequate stationarity concept regarding possible bifurcations.

We characterize the strong stability for C-stationary points by means of first- and second-order information of the defining functions f,h,g,F_1,F_2 under the LICQ (see Theorem 21). The main issue in the strong stability of C-stationary points is related to the so-called biactive Lagrange multipliers (see also Corollary 5). A biactive pair of Lagrange multipliers corresponds to such complementarity constraints which both vanish at a C-stationary point. There are three (degeneracy) possibilities for biactive multipliers:

(a) Both biactive Lagrange multipliers do not vanish (nondegenerate case).
(b) Only one biactive Lagrange multiplier vanishes (first degenerate case).
(c) Both biactive Lagrange multipliers vanish (second degenerate case).

Depending on the kind of possible degeneracy, we use corresponding ideas on strong stability of Kojima (Cases (a) and (b)). Moreover, we describe new unstable phenomena (Case (c)).

We would like to refer to some related papers. In [105], an extension of the stability results of Kojima and Robinson to the MPCC is presented. It refers to the nondegenerate Case (a) of nonvanishing biactive Lagrange multipliers. In [99], the concept of the so-called co-1-singularity for quadratic programs with complementarity constraints (QPCCs) is studied. In our terms they refer to the first degenerate Case (b). We refer the reader to [76] for details.

Notation and Auxiliary Results

From Section 2.3 we recall the following index sets given $\bar{x} \in M$:

$$\begin{aligned} J_0(\bar{x}) &:= \{j \in J \,|\, g_j(\bar{x}) = 0\}, \\ \alpha(\bar{x}) &:= \{m \in \{1,\dots k\} \,|\, F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) > 0\}, \\ \beta(\bar{x}) &:= \{m \in \{1,\dots k\} \,|\, F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) = 0\}, \\ \gamma(\bar{x}) &:= \{m \in \{1,\dots k\} \,|\, F_{1,m}(\bar{x}) > 0, F_{2,m}(\bar{x}) = 0\}. \end{aligned}$$

Without loss of generality, we assume that at the particular point of interest $\bar{x} \in M$ it holds that

$$J_0(\bar{x}) = \{1,\dots,|J_0(\bar{x})|\},\ \alpha(\bar{x}) = \{1,\dots,|\alpha(\bar{x})|\},$$
$$\gamma(\bar{x}) = \{|\alpha(\bar{x})|+1,\dots,|\alpha(\bar{x})|+|\gamma(\bar{x})|\}.$$

We put $s := |I| + |\alpha(\bar{x})| + |\gamma(\bar{x})|$, $p := n - s - |J_0(\bar{x})| - 2|\beta(\bar{x})|$.

We also recall the notions of the LICQ and C-stationarity (see Section 2.3).

The LICQ for MPCC is said to hold at $\bar{x} \in M$ if the vectors

$$\begin{aligned} &Dh_i(\bar{x}),\ i \in I,\ DF_{1,m_\alpha}(\bar{x}),\ m_\alpha \in \alpha(\bar{x}),\ DF_{2,m_\gamma}(\bar{x}),\ m_\gamma \in \gamma(\bar{x}), \\ &Dg_j(\bar{x}),\ j \in J_0(\bar{x}),\ DF_{1,m_\beta}(\bar{x}),\ DF_{2,m_\beta}(\bar{x}),\ m_\beta \in \beta(\bar{x}) \end{aligned}$$

are linearly independent.

A point $\bar{x} \in M$ is called C-stationary for the MPCC (see Definition 10) if there exist real numbers $\bar{\lambda}_i$, $i \in I$, $\bar{\mu}_j$, $j \in J$, $\bar{\sigma}_{1,m}$, $\bar{\sigma}_{2,m}$, $m = 1,\dots,k$ (Lagrange multipliers) such that

$$Df(\bar{x}) = \sum_{i \in I} \bar{\lambda}_i Dh_i(\bar{x}) + \sum_{j \in J} \bar{\mu}_j Dg_j(\bar{x})$$

$$+\sum_{m=1}^{k}\left(\bar{\sigma}_{1,m}DF_{1,m}(\bar{x})+\bar{\sigma}_{2,m}DF_{2,m}(\bar{x})\right), \tag{2.29}$$

$$\bar{\mu}_j \cdot g_j(\bar{x}) = 0,\ j \in J, \tag{2.30}$$

$$\bar{\mu}_j \geq 0 \text{ for all } j \in J_0(\bar{x}), \tag{2.31}$$

$$\bar{\sigma}_{j,m} \cdot F_{j,m}(\bar{x}) = 0,\ j = 1,2,\ m = 1,\dots,k, \tag{2.32}$$

$$\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}). \tag{2.33}$$

The Lagrange function L is defined as follows:

$$L(x,\lambda,\mu,\sigma) := f(x) - \sum_{i\in I}\lambda_i h_i(\bar{x}) - \sum_{j\in J}\mu_j g_j(\bar{x})$$

$$-\sum_{m=1}^{k}\left(\sigma_{1,m}F_{1,m}(\bar{x})+\sigma_{2,m}F_{2,m}(\bar{x})\right). \tag{2.34}$$

Definition 16 (C-stationary pair). A vector $(\bar{x},\bar{\lambda},\bar{\mu},\bar{\sigma}) \in M \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} \times \mathbb{R}^{2k}$ satisfying (2.29)–(2.33) is called a C-stationary pair for the MPCC.

The concept of strong stability is defined by means of an appropriate seminorm. To this aim, let $\bar{x} \in \mathbb{R}^n$, $r > 0$. For defining functions (f,h,g,F_1,F_2) from (2.5), the seminorm $\|(f,h,g,F_1,F_2)\|^{C^2}_{B(\bar{x},r)}$ is defined to be the maximum modulus of the function values and partial derivatives up to order 2 of f,h,g,F_1,F_2.

Definition 17 (Strong stability). A C-stationary point $\bar{x} \in M$ (resp. a C-stationary pair $(\bar{x},\bar{\lambda},\bar{\mu},\bar{\sigma})$), for $MPCC[f,g,h,F_1,F_2]$ is called (C^2)-strongly stable if for some $r > 0$ and each $\varepsilon \in (0,r]$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever

$$\left(\widetilde{f},\widetilde{h},\widetilde{g},\widetilde{F_1},\widetilde{F_2}\right) \in C^2$$

and

$$\left\|\left(f-\widetilde{f},h-\widetilde{h},g-\widetilde{g},F_1-\widetilde{F_1},F_2-\widetilde{F_2}\right)\right\|^{C^2}_{B(\bar{x},r)} \leq \delta,$$

$B(\bar{x},\varepsilon)$ $\left(\text{resp. } B\left(\left(\bar{x},\bar{\lambda},\bar{\mu},\bar{\sigma}\right),\varepsilon\right)\right)$ contains a C-stationary point

$$\widetilde{x}\ \left(\text{resp. a C-stationary pair } \left(\widetilde{x},\widetilde{\lambda},\widetilde{\mu},\widetilde{\sigma}\right)\right) \text{ for MPCC} \left[\widetilde{f},\widetilde{h},\widetilde{g},\widetilde{F_1},\widetilde{F_2}\right]$$

that is unique in $B(\bar{x},r)$ $\left(\text{ resp. unique in } B\left(\left(\widetilde{x},\widetilde{\lambda},\widetilde{\mu},\widetilde{\sigma}\right),r\right)\right)$.

The following lemma establishes the connection between both definitions just introduced (see [82]).

Lemma 6 (C-stationary points and pairs). *The following assertions are equivalent:*

(a) $\bar{x}$ *is a strongly stable C-stationary point for the MPCC that satisfies the LICQ, and* $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$ *is the associated Lagrange multiplier vector.*
(b) $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$ *is a strongly stable C-stationary pair for the MPCC.*

Proof. (a) $\Longrightarrow$ (b) The LICQ remains valid under small perturbations of the defining functions. Hence, the corresponding Lagrange multipliers are unique. Moreover, Remark 10 provides the continuity of Lagrange multipliers w.r.t. perturbations under consideration.

(b) $\Longrightarrow$ (a) The nontrivial part is to prove that LICQ holds at $\bar{x}$. The proof goes along the lines of Theorem 2.3 from [82]. To stress the new aspects, here we assume that there are only biactive constraints (i.e., $I = \emptyset$, $J = \emptyset$, $\alpha(\bar{x}) = \emptyset$, and $\gamma(\bar{x}) = \emptyset$). Let $(\bar{x}, \bar{\sigma})$ be a strongly stable C-stationary pair for the MPCC and let the LICQ not be fulfilled at $\bar{x}$. Then, there exist real numbers δ_{1,m_β}, δ_{1,m_β}, $m_\beta \in \beta(\bar{x})$ (not all vanishing) such that

$$\sum_{m_\beta \in \beta(\bar{x})} \left(\delta_{1,m} DF_{1,m}(\bar{x}) + \delta_{2,m} DF_{2,m}(\bar{x})\right) = 0. \tag{2.35}$$

Setting

$$m_\beta^+(\bar{x}) := \{m_\beta \in \beta(\bar{x}) \,|\, \bar{\sigma}_{1,m},\, \bar{\sigma}_{2,m} \geq 0\},$$

$$m_\beta^-(\bar{x}) := \{m_\beta \in \beta(\bar{x}) \,|\, \bar{\sigma}_{1,m},\, \bar{\sigma}_{2,m} \leq 0\},$$

we define

$$c := -\left[\sum_{m_\beta \in m_\beta^+(\bar{x})} \left(DF_{1,m}(\bar{x}) + DF_{2,m}(\bar{x})\right) - \sum_{m_\beta \in m_\beta^-(\bar{x})} \left(DF_{1,m}(\bar{x}) + DF_{2,m}(\bar{x})\right)\right].$$

For $\varepsilon > 0$, let

$$\sigma_{1,m}(\varepsilon) := \bar{\sigma}_{1,m} + \varepsilon,\ \sigma_{2,m}(\varepsilon) := \bar{\sigma}_{2,m} + \varepsilon \text{ for all } m \in m_\beta^+(\bar{x}),$$

$$\sigma_{1,m}(\varepsilon) := \bar{\sigma}_{1,m} - \varepsilon,\ \sigma_{2,m}(\varepsilon) := \bar{\sigma}_{2,m} - \varepsilon \text{ for all } m \in m_\beta^-(\bar{x}).$$

Putting $\varphi(x) := c \cdot x$, we obtain that

$$(\bar{x}, \sigma(\varepsilon)) \text{ is a C-stationary pair for MPCC}\,[f + \varepsilon \cdot \varphi, F_1, F_2]\,.$$

Moreover, due to the strong stability of $(\bar{x}, \bar{\sigma})$ for MPCC$[f, F_1, F_2]$, we claim that for each sufficiently small $\varepsilon > 0$ the C-stationary pair $(\bar{x}, \sigma(\varepsilon))$ is unique for MPCC$[f + \varepsilon \cdot \varphi, F_1, F_2]$ in some neighborhood U of $(\bar{x}, \bar{\sigma})$.

However, (2.35) and $\sigma_{i,m}(\varepsilon) \neq 0$ for $m \in m_\beta(\bar{x})$, $i = 1, 2$ ensure that, for any sufficiently small real number t, the pair $(\bar{x}, v(\varepsilon, \delta, t))$ with

$$v_{1,m}(\varepsilon, \delta, t) := \sigma_{1,m}(\varepsilon) + \delta_{1,m} t,\ v_{2,m}(\varepsilon, \delta, t) := \sigma_{2,m}(\varepsilon) + \delta_{2,m} t \text{ for all } m \in m_\beta(\bar{x})$$

belongs to U and is a C-stationary pair for MPCC$[f+\varepsilon\cdot\varphi, F_1, F_2]$. Hence, necessarily $\delta=0$, and the LICQ is shown. □

Now we give two guiding examples for instability that may occur at C-stationary points in the second degenerate Case (c).

Example 13 (Unstable minimum/maximum [105]). Consider the MPCC:

$$\min x^2+y^2 \text{ s.t. } x\geq 0,\, y\geq 0,\, x\cdot y=0. \tag{2.36}$$

Obviously, $(0,0)$ is the unique C-stationary point for (2.36) with both vanishing biactive Lagrange multipliers. Consider the following perturbation of (2.36) w.r.t. parameter $t>0$:

$$\min (x-t)^2+(y-t)^2 \text{ s.t. } x\geq 0,\, y\geq 0,\, x\cdot y=0. \tag{2.37}$$

It is easy to see that $(0,0)$, $(0,t)$, and $(t,0)$ are C-stationary points for (2.37). This means that $(0,0)$ is not a strongly stable C-stationary point for (2.36). Analogously, we can treat $-x^2-y^2$ on $\partial\mathbb{H}^2$ at the origin.

Example 14 (Unstable saddle point). Consider the MPCC

$$\min x^2-y^2 \text{ s.t. } x\geq 0,\, y\geq 0,\, x\cdot y=0. \tag{2.38}$$

Obviously, $(0,0)$ is the unique C-stationary point for (2.38) with both vanishing biactive Lagrange multipliers. Consider the following perturbation of (2.38) w.r.t. parameter $t>0$:

$$\min (x-t)^2-(y-t)^2 \text{ s.t. } x\geq 0,\, y\geq 0,\, x\cdot y=0. \tag{2.39}$$

It is easy to see that $(0,t)$ and $(t,0)$ are C-stationary points for (2.39). This means that $(0,0)$ is not a strongly stable C-stationary point for (2.38).

Characterization of strong stability for C-stationary points

Before stating the main result, we define the following index sets at a C-stationary point $\bar{x}\in M$ with Lagrange multipliers $(\bar{\lambda},\,\bar{\mu},\,\bar{\sigma})$ (see Definition 10):

$$J_+ := \left\{ j\in J_0(\bar{x}) \,|\, \bar{\mu}_j>0 \right\},$$
$$p(\bar{x}) := \{m\in\beta(\bar{x}) \,|\, \bar{\sigma}_{1,m}\cdot\bar{\sigma}_{2,m}>0\},$$
$$q(\bar{x}) := \{m\in\beta(\bar{x}) \,|\, \bar{\sigma}_{1,m}>0,\, \bar{\sigma}_{2,m}=0\},$$
$$r(\bar{x}) := \{m\in\beta(\bar{x}) \,|\, \bar{\sigma}_{1,m}=0,\, \bar{\sigma}_{2,m}>0\},$$
$$s(\bar{x}) := \{m\in\beta(\bar{x}) \,|\, \bar{\sigma}_{1,m}<0,\, \bar{\sigma}_{2,m}=0\},$$
$$w(\bar{x}) := \{m\in\beta(\bar{x}) \,|\, \bar{\sigma}_{1,m}=0,\, \bar{\sigma}_{2,m}<0\},$$

$$u(\bar{x}) := \{m \in \beta(\bar{x}) \,|\, \bar{\sigma}_{1,m} = 0,\, \bar{\sigma}_{2,m} = 0\}.$$

Obviously, $p(\bar{x})$, $q(\bar{x})$, $r(\bar{x})$, $s(\bar{x})$, $w(\bar{x})$, and $u(\bar{x})$ constitute a partition of $\beta(\bar{x})$.

For $\bar{J} \subset J$, $K \subset \{1,\dots,k\}$, $j = 1,2$, we write h, $g_{\bar{J}}$, and $F_{j,K}$ for $(h_i \,|\, i \in I)$, $(g_j \,|\, j \in \bar{J})$, and $(F_{j,m} \,|\, m \in K)$, respectively.

Furthermore, for $J_+ \subset \bar{J} \subset J_0(\bar{x})$, $\bar{q} \subset q(\bar{x})$, $\bar{r} \subset r(\bar{x})$, $\bar{s} \subset s(\bar{x})$, $\bar{w} \subset w(\bar{x})$, we define $M_{\bar{J},\bar{q},\bar{r},\bar{s},\bar{w}}(\bar{x})$ to be the block matrix $\begin{pmatrix} C & X \\ Y & 0 \end{pmatrix}$ with

$$X = \left(H^T \;\; G_{\bar{J}}^T \;\; A^T \;\; \Gamma^T \;\; P^T \;\; Q^T \;\; \bar{Q}^T \;\; R^T \;\; \bar{R}^T \;\; S^T \;\; \bar{S}^T \;\; W^T \;\; \bar{W}^T\right),$$

$$Y^T = \left(H^T \;\; -G_{\bar{J}}^T \;\; A^T \;\; \Gamma^T \;\; P^T \;\; Q^T \;\; -\bar{Q}^T \;\; R^T \;\; -\bar{R}^T \;\; S^T \;\; \bar{S}^T \;\; W^T \;\; \bar{W}^T\right),$$

where

$$C = D^2_{xx}L(\bar{x},\bar{\lambda},\bar{\mu},\bar{\sigma}),\, H = Dh(\bar{x}),\, G_{\bar{J}} = Dg_{\bar{J}}(\bar{x}),$$

$$A = DF_{1,\alpha(\bar{x})}(\bar{x}),\, \Gamma = DF_{2,\gamma(\bar{x})}(\bar{x}),\, P = \left(DF_{1,p(\bar{x})}, DF_{2,p(\bar{x})}\right)(\bar{x}),$$

$$Q = DF_{1,q(\bar{x})}(\bar{x}),\, \bar{Q} = DF_{2,\bar{q}}(\bar{x}),\, R = DF_{2,r(\bar{x})}(\bar{x}),\, \bar{R} = DF_{1,\bar{r}}(\bar{x}),$$

$$S = DF_{1,s(\bar{x})}(\bar{x}),\, \bar{S} = DF_{2,\bar{s}}(\bar{x}),\, W = DF_{2,w(\bar{x})}(\bar{x}),\, \bar{W} = DF_{1,\bar{w}}(\bar{x}).$$

Theorem 21 (Characterization of strong stability). *Suppose that the LICQ holds at a C-stationary point $\bar{x} \in M$ with Lagrange multipliers $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$ (see Definition 10). Then, $\bar{x}$ is a strongly stable C-stationary point for the MPCC (see Definition 17) if and only if*

(i) $u(\bar{x}) = \emptyset$ *and*
(ii) *all matrices $M_{\bar{J},\bar{q},\bar{r},\bar{s},\bar{t}}(\bar{x})$ with*

$$J_+ \subset \bar{J} \subset J_0(\bar{x}),\, \bar{q} \subset q(\bar{x}),\, \bar{r} \subset r(\bar{x}),\, \bar{s} \subset s(\bar{x}),\, \bar{w} \subset w(\bar{x})$$

are nonsingular with the same determinant sign.

Proof. By virtue of the LICQ at $\bar{x}$, Lemma 6 allows us to deal equivalently with the strong stability of the C-stationary pair $(\bar{x},\bar{\lambda}, \bar{\mu}, \bar{\sigma})$.

Case 1: $\mathbf{u}(\bar{\mathbf{x}}) = \emptyset$

We consider the following mapping $\mathscr{T} : \mathbb{R}^n \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} \times \mathbb{R}^{2k} \longrightarrow \mathbb{R}^{n+|I|+|J|+2k}$ locally at its zero $(\bar{x},\bar{\lambda}, \bar{\mu}, \bar{\sigma})$:

$$\mathscr{T}(x,\lambda,\mu,\sigma) := \begin{pmatrix} D_x L(x,\lambda,\mu,\sigma) \\ h(x) \\ \min\{\mu,\, g(x)\} \\ F_{1,\alpha(\bar{x})}(x) \\ F_{2,\gamma(\bar{x})}(x) \\ F_{1,p(\bar{x})}(x) \\ F_{2,p(\bar{x})}(x) \\ F_{1,q(\bar{x})}(x) \\ \min\left\{\sigma_{2,q(\bar{x})},\, F_{2,q(\bar{x})}(x)\right\} \\ F_{2,r(\bar{x})} \\ \min\left\{\sigma_{1,r(\bar{x})},\, F_{1,r(\bar{x})}(x)\right\} \\ F_{1,s(\bar{x})} \\ \min\left\{-\sigma_{2,s(\bar{x})},\, F_{2,s(\bar{x})}(x)\right\} \\ F_{2,w(\bar{x})} \\ \min\left\{-\sigma_{1,w(\bar{x})},\, F_{1,w(\bar{x})}(x)\right\} \end{pmatrix}.$$

Note that C-stationary pairs for MPCC in a sufficiently small neighborhood of $(\bar{x},\bar{\lambda},\,\bar{\mu},\,\bar{\sigma})$, are precisely the zeros of $\mathscr{T}$. Moreover, the only difference in $\mathscr{T}$ compared with the case of nonlinear programming is the appearing minus sign in

$$\min\left\{-\sigma_{2,s(\bar{x})},\, F_{2,s(\bar{x})}(x)\right\},\ \min\left\{-\sigma_{1,w(\bar{x})},\, F_{1,w(\bar{x})}(x)\right\}.$$

Since we deal with equality constraints of minimum type, Theorem 4.3 from [82] (characterization of strong stability for KKT points) can be simply adapted here. Indeed, as in Theorem 4.3 from [82], the strong stability for $(\bar{x},\bar{\lambda},\,\bar{\mu},\,\bar{\sigma})$ can be characterized by the fact that all matrices in the Clarke subdifferential $\partial\mathscr{T}(\bar{x},\bar{\lambda},\,\bar{\mu},\,\bar{\sigma})$ are nonsingular. The latter can equivalently be rewritten as condition (ii) (see also [80] and Theorem 9 for the case of nonlinear programming).

Case 2: $\mathbf{u}(\bar{\mathbf{x}}) \neq \emptyset$

Let $\Phi : U \longrightarrow V$ be a standard diffeomorphism according to Definition 14. We put $\bar{f} := f \circ \Phi^{-1}$ on the set $\left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p\right) \cap V$. From now on, we may assume $s = 0$. In view of Remark 10, we have at the origin

(i) $\dfrac{\partial \bar{f}}{\partial y_j} \geq 0,\ j \in J_0(\bar{x}),$

(ii) $\dfrac{\partial \bar{f}}{\partial y_{|J_0|+2m-1}} \cdot \dfrac{\partial \bar{f}}{\partial y_{|J_0|+2m}} \geq 0,\ m = 1,\ldots |\beta(\bar{x})|,$

(iii) $\dfrac{\partial \bar{f}}{\partial y_{l+n-p}} = 0,\ l = 1,\ldots,p.$

Moreover, due to condition $u(\bar{x}) \neq \emptyset$ we may assume w.l.o.g. that

(iv) $\dfrac{\partial \bar{f}}{\partial y_{|J_0|+1}} = 0,\ \dfrac{\partial \bar{f}}{\partial y_{|J_0|+2}} = 0.$

From now on, we denote $\bar{f}$ again by f.

In what follows, we successively perform arbitrarily small perturbations of f such that the origin remains a C-stationary point on $\mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial\mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p$.

(1) As a stabilization step, we add to f an arbitrarily small linear-quadratic term

$$\sum_{j=1}^{|J_0(\bar{x})|} c_j \cdot y_j + \sum_{m=2}^{|\beta(\bar{x})|} \left(c_{|J_0|+2m-1} \cdot y_{|J_0|+2m-1} + c_{|J_0|+2m} \cdot y_{|J_0|+2m}\right) + \sum_{l=1}^{p} c_{l+n-p} y_{l+n-p}^2$$

such that it holds that for the perturbed function (denoted again by f)

(i) $\quad \dfrac{\partial f}{\partial y_j} > 0,\ j \in J_0(\bar{x}),$

(ii) $\quad \dfrac{\partial f}{\partial y_{|J_0|+2m-1}} \cdot \dfrac{\partial f}{\partial y_{|J_0|+2m}} > 0,\ m = 2, \ldots |\beta(\bar{x})|,$

(iii) $\quad \dfrac{\partial f}{\partial y_{l+n-p}} = 0,\ l = 1, \ldots, p,$ and

$$\left(\frac{\partial^2 f}{\partial y_{k_1+n-p} \partial y_{k_2+n-p}}\right)_{1 \le k_1, k_2 \le p} \text{ is a nonsingular matrix,}$$

and

(iv) $\quad \dfrac{\partial f}{\partial y_{|J_0|+1}} = 0,\ \dfrac{\partial f}{\partial y_{|J_0|+2}} = 0.$

(2) We approximate f by means of a C^∞-function in a small C^2-neighborhood of f leaving its value and its first- and second-order derivatives at the origin invariant. This can be done since C^∞-functions lie C^2-dense in C^2-functions. We denote the latter C^∞-approximation again by f.

From the stabilization step (1) and step (2), we can restrict our considerations to the following case:

$$f \in C^\infty\left(\mathbb{R}^2, \mathbb{R}\right), 0 \text{ is a C-stationary point for } f_{|\partial\mathbb{H}^2} \text{ and } \frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0.$$

Now we can write $f(x,y)$ as

$$f(x,y) = g_{1,1}(x,y)x^2 + 2g_{1,2}(x,y)xy + g_{2,2}(x,y)y^2$$

with $g_{1,1}, g_{1,2}, g_{2,2} \in C^\infty\left(\mathbb{R}^2, \mathbb{R}\right)$.

Adding to f an arbitrarily small quadratic term $ax^2 + by^2$, $a, b \in \mathbb{R}$, we can ensure that

$$g_{1,1}(0,0) \neq 0 \text{ and } g_{2,2}(0,0) \neq 0.$$

Hence, $\Psi(x,y) := \begin{pmatrix} x \cdot \sqrt{|g_{1,1}(x,y)|} \\ y \cdot \sqrt{|g_{2,2}(x,y)|} \end{pmatrix}$ is a local C^∞-diffeomorphism leaving $\partial\mathbb{H}^2$ invariant. In new local coordinates induced by Ψ, we obtain

$$f(x,y) = \varepsilon_1 x^2 + G(x,y)xy + \varepsilon_2 y^2,$$

where $\varepsilon_1 = \operatorname{sign}(g_{1,1}(0,0))$, $\varepsilon_2 = \operatorname{sign}(g_{2,2}(0,0))$.

Since $G(x,y)xy = 0$ on $\partial\mathbb{H}^2$, we can perturb f by means of a real parameter as in Example 13 or 14 to get a bifurcation of 0 as a C-stationary point.

Finally, performing all perturbations described above, we ensure that 0 is not a strongly stable C-stationary point. □

The main new issue in the characterization of strong stability of C-stationary points in the MPCC will be clarified in Corollary 5. Its proof follows from Theorem 21 by means of a few elementary calculations.

Corollary 5. *Let $f \in C^2(\mathbb{R}^2, R)$ and suppose that 0 is a C-stationary point for $f_{|\partial\mathbb{H}^2}$. Then, 0 is a strongly stable C-stationary point if and only if*

$$\textit{either } \frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} > 0 \textit{ or } \frac{\partial f}{\partial x} = 0, \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial f}{\partial y} > 0 \textit{ or } \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial f}{\partial x} > 0 \textit{ at } 0.$$

Now, we relate the notion of a nondegenerate C-stationary point to the results in Theorem 21 and Corollary 5.

Corollary 6. *Let $\bar{x} \in M$ be a nondegenerate C-stationary point as in Definition 11. Then, $\bar{x}$ is a strongly stable C-stationary point for the MPCC.*

In the situation of Corollary 5, we claim that 0 is a nondegenerate C-stationary point for $f_{|\partial\mathbb{H}^2}$ if and only if

$$\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} > 0 \text{ at } 0.$$

From this, we get the following degeneracy possibilities of biactive Lagrange multipliers in the situation of Corollary 5 (see Figure 17):

(a) nondegenerate case: both biactive Lagrange multipliers do not vanish,

$$\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} > 0 \longrightarrow \textbf{stability},$$

(b) first degenerate case: only one biactive Lagrange multiplier vanishes,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial f}{\partial y} > 0 \text{ or } \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial f}{\partial x} > 0 \longrightarrow \textbf{stability},$$

(c) second degenerate case: both biactive Lagrange multipliers vanish,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \longrightarrow \textbf{instability}.$$

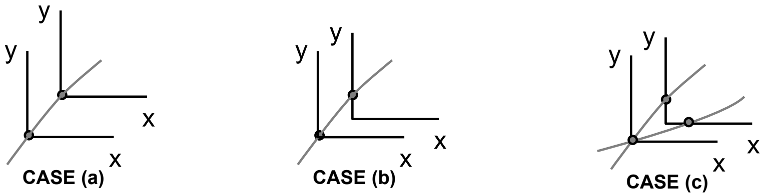

Figure 17 Strong stability in MPCC

On stability w.r.t. different stationarity concepts

For different stationarity concepts (such as A-, M-, S-, and B-stationarity), we refer the reader to Definition 15. Strong stability for A-, M-, S-, and B-stationary points can be defined analogously as in Definition 17. From now on, we assume that the LICQ holds at all points of interest.

It is clear that strongly stable S-stationary points can be characterized by means of Theorem 21. Indeed, each (not) strongly stable S-stationary point corresponds to a (not) strongly stable C-stationary point.

However, the issue becomes different as soon as we consider M-stationary points (or A-stationary points).

Example 15. Consider the MPCC

$$\min \; -x-y^2 \text{ s.t. } x \geq 0,\, y \geq 0,\, x \cdot y = 0. \tag{2.40}$$

Obviously, $(0,0)$ is the unique C-stationary point for (2.40) with biactive Lagrange multipliers $(-1,0)$. Hence, $(0,0)$ is also M-stationary. Moreover, from Corollary 5, $(0,0)$ is a strongly stable C-stationary point. Consider the following perturbation of (2.40) w.r.t. parameter $t > 0$:

$$\min \; -x-(y+t)^2 \text{ s.t. } x \geq 0,\, y \geq 0,\, x \cdot y = 0. \tag{2.41}$$

It is easy to see that $(0,0)$ is the unique C-stationary point for (2.41) with both biactive Lagrange multipliers $(-1,-2t)$ negative. It means that $(0,0)$ is not a strongly stable M-stationary point for (2.40).

Remark 14. We consider once more Example 13. We recall that $(0,0)$ is the unique C-stationary point for (2.36) with both vanishing biactive Lagrange multipliers. Hence, $(0,0)$ is also M-stationary. For the perturbed program (2.37), we have that $(0,0)$, $(0,t)$, and $(t,0)$ are C-stationary. It is easy to see that $(0,0)$ is not M-stationary for (2.37).

We note that adding positive or negative squares to the objective functions above provides higher-dimensional examples with similar features.

Finally, we point out some issues with the MPCC motivated by the strong stability results.

Remark 15. From Example 15 and Remark 14, the concept of C-stationarity is crucial for the numerical treatment of the MPCC via homotopy-based methods. Furthermore, Theorem 21 provides a characterization of the strongly stable C-stationary points. These are solutions of certain stable equations involving first- and second-order information of the defining functions. This fact might be used to establish some nonsmooth versions of the Newton method for the MPCC (see [53, 57]). This is an issue of current research.

Remark 16. In Theorem 21, the strong stability of C-stationary points is characterized under the LICQ. Its characterization in the absence of the LICQ is still open. We point out that this issue might be related to the version of the Mangasarian-Fromovitz condition (MFC) as introduced in [70] (see also Definition 7). The constraint qualification MFC has been introduced in connection with topological stability of the MPCC feasible set. This is still an issue of current research.

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