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Brief paper

Lossless convexification of non-convex optimal control problems for state constrained linear systems[☆]

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ABSTRACT

This paper analyzes a class of finite horizon optimal control problems with mixed non-convex and convex control constraints and linear state constraints. A convex relaxation of the problem is proposed, and it is proved that a solution of the relaxed problem is also a solution of the original problem. This process is called lossless convexification, and its generalization for problems with state constraints is the primary contribution of the paper. Doing so enables the use of interior point methods of convex optimization to obtain global optimal solutions of the original non-convex problem. The approach is also demonstrated on example problems.

1. Introduction

This paper presents lossless convexification for a class of optimal control problems with mixed non-convex and convex control constraints and linear state constraints. Its main contribution is to prove that a global optimal solution of the problem can be obtained by solving a convex relaxation of the problem. Inclusion of the additional convex control constraints and linear state constraints makes this work new and significant. Attempts to directly extend the previous work (Açıkmeşe & Blackmore, 2011) to this new class of problems fall short.

Thus, we use a more comprehensive maximum principle for mixed state/control constraints (Milyutin & Osmolovskii, 1998) and introduce the notions of strongly controllable/observable subspaces (Trentelman, Stoorvogel, & Hautus, 2001; Wonham, 1985) to establish a new lossless convexification result. Utilization of these geometric control concepts increases the mathematical complexity of the new development, but brings broader generality suitable for optimal trajectories on state constraint boundaries.

Perhaps more significant than these theoretical contributions are the immediate impacts on industry applications. In 2012, successful flight tests flew optimal trajectories generated by our convex formulations (Açıkmeşe et al., 2013). In 2013, successful flight tests flew optimal trajectories generated in real-time by our convex formulations (Scharf et al., 2014). These formulations included state constraints, and it is the work presented here that provided the foundations for their real-world success. Thus, the methodology and results herein represent important milestones in lossless convexification and real-time optimization.

Lossless convexification refers to the process of proving that a convex relaxation of a problem can be solved instead of the original non-convex problem to obtain a global optimal solution of the original problem. This result is especially useful since the relaxed convex problem is solvable with interior point methods (IPMs) (Boyd & Vandenberghe, 2004; Nesterov & Nemirovsky, 1994; Toh, Todd, & Tutuncu, 1999) and customized IPM solvers (Mattingley & Boyd, 2012). The use of IPM algorithms guarantees convergence to the global optimal in polynomial time, and the use of customized IPM solvers means that the problems are solvable in real-time—as evidenced by our recent flight tests.

Previous work has been unable to address the state constrained problem. With our new proofs, more difficult problems can be solved and earlier results can be obtained as special cases. The earlier results only handle cases where the optimal state trajectory touches the state constraint boundary at most a finite number of times (Açıkmeşe & Blackmore, 2011; Açıkmeşe & Ploen, 2007; Blackmore, Açıkmeşe, & Carson, 2012; Blackmore, Açıkmeşe, & Scharf, 2010). They do not extend to the case where a state constraint is active continuously over a finite time interval. Active state constraints mean that the states are restricted to a subspace.

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Handling such optimal trajectories is the main motivation for the current work.

Previous convexification results are all tied to controllability of the dynamic system. The generality of our current result is best seen in this light. By bringing the notions of strongly controllable/observable subspaces into the picture (Trentelman et al., 2001; Wonham, 1985), we naturally incorporate the dynamics, control constraints, and restricted state space into the convexification result. The new result states that convexification holds when the state space is a strongly controllable subspace for the system (dynamics and control constraints). When there are no additional control and state constraints, the state space becomes $\mathbb{R}^n$ and strong controllability reduces to the standard notion of controllability. This indicates a deep and useful connection with multivariable systems theory.

There are numerous problems that belong to the class studied here and provide strong motivation for this work. In the planetary landing problem (Açıkmeşe & Ploen, 2005), a spacecraft lands using thrusters, which produce a force vector whose magnitude is bounded above and below. The lower bound introduces a non-convex control constraint. The landing problem also incorporates a number of state constraints, e.g., altitude limits and landing cones, which can be written as linear state constraints. The rendezvous problem in low earth orbit is based on the Clohessy–Wiltshire–Hill (CWH) equations and fits squarely within the class of problems studied here (Clohessy & Wiltshire, 1960). That problem has pointing constraints, thrust magnitude constraints, and more. Additionally, the CWH equations decouple into a double integrator and harmonic oscillator. Integrators and oscillators are prevalent in many mechanical systems, and so our results here apply to many mechanical problems.

Convex optimization, in general, is becoming more popular in applications. The problem of autonomous rendezvous and docking of multiple spacecraft has recently been investigated using convex optimization (Lu & Liu, 2013), and in the case of rendezvous using differential drag, a lossless convexification was proposed for a non-convex, discrete control set (Harris & Açıkmeşe, 2014). Convex optimization has also been used for constrained control of attitude dynamics (Lee & Mesbahi, 2011). Lastly, convexification enables the use of receding horizon model predictive control to obtain a robust feedback control action (Garcia, Prett, & Morari, 1989; Mayne, Rawlings, Rao, & Scokaert, 2000). This paper adds a large class of problems that can be solved as convex optimization problems.

After lossless convexification, the convex problem is still an infinite-dimensional optimal control problem. This difficulty is overcome using direct numerical methods, where the infinite-dimensional control problem is approximated by a finite-dimensional parameter optimization problem (Betts, 1998; Hull, 1997; Vlassenbroeck & Dooren, 1988). Since the relaxed control problem is convex, the resulting parameter optimization problem is also convex. A convex optimization problem, under mild computability and regularity assumptions, is solvable to global optimality in polynomial time with an a priori known upper bound on the number of mathematical operations needed (Boyd & Vandenberghe, 2004; Nesterov & Nemirovsky, 1994; Peng, Roos, & Terlaky, 2001). Hence, the convexification leads to a computationally tractable solution method, which is critically important for autonomous real-time control applications (Açıkmeşe et al., 2013). In contrast, nonlinear programming methods only guarantee convergence to a local optimum and may fail to converge (cf. Buskens and Maurer (2000), Gerdts (2008a,b), Loxton, Teo, Rehbock, and Yiu (2009), and Wu and Teo (2006)). Alternatively, local minimizers can be found by solving a sequence of unconstrained problems using a penalty function approach (Li, Yu, Teo, & Duan, 2011). Convexification also enables the use of model predictive control to obtain a robust feedback control action (Garcia et al., 1989; Mayne et al., 2000).

The structure of the paper is as follows. Section 2 formally states the problem of interest and assumptions on which lossless convexification is proved. A few results from linear systems theory are needed to do so and are detailed in Section 3. The proof of lossless convexification uses a maximum principle for optimal control problems with mixed control/state constraints (Milyutin & Osmolovskii, 1998). Because this is a non-standard result, it is given in Section 4. The relaxed problem is formulated and proof of lossless convexification is given in Section 5. Converting the optimal control problem to a finite dimensional problem is covered in Section 6. Finally, two example problems are solved using Yalmip and SDPT3 (Toh et al., 1999) in Sections 7 and 8.

The following is a partial list of notation used: $A\mathcal{X}$ is the set of all points Ax for every $x \in \mathcal{X}$; im A is the set of all points Ax where $x \in \mathcal{X}$; if $\mathcal{Y}$ is an n dimensioned subspace of $\mathcal{Z}$ and the columns of Z are a basis for $\mathcal{Z}$, then a basis for $\mathcal{Z}$ adapted to $\mathcal{Y}$ is one where the first n columns of Z are a basis for $\mathcal{Y}$; a condition is said to hold almost everywhere in the interval $[a, b]$, a.e. in $[a, b]$, if the set of points in $[a, b]$ where this condition fails to hold is in a set of measure zero; the time derivative of H is denoted $\dot{H}$; the partial derivative of H with respect to t is denoted $\partial_t H$; the partial derivative of H with respect to vector x is denoted $\partial_x H$ and is a column vector.

2. Problem formulation

This section introduces the optimal control problem that is of primary interest. It is labeled as problem P0.

P0 (1)

$$\begin{aligned}
\min \quad & J = m(t_f, x(t_f)) + \int_{t_0}^{t_f} \ell(\kappa(u(t)))\, dt \\
\text{subj. to} \quad & \dot{x}(t) = Ax(t) + Bu(t) + Ew(t),\ x(t_0) = x_0 \\
& 0 < \rho_1 \le \kappa(u(t)) \le \rho_2,\ Cu(t) \le d \\
& x(t) \in \mathcal{X},\ b(t_f, x(t_f)) = 0.
\end{aligned}$$

The system state is $x(t) \in \mathbb{R}^{d_x}$, the control input is $u(t) \in \mathbb{R}^{d_u}$, and the known exogenous input is $w(t) \in \mathbb{R}^{d_w}$. All are defined on the interval $[t_0, t_f]$ where t_0 is the initial time and t_f is the final time. The performance index is J, and it consists of a terminal cost $m : \mathbb{R} \times \mathbb{R}^{d_x} \to \mathbb{R}$ and a running cost $\ell : \mathbb{R} \to \mathbb{R}$ where $\kappa : \mathbb{R}^{d_u} \to \mathbb{R}$. The system dynamics are described by the linear differential equations where A, B, and E are constant matrices. The control input is restricted by the inequalities where $\rho_1, \rho_2 \in \mathbb{R}$, $C \in \mathbb{R}^{d_d \times d_u}$, and $d \in \mathbb{R}^{d_d}$ are constants. The state is restricted to evolve in a linear subspace $\mathcal{X} \subset \mathbb{R}^{d_x}$, and the terminal constraint is $b : \mathbb{R} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_b}$.

In addition, the following qualifications are made:

(i) The controls belong to the set of piecewise continuous functions.
(ii) The function $m(\cdot, \cdot)$ is affine in both arguments.
(iii) The functions $\ell(\cdot)$ and $\kappa(\cdot)$ are convex and strictly positive except possibly at the origin.
(iv) The function $b(\cdot, \cdot)$ is affine in both arguments.

Problem P0 is a non-convex optimal control problem since the control inequality constraint $0 < \rho_1 \le \kappa(u(t)) \le \rho_2$ is non-convex. In two dimensions, this non-convex constraint can be an annulus as in Fig. 1(left).

The shaded region is the set of admissible controls, and it is clearly non-convex. Problem P0 also includes linear inequality constraints on the control of the form $Cu(t) \le d$. In the two-dimensional example, each row of C represents a line cutting through the annulus. For example, when C has two rows C_1 and C_2, the constraints can be illustrated as in Fig. 1(right). The set

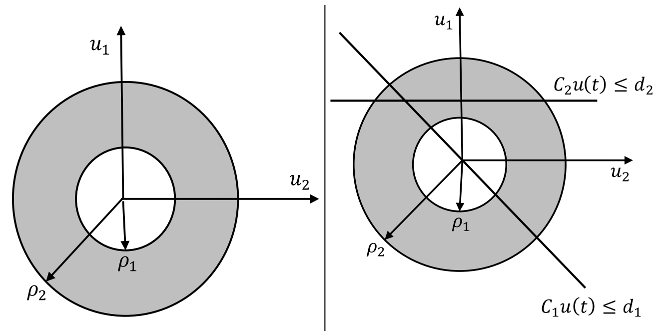

Fig. 1. *Left:* Two-dimensional constraint. *Right:* Two-dimensional constraint with linear inequalities.

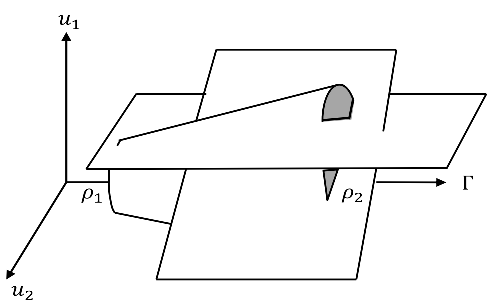

Fig. 2. Three-dimensional constraint.

of admissible controls becomes the shaded region below the C_2 constraint and left of the C_1 constraint. The control set remains non-convex in this case. The purpose of this paper is to show that a convex relaxation of problem P0 can be solved to obtain a solution for P0. The relaxation is motivated by geometric insight. In two dimensions, the annulus of Fig. 1(right) can be lifted to a convex cone by introducing a third dimension and extending the annulus in this direction. See Fig. 2.

This solid cone extends from ρ_1 to ρ_2 along the Γ axis. The two planes that intersect the cone are the two linear inequality constraints that have also been extended in the Γ direction. The set of admissible controls is now all points in the cone that are also below the planes. This particular relaxation has introduced controls that may be inadmissible in P0 since points inside the cone are not necessarily admissible. For example, the point $(u_1, u_2, \Gamma) = (0, 0, \rho_2)$ is admissible in the relaxation although $(u_1, u_2) = (0, 0)$ is not admissible in P0. In lossless convexification, it is shown that this cannot occur.

With this geometric motivation, the relaxed convex problem is stated below as P1.

P1 (2)

$$\begin{aligned} \min \quad & J = m(t_f, x(t_f)) + \int_{t_0}^{t_f} \ell(\Gamma(t))\, dt \\ \text{subj. to} \quad & \dot{x}(t) = Ax(t) + Bu(t) + Ew(t),\ x(t_0) = x_0 \\ & 0 < \rho_1 \le \Gamma(t) \le \rho_2,\ \kappa(u(t)) \le \Gamma(t) \\ & Cu(t) \le d,\ x(t) \in \mathcal{X},\ b(t_f, x(t_f)) = 0. \end{aligned}$$

Because P1 is a convex problem, its finite-dimensional approximation can be solved to global optimality in polynomial time with an upper bound on the number of mathematical operations needed (Boyd & Vandenberghe, 2004; Nesterov & Nemirovsky, 1994). Further, under certain conditions, solutions of P1 are also solutions of P0. Thus, the convex problem can be solved instead of the non-convex problem. These sufficient conditions are stated below in Criterion 1.

Criterion 1. (i) *P0 is time-invariant and there exist friends F and G such that $\mathcal{X}$ is the strongly controllable subspace for the linear system $(A + BF, BG, CF, CG)$; or,*

(ii) *P0 is time-varying, $\mathcal{X}$ is A-invariant, the vector $(\partial_x m[t_f];\ \partial_t m[t_f] + \ell[t_f])$ is linearly independent of $(\partial_x b[t_f];\ \partial_t b[t_f])$, and there exists a friend G such that $\mathcal{X}$ is the strongly controllable subspace for the linear system $(A, BG, 0, CG)$.*

All of these conditions can be checked a priori, and they are natural extensions of the standard controllability conditions required for convexification of problems without state constraints (Açıkmeşe & Blackmore, 2011). Physically, the conditions require that the dynamic system be controllable on the restricted subspace – not the entire state space – meaning the system can transfer between any two points in the subspace without leaving the subspace. The conditions are satisfied for many applications since controllability is designed into the system for practical reasons.

The notions of a "friend" of a linear system and a strongly controllable subspace are introduced in Section 3. The optimal control problem is considered time-invariant if w, m, and b do not depend explicitly on time. It is time-varying if any one of w, m, or b does depend explicitly on time.

Given Criterion 1, the formal statement for lossless convexification of P0 to P1 is Theorem 2. The proof is given in Section 5.

Theorem 2. *If Criterion 1 is satisfied, then optimal solutions of* P1 *are optimal solutions of* P0.

3. Useful concepts from linear system theory

This section presents important concepts from linear systems theory (Trentelman et al., 2001) including those in Criterion 1. Only those concepts that are critical to the proof of lossless convexification in Section 5 are covered in this section.

The first task is to establish the meaning of a "friend". This is done by considering the following linear system, which is the same as that in problems P0 and P1.

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \qquad x(t_0) = x_0. \tag{3}$$

As in those problems, the state $x(t)$ is restricted to evolve in a subspace $\mathcal{X}$. By simple extensions of Trentelman et al. (2001), it can be shown that the state evolves in $\mathcal{X}$ if and only if the control has the form

$$u(t) = Fx(t) + Gv(t) + Hw(t). \tag{4}$$

The matrices F, G, and H are the so-called friends, and they must belong to the following sets.

$$\begin{aligned} \mathcal{F}(\mathcal{X}) &:= \{F : (A + BF)\mathcal{X} \subset \mathcal{X}\} \\ \mathcal{G}(\mathcal{X}) &:= \{G : \text{im } BG \subset \mathcal{X}\} \\ \mathcal{H}(\mathcal{X}) &:= \{H : (E + BH)\mathcal{W} \subset \mathcal{X}\}. \end{aligned} \tag{5}$$

The second task is to establish the meaning of strongly controllable and strongly observable subspaces. To do so, we must introduce the standard output equation

$$y(t) = Rx(t) + Su(t). \tag{6}$$

Together with Eq. (3), this linear system is denoted $\Sigma = (A, B, R, S)$. By definition, an initial point is a strongly controllable point if the origin is instantaneously reachable by an impulsive input. The set of all such points forms the strongly controllable subspace, denoted $\mathcal{C}(\Sigma)$. If the strongly controllable subspace is the entire space, then the system is said to be strongly controllable. Strong observability is defined similarly, but it is sufficient to know that Σ being strongly controllable is equivalent to $\Sigma^T = (A^T, R^T, B^T, S^T)$ being strongly observable (Trentelman et al., 2001). We also have the following theorem from Trentelman et al. (2001).

Theorem 3. *$\mathcal{C}(\Sigma)$ is the smallest subspace $\mathcal{V}$ for which there exists a linear map K such that*

$$(A+KR)\mathcal{V} \subset \mathcal{V} \quad \text{and} \quad \mathrm{im}(B+KS) \subset \mathcal{V}.$$

This theorem leads us to a very important fact used to prove convexification in Section 5. Suppose that $A\mathcal{C}(\Sigma) \subset \mathcal{C}(\Sigma)$, i.e., $\mathcal{C}(\Sigma)$ is A-invariant, $\mathrm{im}B \subset \mathcal{C}(\Sigma)$, and let T be a basis adapted to $\mathcal{C}(\Sigma)$. Then the differential equations for Σ can be written as

$$\begin{bmatrix} \dot{\zeta}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \zeta(t) \\ \sigma(t) \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix} u(t) \tag{7}$$

where the star quantities are possibly non-zero and $\sigma(t) = 0$ if and only if $x(t) \in \mathcal{C}(\Sigma)$. Likewise, the output equation can be written as

$$y(t) = [\tilde{R} \; *] \begin{bmatrix} \zeta(t) \\ \sigma(t) \end{bmatrix} + \tilde{S}u(t). \tag{8}$$

The system of matrices with tilde and starred quantities is denoted $\bar{\Sigma} = (\bar{A}, \bar{B}, \bar{R}, \bar{S})$, and the system with only tilde quantities is denoted $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{R}, \tilde{S})$. With this notation, we can state the following theorem.

Theorem 4. *If $A\mathcal{C}(\Sigma) \subset \mathcal{C}(\Sigma)$ and im $B \subset \mathcal{C}(\Sigma)$, then $\tilde{\Sigma}$ is strongly controllable.*

The proof is beyond the scope of this paper, but it is based on simple extensions of results proved in the text by Trentelman et al. (2001).

4. Maximum principle with mixed constraints

The purpose of this section is to state a maximum principle for optimal control problems with mixed control/state constraints (Milyutin & Osmolovskii, 1998) that is needed in Section 5 for the main convexification result. Because of the mixed constraints, the standard result of Pontryagin does not apply (Pontryagin, Gamkrelidze, Boltyanskii, & Mischenko, 1964). A fairly general optimal control problem with mixed constraints is

OCP

$$\min \quad J = m(t_f, x(t_f)) + \int_{t_0}^{t_f} \ell(t, x(t), u(t))\, dt$$

$$\text{subj. to} \quad \dot{x}(t) = f(t, x(t), u(t)), \;\; x(t_0) = x_0$$

$$g(t, x(t), u(t)) = 0, \;\; h(t, x(t), u(t)) \le 0$$

$$b(t_f, x(t_f)) = 0.$$

The following set notation is also used to denote the control constraints.

$$\Omega(t, x(t)) = \{u(t) : g(t, x(t), u(t)) = 0, \ldots h(t, x(t), u(t)) \le 0\}.$$

Inherent in the statement of this particular maximum principle are three constraint qualifications (cf. Milyutin and Osmolovskii (1998), and Hartl, Sethi, and Vickson (1995)):

- All functions are continuously differentiable.
- The gradients of the active control constraints are linearly independent.
- The gradients of the final constraints are linearly independent.

It is convenient to define the Hamiltonian, Lagrangian, and endpoint functions as follows.

$$\mathcal{H}[t] = p_0 \ell[t] + p(t)^T f[t]$$

$$\mathcal{L}[t] = \mathcal{H}[t] + \lambda(t)^T g[t] + \nu(t)^T h[t]$$

$$\mathcal{G}[t_f] = p_0 m[t_f] + \xi^T b[t_f].$$

The maximum principle with mixed control/state constraints is from Milyutin and Osmolovskii (1998).

Theorem 5. *Let $\{x(\cdot), u(\cdot)\}$ be an optimal pair on the interval $[t_0, t_f]$. Then there exist a constant $p_0 \le 0$, functions $p(\cdot)$, $\lambda(\cdot)$, and $\nu(\cdot)$, and constant ξ such that the following conditions are satisfied:*

(i) *the non-triviality condition*

$$(p_0, p(t)) \ne 0 \quad \forall t \tag{9}$$

(ii) *the pointwise maximum condition*

$$u(t) = \arg \max_{\omega \in \Omega(t, x(t))} \mathcal{H}(t, x(t), \omega, p(t), p_0) \text{ a.e. } t \tag{10}$$

(iii) *the differential equations*

$$\begin{aligned} &\dot{x}(t) = \partial_p \mathcal{L}[t] \\ &-\dot{p}(t) = \partial_x \mathcal{L}[t] \;\; a.e.\, t \\ &\dot{\mathcal{H}}[t] = \partial_t \mathcal{L}[t] \end{aligned} \tag{11}$$

(iv) *the stationary condition*

$$\partial_u \mathcal{L}[t] = 0 \;\; a.e.\, t \tag{12}$$

(v) *the complementary slackness conditions*

$$\begin{aligned} &g[t] = 0, \quad h[t] \le 0 \;\; a.e.\, t \\ &\nu(t) \le 0, \quad \nu(t)^T h[t] = 0 \;\; a.e.\, t \end{aligned} \tag{13}$$

(vi) *the prescribed boundary conditions*

$$x(t_0) = x_0, \qquad b[t_f] = 0 \tag{14}$$

(vii) *the transversality conditions*

$$\begin{aligned} &p(t_f) = \partial_x \mathcal{G}[t_f] \\ &-\mathcal{H}[t_f] = \partial_t \mathcal{G}[t_f]. \end{aligned} \tag{15}$$

5. Proof of lossless convexification

The purpose of this section is to prove Theorem 2, which says that optimal solutions of P1 are optimal solutions of P0. This is the main result of the paper. To complete the proof, two more problems, P2 and P3, must be considered. The strategy here is to prove a property about optimal solutions of P3, link the solutions of P0–P3, and then show that this implies convexification between P0 and P1. The problems P2 and P3 are used only in the proofs. They have no role in the numerical solution process.

The problem P2 is obtained from P1 by replacing the state constraint $x(t) \in \mathcal{X}$ with the mixed control/state constraint specified in Eq. (4). Upon doing so, the closed loop system becomes

$$\dot{x}(t) = A_F x(t) + B_G v(t) + E_H w(t) \tag{16}$$

where $A_F = A + BF$, $B_G = BG$, and $E_H = E + BH$ such that problem P2 is

P2 (17)

$$\min \quad J = m(t_f, x(t_f)) + \int_{t_0}^{t_f} \ell(\Gamma(t))\, dt$$

$$\begin{aligned} \text{subj. to} \quad &\dot{x}(t) = A_F x(t) + B_G v(t) + E_H w(t) \\ &u(t) = Fx(t) + Gv(t) + Ew(t), \; Cu(t) \le d \\ &0 < \rho_1 \le \Gamma(t) \le \rho_2, \;\; \kappa(u(t)) \le \Gamma(t) \\ &x(t_0) = x_0, \; b(t_f, x(t_f)) = 0. \end{aligned}$$

The final problem, labeled P3, is obtained by reducing the state space to $\mathcal{X}$ by using the basis as in Eq. (7). In that case, the evolution of the system on $\mathcal{X}$ can be expressed as follows:

$$\dot{\zeta}(t) = \tilde{A}_F \zeta(t) + \tilde{B}_G v(t) + \tilde{E}_H w(t). \tag{18}$$

Additionally, the control constraint can be expressed as

$$u(t) = \tilde{F}\zeta(t) + Gv(t) + Hw(t). \tag{19}$$

Similar statements hold for the terminal cost and terminal constraint since they are affine by assumption. By using the new state variable ζ, these are written as $\tilde{m}(t_f, \zeta(t_f))$ and $\tilde{b}(t_f, \zeta(t_f))$, respectively.

P3 (20)

$$\begin{aligned}
&\min \quad J = \tilde{m}(t_f, \zeta(t_f)) + \int_{t_0}^{t_f} \ell(\Gamma(t))\, dt \\
&\text{subj. to} \quad \dot{\zeta}(t) = \tilde{A}_F\zeta(t) + \tilde{B}_G v(t) + \tilde{E}_H w(t) \\
&\qquad u(t) = \tilde{F}\zeta(t) + Gv(t) + Ew(t), Cu(t) \le d \\
&\qquad 0 < \rho_1 \le \Gamma(t) \le \rho_2, \quad \kappa(u(t)) \le \Gamma(t) \\
&\qquad \zeta(t_0) = \zeta_0, \ \tilde{b}(t_f, \zeta(t_f)) = 0.
\end{aligned}$$

Results connecting problems P0 through P3 are now given in a sequence of lemmas leading to the main result. Then Theorem 2, the main theorem of the paper, is proved. We first fix some notation used in these lemmas.

Let $\mathcal{F}_i$ denote the set of feasible trajectories for problem Pi, $i = 0, 1, 2, 3$. For example, $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0$ if all the constraints of problem P0 are satisfied. Similarly, let $\mathcal{F}_i^*$ denote the set of global optimal control trajectories for problem Pi. Finally, let J_i^* denote the cost associated with solutions belonging to $\mathcal{F}_i^*$.

Lemma 6 connects the feasible sets for the original problem (P0) and its convex relaxation (P1).

Lemma 6. *(i) If $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0$, then there exists a $\Gamma(\cdot)$ such that $\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1$.(ii) If $\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1$ and $\kappa(u(t)) = \Gamma(t)$ a.e. t, then $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0$.*

Proof. (i) Suppose $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0$ and define $\Gamma(t) = \kappa(u(t))$. Then $\rho_1 \le \Gamma(t) \le \rho_2$ a.e. t such that all constraints of P1 are satisfied and $\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1$. (ii) Since $\rho_1 \le \kappa(u(t)) \le \rho_2$ a.e. t, all constraints of P0 are satisfied and $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0$. □

Lemmas 7 and 8 connect the optimal solutions of problems P2 and P3.

Lemma 7. *$\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1^*$ if and only if there exists a $v(\cdot)$ such that $\{x(\cdot), u(\cdot), v(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_2^*$.*

Proof. The result follows from the facts that (1) the performance indices for P1 and P2 are the same and (2) $x(t) \in \mathcal{X}$ if and only if there exists a $v(t)$ such that $u(t) = Fx(t) + Gv(t) + Hw(t)$ (see Eq. (4) and Trentelman et al. (2001)). □

Lemma 8. *$\{x(\cdot), u(\cdot), v(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_2^*$ if and only if $\{\zeta(\cdot), u(\cdot), v(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_3^*$.*

Proof. The result follows from the fact that the only difference between the problems is a coordinate adaptation (see Eq. (7) and the related discussion). □

We are now finally to the point of synthesizing problem P3 using the optimality conditions given in Theorem 5 of Section 4. The goal is to characterize the optimal solutions with a link back to problems P0 and P1 so that lossless convexification can be proved. To do so, we define the Hamiltonian, Lagrangian, and endpoint functions specifically for P3.

$$\begin{aligned}
\mathcal{H}[t] &= p_0\ell(\Gamma(t)) + p(t)^T(\tilde{A}_F\zeta(t) + \tilde{B}_G v(t) + \tilde{E}_H w(t)) \\
\mathcal{L}[t] &= \mathcal{H}[t] + \lambda(t)^T(u(t) - \tilde{F}\zeta(t) - Gv(t) - Hw(t)) \\
&\quad + \nu_1(t)(\kappa(u(t)) - \Gamma(t)) + \nu_2(t)(\rho_1 - \Gamma(t)) \\
&\quad + \nu_3(t)(\Gamma(t) - \rho_2) + \nu_4(t)^T(Cu(t) - d) \qquad (21) \\
\mathcal{G}[t_f] &= p_0\tilde{m}(t_f, \zeta(t_f)) + \xi^T\tilde{b}(t_f, \zeta(t_f)).
\end{aligned}$$

The adjoint differential equation is

$$\dot{p}(t) = -\tilde{A}_F^T p(t) + \tilde{F}^T\lambda(t). \tag{22}$$

The stationary conditions are

$$\begin{aligned}
&\partial_u\mathcal{L}[t] = \lambda(t) + \nu_1(t)\partial_u\kappa(u(t)) + C^T\nu_4(t) = 0 \\
&\partial_\Gamma\mathcal{L}[t] = p_0\partial_\Gamma\ell(\Gamma(t)) - \nu_1(t) - \nu_2(t) + \nu_3(t) = 0 \\
&\partial_v\mathcal{L}[t] = \tilde{B}_G^T p(t) - G^T\lambda(t) = 0.
\end{aligned} \tag{23}$$

The complementary slackness conditions are

$$\begin{aligned}
&\nu_1(t) \le 0, \quad \nu_1(t)(\kappa(u(t)) - \Gamma(t)) = 0 \\
&\nu_2(t) \le 0, \quad \nu_2(t)(\rho_1 - \Gamma(t)) = 0 \\
&\nu_3(t) \le 0, \quad \nu_3(t)(\Gamma(t) - \rho_2) = 0 \\
&\nu_4(t) \le 0, \quad \nu_4(t)^T(Cu(t) - d) = 0.
\end{aligned} \tag{24}$$

Lemma 9. *If Criterion 1 is satisfied, then the solution $\{\zeta(\cdot), u(\cdot), v(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_3^*$ implies $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$.*

Proof. The proof is done in two parts. Part (i) is for the time-invariant case, and part (ii) is for the time-varying case.

(i) Suppose that $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$ does not hold. Because $u(\cdot)$ and $\Gamma(\cdot)$ are piecewise continuous, there exists an interval $[\tau_1, \tau_2] \subset [t_0, t_f]$ where $\kappa(u(t)) < \Gamma(t)$ for all $t \in [\tau_1, \tau_2]$. Eq. (24) implies that $\nu_1(t) = 0$. Eqs. (22) and (23) become

$$\begin{aligned}
&\dot{p} = -\tilde{A}_F^T p(t) - \tilde{F}^T C^T\nu_4(t) \\
&y := \tilde{B}_G^T p(t) + G^T C^T\nu_4(t) = 0.
\end{aligned} \tag{25}$$

By part (i) of Criterion 1, $\mathcal{X}$ is the strongly controllable subspace for $(A + BF, BG, CF, CG)$. Theorem 4 implies that the system $(\tilde{A}_F, \tilde{B}_G, C\tilde{F}, CG)$ is strongly controllable, i.e., $(\tilde{A}_F^T, \tilde{F}^T C^T, \tilde{B}_G^T, G^T C^T)$ is strongly observable. Strong observability implies $p(\tau_1) = 0$. Because the problem is time-invariant, the Hamiltonian is identically zero. This means that $p_0 = 0$ since $\ell(\Gamma(\tau_1))$ cannot be zero. This violates the non-triviality condition in Theorem 5. Thus, $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$.[1]

(ii) By part (ii) of Criterion 1, $\mathcal{X}$ is A-invariant such that $F = 0$ is a friend. Thus, Eq. (25) becomes

$$\begin{aligned}
&\dot{p} = -\tilde{A}^T p(t) \\
&y := \tilde{B}_G^T p(t) + G^T C^T\nu_4(t) = 0.
\end{aligned} \tag{26}$$

Again, Theorem 4 implies that the system $(\tilde{A}, \tilde{B}_G, 0, CG)$ is strongly controllable, i.e., $(\tilde{A}^T, 0, \tilde{B}_G^T, G^T C^T)$ is strongly observable. Strong observability implies $p(\tau_1) = 0$. Since $p(\cdot)$ is the solution of a homogeneous equation, $p(t_f) = 0$. It follows from the linear independence condition in Criterion 1 and transversality that $(p_0, p(t_f)) = 0$, which violates the non-triviality condition. Thus, $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$. □

Lemma 10. *If Criterion 1 is satisfied, then the solution $\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1^*$ implies $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$.*

Proof. Suppose $\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1^*$ and that $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$ does not hold. Then, from Lemmas 7 and 8, there exists a $v(\cdot)$ such that $\{\zeta(\cdot), u(\cdot), v(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_3^*$ with $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$ not holding. This contradicts Lemma 9. □

[1] For reference, there are cases where the adjoint variables are ill-behaved. Dikusar and Milyutin (1989) and Robbins (1980) provide examples for state constrained problems in which the adjoint variables have countably many jumps. This phenomena is also known to occur in singular optimal control problems, which are intimately connected to state constrained problems (Bell, 1978; Jacobson & Lele, 1969).

The following theorem establishes lossless convexification between P0 and P1. It is a main result of the paper and says that optimal solutions of P1 are also optimal solutions of P0. It is the same as Theorem 2.

Theorem 11. *If Criterion 1 is satisfied, then the solution* $\{x(\cdot), u(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1^*$ *implies* $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0^*$.

Proof. Suppose that $\{x(\cdot), u(\cdot), v(\cdot), \Gamma(\cdot)\} \in \mathcal{F}_1^*$. Lemma 10 implies that $\kappa(u(t)) = \Gamma(t)$ a.e. $t \in [t_0, t_f]$. Consequently, problems P0 and P1 have the same cost function. Lemma 6 implies that $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0$. Thus, $J_0^* \le J_1^*$. Similarly, Lemma 6 also implies that $J_1^* \le J_0^*$. Thus, $J_0^* = J_1^*$. Because the cost functions are the same, the cost of $\{x(\cdot), u(\cdot)\}$ in P0 is $J_1^* = J_0^*$. Thus, $\{x(\cdot), u(\cdot)\} \in \mathcal{F}_0^*$. □

We note that this proof of lossless convexification is the first for problems with both state and control constraints. Previous work has focused on lossless convexification for optimal control problems where the state constraint is not active, i.e., the state at most touches the constraint a finite number of times (Açıkmeşe & Blackmore, 2011).

6. Numerical solution method

Because of lossless convexification, optimal solutions of the convex problem P1 are also optimal solutions of the non-convex problem P0. Nonetheless, P1 is still an infinite-dimensional problem. It can be solved numerically after converting it to a finite dimensional problem by discretizing and enforcing the constraints at the nodes. Since all of the constraints in P1 are first or second-order order cone constraints, the discretized problem is a second-order cone programming problem for which there exist readily available numerical algorithms (Boyd & Vandenberghe, 2004). We briefly describe one type of discretization.

For a given interval $[0, t_f]$, the interval is discretized into $N+1$ nodes separated by Δt.

$$t_i = (i-1)\Delta t, \quad i = 1, \ldots, N+1 \tag{27}$$

The states exist at every node, i.e., $x[i] \in \mathbb{R}^{d_x}$ is the state at t_i for all $i = 1, \ldots, N+1$. The controls exist over every interval, i.e., $u[i] \in \mathbb{R}^{d_u}$ and $\Gamma[i] \in \mathbb{R}$ are the controls in the ith interval for all $i = 1, \ldots, N$. Furthermore, the controls are held constant over every such interval. We now discretize the differential equations using standard techniques leading to

$$x[i+1] = \Phi x[i] + \Theta u[i] + \Psi_i, \quad i = 1, \ldots, N \tag{28}$$

where the matrices are

$$\begin{aligned} \Phi &= e^{A\Delta t}, \qquad \Theta = \int_0^{\Delta t} e^{A\tau} B \, d\tau, \\ \Psi_i &= \int_{t_i}^{t_{i+1}} e^{A\tau} E w(\tau) \, d\tau. \end{aligned} \tag{29}$$

The integral in the performance index can be approximated using any numerical integration technique, e.g., trapezoidal integration:

$$\int_{t_0}^{t_f} \ell(\Gamma(t)) \, dt \approx \frac{\Delta t}{2} \sum_{i=1}^{N} \Big(\ell(\Gamma[i+1]) + \ell(\Gamma[i]) \Big). \tag{30}$$

All other constraints are enforced at the nodes. For example, the state constraints are written as

$$x[i] \in \mathcal{X}, \quad i = 1, \ldots, N+1. \tag{31}$$

Similarly, the control constraints are written as

$$\rho_1 \le \Gamma[i] \le \rho_2, \qquad \kappa(u[i]) \le \Gamma[i], \qquad Cu[i] \le d \tag{32}$$

for all $i = 1, \ldots, N$. Eqs. (27) through (32) represent the discretized version of P1. For the given interval $[0, t_f]$, the discrete problem is a finite-dimensional SOCP, which can be solved to global optimality in polynomial time (Toh et al., 1999).

In many problems, the final time is free. For these problems, we propose a line search, which has been used before for minimum fuel and minimum time problems (Açıkmeşe & Ploen, 2007; Harris & Açıkmeşe, 2014). To summarize, the performance index is evaluated as a function of t_f, and a golden section or bisection type line search method is used to find the optimal t_f.

7. Example 1: minimum fuel planetary landing

An interesting example problem is the planetary, soft landing problem. In the final descent phase, it is assumed that (1) the vehicle is close enough to the surface that gravity is constant, (2) the thrust forces dominate the aerodynamic forces, and (3) a known time-varying disturbance acts on the system. In this case, the equations of motion are

$$\ddot{x}(t) = -g + u(t) + w(t). \tag{33}$$

The first component of x, denoted x_1, is the range. The altitude and cross range are x_2 and x_3, respectively. The components x_4, x_5, and x_6 are the range rate, altitude rate, and cross range rate. Near the surface of Mars, the gravity vector is approximately $g = [0 \;\; -3.71 \;\; 0]$ m/s^2. It is assumed that the disturbance is a sinusoidal function of time of the form $w(t) = [\sin(t) \;\; 0 \;\; \cos(t)]$ m/s^2. The problem is to transfer the vehicle from its initial condition to the landing site with zero final velocity, e.g.,

$$\begin{aligned} x(0) &= [400\;400\;300] \text{ m} \\ \dot{x}(0) &= -[10\;10\;75] \text{ m/s} \\ x(t_f) &= [0\;0\;0] \text{ m} \\ \dot{x}(t_f) &= [0\;0\;0] \text{ m/s}. \end{aligned} \tag{34}$$

For safety reasons, it is also required that the vehicle does not approach the landing site with too steep or too shallow an approach angle. A 45 degree approach in the altitude/range plane is specified and can be written as

$$x_1(t) - x_2(t) = 0. \tag{35}$$

The control magnitude is bounded above and below since the thrusters cannot operate reliably below this bound.

$$2 \le \|u(t)\| \le 10 \text{ m/s}^2. \tag{36}$$

The goal is to achieve the landing and minimize the fuel consumption, i.e.,

$$\min J = \int_0^{t_f} \|u(t)\| \, dt. \tag{37}$$

The optimal control problem is defined by Eqs. (33) through (37) and fits within the structure of P0 given in Eq. (1). Note that no linear control constraints are present, i.e., no C. The convex relaxation is easily obtained by relaxing Eq. (36) to the two constraints

$$\|u(t)\| \le \Gamma(t) \quad \text{and} \quad 2 \le \Gamma(t) \le 10$$

and minimizing $J = \int_0^{t_f} \Gamma(t) \; dt$. For the state to evolve on the plane $x_1(t) - x_2(t) = 0$, the time derivative of the constraint must also be zero, i.e., $x_4(t) - x_5(t) = 0$. Thus, the subspace $\mathcal{X}$ is given as

$$\mathcal{X} = \text{null}\left(\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \right).$$

The friends F, G, and H are

$$F = 0, \qquad G = \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

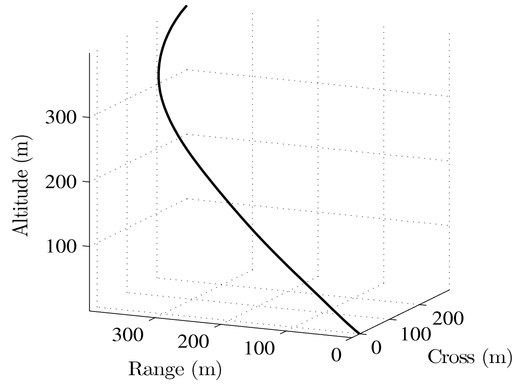

Fig. 3. State trajectory.

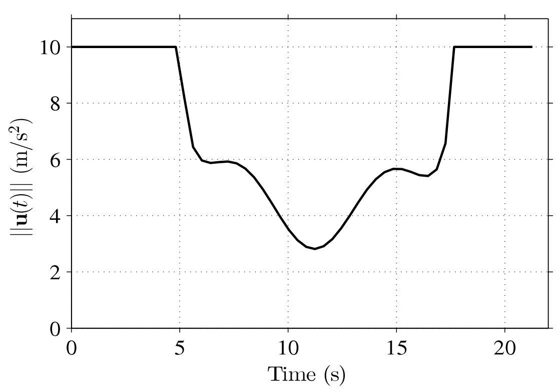

Fig. 4. Control trajectory.

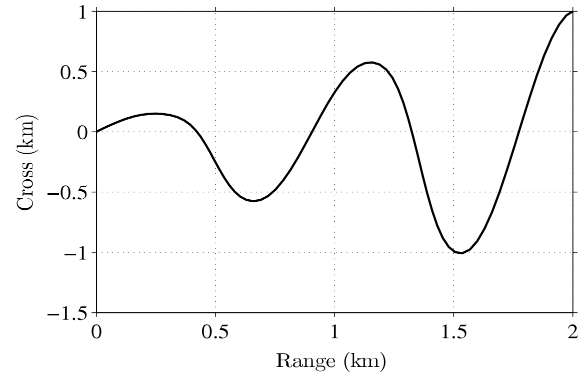

Fig. 5. State trajectory.

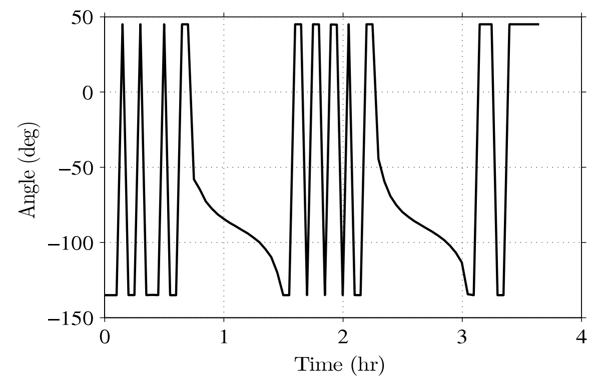

Fig. 6. Control trajectory.

The fact that these are in fact friends and that $\mathcal{X}$ is the strongly controllable subspace for the system $(A + BF, BG, CF, CG)$ can be checked using standard tests in Trentelman et al. (2001).

The numerical simulations are carried out using SDPT3 (Toh et al., 1999). Fig. 3 shows the state trajectory. The trajectory begins at the top center, ends at the origin in the bottom right, and evolves on the 45 degree plane. The thrust magnitude is shown in Fig. 4. The upper constraint is active along the initial and final arcs, and the control is in the interior during the middle arc. The oscillatory movement is caused by the time-varying disturbance.

8. Example 2: minimum time rendezvous

A second example is the rendezvous of two spacecraft at constant altitude using low thrust. It is assumed that the motion of the chaser spacecraft relative to the target spacecraft is accurately described by the Clohessy–Wiltshire–Hill equations (Clohessy & Wiltshire, 1960) so the equations of motion are

$$\ddot{x}(t) = 3\omega^2 x(t) + 2\omega\dot{y}(t) + u_1(t)$$
$$\ddot{y}(t) = -2\omega\dot{x}(t) + u_2(t)$$
$$\ddot{z}(t) = -\omega^2 z(t) + u_3(t)$$

The states x, y, and z are the altitude, range, and cross range, respectively, and $\dot{x}$, $\dot{y}$, $\dot{z}$ are the rates. The orbital mean motion is $\omega = 4\,\mathrm{hr}^{-1}$, and corresponds to a near circular, low earth orbit. The problem is to rendezvous the two vehicles, i.e., bring them together with zero relative velocity. The boundary conditions are

$$[x(0)\ y(0)\ z(0)] = [0\ 2\ 1]\ \mathrm{km}$$
$$[\dot{x}(0)\ \dot{y}(0)\ \dot{z}(0)] = [0\ \ -0.5\ \ -0.25]\ \mathrm{km/s}$$
$$[x(t_f)\ y(t_f)\ z(t_f)] = [0\ 0\ 0]\ \mathrm{km}$$
$$[\dot{x}(t_f)\ \dot{y}(t_f)\ \dot{z}(t_f)] = [0\ 0\ 0]\ \mathrm{km/s}.$$

For safety reasons, viewing angles, etc., it is required that the rendezvous maneuver take place at constant altitude. This constraint is written simply as $x_1(t) = 0$. As in the previous example, the thrust magnitude is bounded above and below

$$3 \le \|u(t)\| \le 5\ \mathrm{km/hr}^2.$$

It is also required that the chaser spacecraft not point in the cross range direction more than $\theta = 45$ degrees.

$$u_3(t) - u_2(t)\tan\theta \le 0.$$

The goal is to achieve the rendezvous and minimize the time of flight so that the cost function is simply $J = \int_{t_0}^{t_f} 1\,dt$. This optimal control problem also fits within the structure of P0 given in Eq. (1), and the convex relaxation of the control constraints is obtained as in the previous example.

For the state to evolve on the plane $x(t) = 0$, the time derivative of the constraint must also be zero, i.e., $\dot{x}(t) = 0$. Thus, the subspace $\mathcal{X}$ is given as

$$\mathcal{X} = \mathrm{null}\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}\right).$$

The friends F, G, and H are

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 & -8 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 9 & 1 & 1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Fig. 5 shows the range and cross range. The trajectory begins in the upper right, oscillates, and terminates at the origin on the left. The thrust angle is shown in Fig. 6, and it is evident that the angle satisfies the point constraint of 45 degrees. The altitude plot is not shown since it is identically zero. The thrust magnitude is not shown, but it is constant along the upper boundary.

9. Summary and conclusions

The theory of lossless convexification has been generalized to optimal control problems with mixed non-convex and convex control and state constraints. This was done by introducing the strongly controllable subspaces and an appropriate maximum principle. As a consequence, a larger class of non-convex problems can be solved as a convex problem. The work is significant because this class of problems includes several important applications, and the work has been a foundation for successful flight tests of planetary landing trajectory optimization.

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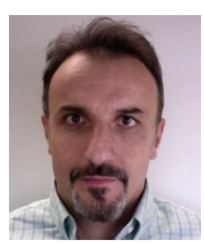

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