

Hierarchical Motion Planning with Kinodynamic Feasibility Guarantees

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Abstract—Motion planning for mobile vehicles involves the solution of two disparate sub-problems: the satisfaction of high-level logical task specifications and the design of low-level vehicle control laws. A hierarchical solution of these two sub-problems is efficient, but may not ensure compatibility between the high-level planner and the dynamic constraints of the vehicle. To guarantee such compatibility, we propose a motion planning framework based on a special interaction between these two levels of planning. In particular, we solve a special shortest path problem on a graph at the higher level of planning, and we use the lower level planner to determine the costs of the paths in that graph. The overall approach hinges on two novel ingredients: a graph-search algorithm that operates on sequences of nodes, and a lower-level planner that ensures consistency between the two levels of hierarchy, by providing meaningful costs for the edge transitions of the higher level planner using dynamically feasible, collision-free trajectories.

Index Terms—Motion planning, autonomous mobile robots, kinodynamic motion planning, graph search, consistency

I. INTRODUCTION

THE problem of motion planning and control for autonomous mobile vehicles deals with finding appropriate control inputs, such that the vehicle's resulting motion satisfies the requirements of a specified task. This problem is inherently complex because it involves two disparate sub-problems: (1) the satisfaction of the vehicular task specifications, which requires tools from combinatorics and/or formal methods, and (2) the design of the vehicle control laws, which requires tools from dynamical systems and control theory. This inherent dichotomy spawns a natural approach to the solution of the motion planning problem: a hierarchical separation of the aforementioned sub-problems.

Consider, for instance, the vehicular task of traversing the environment from a given initial point to a given destination, while avoiding obstacles [1], [2]. For this problem, hierarchical separation has often been used in motion planners to explicitly incorporate the vehicle's kinematic and dynamic constraints [3]–[10]. In such hierarchically separated schemes, the higher level is mainly concerned with obstacle avoidance and with finding a geometric, obstacle-free *path* from the initial point to the destination. We call this level the *geometric path planning* level. The lower level accounts for the kinematic and dynamic constraints of the vehicle: it sufficiently smoothens

the geometric path and imposes a suitable time parametrization along this path to obtain a feasible reference *trajectory*. We call this level the *trajectory planning* level. Additionally, a tracking controller generates the control inputs that enable the vehicle to track this trajectory.

The principal advantages of such hierarchical separation include computational efficiency and simplicity of implementation. Geometric planners typically use a discrete representation of the environment, hence tools from combinatorics and/or formal methods may be applied to ensure satisfaction of the vehicular task specifications [11]. A serious drawback of this hierarchical separation is that the geometric planner has no knowledge of the vehicle's kinematic and dynamical constraints; as a result, it may produce either infeasible paths or unacceptably sub-optimal paths. This inadequacy in providing guarantees of consistency between the higher and lower layers of planning and of optimality of the resultant trajectories has long been identified; consequently, its resolution is of acute interest for the development of autonomous mobile vehicles [11], especially for task specifications more complex than traveling from one point to the other, and for vehicle dynamical constraints that are sufficiently complex such that they cannot be ignored at the geometric planning stage.

In this paper, we propose a hierarchical motion planning framework that addresses the aforementioned inadequacy of hierarchical planners. The proposed framework rests on a novel mode of interaction between the geometric path planner and the vehicle trajectory planner. This interaction is enabled by a special shortest path problem on graphs involving costs defined on *multiple* edge transitions, instead of the usual single-edge transition costs. These transition costs are provided by a low-level trajectory planner. The proposed framework is intended as a basic step towards the design of hierarchically consistent motion planners that combine discrete geometric planning algorithms with trajectory generation algorithms.

A. Related Work

The geometric path planner in this work is based on cell decompositions [12], [1, Ch. 5]. These involve partitioning of the environment into convex, non-overlapping regions called *cells*. A cell is either classified as FREE (if it contains no obstacles), or FULL (if it contains no free space). A graph is associated with the cell decomposition, where each FREE cell is represented by a node, and geometric adjacencies of the FREE cells are represented by edges. A path from a pre-specified initial cell to a pre-specified goal cell in this graph then corresponds to a sequence of FREE cells from the initial cell to the goal cell in this graph.

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Original manuscript submitted May 11, 2010. Revised manuscript submitted January 21, 2011.

Triangular and trapezoidal decompositions [13], [1, Ch.6] are widely used exact cell decomposition techniques for environments with polygonal obstacles, whereas quadtree-based methods [14]–[16] (which employ recursive decompositions of MIXED cells into four square subcells until all cells are either FREE or FULL), are popular approximate cell decomposition techniques. Path planning schemes using *multi-resolution* cell decompositions have also been proposed in [17]–[19].

As mentioned previously, hierarchical approaches are often used in motion planning [3]–[10] to separate the problem into a high-level, discrete, geometric planning level and a low-level, continuous, trajectory planning level. In the context of low-level trajectory planning, references [20]–[23] discuss time-optimal trajectory planning along pre-specified geometric paths for specific vehicle dynamics. Other related works in the literature include [24], which uses a special history-based cost approach that closely matches the approach taken in our work; [25], which deals with kinodynamic planning for robotic manipulators; [26], which uses a hybrid model to describe the motion of a rotorcraft in terms of pre-programmed maneuvers; and [27], which discusses trajectory planning based on the solution of the Hamilton-Jacobi-Bellman equation.

Probabilistic roadmap methods [28]–[31], [1, Ch. 7], and methods that use rapidly exploring trees (RRTs) [32]–[36], are among the most popular, recent works addressing the vehicle’s kinematic and dynamic constraints during motion planning. In these methods, random samples of the obstacle-free space are connected to each other by feasible trajectories, and the resulting graph is searched for a sequence of connected samples from the initial state to the goal state. Sampling-based algorithms require efficient low-level collision detection and trajectory planning algorithms to find collision-free trajectories between different samples [33].

Motion planners that use cell decompositions coupled with feedback control laws [37]–[41] provide methods to generate reference vector fields that guarantee feasible transitions through any given sequence of cells without intersecting any other cell. References [42], [43] and, in a slightly different context, [44], also use this idea to develop solutions that are guaranteed to satisfy both aspects of motion planning and control: vehicular task specifications expressed as temporal logic formulae, and kinematic and dynamic constraints expressed as differential equations. The common idea used in all of these works is to make the geometric planner *independent* of the vehicle dynamics; i.e., to ensure that *any* sequence of cells can be feasibly traversed from *any* initial vehicle state.

B. Contributions of this Work

The principal contribution of this work is a new motion planning framework that maintains a distinction between the discrete planning strategy and the trajectory planning strategy. The advantage of this distinction is that either planner may be developed independently of the other: the discrete planner may be tailored to satisfy vehicular task specifications (finding low-cost, obstacle-free paths is an example of such a task), whereas trajectory planning schemes based on control theoretic ideas may be tailored to cope with complex vehicle

dynamics. However, in contrast to previous similar approaches, we also provide an “interface” between these two planners that maintains compatibility of their solutions. The result of the higher level geometric planner in the proposed framework is a path corresponding to a sequence of cells; consistency between the two levels of planning is ensured by providing a guarantee that this sequence of cells can be feasibly traversed by the vehicle. The proposed framework is thus a step towards the development of motion planners as envisioned in [11]: ones that systematically combine results and techniques from different disciplines, such as formal methods and control theory to generate provably correct control laws that enable the vehicle to satisfy complex tasks specifications.

A secondary contribution of this work is an efficient and flexible algorithm of independent interest that finds a path in a graph minimizing a cost defined on multiple edge transitions. A basic version of this algorithm was originally reported in [45]; in this paper we provide a more general, flexible version of this algorithm, along with detailed numerical simulation results demonstrating its efficiency.

The rest of this paper is organized as follows: in Section II, we introduce the idea of using history-dependent transition costs in path planning. In Section III, we provide an efficient and flexible algorithm that finds paths minimizing history-based transition costs. In Section IV, we discuss the proposed motion planning framework based on the path finding algorithm of Section III. In Section V, we discuss implementations of the TILEPLAN algorithm used in the proposed motion planning framework, and in Section VI, we provide sample numerical simulation results of the proposed approach as well as comparisons with other motion planners. Section VII concludes with a summary of the contributions.

II. HISTORY-BASED TRANSITION COSTS

Geometric path planning algorithms based on workspace cell decompositions provide no guarantees that the resultant channel of cells can be feasibly traversed by a vehicle subject to kinematic and dynamical constraints¹. At first glance, one may argue that this is but an artefact of an inappropriate choice of the edge cost function in the associated graph. We provide a counter-example against this argument.

A. Motivating Example

Consider the path planning problem depicted in Fig. 1, where S denotes the initial position, G denotes the goal, and the dark areas are obstacles. Consider two vehicles A and B , whose minimum radii of turn are kinematically constrained by r_{\min}^A and r_{\min}^B respectively, such that $r_{\min}^A \leq d$ and $r_{\min}^B > d$. Clearly, the dashed path in Fig. 1 is feasible for vehicle A , but not for vehicle B . A path planning algorithm for B ought to result in the bold path shown in Fig. 1.

Figure 2 depicts the same problem with a uniform cell decomposition. The channel containing the dashed path of Fig. 1 is denoted by cells with bold outlines. Such a channel

¹State space decompositions can avoid this problem but are impractical for high dimensional state spaces.

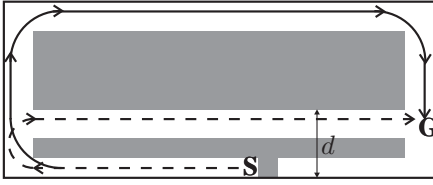


Fig. 1. Counter-example for path planning without kinematic constraints.

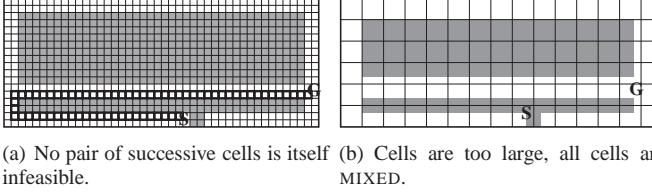


Fig. 2. Problems with geometric path planning using cell decompositions.

is obviously not traversable by vehicle B . However, notice that no *pair of successive cells* is by itself infeasible, i.e., a channel defined by two successive cells alone always contains a feasible path. Stated differently, for any two adjacent cells, there is no cell-dependent property associated with the two adjacent cells that can be penalized by an edge cost function in order to prevent the graph search from generating a channel such as the one shown in Fig. 2(a).

It may be further argued that a feasible path is guaranteed to exist in *any* channel if the dimensions of the cells are large enough². The above counter-example also serves to illustrate that such a choice of cells may be too restrictive in practice. Figure 2(b) shows that large cells may not capture details of the environment, i.e., the number of MIXED cells could be too large for the cell decomposition to be useful for path planning.

In light of the preceding observations, we propose in this paper an approach to find paths in the cell decomposition that minimize a cost defined on multiple edge transitions – called *histories* – instead of costs defined on single edge transitions.

B. Problem Formulation

Consider a cell decomposition of the environment consisting of N cells. For now, we make no assumptions concerning the geometry of the cells involved in the decomposition. The topological graph associated with the cell decomposition is a graph $\mathcal{G} := (V, E)$, such that each element in the set of nodes V corresponds to a unique, obstacle-free cell. Two nodes are *adjacent* if the corresponding cells are geometrically adjacent. The edge set E is the collection of all two-element subsets $\{i, j\} \subset V$ such that the nodes i and j are adjacent.

For given initial and goal nodes $i_S, i_G \in V$, an *admissible path* π in \mathcal{G} is a finite sequence (j_0, j_1, \dots, j_P) of nodes (with no repetition) such that $j_k \in V$, $\{j_{k-1}, j_k\} \in E$, for each $k = 1, \dots, P$, with $j_0 = i_S$, $j_P = i_G$. To formalize the concept of histories, we define, for every integer $H \geq 0$, the set

$$V_H := \{(j_0, \dots, j_H) : \{j_{k-1}, j_k\} \in E, k = 1, \dots, H, \\ j_k \neq j_m, \text{ for } k, m \in \{0, \dots, H\}, \text{ with } k \neq m\}.$$

²Ref. [46] shows that a curvature-bounded path with local curvature less than or equal to $1/r_{\min}$ exists in a polygonal channel if the width w of the channel satisfies $w \geq \tau r_{\min}$, where $\tau \approx 1.55$.

An element of the set V_{H+1} is called an *H-history*. In what follows, we denote by $[I]_k$ the k^{th} element of this $(H+1)$ -tuple, and by $[I]_k^m$ the tuple $([I]_k, [I]_{k+1}, \dots, [I]_m)$, for $k < m \leq H+1$. We associate with each H a non-negative cost function $\tilde{g}_H : V_H \rightarrow \mathbb{R}_+$, and state a shortest path problem with transition costs defined on histories as follows.

Problem 1 (*H-cost shortest paths*): Let $H \geq 0$, and let $i_S, i_G \in V$ be given initial and goal nodes, such that any admissible path in \mathcal{G} contains at least $H+1$ nodes. The H -cost of the path $\pi = (j_0, \dots, j_P)$ is defined by

$$\tilde{\mathcal{J}}_H(\pi) := \sum_{k=H+1}^P \tilde{g}_{H+1}([j_{k-H-1}, j_{k-H}, \dots, j_k]). \quad (1)$$

Find an admissible path π^* in \mathcal{G} such that $\tilde{\mathcal{J}}_H(\pi^*) \leq \tilde{\mathcal{J}}_H(\pi)$ for every admissible path π in \mathcal{G} .

Note that the H -cost of a path is defined as the sum of the costs of H -histories in that path. According to this convention, the 0-cost of a path is the standard notion of cost, i.e., the sum of edge weights, because 0-histories, are precisely the edges in E with $V_1 = E$. In other words, the H -cost shortest path problem for $H = 0$ is the standard shortest path problem on a graph with weighted edges.

It is possible to transform Problem 1 into an equivalent standard shortest path problem on a “lifted” graph \mathcal{G}_H , whose nodes are the elements of V_H . This transformation enables a clear conceptualization of our proposed algorithm to solve Problem 1 in light of the fact that the solution to the standard shortest path problem is well-known. For instance, the so-called *label correcting algorithms* [47] provide efficient solutions to the standard shortest path problem. Well-known examples of label correcting algorithms include the Bellman-Ford, Dijkstra [47], [48], and the A* [2], [49] algorithms.

We define adjacency relations between the elements of V_H as follows. Let $I, J \in V$; then J is adjacent to I if $[I]_k = [J]_{k-1}$, for every $k = 2, \dots, H+1$, and $[I]_1 \neq [J]_{H+1}$. Let E_H denote the edge set of the graph \mathcal{G}_H consisting of all ordered pairs (I, J) , such that J is adjacent to I .

For given initial and terminal nodes $i_S, i_G \in V$, an admissible path Π in \mathcal{G}_H is a finite sequence (J_0, \dots, J_Q) of nodes (with no repetition) such that $(J_{k-1}, J_k) \in E_H$, for each $k = 1, \dots, Q$, with $[J_0]_1 = i_S$, and $[J_Q]_{H+1} = i_G$. Note that every admissible path $\Pi = (J_0, \dots, J_Q)$ in \mathcal{G}_H uniquely corresponds to an admissible path $\pi = (j_0, \dots, j_P)$ in \mathcal{G} , with $P = Q + H$ and $[j_k]_m = [j_{kH+m-1}]$, for each $k = 0, 1, \dots, Q-1$, and $J_Q = (j_{P-H}, \dots, j_P)$. We introduce a non-negative cost function $g_H : E_H \rightarrow \mathbb{R}_+$ defined by

$$g_H((I, J)) := \tilde{g}_{H+1}([I]_1^{H+1}, [J]_{H+1}), \quad (I, J) \in E_H.$$

It follows that Problem 1 is equivalent to the standard shortest path problem on the graph \mathcal{G}_H , where the weight of an edge $(I, J) \in E_H$ given by $g_H((I, J))$.

III. PATH PLANNING WITH HISTORY-DEPENDENT COSTS

Problem 1 may be solved by first transforming it to a standard shortest path problem on the graph \mathcal{G}_H , and then executing a label-correcting algorithm such as Dijkstra’s algorithm. A naïve, brute-force implementation of this approach

Algorithm 1: H -Cost Shortest Path-Finding**Input:** i_S ; **Output:** d, h

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procedure INITIALIZE()
1: for all  $i \in V, m = 1, \dots, T_H$  do
2:    $d(i, m) \leftarrow \infty, h(i, m) \leftarrow \text{NULL}$ 
3:    $\mathcal{P} \leftarrow \{(i, m) : \mathcal{N}_i \neq \emptyset, \text{ and } \text{HISTORY}(i, m) \in \mathcal{N}_i\}$ 
4:   for all  $(i, m) \in \mathcal{P}$  do
5:      $I \leftarrow \text{HISTORY}(i, m)$ 
6:      $d(i, m) \leftarrow \tilde{g}_{H+1}(I), h(i, m) \leftarrow [I]_1^{H+1}$ 

procedure MAIN()
1: INITIALIZE()
2: while  $\mathcal{P} \neq \emptyset$  do
3:    $(i, m) \leftarrow \text{REMOVE}(\mathcal{P})$ 
4:   for all  $j \in V$  such that  $(i, j) \in E$  do
5:      $n \leftarrow \text{INDEX}([h(i, m)]_3^{H+1}, i, j, j)$ 
6:      $J \leftarrow \text{HISTORY}(j, n)$ 
7:      $D_{i,m} \leftarrow d(i, m) + \tilde{g}_{H+1}([h(i, m)]_2, J)$ 
8:     if  $d(j, n) > D_{i,m}$  then
9:        $d(j, n) \leftarrow D_{i,m}$ 
10:       $h(j, n) \leftarrow ([h(i, m)]_2^{H+1}, i)$ 
11:       $\mathcal{P} \leftarrow \text{INSERT}(\mathcal{P}, (j, n))$ 

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Fig. 3. Detailed pseudo-code for the basic version of the proposed algorithm.

is ill-advised because (a) $|V_H|$ and $|E_H|$ grow exponentially with H , and (b) the explicit construction of the graph \mathcal{G}_H may be unnecessary to find the shortest path in \mathcal{G}_H .

In this section, we describe an algorithm that indirectly executes a label-correcting algorithm on the graph \mathcal{G}_H , without constructing the entire graph beforehand. Since $|V_H|$ and $|E_H|$ grow exponentially with H , the execution time of *any* algorithm that solves Problem 1 *exactly* grows exponentially with H . This fact holds true for the proposed algorithm; however, we include in our algorithm a user-specified parameter that can dramatically reduce the execution time at the expense of (exact) optimality of the resultant path. In other words, the proposed algorithm exhibits a flexibility that allows the user to trade off execution time against optimality of the resultant path.

For the sake of clarity, we first present a basic version of our algorithm, namely, one that finds the optimal path and solves Problem 1 exactly. In Section III-D, we introduce the aforementioned user-specified parameter and discuss the effects of this parameter on the algorithm's execution time.

A. Description of Basic Algorithm

Recall that a standard label-correcting algorithm maintains a collection of nodes, called the *fringe* [50] (also referred to as OPEN nodes [47], [48]), whose labels can potentially be reduced. The standard algorithm associates with each node $i \in V$ a *label*, which is an estimate of the least cost of a path from i_S to i , and a *backpointer*, which records the immediate predecessor of each node in the optimal path from i_S to i .

The definition of an admissible path in \mathcal{G}_H from i_S to any node $i \in V$ requires only $[J_Q]_{H+1} = i$, where $J_Q \in V_H$ is

the last node in this path. The first H elements of J_Q are unspecified, which implies that different admissible paths in \mathcal{G}_H may have different terminal nodes in V_H . In the proposed algorithm, we recognize this fact by associating with each node $i \in V$ *multiple* H -histories instead of the backpointer in the standard label-correcting algorithm. Each history of i is a unique element $I \in V_{H+1}$ such that $[I]_{H+2} = i$. The proposed algorithm is a label-correcting algorithm that associates with each history of each node $i \in V$ a label. Accordingly, the fringe in the proposed algorithm is a collection of *pairs*, where each pair consists of a node in V and an index that refers to a particular history of that node.

A detailed pseudo-code of the proposed algorithm appears in Fig. 3. In each iteration, the algorithm updates the label corresponding to a node-index pair, i.e., a particular history. Lines 8–11 update the fringe and the labels, similar to the standard label-correcting algorithm. Line 5 chooses the index corresponding to the particular history (of the newly explored node j) being updated in that iteration. We use the following notation: for each node $i \in V$, \mathcal{T}_i and \mathcal{N}_i are defined by

$$\mathcal{T}_i := \{I \in V_H : [I]_{H+1} = i\}, \quad (2)$$

$$\mathcal{N}_i := \{I \in V_{H+1} : [I]_1 = i_S, [I]_2^{H+2} \in \mathcal{T}_i\}; \quad (3)$$

here \mathcal{P} denotes the fringe; for each node $i \in V$ and $m \in \{1, \dots, T_i\}$, $h(i, m)$ denotes the m^{th} history of i and $d(i, m)$ denotes the label associated with the m^{th} history of i . Finally, the procedure $\text{HISTORY}(i, m)$ returns the m^{th} element of the set \mathcal{T}_i ; and the procedure $\text{INDEX}(I, i)$ returns the index of the history I in the set \mathcal{T}_i if $I \in \mathcal{T}_i$, or returns 0 otherwise. The procedures INSERT and REMOVE depend on the specific data structure used for implementing the fringe; in Section III-B, we consider a list sorted by the current values of labels.

The algorithm terminates when $\mathcal{P} = \emptyset$ (in Section III-B, we note a condition for terminating the algorithm earlier). After termination, the algorithm returns the labels d and the histories h . For every node $i \neq i_S$ in V , we may then calculate the optimal path from i_S to i by recursively tracing the $(H+1)^{\text{th}}$ element of histories recorded by h : a process similar to recursively tracing the backpointer in the standard label-correcting algorithm. We illustrate the execution of the algorithm with a simple example.

Example 1: Consider the graph shown in Fig. 4(a), where $i_S = 1$. Let $H = 1$. Let \tilde{g}_2 be a non-negative cost function given by the lookup table in Fig. 4(b) (for brevity, the values of some of the elements of V_2 are not shown). Note that

$$|\mathcal{T}_i| = \begin{cases} 2, & i \in \{1, 4, 13, 16\}, \\ 3, & i \in \{2, 3, 5, 8, 9, 12, 14, 15\}, \\ 4, & \text{otherwise,} \end{cases}$$

i.e., the algorithm maintains at most 4 histories and labels for each node. Also note that $\mathcal{N}_3 = \{(1, 2, 3)\}$, $\mathcal{N}_6 = \{(1, 5, 6), (1, 2, 6)\}$, $\mathcal{N}_9 = \{(1, 5, 9)\}$, and $\mathcal{N}_i = \emptyset$ for $i \in \{1, \dots, 16\} \setminus \{3, 6, 9\}$.

We index the elements of \mathcal{T}_j as 1, 2, 3, 4 corresponding to UP, RIGHT, DOWN, LEFT edges of j , with reference to Fig. 4(a). If a particular edge is absent, the corresponding index applies to the next edge in the order listed above. For

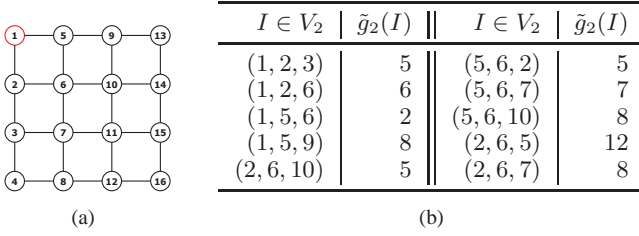


Fig. 4. The graph and the history-based cost function used in Example 1.

example, the indices of $(5, 6), (10, 6), (7, 6), (2, 6) \in \mathcal{T}_6$ are 1, 2, 3, and 4 respectively, while the indices of $(5, 1), (2, 1) \in \mathcal{T}_1$ are 1 and 2 respectively.

Line 3 of procedure INITIALIZE results in

$$\mathcal{P} = \{(3, 1), (6, 1), (6, 4), (9, 3)\}.$$

By Fig. 4(b), Line 6 of procedure INITIALIZE results in

$$\begin{aligned} d(3, 1) &= 5, & d(6, 1) &= 2, & d(6, 4) &= 6, & d(9, 3) &= 8, \\ h(3, 1) &= (1, 2), & h(6, 1) &= (1, 5), \\ h(6, 4) &= (1, 2), & h(9, 3) &= (1, 5). \end{aligned}$$

Next, suppose we remove $(i, m) = (6, 1)$ from the fringe in Line 3 of procedure MAIN. Then $\mathcal{P} = \{(3, 1), (6, 4), (9, 3)\}$, and the **for** loop in Line 4 is executed for nodes 2, 5, 7 and 10. In particular, for node $j = 2$, Lines 5 and 6 result in $n = 2$, and $J = (6, 2)$, respectively. By Table 4(b), it follows that $d(6, 1) + \tilde{g}_2([h(6, 1)]_2, 6, 2) = 2 + 5 = 7 < d(2, 2) = \infty$ (by Line 2 of procedure INITIALIZE). Hence, Lines 9 and 10 result in $d(2, 2) = 7$, and $h(2, 2) = ([h(6, 1)]_2, 6) = (5, 6)$, while Line 11 results in $\mathcal{P} = \{(3, 1), (6, 4), (9, 3), (2, 2)\}$.

Similarly, the execution of the **for** loop in Line 4 of procedure MAIN for nodes 5, 7, and 10, results in

$$\mathcal{P} = \{(3, 1), (6, 4), (9, 3), (2, 2), (7, 1), (10, 4), (5, 2)\}.$$

The labels and histories at the end of the first iteration are

$$\begin{aligned} d(5, 2) &= 18, & d(7, 1) &= 9, & d(10, 4) &= 10, \\ h(5, 2) &= (2, 6), & h(7, 1) &= (5, 6), & h(10, 4) &= (5, 6). \end{aligned}$$

B. Optimality and Performance

Different instances of label-correcting algorithms are obtained by implementing the fringe using different data structures. For example, implementing the fringe as a LIFO stack results in a breadth-first search; implementing the fringe as a list sorted by current labels results in Dijkstra's algorithm. In this section, we consider an instance of the proposed algorithm with the fringe implemented as a list sorted by the current labels, i.e., the procedure REMOVE returns $(i, m) = \arg \min\{d(i, m) : (i, m) \in \mathcal{P}\}$.

Proposition 1: For every node $i \in V$, suppose there exists at least one admissible path from i_S to i containing HISTORY(i, m), for any $m \in \{1, \dots, |\mathcal{T}_i|\}$. Let π^* be such an admissible path in \mathcal{G} with the least cost. Then the proposed algorithm terminates with $d(i, m) = \tilde{J}_H(\pi^*)$. Otherwise, the algorithm terminates with $d(i, m) = \infty$.

Proof: See [45]. ■

Proposition 1 asserts that the algorithm computes the minimum H -cost of paths from i_S to every node $i \in V$ and every

history in \mathcal{T}_i . However, because we need to compute only the minimum H -cost from i_S to i_G for any history in \mathcal{T}_{i_G} , it is possible to terminate the algorithm earlier, as shown by the following result.

Proposition 2: Each pair (j, m) , $j \in V$, $m = 1, \dots, |\mathcal{T}_j|$ enters the set \mathcal{P} at most once during the execution of the algorithm.

Proof: See [45]. ■

The conditions in Lines 8 and 11 of the procedure MAIN imply that a pair (j, m) is inserted in \mathcal{P} only when the value of $d(j, m)$ can be reduced. It follows from Proposition 2 that once a pair (j, m) is removed from \mathcal{P} , the value of $d(j, m)$ cannot be further reduced. The implication of this fact is that we may terminate the algorithm after Line 3 if $i = i_G$.

Note that Proposition 2 closely resembles a similar, known result regarding the execution of Dijkstra's algorithm: namely, that each vertex enters the fringe at most once (see, for instance, [47], [48]). We may use Proposition 2 to characterize the execution time of the basic version of the proposed algorithm as follows: in the worst case, every pair (j, m) enters the set \mathcal{P} exactly once. Thus, the maximum number of iterations (of the **while** loop in procedure MAIN) is upper bounded by $\sum_{j=1}^{|V|} |\mathcal{T}_j| = |V_H| = O(|V|^H)$.

C. Numerical Simulation Results

Table I shows sample examples comparing the times required for constructing the lifted graph \mathcal{G}_H and then executing Dijkstra's algorithm to the execution times of the proposed algorithm. In particular, the fourth, fifth, and sixth columns of Table I show, respectively, the absolute values of the maximum, the minimum, and average ratios of execution times. The graph \mathcal{G} used in these simulations is the graph arising out of a uniform cell decomposition with 4-connectivity, i.e., a graph of the form shown in Fig. 4(a). For each combination of H and $|\mathcal{G}|$, 30 trials were performed. In each of these trials, the structure of the graph \mathcal{G} was kept constant and the costs of transitioning H -histories, i.e., the costs of edge transitions in \mathcal{G}_H , were assigned randomly. The initial and goal nodes i_S and i_G were randomly assigned in each trial.

Table I indicates that the proposed algorithm executes up to three orders of magnitude faster, on average, and may execute up to four orders of magnitude faster in the best-case, than the alternative approach of first constructing the lifted graph and then executing the search. Furthermore, the memory required to store the graph \mathcal{G}_H is approximately K times the memory required by the proposed algorithm to store multiple histories of each node $j \in V$, where K is the valency of the graph \mathcal{G} .

D. Modifications for Further Efficiency

As mentioned previously, the execution time of the basic version of the proposed algorithm increases exponentially with H , which may slow down the algorithm for large values of H . To address this issue, we present in this section a simple modification of the basic version of the algorithm that dramatically reduces its execution time at the expense of optimality of the resultant path.

TABLE I
EXECUTION TIME RATIOS: SAMPLE VALUES

$ \mathcal{G} $	H	$ \mathcal{G}_H $	Max. ratio	Min. ratio	Avg. ratio
6,400	1	25,280	95.49	0.937	5.039
10,000	1	39,600	56.78	0.993	4.756
14,400	1	57,120	17.07	1.017	3.726
22,500	1	89,400	36.19	1.221	4.574
6,400	2	74,888	43.01	1.199	4.646
10,000	2	117,608	59.43	1.249	5.901
14,400	2	169,928	222.4	1.396	13.15
16,900	2	199,688	73.27	1.322	6.901
2,500	3	85,056	141.3	1.263	8.831
4,900	3	169,456	238.4	1.215	11.26
6,400	3	222,456	630.7	1.231	29.87
10,000	3	350,056	359.7	1.182	21.47
900	4	79,472	155.1	1.440	16.63
1,600	4	145,872	1264	1.410	75.14
2,500	4	232,272	399.8	1.287	20.57
625	5	147,952	1294	1.788	93.06
1,225	5	306,072	2 761	0.834	125.2
225	6	120,532	1399	3.604	267.1
400	6	237,232	2091	1.697	226.6

TABLE II
COMPARISONS OF EXEC. TIME AND SUB-OPTIMALITY: SAMPLE VALUES

$ \mathcal{G} $	H	L	$\max_i \{ \mathcal{T}_i \}$	Time ratio	Cost diff. (%)
22,500	1	1	4	2.424	18.13
22,500	1	2	4	1.434	0.144
14,400	2	1	12	8.055	34.43
14,400	2	3	12	2.645	0.108
14,400	2	5	12	1.698	0.000
10,000	3	2	36	15.62	2.379
10,000	3	4	36	7.119	0.146
10,000	3	5	36	5.521	0.036
2,500	4	3	100	31.90	0.662
2,500	4	5	100	17.75	0.155
2,500	4	7	100	12.20	0.038
1,225	5	3	284	121.7	1.170
1,225	5	5	284	69.11	0.543
1,225	5	7	284	47.15	0.243
625	6	2	780	681.0	4.727
625	6	5	780	257.6	0.496
625	6	11	780	107.1	0.148

The algorithm presented in Fig. 3 maintains, for each node $j \in V$, a record of the costs-to-come to that node through each of its histories. To reduce the execution time of the algorithm, we may modify the algorithm such that it maintains, for each node $j \in V$, the costs-to-come through a fixed number L of histories. In particular, we modify the proposed algorithm by inserting the following statements between Lines 5 and 8 in procedure MAIN:

```

 $\mathcal{L}_i \leftarrow \{d(i, m) < \infty : m = 1, \dots, |\mathcal{T}_i|\}$ 
if  $|\mathcal{L}_i| = L$  and  $D_{i,m} \geq \max\{\mathcal{L}_i\}$  then
  continue

```

Accordingly, we delete from the set \mathcal{N}_i defined in (3) all but the first L histories, ranked by increasing H -costs.

Table II shows a few sample ratios of the execution times of the proposed algorithm with the above modifications to the execution times without these modifications, and the corresponding average sub-optimality of the resultant paths of the percentage increases in cost. For each combination of H , $|\mathcal{G}|$, and L , 30 trials were performed. In each of these trials, the structure of the graph \mathcal{G} was kept constant and the costs of

transitioning H -histories, i.e., the costs of edge transitions in \mathcal{G}_H , were assigned randomly. The initial and goal nodes were kept fixed in each trial: in particular, $i_S = 1$ and $i_G = |\mathcal{G}|$ were assigned.

Table II indicates that relatively small values of L speed up the original algorithm by up to three orders of magnitude, with relatively low increases in the cost of resultant paths. A similar observation has been reported in [24], for the specific case of $H = 1$ and $L = 1$.

IV. MOTION PLANNING WITH KINODYNAMIC FEASIBILITY GUARANTEES

In this section, we present a hierarchical motion planning framework based on the H -cost shortest paths defined in the previous section. In this framework, the high-level geometric path planner searches for H -cost shortest paths in the cell decomposition graph \mathcal{G} . This geometric planner repeatedly invokes a special trajectory generation algorithm, called the *tile motion planner*, to determine the costs of H -histories. Due to the interaction between the geometric planner and the trajectory planner, the search for a channel of cells from the initial cell to the goal cell progresses *simultaneously* with the (piecewise) construction of a trajectory from the initial vehicle state to the goal state.

In this paper, we consider only vehicle dynamical models where the vehicle configuration is described by its position in the plane and its orientation, i.e., the configuration space $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$. This class of vehicle models includes all navigational models of aerial and terrestrial vehicles moving in the horizontal plane. Note that the vehicle *state space* \mathcal{D} may be larger than the configuration space.

We consider a vehicle model described as follows. Let $(x, y, \theta) \in \mathcal{C}$ denote the position coordinates of the vehicle in a pre-specified Cartesian axis system, and let ψ denote any additional state variables required to describe the state of the vehicle. We assume that $\psi \in \Psi$, where Ψ is a n -dimensional smooth manifold. The state of the vehicle is thus described by $\xi := (x, y, \theta, \psi) \in \mathcal{D} = \mathcal{C} \times \Psi$. Let $U \in \mathbb{R}^m$ denote the set of admissible control values; and for $t > 0$, let \mathcal{U}_t denote the set of piecewise continuous functions defined on the interval $[0, t]$ that take values in U . We assume that the evolution of the vehicle state ξ over a given time interval $[0, t_f]$ is described by the differential equation $\dot{\xi}(t) = f(\xi(t), u(t))$ for all $t \in [0, t_f]$, where $u \in \mathcal{U}_{t_f}$ is an admissible control input, and f is sufficiently smooth to guarantee global existence and uniqueness of solutions. We denote by $\xi(\cdot; \xi_0, u)$ the state trajectory that is the unique solution to the preceding differential equation with initial condition $\xi(0) = \xi_0$. Finally, we denote by $\mathbf{x}(\xi)$ the projection of a state ξ on \mathbb{R}^2 , i.e., the position components of ξ .

We introduce a special state trajectory planner called the *tile motion planner* as follows. Consider two nodes $I, J \in V_H$ such that $(I, J) \in E_H$. We define the *tile* associated with the edge (I, J) as the sequence of cells associated with the tuple $([I]_1, [J]_1^{H+1})$ of nodes in \mathcal{G} . In what follows, we use the symbol τ to denote a tile, and we denote by (I^τ, J^τ) the edge in E_H associated with this tile. TILEPLAN is then defined as

Algorithm 2: Tile Motion Planning Algorithm**Input:** τ, ξ_0 ; **Output:** t_1, ξ_1, u, Λ procedure TILEPLAN(τ, ξ_0)

- 1: Determine if there exist $t_f \in \mathbb{R}$ and admissible control input $u \in \mathcal{U}_{t_f}$ such that $\xi(\cdot; \xi_0, u)$ satisfies

$$x(\xi(t; \xi_0, u)) \in \bigcup_{k=1}^H \text{cell}([J^\tau]_k), \quad t \in (0, t_f), \quad (4)$$

$$x(\xi(t_f; \xi_0, u)) \in \text{cell}([J^\tau]_H) \cap \text{cell}([J^\tau]_{H+1}) \quad (5)$$

- 2: **if** $\exists t_f$ and $\exists u$ **then**

- 3: Find t_1 such that

$$x(\xi(t_1; \xi_0, u)) \in \text{cell}([J^\tau]_1) \cap \text{cell}([J^\tau]_2) \quad (6)$$

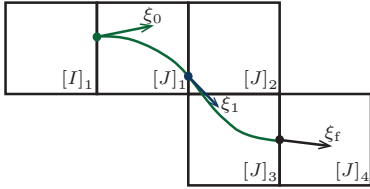
- 4: Return $t_1, u_{[0, t_1]}, \xi_1 := \xi(t_1; \xi_0, u)$, and

$$\Lambda := \int_0^{t_1} \ell(\xi(t; \xi_0, u), u, t) dt \quad (7)$$

- 5: **else**

- 6: Return $\Lambda = \infty$

Fig. 5. General form of the tile motion planning algorithm.

Fig. 6. Illustration of tile motion planning for square cells, with $H = 3$.

any algorithm which determines if a given tile may be feasibly traversed by the vehicle from a specific initial condition. The cost of traversal of a tile is the integral along the state trajectory of a pre-specified incremental cost $\ell(\xi, u, t)$. The general form of TILEPLAN is given in Fig. 5; in Section V, we outline a specific implementation of TILEPLAN, when the workspace is decomposed into square cells.

Briefly, TILEPLAN determines if there exists a finite time t_f and a control $u \in \mathcal{U}_{t_f}$ such that the corresponding vehicle state trajectory satisfies the constraints (4) and (5). The constraint (4) states the requirement that the position components of the vehicle state trajectory remains within the tile τ at all times in the interval $(0, t_f)$, whereas the constraint (5) states the requirement that the position components leave the tile in a finite time t_f . The algorithm returns the time t_1 required to traverse the first cell of the tile τ , the time history $u_{[0, t_1]}$ over the interval $[0, t_1]$ of the control input u that enables traversal of the tile, the vehicle state ξ_1 at the boundary between the first and second cells of the tile τ (see Fig. 6), and the cost Λ of traversal of the first cell.

Note that expression (4) does not depend explicitly on θ or ψ . This observation is consistent with our observation in the introductory discussion that the higher level planner typically operates on a discrete representation of the vehicle's workspace (\mathbb{R}^2), not its state space. In practice, θ or ψ may be subject to other constraints: for example, if $\psi = (\dot{x}, \dot{y})$

Algorithm 3: Overall Motion Planning Framework**Input:** I_S, ξ_0 ; **Output:** $d, b, \Xi, \Theta, \Upsilon$ procedure INITIALIZE(I_S, ξ_0)

- 1: $\mathcal{P} \leftarrow \{I_S\}, d(I_S) \leftarrow 0$;

- 2: **for all** $I \in V_H \setminus \{I_S\}$ **do**

- 3: $d(I) = \infty$;

- 4: $\Xi(I) = \xi_0, \Theta(I) = 0$

procedure MAIN

- 1: INITIALIZE(I_S, ξ_0)

- 2: **while** $\mathcal{P} \neq \emptyset$ **do**

- 3: $I \leftarrow \text{REMOVE}(\mathcal{P})$

- 4: **for all** $J \in V_H$ such that $(I, J) \in E_H$ **do**

- 5: $\tau \leftarrow \text{TILE}([I]_1, [J]_1^{H+1})$

- 6: $(t_1, u_{[0, t_1]}, \xi_1, \Lambda) \leftarrow \text{TILEPLAN}(\tau, \Xi(I))$

- 7: **if** $d(I) + \Lambda < d(J)$ **then**

- 8: $d(J) \leftarrow d(I) + \Lambda$

- 9: $b(J) \leftarrow I, \Xi(J) \leftarrow \xi_1$

- 10: $\Theta(J) \leftarrow t_1, \Upsilon(J) \leftarrow u_{[0, t_1]}$

- 11: INSERT(\mathcal{P}, J)

Fig. 7. Pseudo-code for the overall motion planner.

represents the vehicle's velocity, then $\|\psi(t)\|$ may be subject to lower and upper bounds. However, these constraints are of no concern at the geometric planning level; their satisfaction will be ensured internally by TILEPLAN.

Suppose, for now, that a tile motion planning algorithm satisfying the above requirements is available. Figure 7 then describes the overall motion planning framework. It consists of a geometric path planner that repeatedly invokes TILEPLAN to determine H -costs of histories. The proposed motion planner associates with each node $I \in V_H$ (in addition to the label $d(I)$ and backpointer $b(I)$ of the standard label correcting algorithm) a vehicle state $\Xi(I) \in \mathcal{D}$, a time of traversal $\Theta(I) \in \mathbb{R}_+$, and an admissible control input $\Upsilon(I) \in \mathcal{U}_{\Theta(I)}$.

Informally, the proposed planner searches for a path in the graph \mathcal{G}_H by traversing one edge during each iteration, while simultaneously propagating the vehicle state forward. As previously mentioned, the choice of an appropriate control input for propagating the vehicle state are left to TILEPLAN.

The proposed planner produces a path $\Pi^* = (J_0, \dots, J_P)$ in \mathcal{G}_H , where $[J_0]_1 = i_S$ and $[J_P]_{H+1} = i_G$. As discussed previously, Π^* corresponds to a sequence of cells, and the control input for traversing this sequence of cells is given by

$$u(t) := \Upsilon(J_k), \quad t \in \left[\sum_{m=1}^{k-1} \Theta(J_m), \sum_{m=1}^{k-1} \Theta(J_m) + \Theta(J_k) \right), \quad (8)$$

for each $k = 1, \dots, P$.

In Appendix A, we show that, with increasing H , the costs of trajectories resulting from the proposed motion planner are non-increasing. An informal interpretation of this result is that as H is increased, the proposed motion planner erroneously rejects fewer admissible paths in \mathcal{G} as infeasible. This result guides the selection of the value of H for implementing the proposed motion planner, in that it assures benefits (in terms of

optimality of the resultant trajectory) in return for expending computational resources for using larger values of H .

V. TILE MOTION PLANNING IMPLEMENTATIONS

The low-level trajectory planner TILEPLAN plays a crucial role in the proposed motion planning framework. In this section, we present two specific implementations of TILEPLAN for nonholonomic vehicles.

The implementation of TILEPLAN is difficult mainly because (4) imposes a non-convex constraint on the state trajectory. To alleviate this difficulty, we take advantage of the fact that each cell in the sequence of cells associated with a tile is a convex region, using the idea of *effective target sets*, first introduced in [51]. As described next, effective target sets enable the use of MPC-based techniques for implementing TILEPLAN by transforming the constraint in (4) to a convex constraint defined over a single cell.

Let τ be a tile corresponding to the edge $(I, J) \in E_H$. We define a sequence $\{\mathcal{X}_k\}_{k=1}^{H+1}$ of subsets of the vehicle state space called the *effective target sets* as follows. Let

$$\mathcal{X}_H := (\text{cell}([J]_H) \cap \text{cell}([J]_{H+1})) \times [-\pi, \pi] \times \Psi.$$

For each $k = 1, \dots, H-1$, we define the effective target set \mathcal{X}_k as the set of all states $\xi_k \in \mathcal{D}$ such that

$$x(\xi_k) \in \text{cell}([J]_k) \cap \text{cell}([J]_{k+1}), \quad (9)$$

and such that there exists $t_{k+1} \in \mathbb{R}_+$ and an admissible control input $u_{k+1} \in \mathcal{U}_{t_{k+1}}$ such that the state trajectory $\xi(\cdot; \xi_k, u_{k+1})$ satisfies

$$x(\xi(t; \xi_k, u_{k+1})) \in \text{cell}([J]_{k+1}), \quad t \in (0, t_{k+1}), \quad (10)$$

$$\xi(t_{k+1}; \xi_k, u_{k+1}) \in \mathcal{X}_{k+1}. \quad (11)$$

The preceding definition of effective target sets allows a simplification of the tile motion planning problem as follows. Suppose there exist a time t_1 and a control $u_1 \in \mathcal{U}_{t_1}$ such that the resultant state trajectory $\xi(\cdot; \xi_0, u_1)$ satisfies

$$x(\xi(t; \xi_0, u_1)) \in \text{cell}([J]_1), \quad t \in (0, t_1), \quad (12)$$

$$\xi_1 := \xi(t_1; \xi_0, u_1) \in \mathcal{X}_1. \quad (13)$$

Note that, due to (9), the conditions (12)-(13) imply the satisfaction of (4)-(5) for $H = 1$. Next, because $\xi_1 \in \mathcal{X}_1$, it follows by (10)-(11) that there exists a $t_2 \in \mathbb{R}_+$ and an admissible control input $u_2 \in \mathcal{U}_{t_2}$ such that

$$x(\xi(t; \xi_1, u_2)) \in \text{cell}([J]_2), \quad t \in (0, t_2),$$

$$\xi(t_2; \xi_1, u_2) \in \mathcal{X}_2.$$

In other words, the admissible control input u_{1-2} defined as the concatenation of the inputs u_1 and u_2 by

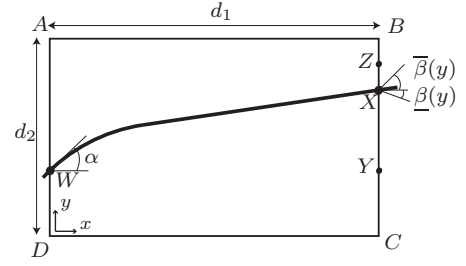
$$u_{1-2}(t) := \begin{cases} u'(t), & t \in [0, t_1], \\ u_1(t), & t \in [t_1, T_2], \end{cases}$$

$$\text{where } T_2 := t_1 + t_2,$$

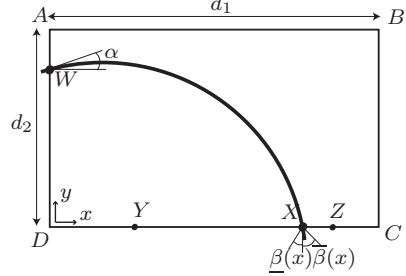
enables the vehicle's traversal through the cells corresponding to the nodes $[J]_1$ and $[J]_2$, i.e.,

$$x(\xi(t; \xi_0, u_{1-2})) \in \text{cell}([J]_1) \cup \text{cell}([J]_2), \quad t \in (0, T_2), \quad (14)$$

$$\xi(T_2; \xi_0, u_{1-2}) \in \mathcal{X}_2. \quad (15)$$



(a) Type 1 admissible path.



(b) Type 2 admissible path.

Fig. 8. Setup for Problems 2 and 3, which is used in the computation of effective target configuration sets.

Note that, due to (9), the conditions (14)-(15) imply the satisfaction of (4)-(5) for $H = 2$. Continuing recursively the preceding arguments, it follows that, for each $H \geq 2$, there exist $t_{k+1} \in \mathbb{R}_+$ and admissible inputs $u_{k+1} \in \mathcal{U}_{t_{k+1}}$, for $k = 1, \dots, H-1$, such that the admissible input u defined by

$$u(t) := \begin{cases} u_1(t), & t \in [0, T_1], \\ u_2(t), & t \in [T_1, T_2], \\ \vdots & \vdots \\ u_H(t), & t \in [T_{H-1}, T_H], \end{cases} \quad (16)$$

$$\text{where } T_k := \sum_{m=1}^k t_m,$$

solves the tile motion planning problem.

Thus, if the effective target sets \mathcal{X}_k , the corresponding times of traversal t_{k+1} and the control inputs u_{k+1} in (16) are known for each $k = 1, \dots, H-1$, then the tile motion planning problem is equivalent to the problem of finding u_1 and t_1 as described above. Crucially, (12) constrains the position components of the state trajectory to lie within a convex set. Furthermore, we may replace \mathcal{X}_1 in (13) by an interior convex approximating set $\tilde{\mathcal{X}}_1 \subset \mathcal{X}_1$ thus transforming the tile motion planning problem into the problem of finding u_1 and t_1 subject to convex constraints.

A. Computation of Effective Target Sets

Guidelines for constructing the effective target sets are provided in [51]; however, specific constructions may be tailored to the specific dynamical model. In light of the fact that the vehicle state includes the configuration (x, y, θ) , we consider first the computation of the intersections of the effective target sets with the configuration space $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$. To this end, we define the *effective target configuration sets* by $\mathcal{C}_k := \mathcal{X}_k \cap \mathcal{C}$, and, in what follows, we outline a geometric scheme of computing the sets \mathcal{C}_k .

Assumption 1: The geometric curves in the plane that can be feasibly traversed by the vehicle satisfy a local upper bound on their curvatures.

We will comment on the validity of Assumption 1 in Section VI. Here, we use this Assumption to compute the sets \mathcal{C}_k by solving the following problems in plane geometry.

Let $ABCD$ be a rectangle. We attach a Cartesian axes system as shown in Fig. 8. Let the dimensions of the rectangle be d_1 and d_2 , and let $r > 0$ be fixed.

Definition 1: Let $\beta(x), \bar{\beta}(x)$, $x \in [0, d_2]$ be functions such that $-\frac{\pi}{2} \leq \beta(x) \leq \bar{\beta}(x) \leq \frac{\pi}{2}$. Let $Y = (d_1, y)$, $Z = (d_1, z)$ be points on the segment BC with $y \leq z$. A path Π is a *Type 1 admissible path* if it satisfies the following properties:

- 1) (Curvature Boundedness): The curvature at any point on Π is at most r^{-1} ,
- 2) (Containment): Π intersects the segment BC in exactly one point $X = (d_1, x)$ such that $x \in [y, z]$, and it may intersect segment AB and/or segment CD in at most one point each, and
- 3) (Terminal Orientation): $\Pi'(X) \in [\beta(x), \bar{\beta}(x)]$.

A Type 2 admissible path is defined analogously for traversal across adjacent edges.

Definition 2: Let $\beta(x), \bar{\beta}(x)$, $x \in [0, d_1]$ be functions such that $-\frac{\pi}{2} \leq \beta(x) \leq \bar{\beta}(x) \leq \frac{\pi}{2}$. Let $Y = (y, 0)$, $Z = (z, 0)$ be points on the segment CD with $y \leq z$. A path Π is a *Type 2 admissible path* if it satisfies the following properties:

- 1) (Curvature Boundedness): The curvature at any point on Π is at most r^{-1} ,
- 2) (Containment): Π intersects the segment CD in exactly one point $X = (x, 0)$ such that $x \in [y, z]$, and it may intersect segment AB and/or segment BC in at most one point each, and
- 3) (Terminal Orientation): $\Pi'(X) + \frac{\pi}{2} \in [\beta(x), \bar{\beta}(x)]$.

We state two geometric problems as follows. Let $\beta, \bar{\beta}$, Y , and Z be as in the preceding definitions. Let $W = (0, w)$ and $r > 0$ be fixed.

Problem 2 (Traversal across parallel edges): Find bounds $\underline{\alpha}, \bar{\alpha}$ such that for all $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, there exists a Type 1 admissible path with initial configuration (W, α) .

Problem 3 (Traversal across adjacent edges): Find bounds $\underline{\alpha}, \bar{\alpha}$ such that for all $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, there exists a Type 2 admissible path with initial configuration (W, α) .

Problems 2 and 3 appear in the recursive computation of effective target configurations as follows. Suppose that the effective target configuration set \mathcal{C}_{k+1} is known, for $k \in \{1, \dots, H-1\}$. Then we may express \mathcal{C}_{k+1} as the product set of a line segment on the boundary between cells $\text{cell}([J]_{k+1})$ and $\text{cell}([J]_{k+2})$ and a set of allowable orientations on this line segment. In other words, we may express \mathcal{C}_{k+1} in terms of the points Y, Z and the functions $\beta, \bar{\beta}$ used in Definitions 1 and 2. We may then solve Problem 2 or 3, as applicable for the cell $\text{cell}([J]_{k+1})$, for each point on the line segment forming the boundary between cells $\text{cell}([J]_k)$ and $\text{cell}([J]_{k+1})$ and obtain allowable orientations for each point on this line segment. The product set of these allowable orientations and this line segment is precisely the set \mathcal{C}_k . The effective

target configuration sets may thus be computed recursively by repeatedly solving Problems 2 and 3 as applicable for each cell, with $\mathcal{C}_H := (\text{cell}([J]_H) \cap \text{cell}([J]_{H+1})) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The solutions to Problems 2 and 3 are based on detailed plane geometrical analysis, and are beyond the scope of this paper. We refer the reader to [52] for the analysis involved in the solution of Problems 2 and 3. Here, we present a procedure that uses the solutions to Problems 2 and 3 to recursively compute the effective target configuration sets. Note that the sequence of cells associated with a tile is, in general, a *rectangular channel*, that is, a finite sequence $\{R_n\}_{n=1}^C$, of disjoint rectangles of arbitrary dimensions such that every pair of successive rectangles has a common edge.

We attach a coordinate axes system to each rectangle R_n in a manner consistent with the axes system used in the statement of Problems 2 and 3 (see Fig. 10). The dimensions of each rectangle along the x and y axes are denoted, respectively, as $d_{n,1}$ and $d_{n,2}$.

For each rectangle R_n , we refer to the segments formed by the intersections $R_{n-1} \cap R_n$ and $R_n \cap R_{n+1}$, respectively, as the *entry and exit segments*. For the rectangle R_1 (resp. rectangle R_C), the entry segment (resp. exit segment) is specified arbitrarily. We denote the endpoints of the entry segment by U_n and V_n , and the endpoints of the exit segment by Y_n and Z_n . We specify the coordinates of the points U_n, V_n, Y_n, Z_n , by the corresponding lower case letters, i.e., $V_n = (0, v_n)$, etc.

For every point $Q = (q, 0)$ (or $Q = (d_{n,1}, q)$, as applicable), with $q \in [y_n, z_n]$, on the segment $Y_n Z_n$, we denote by $\underline{\beta}_n(q)$ and $\bar{\beta}_n(q)$, respectively, the lower and upper bounds on orientation of the effective target configuration sets. Similarly, for every point $P = (0, p)$, $p \in [u_n, v_n]$, on the segment $U_n V_n$, we denote by $\underline{\alpha}_n(p)$ and $\bar{\alpha}_n(p)$, respectively, the upper and lower bounds resulting from the solution of Problem 2 (or Problem 3, as applicable). Note that the angles $\underline{\alpha}_n(\cdot), \bar{\alpha}_n(\cdot), \underline{\beta}_n(\cdot)$, and $\bar{\beta}_n(\cdot)$ are all measured with respect to the local coordinate axes system attached to R_n . Finally, we denote by ϱ_n the number of reflection operations involved in the geometric transformation (consisting of rotation and reflection operations) required to align the entry and exit segments of R_n to the segments AD and BC , respectively, for traversal across parallel edges, or to segments AD and CD , respectively, for traversal across adjacent edges.

The recursive analysis for determining the existence of curvature-bounded paths in rectangular channels is described using the above notation in Fig. 9. To better explain the procedure in Fig. 9, we illustrate its execution by an example.

Example 2: Let $\bar{\mathcal{R}}^4 = \{R_n\}_{n=1}^4$ be a rectangular channel with four rectangles, as shown in Fig. 10, and let $r_n > 0$, $n = 1, \dots, 4$, be given. The points U_n, V_n , $n = 1, \dots, 4$ and the points Y_4, Z_4 are shown in Fig. 10. We note that $Y_1 = U_2$, $Z_1 = V_2$, $Y_2 = V_3$, $Z_2 = U_3$, $Y_3 = V_4$, and $Z_3 = U_4$.

Following the procedure in Fig. 9, we note that the last rectangle, R_4 , involves traversal across parallel edges, and we initialize $\bar{\alpha}_5$ and $\underline{\alpha}_5$ as

$$\bar{\alpha}_5(q) = \frac{\pi}{2}, \quad \underline{\alpha}_5(q) = -\frac{\pi}{2}, \quad q \in [0, d_{4,2}].$$

Algorithm 4: Effective Configuration Sets

-
- 1: $\bar{\alpha}_{C+1}(q) \leftarrow \frac{\pi}{2}, \underline{\alpha}_{C+1}(q) = -\frac{\pi}{2}, q \in [0, d_{C,2}]$ if R_C involves traversal across parallel edges, or otherwise, for $q \in [0, d_{C,1}]$ if R_C involves traversal across adjacent edges
 - 2: $\varrho_{C+1} \leftarrow 0$
 - 3: **for** $n = C$ to 1 **do**
 - 4: **if** $\varrho_n + \varrho_{n+1}$ **even then**
 - 5: $\underline{\beta}_n \leftarrow \underline{\alpha}_{n+1}, \bar{\beta}_n \leftarrow \bar{\alpha}_{n+1}$
 - 6: **else**
 - 7: $\underline{\beta}_n(q) \leftarrow -\bar{\alpha}_{n+1}(y_n - (q - v_{n+1}))$, for $q \in [y_n, z_n]$
 - 8: $\bar{\beta}_n(q) \leftarrow -\underline{\alpha}_{n+1}(y_n - (q - v_{n+1}))$, for $q \in [y_n, z_n]$
 - 9: $\underline{\alpha}_n(q), \bar{\alpha}_n(q) \leftarrow$ solution of Problem 2 or Problem 3, as applicable to R_n for $q \in [u_n, v_n]$
-

Fig. 9. Recursive computation of effective target configuration sets.

Next, we execute Line 4 of the algorithm, and we note that the entry and exit segments of rectangle R_4 are aligned with segments AD and BC , respectively, of Fig. 8. Thus, the total number of reflections occurring in the transformations required for R_4 and (the fictitious rectangle) R_5 is zero, and we set

$$\bar{\beta}_4 = \bar{\alpha}_5 = \frac{\pi}{2}, \quad \underline{\beta}_4 = \underline{\alpha}_5 = -\frac{\pi}{2}.$$

For executing Line 9, we solve Problem 2 for each point $Q = (0, q)$, $q \in [0, d_{4,2}]$ on the segment U_4V_4 , and we obtain the values taken by the functions $q \mapsto \underline{\alpha}_4(q)$ and $q \mapsto \bar{\alpha}_4(q)$.

Now we repeat Line 4 for the rectangle R_3 . Rectangle R_3 involves traversal across adjacent edges, and the entry and exit segments of R_3 may be aligned with segments AD and DC of Fig. 8 after a reflection about an axis parallel to the segment U_4V_4 , followed by a rotation through $\frac{\pi}{2}$ rad. Thus, the total number of reflections occurring in the transformations required for R_3 and R_4 is one (odd), and we set

$$\begin{aligned} \bar{\beta}_3(q) &= -\underline{\alpha}_4(y_3 - (q - v_4)), & q \in [y_3, z_3], \\ \underline{\beta}_3(q) &= -\bar{\alpha}_4(z_3 - (q - u_4)), \end{aligned}$$

where $z_3 = d_{3,1}$, $y_3 = \ell(U_3V_4)$, $v_4 = d_{4,2}$, and $u_4 = 0$ (see Fig. 10). For executing Line 9, we solve Problem 3 for each point $P = (0, p)$, $p \in [0, d_{3,2}]$ on the segment U_3V_3 to obtain values taken by the functions $p \mapsto \underline{\alpha}_3(p)$ and $p \mapsto \bar{\alpha}_3(p)$. Proceeding further in a similar manner, we may obtain the values taken by the functions $\underline{\beta}_2, \bar{\beta}_2$, and by the functions $\underline{\beta}_1, \bar{\beta}_1$. As discussed previously, the effective target configuration sets \mathcal{C}_n may then be expressed in terms of the functions $\underline{\beta}_n$, and $\bar{\beta}_n$, for each $n = 1, \dots, 4$.

B. TILEPLAN for the Dubins Car

In this section, we discuss an implementation of TILEPLAN for the Dubins car kinematic model described by

$$\dot{x}(t) = v \cos \theta(t), \quad \dot{y}(t) = v \sin \theta(t), \quad \dot{\theta}(t) = u(t),$$

where x, y , and θ are, respectively, the position coordinates and the orientation of the vehicle with respect to a pre-specified inertial axes system; $v > 0$ is the (fixed) forward speed of the vehicle; and u is the steering control input. The

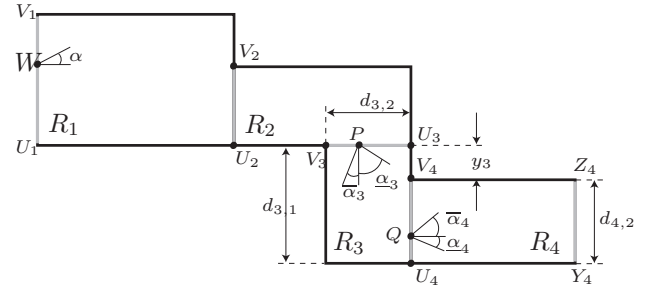
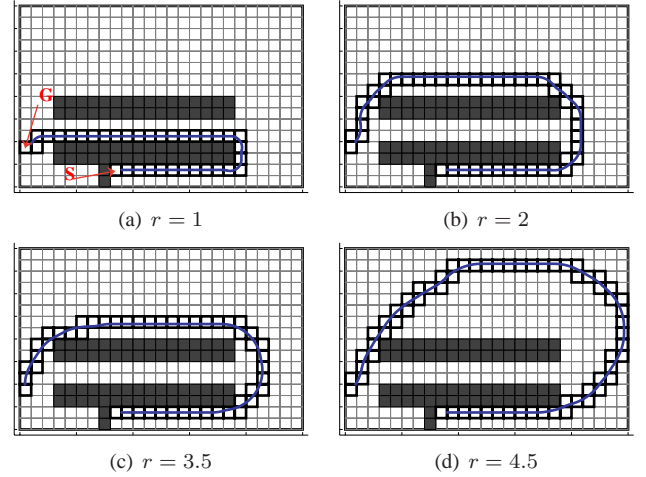


Fig. 10. Illustrative example of the computation of effective target configuration sets.

Fig. 11. Making U-turns with different curvature bounds. The curvature bound in each case is r^{-1} . Here, $H = 3$.

set of admissible control inputs is $U := [-1/r, 1/r]$, for a pre-specified $r > 0$. As shown in Section VI, the upper bound on the curvature of feasible geometric paths is $\kappa_{\max} = (rv)^{-1}$.

Note that when the state space \mathcal{D} is the same as the configuration space \mathcal{C} , the effective target sets coincide with the effective target configurations sets, which can be computed via pure geometric analysis as described in the previous section. In addition, the feasible paths specified in TILEPLAN can also be constructed geometrically, as discussed in [52], thus enabling a solution to the motivating example of Section II.

Figure 11 shows results of the simulations of the proposed algorithm for a problem similar to the motivating example of Section II. Note, in particular, that different channels are obtained for different bounds on the curvature, whereas any cost function defined on the edges E of the cell decomposition graph \mathcal{G} will result in the channel shown in Fig. 11(a). For this example, the cost of traversal of a tile was chosen as the time of traversal, i.e., $\ell(\xi, u, t) := 1$ in (7).

VI. RESULTS AND COMPARATIVE DISCUSSIONS

In this section, we present the results of implementation of the proposed motion planning framework for general vehicle models, and we compare our results to those obtained using randomized sampling-based (RRT-based) motion planners. We implemented the fringe as a list sorted by the sum of the current label and a heuristic; specifically, we used the Manhattan distance to the goal cell as a heuristic. We implemented

TILEPLAN using a trajectory generation scheme based on model predictive control, similar to that reported in [53]. Before discussing the results, we comment on Assumption 1, which was used for geometrically computing the effective target configuration sets.

The expression for the local curvature of the geometric path corresponding to feasible state trajectories is given by

$$\kappa(t) = \left| \dot{\theta}(t)/v(t) \right|. \quad (17)$$

An upper bound for the local curvature may then be computed based on the specific vehicle model. For the Dubins car model for example, $|\dot{\theta}| = |u| \leq 1/r$, and by (17) it follows that $\kappa(t) \leq (rv)^{-1}$ for all $t \geq 0$, i.e., the upper bound on the curvature of feasible geometric paths is $\kappa_{\max} = (rv)^{-1}$. Similarly, for the dynamical model in Section VI-A below, note that $|\dot{\theta}(t)| \leq |u_2(t)| \leq f_{\max}^r/v(t)$. It follows by (17) that $\kappa(t) \leq f_{\max}^r/v^2(t)$. Thus, an upper bound on the curvature is f_{\max}^r/v_{\max}^2 ; however, for each tile one may compute a local bound \bar{v} on the speed valid for traversal across the tile, and use the less conservative upper bound $f_{\max}^r/(\min\{\bar{v}, v_{\max}\})^2$ for implementing TILEPLAN.

A. Optimality of Resultant Trajectories

As discussed in Section I-A, randomized sampling-based algorithms based on RRTs [33] represent the state-of-the-art in kinodynamic motion planning. We consider a vehicle dynamical model described by

$$\begin{aligned} \dot{x}(t) &= v(t) \cos \theta(t), & \dot{y}(t) &= v(t) \sin \theta(t), \\ \dot{\theta}(t) &= u_2(t), & \dot{v}(t) &= u_1(t), \end{aligned}$$

where $v > 0$ is the forward speed of the vehicle; u_1 is the acceleration input, and u_2 is the steering input. The speed v is constrained to lie within pre-specified bounds v_{\min} and v_{\max} ; these bounds may be different for different regions of the workspace. The set of admissible control inputs is

$$U := \left\{ (a, \omega) : \left(\frac{v\omega}{f_{\max}^r} \right)^2 + \left(\frac{a}{f_{\max}^t} \right)^2 \leq 1 \right\}, \quad (18)$$

where f_{\max}^r and f_{\max}^t are pre-specified. The input constraint defined by (18) is an example of a “friction ellipse” constraint that models the limited tire frictional forces available for acceleration and steering of the vehicle.

Figure 12(a) shows the first of two environments used in the numerical examples. This environment consists of “lanes” separated by obstacles (black regions), with a different upper bound on the allowable speed of the vehicle (lighter areas represent higher upper bounds). The “friction ellipse” parameters were fixed at $f_{\max}^r = 1$, $f_{\max}^t = 0.25$ over the entire environment. The initial and goal cells are marked in Fig. 12(a); as before, the objective was to find a minimum time trajectory from the initial cell to the goal cell. We compared the proposed motion planner against the following two RRT-based planners: (1) the standard RRT-based planner³ as reported

³Several improvements to the standard RRT planner of [33] have been reported (see [36], [54] and references therein); however, with the exception of [54], these improvements focus mainly on the efficiency of the motion planner. We choose to compare against the standard RRT planner as we are mainly interested in the comparison of costs of resultant trajectories.

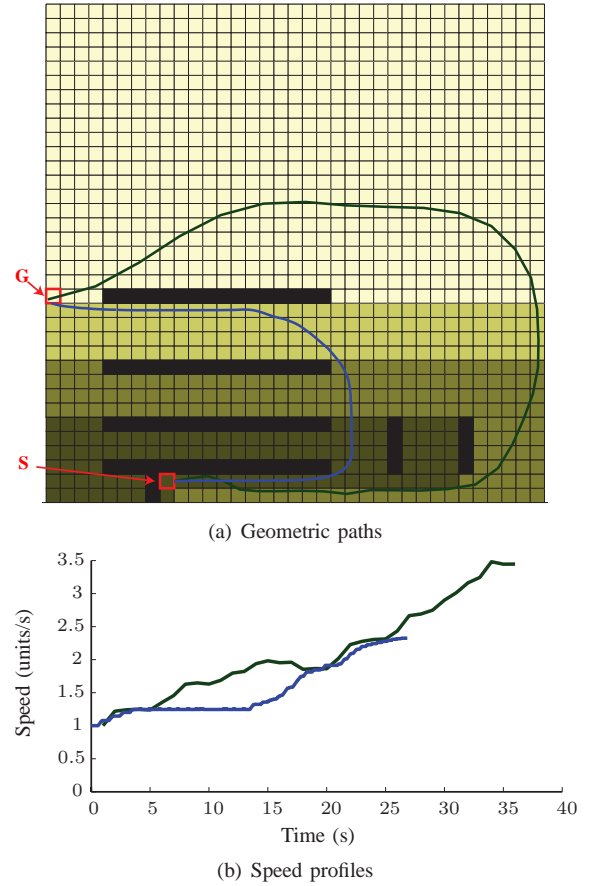


Fig. 12. The “lanes” environment. Black colored regions are obstacles; areas with other colors represent different speed limits: $v_{\max} = 1.25$ units/s for the darkest area, $v_{\max} = 2$ units/s, $v_{\max} = 2.5$ units/s, and $v_{\max} = 3.5$ units/s, respectively, for progressively lighter areas. The dark green curve is the geometric path corresponding to a sample trajectory returned by the T-RRT algorithm, with $\delta = 1$ s. The blue curve is the geometric path corresponding to the trajectory returned by the proposed approach, with $H = 6$.

in [33], and (2) the T-RRT planner recently reported in [54]. The T-RRT planner finds low-cost trajectories with respect to a pre-specified state space⁴ cost map. Note that the minimum-time criterion cannot be expressed as a state space cost map; therefore, we execute the T-RRT planner with the objective “travel as fast as possible,” which is immediately defined by the state space cost map $c(x, y, \theta, v) = v$.

As mentioned previously, linear interpolation between two states does not, in general, correspond to a feasible state trajectory. Hence, for extending known states towards randomly selected new states, the RRT-based planners were programmed to randomly select an input vector from the set of admissible inputs and integrate the vehicle model for a fixed time δ , as recommended in [33]. For the “lanes” environment, we used three different values of δ , namely, $\delta = 0.5$ s, $\delta = 1$ s, and $\delta = 1.5$ s, and we conducted 30 trials of both algorithms (standard RRT and T-RRT) for each value of δ . For comparison, we executed the proposed algorithm on the same environment with three different values of H , namely, $H = 4$, $H = 5$, and $H = 6$, with $L = 10$ in each case.

⁴In [54], the authors deal with a configuration space cost map, but their approach extends easily to state spaces.

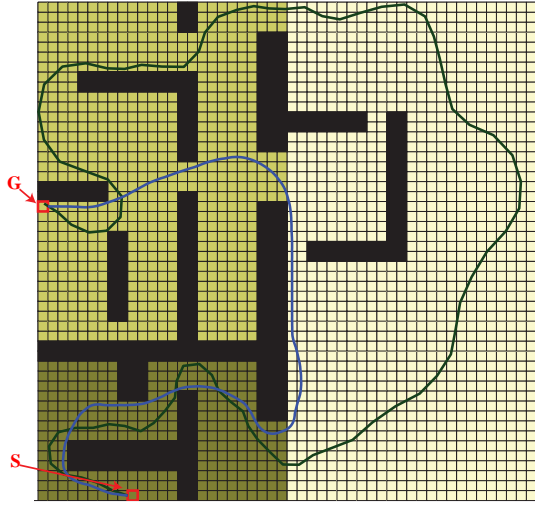


Fig. 13. The “maze” environment. Black colored regions are obstacles; areas with other colors represent different speed limits: $v_{\max} = 1.25$ units/s for the darkest area, $v_{\max} = 2$ units/s, and $v_{\max} = 2.25$ units/s, respectively, for progressively lighter areas. The dark green curve is the geometric path corresponding to a sample trajectory returned by the RRT-based planner, with $\delta = 1.5$ s. The blue curve is the geometric path corresponding to the trajectory returned by the proposed approach, with $H = 5$.

Figure 14(a) shows comparative data for the trajectory costs (i.e., time of traversal) resulting from the simulations described above. The proposed motion planner returned trajectories with almost identical costs for each H , in particular, the trajectory cost corresponding to $H = 6$ was 26.626 s. On the other hand, both the standard RRT and T-RRT planners returned, on an average, significantly costlier trajectories. For instance, the trajectory costs returned by the standard RRT planner with $\delta = 1$ were, in the best case 24% higher, on an average 78% higher, and in the worst case 181% higher. Similarly, the trajectory costs returned by the standard RRT planner with $\delta = 1$ were, in the best case 8.9% higher, on an average 29% higher, and in the worst case 46% higher.

Figure 12(a) shows the geometric path corresponding to the trajectory returned by the proposed planner with $H = 6$ (blue curve) in comparison to the geometric path corresponding to a trajectory returned by the T-RRT planner in one of the 30 trials with $\delta = 1$ (green curve); Fig. 12(b) shows the speed profiles corresponding to these two trajectories. This example illustrates that the “travel as fast as possible” objective is not always a practically acceptable alternative to the minimum-time criterion: Figure 12(b) shows that the vehicle achieves higher speeds along the T-RRT trajectory but the travel time is 35.2% higher than the trajectory found by the proposed planner. This result is a consequence of the input constraint (18), which forces the vehicle to traverse paths of lower curvature at higher speeds, thus producing longer geometric paths.

Figure 13 shows the second, maze-like environment used for our comparative analysis. As before, different upper bounds on the speed were assigned to different areas in the environment, and the friction ellipse parameters were fixed at $f_{\max}^r = 1$, $f_{\max}^t = 0.25$ over the entire environment. As before, the

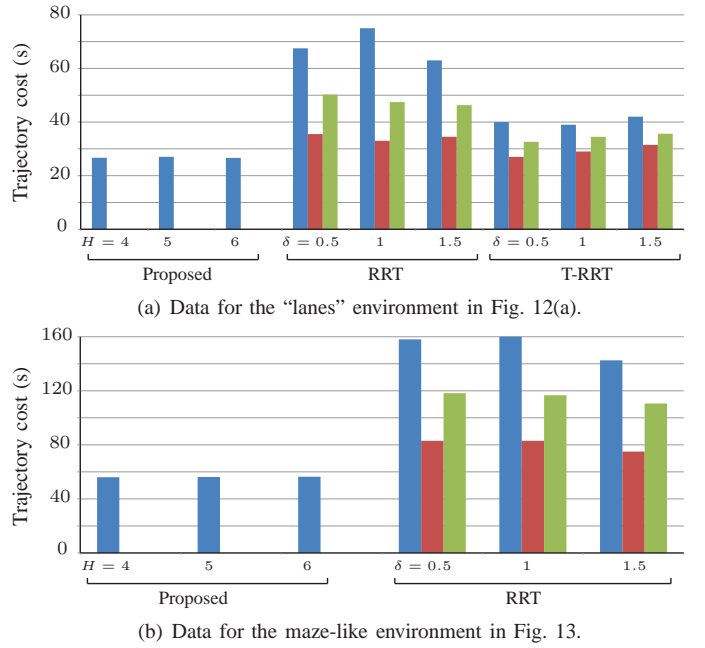


Fig. 14. Comparison of trajectory costs: for the RRT and T-RRT data, the blue (left), red (middle), and green (right) bars represent, respectively, the maximum, the minimum, and the average values over 30 trials.

TABLE III
EXECUTION TIMES FOR THE PROPOSED PLANNER

	$H = 4$	$H = 5$	$H = 6$
“Lanes”	31.89 s	52.02 s	55.12 s
Maze	129.6 s	303.7 s	378.2 s

objective is to find a minimum-time trajectory from the initial cell to the goal cell. We compared the proposed motion planner against the standard RRT planner alone, because the T-RRT planner was found to be impractically slow for this case. As shown in Fig. 13, the environment has a narrow “short-cut” between the initial cell and the goal cell.

Figure 14(b) shows comparative data for the trajectory costs for this maze-like environment. The proposed motion planner returned trajectories with almost identical costs for each H ; in particular, the trajectory cost corresponding to $H = 5$ was 56.23 s. The trajectory costs returned by the standard RRT planner were significantly higher, mainly because it failed to traverse the aforementioned “short-cut” on several occasions, as illustrated in Fig. 13. For instance, the trajectory costs returned by the standard RRT planner with $\delta = 1$ were, in the best case 48% higher, on an average 107% higher, and in the worst case 185% higher. Clearly, the average costs of trajectories returned by RRT-based planners may be further worsened in environments where the differences between the costs of trajectories corresponding to “short-cuts” and the costs of alternative trajectories is larger.

B. Performance of the Proposed Motion Planner

Table III presents the execution times of the simulations of the proposed planner for the examples discussed in the previous section. The simulations were implemented in the

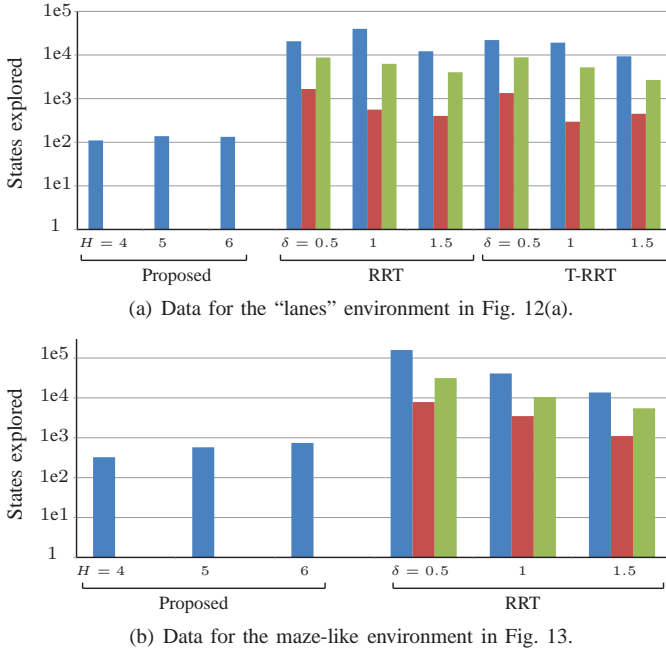


Fig. 15. Comparison of number of states explored: for the RRT and T-RRT data, the blue (left), red (middle), and green (right) bars represent, respectively, the maximum, the minimum, and the average values over 30 trials.

MATLAB programming language; implementations in lower-level languages will execute much faster.

Figure 15(a) shows on a logarithmic scale the number of states explored by each of the algorithms discussed in the previous section for the "lanes" environment; Fig. 15(b) shows similar data for the maze-like environment. In both cases, the number of states explored by the RRT-based planners were higher by at least an order of magnitude. However, the time required for the RRT-based planners to explore a new state (including the nearest neighbor search and collision checking) was found to be approximately an order of magnitude lower than the execution time of the MPC-based TILEPLAN. Consequently, the comparison of the performance of the proposed planner against the aforementioned RRT-based planners showed no conclusive evidence of the superiority of either planner over the other in that respect. Direct comparisons of the execution times of these algorithms corroborated this observation. However, Fig. 15 shows that the proposed planner is preferable in cases where the exploration of new states is expensive due to complex dynamics or due to expensive collision checking.

C. Further Comparative Discussion

1) *Comparisons with randomized sampling-based motion planners:* The exploration of the state space is difficult with standard RRT-based motion planners when the states and control inputs are coupled via complex, nonlinear differential equations [36], because linear interpolation between two states no longer corresponds, in general, to an admissible state trajectory. Whereas [36] and similar earlier works focus on aiding the *efficiency* of sampling-based algorithms using a discrete search, we focus on the complementary aspect of

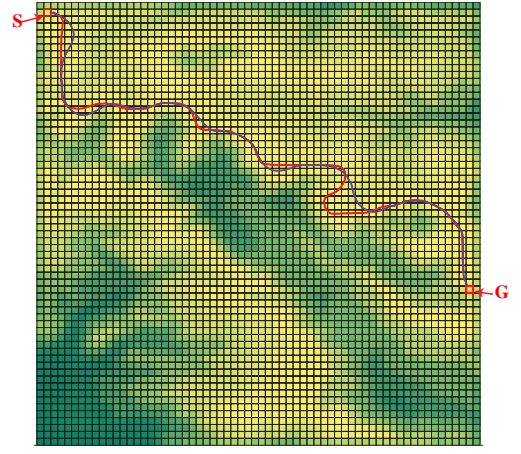


Fig. 16. Application of proposed motion planning framework for finding "low-elevation" paths: lighter areas represent more favorable regions of the workspace. The Dubins car kinematical model is used in TILEPLAN. The red curve represents a path with $r = 1$, whereas the blue curve represents a path with $r = 2$.

optimality by ensuring that the result of a discrete shortest-path search remains compatible with the vehicle dynamics.

In addition to the benefits of the proposed planner over randomized sampling-based planners in terms of optimality, the proposed planner also offers the benefit of a clear distinction between the discrete and continuous layers of motion planning. The idea of planning on the lifted graph, introduced in Section II, allows this distinction to be maintained, while providing guarantees of consistency between the two levels of planning. Consequently, changes to the discrete planning strategy and/or the (continuous) tile motion planning may be incorporated with relative ease. In this paper, we used the shortest-path search as a concrete, important example of a discrete search strategy; in the future, we envision extensions of the proposed planner where the discrete planner attempts to satisfy vehicular tasks specified as formulae of predicate or temporal logic [11] instead of solving a shortest path problem. On the other hand, complex vehicle dynamics can be easily incorporated in TILEPLAN without affecting the discrete planning strategy.

In the context of the shortest path problem alone, different trajectory quality criteria can be incorporated easily to meet different motion planning objectives. For example, Fig. 16 shows the result of simulating the proposed planner by defining the H -cost as a weighted sum of the time of traversal and the terrain elevation. Consequently, the planner finds paths that traverse low-elevation portions of the terrain (lighter regions in Fig. 16), while ensuring kinematic feasibility guarantees of the resultant paths. The Dubins car model was used for this simulation; the two curves in Fig. 16 indicate the resultant paths for different curvature constraints. It should be noted that the problem of finding low cost trajectories with respect to configuration space cost maps using randomized sampling-based methods has been addressed in [54]; however, many important trajectory quality criteria such as time optimality (considered in the preceding section) and fuel optimality cannot be expressed as configuration space cost maps.

2) *Comparisons with feedback-based motion planners:* An underlying assumption in the feedback-based motion planning approach described in Section I-A is the complete controllability of the vehicle dynamical model in the presence of obstacles, i.e., the assumption that there exists a feasible, obstacle-free trajectory from any initial state to any goal state. In the context of mobile vehicles, complete controllability in the presence of obstacles is a strong assumption: fixed-wing aircraft moving in the horizontal plane do not satisfy this assumption; terrestrial vehicles constrained to move only forward, or high-speed vehicles for which stopping and reversing the direction of motion may not be desirable, also do not satisfy this assumption. In contrast with the planners presented in [37]–[41], the proposed motion planning framework does not assume complete controllability in the presence of obstacles.

When the complete controllability assumption is violated, the central tenet of the preceding feedback-based motion planning schemes is no longer valid: arbitrary sequences of cell transitions cannot *in principle* be guaranteed from arbitrary initial states. A simple example of a vehicle kinematic model that violates the complete controllability assumption is the Dubins car model. For any given sequence of cell transitions in the workspace, there exists a set of initial states of the vehicle from which it is impossible for the vehicle to execute that sequence. The proposed framework does not require this assumption because the geometric planner ensures by computing an admissible control input, given in (8), the feasibility of traversal of its resultant path (i.e., the sequence of cell transitions from the initial position to the goal).

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented a hierarchical motion planning framework comprising of a high-level (discrete) path planner and a low-level continuous trajectory planner. In light of the fact that obstacles are typically defined in the lower-dimensional workspace, we emphasize the discretization of only the workspace, while allowing the low-level trajectory planner to apply local control algorithms specific to the vehicle model to find control inputs that enable the vehicle's desired motion.

The proposed framework involves a precise characterization of the interaction between the two levels of planning. Specifically, the high level planner solves a special path optimization problem on a graph – finding a path in the cell decomposition graph that minimizes costs defined over multiple edge transitions (histories) – and the low level trajectory planner determines the feasibility of traversal and the cost of traversal (if feasibility is ensured) of the sequence of cells corresponding to the given history of edge transitions. We provide an algorithm to solve exactly the aforementioned path optimization problem, along with a modification of the algorithm that executes faster, albeit at the expense of optimality of the resultant path. Using a four-state, two-input vehicle dynamical model, we compare the proposed motion planner against two RRT-based planners and provide numerical data demonstrating the superiority of the proposed planner in terms of the quality of the resultant trajectories.

Future work will deal with the implementation of the proposed framework using multi-resolution cell decompositions, and with the development of tile motion planning algorithms for more realistic vehicle dynamic models.

APPENDIX A

DEPENDENCE OF PATH OPTIMALITY ON H

We assume here that the proposed motion planner solves an H -cost shortest path problem, where the H -cost of an edge in \mathcal{G}_H is determined by the tile motion planning algorithm. We discuss the variation of the minimum H -cost with respect to H via the following results; we denote by \bar{P} the maximum number of nodes in any path in \mathcal{G} from i_S to i_G .

Lemma A.1: Let $\pi = (j_0, \dots, j_P)$ be an admissible path in \mathcal{G} . Then for each $H \in \mathbb{N}$,

$$\tilde{J}_{H+1}(\pi) \leq \tilde{J}_H(\pi), \quad (\text{A.1})$$

Proof: See [55]. ■

Proposition 3: Let \tilde{J}_H^* denote the minimum H -cost of paths in \mathcal{G} . Then $\{\tilde{J}_H^*\}_{H=1}^{\bar{P}}$ is a non-increasing sequence.

Proof: Let π be an admissible path in \mathcal{G} . By Lemma A.1,

$$\tilde{J}_{\bar{P}}(\pi) \leq \dots \leq \tilde{J}_1(\pi). \quad (\text{A.2})$$

For $H \in \{1, \dots, \bar{P}\}$, let π_H^* denote the H -cost shortest path in \mathcal{G} . Then for each admissible path π , $\tilde{J}_H^* = \tilde{J}_H(\pi_H^*) \leq \tilde{J}_H(\pi)$ by optimality. In particular, for $\pi = \pi_{H-1}^*$,

$$\begin{aligned} \tilde{J}_H^* = \tilde{J}_H(\pi_H^*) &\leq \tilde{J}_H(\pi_{H-1}^*) \\ &\leq \tilde{J}_{H-1}(\pi_{H-1}^*) = \tilde{J}_{H-1}^* \quad (\text{due to (A.2)}), \end{aligned}$$

and the result follows. ■

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