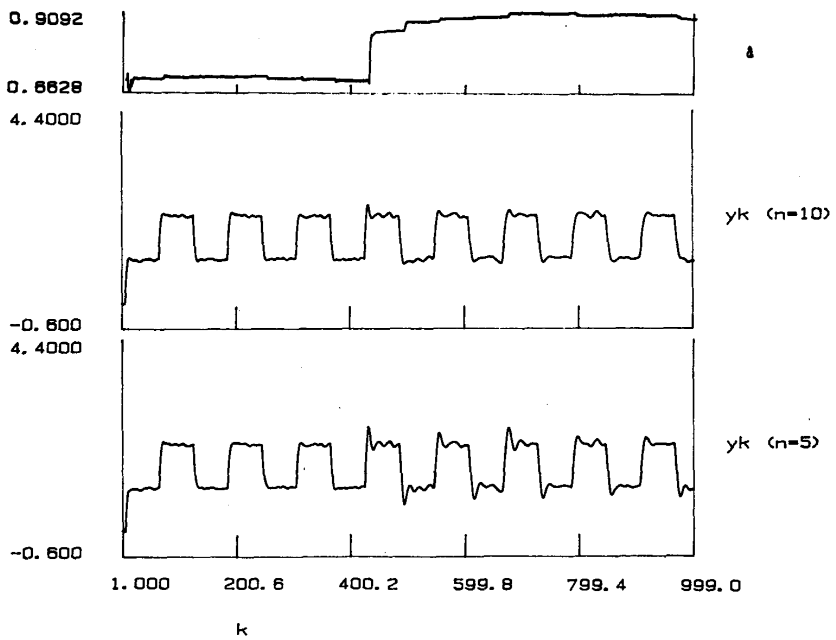

Fig. 8. System responses for different control horizons.

as they have few parameters to be estimated as the computer memory required increases rapidly with the number of estimated parameters.

The use of a predictor makes the controller coefficients, and the closed-loop poles and zeros, independent of the deadtime.

Acknowledgment

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References

[1] D. W. Clarke, C. Mohtadi, and P. S. Tuffs, "Generalized predictive control—Part I: The basic algorithm," *Automatica*, vol. 23, no. 2, pp. 137-148, 1987.
[2] ——, "Generalized predictive control—Part II: Extensions and interpretations," *Automatica*, vol. 23, no. 2, pp. 149-160, 1987.
[3] M. A. Lelic and M. B. Zarrop, "Generalized pole placement self tuning controller—Part 1: Basic algorithm," *Int. J. Contr.*, vol. 46, no. 2, pp. 547-568, 1987.
[4] M. A. Lelic and P. E. Wellstead, "Generalized pole placement self tuning controller—Part 2: Application to robot manipulator control," *Int. J. Contr.*, vol. 46, no. 2, pp. 569-601, 1987.
[5] A. Y. Allidina and F. M. Hughes, "Generalised self-tuning controller with pole assignment," *Proc. IEE*, vol. 127-D, pp. 13-18, 1980.
[6] P. E. Wellstead, D. Prager, and P. Zanker, "A pole assignment self-tuning regulator," *Proc. IEE*, vol. 126-8, pp. 781-787, 1978.
[7] F. R. Rubio, F. M. Hughes, and E. F. Camacho, "Self tuning PI control of a solar power plant," in *Proc. 3rd IFAC Symp. Adaptive Syst. Contr. Signal Processing*, 1989, pp. 335-340.
[8] R. Isermann, *Digital Control Systems*. New York: Springer-Verlag, 1981.
[9] C. A. Hearn and L. R. Seward, *REDUCE Users Guide*. Rand publication CP97, 1985.
[10] G. Bierman, *Factorization Methods for Discrete Estimation*. New York: Academic, 1977.

On the Linear Quadratic Minimum-Time Problem

E. I. Verriest and F. L. Lewis

***Abstract*—We investigate a nontraditional minimum-time problem that includes quadratic state and control weighting terms in the performance index. Using this formulation we are able to provide a convenient solution to the problem that uses the solution to the Riccati equation to compute the optimal feedback gain and the optimal time. In some cases the latter is simply found using the derivative of the Riccati equation solution.**

I. Introduction

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad x_0 \text{ given} \tag{1.1}$$

with $x(t) \in R^n$, $u(t) \in R^m$. It is well known that the minimum-time control problem of selecting the control $u(t)$ that minimizes the performance index (PI)

$$J = \int_0^T 1\, dt, \tag{1.2}$$

with the final time T free, is ill posed in the sense that the solution is to select $u(t)$ infinite. To make the problem well posed, it is traditional to impose constraints on the magnitude of $u(t)$, which results in the familiar bang-bang control law.

This bang-bang solution generally places the optimal control input

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at the boundary of a polyhedron. In fact, the minimum-time problem has no solution if the control magnitude constraints are strict inequalities of the form $|u| < u_{\max}$. The fast switching that occurs when the control changes facets on the polyhedron is undesirable in many applications. That is, bang-bang control is often impractical.

In this note, we shall take a different approach to minimum-time control that results in smooth control inputs by considering PI's of the form

$$J = x^T(T)S_T x(T) + \int_0^T (\rho + x^TQx + u^TRu)\, dt \quad (1.3)$$

with $S_T \geq 0$, $Q \geq 0$, $R > 0$, $\rho > 0$, and T free. We take S_T, Q, ρ, and R to be constant.

By selecting the magnitudes of Q and R, one may balance off the requirement for minimum time versus the requirement for keeping the state and inputs small over the interval $[0, T]$. Due to the fact that J contains both the final time T and quadratic components of $x(t)$ and $u(t)$, we shall call J a *linear quadratic minimum-time* (LQMT) PI.

Although this PI has been considered before (e.g., [1], [3]), it has not been fully exploited, nor has an analytic solution been given. As we shall see, it results in a convenient and practical design technique. Indeed, in the free terminal conditions case the minimum-time T may be simply computed using x_0 and the derivative of the Riccati solution. The same is true in the fixed terminal condition case if $x_0 = 0$; otherwise, another more general, but still not inconvenient test, must be used.

II. Fixed Terminal Condition Design

In this section, the objective is to find the open-loop control $u(t)$ that minimizes

$$J = \int_0^T (\rho + u^TRu)\, dt = \rho T + \int_0^T u^TRu\, dt \quad (2.1)$$

with T free, while driving (1.1) from the given initial state of $x(0) = x_0$ to a prescribed final state of $x(T) = x_f$.

The solution to this problem is provided in the next theorem.

Theorem 2.1: Let (A, B) be reachable and R nonsingular. Then the solution to the fixed terminal condition LQMT-problem exists. The optimal time is either zero or satisfies

$$d/dT\{d_T^T G_T^{-1} d_T\} = -\rho \quad (2.2)$$

or equivalently,

$$x_f^T A^T S d_T + d_T^T S B R^{-1} B^T S d_T + d_T^T S A x_f = \rho \quad (2.3)$$

where

$$d_T = x_f - e^{AT}x_0 \quad (2.4)$$

G_T is the Grammian

$$G_T = \int_0^T e^{At}BR^{-1}B^Te^{A^Tt}\, dt \quad (2.5)$$

and

$$S(t) = G_t^{-1}. \quad (2.6)$$

Moreover, the optimal control is given by

$$u(t)^* = R^{-1}B^Te^{A^T(T-t)}G_T^{-1}d_T, \qquad 0 \leq t \leq T \quad (2.7)$$

or in feedback form by

$$u(t)^* = R^{-1}B^TG_t^{-1}[d_t + x(t) - x_f], \qquad 0 \leq t \leq T. \quad (2.8)$$

■

Remark 1: The reachability condition is sufficient, but not necessary. This is clear if the system is not completely reachable but x_f is chosen in the reachable set.

Remark 2: If x_0 is zero, then $d_T = x_f$, and the local minima are found by

$$x_f^T[(d/dT)G_T^{-1}]x_f = -\rho \quad (2.9)$$

or, equivalently

$$x_f^T \frac{dS(T)}{dT} x_f = -\rho \quad (2.10)$$

which is easy to check. Indeed, it is well known that $S(t)$ satisfies the Riccati equation.

$$-\dot{S} = SA + A^TS + SBR^{-1}B^TS. \quad (2.11)$$

Thus, one can approximately find the minimum time T by integrating the Riccati equation forward in time from some large (theoretically infinite) initial value $S(0)$ and checking $\dot{S}$ at each time using (2.10).

Proof of the Theorem:

1) Consider first the problem of optimization of the quadratic part.

$$J_T = \int_0^T u^T(t)Ru(t)\, dt \quad (2.12)$$

for a fixed value of T. The derivation of the optimal solution relies on standard theory for the optimal control problem. (See, for example [2].) One finds that

$$u^*(t, T) = R^{-1}B^Te^{A^T(T-t)}G_T^{-1}d_T \quad (2.13)$$

with d_T and G_T, respectively, given by (2.4) and (2.5), transfers the initial state x_0 at $t = 0$ to the final state x_f at T. The associated minimal quadratic cost can be expressed by [4]

$$J_T^* = d_T^TG_T^{-1}d_T. \quad (2.14)$$

2) For T fixed, the total cost for the original performance index (2.1) is

$$J(t, \rho) = \rho T + J_T^* \quad (2.15)$$

and is minimal along all controls achieving $x(T) = x_f$. The role of the additional parameter ρ has been introduced as a relative weight between the (quadratic) control energy and the elapsed time. Hence, the optimal transit time T^* is found by minimizing the parametrized family $J(T, \rho)$ over $T \geq 0$.

3) To show that a solution exists for the optimal time T^*, we investigate the function $J(T, \rho)$. If $x_f = x_0$, the problem is trivial with solution $T^* = 0$ and $J^* = J(T^*, \rho) = J(0, \rho) = 0$. If $x_f \neq x_0$, the proof proceeds in several steps. First, note that by continuity of $d_t = x_f - e^{At}x_0$ and the fact that $d_0 = x_f - x_0 \neq 0$, there exists for all $\delta < \|d_0\|$ a T depending on δ, such that for all $t < T$, $\|d_t\| > \delta$. Next, the largest eigenvalue of G_t, which equals the norm of G_t as an operator from $\mathbf{R}^n$ to $\mathbf{R}^n$, satisfies

$$\lambda_{\max}(G_t) = \|G_t\| \leq \int_0^T \|e^{A\tau}BR^{-1}B^Te^{A^T\tau}\|\, d\tau$$

$$\leq \int_0^t \max_{\tau\in[0,t]} \|e^{A\tau}\|^2 \|BR^{-1}B^T\|\, d\tau.$$

Now, $\|e^{At}\|$ is continuous and for $t = 0$, $\|e^{At}\| = 1$. Hence, for all δ', there exists a T', depending on δ', such that for all $t < T'$ we have $\|e^{At}\| < 1 + \delta'$. It follows that

$$\lambda_{\max}(G_t) \leq (1 + \delta')^2 \|BR^{-1}B^T\| t \qquad \text{for all } t \text{ in } [0, T'].$$

Since

$$d_\epsilon^T G_\epsilon^{-1} d_\epsilon \geq \|d_\epsilon\|^2 \lambda_{\min}(G_\epsilon^{-1})$$
$$\geq \|d_\epsilon\|^2 \lambda_{\max}^{-1}(G_\epsilon)$$

we get from the previously obtained bounds that

$$d_\epsilon^T G_\epsilon^{-1} d_\epsilon \geq \delta^2 / \left[(1+\delta')^2 \|BR^{-1}B^T| \epsilon\right]$$

for all $\epsilon < \min(T, T')$. Clearly then J_ϵ^* and therefore also $J(\epsilon, \rho)$ diverge to infinity if ϵ decreases to zero.

Since $J(T, \rho)$ is lower bounded by ρT, once a value $J(T_o, \rho)$ for arbitrary T_o is found, the minimum must occur in the interval $[0, T_1]$, where $T_1 = J(T_o, \rho)/\rho$. Similarly, for $\epsilon < \min(T, T') := \epsilon_o$

$$J(\epsilon, \rho) \geq \epsilon\rho + K_o/\epsilon \qquad \text{for some } K_o > 0$$

and $J(t, \rho)$ is monotonically decreasing in $(o, \epsilon']$, where

$$\epsilon' := \min\left(\epsilon_o, \epsilon_1 := (\rho/K_o)^{1/2}\right).$$

By Weierstrass' theorem, the closed interval $[\epsilon', T_1]$ contains the absolute minimum of $J(T, \rho)$.

Three possibilities remain.

i) The optimum occurs inside $[\epsilon', T_1]$. Since J is differentiable, it is then necessary that

$$(d/dT)\left(d_T^T G_T^{-1} d_T\right) = -\rho.$$

ii) The optimum occurs at the boundary T_1. Since $J(T_1, \rho) \geq \rho T_1 = J(T_o, \rho)$, it must then also occur at $T_o < T_1$.

iii) The optimum occurs at ϵ'. But since it was derived that $J(T, \rho)$ strictly decreases for $0 < T < \epsilon'$, invoking the differentiability, one must again conclude that

$$(d/dT)\left(d_T^T G_T^{-1} d_T\right) = -\rho \qquad \text{for } T = \epsilon'.$$

Hence, one concludes that the optimum occurs either at $T = 0$, or a point where the derivative of $(d_T^T G_T^{-1} d_T)$ crosses the level $-\rho$.

Note that several local minima may occur.

4) Finally, noting that with the control $u^*(t, T)$ the states are given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}BR^{-1}B^T e^{A^T(T-\tau)} d\tau G_T^{-1} d_T$$
$$= e^{At}x_0 + \int_0^t e^{A(t-\tau)}BR^{-1}B^T e^{A^T(t-\tau)} d\tau e^{A^T(T-t)} G_T^{-1} d_T$$
$$= e^{At}x_0 + G_t e^{A^T(T-t)} G_T^{-1} d_T$$

from which

$$G_T^{-1} d_T = e^{-A^T(T-t)} G_t^{-1}\left[x(t) - e^{At}x_0\right]$$

substituting in (2.7)

$$u(t) = R^{-1}B^T G_T^{-1}\left[d_t + x(t) - x_f\right]$$

which gives the feedback form (2.8).

Since $S(t)$ satisfies the Riccati equation (2.11) and

$$(d/dt)d_t = Ad_t - Ax_f,$$

there follows easily the equivalent condition (2.3). ■

The LQMT solution can also be obtained directly from the transversality condition for the time optimal control. The indirect route was preferred since it led to the existence of the optimal solution as well.

The proof depends critically in part 3) on the fact that $\rho > 0$. In the limiting case $\rho = 0$, we retrieve the familiar LQR with free final time and fixed final state. It is well known that the optimal solution for this problem may require an infinite time T. The following example shows this.

Example: Let $\dot{x} = ax + u$, with PI

$$J_T = \int_0^T u^2\, dt$$

and $0 \neq x_0$ given and constraint $x_f = 0$. From (2.14), the optimal solution for a fixed choice of T is

$$J_T^* = \frac{2ax_0^2}{1 - e^{-2aT}}.$$

In this case, J_T^* converges monotonically to $2ax_0^2$ when $a > 0$ and to 0 when $a < 0$. In both cases, an optimum is obtained for infinite T.

III. Free Terminal Condition Design

In this section, the objective is to find the feedback gain in

$$u(t) = -K(t)x(t) \tag{3.1}$$

that minimizes

$$J = x^T(T)S_f x(T) + \int_0^T \left(\rho + x^TQx + u^TRu\right) dt \tag{3.2}$$

with $S_f \geq 0$, $Q \geq 0$, $R > 0$, T free, $x(0) = x_0$ given, and no constraint on the final state $x(T)$. Design parameter ρ trades off the weighting on control energy and control time. The solution to the free final state LQMT problem is provided in the next theorem.

Theorem 3.1: The solution to the free final condition LQMT problem exists. The optimal final time T is either zero, or satisfies the condition

$$x_0^T\left[(d/dt)S(t, T)\right]_{t=0} x_0 = \rho \tag{3.3}$$

where $S(t, T)$ is the solution to the Riccati equation

$$-(d/dt)S(t, T) = A^TS(t, T) + S(t, T)A - S(t, T)BR^{-1}B^TS(t, T) + Q,$$
$$\text{for } t \leq T,\ S(T, T) = S_f. \tag{3.4}$$

Moreover, the optimal feedback gain is given by

$$K(t) = R^{-1}B^TS(t, T). \tag{3.5}$$

■

Remark: The straightforward solution procedure suggested by this theorem is to integrate the Riccati equation backwards from some time τ using as the final condition $S(\tau, \tau) = S_f$. At each time t, the left-hand side of (3.3) is computed using the known initial state x_0 and the derivative of $S(t, \tau)$ with respect to t. Then the optimal final time T is equal to $(\tau - t)$ where t is the time for which (3.3) first holds. Note that equality may hold for several values of t, as only necessary conditions are expressed.

Proof of the Theorem:

1) Minimize first for a fixed time T the quadratic PI

$$x^T(T)S_f x(T) + \int_0^T \left(x^TQx + u^TRu\right) dt.$$

The solution (3.5) is standard. Along the optimal path x^*

$$J_T^* = x^T(T)^* S_f x(T)^* + \int_0^T x^{*T}\left(Q + K^TRK\right)x^*\, dt.$$

Noting that along the optimal path, invoking the Riccati equation

$$\begin{aligned}\int_0^T x^T(Q+KRK^T)x\,dt &= \int_0^T x^T[-(d/dt)S-(A-BK)^TS\\&\quad -S(A-BK)]\,x\,dt\\&= -\int_0^T (d/dt)[x^TS(t,T)x]\,dt\\&= -x_T^TS_fx_T + x_0^TS(0,T)x_0\end{aligned}$$

hence,

$$J_T^* = x_0^TS(0,T)x_0$$

where $S(t, T)$ satisfies (3.4).

2) Next, the minimization over T of the following continuous function follows

$$J(T,\rho) = \rho T + J_T^* = \rho T + x_0^TS(0,T)x_0$$

since obviously $J(T,\rho) \geq \rho T$, by Weierstrass' theorem, an absolute minimum of $J(T,\rho)$ exists in $[0, T_0(x_0)]$, where $T_0(x_0) = x_0^TS_fx_0/\rho$, where the fact that

$$\lim_{\epsilon\to\infty} J^*(\epsilon) = \lim\{\rho\epsilon + x_0^TS(0,\epsilon)x_0\} = x_0^TS_fx_0$$

has been used. Hence, for any infinite x_0, a minimum-time control exists with $T \leq T_0(x_0) = x_0^TS_fx_0$. A necessary condition is either that the minimum occurs at the boundary $T = 0$ (if $x_0^T(d/dT)S(t,T|_{t=0}x_0 + \rho > 0)$, or the boundary $T = T_0$ (if $x_0^T(d/dT)S(t,T)|_{t=0}x_0 + \rho < 0$), or at the point where $x_0^T(d/dT)S(t,T)|_{t=0}x_0 + \rho = 0$.

Since $(d/dt)S(t,T) = -(d/dT)S(t,T)$ one obtains from the Riccati equation at once that a necessary condition for the optimal time is either that $T = 0$ or (3.3). ■

Using the Riccati equation (3.4), an alternative form of (3.3) is given by

$$\rho + x_0^T[A^TS + SA + Q - SBR^{-1}B^TS]_{t=0}x_0 = 0 \quad (3.6)$$

IV. Examples

A. Fixed Terminal Condition Design for Newton's System

For a particle obeying Newton's laws

$$\dot{x} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u \quad (4.1)$$

let us determine the open-loop control that minimizes the PI

$$J = \frac{1}{2}\int_0^T (1+ru^2)\,dt \quad (4.2)$$

while driving $x(t)$ from $x(0) = 0$ to $x(T) = [d \;\; v]^T$, with d, v, respectively, the desired final position and velocity. The final time T is free.

It is well known [1] that the reachability Grammian is

$$G(t) = \begin{bmatrix}t^3/3r & t^2/2r\\t^2/2r & t/r\end{bmatrix} \quad (4.3)$$

whence the solution to Riccati equation (2.11) is

$$S(t) = G^{-1}(t) = \begin{bmatrix}12r/t^3 & -6r/t^2\\-6r/t^2 & 4r/t\end{bmatrix}. \quad (4.4)$$

Therefore

$$\dot{S}(t) = \begin{bmatrix}-36r/t^4 & 12r/t^3\\12r/t^3 & -4r/t^2\end{bmatrix} \quad (4.5)$$

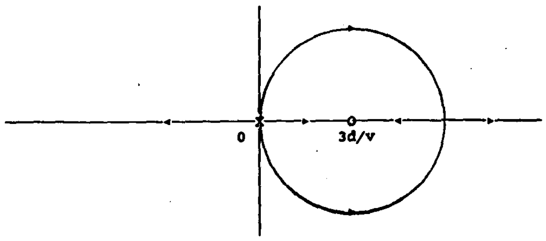

Fig. 1. Root locus versus r for minimum-time polynomial.

and condition (2.2), (2.10) for the final time T becomes

$$\begin{aligned}-1 &= x_T^t\dot{S}(T)x_T,\\&= \frac{-36rd^2}{T^4} + \frac{24rdv}{T^3} - \frac{4rv^2}{T^2},\end{aligned}$$

or

$$T^4 - 4rv^2T^2 + 24rdvT - 36rd^2 = 0. \quad (4.6)$$

This is a polynomial equation of the conventional sort for the final time T. To examine it further, write it in the form

$$1 - 4rv^2\frac{(T-3d/v)^2}{Y^4} = 0. \quad (4.7)$$

It is now easy to sketch a root locus versus r. This appears in Fig. 1 for the case $d/v > 0$. Since there is always a positive-real root, the optimal final time T exists for all d and v. Note that, for larger values of r there may be more than one positive-real root to (4.6); the optimal time T is the smaller root. The root locus for $d/v < 0$ is the reflection of Fig. 1 in the imaginary axis.

B. Free Terminal Condition Design for a Scalar System

With the scalar system

$$\dot{x} = ax + bu \quad (4.8)$$

$x(0) = x_0$, let us associate the PI

$$J = S_fx^2(T) + \int_0^T (\rho + qx^2 + ru^2)\,dt. \quad (4.9)$$

It is desired to minimize J given x_0 with no restrictions on $x(T)$.

The Riccati equation is

$$-s = 2as - \frac{b^2s^2}{r} + q, \qquad s(T) = s_f. \quad (4.10)$$

The steady-state solution is

$$s_\infty = \frac{q}{\gamma}\left(1+\sqrt{1+\gamma/a}\right) \quad (4.11)$$

with

$$\gamma = \frac{b^2q}{ar} \quad (4.12)$$

a "control-effectiveness-to-plant-inertia ratio." As long as $b \neq 0$, $q \neq 0$, s_∞ exists and is nonzero.

If $x_0 = 0$ the minimum time is $T = 0$. Otherwise, the LQMT condition (3.3) for the minimum time T is

$$\dot{s}(0)x_0^2 = \rho. \quad (4.13)$$

At steady state, we have

$$\dot{s}_\infty x_0^2 = 0. \quad (4.14)$$

Since $s(t)$ is continuous, a necessary and sufficient condition for

(4.13) to hold for some $t \leq T$ is

$$\dot{s}_f x_0^2 \geq \rho. \tag{4.15}$$

According to (4.10), this is equivalent to

$$\left[\frac{b^2 s_f^2}{r} - 2as_f - q\right] x_0^2 \geq \rho$$

or

$$g(s_f) \equiv \frac{b^2 s_f^2}{r} - 2as_f - \left(q + \frac{\rho}{x_0^2}\right) \geq 0. \tag{4.16}$$

The largest root of $g(s_f) = 0$ is given by

$$\sigma = \frac{1}{\delta}\left[1 + \sqrt{1 + \frac{\delta}{a}\left(q + \frac{\rho}{x_0^2}\right)}\right] \tag{4.17}$$

with

$$\delta = b^2/ar. \tag{4.18}$$

Thus, for (4.15) to hold it is necessary and sufficient that

$$S_f \geq \frac{1}{\delta}\left[1 + \sqrt{1 + \frac{\delta}{a}\left(q + \frac{\rho}{x_0^2}\right)}\right]. \tag{4.19}$$

If this condition holds, the minimum-time T is greater than zero. Otherwise, it is not worthwhile to move the state from x_0 and T is equal to 0. That is, the minimum-time T is nonzero if $x_0 \neq 0$ and we weight the final state x_T sufficiently in the PI.

Some insight may be gained by noting the following points in connection with (4.19).

1) Parameters q, ρ, and $1/x_0^2$ have a similar effect. In fact, we could define

$$\kappa = q + \frac{\rho}{x_0^2}. \tag{4.20}$$

Then as κ increases, a larger final state weighting s_f is required for $T > 0$. As κ decreases, a smaller s_f suffices for $T > 0$.

2) As r increases, δ decreases and the influence of parameter κ wanes. Moreover, a larger s_f is required for $T > 0$. As the control weight r decreases, the influence of κ increases, however, a smaller final weighting s_f is sufficient to make $T > 0$. ■

V. Conclusion

We have defined and studied the linear quadratic minimum-time (LQMT) control problem. As in the conventional LQR problem, the control input is given in terms of a Riccati equation solution $S(t)$. We have given an extra condition that yields the minimum-time T. In free final state design, as well as in fixed final state design with $x(0) = 0$, the minimum-time depends on $\dot{S}(t)$, and is easy to check.

References

[1] M. Athans and P. L. Falb, *Optimal Control.* New York: McGraw-Hill, 1966.
[2] F. L. Lewis, *Optimal Control.* New York: Wiley, 1986.
[3] H. M. Schwartz and R. Gourdeau, "Optimal control of a robot manipulator using a weighted time-energy cost function," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1989, pp. 1628–1631.
[4] E. I. Verriest, "Reduced order LQG design: Conditions for feasibility," in *Proc. 25th IEEE Conf. Decision Contr.*, Athens, Greece, Dec. 1986, pp. 1765–1769.

On Perturbation Analysis of Queueing Networks with Finitely Supported Service Time Distributions

Y. Wardi, M. W. McKinnon, and R. Schuckle

Abstract—**Infinitesimal perturbation analysis (IPA) has emerged as an efficient tool for estimating the gradient of a function defined on the steady state of a queueing network. Differentiability of such functions is often assumed due to difficulties in proving it. In this note, we point out that such functions may be nondifferentiable at an infinite set of points, dense in a given interval, for a large class of realistic system models. Such a phenomenon has not been suspected in the literature on perturbation analysis, it is largely due to correlation of traffic patterns on the links of a network, and to the presence of atoms in the distributions of their service times. This issue of nondifferentiability goes beyond IPA, to any method for estimating the gradients of functions in steady state.**

I. Introduction

Infinitesimal perturbation analysis (IPA) has emerged as an efficient tool for estimating the derivative of a function defined on the steady state of a queueing system, with respect to a Euclidean variable (see [1], [2] and the references therein). Successful practical applications of IPA were reported on in [3] and [4], for communications and manufacturing systems, respectively, and also in many references in [1], [2]. Consider such a system, parameterized by x, and a simulation run of it. The simulation output includes a sequence of scalar observables $y_n(x)$, $n = 1, 2, \cdots$. This sequence can be thought of as a family of stochastic processes, parameterized by x, and defined on a common probability space (Ω, F, p). For a given x, an $\omega \in \Omega$ determines a realization of $y_n(x)$, $n = 1, 2, \cdots$, also to be denoted by $y_n(x)$. If for every x the process is ergodic, then a.s. $\sum_{n=1}^{N} y_n(x)/N \to Y(x)$, for a function $Y(x)$. For a given $\omega \in \Omega$, the functions $y_n(x)$ $n = 1, 2, \cdots$, are called the sample performance functions, and $Y(x)$ is called the steady-state function. For the purpose of our discussion, let x be a scalar, and let "prime" denote derivative with respect to x. Given an x, the objective of IPA is to estimate $Y'(x)$. An IPA algorithm is typically applied under the assumptions that $Y'(x)$ exists, and that for every $n = 1, 2, \cdots$, a.s. $y_n'(x)$ exists. In this case, it computes realizations of $\sum_{n=1}^{N} y_n'(x)/N$, for a given N over a single simulation-run. The latter term can be used to approximate $Y'(x)$ if a.s.

$$\sum_{n=1}^{N} y_n'(x)/N \to Y'(x). \tag{1}$$

If so, IPA is said to be strongly consistent [5]. Equation (1) was proved for a number of systems: single queues [6]–[10] and closed Jackson networks [11], often for the delay or throughput as a function of parameters of the service time distributions. It was generally noticed that for given n and $\omega \in \Omega$ the function $x \to y_n(x)$ was piecewise differentiable in a given interval, but the nondifferentiabilities disappeared in steady state, and $Y(\cdot)$ was differentiable in the entire interval.

Many realistic system-models consist of non-Markovian networks with correlated traffic patterns and correlated service times. For such systems, proofs of (1) are generally unavailable, and obtaining

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