Sharp Maximal Function Estimates and Boundedness for Commutator Related to Generalized Fractional Integral Operator

Guo Sheng, Huang Chuangxia, and Liu Lanzhe

Abstract.
In this paper, we prove some sharp maximal function estimates for the commutators related to certain generalized fractional integral operator with general kernel and the BMO and Lipschitz functions. As an application, we obtain the boundedness of the commutators on Lebesgue, Morrey and Triebel-Lizorkin spaces. The operator includes fractional Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

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1 Introduction

As the development of singular integral operators(see [9][24][25]), their commutators have been well studied(see [6][22][23]). In [8][22][23], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the

fractional integral operators. In [3][10][19], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained(see [1][11]). In this paper, certain generalized fractional integral operators with general kernel are introduced, and the sharp maximal function estimates for the commutator related to the operator and the BMO and Lipschitz functions are obtained, as an application, we obtain the boundedness of the commutator on Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes fractional Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

2 Preliminaries

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [9][24])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||M^{\#}(f)||_{L^\infty}$. It has been known that (see [20])

$$||f - f_{2^k Q}||_{BMO} \leq Ck||f||_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.
For $0 < \eta < n$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

The A_p weight is defined by (see [9])

$$A_p = \left\{ w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\},$$

for $1 < p < \infty$ and

$$A_1 = \{ w \in L^p_{loc}(R^n) : M(w)(x) \leq Cw(x), a.e. \}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}^{\beta,\infty}_p(R^n)$ be the homogeneous Triebel-Lizorkin space(see [19]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

In this paper, we will study some integral operators as following(see [1]).

Definition 1. Fixed $0 \leq \delta < n$. Let $F^{\delta,t}(x,y)$ be the function defined on $R^n \times R^n \times [0, +\infty)$ and b be a locally integrable function on R^n, set

$$F^{\delta,t}(f)(x) = \int_{R^n} F^{\delta,t}(x,y)f(y)dy$$

and

$$F^{\delta,t}_b(f)(x) = \int_{R^n} (b(x) - b(y))F^{\delta,t}(x,y)f(y)dy$$

for every bounded and compactly supported function f.

Let H be the Banach space $H = \{h : ||h|| < \infty\}$. For each fixed $x \in R^n$, we view $F^{\delta,t}(f)(x)$ and $F^{\delta,t}_b(f)(x)$ as the mappings from $[0, +\infty)$ to H. Set

$$T^\delta(f)(x) = ||F^{\delta,t}(f)(x)||,$$

which T_δ is bounded on $L^2(R^n)$. The commutator related to $F^{\delta,t}$ is defined by

$$T^\delta_b(f)(x) = ||F^{\delta,t}_b(f)(x)||$$

and $F^{\delta,t}$ satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} (||F^{\delta,t}(x,y) - F^{\delta,t}(x,z)|| + ||F^{\delta,t}(y,x) - F^{\delta,t}(z,x)||)dx \leq C,$$

and

$$\left(\int_{2^k|z-y|\leq|x-y|<2^{k+1}|z-y|} (||F^{\delta,t}(x,y) - F^{\delta,t}(x,z)|| + ||F^{\delta,t}(y,x) - F^{\delta,t}(z,x)||)^q dy \right)^{1/q}$$

$$\leq C_k(2^k|z-y|)^{-n/q'+\delta},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Definition 2. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \ \text{for} \ t \geq 0.$$

Let f be a locally integrable function on R^n. Set, for $1 \leq p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x \in R^n, \ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x - y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [20][21]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n)$, which is the Lebesgue spaces.

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [4][7][8][12][17]).

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [22][23]). In [23], Pérez and Trujillo-Gonzalez prove a sharp estimate for the commutator. The main purpose of this paper is to prove the sharp maximal inequalities for the commutator T_b^δ. As the application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the commutator.

3   Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. Let T^δ be the integral operator as **Definition 1**, the sequence $\{C_k\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in Lip_\beta(R^n)$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#(T_b^\delta(f))(\tilde{x}) \leq C||b||_{Lip_\beta} \left(M_{\beta+\delta,s}(f)(\tilde{x}) + M_{\beta,s}(T^\delta(f))(\tilde{x}) \right).$$

Theorem 2. Let T^δ be the integral operator as **Definition 1**, the sequence $\{2^{k\beta}C_k\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in Lip_\beta(R^n)$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T_b^\delta(f)(x) - c \right| dx \leq C\|b\|_{Lip_\beta} \left(M_{\delta,s}(f)(\tilde{x}) + M_s(T^\delta(f))(\tilde{x}) \right).$$

Theorem 3. Let T^δ be the integral operator as **Definition 1**, the sequence $\{kC_k\} \in l^1$, $q' \leq s < \infty$ and $b \in BMO(R^n)$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#(T_b^\delta(f))(\tilde{x}) \leq C\|b\|_{BMO} \left(M_{\delta,s}(f)(\tilde{x}) + M_s(T^\delta(f))(\tilde{x}) \right).$$

Theorem 4. Let T^δ be the integral operator as **Definition 1**, the sequence $\{C_k\} \in l^1$, $0 < \beta < \min(1, n - \delta)$, $q' < p < n/(\beta + \delta)$, $1/r = 1/p - (\beta + \delta)/n$ and $b \in Lip_\beta(R^n)$. Then T_b^δ is bounded from $L^p(R^n)$ to $L^r(R^n)$.

Theorem 5. Let T^δ be the integral operator as **Definition 1**, the sequence $\{C_k\} \in l^1$, $0 < D < 2^n$, $0 < \beta < \min(1, n - \delta)$, $q' < p < n/(\beta + \delta)$, $1/r = 1/p - (\beta + \delta)/n$ and $b \in Lip_\beta(R^n)$. Then T_b^δ is bounded from $L^{p,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.

Theorem 6. Let T^δ be the integral operator as **Definition 1**, the sequence $\{2^{k\beta}C_k\} \in l^1$, $0 < \beta < 1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in Lip_\beta(R^n)$. Then T_b^δ is bounded from $L^p(R^n)$ to $\dot{F}_r^{\beta,\infty}(R^n)$.

Theorem 7. Let T^δ be the integral operator as **Definition 1**, the sequence $\{kC_k\} \in l^1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in BMO(R^n)$. Then T_b^δ is bounded from $L^p(R^n)$ to $L^r(R^n)$.

Theorem 8. Let T^δ be the integral operator as **Definition 1**, the sequence $\{kC_k\} \in l^1$, $0 < D < 2^n$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in BMO(R^n)$. Then T_b^δ is bounded from $L^{p,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.

To prove the theorems, we need the following lemma.

Lemma 1.(see [1]) Let T^δ be the integral operator as **Definition 1**. Then T^δ is bounded from $L^p(R^n)$ to $L^r(R^n)$ for $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$.

Lemma 2.(see [19]). For $0 < \beta < 1$ and $1 < p < \infty$, we have

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \approx$$

$$\left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}.$$

Lemma 3.(see [9]). Let $0 < p < \infty$ and $w \in \cup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M(f)(x)^p w(x) dx \leq C \int_{R^n} M^\#(f)(x)^p w(x) dx.$$

Lemma 4.(see [2]) Suppose that $0 < \eta < n$, $1 < s < p < n/\eta$ and $1/q = 1/p - \eta/n$. Then
$$||M_{\eta,s}(f)||_{L^q} \leq C||f||_{L^p}.$$

Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,

$$||M(f)||_{L^{p,\varphi}} \leq C||M^{\#}(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [5]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx$$

$$\leq \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx$$

$$\leq C \int_{R^n} M^{\#}(f)(x)|^p M(\chi_Q)(x) dx$$

$$= C \left(\int_Q M^{\#}(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p M(\chi_Q)(x) dx \right)$$

$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \right)$$

$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p 2^{-kn} dy \right)$$

$$\leq C||M^{\#}(f)||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d)$$

$$\leq C||M^{\#}(f)||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d)$$

$$\leq C||M^{\#}(f)||_{L^{p,\varphi}}^p \varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)} \int_Q M(f)(x)^p dx \right)^{1/p} \leq C \left(\frac{1}{\varphi(d)} \int_Q M^{\#}(f)(x)^p dx \right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leq C||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof.

Lemma 6. Let T^δ be the integral operator as **Definition 1**, $0 < D < 2^n$, $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$. Then

$$||T^\delta(f)||_{L^{r,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

Lemma 7. Let $0 < D < 2^n$, $1 \le s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then

$$||M_{\eta,s}(f)||_{L^{r,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

The proofs of two Lemmas are similar to that of Lemma 5 by Lemma 1 and 4, we omit the details.

4   Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|} \int_Q \left| T_b^\delta(f)(x) - C_0 \right| dx \le C||b||_{Lip_\beta} \left(M_{\beta+\delta,s}(f)(\tilde{x}) + M_{\beta,s}(T^\delta(f))(\tilde{x}) \right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_b^{\delta,t}(f)(x) = (b(x) - b_{2Q})F^{\delta,t}(f)(x) - F^{\delta,t}((b - b_{2Q})f_1)(x) - F^{\delta,t}((b - b_{2Q})f_2)(x).$$

Then

$$\frac{1}{|Q|} \int_Q ||F_b^{\delta,t}(f)(x) - F^{\delta,t}((b_{2Q} - b)f_2)(x_0)||dx$$

$$\le \frac{1}{|Q|} \int_Q ||(b(x) - b_{2Q})F^{\delta,t}(f)(x)||dx + \frac{1}{|Q|} \int_Q ||F^{\delta,t}((b - b_{2Q})f_1)(x)||dx$$

$$+ \frac{1}{|Q|} \int_Q ||F^{\delta,t}((b - b_{2Q})f_2)(x) - F^{\delta,t}((b - b_{2Q})f_2)(x_0)||dx$$

$$= I_1 + I_2 + I_3.$$

For I_1, by Hölder's inequality, we obtain

$$I_1 \le \frac{1}{|Q|} \sup_{x \in 2Q} |b(x) - b_{2Q}| \int_Q |T^\delta(f)(x)|dx$$

$$\le \frac{1}{|Q|} \sup_{x \in 2Q} |b(x) - b_{2Q}||Q|^{1-1/s} \left(\int_Q |T^\delta(f)(x)|^s dx \right)^{1/s}$$

$$\le C||b||_{Lip_\beta}|Q|^{-1/s}|2Q|^{\beta/n}|Q|^{1/s-\beta/n} \left(\frac{1}{|Q|^{1-s\beta/n}} \int_Q |T^\delta(f)(x)|^s dx \right)^{1/s}$$

$$\le C||b||_{Lip_\beta} M_{\beta,s}(T^\delta(f))(\tilde{x}).$$

For I_2, choose $1 < r < \infty$ such that $1/r = 1/s - \delta/n$, by (L^s, L^r)-boundedness of T^δ (see Lemma 1), we get

$$
\begin{aligned}
I_2 &\leq \left(\frac{1}{|Q|} \int_{R^n} |T^\delta((b - b_{2Q})f_1)(x)|^r dx \right)^{1/r} \\
&\leq C|Q|^{-1/r} \left(\int_{R^n} |(b(x) - b_{2Q})f_1(x)|^s dx \right)^{1/s} \\
&\leq C|Q|^{-1/r} ||b||_{Lip_\beta} |2Q|^{\beta/n} |2Q|^{1/s - (\beta+\delta)/n} \left(\frac{1}{|2Q|^{1-s(\beta+\delta)/n}} \int_{2Q} |f(x)|^s dx \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} M_{\beta+\delta,s}(f)(\tilde{x}).
\end{aligned}
$$

For I_3, recalling that $s > q'$, we have

$$
\begin{aligned}
I_3 &\leq \frac{1}{|Q|} \int_Q \int_{(2Q)^c} |b(y) - b_{2Q}||f(y)|||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)||dydx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^\infty \int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)|| \\
&\qquad\qquad\qquad\qquad\qquad \times |b(y) - b_{2^{k+1}Q}||f(y)|dydx \\
&\quad + \frac{1}{|Q|} \int_Q \sum_{k=1}^\infty \int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)|| \\
&\qquad\qquad\qquad\qquad\qquad \times |b_{2^{k+1}Q} - b_{2Q}||f(y)|dydx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C}{|Q|} \int_Q \sum_{k=1}^\infty \left(\int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)||^q dy \right)^{1/q} \\
&\quad \times \sup_{y \in 2^{k+1}Q} |b(y) - b_{2^{k+1}Q}| \left(\int_{2^{k+1}Q} |f(y)|^{q'} dy \right)^{1/q'} dx \\
&\quad + \frac{C}{|Q|} \int_Q \sum_{k=1}^\infty |b_{2^{k+1}Q} - b_{2Q}| \left(\int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)||^q dy \right)^{1/q} \\
&\quad \times \left(\int_{2^{k+1}Q} |f(y)|^{q'} dy \right)^{1/q'} dx
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & C\sum_{k=1}^{\infty} C_k(2^k d)^{-n/q'+\delta}|2^{k+1}Q|^{\beta/n}||b||_{Lip_\beta}|2^{k+1}Q|^{1/q'-1/s}|2^{k+1}Q|^{1/s-(\beta+\delta)/n} \\
& \times\left(\frac{1}{|2^{k+1}Q|^{1-s(\beta+\delta)/n}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} \\
& +C\sum_{k=1}^{\infty}||b||_{Lip_\beta}|2^k Q|^{\beta/n}C_k(2^k d)^{-n/q'\delta}|2^{k+1}Q|^{1/q'-1/s}|2^{k+1}Q|^{1/s-(\beta+\delta)/n} \\
& \times\left(\frac{1}{|2^{k+1}Q|^{1-s(\beta+\delta)/n}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} \\
\leq\ & C||b||_{Lip_\beta}M_{\beta+\delta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}C_k \\
\leq\ & C||b||_{Lip_\beta}M_{\beta+\delta,s}(f)(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$
\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T_b^\delta(f)(x)-C_0\right|dx\leq C||b||_{Lip_\beta}\left(M_{\delta,s}(f)(\tilde{x})+M_s(T^\delta(f))(\tilde{x})\right).
$$

Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Write, for $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$,

$$
F_b^{\delta,t}(f)(x)=(b(x)-b_{2Q})F^{\delta,t}(f)(x)-F^{\delta,t}((b-b_{2Q})f_1)(x)-F_b^{\delta,t}((b-b_{2Q})f_2)(x).
$$

Then

$$
\begin{aligned}
& \frac{1}{|Q|^{1+\beta/n}}\int_Q||F_b^{\delta,t}(f)(x)-F^{\delta,t}((b_{2Q}-b)f_2)(x_0)||dx \\
\leq\ & \frac{1}{|Q|^{1+\beta/n}}\int_Q||(b(x)-b_{2Q})F^{\delta,t}(f)(x)||dx \\
& +\frac{1}{|Q|^{1+\beta/n}}\int_Q||F^{\delta,t}((b-b_{2Q})f_1)(x)||dx \\
& +\frac{1}{|Q|^{1+\beta/n}}\int_Q||F^{\delta,t}((b-b_{2Q})f_2)(x)-F^{\delta,t}((b-b_{2Q})f_2)(x_0)||dx \\
=\ & II_1+II_2+II_3.
\end{aligned}
$$

By using the same argument as in the proof of Theorem 1, we get, for

$1 < r < \infty$ with $1/r = 1/s - \delta/n$,

$$
\begin{aligned}
II_1 &\leq \frac{C}{|Q|^{1+\beta/n}} \sup_{x \in 2Q} |b(x) - b_{2Q}| |Q|^{1-1/s} \left(\int_Q |T^\delta(f)(x)|^s dx \right)^{1/s} \\
&\leq C|Q|^{-1-\beta/n} ||b||_{Lip_\beta} |2Q|^{\beta/n} |Q|^{1-1/s} |Q|^{1/s} \left(\frac{1}{|Q|} \int_Q |T^\delta(f)(x)|^s dx \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} M_s(T^\delta(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
II_2 &\leq \frac{1}{|Q|^{1+\beta/n}} |Q|^{1-1/r} \left(\int_{R^n} |T^\delta((b-b_{2Q})f_1)(x)|^r dx \right)^{1/r} \\
&\leq \frac{C}{|Q|^{1+\beta/n}} |Q|^{1-1/r} \left(\int_{R^n} |(b(x) - b_{2Q})f_1(x)|^s dx \right)^{1/s} \\
&\leq \frac{C}{|Q|^{1+\beta/n}} |Q|^{1-1/r} ||b||_{Lip_\beta} |2Q|^{\beta/n} |2Q|^{1/s-\delta/n} \left(\frac{1}{|2Q|^{1-s\delta/n}} \int_{2Q} |f(x)|^s dx \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} M_{\delta,s}(f)(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
II_3 &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \int_{(2Q)^c} |b(y) - b_{2Q}| |f(y)| ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)|| dy dx \\
&\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^\infty \int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)|| \\
&\qquad\qquad\qquad\qquad \times |b(y) - b_{2^{k+1}Q}| |f(y)| dy dx \\
&\quad + \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^\infty \int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)|| \\
&\qquad\qquad\qquad\qquad \times |b_{2^{k+1}Q} - b_{2Q}| |f(y)| dy dx \\
&\leq \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^\infty \left(\int_{2^k d \leq |y-x_0| < 2^{k+1}d} ||F^{\delta,t}(x,y) - F^{\delta,t}(x_0,y)||^q dy \right)^{1/q} \\
&\qquad \times \sup_{y \in 2^{k+1}Q} |b(y) - b_{2^{k+1}Q}| \left(\int_{2^{k+1}Q} |f(y)|^{q'} dy \right)^{1/q'} dx
\end{aligned}
$$

$$+\frac{C}{|Q|^{1+\beta/n}}\int_Q\sum_{k=1}^\infty|b_{2^{k+1}Q}-b_{2Q}|$$

$$\times\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}||F^{\delta,t}(x,y)-F^{\delta,t}(x_0,y)||^qdy\right)^{1/q}$$

$$\times\left(\int_{2^{k+1}Q}|f(y)|^{q'}dy\right)^{1/q'}dx$$

$$\le\ C|Q|^{-\beta/n}\sum_{k=1}^\infty C_k(2^kd)^{-n/q'+\delta}|2^{k+1}Q|^{\beta/n}||b||_{Lip_\beta}|2^{k+1}Q|^{1/q'-\delta/n}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-s\delta/n}}\int_{2^{k+1}Q}|f(y)|^sdy\right)^{1/s}$$

$$+C|Q|^{-\beta/n}\sum_{k=1}^\infty||b||_{Lip_\beta}|2^kQ|^{\beta/n}C_k(2^kd)^{-n/q'+\delta}|2^{k+1}Q|^{1/q'-\delta/n}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-s\delta/n}}\int_{2^{k+1}Q}|f(x)|^sdx\right)^{1/s}$$

$$\le\ C||b||_{Lip_\beta}M_{\delta,s}(f)(\tilde x)\sum_{k=1}^\infty 2^{k\beta}C_k$$

$$\le\ C||b||_{Lip_\beta}M_{\delta,s}(f)(\tilde x).$$

This completes the proof of Theorem 2.

Proof of Theorem 3. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q\left|T_b^\delta(f)(x)-C_0\right|dx\le C||b||_{BMO}\left(M_{\delta,s}(f)(\tilde x)+M_s(T^\delta(f))(\tilde x)\right).$$

Fix a cube $Q=Q(x_0,d)$ and $\tilde x\in Q$. Write, for $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$,

$$F_b^{\delta,t}(f)(x)=(b(x)-b_{2Q})F^{\delta,t}(f)(x)-F^{\delta,t}((b-b_{2Q})f_1)(x)-F^{\delta,t}((b-b_{2Q})f_2)(x).$$

Then

$$\frac{1}{|Q|}\int_Q||F_b^{\delta,t}(f)(x)-F^{\delta,t}((b_{2Q}-b)f_2)(x_0)||dx$$

$$\le\ \frac{1}{|Q|}\int_Q||(b(x)-b_{2Q})F^{\delta,t}(f)(x)||dx+\frac{1}{|Q|}\int_Q||F^{\delta,t}((b-b_{2Q})f_1)(x)||dx$$

$$+\frac{1}{|Q|}\int_Q||F^{\delta,t}((b-b_{2Q})f_2)(x)-F^{\delta,t}((b-b_{2Q})f_2)(x_0)||dx$$

$$=\ III_1+III_2+III_3.$$

For III_1, by Hölder's inequality, we get

$$
\begin{aligned}
III_1 &\leq \left(\frac{1}{|Q|}\int_Q |b(x)-b_{2Q}|^{s'}dx\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q |T^\delta(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{BMO}M_s(T^\delta(f))(\tilde{x}).
\end{aligned}
$$

For III_2, choose $1<p<s$ such that $1/r=1/p-\delta/n$, by Hölder's inequality and (L^p,L^r)-boundedness of T_δ, we obtain

$$
\begin{aligned}
III_2 &\leq \left(\frac{1}{|Q|}\int_{R^n}|T^\delta((b-b_{2Q})f_1)(x)|^r dx\right)^{1/r}\\
&\leq C|Q|^{-1/r}\left(\int_{R^n}|(b-b_{2Q})f_1(x)|^p dx\right)^{1/p}\\
&\leq C|Q|^{-1/r}\left(\int_{2Q}|b(x)-b_{2Q}|^{sp/(s-p)}dx\right)^{(s-p)/sp}\left(\int_{2Q}|f(x)|^s dx\right)^{1/s}\\
&\leq C|Q|^{-1/r}|2Q|^{(s-p)/sp}\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{sp/(s-p)}dx\right)^{(s-p)/sp}\\
&\quad\times |2Q|^{1/s-\delta/n}\left(\frac{1}{|2Q|^{1-s\delta/n}}\int_{2Q}|f(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{BMO}|Q|^{-1/r+1/p-\delta/n}\left(\frac{1}{|2Q|^{1-s\delta/n}}\int_{2Q}|f(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{BMO}M_{\delta,s}(f)(\tilde{x}).
\end{aligned}
$$

For III_3, recalling that $s>q'$, taking $1<p<\infty$ with $1/p+1/q+1/s=1$, by Hölder's inequality, we obtain

$$
\begin{aligned}
III_3 &\leq \frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^k d\leq|y-x_0|<2^{k+1}d}||F^{\delta,t}(x,y)-F^{\delta,t}(x_0,y)||\\
&\qquad\qquad\qquad\times|b(y)-b_{2Q}||f(y)|dydx\\
&\leq \frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\left(\int_{2^k d\leq|y-x_0|<2^{k+1}d}||F^{\delta,t}(x,y)-F^{\delta,t}(x_0,y)||^q dy\right)^{1/q}\\
&\quad\times\left(\int_{2^{k+1}Q}|b(x)-b_{2Q}|^p dx\right)^{1/p}\left(\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}dx
\end{aligned}
$$

$$\leq\ C\sum_{k=1}^{\infty}C_k(2^kd)^{-n/q'+\delta}k(2^kd)^{n/p}||b||_{BMO}(2^kd)^{n(1/s-\delta/n)}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-s\delta/n}}\int_{2^{k+1}Q}|f(y)|^sdy\right)^{1/s}$$

$$\leq\ C||b||_{BMO}\sum_{k=1}^{\infty}kC_k(2^kd)^{n(-1+1/q+1/p+1/s)}\left(\frac{1}{|2^{k+1}Q|^{1-s\delta/n}}\int_{2^{k+1}Q}|f(y)|^sdy\right)^{1/s}$$

$$\leq\ C||b||_{BMO}M_{\delta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}kC_k$$

$$\leq\ C||b||_{BMO}M_{\delta,s}(f)(\tilde{x}).$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $q'<s<p$ in Theorem 1 and let $1/t=1/p-\delta/n$, then $1/r=1/t-\beta/n$, thus, we have, by Lemma 1, 3 and 4,

$$||T_b^{\delta}(f)||_{L^r}\leq||M(T_b^{\delta}(f))||_{L^r}\leq C||M^{\#}(T_b^{\delta}(f))||_{L^r}$$

$$\leq\ C||b||_{Lip_{\beta}}\left(||M_{\beta,s}(T^{\delta}(f))||_{L^r}+||M_{\beta+\delta,s}(f)||_{L^r}\right)$$

$$\leq\ C||b||_{Lip_{\beta}}(||T^{\delta}(f)||_{L^t}+||f||_{L^p})$$

$$\leq\ C||b||_{Lip_{\beta}}||f||_{L^p}.$$

This completes the proof of Theorem 4.

Proof of Theorem 5. Choose $q'<s<p$ in Theorem 1 and let $1/t=1/p-\delta/n$, then $1/r=1/t-\beta/n$, thus, we have, by Lemma 5-7,

$$||T_b^{\delta}(f)||_{L^{r,\varphi}}\leq||M(T_b^{\delta}(f))||_{L^{r,\varphi}}\leq C||M^{\#}(T_b^{\delta}(f))||_{L^{r,\varphi}}$$

$$\leq\ C||b||_{Lip_{\beta}}\left(||M_{\beta,s}(T_b^{\delta}(f))||_{L^{r,\varphi}}+||M_{\beta+\delta,s}(f)||_{L^{r,\varphi}}\right)$$

$$\leq\ C||b||_{Lip_{\beta}}(||T^{\delta}(f)||_{L^{t,\varphi}}+||f||_{L^{p,\varphi}})$$

$$\leq\ C||b||_{Lip_{\beta}}||f||_{L^{p,\varphi}}.$$

This completes the proof of Theorem 5.

Proof Theorem 6. Choose $q'<s<p$ in Theorem 2. By using Lemma 2, we obtain

$$||T_b^{\delta}(f)||_{\dot{F}_r^{\beta,\infty}}\leq\ C\left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T_b^{\delta}(f)(x)-T^{\delta}((b_{2Q}-b)f_2)(x_0)\right|dx\right|\right|_{L^r}$$

$$\leq\ C||b||_{Lip_{\beta}}\left(||M_s(T^{\delta}(f))||_{L^r}+||M_{\delta,s}(f)||_{L^r}\right)$$

$$\leq\ C||b||_{Lip_{\beta}}(||T^{\delta}(f)||_{L^r}+||f||_{L^p})$$

$$\leq\ C||b||_{Lip_{\beta}}||f||_{L^p}.$$

This completes the proof of Theorem 6.

Proof of Theorem 7. Choose $q' \leq s < p$ in Theorem 3 and similar to the proof of Theorem 4, we have

$$
\begin{aligned}
||T_b^\delta(f)||_{L^r} &\leq ||M(T_b^\delta(f))||_{L^r} \leq C||M^\#(T_b^\delta(f))||_{L^r} \\
&\leq C||b||_{BMO}\left(||M_s(T^\delta(f))||_{L^r} + ||M_{\delta,s}(f)||_{L^r}\right) \\
&\leq C||b||_{BMO}(||T^\delta(f)||_{L^r} + ||f||_{L^p}) \\
&\leq C||b||_{BMO}||f||_{L^p}.
\end{aligned}
$$

This completes the proof of the theorem.

Proof of Theorem 8. Choose $q' \leq s < p$ in Theorem 3 and similar to the proof of Theorem 5, we have

$$
\begin{aligned}
||T_b^\delta(f)||_{L^{r,\varphi}} &\leq ||M(T_b^\delta(f))||_{L^{r,\varphi}} \leq C||M^\#(T_b^\delta(f))||_{L^{r,\varphi}} \\
&\leq C||b||_{BMO}\left(||M_s(T^\delta(f))||_{L^{r,\varphi}} + ||M_{\delta,s}(f)||_{L^{r,\varphi}}\right) \\
&\leq C||b||_{BMO}(||T^\delta(f)||_{L^{r,\varphi}} + ||f||_{L^{p,\varphi}}) \\
&\leq C||b||_{BMO}||f||_{L^{p,\varphi}}.
\end{aligned}
$$

This completes the proof of the theorem.

5   Applications

In this section we shall apply the Theorems 1-8 of the paper to some particular operators such as the Littlewood-Paley operators, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1. Littlewood-Paley operators.

Fixed $0 \leq \delta < n$, $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley commutators are defined by

$$
g_\psi^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t}\right)^{1/2},
$$

$$
S_\psi^b(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}
$$

and

$$g_\mu^b(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t^b(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^b(f)(x) = \int_{R^n} (b(x) - b(y)) \psi_t(x - y) f(y) dy,$$

$$F_t^b(f)(x, y) = \int_{R^n} (b(x) - b(z)) f(z) \psi_t(y - z) dz$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which are the Littlewood-Paley operators (see [25]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(y, t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^b(f)(x)$ and $F_t^b(f)(x, y)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi^b(f)(x) = ||F_t^b(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^b(f)(x) = \left|\left| \chi_{\Gamma(x)} F_t^b(f)(x, y) \right|\right|, \quad S_\psi(f)(x) = \left|\left| \chi_{\Gamma(x)} F_t(f)(y) \right|\right|$$

and

$$g_\mu^b(f)(x) = \left|\left| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^b(f)(x, y) \right|\right|,$$

$$g_\mu(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right\|.$$

It is easily to see that g_ψ^b, S_ψ^b and g_μ^b satisfy the conditions of Theorems 1-8(see [13-15]), thus theorems 1-8 hold for g_ψ^b, S_ψ^b and g_μ^b.

Application 2. Marcinkiewicz operators.

Fixed $0 \le \delta < n$, $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz commutators are defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S^b(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\mu_\lambda^b(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^b(f)(x) = \int_{|x-y|\le t} (b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy$$

and

$$F_t^b(f)(x,y) = \int_{|y-z|\le t} (b(x)-b(z))\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators(see [26]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(y,t)|^2 dy dt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^b(f)(x) = ||F_t^b(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^b(f)(x) = \left|\left| \chi_{\Gamma(x)} F_t^b(f)(x,y) \right|\right|, \quad \mu_S(f)(x) = \left|\left| \chi_{\Gamma(x)} F_t(f)(y) \right|\right|$$

and

$$\mu_\lambda^b(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^b(f)(x,y) \right|\right|,$$

$$\mu_\lambda(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right|\right|.$$

It is easily to see that μ_Ω^b, μ_S^b and μ_λ^b satisfy the conditions of Theorems 1-8 (see [13-15][26]), thus Theorems 1-8 hold for μ_Ω^b, μ_S^b and μ_λ^b.
Application 3. Bochner-Riesz operator.
Let $\delta > (n-1)/2$, $\widehat{B_t^\delta(f)}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^b(f)(x) = \int_{R^n} (b(x) - b(y)) B_t^\delta(x-y) f(y) dy,$$

The maximal Bochner-Riesz commutator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |B_{\delta,t}^b(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator(see [16]). Let H be the space $H = \{ h : ||h|| = \sup_{t>0} |h(t)| < \infty \}$, then

$$B_{\delta,*}^b(f)(x) = ||B_{\delta,t}^b(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^b$ satisfies the conditions of Theorems 1-8 with $\delta = 0$(see [13]), thus Theorems 1-8 hold for $B_{\delta,*}^b$.

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Guo Sheng

Department of Mathematics
Changsha University of Science and Technology
Changsha 410077, P. R. of China
E-mail: `lanzheliu@163.com`

Huang Chuangxia

Department of Mathematics
Changsha University of Science and Technology
Changsha 410077, P. R. of China

Liu Lanzhe

Department of Mathematics
Changsha University of Science and Technology
Changsha 410077, P. R. of China