1 σ algebra

a. Sigma algebra definition
Given a non-empty set Ω, A sigma algebra is defined
1) Include empty set and whole set
2) Include the complement of any element itself
3) Closed under countable union

b. Sigma algebra example by tossing a coin
Tossing 0 time
Check 1)
$F_0 =$ 0,
Tossing once
Check 1 Ω = H, T
Check 2 H_c = T, T_c = H
Check 3 H U T = Ω
So
F_1 = 0, Ω, H, T
Tossing twice
Check 1 Ω = HH, HT, TH, TT
Check 2 HH_c, HT_c, TH_c, TT_c
Check 3 HH U HT, HH U TH, HH U TT, HT U TH, HT U TT, TH U TT
HH U HT U TH = TT_c,
HH U HT U TT = TH_c,
HH U TH U TT = HT_c,
HT U TH U TT = HH_c
So
$F_2 =$ 0, Ω, HH, HT, TH, TT,
HH_c, HT_c, TH_c, TT_c
HH U HT, HH U TH, HH U TT, HT U TH, HT U TT, TH U TT,
TT_c, TH_c, HT_c, HH_c

c. Why define sigma algebra?
On top of the sigma algebra, we can define the probability, because the object that probability measure takes is the sigma algebra.

2 Filtration

Consider a sequence of coin toss
For the first toss, we get F_1
For the first and second toss, we get F_2
For the first n tosses, we get F_n
The collection of sigma algebra F_1, F_2 F_n is called a Filtration.

3 Random variable

a. Definition
A random variable is function from Ω -¿ R, which satisfies for all of the subsets ω in Ω, X in Borel is in σ-algebra F.

b. example
Consider 3 toss case, H with prob p, T with prob q
Def. random variable S $S_0(w_0) = 4$ for all ω

$$\begin{aligned} S_{n+1}(w_{n+1}) =& 2S_n(w_n) \ \text{ if } \ w_{n+1} = H \\ & \frac{1}{2}S_n(w_n) \ \text{ if } \ w_{n+1} = T \end{aligned}$$

so
$S_0(w_1w_2w_3) = 4$ for all w_i
$S_1(w_1w_2w_3) = 8$ if w_1 = H
$S_1(w_1w_2w_3) = 2$ if w_1 = T
$S_2(w_1w_2w_3) = 16$ if $w_1 = w_2 = H$
$S_2(w_1w_2w_3) = 4$ if $w_1 \neq w_2$
$S_2(w_1w_2w_3) = 1$ if $w_1 = w_2 = T$

4 σ algebra generated by a random variable and measurable function

Give consider a random variable S: Ω -¿ R, for every open set in R, the collection of their inverse image forms an sigma algebra, and it is called the sigma algebra generated by S. And S is called F-measurable.