Author: Dr. Shi Guo Email: guoshi1984@hotmail.com

1 Basic of Fourier Transform

Fourier Series
If $x(t) = x(t+T)$ then $x(t)$ can be written as

$$x(t) = \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikt}{T}}$$

i is the imaginary unit, and k is an integer. The above expression is eligible because $e^{\frac{2\pi ikt}{T}}$ is a periodic function

$$e^{\frac{2\pi ikt}{T}} = e^{\frac{2\pi ik(t+T)}{T}}$$

Each basis $e^{\frac{2\pi ikt}{T}}$ represents a signal with frequency $f_k = \frac{k}{T}$. So the interval between each adjacent frequency $\Delta f = \frac{1}{T}$. Based on orthogonality, we can get c_k

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-i\frac{2\pi kt}{T}}dt$$

Fourier Series: Example

$$x(t) = cos(2\pi f_0 t) = \frac{1}{2}(e^{i2\pi f_0 t} + e^{-i2\pi f_0 t})$$

where $f_0 = \frac{1}{T}$

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T x(t)e^{-i\frac{2\pi kt}{T}}dt \\ &= \frac{1}{T}\int_0^T \frac{1}{2}(e^{\frac{2i\pi t}{T}} + e^{\frac{-2i\pi t}{T}})e^{-i\frac{2\pi kt}{T}}dt \end{aligned}$$

Only terms with k = +/-1 in the above expression can survive, so

$$c_1 = \frac{1}{T}\int_0^T \frac{1}{2}dt = \frac{1}{2}$$

Similarly, $c_{-1} = \frac{1}{2}$.

Fourier Transform
We can generalize the Fourier series to non-periodic functions. We define the Fourier transform as

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi ift}dt$$

With the inverse Fourier transform defined as

$$x(t) = \int_{-\infty}^{\infty} \mathcal{F}(f) e^{2\pi ift} df$$

To see why the above makes sense, it is easy to prove the identity.

$$\begin{aligned}
x(t^{'}) &= \int_{-\infty}^{\infty} \mathcal{F}(f) e^{2\pi ift^{'}} df \\
&= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} x(t) e^{-2\pi ift} dt) e^{2\pi ift^{'}} df \\
&= \int_{-\infty}^{\infty} x(t) (\int_{-\infty}^{\infty} e^{-2\pi ift} e^{2\pi ift^{'}}) df \\
&= \int_{-\infty}^{\infty} x(t) \delta(t - t^{'}) dt \\
&= x(t^{'})
\end{aligned}$$

Fourier Transform: Example
1. Constant Function

$$x(t) = 1$$

$$\begin{aligned}
\mathcal{F}(f) &= \int_{-\infty}^{\infty} x(t) e^{-2\pi ift} dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi ift} dt \\
&= \lim_{a\to\infty} \int_{-a}^{a} e^{-2\pi ift} dt \\
&= \lim_{a\to\infty} \frac{1}{-2\pi ift} e^{-2\pi ift}|_{-a}^{a} \\
&= \lim_{a\to\infty} \frac{1}{-2\pi ifa} (e^{-2\pi ifa} - e^{2\pi ifa}) \\
&= \lim_{a\to\infty} \frac{1}{2\pi ifa} (e^{2\pi ifa} - e^{-2\pi ifa}) \\
&= \lim_{a\to\infty} 2 \frac{sin(2\pi fa)}{2\pi fa} \\
&= 2 \lim_{a\to\infty} \frac{sin(2\pi fa)}{2\pi fa} \\
&= \delta(f)
\end{aligned}$$

2. Trigeometic Function Take the same x(t) as above in the discrete case

$$x(t) = cos(2\pi f_0 t) = \frac{1}{2}(e^{2i\pi f_0 t} + e^{-2i\pi f_0 t})$$

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} \frac{1}{2}(e^{2i\pi f_0 t} + e^{-2i\pi f_0 t})e^{-2i\pi f t}dt \quad = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Discrete Fourier Series
The above is the Fourier transform in continuous case, in discrete case If $x = n\Delta t$, where n = 1...N, and $T = N\Delta t$, then the Fourier series can be written as

$$\begin{aligned} x(n) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k n \Delta t}{N\Delta t}} \\ &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k n}{N}} \end{aligned}$$

$$c_k = \frac{1}{N\Delta t}\sum_{n=1}^{N} f(n\Delta t)e^{-i2\pi k \frac{1}{N\Delta t} n\Delta t} d(n\Delta t) = \frac{1}{N}\sum_{n=1}^{N} f(n)e^{-i2\pi k \frac{n}{N}}$$

This is the discrete Fourier series.
The interval in the frequency domain is

$$\Delta f = f_{k+1} - f_k = \frac{k+1}{T} - \frac{k}{T} = \frac{1}{T} = \frac{1}{N\Delta t}$$

Discrete Fourier Transform
In the discrete case, suppose we sample a signal N times within time T. We divide time T into N time intevals with length being $\Delta t = N/T$. Then we can let $t = n\Delta t$, the integral in the Fourier transform becomes a summation. So we write the Fourier transform as

$$\mathcal{F}(f) = \sum_{0}^{N} x(n\Delta t)e^{-2\pi i f n \Delta t}\frac{T}{N}$$

In frequency domain, the frequency also becomes discrete, and same as the case in discrete Fourier series, $\Delta f = \frac{1}{T}$. Another way of seeing Δf is when we confine the length of time domain to T, the function in time domain has to be periodic function with period T. So we have

$$e^{2\pi f(n\Delta t + T)} = e^{2\pi f n \Delta t}$$

This requires

$$2\pi f T = 2\pi k$$

where k is integer. This leads discrete frequencies

$$f = \frac{k}{T}$$

and

$$\Delta f = \frac{1}{T}$$

Using $f = k\Delta f$, we can rewrite our Fourier transform

$$\begin{aligned}\mathcal{F}(k\Delta f) &= \sum_0^N x(n\Delta t)e^{-2\pi ik\Delta f n\Delta t}\frac{T}{N} \\ &= \sum_0^N x(n\Delta t)e^{-2\pi ikn/N}\frac{T}{N}\end{aligned}$$

Then we work out the Fourier transform and inverse Fourier transform identity

$$\begin{aligned}x(n^{'}\Delta t) &= \sum_{k=0}^N \mathcal{F}(k\Delta f)e^{2\pi ikn^{'}/N}\Delta f \\ &= \sum_{k=0}^N (\sum_{n=0}^N x(n\Delta t)e^{-2\pi ikn/N}\frac{T}{N})e^{2\pi ikn^{'}/N}\Delta f \\ &\sum_{k=0}^N \frac{1}{N}(\sum_{n=0}^N x(n\Delta t)e^{-2\pi ikn/N})e^{2\pi ikn^{'}/N}\end{aligned}$$

So we define discrete Fourier transform

$$\mathcal{F}(k) = \sum_{n=0}^N x(n)e^{-2\pi ikn/N}$$

and the discrete inverse Fourier transform

$$x(n) = \frac{1}{N}\sum_{n=0}^N \mathcal{F}(k)e^{2\pi ikn/N}$$

Example
Let N = 4, and

$$x(n) = cos(2\pi\frac{n}{4}) = \frac{1}{2}(e^{i2\pi\frac{n}{4}} + e^{-i2\pi\frac{n}{4}})$$

$$\mathcal{F}(k) = \frac{1}{4}\sum_{n=1}^4 \frac{1}{2}(e^{i2\pi\frac{n}{4}} + e^{-i2\pi\frac{n}{4}})e^{-i\frac{2\pi kn}{4}}$$

Similary to the continuous case, only terms with k = +/-1 in the above expression can survive, when $k = 1$

$$\begin{aligned}\mathcal{F}(1) &= \frac{1}{4}\sum_{n=1}^4 \frac{1}{2}e^{i2\pi\frac{n}{4}}e^{-i\frac{2\pi n}{4}} \\ &= \frac{1}{4}\frac{1}{2}4 \\ &= \frac{1}{2}\end{aligned}$$

What about case for $k = -1$? We define k = 1, 2, 3, 4 so $k = -1$ is not defined. However, in discrete case we note $c_{-1} = c_3$ due to the periodicity. Similarly, we can calculate $\mathcal{F}(3) = \frac{1}{2}$.
N is the total sample within time T.

Properties

1) To be eligible, f(x) has to be a period function with time T(with frequency $F = \frac{1}{T}$) in both continuous case and discrete case. The requirement in discrete case leads to uniform sampling theorem used in signal processing. The total sampling time $T_{sampling}$ has to be an integer multiple of T.

$$T_{sampling} = MT$$

while $T = \frac{N}{F_s}$ So

$$MT = N\Delta t$$

if we let $\Delta t = \frac{1}{F_s}$, where F_s is the sampling frequency, and $T = \frac{1}{F}$, we have

$$\frac{M}{F} = \frac{N}{F_s}$$

2) If $x(n)$ is real, which means $x(n) = x^*(n)$. We then substitute Fourier series for both $x(n)$ and $x * (n)$,

$$\sum_{-\infty}^{+\infty} c_k e^{2\pi i \frac{1}{T} kx} = \sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} \tag{1}$$

Since the summation on the right hand side is from $-\infty$ to ∞, it is eligible to replace k with k.

$$\sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} = \sum_{\infty}^{-\infty} c_{-k}^* e^{2\pi i \frac{1}{T} kx} \tag{2}$$

Combine the above two equations 1 and 2, we can see $c_k = c_{-k}^*$. This means they are complex conjugate: their magnitude are equal, their phase are opposite. Namely $||c_k|| = ||c_{-k}||$, $\phi(c_k) = \phi(c_{-k})$. Similarly, for discrete Fourier transform, $||\mathcal{F}_k|| = ||\mathcal{F}_{-k}||$
3) Connection between complex representation and real representation.
We have shown that for real signal $c_k = c_{-k}^*$ and $c_k = |c_k|e^{j\theta_k}$, $c_{-k} = |c_k|e^{-j\theta_k}$. And in complex representation, we can combine the term with index k and -k,

$$c_k e^{j2\pi k F_0 t} + c_{-k} e^{-j2\pi k F_0 t} = 2|c_k| cos(2\pi k F_0 t + \theta_k)$$

$$\begin{aligned}
f(x) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k x}{T}} \\
&= c_0 + 2\sum_{k=1}^{\infty} |c_k| cos(2\pi k F_0 t + \theta_k) \\
&= a_0 + \sum_{k=1}^{\infty} (a_k cos(2\pi k F_0 t) - b_k sin(2\pi k F_0 t))
\end{aligned}$$

where $a_0 = c_0$, $a_k = 2|c_k|cos\theta_k$, $b_k = 2|c_k|sin\theta_k$.
4) $\mathcal{F}(k) = \mathcal{F}(k+N)$, which means $\mathcal{F}(k)$ is periodic with period N. We remember $\Delta f = \frac{1}{T}$, so the period of N correspond to time length of $\frac{N}{T}$. Therefore, it is sufficient enough for us to confine k to be within the range $-N/2 < k \leq N/2$. For all the integers of $k^{'}$ which are beyond this range, we can find an equivalent integer of k which is within $-N/2 < k \leq N$ that satisfies $\mathcal{F}(k^{'}) = \mathcal{F}(k)$. With sample frequency $F_s = \frac{N}{T}$, the maximum frequecy of the signal(bandwidth B) we can tell is $B = \frac{N}{2T} = \frac{F_s}{2}$. In other words, in order to capture the whole bandwidth B of the signal, we must have $F_s \geq 2B$. This is **Nyquist sampling theorem**.
5) Power density

$$\begin{aligned} P_x &= \frac{1}{T}\int |x(t)|^2 dt \\ &= \frac{1}{T}\int x(t)\sum_{-\infty}^{\infty} c_k^* e^{-j2\pi k F_0 t} \\ &= \sum_{-\infty}^{\infty} c_k^* [\frac{1}{T}\int x(t)e^{-j2\pi k F_0 t}] \\ &= \sum_{-\infty}^{\infty} |c_k|^2 \end{aligned}$$

When signal is real, then

$$\begin{aligned} P_x &= \sum_{-\infty}^{\infty} |c_k|^2 \\ &= a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \end{aligned}$$

2 Fast Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

let

$$u_k = e^{-i2\pi k \frac{n}{N}}$$

then we have the basis orthogonality

$$u_{k1}^T u_{k2} = N\delta_{k_1,k_2}$$

We recognize we can write X_k with even index terms and odd index terms

$X_k =$ Even index parts $+$ Odd index parts

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N} 2mk} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of N/2 even indexed points, where k is 0,1N/2)

$$+ e^{-\frac{2\pi i}{N} k}$$

$$\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of N/2 odd indexed points, where k is 0,1N/2)

(Since each part is a Fourier transform of N/2 points, k has to be smaller than N/2)

$$= E_k + e^{-\frac{2\pi i}{N} k} O_k$$

As noted, the above derivation is for $k < N/2$, a very similar derivation for $N/2 <= k < N$ leads to

$$X_{k+N/2} = E_k - e^{-\frac{2\pi i}{N} k} O_k$$

Now we have divided the FFT of N points to two FFT with N/2 points Keep going till we reach the size to one, then combine together recursively.

3 Fourier Transform of Useful Functions

The Fourier Transform of Step Function

Let $u(t)$ be a step function: $u(t) = 1$ when $t \geq 0$, $u(t) = 0$ when $t < 0$. And its derivative is a delta function

$$\frac{d\,\mathrm{u}(t)}{dt} = \delta(t)$$

Taking Fourier transform on both sides yields

$$2\pi i f \mathcal{F}(f) = 1$$

So

$$\mathcal{F}(f) = \frac{1}{2\pi i f}|_{f\neq 0} + \mathcal{F}(f)|_{f=0}$$

Since any function with a different constant can have the same derivative, the Fourier transform of the original function has to have a constant, which corresponds to zero frequency component $F(0)$. The constant component of function $u(t)$ is its offset to zero, which is $1/2$. so

$$F(f) = \frac{1}{2\pi i f}|_{f\neq 0} + \frac{1}{2}\delta(f)$$

The Fourier Transform of a Shifted Step Function
Let $u(t)$ be a step function: $u(t-\tau) = 1$ when $t \geq \tau$, $u(t-\tau) = 0$ when $t < \tau$. Then

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} u(t-\tau)e^{-2\pi f t}dt$$

Let $t^{'} = t - \tau$, then

$$\mathcal{F}(f) = e^{-2\pi i f \tau}\int_{-\infty}^{\infty} u(t^{'})e^{-2\pi f t^{'}}dt^{'}$$

So we see this is a factor times Fourier transform of step function, therefore

$$\begin{aligned}\mathcal{F}(f) &= e^{-2\pi i f \tau}(\frac{1}{2\pi i f}|_{f\neq 0} + \frac{1}{2}\delta(f)) \\ &= e^{-2\pi i f \tau}\frac{1}{2\pi i f}|_{f\neq 0} + \frac{1}{2}\delta(f)\end{aligned}$$

The Fourier Transform of Gaussian

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}$$

$$\mathcal{F}(f) = e^{-2\pi^2\sigma^2 f^2}$$

So the Fourier transform of a Gaussian function is another Gaussian function but with different width.

The Fourier Transform of Dirac Comb

$$x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT)$$

It is clearly that x(t) is periodic with period T. So we can expand that into Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t/T}$$

Where

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-i\frac{2\pi kt}{T}}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(0)e^{-i\frac{2\pi kt}{T}} \\
&= \frac{1}{T}
\end{aligned}
$$

So

$$
x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}e^{2\pi ikt/T}
$$

On the other hand, based on the formula of Fourier transform

$$
\mathcal{F}(f) = \int \sum_{n=-\infty}^{\infty} \delta(t-nT)e^{-2\pi ift}dt \quad = \sum_{n=-\infty}^{\infty} e^{-2\pi inTf} = \sum_{n=-\infty}^{\infty} e^{-2\pi inf/f_0}
$$

Comparing the Fourier series of x(t) and the expression of F(f), they are the same except T being changed to f_0. Therefore we can conclude that F(f) itself is also a Dirac comb, which is

$$
\mathcal{F}(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f-nf_0)
$$

The Fourier Transform of White Noise
Assuming noise we sample in time is $n[m]$, where m = 0,... M-1. $n[m]$ is a Gaussian random variable with zero mean and variance σ^2. The the FFT of $n[m]$ is

$$
\begin{aligned}
N[k] &= \frac{1}{M}\sum_{m=0}^{M-1} n[m]e^{-i2\pi mk/M} \\
&= \frac{1}{M}\sum_{m=0}^{M-1} n[m](cos(2\pi mk/M) - i\ n[m]sin(2\pi mk/M))
\end{aligned}
$$

The expected value is

$$
\begin{aligned}
E[N[k]] &= E[\frac{1}{M}\sum_{0}^{M-1} n[m]e^{-i2\pi mk/M}] \\
&= \frac{1}{M}\sum_{0}^{M-1} E[n[m]]e^{-i2\pi mk/M}] \\
&= 0(\text{because E[n[m]] = 0})
\end{aligned}
$$

The variance of the real part is

$$
\begin{aligned}
Var[R[N[k]]] &= E[(\frac{1}{M}\sum_{m=0}^{M-1} n[m](cos(2\pi mk/M)) * (\frac{1}{M}\sum_{p=0}^{M-1} n[p](cos(2\pi pk/M))] \\
&= \frac{1}{M^2}E[\sum_{m=0}^{M-1} n[m]n[p]\delta(n-p)cos(2\pi mk/M) * cos(2\pi pk/M)] \\
&= \frac{1}{M^2}\sum_{m=0}^{M-1} E[n[m]^2]cos^2(2\pi mk/M) \\
&= \frac{1}{M^2}\sigma^2(\sum_{m=0}^{M-1} cos^2(2\pi mk/M)) \\
&= \frac{1}{M^2}\sigma^2(\frac{M}{2} + \frac{cos((M+1)2\pi k/M)sin(2\pi Mk/M)}{2sin(2\pi k/M)}) \\
&= \frac{1}{M}\frac{\sigma^2}{2}
\end{aligned}
$$

The same derivation applies for the imaginary part. So the FFT is Gaussian noise with mean zero and variance σ^2.

4 Connection with Uncertainty Principle

Relationship between time length and frequency bandwidth
We consider a few examples
1) We consider a function g(t) which is infinitely long in time domain

$$g(t) = cos(2\pi f_0 t)$$

Its Fourier transform is

$$
\begin{aligned}
F(f) &= \int \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2} e^{i2\pi ft} dt \\
&= \int \frac{1}{2}e^{i2\pi t(f_0+f)}dt + \int \frac{1}{2}e^{i2\pi t(f-f_0)}dt \\
&= \frac{1}{2}\delta(f+f_0) + \frac{1}{2}\delta(f-f_0)
\end{aligned}
$$

The last line is based on $\int_{-\infty}^{\infty} e^{i2\pi ft} = \delta(f)$.
Since the delta function has width zero, so the the bandwidth in frequency domain is zero. We see a signal which is infinitely long in time domain has zero bandwidth in frequency domain.
2) We consider a function g(t) which has zero width in time, namely an impulse function.

$$g(t) = \delta(t)$$

Since this function is not a periodic function, we assume its period is infinity. Its Fourier transform is

$$F(f) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi f t} = 1$$

Now we see a signal which has zero width in time has infinitely long frequency bandwidth. Typically, for a signal, the width in its time domain and the width in its frequency domain can not shrink to zero simultaneously. This leads to the uncertainty principle.

Uncertainty Principle

In quantum mechanics, if there is a particle with position x and momentum p, then uncertainty principle states

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Similar relationship holds for time t and Energy.

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

We can modify this expression to get the time and frequency relationship in our Fourier transform. Since $E = \hbar\omega$. Then

$$\Delta t \Delta \omega \geq \frac{1}{2}$$

5 Connection to Bloch Theorem in Solid State Physics

In solid state physics, the crystal lattice is periodic so as the periodic potential. The wavefunction Ψ at the presence of a periodic potential has the following property.

$$\Psi(\mathbf{r} + \mathbf{R}_n) = e^{i\mathbf{k}\cdot\mathbf{R}_n}\Psi(\mathbf{r})$$

This is the Bloch Theorem. And the wavefunction $\Psi(r)$ can be written as

$$\Psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u(\mathbf{r})$$

Where $u(\mathbf{r})$ is periodic too with $u(\mathbf{r}) = u(\mathbf{r} + \mathbf{R}_n)$. In above, $\mathbf{R}_n$ is the crystal translation vector, and $\mathbf{k}$ is a vector. For simplicity, we consider one dimension crystal and the lattice basis vector is $\mathbf{a}$, therefore $\mathbf{R}_n = n\mathbf{a}$.

We note for a k vector which holds the Bloch theorem, $k + l\frac{2\pi}{a}$ (where l is an integer) can also holds the Bloch theorem. As in the discrete Fourier transform, where $\mathcal{F}(k)$ is periodic function with period N and we confine the frequency to be in the range of $[-N/2T, N/2T]$, the function $\Psi(k)$ here is also a periodic function with $\frac{2\pi}{a}$. So to make k unique, we usually confine k to be within the range$-\frac{2\pi}{a}, \frac{2\pi}{a}$, and we call this the First Brillioun zone.