Author: Dr. Shi Guo   Email: guoshi1984@hotmail.com

1   Black Scholes Merton Equation

We assume the stock prices following a geometric Brownian motion
1) Stock price:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

2) We have a portfolio X(t) which consists of $\Delta(t)$ share of stock $\Delta(t)S(t)$, and $(X(t) - \Delta(t)S(t))$ money market account with interest rate r.

$$X(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

3) Change of the portfolio with respect to time

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$
$$= rX(t)dt + \Delta(t)(\alpha - r)S(t) + \Delta(t)\sigma S(t)dW(t)$$

4) Change of the present value of the stock with respect to time

$$d(e^{-rt}S(t)) = (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$$

5) With a few steps, we get change of the present value of the portfolio with respect to time

$$d(e^{-rt}X(t))$$
$$= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t)$$

6) Assume the option value is $c(t, S(t))$ and we apply Ito's formula

$$d(e^{-rt}c(t, S(t))$$
$$= e^{-rt}[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)\frac{\partial c(t,S(t))}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c(t,S(t))}{\partial S^2(t)}]dt$$
$$+ e^{-rt}\sigma S(t)\frac{\partial c(t,S(t))}{\partial S(t)}dW(t)$$

7)Now equate Equation in 5) and 6), we get
dW(t) term:

$$\Delta(t) = \frac{\partial c(t, S(t))}{\partial S(t)}$$

dt term:

$$rc(t, S) = c_t(t, S(t)) + rS(t) + \frac{1}{2}\sigma^2 S(t)\frac{\partial c(t, S(t))}{\partial S^2(t)}$$

which is known as Black-Scholes-Merton partial differential equation.

2 Connection to Faynman-Kac formula

In risk-neutral measure, we write the stock price as

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

Where $W(t)$ is a standard Brownian motion under risk-neutral measure.
According to the risk-neutral pricing formula, the price of the derivative security
at time t is

$$V(t) = \tilde{E}[e^{-r(T-t)}V(T)|F(t)] = \tilde{E}[e^{-r(T-t)}h(S(T))|F(t)]$$

Since the stock price is Markov and the payoff is a function fo the stock price
alone, based on Faynman-Kac formula, there is a function $v(t, x)$ such that
$V(t) = v(t, S(t))$, and $v(v, S(t))$ must satisfy discounted partial differential equa-
tion

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$