

1 σ algebra

a. Sigma algebra definition

Given a non-empty set Ω , a sigma algebra is defined

- 1) Include empty set and whole set
- 2) Include the complement of any element itself
- 3) Closed under countable union

b. Sigma algebra example by tossing a coin

Tossing 0 time

Check 1)

$$F_0 = \emptyset,$$

Tossing once

Check 1 $\Omega = H, T$

Check 2 $H_c = T, T_c = H$

Check 3 $H \cup T = \Omega$

So

$$F_1 = \emptyset, \Omega, H, T$$

Tossing twice

Check 1 $\Omega = HH, HT, TH, TT$

Check 2 HH_c, HT_c, TH_c, TT_c

Check 3 $HH \cup HT, HH \cup TH, HH \cup TT, HT \cup TH, HT \cup TT, TH \cup TT$

$$HH \cup HT \cup TH = TT_c,$$

$$HH \cup HT \cup TT = TH_c,$$

$$HH \cup TH \cup TT = HT_c,$$

$$HT \cup TH \cup TT = HH_c$$

So

$$F_2 = \emptyset, \Omega, HH, HT, TH, TT,$$

$$HH_c, HT_c, TH_c, TT_c$$

$$HH \cup HT, HH \cup TH, HH \cup TT, HT \cup TH, HT \cup TT, TH \cup TT,$$

$$TT_c, TH_c, HT_c, HH_c$$

c. Why define sigma algebra?

On top of the sigma algebra, we can define the probability, because the object that probability measure takes is the sigma algebra.

2 Filtration

Consider a sequence of coin toss

For the first toss, we get F_1

For the first and second toss, we get F_2

For the first n tosses, we get F_n

The collection of sigma algebra F_1, F_2, F_n is called a Filtration.

3 Random variable

a. Definition

A random variable is function from Ω to \mathbb{R} , which satisfies for all of the subsets ω in Ω , X in Borel is in σ -algebra \mathcal{F} .

b. example

Consider 3 toss case, H with prob p , T with prob q

Def. random variable S $S_0(w_0) = 4$ for all ω

$$S_{n+1}(w_{n+1}) = \begin{cases} 2S_n(w_n) & \text{if } w_{n+1} = H \\ \frac{1}{2}S_n(w_n) & \text{if } w_{n+1} = T \end{cases}$$

so

$$S_0(w_1w_2w_3) = 4 \text{ for all } w_i$$

$$S_1(w_1w_2w_3) = 8 \text{ if } w_1 = H$$

$$S_1(w_1w_2w_3) = 2 \text{ if } w_1 = T$$

$$S_2(w_1w_2w_3) = 16 \text{ if } w_1 = w_2 = H$$

$$S_2(w_1w_2w_3) = 4 \text{ if } w_1 \neq w_2$$

$$S_2(w_1w_2w_3) = 1 \text{ if } w_1 = w_2 = T$$

4 σ algebra generated by a random variable and measurable function

Give consider a random variable $S: \Omega \rightarrow \mathbb{R}$, for every open set in \mathbb{R} , the collection of their inverse image forms an sigma algebra, and it is called the sigma algebra generated by S . And S is called \mathcal{F} -measurable.

5 Conditional Expectation

a. Definition

1) $E[X|G]$ is G measurable, which means the value of $E[X]$ is completely determined by G

2) $\int_A E[X|G](w) dP(w) = \int_A X(w) dP(w)$ for all A which belongs to G

b. Example to understand 2)

Consider 3 toss case, H with prob p , T with prob q

Define random variable S

$$S_0(w) = 4 \text{ for all } w$$

$$S_{n+1}(w) = 2S_n(w) \text{ if } w_{n+1} = H$$

$$S_{n+1}(w) = \frac{1}{2}S_n(w) \text{ if } w_{n+1} = T$$

Expectation of 3 tosses random variable S_3 give the first two is HH

$$\begin{aligned}
E_2(S_3|HH) &= pS_3(HHH) + qS_3(HHT) \\
E_2(S_3|HT) &= pS_3(HTH) + qS_3(HTT) \\
E_2(S_3|TH) &= pS_3(THH) + qS_3(THT) \\
E_2(S_3|TT) &= pS_3(TTH) + qS_3(TTT) \\
E_2(S_3|HH)P(HH) &= \text{prob}(HHH)S_3(HHH) + \text{prob}(HHT)S_3(HHT) \\
E_2(S_3|HT)P(HT) &= \text{prob}(HTH)S_3(HTH) + \text{prob}(HTT)S_3(HTT) \\
E_2(S_3|TH)P(TH) &= \text{prob}(THH)S_3(THH) + \text{prob}(THT)S_3(THT) \\
E_2(S_3|TT)P(TT) &= \text{prob}(TTH)S_3(TTH) + \text{prob}(TTT)S_3(TTT)
\end{aligned}$$

This confirms def 2), for A = HH or HT or TH or TT
 $\int_2(S_3|G)(w)dP(w) = \int_A X(w)dP(w)$

c. Properties

- 1) The conditional expectation is a random variable. Because the value is dependent on G.
- 2) If X is G measurable, then $E[X|G] = X$.
- 3) If X is G measurable $E[XY|G] = XE[Y|G]$, this is to take out what is known.
- 4) If X is independent of G

$$E[X|G] = EX$$

To understand 2) and 3), consider two extreme cases

Define random variable S

$$S_0(w) = 4 \text{ for all } w$$

$$S_{n+1}(w) = 2S_n(w) \text{ if } w_{n+1} = H$$

$$S_{n+1}(w) = \frac{1}{2}S_n(w) \text{ if } w_{n+1} = T$$

Then a condition expectation can be defined as

$$E[S_n|F_t] = E[S_n|\omega_1, \omega_2, \dots, \omega_t]$$

if t=n, then $E[S_n|F_n] = S_n$, this is because when F_n is known, then S_n is known, there is nothing to average.

if t=0, then $E[S_n|F_0] = E[S_n]$, this is because F_0 provides no restriction to average S_n , the conditional expectation needs to average all possible cases, it is a general expectation.

- 5) If G is a subset of H

$$E[E[X|G|H]] = E[X|H]$$