

1 Black Scholes Merton Equation: Introduction

We assume the stock prices following a geometric Brownian motion

1) Stock price:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

2) We have a portfolio $X(t)$ which consists of $\Delta(t)$ share of stock $\Delta(t)S(t)$, and $(X(t) - \Delta(t)S(t))$ money market account with interest rate r .

$$X(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

3) Change of the portfolio with respect to time

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

4) Change of the present value of the stock with respect to time

$$d(e^{-rt}S(t)) = (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$$

5) With a few steps, we get change of the present value of the portfolio with respect to time

$$\begin{aligned} d(e^{-rt}X(t)) &= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \end{aligned}$$

6) Assume the option value is $c(t, S(t))$ and we apply Ito's formula

$$\begin{aligned} d(e^{-rt}c(t, S(t))) &= e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)\frac{\partial c(t, S(t))}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c(t, S(t))}{\partial S^2(t)}]dt \\ &\quad + e^{-rt}\sigma S(t)\frac{\partial c(t, S(t))}{\partial S(t)}dW(t) \end{aligned}$$

7) Now equate Equation in 5) and 6), we get $dW(t)$ term:

$$\Delta(t) = \frac{\partial c(t, S(t))}{\partial S(t)}$$

dt term:

$$rc(t, S) = c_t(t, S(t)) + rS(t) + \frac{1}{2}\sigma^2 S(t) \frac{\partial^2 c(t, S(t))}{\partial S^2(t)}$$

which is known as Black-Scholes-Merton partial differential equation.

2 Connection to Feynman-Kac formula

In risk-neutral measure, we write the stock price as

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

Where $\tilde{W}(t)$ is a standard Brownian motion under risk-neutral measure. According to the risk-neutral pricing formula, the price of the derivative security at time t is

$$V(t) = \tilde{E}[e^{-r(T-t)}V(T)|F(t)] = \tilde{E}[e^{-r(T-t)}h(S(T))|F(t)]$$

Since the stock price is Markov and the payoff is a function of the stock price alone, based on Feynman-Kac formula, there is a function $v(t, x)$ such that $V(t) = v(t, S(t))$, and $v(t, S(t))$ must satisfy discounted partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

Now we have seen two ways of showing the Black-Scholes-Merton (BSM) equation. One way is to reproduce the payoff of the option using a portfolio that consists of a saving account. Another way is based on the risk-neutral pricing formula and Feynman-Kac formula. These two ways are equivalent. Because under risk-neutral measure, the payoff of a derivative is the same as a saving account, which implies we are able to reproduce the payoff using a portfolio that consists of a saving account.

3 Black-Scholes-Merton Model: Analytic Solution for European option

1. European call option

For European call option with payoff to be $V(T) = S(T) - K$, with K as strike price, let us assume constant volatility σ , and constant interest rate r . Then we can obtain the solution to the BSM equation with martingale property without

bothering solving the complex partial differential equation. The call option value satisfies

$$c(t, S(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ | F(t)]$$

We write

$$\begin{aligned} S(T) &= S(t) \exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)\tau\} \\ &= S(t) \exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} \end{aligned}$$

Where Y is the standard normal random variable and $\tau = T - t$ is the time to expiration. We see we write S(T) as a product of S(t), which is $F(t)$ measurable, and a random variable independent of $F(t)$. Then solving $c(t, S(t))$ becomes solving an expectation value of a random variable composed of a standard normal random variable.

$$c(t, x) = \tilde{E}[e^{-r\tau}(x \exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^+]$$

Where Y is a standard normal distribution under \hat{P} .

After a little bit of math with integration, we have the solution to the Black-Scholes-Merton model for European call option

$$c(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - e^{-r\tau}KN(d_-(\tau, x))$$

Where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

$N()$ is the cumulative distribution function of the standard normal distribution

2. European put option The payoff for the European put option is $V(T) = K - S(T)$, we can follow a similar derivation and get the formula for put option

$$p(t, x) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)x$$

3. Alternative formulation

$$\begin{aligned} C(F, \tau) &= D[N(d_+)F - N(d_-)K] \\ P(F, \tau) &= D[N(-d_-)K - N(-d_+)F] \\ d_{+/-} &= \frac{1}{\sigma\sqrt{\tau}}[\ln(\frac{F}{K}) + / - \frac{1}{2}\sigma^2\tau] \end{aligned}$$

The variables are:

$\tau = T - t$ is the time to expiry

$D = e^{-r\tau}$ is the discount factor

$F = e^{r\tau}S$ is the forward price of the asset S.