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1 Option Pricing Models and Solutions

1.1 Black Scholes Merton Equation: Introduction

We assume the stock prices following a geometric Brownian motion
1) Stock price:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

2) We have a portfolio X(t) which consists of $\Delta(t)$ share of stock $\Delta(t)S(t)$, and $(X(t) - \Delta(t)S(t))$ money market account with interest rate r.

$$X(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

3) Change of the portfolio with respect to time

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t) + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

4) Change of the present value of the stock with respect to time

$$d(e^{-rt}S(t)) = (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$$

5) With a few steps, we get change of the present value of the portfolio with respect to time

$$\begin{aligned} &d(e^{-rt}X(t)) \\ =&\Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \end{aligned}$$

6) Assume the option value is $c(t, S(t))$ and we apply Ito's formula

$$\begin{aligned} &d(e^{-rt}c(t, S(t)) \\ =&e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)\frac{\partial c(t, S(t))}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c(t, S(t))}{\partial S^2(t)}]dt \\ &+ e^{-rt}\sigma S(t)\frac{\partial c(t, S(t))}{\partial S(t)}dW(t) \end{aligned}$$

7)Now equate Equation in 5) and 6), we get
dW(t) term:

$$\Delta(t) = \frac{\partial c(t, S(t))}{\partial S(t)}$$

dt term:

$$rc(t, S) = c_t(t, S(t)) + rS(t) + \frac{1}{2}\sigma^2 S(t)\frac{\partial c(t, S(t))}{\partial S^2(t)}$$

which is known as Black-Scholes-Merton partial differential equation.
The terminal condition the equation satisfies for call option is

$$c(T, S) = (S(T) - K)^+$$

Similarly, for put option

$$p(T, S) = (K - S(T))^+$$

1.2 Connection to Faynman-Kac formula

In risk-neutral measure, we write the stock price as

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

Where $\tilde{W}(t)$ is a standard Brownian motion under risk-neutral measure.
According to the risk-neutral pricing formula, the price of the derivative security at time t is

$$V(t) = \tilde{E}[e^{-r(T-t)}V(T)|F(t)] = \tilde{E}[e^{-r(T-t)}h(S(T))|F(t)] \tag{1}$$

Since the stock price is Markov and the payoff is a function fo the stock price alone, based on Faynman-Kac formula, there is a function $v(t, x)$ such that $V(t) = v(t, S(t))$, and $v(v, S(t))$ must satisfy discounted partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

Now we have seen two ways of showing the Black-Scholes-Merton(BSM) equation. One way is to reproduce the payoff of the option using a portfolio that consists of a saving account. Another way is based on the risk-neutral pricing formula and Feynman-Kac formula. These two ways are equivalent. Because under risk-neutral measure, the payoff of a derivative is the same as a saving account, which imply we are able to reproduce the payoff using portfolio that consisting of a saving account.

1.3 Black-Scholes-Merton Model: Analytic Solution for European Option

1. European call option
For European call option with payoff to be $V(T) = S(T) - K$, with K as strike price, let us assume constant volatility σ, and constant interest rate r. Then we can obtain the solution to the BSM equation with martingale property without bothering solving the complex parital differential equation. The call option value satisfies

$$c(t, S(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^{+}|F(t)]$$

We write

$$\begin{aligned} S(T) &= S(t)exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)\tau\} \\ &= S(t)exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} \end{aligned}$$

Where Y is the stardard normal random variable and $\tau = T - t$ is the time to expiration.

$$Y = -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{T-t}}$$

So we see that $S(T)$ is the product of the $F(t)$ measurable random variable S(t) and random variable

$$exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\}$$

Which is independent of $F(t)$. Therefore based on risk-neutral pricing formula[1]

$$c(t, x) = \tilde{E}[e^{-r\tau}(xexp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^{+}]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\tau r}(xexp\{-\sigma\sqrt{\tau}y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^{+}e^{-\frac{1}{2}y^2}dy$$

After a little bit of math with integration, we have the solution to the Black-Scholes-Merton model for European call option

$$c(\tau, x; K, r, \sigma) = xN(d_{+}(\tau, x)) - e^{-r\tau}KN(d_{-}(\tau, x))$$

Where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}[ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})\tau]$$
$$d_2 = d_1 - \sigma\sqrt{\tau}$$

N() is the cumulative distribution function of the standard normail distribution
2. European put option
The payoff for the European put option is $V(T) = K - S(T)$, we can follow a similiar derivation and get the formula for put option

$$p(t, x) = N(-d_2)Ke^{-r\tau} - N(-d_1)x$$

3. Boundary conditons

Using the solution $c(t,x)$ and $p(t,x)$, we can easily check the boundary conditions when time t approaches to expiration time T.

As we know

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} ln(\frac{S}{K}) + \frac{1}{\sigma}(r + \frac{\sigma^2}{2}\sqrt{\tau})$$

When $\tau \to 0$, the second term decays much faster, so it vanishes. When $S > K$, d_1 goes to infinity, when $S < K$, d_1 goes to negative infinity. Therefore, when $S > K$

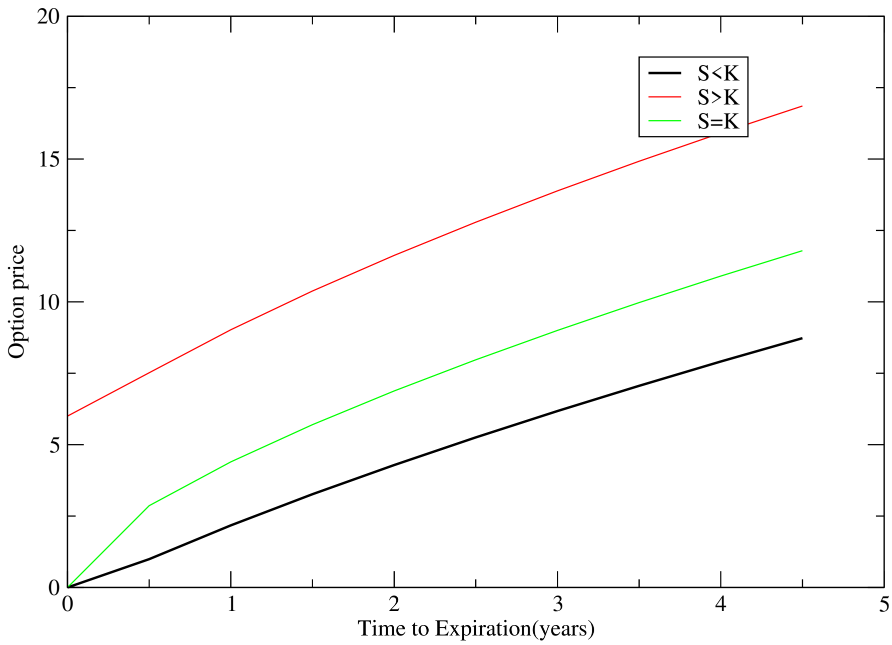
$$c(t,x) = S * N(+\infty) - K * N(+\infty) = S - K$$

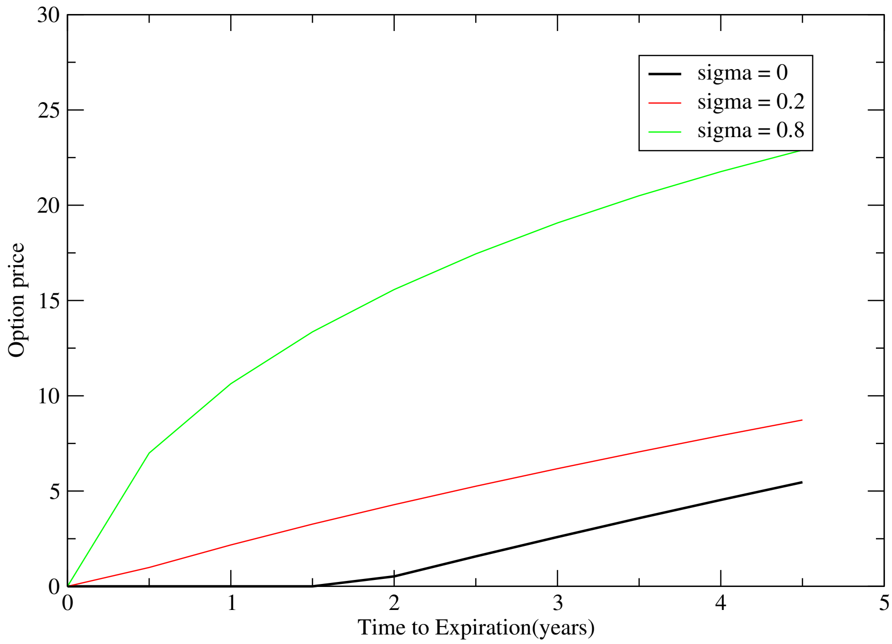
Therefore, when $S < K$

$$c(t,x) = S * N(-\infty) - K * N(-\infty) = 0 - 0 = 0$$

4. Examples

The following graphs show the change of option price with respect to different





5. Alternative formulation
If we introduce $F = e^{r\tau}S$, which is the forward price of the asset S. Then the equation pricing equation becomes

$$C(F,\tau) = D[N(d_+)F - N(d_-)K]$$
$$P(F,\tau) = D[N(-d_-)K - N(-d_+)F]$$
$$d_{+/-} = \frac{1}{\sigma\sqrt{\tau}}[ln(\frac{F}{K}) + / - \frac{1}{2}\sigma^2\tau]$$

The variables are:
$\tau = T - t$ is the time to expiry
$D = e^{-r\tau}$ is the discount factor

1.4 Heston Stochasic Volatility Model

The Black-Scholes equation assumes the volatility is constant, which is the ideal case and not practical in the real market. The Heston model assumes the volatility to follow a stochastic process. The following content refers to the paper: "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options" by Steven Heston. Suppose a stock price under risk-neutral measure is governed by

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)\tilde{d}W_1(t) \tag{2}$$

and the volatility itself is governed by the equation

$$dV(t) = (a - bV(t))dt + \sigma\sqrt{V(t)}\tilde{d}W_2(t) \tag{3}$$

Where

$$\tilde{d}W_1(t)\tilde{d}W_2(t) = \rho dt$$

At time t, the risk-neutral price of a call expiring at time $T \geq t$ in this model is

$$c(t, S(t), V(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+|F(t)]$$

If we move the term c^{rt} to the left hand side, we see

$$e^{-rt}c(t, S(t), V(t)) = \tilde{E}[e^{-rT}(S(T) - K)^+|F(t)] \tag{4}$$

which satisfies the martingale property. Then we take the differentiation of $e^{-rt}c(t, S(t), V(t))$. We get

$$\begin{aligned}
&d(e^{-rt}c(t, S(t), V(t)) \\
=&\frac{\partial e^{-rt}}{\partial t}c(t, S(t), V(t)) + e^{-rt}\frac{\partial c(t, S(t), V(t))}{\partial t}dt \\
=&-re^{-rt}c(t, S(t), V(t))dt \ \ (1) \\
+&e^{-rt}\frac{\partial c}{\partial t}dt \ \ (2) \\
+&e^{-rt}\frac{\partial c}{\partial S}dS \ \ (3) \\
+&e^{-rt}\frac{\partial^2 c}{\partial S^2}dSdS \ \ (4) \\
+&e^{-rt}\frac{\partial c}{\partial V}dV \ \ (5) \\
+&e^{-rt}\frac{\partial^2 c}{\partial V^2}dVdV \ \ (6) \\
+&e^{-rt}\frac{\partial^2 c}{\partial V \partial S}dVdS \ \ (7)
\end{aligned}$$

As we are interested in only the dt terms, we find out the dt terms from (1) to (7) the dt term in (1) is

$$-rc(t, S(t), V(t))e^{-rt}dt$$

the dt term in (2) is

$$\frac{\partial c}{\partial t}e^{-rt}dt$$

the dt term in (3) is

$$\frac{\partial c}{\partial S}rSe^{-rt}dt$$

the dt term in (4) is

$$\frac{1}{2}\frac{\partial^2 c}{\partial S^2}VS^2e^{-rt}dt$$

the dt term in (5) is

$$\frac{\partial c}{\partial V}(a - bV(t))e^{-rt}dt$$

the dt term in (6) is

$$\frac{1}{2}\frac{\partial^2 c}{\partial V^2}V\sigma^2 e^{-rt}dt$$

the dt term in (7) is

$$\frac{\partial^2 c}{\partial V \partial S}VS\sigma e^{-rt}dt$$

Collect all the dt terms and let those terms equal to zero, we get

$$c_t + rsc_s + (a - bv)c_v + \frac{1}{2}s^2 vc_{ss} + \rho\sigma svc_{sv} + \frac{1}{2}\sigma^2 vc_{vv} = rc \tag{5}$$

The function $c(t, s, v)$ satisfies boundary condition

$$\begin{aligned} c(T, s, v) &= (s - K)^+ \\ c(t, 0, v) &= 0 \\ c(t, s, 0) &= (s - e^{-r(T-t)}K)^+ \\ lim_{s\to\infty}\frac{c(t, s, v)}{s - K} &= 1 \\ lim_{v\to\infty}c(t, s, v) &= s \end{aligned}$$

Based on the solution to the BSM equation, we can guess that the solution has the following form

$$c(t, s, v) = sf(t, logs, v) - e^{-r(T-t)}Kg(t, logs, v) \tag{6}$$

Where f and g can be intepreted as a cumulative distribution function. Then since c(t,s,v) satisfies the partial differential equation 5, we can show that f and g satisfy the following

$$f_t + (r + \frac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v + \frac{1}{2}vf_{xx} + \rho\sigma vf_{xv} + \frac{1}{2}\sigma^2 vf_{vv} = 0 \tag{7}$$

$$\tag{8}$$

$$g_t + (r - \frac{1}{2}v)g_x + (a - bv)g_v + \frac{1}{2}vg_{xx} + \rho\sigma vg_{xv} + \frac{1}{2}\sigma^2 vg_{vv} = 0 \tag{9}$$

$$\tag{10}$$

The derivation is straightforward but one needs to keep in mind here we treat x and v as two independent variables. The above PDE for f and g satisfy boundary condition

$$\begin{aligned} f(T, x, v) &= 1_{x\geq logK} \\ g(T, x, v) &= 1_{x\geq logK} \end{aligned}$$

This implies that f and g can be intepreted as "Probabilities". We can define

$$f(t,x,v) = E^{t,x,v} 1_{x \geq logK}$$

We suppose a pair of stochastic process X(t), V(t) given by the following expression

$$dX(t) = (r + \frac{1}{2}V(t))dt + \sqrt{V(t)}dW_1(t)dV(t) = (a - bV(t) + \rho\sigma V(t))dt + \sigma\sqrt{V(t)}dW_2(t)$$

By F-K formula, we can show that f satisfies the PDE above. Similarly, we have

$$g(t,x,v) = E^{t,x,v} 1_{x \geq logK}$$

and the stochastic process of X(t) and V(t) are

$$dX(t) = (r - \frac{1}{2}V(t))dt + \sqrt{V(t)}dW_1(t)$$
$$dV(t) = (a - bV(t))dt + \sigma\sqrt{V(t)}dW_2(t)$$

To find the analytical solution of f(t, x, v) and g(t, x, v) is not an easy task. Instead, we do Fourier transform of f and g. First we work out the function f(t, x, v.). Let $\tau = T - t$

$$\tilde{f}(k,v,\tau) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x,v,\tau)$$

The inverse Fourier transform is

$$f(x,v,\tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k,v,\tau)$$

Substitute this into equation 7, then

$$-\frac{\partial \tilde{f}}{\partial \tau} + (r + \frac{1}{2})ik\tilde{f} + (a - bv + \rho\sigma v)\frac{\partial \tilde{f}}{\partial v} - \frac{1}{2}vk^2\tilde{f} + \rho\sigma vik\frac{\partial \tilde{f}}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \tilde{f}}{\partial v^2} = 0 \tag{11}$$

(12)

Now the problem is to solve for $\tilde{f}$. We note when $\tau = 0$,

$$\tilde{f}(k,v,0) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x,v,\tau = 0) = \int_{-\infty}^{\infty} dx e^{-ikx} 1_{x \geq logK} = \int_{0}^{\infty} dx e^{-ikx} = \pi\delta(k) + \frac{1}{ik}$$

when $\tau != 0$, we guess a general solution which has the following form

$$\tilde{f}(k,v,\tau) = exp(C\tau + D\tau v)\tilde{f}(k,v,0)$$

From above we easily see it match the terminal condition at $\tau \to 0$. With inverse Fourier transform

$$
\begin{aligned}
f(x, v, \tau, x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k, v, \tau) && (13)\\
&= \int_{0}^{\infty} \frac{dk}{\pi} e^{ikx} exp(C\tau + D\tau v) \tilde{f}(k, v, 0) && (14)\\
&= \int_{0}^{\infty} \frac{dk}{\pi} e^{ikx} exp(C\tau + D\tau v)(\pi\delta(k) + \frac{1}{ik}) && (15)\\
&= \frac{1}{2} + \int_{0}^{\infty} \frac{dk}{\pi} e^{ikx} exp(\frac{C\tau + D\tau v}{ik}) && (16)
\end{aligned}
$$

Now the only remaining task is to find C and D. If we substitute 13 into 11, we can get the expression C and D.

1.5 Jump Diffusion Model

The BSM model and the Heston model assumes the asset prices follows the stochastic process driven by Brownian motion. Under this assumption, the asset price is continuous in time. The jump diffusion model extends the BSM model and Heston model by assuming asset price has discrete jump in time. Therefore in jump diffusion model, the stochastic process contains a continuous Brownian motion process and a discrete jump process.

1. Number of jumps during per unit time: Possion distribution
The number of jump should be either zero or a positive integer. Let N be the number of jump per unit of time and k be a nonnegative integer. Then the probability of $N = k$ should satisfies

$$\sum_{k=0}^{\infty} P(N = k) = 1$$

In order to find the distribution, we borrow the idea of Taylor's expansion. We consider a function e^{λ}, its Taylor's expansion is

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

Dividing e^{λ} on both sides, we can

$$1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}$$

So we can assign the probability

$$P(N = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

This is the well-known Possion distribution. It is easy to show Possion distribution has mean and variance equal to λ. So λ can be intepreted as the average number of jumps per unit of time.

2. Number of jumps during time interval Δt: Possion process
Now we consider the number of jumps in a given time interval Δt, and we call it ΔN. The average jump during Δt is $\lambda \Delta t$. So we can get the probability distribution by substitude $\lambda \Delta t$ to λ

$$P(\Delta N = k) = \frac{(\lambda \Delta t)^k}{k!} e^{-\lambda \Delta t}$$

The mean of ΔN is $\lambda \Delta t$.

Let Δt be really small time interval dt, and from above we see

$$\begin{aligned} P(dN = 0) &= e^{-\lambda dt} \approx 1 - \lambda dt \\ P(dN = 1) &= \lambda dt e^{-\lambda dt} \approx \lambda dt(1 - \lambda dt) \approx \lambda dt \\ P(dN = 2) &= (-\lambda dt)^2 e^{-\lambda dt} \approx 0 \end{aligned}$$

This makes perfectly sense, as during an infinitesimal time interval, the jump can either not happen or happen only once.

We define S_k is the time when the kth jump occurs and we assume $S_0 = 0$. Then the Possion process is

$$N(t) = k \text{ if } S_k \leq t < S_{k+1}$$

With $E[N(t)] = \lambda t$. Note the Poisson process defined above is right-continuous.

3. Compensated Poisson process
The Poisson process has a mean that is a function of time. As time evolves, the process has a non-zero drift so it is not a martingale. We define a compensated Poission process $M(t)$ which obeys the martingale property

$$\begin{aligned} M(t) &= N(t) - \lambda t \\ E[M(t)|F(s)] &= M(s) \end{aligned}$$

4. Compound Poisson process
In the above definition, we assume the jump size in the Possion process is 1. Now we allow the jump size to be random. Let N(t) be a Poisson process with mean λ and Y_i be a sequence of random variable with mean $E[Y] = m$. We define compound Poisson process

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

The number of jumps follows Poisson distribution, and at each jump, the size of the jump is Y_i. In other words, the first jump size is Y_1 and the second jump size is Y_2, etc.

The mean of the compound Poisson process is the product of the number of jumps and the size of the jumps, as these two are independent.

$$EQ(t) = m\lambda t$$

We can also define a compensated compound Poisson process

$$\tilde{M}(t) = Q(t) - m\lambda t$$

And this is a martingale.

5. Geometric Possion process with constant jump size
In the compound Possion process, let the jump size be a constant y. When jump occurs, we have

$$Q(t_i) = Q(t_{i-}) + y$$

The asset value in the financial market follows the geometric Possion process. Let S(t) be the asset price, let

$$\begin{aligned}
&S(t_i) = S(t_{i-})y \\
&\frac{S(t_i) - S(t_{i-})}{S(t_{i-})} = y - 1 \\
&\Delta S(t_i) = (y-1)S(t_{i-})\Delta N(t)
\end{aligned}$$

Where $\Delta N(t) = 1$ when jump occurs, 0 otherwise. When $N(t) = 1$, there is one jump from time zero up to t, so $S(t) = yS(0)$. When $N(t) = 2$, $S(t) = y^2S(0)$. So for any abitrary $N(t)$

$$S(t) = S(0)y^{N(t)}$$

In order to fulfill the martingale condition, we write

$$dS(t) = (y-1)S(t)dN(t) - (y-1)\lambda S(t)dt$$

Its integral form is

$$S(t) = S(0)y^{N(t)}exp(-(y-1)\lambda t)$$

6. Geometric Poisson process with log-normal jump size: Bates jump diffusion model
If the jump size Y is random in the geometric Poisson process, then to model the asset price, we need to modify the last equation by changing $y^{N(t)}$ to $\Pi_{i=1}^{N(t)}Y_i$, and $y-1$ to E[Y] - 1.

$$S(t) = S(0)\Pi_{i=1}^{N(t)}Y_i exp(-(E[Y]-1)\lambda t)$$

We need to model the jump size Y_i. Y_i is a strictly positive random variable so a good candidate is the exponential of normal random variable which is log-nomal random variable. If we have a normal distributed random variable $J \sim N(\nu, \delta^2)$, and let Y be e^J. Then Y follows log-normal distribution with mean $e^{\nu+\frac{1}{2}\delta^2}$. We can write S(t) using J

$$S(t) = S(0)exp\sum_{i}^{N(t)} J_i exp(-(e^{\nu+\frac{1}{2}\delta^2} - 1)\lambda t)$$

Let $Z = \sum_i^{N(t)} J_i$, which is a sum of normal random variable. Then $Z \sim Normal(N(t)\nu, N(t)\delta^2)$. So

$$S(t) = S(0)exp(Z - (e^{\nu+\frac{1}{2}\delta^2} - 1)\lambda t)$$

Where $Z \sim Normal(N(t)\nu, N(t)\delta^2)$ given $N(t) \sim Poisson(\lambda t)$.

2 Simulation Processes

In simulation, we decompose the time evolvement of an asset price into three components: drift, diffusion, jump. The drift term is dt term, the diffusion term has Brownian motion component, the jump term has Possion component. Let us look at the following examples.
1. BSM process
In BSM process, the asset price follows

$$S(t) = S(0)exp((r - \frac{1}{2}\sigma^2)dt + \sigma dW(t))$$

The drift term is clearly $r - \frac{1}{2}\sigma^2$ for any given dt. For the diffusion term, we need to generate W(t). We know the W(t) is a Normal random variable with mean 0 and variance t. So suppose we generate a Gaussian random variable $\hat{W}$, it turns out $W = \sqrt{t}\hat{W}$ is normally distributed with variance t.

$$\begin{aligned} &\mu = r - \frac{1}{2}\sigma^2 \\ &S^{(t+dt)} = S^{(t)} exp(\mu dt + \sigma\sqrt{dt}\hat{W}) \end{aligned}$$

2. Heston process
The Heston model states

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{V(t)}S(t)\tilde{d}W_1(t) \\ dV(t) &= (a - bV(t))dt + \sigma\sqrt{V(t)}\tilde{d}W_2(t) \\ \tilde{d}W_1(t)\tilde{d}W_2(t) &= \rho dt \end{aligned}$$

a. Generating two Brownian motions with correlation.
To simulate Heston process needs to generate two Gaussian random variables that has correlation ρ, we do this by first generate two independent Gaussian random variables W_1, W_2, then

$$W_1^{'} = W_1 W_2^{'} = \rho W_1 + \sqrt{1-\rho^2}W_2$$

We can easily check the mean of $W_2^{'}$

$$E[W_2^{'}] = \rho E[W_1] + \sqrt{1-\rho^2}E[W_2] = 0$$
$$Var[W_2^{'}] = \rho^2 Var[W_1] + (1-\rho^2)Var[W_2] = \rho^2 + 1 - \rho^2 = 1$$

$$\begin{aligned}
cor(W_1^{'}, W_2^{'}) &= \frac{E[W_1^{'}W_2^{'}]}{\sqrt{Var[W_1^{'}]}\sqrt{Var[W_2^{'}]}} \\
&= E[W_1(\rho W_1 + \sqrt{1-\rho^2})W_2] \\
&= \rho E[W_1^2] + \sqrt{1-\rho^2}E[W_1W_2] \\
&= \rho
\end{aligned}$$

The last step uses the fact W_1 and W_2 are independent, so $E[W_1W_2] = E[W_1]E[W_2] = 0$.

b. Coping with negative V(t).

The Heston model setting provides a non-negative V(t). This can be seen when V(t) is positively moving close to zero, the drift term adt pulls the V(t) up and avoid $V(t)$ to cross zero. However, in discretized version $V(t)$ may attain negative values. There exists several methods to deal with this issue.

(1) Absorption: Keep positive part of the previous $V(t)$ and use it to calculate next step $V(t+dt)$ and X(t+dt)

$$V^{(t+dt)} = V^{+(t)} + (a - bV^{+(t)})\sqrt{dt} + \sigma\sqrt{V^{+(t)}}dt(\rho\hat{W}_1 + \sqrt{1-\rho^2}\hat{W}_2)$$

(2) Relection: Keep the absolute value of the previous $V(t)$ and use it to calculate next step $V(t+dt)$ and X(t+dt)

$$V^{(t+dt)} = |V^{(t)}| + (a - b|V^{(t)}|)\sqrt{dt} + \sigma\sqrt{|V^{(t)}|}dt(\rho\hat{W}_1 + \sqrt{1-\rho^2}\hat{W}_2)$$

(3) Partial truncation: Using the posivive part in diffusion term of $V(t+dt)$ and in $X(t+dt)$

$$V^{(t+dt)} = V^{(t)} + (a - bV^{(t)})\sqrt{dt} + \sigma\sqrt{V^{+(t)}}dt(\rho\hat{W}_1 + \sqrt{1-\rho^2}\hat{W}_2)$$

(4) Full truncation: Using the positive part in drift and diffusion terms of $V(t+dt)$, and in $X(t+dt)$. This scheme is proved to have the lowest bias. Reference: A comparison of biased simulation schemes for stochastic volatility models by Roger Lord, Quantitative Finance,2010.

$$V^{(t+dt)} = V^{(t)} + (a - bV^{+(t)})\sqrt{dt} + \sigma\sqrt{V^{+(t)}}dt(\rho\hat{W}_1 + \sqrt{1-\rho^2}\hat{W}_2)$$

Together with formula for the drift and asset price, we write

$$\mu = r - \frac{1}{2}(V^{+(t)})$$
$$S^{(t+dt)} = S^{(t)}exp(\mu dt + \sqrt{V^{+(t)}}\sqrt{dt}W_1)$$
$$V^{(t+dt)} = V^{(t)} + (a - bV^{+(t)})\sqrt{dt} + \sigma\sqrt{V^{+(t)}}dt(\rho\hat{W}_1 + \sqrt{1-\rho^2}\hat{W}_2)$$

3. Bates process

$$
\begin{aligned}
&\mu = r - \frac{1}{2}(V^{+(t)}) - \lambda(e^{\nu+\frac{1}{2}\delta^2} - 1)\\
&S^{(t+dt)} = S^{(t)} exp(\mu dt + \sqrt{V^{+(t)}}\sqrt{dt} dW_1 + (\nu_* N + \delta^2 * \sqrt{N}) * dW_3)\\
&V^{(t+dt)} = V^{(t)} + (a - bV^{+(t)})dt + \sigma\sqrt{V^{+(t)}} dt(\rho\hat{W}_1 + \sqrt{1-\rho^2}\hat{W}_2)
\end{aligned}
$$