THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

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Abstract

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

Dr. J. B. Johnson[1] has reported the discovery and measurement of an electromotive force in conductors which is related in a simple manner to the temperature of the conductor and which is attributed by him to the thermal agitation of the carriers of electricity in the conductors. The work to be resported in the present paper was undertaken after Johnson's results were available to the writer and consists of a theoretical deduction of the electromotive force in question from thermodynamics and statistical mechanics.[2]

Consider two conductors each of resistance R and of the same uniform temperature T connected in the manner indicated in Fig. 1. The electromotive force due to thermal agitation in conductor I causes a current to be set up in the circuit whose value is obtained by dividing the electromotive force by $2R$. This current causes a heating or absorption of power in conductor II, the absorbed power being equal to the product of R and the square of the current. In other words power is transferred from conductor I to conductor II. In precisely the same manner it can be deduced that power is transferred from conductor II to conductor I. Now since the two conductors are at the same temperature it follows directly from the second law of thermodynamics that the power flowing in one direction is exactly equal to that flowing in the other direction. It will be noted that no assumption has been made as to the nature of the two conductors. One may be made of silver and the other of lead, or one may be metallic and the other electrolytic, etc.

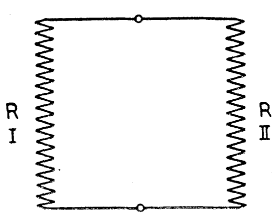

Fig. 1.

It can be shown that this equilibrium condition holds not only for the total power exchanged by the conductors under the conditions assumed, but also for the power exchanged within any frequency. For, assume that this is not so and let A denote a frequency range in which conductor I delivers more power than it receives. Connect a non-dissipative network between the two conductors so designed as to interfere more with the transfer of energy

* A preliminary report of this work was presented before the Physical Society in February, 1927.

[1] See preceding paper.

[2] Cf. W. Schottky, Ann. d. Physik 57, 541 (1918).

in range A than in any other range, for example, a resonant circuit connected as indicated in Fig. 2 and resonant within the range A. Since there is equilibrium between the amounts of power transferred in the two directions before inserting the network, it follows that after the network is inserted more power would be transferred from conductor II to the conductor I than in the opposite direction. But since the conductors are at the same temperature, this would violate the second law of thermodynamics. We arrive, therefore, at the important conclusion that the electromotive force due to thermal agitation in conductors is a universal function of frequency, resistance and temperature and of these variables only.[3]

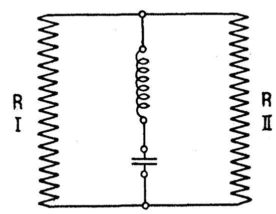

Fig. 2.

To determine the form of this function consider again two conductors each of resistance R connected as shown in Fig. 3 by means of a long non-dissipative transmission line, having an inductance L and a capacity C per unit length so chosen that $(L/C)^{1/2}=R$. In order to avoid radiation one conductor may be internal to the other. Under these conditions the lines has the characteristic impedance R, that is to say the impedance of any length of line when terminated at the far end in the impedance R presents the impedance R at the near end and consequently there is no reflection at either end of the line. Let the length of the line by l and the velocity of propagation

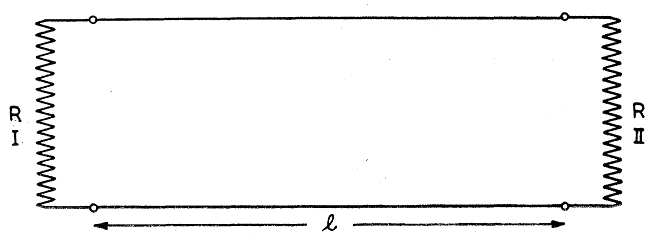

Fig. 3.

v. After thermal equilibrium has been established, let the absolute temperature of the system be T. There are then two trains of energy traversing the transmission line, one from left to right in the figure, being the power delivered by conductor I and absorbed by the conductor II, and another train in the reverse direction.

At any instant after equilibrium has been established, let the line be isolated from the conductors, say, by the application of short circuits at the two ends. Under these conditions there is complete reflection at the two ends and the energy which was on the line at the time of isolation remains trapped. Now, instead of describing the waves on the line as two trains traveling in opposite directions, it is permissible to describe the line as vibrating at its natural frequencies. Corresponding to the lowest frequency

[3] For a general treatment of the principle underlying the discussion of this paragraph reference is made to P. W. Bridgman, Phys. Rev. **31**, 101 (1928).

the voltage wave has a node at each end and no intermediate nodes. The frequency corresponding to this mode of vibration is $v/2l$. The next higher natural frequency is $2v/2l$. For this mode of vibration there is a node at each end and one in the middle. Similarly there are natural frequencies $3v/2l$, $4v/2l$, etc. Consider a frequency range extending from ν cycles per second to $\nu+d\nu$ cycles per second, i.e., a frequency range of width $d\nu$. The number of modes of vibration, or degrees of freedom, lying within this range may be taken to be $2ld\nu/v$, provided l is taken sufficiently large to make this expression a great number. Under this condition it is permissible to speak of the average energy per degree of freedom as a definite quantity. To each degree of freedom there corresponds an energy equal to kT on the average, on the basis of the equipartition law, where k is the Boltzmann constant. Of this energy, one-half is magnetic and one-half is electric. The total energy of the vibrations within the frequency interval $d\nu$ is then seen to be $2lkTd\nu/v$. But since there is no reflection this is the energy within that frequency interval which was transferred from the two conductors to the line during the time of transit l/v. The average power, transferred from each conductor to the line within the frequency interval $d\nu$ during the time interval l/v is therefore $kTd\nu$.

It was pointed out above that the current in the circuit of Fig. 1 due to the electromotive force of either conductor is obtained by dividing the electromotive force by $2R$, and that the power transferred to the other conductor is obtained by multiplying the square of the current by R. If the square of the voltage within the interval $d\nu$ be denoted by $E^2d\nu$ we have, therefore

$$E^2d\nu = 4RkTd\nu \tag{1}$$

This is the expression for the thermal electromotive force in a conductor of pure resistance R and of temperature T. Let it next be required to find the corresponding expression for any network built up of impedance members of the common temperature T. Let the resistance R be connected, as shown in Fig. 4, to any such network having the impedance $R_\nu+iX_\nu$, where R_ν and X_ν may be any function of frequency. By reasoning entirely similar to that used above it is deduced that the power transferred from the conductor to the impedance network is equal to that transferred in the opposite direction. But the former is shown by simple circuit theory to be equal to

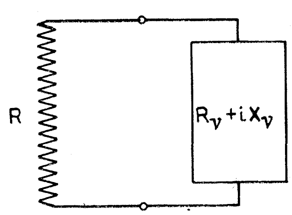

Fig. 4.

$$E^2R_\nu d\nu/[(R+R_\nu)^2+X_\nu^2] \tag{2}$$

and the latter is similarly equal to

$$E_\nu^2Rd\nu/[(R+R_\nu)^2+X_\nu^2] \tag{3}$$

where $E_\nu^2d\nu$ is the square of the voltage within the frequency range $d\nu$. It follows that

$$E_\nu^2 d\nu = 4R_\nu kT d\nu \tag{4}$$

for any network.

To put this relation in a form suitable for comparison with measurements let $Y(\omega)$ be the transfer admittance of any network from the member in which the electromotive force in question originates to a member in which the resulting current is measured. Let $\omega = 2\pi\nu$ and let $R(\omega) = R_\nu$ be the resistance of the member in which the electromotive force is generated. We have then for the square of the measured current within the interval $d\nu$

$$I^2 d\nu = E_\nu^2 \mid Y(\omega) \mid^2 d\nu = (2/\pi) kTR(\omega) \mid Y(\omega) \mid^2 d\omega \tag{5}$$

Integrating from 0 to ∞

$$I^2 = (2/\pi) kT \int_0^\infty R(\omega) \mid Y(\omega) \mid^2 d\omega \tag{6}$$

which is Eq. (1) in Johnson's paper.

It will be noted that such quantities as charge, number, and mass of the carriers of electricity do not appear explicitly in the formula for electromotive force These quantities influence R_ν, however, and, therefore, enter indirectly.

It is instructive to consider the equilibrium between the thermal agitation of the carriers of electricity in a conductor and the thermal agitation of molecules in a gas. Consider a semi-infinite tube filled with gas of temperature T and let the end be closed in a weightless inflexible piston forming the diaphragm of an ideal non-dissipative telephone receiver having no magnetic leakage. Such a receiver presents an electrical impedance which is a function of the mechanical impedance of the gas in the tube and which may be taken as R by choosing a suitable number of turns for the receiver element. Due to the bombardment of the diaphragm by the molecules in the gas, there will be an electromotive force at the terminals of the receiver. This electromotive force is, of course, in statistical equilibrium with that due to thermal agitation in a conductor of resistance R. It follows that it should be possible to calculate that electromotive force from the kinetic theory of gases, but this calculation would not be so direct as that given above, making use of a transmission line.

In what precedes the equipartition law has been assumed, assigning a total energy per degree of freedom of kT. If the energy per degree of freedom be taken

$$h\nu/(e^{h\nu/kT} - 1) \tag{7}$$

where h is the Planck constant, the expression for the electromotive force in the interval $d\nu$ becomes

$$E_\nu^2 d\nu = 4R_\nu h d\nu/(e^{h\nu/kT} - 1). \tag{8}$$

Within the ranges of frequency and temperature where experimental information is available this expression is indistinguishable from that obtained from the equipartition law.

American Telephone and Telegraph Company,
April, 1928.