1 Jacobi Method for Solving Eigenvalues

a. Intuition
One can understand the Jacobi Method by rotation matrix. From geometry perspective, a diagonal matrix can be rotated to an non-diagonal matrix. So it is natural to think for a non-diagonal matrix, we can rotate it back to a diagonal matrix. This can be done by similar transformation using rotation matrix.
b. Eligibility
Since we apply similar transformation by rotation matrix and eventually we can the diagonal matrix which is symmetric, the original matrix has to be symmetrical.
c. Algorithm
The Jacobi iteration for a matrix A is

$$A^{(k)} = G^T_{pkqk}(\theta_k)A^{k-1}G_{pkqk}(\theta_k)$$

Where

$$G_{pq}(\theta) = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

It is an Identity matrix replaced by an rotation matrix on pth and qth columns and rows. The iteration is chosen to reduce the sum of the squares of the off-diagonal elements, which for any square matrix A is

$$||A||_F^2 - \sum_i a_{ii}^2$$

The orthogonal similarity transforms preserve the Frobenius norm

$$||A^{(k)}||_F = ||A^{(k-1)}||_F$$

Because the rotation matrix change only (p,p), (q,q), (p,q), (q,p) positions. We have

$$(a_{pp}^{(k)})^2 + (a_{qq}^{(k)})^2 + 2(a_{pq}^{(k)})^2 = (a_{pp}^{(k-1)})^2 + (a_{qq}^{(k-1)})^2 + 2(a_{pq}^{(k-1)})^2$$

The off-diagonal sum of squares at the kth stage in terms of that at k-1 th stage is

$$\begin{aligned} &||A^{(k)}||_F^2 - \sum_i (a_{ii}^{(k)}) \\ =&||A^{(k)}||_F^2 - \sum_{i \neq p,q} (a_{ii}^{(k)}) - ((a_{pp}^{(k)})^2 + (a_{qq}^{(k)})^2) \\ =&||A^{(k)}||_F^2 - \sum_i (a_{ii}^{(k-1)}) - 2(a_{pq}^{(k-1)})^2 + 2(a_{pq}^{(k)})^2 \end{aligned}$$

In order to minimize this, we need

$$a_{pq}^{(k)} = 0$$
$$a_{pq}^{(k-1)} = max_{i<j}|a_{ij}^{(k-1)}|$$

This implies

$$a_{pq}^{(k-1)}(cos^2\theta - sin^2\theta) + (a_{pp}^{k-1} - a_{qq}^{k-1})cos\theta sin\theta = 0$$

Solve for θ

$$tan(2\theta) = \frac{2a_{pq}^{(k-1)}}{a_{pp}^{k-1} - a_{qq}^{k-1}} tan(\theta) = \frac{tan(2\theta)}{1 + \sqrt{1 + tan^2(2\theta)}} cos\theta = \frac{1}{\sqrt{1 + tan^2\theta}} sin\theta = cos\theta tan\theta$$