1 Transmission Line

The transmission line in general refers two conductor line with some kind of dielectric materials in between. The two conductor lines build oscillating voltage across each other and a E-M waves is transmitted inside the dielectric materials. When the oscillating voltage signal has a frequency high enough such that the length of conducting wire is comparable to signal's wavelength, we have to use the transmission line to transmit the signal. The reason is
(1) When at high frequency, due to skin effect, the signal only lies on and near the surface of the conduction, inside the conductor, the amplitude of the signal decays very rapidly. This effect cause the signal to lose most of the power when being conducting use wires.
(2) When at high frequency, the wire has a high inductance impedance, the conductor can not be treated as it is in the case of DC.

1.1 Transmission Line Model

a. Wave equation
For most common transmission lines, they can be considered as a set of series inductors, shunt capacitance and resistors:
The distributed resistance R of the conductors is represented by a series resistor (expressed in ohms per unit length).
The distributed inductance L (due to the magnetic field around the wires, self-inductance, etc.) is represented by a series inductor (in henries per unit length).
The capacitance C between the two conductors is represented by a shunt capacitor (in farads per unit length).
The conductance G of the dielectric material separating the two conductors is represented by a shunt resistor between the signal wire and the return wire (in siemens per unit length).
We consider an ideal case in which the resistance R and G are negligible meaning the transmission line is lossless. And we also assume the inductor and capacitor both have unit length. Based on Maxwell's equation, at a certain point z of the transmission line, the voltage across the two conductor line V and the current I satisfy the following equation

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$
$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

It is possible to show that $V(z)$ and $I(z)$ satisfy

$$V(z) = V_1 e^{-jkz} + V_2 e^{+jkz}$$
$$I(z) = \frac{V_1}{Z_0} e^{-jkz} - \frac{V_2}{Z_0} e^{+jkz}$$

Where $Z_0 = \sqrt{\frac{L}{C}}$
b. Characteristic impedance
In DC circuit, the well-known Ohm's law tells us there is a simple and linear

relationship between the current and voltage. In an AC circuit with capacitors and inductors, however, both voltage and current change over time, which prevents us to apply Ohm's law directly as the current voltage relationship are not linear anymore. That is why we define the characteristic impedance to generalize the Ohm's in AC circuit.
Based on the derivation above, the characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}}$$

it is good to note that the characteristic impedance is it does not depend on length. From the previous derivation we assume the capacitance and inductance take the unit length. If they have real length Δz, then the characteristic impedance becomes

$$Z_0 = \sqrt{\frac{L\Delta z}{C\Delta z}} = \sqrt{\frac{L}{C}}$$

As we see the lengths are cancelled out leaving the impedance length independent.

1.2 Reflection Coefficients and Impedance Matching

a. Reflection coefficients
When the voltage propagates along the transmission line, eventually it will hit the end of the transmission line. What happens next and how do we solve it? In general, every EM problem can be answered by solving Maxwell's equations plus boundary conditions. In this particular case, we can solve it using wave equation derived above based on Maxwell's equations plus the Kirchhoff's law as our boundary conditions. When a wave reaches the boundary of two different media, there usually exists reflection wave. Here we define the reflection coefficient as ratio of refection voltage to the incident voltage. If a resistor Z_L is connected across two conductor wires at the end of the transmission line, then the refection coefficients can be derived

$$\Gamma = \frac{Z_L + Z_0}{Z_L - Z_0}$$

b. Refection examples in special cases
(1) When $Z_L = \infty$
When $Z_L = \infty$, the end is open. When a current flows to the end point, the Kirchhoff's law requires that there must be an current with same magnitude but in opposite direction. Therefore, the voltage of reflection wave V_r is equal to the voltage of the incident wave V_i

$$V_r = V_i$$

In this case $\Gamma = 1$. We can check this by the definition of Γ

$$\Gamma = \frac{Z_L + Z_0}{Z_L - Z_0} = \frac{\infty + Z_0}{\infty - Z_0} = 1$$

(2) When $Z_L = 0$
When $Z_L = 0$, at the end point, the voltage is always zero. By Kirchhoff's law,

there must be a refection wave that cancels the incident wave such that the total voltage vanishes. So

$$V_r = -V_i$$

In this case, $\Gamma = -1$. We can also check this by the definition of Γ

$$\Gamma = \frac{Z_L + Z_0}{Z_L - Z_0} = \frac{0 + Z_0}{0 - Z_0} = -1$$

c. Smith Chart The reflection coefficient Γ is a complex number and we can write

$$\Gamma = |\Gamma| e^{j\theta}$$

If we define a normalized coefficient $z = \frac{Z_L}{Z_0}$, then

$$z = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{j\theta}}$$

Let $\Gamma = \Gamma_r + j\Gamma_i$, and $z = r + jx$, we have

$$r + jx = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - \Gamma_i}$$

If we equal the real part and imaginary part on both sides,

$$(\Gamma_r - \frac{r}{1+r})^2 + \Gamma_i^2 = (\frac{1}{1+r})^2$$
$$(\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{x})^2 = (\frac{1}{x})^2$$

If we plot the $\Gamma = 1$ circle and two above circle in a complex coordinate graph, we get **Smith Chart**. Any arbitrary point on the circle represent a value of z.

2 S parameters

Consider a two port system with port 1 and port 2. For example, the system could be a band pass filter with one port as input, and the other port as output. We define a_1 and a_2 to be the incident waves and the b_1 and b_2 to be the reflected waves with the subscript being port number. In this case the relationship between the reflected, incident power waves and the S-parameter matrix is given by:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

From definition above, we clearly see
when $a_2 = 0$, then S_{11} is the reflection coefficient of port 1. when $a_1 = 0$, then S_{22} is the reflection coefficient of port 2.