1 Linear Regression

1.1 Linear regression and Least Square Solution

$$Y = X\boldsymbol{\beta} + \epsilon$$

Where Y is a n × 1 matrix, X is a n × k matrix, beta is k × 1 vector and ϵ is nx1 vector with ϵ_i begin iid with normal distribution.

Assumptions

1. Linear
2. X matrix has full rank. In other words, no multicollinearity.
2. error term has zero mean $E[\epsilon|X] = 0$
3. Homescedasticity or equal variance of ϵ. In other words, no autocorrelation between disturbances.$cov(\epsilon_i, \epsilon_j) = 0$.
6. Number of obsearvations n must be greater than the number of parameters.

Least Square Solution

The cost function is given by

$$f(\boldsymbol{\beta}) = ||Y - X\beta||^2 = (Y - X\beta)^T(Y - X\beta) \quad = Y^TY - Y^TX\beta - \beta^TX^TY + \beta^TX^TX\beta$$

Since third term are scalar,

$$\beta^TX^TY = (\beta^TX^TY)^T = Y^TX\beta$$

$$f(\beta) = Y^TY - 2Y^TX\beta - \beta^TX^TX\beta = Y^TY - 2(X^TY)^T\beta + \beta^TX^TX\beta$$

The first term is a constant and its derivative is zero.

The deriviative of 2nd term

Consider the derivative of $\alpha^T\beta$ with respect to β.

$$\boldsymbol{\alpha}^T\boldsymbol{\beta} = \Sigma\alpha_i\beta_i$$
$$\frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_i} = \alpha_i$$

Write the derivative in matrix form

$$\begin{pmatrix} \frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_1} \\ \frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_2} \\ \cdots \\ \frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_3} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_p \end{pmatrix}$$

So if we let $\alpha = X^TY$, we have

$$\frac{\partial 2(X^TY)^T\beta}{\partial\beta} = 2X^TY$$

The derivative of 3rd term

let $A = X^TX$,

$$\beta^TX^TX\beta = \beta^T\begin{pmatrix} \Sigma_iA_{1k}\beta_k \\ \Sigma_iA_{2k}\beta_k \\ \cdots \\ \Sigma_kA_{pk}\beta_k \end{pmatrix} = \Sigma_j\beta_j(\Sigma_kA_{jk}\beta_k)$$

To calculate the derivative of $f(\beta)$, we note there are only 3 cases that the derivative does not vanish
1) l = j = k

$$\frac{f(\boldsymbol{\beta})}{\partial \beta_l} = 2A_{ll}\beta_l$$

2) l=j, j $\neq$ k

$$\frac{f(\boldsymbol{\beta})}{\partial \beta_l} = \Sigma_{k,k\neq l} A_{lk}\beta_k$$

3) l=k, j $\neq$ k

$$\frac{f(\boldsymbol{\beta})}{\partial \beta_l} = \Sigma_{j,j\neq l} A_{jl}\beta_j = \Sigma_{j,j\neq l} A^T_{lj}\beta j$$

Therefore

$$\begin{aligned}\frac{f(\boldsymbol{\beta})}{\partial \beta_l} &= A_{ll}\beta_l + \Sigma_{k,k\neq l} A_{lk}\beta_k + A_{ll}\beta_l + \Sigma_{j,j\neq l} A^T_{lj}\beta j \\ &= \Sigma_k A_{lk}\beta_k + \Sigma_j A^T_{lj}\beta j\end{aligned}$$

The first term is the lth row of vector $A\beta = X^T X\beta$, and the 2nd term is the lth row of vector$A^T\beta = X^T X\beta$. So we put the whole derivative in matrix form

$$\frac{f(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2X^T Y + 2X^T X\beta$$

which is a px1 vector with each row corresponding to the derivative with respect to β_i letting the derivative equal to zero yields the **normal equation** and the estimation of β
Normal equation

$$(X^T X)\hat{\beta} = X^T Y$$

Estimator of β

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Least Square Estimator for Simple Linear Regression

$$y = \beta_o + \beta_1 X + \epsilon$$

$$\begin{aligned}&\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \\ &= (X^T X)^{-1} X^T Y \\ &= \left(\begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_n \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix} \\ &= \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{pmatrix} \Sigma_i x_i^2 & -\Sigma_i x_i \\ -\Sigma_i x_i & n \end{pmatrix} \begin{pmatrix} \Sigma_i y_i \\ -\Sigma x_i y_i \end{pmatrix}\end{aligned}$$

So

$$\beta_1 = \frac{\Sigma x_i^2 \Sigma y_i - \Sigma x_i(\Sigma x_i y_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$
$$\beta_2 = \frac{-\Sigma x_i \Sigma y_i + n\Sigma x_i y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

1.2 Projection matrix

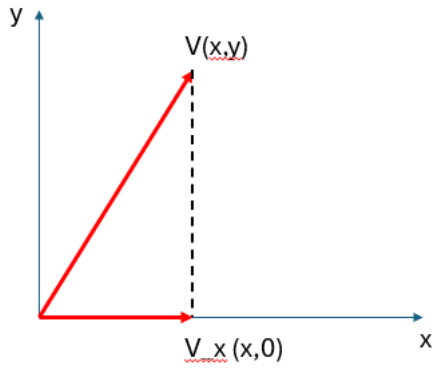
Given $\hat{\beta} = (X^T X)^{-1} X^T Y$, we have the predictor value of $y = X\beta$

$$\hat{y} = X(X^T X)^{-1} X^T y$$

The matrix $P = X(X^T X)^{-1} X^T$ is a projection matrix. It projects the vector of y into the column space of X.

Understand the word projection

Let us understand this first through geometry point of view. Consider a vector on 2 dimensional space, $V_1 = (x_1, y_1)^T$, where x_1 and y_1 are the x and y component, respectively. If we project the vector V into x-line, then apparently we get $V_x = (x_1, 0)^T$, see graph below.



If we have a vector that is along the x axis

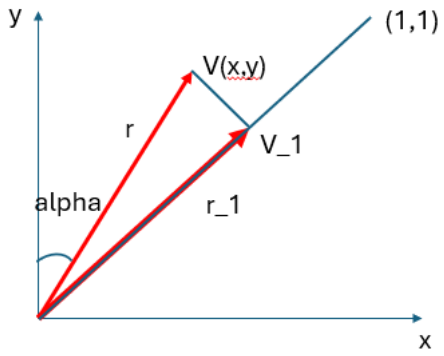
$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The projection matrix of a vector into x line is

$$\begin{aligned} P_x &= x(x^T x)^{-1} x^T \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Applying this projection matrix to any 2 dimensional vector V gives $(V_x, 0)^T$. So it projects the vector into x line. Let us take another example. Imagine V_1

is vector if we project V onto the line that has 45 degree angle with x axis. See below.



In order to calculate V_1, we see

$$r_1 = rcos(\pi/4 - alpha) = r(\frac{\sqrt{2}}{2}\frac{y}{r} + \frac{\sqrt{2}}{2}\frac{x}{r}) = \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}x$$

$$V_{1x} = r_1 cos(\pi/4) = \frac{x+y}{2}$$
$$V_{1y} = r_1 sin(\pi/4) = \frac{x+y}{2}$$

After we understand this using geometry point of view, we can workout from algebra point of view. The vector we want to project onto is

$$i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The projection matrix of a vector into x line is

$$\begin{aligned} P_x &= x(x^T x)^{-1} x^T \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Therefore we easily see

$$V_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x+y) \end{pmatrix}$$

which is the same as what we get based on geometry. For n dimensional vector y, if our X matrix has rank of k, then the projection matrix P projects the vector

y into k dimensional hyperplane. For example, if we define

$$i_N = \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix}$$

The projection matrix P is

$$P = i\frac{1}{N}i^T = \frac{1}{N}\begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ ... & ... & ... & ... \\ 1 & 1 & ... & 1 \end{pmatrix}$$

Projection matrix into null space
If P is a projection matrix, the matrix $I - P$ is also a projection matrix. In linear regression model

$$\begin{aligned} y &= X\beta + \epsilon \\ P &= X(X^TX)^{-1}X^T \\ \hat{\epsilon} &= (I-P)y = (I - X(X^TX)^{-1}X^T)y \end{aligned}$$

For the above example, we define $M = I - \frac{1}{N}ii^T$, and My express the mean deviations of a vector.
Idempotent property of projection matrix
Consider the previous example that we project a vector V onto x axis, how about we do this projection twice, we would end up the same vector V_x. Using a little matrix algebra, it is easy to prove that for any project matrix P, we have $PP = P$.

1.3 Partitioned Regression and Partial Regression

$$y = X\boldsymbol{\beta} + \epsilon = X_1\beta_1 + X_2\beta_2 + \epsilon$$

The normal equation is

$$\begin{pmatrix} X_1^TX_1 & X_1^TX_2 \\ X_2^TX_1 & X_2^TX_2 \end{pmatrix}\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1^Ty \\ X_2^Ty \end{pmatrix}$$

If X1 and X2 are orthogonal, namely, $X_1^TX_2 = 0$, then

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{\mathbf{1}} &= (X_1^TX_1)^{-1}X_1^Ty \\ \hat{\boldsymbol{\beta}}_{\mathbf{2}} &= (X_2^TX_2)^{-1}X_2^Ty \end{aligned}$$

$$\begin{aligned} \hat{\beta}_2 &= [X_2^T(I - X_1(X_1^TX_1)^{-1}X_1^T)X_2]^{-1}[X_2(I - X_1(X_1^TX_1)^{-1}X_1^T)y] \\ &= (X_2^TM_1M_1X_2)^{-1}(X_2^TM_1M_1y) \\ &= (X_2^TM_1^TM_1X_2)^{-1}(X_2^TM_1^TM_1y) \\ &= ((M_1X_2)^TM_1X_2)^{-1}((M_1X_2)^TM_1y) \end{aligned}$$

The above uses the property that $M_1^T = M_1$ and $M_1 M_1 = M_1$
The $\hat{\beta}_2$ is also the solution of

$$M_1 y = M_1 X_2 \beta + \epsilon$$

where $M_1 y$ is the residual of y regressed on X_1 and $M_1 X_2$ is the residual of X_2 regressed on X_1.

1.4 Variance of $\hat{\beta}$ and σ^2 estimation

$$\begin{aligned} Var(\hat{\beta}) = Var((X^T X)^{-1} X^T \epsilon) &= (X^T X)^{-1} X^T Var(\epsilon)((X^T X)^{-1} X^T)^T \\ &= \sigma^2 (X^T X)^{-1} X^T X (X^T X)^{-1} = \sigma^2 (X^T X)^{-1} \end{aligned}$$

For simple linear regression

$$Var(\hat{\beta}) = \frac{\sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{pmatrix} \Sigma_i x_i^2 & -\Sigma_i x_i \\ -\Sigma_i x_i & n \end{pmatrix}$$

$$Var(\hat{\beta}_0) = \frac{\Sigma x_i^2 \sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

$$Var(\hat{\beta}_1) = \frac{n\sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

Try

$$\begin{aligned} \Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2 - 2\bar{x}x_i + \bar{x}^2) &= \Sigma_i (x_i^2 - 2(\Sigma_j \frac{x_j}{n}) x_i + \frac{(\Sigma_j x_j)^2}{n^2}) \\ &= \Sigma_i x_i^2 - \frac{2}{n}(\Sigma_i x_i)^2 + \frac{(\Sigma_i x_i)^2}{n} = \Sigma_i x_i^2 - \frac{1}{n}(\Sigma x_i)^2 \end{aligned}$$

So

$$Var(\hat{\beta}_0) = \frac{\Sigma x_i^2 \sigma^2}{n\Sigma(x_i - \bar{x})^2}$$

$$Var(\hat{\beta}_1) = \frac{n\sigma^2}{n\Sigma(x_i - \bar{x})^2} = \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}$$

$$\begin{aligned} SSE &= \Sigma_i (y - \hat{y}_i)^2 \\ &= (Y - X\beta)^T (Y - X\beta) \\ &= (Y - X(X^T X)^{-1} X^T Y)^T (Y - X(XTX) - 1X^T Y) \\ &= (Y - PY)^T (Y - PY) \\ &= Y^T (1 - P)^T (1 - P) Y = Y^T (1 - P) Y \\ &= (X\beta + \epsilon)^T (1 - P)(X\beta + \epsilon) \\ &= \beta^T X^T (1 - P) X\beta + 2\beta^T X^T X^T (I - P)\epsilon + \epsilon^T (I - H)\epsilon \end{aligned}$$

$$E[SSE] = E[\epsilon^T(I-P)\epsilon] = E[\epsilon^T\epsilon]trace(I-H) = \sigma^2(n-k)$$

We obtain the unbiased estimator of σ^2

$$\hat{\sigma}^2 = \frac{SSE}{n-k}$$

Therefore the estimator of variance of β

$$\hat{V}ar(\hat{\beta}_i) = \hat{\sigma}^2(X^TX)^{-1}_{ii}$$

and the stanard error of β_i is

$$SE(\hat{\beta}_i) = \sqrt{\hat{\sigma}^2(X^TX)^{-1}_{ii}}$$

2 Properties of Least Square Estimators

2.1 Unbiasness

When we have a estimator, we need to ask ourselves two questions, 1)how accurate is our estimator, can the estimator give us true value of β, 2) Does it converge to the real value with reasonable speed as sample size increases? The metric we use to evaluate accuracy is unbiasness, and the metric we measure the convergence speed is the efficiency.
Unbiased

$$\begin{aligned}\hat{\beta} &= (X^TX)^{-1}X^TY \\ &= (X^TX)^{-1}X^T(X\beta+\epsilon) \\ &= (X^TX)^{-1}X^TX\beta + (X^TX)^{-1}X^T\epsilon \\ &= \beta + (X^TX)^{-1}X^T\epsilon\end{aligned}$$

Then the expectation of $\hat{\beta}$ condition on X is

$$E[\hat{\beta}|X] = \beta + (X^TX)^{-1}X^TE(\epsilon|X)$$

The last term is zero by assuption of linear regression. So

$$E[\hat{\beta}] = \beta$$

The expectation of the estimator is the same as true value, this is called **unbiased**.
Bias due to omission of relevant variables
Suppose we have a model

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

If we regression y on X_1 only, our estimator is

$$\hat{\beta}_1 = (X_1^T X_1)^{-1} X_1^T y = \beta_1 + (X_1^T X_1)^{-1} X_1^T X_2 \beta_2 + (X_1^T X_1)^{-1} X_1^T \epsilon$$

On the second term, we see unless 1)X_1 and X_2 are orthogonal, or 2)β_2 =0, β_1 is biased.

2.2 Consistency

We know

$$\hat{\beta} = \beta + (X^T X)^{-1} X^T \epsilon$$

$$\begin{aligned}
X^T X &= \Sigma_{i=1}^N \begin{pmatrix} x_{1i}^T x_{i1} & x_{1i}^T x_{i2} & ... & x_{1i}^T x_{ik} \\ ... & ... & ... & \\ x_{ki}^T x_{i1} & x_{ki}^T x_{k2} & ... & x_{ki}^T x_{ik} \end{pmatrix} \\
&= \Sigma_{i=1}^N \begin{pmatrix} x_{i1} x_{i1} & x_{i1} x_{i2} & ... & x_{i1} x_{ik} \\ ... & ... & ... & \\ x_{ik} x_{i1} & x_{ik} x_{k2} & ... & x_{ik} x_{ik} \end{pmatrix} \\
&= \Sigma_{i=1}^N \begin{pmatrix} x_{i1} \\ ... \\ x_{ik} \end{pmatrix} \begin{pmatrix} x_{i1} & ... & ...x_{ik} \end{pmatrix} \\
&= \Sigma_{i=1}^N X_i X_i^T
\end{aligned}$$

$$\begin{aligned}
\hat{\beta} &= \beta + (X^T X)^{-1} X^T \epsilon \\
&= \beta + (\Sigma_{i=1}^N X_i X_i^T)^{-1} X^T \epsilon \\
&= \beta + (\Sigma_{i=1}^N \frac{1}{N} X_i^T X_i)^{-1} (\frac{X^T \epsilon}{n})
\end{aligned}$$

If X_is are iid, then by law of large numbers

$$\Sigma_{i=1} \frac{1}{n} X_i^T X_i$$

converges to Q in probability.
There are certain conditions in which the estimators become inconsitent.
1) X is not full rank, or X has multicollinearity 2) $cov[X, \epsilon] \neq 0$

2.3 Multicollinearity

Suppose we have a regression model that contains two parameters

$$y = \beta_0 + X_1 \beta_1 + X_2 \beta_2$$

From above, we know variance of $\hat{\beta}$ is

$$Var(\hat{\beta}) = \frac{\sigma^2}{(X^T X)^{-1}}$$

When X only contains 2 variables, $X = (X_1, X_2)$

$$Var(\hat{\beta}_1) = \sigma^2 \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{1}{S_{11}(1 - \frac{S_{12}^2}{S_{11}S_{22}})} = \frac{1}{S_{11}(1 - r_{12}^2)}$$
$$Var(\hat{\beta}_2) = \sigma^2 \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{1}{S_{22}(1 - \frac{S_{12}^2}{S_{11}S_{22}})} = \frac{1}{S_{22}(1 - r_{12}^2)}$$

Where

$$\begin{aligned} S_{11} &= \Sigma(x_{1i} - \hat{x}_1)^2 \\ S_{22} &= \Sigma(x_{2i} - \hat{x}_2)^2 \\ S_{12} &= \Sigma(x_{1i} - \hat{x}_1)(x_{2i} - \hat{x}_2) \\ r_{12} &= \frac{S_{12}}{\sqrt{S_{11}S_{22}}} \end{aligned}$$

r_{12} is the correlation coefficient. In extreme case, when X_1 and X_2 are perfectly correlated, the variance becomes infinite.

2.4 Model Testing

Lagrange Multiplie(LM) test
Suppose we have two models, one is restriced, the other is unrestricted: Restricted(R): $y = X_1\beta_1 + \epsilon$
Unrestricted (U): $y = X_1\beta_1 + X_2\beta_2 + \epsilon$
Given the unrestriced model, the likehood function is

$$L(\beta_1, \beta_2, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} exp(-\frac{1}{2\sigma^2}(y - X_1\beta_1 - X_2\beta_2)^T(y - X_1\beta_1 - X_2\beta_2))$$

$$S_2 = \frac{\partial L}{\partial \beta_2} = \frac{1}{\sigma^2} X_2^T(y - X_1\beta_1 - X_2\beta_2)$$

When $\beta = 0$, define

$$M_1 = I - X_1(X_1^T X_1)^{-1} X_1^T$$

and $M_1 X_1 = 0$.

$$S_2 = \frac{1}{\sigma^2} X_2^T M_1 y = \frac{1}{\sigma^2} X_2^T M_1(X_1\beta_1 + \epsilon) = \frac{1}{\sigma^2} X_2^T M_1 \epsilon$$

The last equal sign uses the fact $M_1 X_1 = 0$.

$$\begin{aligned} Var(X_2^T M_1 y) &= Var(X_2^T M_1 \epsilon) \\ &= X_2^T M_1 Var(\epsilon)(X_2^T M_1)^T \\ &= X_2^T M_1 M_1^T X_2 Var(\epsilon) = \sigma^2 X_2^T M_1 X_2 \end{aligned}$$

So $X_2^T M_1 \epsilon$ follows normal distribution with mean 0 and variance $\sigma^2 X_2^T M_1 X_2$. Define

$$Z = \frac{X_2^T M_1 \epsilon}{\sqrt{\sigma^2 X_2^T M_1 X_2}}$$

then Z follows standard normal distribution. The **Lagrange Multiplier (LM) test** is defined

$$LM = Z^2 = \frac{(X_2^T M_1 \epsilon)^2}{\sigma^2 X_2^T M_1 X_2}$$

which follows χ^2 distribution with degree of freedom 1.
F test
We define

$$SSE_U = ||y - X_1\hat{\beta}_1 - X_2\hat{\beta}_2||^2$$
$$SSE_R = ||y - X_1\hat{\beta}_1||^2$$

F test is defined as

$$F = \frac{\frac{Extra\ expalined\ variation}{Degree\ of\ Freedom}}{\frac{Remaining\ unexplained\ variation}{Degree\ of\ Freedom}} = \frac{SSE_R - SSE_U}{\frac{SSE_U}{n-1}}$$

Let $X = (X_1, X_2)$, and we define two projection matrices

$$P_U = X(X^T X)^{-1} X^T$$
$$P_R = X_1(X_1^T X_1)^{-1} X_1^T$$

$$SSE_R - SSE_U = y^T(I - P_R)y - y^T(I - P_U)y = y^T(P_U - P_R)y$$

recall

$$\hat{\beta}_2 = (X_2^T M_1 X_2)^{-1} X_2^T M_1 y$$

So

$$P_U - P_R = X_2\beta = X_2(X_2^T M_1 X_2)^{-1} X_2^T M_1 y$$
$$SSR_R - SSR_U = (X_2^T M_1 y)^T (X_2^T M_1 X_2)^{-1} X_2^T M_1 y$$

$$F = \frac{SSR_R - SSR_U}{\frac{SSR_U}{n-k}} = \frac{(X_2 M_1 y)^T (X_2^T M_1 y)}{\hat{\sigma}^2 (X_2^T M_1 X_2)} = LM$$

We see that F test and LM test are equivalent.