1 Basic of Fourier Transform

If $f(x) = f(x+T)$ then $f(x)$ can be written as

$$f(x) = \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k x}{T}}$$

because

$$e^{\frac{2\pi i k x}{T}} = e^{\frac{2\pi i k(x+T)}{T}}$$

Based on orthogonality,

$$c_k = \frac{1}{T}\int_0^T f(x) e^{-i\frac{2\pi k x}{T}} dx$$

The above is the Fourier transform in continuous case, in discrete case If $x = n\Delta t$, where n = 1...N, and $T = N\Delta t$,

$$c_k = \frac{1}{N\Delta t}\sum_{n=1}^{N} f(n\Delta t) e^{-i2\pi k \frac{1}{N\Delta t} n\Delta t} d(n\Delta t) = \frac{1}{N}\sum_{n=1}^{N} f(n) e^{-i2\pi k \frac{n}{N}}$$

This is the discrete Fourier transform.

$$\Delta f = \frac{1}{T} = \frac{1}{N\Delta t} = \frac{F_s}{N}$$

Where F_s, N are the sample frequency and number of samples.
Properties
1) To be eligible, f(x) has to be a period function with time T. This leads to uniform sampling theorem used in signal processing. The uniform sampling theorem states w
2) If $f(x)$ is real, which means $f(x) = f^*(x)$.

$$\sum_{-\infty}^{+\infty} c_k e^{2\pi i \frac{1}{T} kx} = \sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} = \sum_{-\infty}^{+\infty} c_{-k} e^{2\pi i \frac{1}{T} kx}$$

so$c_k = c_{-k}^*$, $||c_k|| = ||c_{-k}||$.
3) $c_k = c_{k+N}$. So when a signal contains frequency component no larger than B, in other words, the bandwidth of the signal is $2B(-B(to)B)$, then in order to capture the whole bandwidth of the signal, $N\Delta f > 2B$. This leads to Nyquist sampling theorem $F_s > 2B(bandwidth)$.

2 Fast Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

let

$$u_k = e^{-i2\pi k \frac{n}{N}}$$

then we have the basis orthogonality

$$u_{k1}^T u_{k2} = N\delta_{k_1,k_2}$$

We recognize we can write X_k with even index terms and odd index terms

$X_k =$ Even index parts $+$ Odd index parts

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N} 2mk} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of N/2 even indexed points, where k is 0,1N/2)

$$+ e^{-\frac{2\pi i}{N} k}$$

$$\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of N/2 odd indexed points, where k is 0,1N/2)

(Since each part is a Fourier transform of N/2 points, k has to be smaller than N/2)

$$= E_k + e^{-\frac{2\pi i}{N} k} O_k$$

As noted, the above derivation is for $k < N/2$, a very similar derivation for $N/2 <= k < N$ leads to

$$X_{k+N/2} = E_k - e^{-\frac{2\pi i}{N} k} O_k$$

Now we have divided the FFT of N points to two FFT with N/2 points Keep going till we reach the size to one, then combine together recursively.