

## 1 Basic of Fourier Transform

If  $f(x) = f(x + T)$  then  $f(x)$  can be written as

$$f(x) = \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k x}{T}}$$

$i$  is the imaginary unit, and  $k$  is an integer. The above expression is eligible

$$e^{\frac{2\pi i k x}{T}} = e^{\frac{2\pi i k (x+T)}{T}}$$

Each basis  $e^{\frac{2\pi i k x}{T}}$  represents a signal with frequency  $F_k = \frac{k}{T}$ . So the interval between each adjacent frequency  $\Delta F = \frac{1}{T}$ .

Based on orthogonality,

$$c_k = \frac{1}{T} \int_0^T f(x) e^{-i \frac{2\pi k x}{T}} dx$$

The above is the Fourier transform in continuous case, in discrete case If  $x = n\Delta t$ , where  $n = 1 \dots N$ , and  $T = N\Delta t$ , then the Fourier series can be written as

$$\begin{aligned} f(n) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k n \Delta t}{N \Delta t}} \\ &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k n}{N}} \end{aligned}$$

$$c_k = \frac{1}{N \Delta t} \sum_{n=1}^N f(n \Delta t) e^{-i 2\pi k \frac{1}{N \Delta t} n \Delta t} d(n \Delta t) = \frac{1}{N} \sum_{n=1}^N f(n) e^{-i 2\pi k \frac{n}{N}}$$

This is the discrete Fourier transform.

$$\Delta F = \frac{1}{T} = \frac{1}{N \Delta t}$$

$N$  is the total sample within time  $T$ .

### Properties

1) To be eligible,  $f(x)$  has to be a period function with time  $T$  (with frequency  $F = \frac{1}{T}$ ) in both continuous case and discrete case. The requirement in discrete case leads to uniform sampling theorem used in signal processing. The total sampling time  $T_{sampling}$  has to be an integer multiple of  $T$ .

$$T_{sampling} = MT$$

while  $T = \frac{N}{F_s}$  So

$$MT = N \delta t$$

if we let  $\delta t = \frac{1}{F_s}$ , where  $F_s$  is the sampling frequency, and  $T = \frac{1}{F}$ , we have

$$\frac{M}{F} = \frac{N}{F_s}$$

2) If  $f(x)$  is real, which means  $f(x) = f^*(x)$ . We then substitute Fourier series for both  $f(x)$  and  $f^*(x)$ ,

$$\sum_{-\infty}^{+\infty} c_k e^{2\pi i \frac{1}{T} k x} = \sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} k x} \quad (1)$$

Since the summation on the right hand side is from  $-\infty$  to  $\infty$ , it is eligible to replace  $k$  with  $-k$ .

$$\sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} k x} = \sum_{\infty}^{-\infty} c_{-k}^* e^{2\pi i \frac{1}{T} k x} \quad (2)$$

Combine the above two equations 1 and 2, we can see  $c_k = c_{-k}^*$ . This means they are complex conjugate: their magnitude are equal, their phase are opposite. Namely  $|c_k| = |c_{-k}|$ ,  $\phi(c_k) = \phi(c_{-k})$ .

3) Connection between complex representation and real representation.

We have shown that for real signal  $c_k = c_{-k}^*$  and  $c_k = |c_k|e^{j\theta_k}$ ,  $c_{-k} = |c_k|e^{-j\theta_k}$ . And in complex representation, we can combine the term with index  $k$  and  $-k$ ,

$$c_k e^{j2\pi k F_0 t} + c_{-k} e^{-j2\pi k F_0 t} = 2|c_k| \cos(2\pi k F_0 t + \theta_k)$$

$$\begin{aligned} f(x) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k x}{T}} \\ &= c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k) \\ &= a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k F_0 t) - b_k \sin(2\pi k F_0 t)) \end{aligned}$$

where  $a_0 = c_0$ ,  $a_k = 2|c_k| \cos \theta_k$ ,  $b_k = 2|c_k| \sin \theta_k$ . 4)  $c_k = c_{k+N}$ . So when a signal contains frequency component no larger than  $B$ , in other words, the bandwidth of the signal is  $2B(-B \text{ to } B)$ , then in order to capture the whole bandwidth of the signal,  $N\Delta f > 2B$ . This leads to Nyquist sampling theorem  $F_s > 2B(\text{bandwidth})$ .

5) Power density

$$\begin{aligned}
 P_x &= \frac{1}{T} \int |x(t)|^2 dt \\
 &= \frac{1}{T} \int x(t) \sum_{-\infty}^{\infty} c_k^* e^{-j2\pi k F_0 t} \\
 &= \sum_{-\infty}^{\infty} c_k^* \left[ \frac{1}{T} \int x(t) e^{-j2\pi k F_0 t} \right] \\
 &= \sum_{-\infty}^{\infty} |c_k|^2
 \end{aligned}$$

When signal is real, then

$$\begin{aligned}
 P_x &= \sum_{-\infty}^{\infty} |c_k|^2 \\
 &= a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)
 \end{aligned}$$

## 2 Fast Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

let

$$u_k = e^{-i2\pi k \frac{n}{N}}$$

then we have the basis orthogonality

$$u_{k_1}^T u_{k_2} = N \delta_{k_1, k_2}$$

We recognize we can write  $X_k$  with even index terms and odd index terms

$X_k = \text{Even index parts} + \text{Odd index parts}$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N} 2mk} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N} (2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of  $N/2$  even indexed points, where  $k$  is  $0, 1N/2$ )

$$+ e^{-\frac{2\pi i}{N} k}$$

$$\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of  $N/2$  odd indexed points, where  $k$  is  $0, 1N/2$ )

(Since each part is a Fourier transform of  $N/2$  points,  $k$  has to be smaller than  $N/2$ )

$$= E_k + e^{-\frac{2\pi i}{N} k} O_k$$

As noted, the above derivation is for  $k < N/2$ , a very similar derivation for  $N/2 \leq k < N$  leads to

$$X_{k+N/2} = E_k - e^{-\frac{2\pi i}{N} k} O_k$$

Now we have divided the FFT of  $N$  points to two FFT with  $N/2$  points. Keep going till we reach the size to one, then combine together recursively.

## 2.1 Connection with Uncertainty Principle

### Relationship between time length and frequency bandwidth

We consider a few examples

1) We consider a function  $g(t)$  which is infinitely long in time domain

$$g(t) = \cos(2\pi f_0 t)$$

Its Fourier transform is

$$F(f) = \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0)$$

Since the delta function has width zero, so the bandwidth in frequency domain is zero. We see a signal which is infinitely long in time domain has zero bandwidth in frequency domain. 2) We consider a function  $g(t)$  which has zero width in time, namely an impulse function.

$$g(t) = \delta(t)$$

Since this function is not a periodic function, we assume its period is infinity. Its Fourier transform is

$$F(f) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi f t} = 1$$

Now we see a signal which has zero width in time has infinitely long frequency bandwidth. This leads to the uncertainty principle.

**Uncertainty Principle** In quantum mechanics, if there is a particle with position  $x$  and momentum  $p$ , then uncertainty principle states

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Similar relationship holds for time  $t$  and Energy.

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

We can modify this expression to get the time and frequency relationship in our Fourier transform. Since  $E = \hbar\omega$ . Then

$$\Delta t \Delta \omega \geq \frac{1}{2}$$