

# 1 Basic of Fourier Transform

## Fourier Series

If  $x(t) = x(t + T)$  then  $x(t)$  can be written as

$$x(t) = \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k t}{T}}$$

$i$  is the imaginary unit, and  $k$  is an integer. The above expression is eligible because  $e^{\frac{2\pi i k t}{T}}$  is a periodic function

$$e^{\frac{2\pi i k t}{T}} = e^{\frac{2\pi i k (t+T)}{T}}$$

Each basis  $e^{\frac{2\pi i k t}{T}}$  represents a signal with frequency  $f_k = \frac{k}{T}$ . So the interval between each adjacent frequency  $\Delta f = \frac{1}{T}$ . Based on orthogonality, we can get  $c_k$

$$c_k = \frac{1}{T} \int_0^T x(t) e^{-i \frac{2\pi k t}{T}} dt$$

## Fourier Series: Example

$$x(t) = \cos(2\pi f_0 t) = \frac{1}{2}(e^{i2\pi f_0 t} + e^{-i2\pi f_0 t})$$

where  $f_0 = \frac{1}{T}$

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T x(t) e^{-i \frac{2\pi k t}{T}} dt \\ &= \frac{1}{T} \int_0^T \frac{1}{2} (e^{\frac{2i\pi t}{T}} + e^{-\frac{2i\pi t}{T}}) e^{-i \frac{2\pi k t}{T}} dt \end{aligned}$$

Only terms with  $k = \pm 1$  in the above expression can survive, so

$$c_1 = \frac{1}{T} \int_0^T \frac{1}{2} dt = \frac{1}{2}$$

Similarly,  $c_{-1} = \frac{1}{2}$ .

## Fourier Transform

We can generalize the Fourier series to non-periodic functions. We define the Fourier transform as

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt$$

With the inverse Fourier transform defined as

$$x(t) = \int_{-\infty}^{\infty} \mathcal{F}(f) e^{2\pi i f t} df$$

To see why the above makes sense, it is easy to prove the identity.

$$\begin{aligned} x(t') &= \int_{-\infty}^{\infty} \mathcal{F}(f) e^{2\pi i f t'} df \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt \right) e^{2\pi i f t'} df \\ &= \int_{-\infty}^{\infty} x(t) \left( \int_{-\infty}^{\infty} e^{-2\pi i f t} e^{2\pi i f t'} df \right) dt \\ &= \int_{-\infty}^{\infty} x(t) \delta(t - t') dt \\ &= x(t') \end{aligned}$$

### Fourier Transform: Example

#### 1. Constant Function

$$x(t) = 1$$

$$\begin{aligned} \mathcal{F}(f) &= \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt \\ &= \int_{-\infty}^{\infty} e^{-2\pi i f t} dt \\ &= \lim_{a \rightarrow \infty} \int_{-a}^a e^{-2\pi i f t} dt \\ &= \lim_{a \rightarrow \infty} \frac{1}{-2\pi i f} e^{-2\pi i f t} \Big|_{-a}^a \\ &= \lim_{a \rightarrow \infty} \frac{1}{-2\pi i f a} (e^{-2\pi i f a} - e^{2\pi i f a}) \\ &= \lim_{a \rightarrow \infty} \frac{1}{2\pi i f a} (e^{2\pi i f a} - e^{-2\pi i f a}) \\ &= \lim_{a \rightarrow \infty} 2 \frac{\sin(2\pi f a)}{2\pi f a} \\ &= 2 \lim_{a \rightarrow \infty} \frac{\sin(2\pi f a)}{2\pi f a} \\ &= \delta(f) \end{aligned}$$

#### 2. Trigeometric Function Take the same x(t) as above in the discrete case

$$x(t) = \cos(2\pi f_0 t) = \frac{1}{2}(e^{2i\pi f_0 t} + e^{-2i\pi f_0 t})$$

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} \frac{1}{2}(e^{2i\pi f_0 t} + e^{-2i\pi f_0 t})e^{-2i\pi f t} dt = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

### Discrete Fourier Series

The above is the Fourier transform in continuous case, in discrete case If  $x = n\Delta t$ , where  $n = 1 \dots N$ , and  $T = N\Delta t$ , then the Fourier series can be written as

$$\begin{aligned} x(n) &= \sum_{k=-\infty}^{+\infty} c_k e^{\frac{2\pi i k n \Delta t}{N\Delta t}} \\ &= \sum_{k=-\infty}^{+\infty} c_k e^{\frac{2\pi i k n}{N}} \end{aligned}$$

$$c_k = \frac{1}{N\Delta t} \sum_{n=1}^N f(n\Delta t) e^{-i2\pi k \frac{1}{N\Delta t} n\Delta t} d(n\Delta t) = \frac{1}{N} \sum_{n=1}^N f(n) e^{-i2\pi k \frac{n}{N}}$$

This is the discrete Fourier series.

The interval in the frequency domain is

$$\Delta f = f_{k+1} - f_k = \frac{k+1}{T} - \frac{k}{T} = \frac{1}{T} = \frac{1}{N\Delta t}$$

### Discrete Fourier Transform

In the discrete case, suppose we sample a signal  $N$  times within time  $T$ . We divide time  $T$  into  $N$  time intervals with length being  $\Delta t = T/N$ . Then we can let  $t = n\Delta t$ , the integral in the Fourier transform becomes a summation. So we write the Fourier transform as

$$\mathcal{F}(f) = \sum_0^N x(n\Delta t) e^{-2\pi i f n \Delta t} \frac{T}{N}$$

In frequency domain, the frequency also becomes discrete, and same as the case in discrete Fourier series,  $\Delta f = \frac{1}{T}$ . Another way of seeing  $\Delta f$  is when we confine the length of time domain to  $T$ , the function in time domain has to be periodic function with period  $T$ . So we have

$$e^{2\pi i f(n\Delta t + T)} = e^{2\pi i f n \Delta t}$$

This requires

$$2\pi f T = 2\pi k$$

where  $k$  is integer. This leads discrete frequencies

$$f = \frac{k}{T}$$

and

$$\Delta f = \frac{1}{T}$$

Using  $f = k\Delta f$ , we can rewrite our Fourier transform

$$\begin{aligned}\mathcal{F}(k\Delta f) &= \sum_0^N x(n\Delta t) e^{-2\pi i k \Delta f n \Delta t} \frac{T}{N} \\ &= \sum_0^N x(n\Delta t) e^{-2\pi i k n / N} \frac{T}{N}\end{aligned}$$

Then we work out the Fourier transform and inverse Fourier transform identity

$$\begin{aligned}x(n' \Delta t) &= \sum_{k=0}^N \mathcal{F}(k\Delta f) e^{2\pi i k n' / N} \Delta f \\ &= \sum_{k=0}^N \left( \sum_{n=0}^N x(n\Delta t) e^{-2\pi i k n / N} \frac{T}{N} \right) e^{2\pi i k n' / N} \Delta f \\ &= \sum_{k=0}^N \frac{1}{N} \left( \sum_{n=0}^N x(n\Delta t) e^{-2\pi i k n / N} \right) e^{2\pi i k n' / N}\end{aligned}$$

So we define discrete Fourier transform

$$\mathcal{F}(k) = \sum_{n=0}^N x(n) e^{-2\pi i k n / N}$$

and the discrete inverse Fourier transform

$$x(n) = \frac{1}{N} \sum_{k=0}^N \mathcal{F}(k) e^{2\pi i k n / N}$$

### Example

Let  $N = 4$ , and

$$x(n) = \cos(2\pi \frac{n}{4}) = \frac{1}{2} (e^{i2\pi \frac{n}{4}} + e^{-i2\pi \frac{n}{4}})$$

$$\mathcal{F}(k) = \frac{1}{4} \sum_{n=1}^4 \frac{1}{2} (e^{i2\pi \frac{n}{4}} + e^{-i2\pi \frac{n}{4}}) e^{-i \frac{2\pi k n}{4}}$$

Similar to the continuous case, only terms with  $k = \pm 1$  in the above expression can survive, when  $k = 1$

$$\begin{aligned}\mathcal{F}(1) &= \frac{1}{4} \sum_{n=1}^4 \frac{1}{2} e^{i2\pi \frac{n}{4}} e^{-i \frac{2\pi n}{4}} \\ &= \frac{1}{4} \frac{1}{2} 4 \\ &= \frac{1}{2}\end{aligned}$$

What about case for  $k = -1$ ? We define  $k = 1, 2, 3, 4$  so  $k = -1$  is not defined. However, in discrete case we note  $c_{-1} = c_3$  due to the periodicity. Similarly, we can calculate  $\mathcal{F}(3) = \frac{1}{2}$ .

$N$  is the total sample within time  $T$ .

### Properties

1) To be eligible,  $f(x)$  has to be a period function with time  $T$  (with frequency  $F = \frac{1}{T}$ ) in both continuous case and discrete case. The requirement in discrete case leads to uniform sampling theorem used in signal processing. The total sampling time  $T_{sampling}$  has to be an integer multiple of  $T$ .

$$T_{sampling} = MT$$

while  $T = \frac{N}{F_s}$  So

$$MT = N\Delta t$$

if we let  $\Delta t = \frac{1}{F_s}$ , where  $F_s$  is the sampling frequency, and  $T = \frac{1}{F}$ , we have

$$\frac{M}{F} = \frac{N}{F_s}$$

2) If  $x(n)$  is real, which means  $x(n) = x^*(n)$ . We then substitute Fourier series for both  $x(n)$  and  $x^*(n)$ ,

$$\sum_{-\infty}^{+\infty} c_k e^{2\pi i \frac{1}{T} kx} = \sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} \quad (1)$$

Since the summation on the right hand side is from  $-\infty$  to  $\infty$ , it is eligible to replace  $k$  with  $-k$ .

$$\sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} = \sum_{\infty}^{-\infty} c_{-k}^* e^{2\pi i \frac{1}{T} kx} \quad (2)$$

Combine the above two equations 1 and 2, we can see  $c_k = c_{-k}^*$ . This means they are complex conjugate: their magnitude are equal, their phase are opposite. Namely  $|c_k| = |c_{-k}|$ ,  $\phi(c_k) = \phi(c_{-k})$ . Similarly, for discrete Fourier transform,  $|\mathcal{F}_k| = |\mathcal{F}_{-k}|$

3) Connection between complex representation and real representation.

We have shown that for real signal  $c_k = c_{-k}^*$  and  $c_k = |c_k|e^{j\theta_k}$ ,  $c_{-k} = |c_k|e^{-j\theta_k}$ . And in complex representation, we can combine the term with index  $k$  and  $-k$ ,

$$c_k e^{j2\pi k F_0 t} + c_{-k} e^{-j2\pi k F_0 t} = 2|c_k| \cos(2\pi k F_0 t + \theta_k)$$

$$\begin{aligned} f(x) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k x}{T}} \\ &= c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k) \\ &= a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k F_0 t) - b_k \sin(2\pi k F_0 t)) \end{aligned}$$

where  $a_0 = c_0$ ,  $a_k = 2|c_k|\cos\theta_k$ ,  $b_k = 2|c_k|\sin\theta_k$ .

4)  $f(k) = f(k + N)$ . So when a signal contains frequency component no larger than  $B$ , in other words, the bandwidth of the signal is  $2B(-B \text{ to } B)$ , then in order to capture the whole bandwidth of the signal,  $N\Delta f > 2B$ . This leads to Nyquist sampling theorem  $F_s > 2B(\text{bandwidth})$ .

5) Power density

$$\begin{aligned} P_x &= \frac{1}{T} \int |x(t)|^2 dt \\ &= \frac{1}{T} \int x(t) \sum_{-\infty}^{\infty} c_k^* e^{-j2\pi k F_0 t} \\ &= \sum_{-\infty}^{\infty} c_k^* \left[ \frac{1}{T} \int x(t) e^{-j2\pi k F_0 t} \right] \\ &= \sum_{-\infty}^{\infty} |c_k|^2 \end{aligned}$$

When signal is real, then

$$\begin{aligned} P_x &= \sum_{-\infty}^{\infty} |c_k|^2 \\ &= a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \end{aligned}$$

## 2 Fast Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

let

$$u_k = e^{-i2\pi k \frac{n}{N}}$$

then we have the basis orthogonality

$$u_{k1}^T u_{k2} = N \delta_{k1, k2}$$

We recognize we can write  $X_k$  with even index terms and odd index terms

$$X_k = \text{Even index parts} + \text{Odd index parts}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N} 2mk} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N} (2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of  $N/2$  even indexed points, where  $k$  is  $0, 1N/2$ )

$$+ e^{-\frac{2\pi i}{N} k}$$

$$\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2} mk}$$

(We can view this as Fourier Transform of  $N/2$  odd indexed points, where  $k$  is  $0, 1N/2$ )

(Since each part is a Fourier transform of  $N/2$  points,  $k$  has to be smaller than  $N/2$ )

$$= E_k + e^{-\frac{2\pi i}{N} k} O_k$$

As noted, the above derivation is for  $k < N/2$ , a very similar derivation for  $N/2 \leq k < N$  leads to

$$X_{k+N/2} = E_k - e^{-\frac{2\pi i}{N} k} O_k$$

Now we have divided the FFT of  $N$  points to two FFT with  $N/2$  points. Keep going till we reach the size to one, then combine together recursively.

### 3 Fourier Transform of Useful Functions

#### The Fourier Transform of Step Function

Let  $u(t)$  be a step function:  $u(t) = 1$  when  $t \geq 0$ ,  $u(t) = 0$  when  $t < 0$ . And its derivative is a delta function

$$\frac{du(t)}{dt} = \delta(t)$$

Taking Fourier transform on both sides yields

$$2\pi i f \mathcal{F}(f) = 1$$

So

$$\mathcal{F}(f) = \frac{1}{2\pi i f} |_{f \neq 0} + \mathcal{F}(f)|_{f=0}$$

Since any function with a different constant can have the same derivative, the Fourier transform of the original function has to have a constant, which corresponds to zero frequency component  $F(0)$ . The constant component of function  $u(t)$  is its offset to zero, which is  $1/2$ . so

$$F(f) = \frac{1}{2\pi i f} \Big|_{f \neq 0} + \frac{1}{2} \delta(f)$$

### The Fourier Transform of a Shifted Step Function

Let  $u(t)$  be a step function:  $u(t - \tau) = 1$  when  $t \geq \tau$ ,  $u(t - \tau) = 0$  when  $t < \tau$ . Then

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} u(t - \tau) e^{-2\pi f t} dt$$

Let  $t' = t - \tau$ , then

$$\mathcal{F}(f) = e^{-2\pi i f \tau} \int_{-\infty}^{\infty} u(t') e^{-2\pi f t'} dt'$$

So we see this is a factor times Fourier transform of step function, therefore

$$\begin{aligned} \mathcal{F}(f) &= e^{-2\pi i f \tau} \left( \frac{1}{2\pi i f} \Big|_{f \neq 0} + \frac{1}{2} \delta(f) \right) \\ &= e^{-2\pi i f \tau} \frac{1}{2\pi i f} \Big|_{f \neq 0} + \frac{1}{2} \delta(f) \end{aligned}$$

### The Fourier Transform of Gaussian

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$$

$$\mathcal{F}(f) = e^{-2\pi^2 \sigma^2 f^2}$$

So the Fourier transform of a Gaussian function is another Gaussian function but with different width.

### The Fourier Transform of Dirac Comb

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

It is clearly that  $x(t)$  is periodic with period  $T$ . So we can expand that into Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t / T}$$



Where

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-i \frac{2\pi k t}{T}} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(0) e^{-i \frac{2\pi k t}{T}} \\ &= \frac{1}{T} \end{aligned}$$

So

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{2\pi i k t / T}$$

On the other hand, based on the formula of Fourier transform

$$\mathcal{F}(f) = \int \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-2\pi i f t} dt = \sum_{n=-\infty}^{\infty} e^{-2\pi i n T f} = \sum_{n=-\infty}^{\infty} e^{-2\pi i n f / f_0}$$

Comparing the Fourier series of  $x(t)$  and the expression of  $F(f)$ , they are the same except  $T$  being changed to  $f_0$ . Therefore we can conclude that  $F(f)$  itself is also a Dirac comb, which is

$$\mathcal{F}(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

### The Fourier Transform of White Noise

Assuming noise we sample in time is  $n[m]$ , where  $m = 0, \dots, M-1$ .  $n[m]$  is a Gaussian random variable with zero mean and variance  $\sigma^2$ . The the FFT of  $n[m]$  is

$$\begin{aligned} N[k] &= \frac{1}{M} \sum_{m=0}^{M-1} n[m] e^{-i 2\pi m k / M} \\ &= \frac{1}{M} \sum_{m=0}^{M-1} n[m] (\cos(2\pi m k / M) - i n[m] \sin(2\pi m k / M)) \end{aligned}$$

The expected value is

$$\begin{aligned} E[N[k]] &= E\left[\frac{1}{M} \sum_0^{M-1} n[m] e^{-i 2\pi m k / M}\right] \\ &= \frac{1}{M} \sum_0^{M-1} E[n[m]] e^{-i 2\pi m k / M} \\ &= 0 (\text{because } E[n[m]] = 0) \end{aligned}$$

The variance of the real part is

$$\begin{aligned}
Var[R[N[k]]] &= E[(\frac{1}{M} \sum_{m=0}^{M-1} n[m](\cos(2\pi mk/M)) * (\frac{1}{M} \sum_{p=0}^{M-1} n[p](\cos(2\pi pk/M)))] \\
&= \frac{1}{M^2} E[\sum_{m=0}^{M-1} n[m]n[p]\delta(n-p)\cos(2\pi mk/M) * \cos(2\pi pk/M)] \\
&= \frac{1}{M^2} \sum_{m=0}^{M-1} E[n[m]^2]\cos^2(2\pi mk/M) \\
&= \frac{1}{M^2} \sigma^2 (\sum_{m=0}^{M-1} \cos^2(2\pi mk/M)) \\
&= \frac{1}{M^2} \sigma^2 (\frac{M}{2} + \frac{\cos((M+1)2\pi k/M)\sin(2\pi Mk/M)}{2\sin(2\pi k/M)}) \\
&= \frac{1}{M} \frac{\sigma^2}{2}
\end{aligned}$$

The same derivation applies for the imaginary part. So the FFT is Gaussian noise with mean zero and variance  $\sigma^2$ .

## 4 Connection with Uncertainty Principle

### Relationship between time length and frequency bandwidth

We consider a few examples

1) We consider a function  $g(t)$  which is infinitely long in time domain

$$g(t) = \cos(2\pi f_0 t)$$

Its Fourier transform is

$$\begin{aligned}
F(f) &= \int \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2} e^{i2\pi f t} dt \\
&= \int \frac{1}{2} e^{i2\pi t(f_0+f)} dt + \int \frac{1}{2} e^{i2\pi t(f-f_0)} dt \\
&= \frac{1}{2} \delta(f+f_0) + \frac{1}{2} \delta(f-f_0)
\end{aligned}$$

The last line is based on  $\int_{-\infty}^{\infty} e^{i2\pi f t} dt = \delta(f)$ .

Since the delta function has width zero, so the the bandwidth in frequency domain is zero. We see a signal which is infinitely long in time domain has zero bandwidth in frequency domain.

2) We consider a function  $g(t)$  which has zero width in time, namely an impulse function.

$$g(t) = \delta(t)$$

Since this function is not a periodic function, we assume its period is infinity. Its Fourier transform is

$$F(f) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi f t} = 1$$

Now we see a signal which has zero width in time has infinitely long frequency bandwidth. This leads to the uncertainty principle.

**Uncertainty Principle** In quantum mechanics, if there is a particle with position  $x$  and momentum  $p$ , then uncertainty principle states

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Similar relationship holds for time  $t$  and Energy.

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

We can modify this expression to get the time and frequency relationship in our Fourier transform. Since  $E = \hbar\omega$ . Then

$$\Delta t \Delta \omega \geq \frac{1}{2}$$