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1 Black Scholes Merton Equation: Introduction

We assume the stock prices following a geometric Brownian motion
1) Stock price:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

2) We have a portfolio X(t) which consists of $\Delta(t)$ share of stock $\Delta(t)S(t)$, and $(X(t) - \Delta(t)S(t))$ money market account with interest rate r.

$$X(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

3) Change of the portfolio with respect to time

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t) + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

4) Change of the present value of the stock with respect to time

$$d(e^{-rt}S(t)) = (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$$

5) With a few steps, we get change of the present value of the portfolio with respect to time

$$\begin{aligned} &d(e^{-rt}X(t)) \\ =&\Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \end{aligned}$$

6) Assume the option value is $c(t, S(t))$ and we apply Ito's formula

$$\begin{aligned} &d(e^{-rt}c(t, S(t)) \\ =&e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)\frac{\partial c(t, S(t))}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c(t, S(t))}{\partial S^2(t)}]dt \\ &+ e^{-rt}\sigma S(t)\frac{\partial c(t, S(t))}{\partial S(t)}dW(t) \end{aligned}$$

7)Now equate Equation in 5) and 6), we get
dW(t) term:

$$\Delta(t) = \frac{\partial c(t, S(t))}{\partial S(t)}$$

dt term:

$$rc(t, S) = c_t(t, S(t)) + rS(t) + \frac{1}{2}\sigma^2 S(t)\frac{\partial c(t, S(t))}{\partial S^2(t)}$$

which is known as Black-Scholes-Merton partial differential equation.
The terminal condition the equation satisfies for call option is

$$c(T, S) = (S(T) - K)^+$$

Similarly, for put option

$$p(T, S) = (K - S(T))^+$$

2 Connection to Faynman-Kac formula

In risk-neutral measure, we write the stock price as

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

Where $\tilde{W}(t)$ is a standard Brownian motion under risk-neutral measure.
According to the risk-neutral pricing formula, the price of the derivative security at time t is

$$V(t) = \tilde{E}[e^{-r(T-t)}V(T)|F(t)] = \tilde{E}[e^{-r(T-t)}h(S(T))|F(t)]$$

Since the stock price is Markov and the payoff is a function fo the stock price alone, based on Faynman-Kac formula, there is a function $v(t, x)$ such that $V(t) = v(t, S(t))$, and $v(v, S(t))$ must satisfy discounted partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

Now we have seen two ways of showing the Black-Scholes-Merton(BSM) equation. One way is to reproduce the payoff of the option using a portfolio that consists of a saving account. Another way is based on the risk-neutral pricing formula and Feynman-Kac formula. These two ways are equivalent. Because under risk-neutral measure, the payoff of a derivative is the same as a saving account, which imply we are able to reproduce the payoff using portfolio that consisting of a saving account.

3 Black-Scholes-Merton Model: Analytic Solution for European Option

1. European call option
For European call option with payoff to be $V(T) = S(T) - K$, with K as strike price, let us assume constant volatility σ, and constant interest rate r. Then we can obtain the solution to the BSM equation with martingale property without bothering solving the complex parital differential equation. The call option value satisfies

$$c(t, S(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^{+}|F(t)]$$

We write

$$\begin{aligned} S(T) &= S(t)exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)\tau\} \\ &= S(t)exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} \end{aligned}$$

Where Y is the stardard normal random variable and $\tau = T - t$ is the time to expiration. We see we write S(T) as a product of S(t), which is F(t) measurable, and a random variable independent of F(t). Then solving c(t, S(t)) becomes solving an expectation value of a random varible composed of a standard normal random variable.

$$c(t, x) = \tilde{E}[e^{-\tau r}(xexp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^{+}]$$

Where Y is a standard normal distribution under $\hat{P}$.
After a little bit of math with integration, we have the solution to the Black-Scholes-Merton model for European call option

$$c(\tau, x; K, r, \sigma) = xN(d_{+}(\tau, x)) - e^{-r\tau}KN(d_{-}(\tau, x))$$

Where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}[ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})\tau]$$
$$d_2 = d_1 - \sigma\sqrt{\tau}$$

N() is the cumulative distribution function of the standard normail distribution
2. European put option
The payoff for the European put option is $V(T) = K - S(T)$, we can follow a similiar derivation and get the formula for put option

$$p(t, x) = N(-d_2)Ke^{-r\tau} - N(-d_1)x$$

3. Boundary conditons
Using the solution $c(t, x)$ and $p(t, x)$, we can easily check the boundary conditions when time t approaches to expiration time T.
As we know

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}ln(\frac{S}{K}) + \frac{1}{\sigma}(r + \frac{\sigma^2}{2}\sqrt{\tau})$$

When $\tau \to 0$, the second term decays much faster, so it vanishes. When $S > K$, d_1 goes to infinity, when $S < K$, d_1 goes to negative infinity. Therefore, when $S > K$

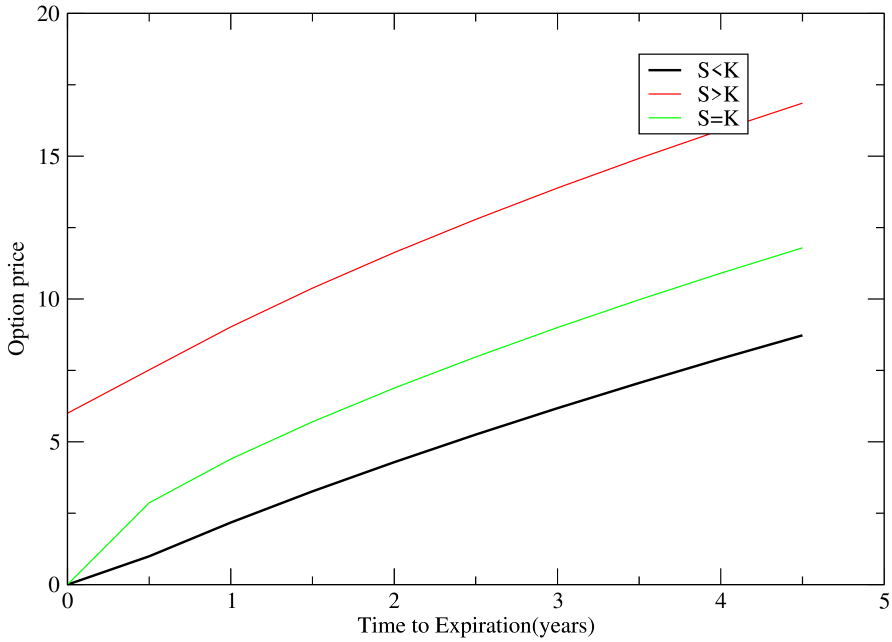
$$c(t, x) = S * N(+\infty) - K * N(+\infty) = S - K$$

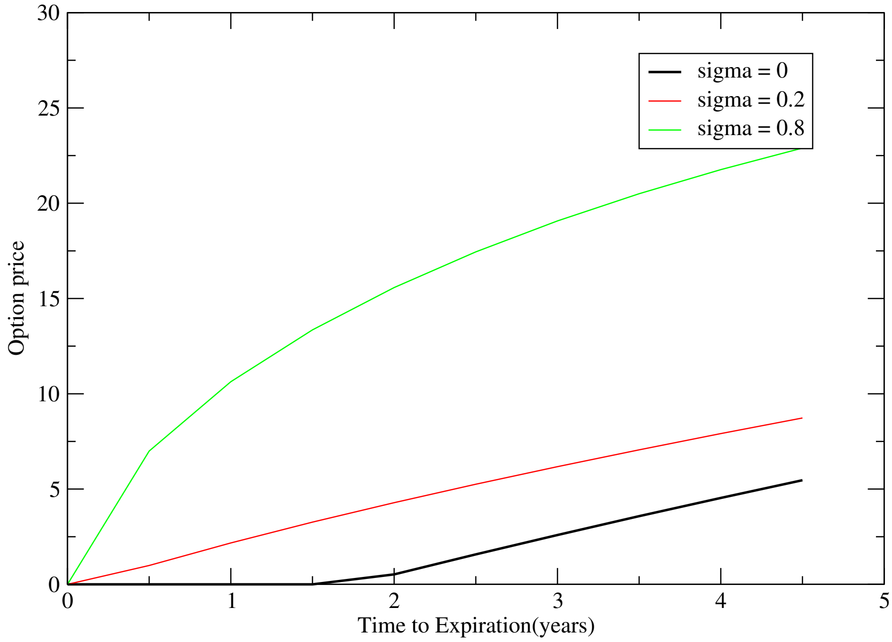
Therefore, when $S < K$

$$c(t, x) = S * N(-\infty) - K * N(-\infty) = 0 - 0 = 0$$

4. Examples

The following graphs show the change of option price with respect to different





5. Alternative formulation
If we introduce $F = e^{r\tau}S$, which is the forward price of the asset S. Then the equation pricing equation becomes

$$C(F, \tau) = D[N(d_+)F - N(d_-)K]$$
$$P(F, \tau) = D[N(-d_-)K - N(-d_+)F]$$
$$d_{+/-} = \frac{1}{\sigma\sqrt{\tau}}[ln(\frac{F}{K}) + / - \frac{1}{2}\sigma^2\tau]$$

The variables are:
$\tau = T - t$ is the time to expiry
$D = e^{-r\tau}$ is the discount factor

4 Heston Stochasic Volatility Model

4.1 Introduction

The Black-Scholes equation assumes the volatility is constant, which is the ideal case and not practical in the real market. The Heston model assumes the volatility to follow a stochastic process. Suppose a stock price under risk-neutral measure is governed by

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)\tilde{d}W_1(t)$$

and the volatility itself is governed by the equation

$$dV(t) = (a - bV(t))dt + \sigma\sqrt{V(t)}\tilde{d}W_2(t)$$

Where

$$\tilde{d}W_1(t)\tilde{d}W_2(t) = \rho dt$$

At time t, the risk-neutral price of a call expiring at time $T \geq t$ in this model is

$$c(t, S(t), V(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+|F(t)]$$

If we move the term c^{rt} to the left hand side, we see

$$e^{-rt}c(t, S(t), V(t)) = \tilde{E}[e^{-rT}(S(T) - K)^+|F(t)]$$

which satisfies the martingale property. Then we take the differentiation of $e^{-rt}c(t, S(t), V(t))$. We get

$$\begin{aligned}
&d(e^{-rt}c(t, S(t), V(t))\\
=&\frac{\partial e^{-rt}}{\partial t}c(t, S(t), V(t)) + e^{-rt}\frac{\partial c(t, S(t), V(t))}{\partial t}dt\\
=&-re^{-rt}c(t, S(t), V(t))dt \quad (1)\\
+&e^{-rt}\frac{\partial c}{\partial t}dt \quad (2)\\
+&e^{-rt}\frac{\partial c}{\partial S}dS \quad (3)\\
+&e^{-rt}\frac{\partial^2 c}{\partial S^2}dSdS \quad (4)\\
+&e^{-rt}\frac{\partial c}{\partial V}dV \quad (5)\\
+&e^{-rt}\frac{\partial^2 c}{\partial V^2}dVdV \quad (6)\\
+&e^{-rt}\frac{\partial^2 c}{\partial V\partial S}dVdS \quad (7)
\end{aligned}$$

As we are interested in only the dt terms, we find out the dt terms from (1) to (7) the dt term in (1) is

$$-rc(t, S(t), V(t))e^{-rt}dt$$

the dt term in (2) is

$$\frac{\partial c}{\partial t}e^{-rt}dt$$

the dt term in (3) is

$$\frac{\partial c}{\partial S}rSe^{-rt}dt$$

the dt term in (4) is

$$\frac{1}{2}\frac{\partial^2 c}{\partial S^2}VS^2e^{-rt}dt$$

the dt term in (5) is

$$\frac{\partial c}{\partial V}(a - bV(t))e^{-rt}dt$$

the dt term in (6) is

$$\frac{1}{2}\frac{\partial^2 c}{\partial V^2}V\sigma^2 e^{-rt}dt$$

the dt term in (7) is

$$\frac{\partial^2 c}{\partial V \partial S}VS\sigma e^{-rt}dt$$

Collect all the dt terms and let those terms equal to zero, we get

$$c_t + rsc_s + (a - bv)c_v + \frac{1}{2}s^2 vc_{ss} + \rho\sigma svc_{sv} + \frac{1}{2}\sigma^2 vc_{vv} = rc$$

The function $c(t, s, v)$ satisfies boundary condition

$$\begin{aligned} c(T, s, v) &= (s - K)^+ \\ c(t, 0, v) &= 0 \\ c(t, s, 0) &= (s - e^{-r(T-t)}K)^+ \\ lim_{s\to\infty}\frac{c(t, s, v)}{s - K} &= 1 \\ lim_{v\to\infty}c(t, s, v) &= s \end{aligned}$$

If we guess the solution has the following form

$$c(t, s, v) = sf(t, logs, v) - e^{-r(T-t)}Kg(t, logs, v)$$

Then since c(t,s,v) satisfies the partial differential equation above, we can show that f_0 and f_1 satisfy the following

$$f_t + (r + \frac{1}{2})f_x + (a - bv + \rho\sigma v)f_v + \frac{1}{2}vc_{vv} + \rho\sigma vf_{xv} + \frac{1}{2}\sigma^2 vf_{vv} = 0$$

$$g_t + (r - \frac{1}{2})g_x + (a - bv)g_v + \frac{1}{2}vg_{vv} + \rho\sigma vg_{xv} + \frac{1}{2}\sigma^2 vg_{vv} = 0$$