

1 Ito Integral

a. Ito integral Definition

1) For simple integrand

Given t_0, t_1, \dots, t_n and Δt is a constant in between any $[t_k, t_{k+1}]$

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

We can also rewrite $I(t) = \int_0^t \Delta(u) dW(u)$.

b. Properties of Ito integral

1) Ito Integral is a martingale

2) Isometry

$$EI^2(t) = E \int_0^t \Delta^2(u) du$$

c. For general integrand

Choose $\Delta_n(t)$ such that when $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} E \int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0$$

Define Ito integral

$$\int_0^t \Delta(u) dW(u) = \lim_{n \rightarrow \infty} \int_0^t \Delta_n(u) dW(u)$$

2 Ito formula

a. Ito formula Suppose $dX_t = u dt + \sigma dB_t$

If $g(t, X)$ is twice continuously differentiable $Y_t = g(t, X_t)$

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial X} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2$$

Where $(dX)^2 = d(X_t)d(X_t) = u^2(dt)^2 + 2u\sigma dt dB_t + \sigma^2 (dB_t)^2$ $dt * dt = 0$
 $dt * dB_t = dB_t * dB_t = 0$ $dB_t dB_t = dt$

b. Example: Geometric Brownian motion

The geometric Brownian motion satisfies $dN_t/N_t = r dt + \sigma dB_t$ to solve this we consider

$$\begin{aligned} d(\ln N_t) &= \frac{\partial \ln N_t}{\partial N_t} dN_t + \frac{1}{2} \frac{\partial^2 \ln N_t}{\partial N_t^2} (dN_t)^2 \\ &= \frac{1}{N_t} dN_t + \left(-\frac{1}{N_t^2}\right) (dN_t)^2 \\ (dN_t)^2 &= r^2 N_t^2 (dt)^2 + r N_t dt \sigma dB_t + \sigma^2 N_t^2 d^2 B_t \\ &= 0 + 0 + \sigma^2 N_t^2 dt \end{aligned}$$

So