Author: Dr. Shi Guo Email: guoshi1984@hotmail.com

1 Change of Measure for Random Variable

1.1 Change of measure for a discrete random variable

1) Give a binomial random variable S associated with a coin toss, define S = 1 if ω = Head, S = -1 if ω = Tail.

2)We define the probability measure P. In this measure getting a head has prob $\frac{1}{2}$, and getting a tail has $\frac{1}{2}$. We can also define the probability measure $P^{'}$, under $P^{'}$, getting a head has prob $\frac{2}{3}$, and getting a tail has $\frac{1}{3}$.

3)We can define change of measure to connect P and $P^{'}$. Consider the transformation $Z(\omega) = \frac{P^{'}(\omega)}{P(\omega)}$, so $Z(H) = \frac{4}{3}$, $Z(T) = \frac{2}{3}$. This is the change of measure for a discrete random variable.

1.2 Change of measure for a continuously random variable

The probability measure defined on the sigma algebra does not have to be unique. Give a random variable X(ω)=x where x in [0,1]
1)
Define probability measure P
P(a <x <b) = b - a, the pdf is $p(x) = 1$. This is a uniform measure.

2)
Define another probability measure P$^{'}$
$P^{'}(a<x<b)$ =b^2 - a^2, the pdf is $p(x) = 2x$. So this is non-uniform measure

3) To justify they both are probability measure
Check P[0,1] = 1; P(0)=0;
P$^{'}[0,1] = 1; P^{'}(0) = 0;$

4) we can define change of measure to connect P and $P^{'}$
Consider the transformation
$P^{'}(a < X(\omega) = x < b) = \int_a^b 2xdx = \int_a^b 2xdx = \int_a^b 2xdP(X(\omega))$
$dP(X(\omega)) = Z(X(\omega))dP(X(\omega))$ where $Z(X(\omega)) = 2x$
This is the change of measure for a uniformly distributed random variable.

1.3 Change of measure for normal distributed random variable

We show an example of change of measure in normal distribution. If X is N(0,1), let Y = X+ u, so Y is N(u,1), so the random variable Y does not have mean 0. However, based on the definition of expectation

$$E(Y(\omega)) = \int Y(\omega)dP(\omega)$$

we can change the probability measure $P(\omega)$, such that E(Y) becomes zero. Define $Z(w) = exp(-uX(\omega) - \frac{1}{2}u^2)$ We are able to show two things
1 Z >0
2 E(Z) =1 i.e. $\int Z(w)dP(X(w)) = 1$
Because

$$\begin{aligned}
E(Z) &= \int exp(-ux - 1/2u^2)\frac{1}{\sqrt{2\pi}}exp(-1/2x^2)dx \\
&= \frac{1}{\sqrt{2\pi}}\int exp(-1/2(x+u)^2)dx \\
&= \frac{1}{\sqrt{2\pi}}\int exp(-1/2(y)^2)dy \\
&= 1
\end{aligned}$$

So $P^{'}(w) = \int Z(w)dP(w)$ is a new probability measure
The pdf of Y under the new measure is

$= \int_{Y(\omega)<=b} Z(\omega)dP(\omega)$

$= \int 1_{X(\omega)<=b-u} exp(-uX - \frac{1}{2}u^2)dP(\omega)$

$=\int 1_{X(\omega)<=b-u} exp(-uX - \frac{1}{2}u^2)pdf(N(0,1))dx$

$= \sqrt{2\pi}^{-1} \int_{-\infty}^{b-u} exp(-ux - \frac{1}{2}u^2 - 1/2x^2)dx$

$= \sqrt{2\pi}^{-1} \int_{-\infty}^{b-u} exp(-\frac{1}{2}(x+u)^2)dx$

(change x back to y)
$= \sqrt{2\pi}^{-1} \int_{-\infty}^{b} exp(-\frac{1}{2}(y)^2)dy$

= cdf of N(0,1)
This shows it is a standard normal distribution with mean 0.

2 Change of Measure for a Filtration(Series of Events in Time)

2.1 Change of measure for Stock under binomial model - Risk neutral measure

Suppose we have the following stock S_0 at t=0. At t=1, we can associate the value of S_1 to outcome of tossing a coin. When we toss a coin and if the coin is fair, we can get Head and Tail and each has 50% probability. If we get a head, the stock moves to $S_1(H)$, and if we get a tail, the stock moves to $S_1(T)$. Clearly, the stock has 50% to move up, and 50% to move down.

$$S_1(H) = (1 + \alpha + \sigma)S_0$$
$$S_1(T) = (1 + \alpha - \sigma)S_0$$

In the sense of risk neutral pricing, we would like to have the stock values grows as the same as a saving account with interest rate r. Namely, we need

$$S_0(1 + r) = \frac{1}{2}S_1(H) + \frac{1}{2}S_1(T)$$

Simply plug in the definition of S_1, we easily see the equation does not hold except the special case when $\alpha = r$.

When α does not equal to r, we artificially create two probabilities p and q with p + q =1, define

$$S_0(1 + r) = pS_1(H) + qS_1(T)$$

Then solve for p and q, we have

$$p = \frac{r - \alpha + \sigma}{2\sigma}$$
$$q = \frac{\alpha - r + \sigma}{2\sigma}$$

We call this risk-neutral measure. Under this measure, the expectation of the stock return is the same as the return of saving account. We define this as risk neutral measure. To understand this measure, we can see when $\alpha > r$ then $q(H) < q(T)$, so we lower the prob of stock moving up and raise the prob of the stock moving down such that the return is 1+r. The same argument holds for $r < \alpha$.

2.2 Girsanov's Theorem

Define change of measure for continous variable

For (Ω, F, P), given random variable Z with $E(Z) = 1$, define new probability measure

$$P' = \int_A Z(\omega)dP(\omega) \tag{1}$$

$$\tag{2}$$

We have two expectation defined, one is under P, the other under P$^{'}$,E$^{'}[X] = E[XZ]$, $dP^{'}(\omega) = Z(\omega)dP(\omega)$, and $Z(\omega) = \frac{d^{'}P(\omega)}{dP(\omega)}$
Define change of measure for filtration
$E[Z] = 1$ and $Z(t) = E[Z|F(t)]$

Properties of Z(t)
1) Martingale
Given $0 <= s <= t <= T$
$E[Z(t)|F(s)] = E[E[Z|F(t)]|F(s)] = E[Z|F(s)] = Z(s)$
2) $E^{'}[Y] = E[YZ(t)]$
$E^{'}[Y] = E[YZ] = E[E[YZ|F(t)]] = E[YE[Z|F(t)]] = E[YZ(t)]$
3) Given $0 <= s <= t <= T$, Y is F(t)- measurable, then
E$^{'}[Y|F(s)] = \frac{1}{Z(s)}E[YZ(t)|F(s)]$

Girsanov's Theorem
Suppose w(t) is Brownian Motion given Ω, F, P F(t) is the filtration. Let $\Theta(t)$, $0 <= t <= T$ is adapted process, define $Z(t) = exp(-\int_0^t \Theta(u)dW(u))$, and $W^{'}(t) = W(t) + \int_0^t \Theta(u)du$, s.t. $E[\int_0^T \Theta^2(u)Z^2(u)du < \infty]$. Then $E[Z] = 1$, and under $P^{'}$, $W^{'}(t)$ is Brownian motion.

2.3 Risk neutral measure in filtration for stock price

We model the stock price using geometric Brownian motion

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$$

Its integrated form is

$$S(t) = S(0)exp(\int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds)$$

define $D(t) = exp(-\int_0^t R(s)ds)$, then $dD(t) = R(t)D(t)dt$
Discounted stock price

$$D(t)S(t) = S(0)exp(\int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s))ds)$$

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)]$$

where $\Theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$ let $dW^{'}(t) = dW(t) + \theta(t)dt$, then $d(D(t)S(t)) = \sigma(t)D(t)S(t)dW^{'}(t)$, **which implies martingale**, then $dS(t) = R(t)S(t)dt + \sigma(t)S(t)dW^{'}(t)$. The drift term now change from $\alpha torisk - freeinterestRateR$.