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1 σ algebra

a. Sigma algebra definition
Given a non-empty set Ω, a sigma algebra is defined
1) Include empty set and whole set
2) Include the complement of any element itself
3) Closed under countable union

b. Sigma algebra example by tossing a coin
Tossing 0 time
Check 1)
$F_0 = 0$,
Tossing once
Check 1 Ω = H, T
Check 2 H_c = T, T_c = H
Check 3 H U T = Ω
So
$F_1 = 0$, Ω, H, T
Tossing twice
Check 1 Ω = HH, HT, TH, TT
Check 2 HH_c, HT_c, TH_c, TT_c
Check 3 HH U HT, HH U TH, HH U TT, HT U TH, HT U TT, TH U TT
HH U HT U TH = TT_c,
HH U HT U TT = TH_c,
HH U TH U TT = HT_c,
HT U TH U TT = HH_c
So
$F_2 = 0$, Ω, HH, HT, TH, TT,
HH_c, HT_c, TH_c, TT_c
HH U HT, HH U TH, HH U TT, HT U TH, HT U TT, TH U TT,
TT_c, TH_c, HT_c, HH_c

c. Why define sigma algebra?
On top of the sigma algebra, we can define the probability, because the object
that probability measure takes is the sigma algebra.

2 Filtration

Consider a sequence of coin toss
For the first toss, we get F_1
For the first and second toss, we get F_2
For the first n tosses, we get F_n
The collection of sigma algebra F_1, F_2 F_n is called a Filtration.

3   Random variable

a. Definition
A random variable is function from Ω -¿ R, which satisfies for all of the subsets ω in Ω, X in Borel is in σ-algebra F.

b. example
Consider 3 toss case, H with prob p, T with prob q
Def. random variable S $S_0(w_0) = 4$ for all ω

$$S_{n+1}(w_{n+1}) = 2S_n(w_n) \ \text{if} \ w_{n+1} = H$$
$$\frac{1}{2}S_n(w_n) \ \text{if} \ w_{n+1} = T$$

so
$S_0(w_1 w_2 w_3) = 4$ for all w_i
$S_1(w_1 w_2 w_3) = 8$ if $w_1 = $ H
$S_1(w_1 w_2 w_3) = 2$ if $w_1 = $ T
$S_2(w_1 w_2 w_3) = 16$ if $w_1 = w_2 = H$
$S_2(w_1 w_2 w_3) = 4$ if $w_1 \neq w_2$
$S_2(w_1 w_2 w_3) = 1$ if $w_1 = w_2 = T$

4 σ algebra generated by a random variable and measurable function

Give consider a random variable S: Ω -¿ R, for every open set in R, the collection of their inverse image forms an sigma algebra, and it is called the sigma algebra generated by S. And S is called F-measurable.

5   Conditional Expectation

a. Definition
1) $E[X|G]$ is G measurable, which means the value of $E[X]$ is completely determined by G
2) $\int_A E[X|G](w)dP(w) = \int_A X(w)dP(w)$ for all A which belongs to G

b. Example to understand 2)
Consider 3 toss case, H with prob p, T with prob q
Define random variable S
$S_0(w) = 4$ for all w
$S_{n+1}(w) = 2S_n(w)$ if $w_{n+1} = H$
$S_{n+1}(w) = \frac{1}{2}S_n(w)$ if $w_{n+1} = T$
Expectation of 3 tosses random variable S_3 give the first two is HH

$$E_2(S_3|HH) = pS_3(HHH) + qS_3(HHT)$$
$$E_2(S_3|HT) = pS_3(HTH) + qS_3(HTT)$$
$$E_2(S_3|TH) = pS_3(THH) + qS_3(THT)$$
$$E_2(S_3|TT) = pS_3(TTH) + qS_3(TTT)$$
$$E_2(S_3|HH)P(HH) = prob(HHH)S_3(HHH) + prob(HHT)S_3(HHT)$$
$$E_2(S_3|HT)P(HT) = prob(HTH)S_3(HTH) + prob(HTT)S_3(HTT)$$
$$E_2(S_3|TH)P(TH) = prob(THH)S_3(THH) + prob(THT)S_3(THT)$$
$$E_2(S_3|TT)P(TT) = prob(HTH)S_3(TTH) + prob(TTT)S_3(TTT)$$

This confirms def 2), for A = HH or HT or TH or TT
$$\int_2 (S_3|G)(w)dP(w) = \int_A X(w)dP(w)$$

c. Properties

1) The conditional expectation is a random variable. Because the value is dependent on G.

2) If X is G measurable, then $E[X|G] = X$.

3) If X is G measurable $E[XY|G] = XE[Y|G]$, this is to take out what is known.

4) If X is independent of G
$$E[X|G] = EX$$
To understand 2) and 3), consider two extreme cases
Define random variable S
$S_0(w) = 4$ for all w
$S_{n+1}(w) = 2S_n(w)$ if $w_{n+1} = H$
$S_{n+1}(w) = \frac{1}{2}S_n(w)$ if $w_{n+1} = T$
Then a condition expectation can be defined as
$$E[S_n|F_t] = E[S_n|\omega_1, \omega_2, ..., \omega_t]$$
if t=n, then $E[S_n|F_n] = S_n$, this is because when F_n is known, then S_n is known, there is nothing to average.
if t=0, then $E[S_n|F_0] = E[S_n]$, this is because F_0 provides no restriction to average S_n, the conditional expectation needs to average all possible cases, it is a general expectation.

5) If G is a subset of H
$$E[E[X|G|H]] = E[X|H]$$

3