1 The Fourier Transform of white noise is white noise

Assuming noise we sample in time is $n[m]$, where m = 0,... M-1. $n[m]$ is a Gaussian random variable with zero mean and variance σ^2. The the FFT of $n[m]$ is

$$\begin{aligned} N[k] &= \frac{1}{M}\sum_{m=0}^{M-1} n[m]e^{-i2\pi mk/M} \\ &= \frac{1}{M}\sum_{m=0}^{M-1} n[m](cos(2\pi mk/M) - i\ n[m]sin(2\pi mk/M)) \end{aligned}$$

The expected value is

$$\begin{aligned} E[N[k]] &= E[\frac{1}{M}\sum_{0}^{M-1} n[m]e^{-i2\pi mk/M}] \\ &= \frac{1}{M}\sum_{0}^{M-1} E[n[m]]e^{-i2\pi mk/M}] \\ &= 0(\text{because E[n[m]] = 0}) \end{aligned}$$

The variance of the real part is

$$\begin{aligned} Var[R[N[k]]] &= E[(\frac{1}{M}\sum_{m=0}^{M-1} n[m](cos(2\pi mk/M)) * (\frac{1}{M}\sum_{p=0}^{M-1} n[p](cos(2\pi pk/M))] \\ &= \frac{1}{M^2}E[\sum_{m=0}^{M-1} n[m]n[p]\delta(n-p)cos(2\pi mk/M) * cos(2\pi pk/M)] \\ &= \frac{1}{M^2}\sum_{m=0}^{M-1} E[n[m]^2]cos^2(2\pi mk/M) \\ &= \frac{1}{M^2}\sigma^2(\sum_{m=0}^{M-1} cos^2(2\pi mk/M)) \\ &= \frac{1}{M^2}\sigma^2(\frac{M}{2} + \frac{cos((M+1)2\pi k/M)sin(2\pi Mk/M)}{2sin(2\pi k/M)}) \\ &= \frac{1}{M}\frac{\sigma^2}{2} \end{aligned}$$

The same derivation applies for the imaginary part. So the FFT is Gaussian noise with mean zero and variance σ^2.