1 Basic of Fourier Transform

If $f(x) = f(x+T)$ then $f(x)$ can be written as

$$f(x) = \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikx}{T}}$$

because

$$e^{\frac{2\pi ikx}{T}} = e^{\frac{2\pi ik(x+T)}{T}}$$

Based on orthogonality,

$$c_k = \frac{1}{T}\int_0^T f(x)e^{-i\frac{2\pi kx}{T}}dx$$

The above is the Fourier transform in continuous case, in discrete case If $x = n\Delta t$, where n = 1...N, and $T = N\Delta t$, then the Fourier series can be written as

$$\begin{aligned} f(n) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikn\Delta t}{N\Delta t}} \\ &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikn}{N}} \end{aligned}$$

$$c_k = \frac{1}{N\Delta t}\sum_{n=1}^{N} f(n\Delta t)e^{-i2\pi k\frac{1}{N\Delta t}n\Delta t}d(n\Delta t) = \frac{1}{N}\sum_{n=1}^{N} f(n)e^{-i2\pi k\frac{n}{N}}$$

This is the discrete Fourier transform.

$$\Delta f = \frac{1}{T} = \frac{1}{N\Delta t} = \frac{F_s}{N}$$

Where F_s, N are the sample frequency and number of samples.

Properties

1) To be eligible, f(x) has to be a period function with time T. This leads to uniform sampling theorem used in signal processing. The uniform sampling theorem states w

2) If $f(x)$ is real, which means $f(x) = f^*(x)$. We then substitute Fourier series for both $f(x)$ and $f*(x)$,

$$\sum_{-\infty}^{+\infty} c_k e^{2\pi i\frac{1}{T}kx} = \sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i\frac{1}{T}kx} \tag{1}$$

Since the summation on the right hand side is from $-\infty$ to ∞, it is eligible to replace k with k.

$$\sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i\frac{1}{T}kx} = \sum_{\infty}^{-\infty} c_{-k}^* e^{2\pi i\frac{1}{T}kx} \tag{2}$$

Combine the above two equations 1 and 2, we can see $c_k = c_{-k}^*$. This means they are complex conjugate: their magnitude are equal, their phase are opposite. Namely $||c_k|| = ||c_{-k}||$, $\phi(c_k) = \phi(c_{-k})$.
3) Connection between complex representation and real representation.
We have shown that for real signal $c_k = c_{-k}^*$ and $c_k = |c_k|e^{j\theta_k}$, $c_{-k} = |c_k|e^{-j\theta_k}$.
And in complex representation, we can combine the term with index k and -k,

$$c_k e^{j2\pi kF_0t} + c_{-k}e^{-j2\pi kF_0t} = 2|c_k|cos(2\pi kF_0t + \theta_k)$$

$$\begin{aligned} f(x) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikx}{T}} \\ &= c_0 + 2\sum_{k=1}^{\infty}|c_k|cos(2\pi kF_0t + \theta_k) \\ &= a_0 + \sum_{k=1}^{\infty}(a_k cos(2\pi kF_0t) - b_k sin(2\pi kF_0t)) \end{aligned}$$

where $a_0 = c_0$, $a_k = 2|c_k|cos\theta_k$, $b_k = 2|c_k|sin\theta_k$. 4) $c_k = c_{k+N}$. So when a signal contains frequency component no larger than B, in other words, the bandwidth of the signal is $2B(-B(to)B)$, then in order to capture the whole bandwidth of the signal, $N\Delta f > 2B$. This leads to Nyquist sampling theorem $F_s > 2B(bandwidth)$.
5) Power density

$$\begin{aligned} P_x &= \frac{1}{T}\int |x(t)|^2 dt \\ &= \frac{1}{T}\int x(t)\sum_{-\infty}^{\infty} c_k^* e^{-j2\pi kF_0t} \\ &= \sum_{-\infty}^{\infty} c_k^*[\frac{1}{T}\int x(t)e^{-j2\pi kF_0t}] \\ &= \sum_{-\infty}^{\infty}|c_k|^2 \end{aligned}$$

When signal is real, then

$$\begin{aligned} P_x &= \sum_{-\infty}^{\infty}|c_k|^2 \\ &= a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \end{aligned}$$

2 Fast Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

let

$$u_k = e^{-i2\pi k \frac{n}{N}}$$

then we have the basis orthogonality

$$u_{k1}^T u_{k2} = N\delta_{k_1,k_2}$$

We recognize we can write X_k with even index terms and odd index terms

$X_k =$ Even index parts $+$ Odd index parts

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}2mk} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2}mk}$$

(We can view this as Fourier Transform of N/2 even indexed points, where k is 0,1N/2)

$$+ e^{-\frac{2\pi i}{N}k}$$

$$\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2}mk}$$

(We can view this as Fourier Transform of N/2 odd indexed points, where k is 0,1N/2)

(Since each part is a Fourier transform of N/2 points, k has to be smaller than N/2)

$$= E_k + e^{-\frac{2\pi i}{N}k} O_k$$

As noted, the above derivation is for $k < N/2$, a very similar derivation for $N/2 <= k < N$ leads to

$$X_{k+N/2} = E_k - e^{-\frac{2\pi i}{N}k} O_k$$

Now we have divided the FFT of N points to two FFT with N/2 points Keep going till we reach the size to one, then combine together recursively.