

# 1 Black Scholes Merton Equation: Introduction

We assume the stock prices following a geometric Brownian motion

1) Stock price:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

2) We have a portfolio  $X(t)$  which consists of  $\Delta(t)$  share of stock  $\Delta(t)S(t)$ , and  $(X(t) - \Delta(t)S(t))$  money market account with interest rate  $r$ .

$$X(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

3) Change of the portfolio with respect to time

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

4) Change of the present value of the stock with respect to time

$$d(e^{-rt}S(t)) = (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$$

5) With a few steps, we get change of the present value of the portfolio with respect to time

$$\begin{aligned} d(e^{-rt}X(t)) \\ = \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \end{aligned}$$

6) Assume the option value is  $c(t, S(t))$  and we apply Ito's formula

$$\begin{aligned} d(e^{-rt}c(t, S(t))) \\ = e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)\frac{\partial c(t, S(t))}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c(t, S(t))}{\partial S^2(t)}]dt \\ + e^{-rt}\sigma S(t)\frac{\partial c(t, S(t))}{\partial S(t)}dW(t) \end{aligned}$$

7) Now equate Equation in 5) and 6), we get  
dW(t) term:

$$\Delta(t) = \frac{\partial c(t, S(t))}{\partial S(t)}$$

dt term:

$$rc(t, S) = c_t(t, S(t)) + rS(t) + \frac{1}{2}\sigma^2 S(t) \frac{\partial^2 c(t, S(t))}{\partial S^2(t)}$$

which is known as Black-Scholes-Merton partial differential equation.  
The terminal condition the equation satisfies for call option is

$$c(T, S) = (S(T) - K)^+$$

Similarly, for put option

$$p(T, S) = (K - S(T))^+$$

## 2 Connection to Feynman-Kac formula

In risk-neutral measure, we write the stock price as

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

Where  $\tilde{W}(t)$  is a standard Brownian motion under risk-neutral measure.  
According to the risk-neutral pricing formula, the price of the derivative security at time t is

$$V(t) = \tilde{E}[e^{-r(T-t)}V(T)|F(t)] = \tilde{E}[e^{-r(T-t)}h(S(T))|F(t)] \quad (1)$$

Since the stock price is Markov and the payoff is a function of the stock price alone, based on Feynman-Kac formula, there is a function  $v(t, x)$  such that  $V(t) = v(t, S(t))$ , and  $v(t, S(t))$  must satisfy discounted partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

Now we have seen two ways of showing the Black-Scholes-Merton(BSM) equation. One way is to reproduce the payoff of the option using a portfolio that consists of a saving account. Another way is based on the risk-neutral pricing formula and Feynman-Kac formula. These two ways are equivalent. Because under risk-neutral measure, the payoff of a derivative is the same as a saving account, which implies we are able to reproduce the payoff using portfolio that consists of a saving account.

### 3 Black-Scholes-Merton Model: Analytic Solution for European Option

#### 1. European call option

For European call option with payoff to be  $V(T) = S(T) - K$ , with  $K$  as strike price, let us assume constant volatility  $\sigma$ , and constant interest rate  $r$ . Then we can obtain the solution to the BSM equation with martingale property without bothering solving the complex partial differential equation. The call option value satisfies

$$c(t, S(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t)]$$

We write

$$\begin{aligned} S(T) &= S(t) \exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)\tau\} \\ &= S(t) \exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} \end{aligned}$$

Where  $Y$  is the standard normal random variable and  $\tau = T - t$  is the time to expiration.

$$Y = -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{T - t}}$$

So we see that  $S(T)$  is the product of the  $\mathcal{F}(t)$  measurable random variable  $S(t)$  and random variable

$$\exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\}$$

Which is independent of  $\mathcal{F}(t)$ . Therefore based on risk-neutral pricing formula[1]

$$\begin{aligned} c(t, x) &= \tilde{E}[e^{-r\tau}(x \exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^+] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-r\tau}(x \exp\{-\sigma\sqrt{\tau}y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^+ e^{-\frac{1}{2}y^2} dy \end{aligned}$$

After a little bit of math with integration, we have the solution to the Black-Scholes-Merton model for European call option

$$c(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - e^{-r\tau}KN(d_-(\tau, x))$$

Where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{\tau}}[\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})\tau] \\ d_2 &= d_1 - \sigma\sqrt{\tau} \end{aligned}$$

$N(\cdot)$  is the cumulative distribution function of the standard normal distribution

#### 2. European put option

The payoff for the European put option is  $V(T) = K - S(T)$ , we can follow a similar derivation and get the formula for put option

$$p(t, x) = N(-d_2)Ke^{-r\tau} - N(-d_1)x$$

### 3. Boundary conditons

Using the solution  $c(t, x)$  and  $p(t, x)$ , we can easily check the boundary conditions when time  $t$  approaches to expiration time  $T$ .

As we know

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{S}{K}\right) + \frac{1}{\sigma} \left(r + \frac{\sigma^2}{2}\right) \sqrt{\tau}$$

When  $\tau \rightarrow 0$ , the second term decays much faster, so it vanishes. When  $S > K$ ,  $d_1$  goes to infinity, when  $S < K$ ,  $d_1$  goes to negative infinity. Therefore, when  $S > K$

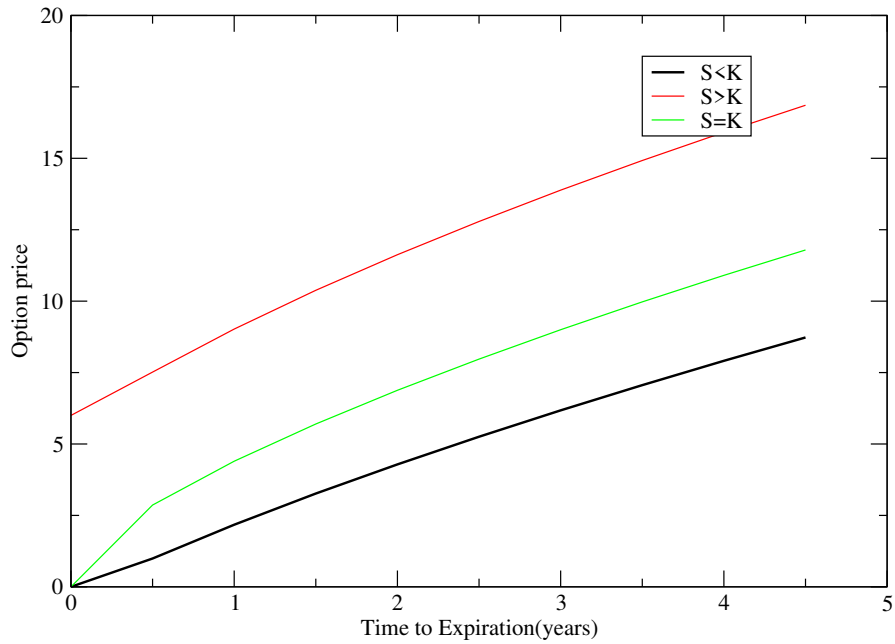
$$c(t, x) = S * N(+\infty) - K * N(+\infty) = S - K$$

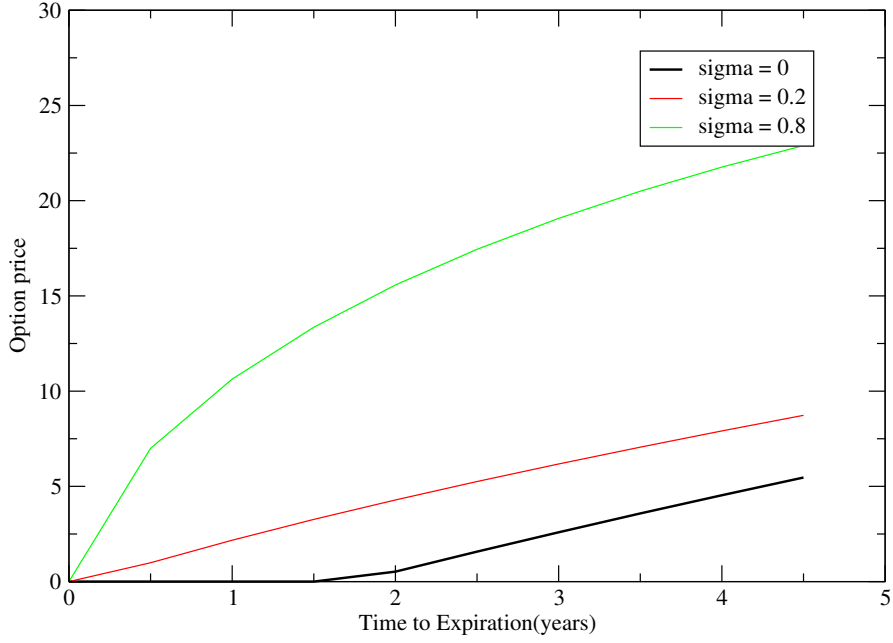
Therefore, when  $S < K$

$$c(t, x) = S * N(-\infty) - K * N(-\infty) = 0 - 0 = 0$$

### 4. Examples

The following graphs show the change of option price with respect to different





## 5. Alternative formulation

If we introduce  $F = e^{r\tau}S$ , which is the forward price of the asset  $S$ . Then the equation pricing equation becomes

$$\begin{aligned}
 C(F, \tau) &= D[N(d_+)F - N(d_-)K] \\
 P(F, \tau) &= D[N(-d_-)K - N(-d_+)F] \\
 d_{+/-} &= \frac{1}{\sigma\sqrt{\tau}}[\ln(\frac{F}{K}) + / - \frac{1}{2}\sigma^2\tau]
 \end{aligned}$$

The variables are:

$\tau = T - t$  is the time to expiry

$D = e^{-r\tau}$  is the discount factor

## 4 Heston Stochastic Volatility Model

### 4.1 Introduction

The Black-Scholes equation assumes the volatility is constant, which is the ideal case and not practical in the real market. The Heston model assumes the volatility to follow a stochastic process. Suppose a stock price under risk-neutral measure is governed by

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)\tilde{d}W_1(t) \quad (2)$$

and the volatility itself is governed by the equation

$$dV(t) = (a - bV(t))dt + \sigma\sqrt{V(t)}\tilde{d}W_2(t) \quad (3)$$

Where

$$\tilde{d}W_1(t)\tilde{d}W_2(t) = \rho dt$$

At time  $t$ , the risk-neutral price of a call expiring at time  $T \geq t$  in this model is

$$c(t, S(t), V(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ | F(t)]$$

If we move the term  $e^{-rt}$  to the left hand side, we see

$$e^{-rt}c(t, S(t), V(t)) = \tilde{E}[e^{-rT}(S(T) - K)^+ | F(t)] \quad (4)$$

which satisfies the martingale property. Then we take the differentiation of  $e^{-rt}c(t, S(t), V(t))$ . We get

$$\begin{aligned} & d(e^{-rt}c(t, S(t), V(t))) \\ &= \frac{\partial e^{-rt}}{\partial t}c(t, S(t), V(t)) + e^{-rt} \frac{\partial c(t, S(t), V(t))}{\partial t} dt \\ &= -re^{-rt}c(t, S(t), V(t))dt \quad (1) \\ &+ e^{-rt} \frac{\partial c}{\partial t} dt \quad (2) \\ &+ e^{-rt} \frac{\partial c}{\partial S} dS \quad (3) \\ &+ e^{-rt} \frac{\partial^2 c}{\partial S^2} dS dS \quad (4) \\ &+ e^{-rt} \frac{\partial c}{\partial V} dV \quad (5) \\ &+ e^{-rt} \frac{\partial^2 c}{\partial V^2} dV dV \quad (6) \\ &+ e^{-rt} \frac{\partial^2 c}{\partial V \partial S} dV dS \quad (7) \end{aligned}$$

As we are interested in only the  $dt$  terms, we find out the  $dt$  terms from (1) to (7) the  $dt$  term in (1) is

$$-rc(t, S(t), V(t))e^{-rt}dt$$

the  $dt$  term in (2) is

$$\frac{\partial c}{\partial t}e^{-rt}dt$$

the  $dt$  term in (3) is

$$\frac{\partial c}{\partial S}rSe^{-rt}dt$$

the  $dt$  term in (4) is

$$\frac{1}{2} \frac{\partial^2 c}{\partial S^2} V S^2 e^{-rt} dt$$

the dt term in (5) is

$$\frac{\partial c}{\partial V}(a - bV(t))e^{-rt}dt$$

the dt term in (6) is

$$\frac{1}{2} \frac{\partial^2 c}{\partial V^2} V \sigma^2 e^{-rt} dt$$

the dt term in (7) is

$$\frac{\partial^2 c}{\partial V \partial S} V S \sigma e^{-rt} dt$$

Collect all the dt terms and let those terms equal to zero, we get

$$c_t + rsc_s + (a - bv)c_v + \frac{1}{2}s^2vc_{ss} + \rho\sigma sv c_{sv} + \frac{1}{2}\sigma^2vc_{vv} = rc \quad (5)$$

The function  $c(t, s, v)$  satisfies boundary condition

$$\begin{aligned} c(T, s, v) &= (s - K)^+ \\ c(t, 0, v) &= 0 \\ c(t, s, 0) &= (s - e^{-r(T-t)}K)^+ \\ \lim_{s \rightarrow \infty} \frac{c(t, s, v)}{s - K} &= 1 \\ \lim_{v \rightarrow \infty} c(t, s, v) &= s \end{aligned}$$

Based on the solution to the BSM equation, we can guess that the solution has the following form

$$c(t, s, v) = sf(t, \log s, v) - e^{-r(T-t)}Kg(t, \log s, v) \quad (6)$$

Where f and g can be interpreted as a cumulative distribution function. Then since  $c(t, s, v)$  satisfies the partial differential equation 5, we can show that  $f$  and  $g$  satisfy the following

$$f_t + (r + \frac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v + \frac{1}{2}vf_{xx} + \rho\sigma vf_{xv} + \frac{1}{2}\sigma^2vf_{vv} = 0 \quad (7)$$

$$(8)$$

$$g_t + (r - \frac{1}{2}v)g_x + (a - bv)g_v + \frac{1}{2}vg_{xx} + \rho\sigma vg_{xv} + \frac{1}{2}\sigma^2vg_{vv} = 0 \quad (9)$$

$$(10)$$

The derivation is straightforward but one needs to keep in mind here we treat  $x$  and  $v$  as two independent variables. The above PDE for  $f$  and  $g$  satisfy boundary condition

$$\begin{aligned} f(T, x, v) &= 1_{x \geq \log K} \\ g(T, x, v) &= 1_{x \geq \log K} \end{aligned}$$

This implies that  $f$  and  $g$  can be interpreted as "Probabilities". We can define

$$f(t, x, v) = E^{t,x,v} 1_{x \geq \log K}$$

We suppose a pair of stochastic process  $X(t)$ ,  $V(t)$  given by the following expression

$$dX(t) = (r + \frac{1}{2}V(t))dt + \sqrt{V(t)}dW_1(t) \quad dV(t) = (a - bV(t) + \rho\sigma V(t))dt + \sigma\sqrt{V(t)}dW_2(t)$$

By F-K formula, we can show that  $f$  satisfies the PDE above. Similarly, we have

$$g(t, x, v) = E^{t,x,v} 1_{x \geq \log K}$$

and the stochastic process of  $X(t)$  and  $V(t)$  are

$$\begin{aligned} dX(t) &= (r - \frac{1}{2}V(t))dt + \sqrt{V(t)}dW_1(t) \\ dV(t) &= (a - bV(t))dt + \sigma\sqrt{V(t)}dW_2(t) \end{aligned}$$

To find the analytical solution of  $f(t, x, v)$  and  $g(t, x, v)$  is not an easy task. Instead, we do Fourier transform of  $f$  and  $g$ . First we work out the function  $f(t, x, v)$ . Let  $\tau = T - t$

$$\tilde{f}(k, v, \tau) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x, v, \tau)$$

The inverse Fourier transform is

$$f(x, v, \tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k, v, \tau)$$

Substitute this into equation 7, then

$$-\frac{\partial \tilde{f}}{\partial \tau} + (r + \frac{1}{2})ik\tilde{f} + (a - bv + \rho\sigma v)\frac{\partial \tilde{f}}{\partial v} - \frac{1}{2}vk^2\tilde{f} + \rho\sigma vik\frac{\partial \tilde{f}}{\partial v} + \frac{1}{2}\sigma^2v\frac{\partial^2 \tilde{f}}{\partial v^2} = 0$$

Now the problem is to solve for  $\tilde{f}$ . We note when  $\tau = 0$ ,

$$\tilde{f}(k, v, 0) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x, v, \tau = 0) = \int_{-\infty}^{\infty} dx e^{-ikx} 1_{x \geq \log K} = \int_0^{\infty} dx e^{-ikx} = \pi\delta(k) + \frac{1}{ik}$$

when  $\tau \neq 0$ , we guess a general solution which has the following form

$$\tilde{f}(k, v, \tau) = \exp(C\tau + D\tau v)\tilde{f}(k, v, 0)$$



From above we easily see it match the terminal condition at  $\tau \rightarrow 0$ . With inverse Fourier transform

$$\begin{aligned}
f(x, v, \tau, x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k, v, \tau) \\
&= \int_0^{\infty} \frac{dk}{2\pi} e^{ikx} \exp(C\tau + D\tau v) \tilde{f}(k, v, 0) \\
&= \int_0^{\infty} \frac{dk}{2\pi} e^{ikx} \exp(C\tau + D\tau v) (\pi\delta(k) + \frac{1}{ik}) \\
&= \frac{1}{2} + \int_0^{\infty} \frac{dk}{2\pi} e^{ikx} \exp(\frac{C\tau + D\tau v}{ik})
\end{aligned}$$

Now the only remaining task is to find C and D.