1 Gaussian Random Number Generator

Given U_1 and U_2 belong $[-1, 1]$, the following is to find out random variables that are Gaussian distributed.

a. Principle
1) Start from an cdf of joint Gaussian random variable x and y

$$\begin{aligned}&\int\int exp(-1/2x^2)exp(-1/2y^2)dxdy\\=&\int d\theta\int rexp(-r^2/2)dr\\=&\int pdf(\theta)*\int pdf(r)\end{aligned}$$

So if we are able to fine r and θ, then by taking x = r cos θ, y = r sin θ, we get x and y
2) To fine θ is easy as we can take $\theta = 2\pi U_2$
3) Suppose $R = \sqrt{-2lnU_1}$
The cdf of R is

$$\begin{aligned}&P(R <= r)\\=&P(\sqrt{-2U_1} <= r)\\=&P(U_1 >= exp(-1/2r^2))\\=&\int_{exp(1/2r^2)}^{1} du_1\\=&1 - exp(-1/2r^2)\end{aligned}$$

So the $pdf = rexp(-1/2r^2)$
4) Therefore

$$\begin{aligned}X &= \sqrt{-2lnU_1}cos(2\pi U_2)\\Y &= \sqrt{-2lnU_1}sin(2\pi U_2)\end{aligned}$$

b. Efficient way of evaluating cos and sin function
In most computers, evaluating cos and sine is not quite straightforward, so the above method is not practical to use. So the following alternative avoids evaluating sine and cos functions directly.

1) We generate new random variable $S = U_1^2 + U_2^2$

$$\begin{aligned}
P(S < s) &= P(U_1^2 + U_2^2 < s) \\
&= \int\int_{u_1^2+u_2^2} 1dxdy \\
&= \int_0^{\sqrt{s}} rdr * 2\pi \\
&= \pi r^2|_0^s \\
&= \pi s
\end{aligned}$$

So pdf of s is 1/pi, it is uniform distribution.
2) Now if we let $S < 1$, by taking S that satisfies only $U_1^2 + U_2^2 < 1$, then throwing away others, then S is uniform in $(0,\pi)$, which corresponds r in(0, 1) Then a.4) becomes

$$\begin{aligned}
X &= \sqrt{-2lnS}cos(2\pi S) = \sqrt{-2lnS}U_1/\sqrt{S} \\
Y &= \sqrt{-2lnS}sin(2\pi S) = \sqrt{-2lnS}U_2/\sqrt{S}
\end{aligned}$$

c. Final algorithm

1) Generate two uniform random variables U_1 U_2 y belongs to [-1, 1]
2) If $U_1^2 + U_2^2 >= 1$, throw away and regenerate.
3) Return

$$U_1 * \sqrt{-2log(U_1^2 + U_2^2)/(U_1^2 + U_2^2)}$$
$$U_2 * \sqrt{-2log(U_1^2 + U_2^2)/(U_1^2 + U_2^2)}$$