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1 Black Scholes Merton Equation

We assume the stock prices following a geometric Brownian motion
1) Stock price:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

2) We have a portfolio X(t) which consists of δ t share of stock $\Delta(t)S(t)$, and $(X(t) - \Delta(t)S(t))$ money market account with interest rate r.

3) Change of the portfolio with respect to time

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t) + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

4) Change of the present value of the portfolio with respect to time

$$d(e^{-rt}S(t)) = (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$$

5) With a few steps, we get

$$\begin{aligned} & d(e^{-rt}X(t)) \\ =& \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \\ =& \Delta(t)d(e^{-rt}S(t)) \end{aligned}$$

6) Assume the option value is c(t,S(t)) and we apply Ito's formula

$$\begin{aligned} & d(e^{-rt}c(t, S(t)) \\ =& e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_S(t, S(t)) + 1/2\sigma^2 S^2(t)c_S S(t, S(t))]dt \\ & + e^{-rt}\sigma S(t)c_S(t, S(t))dW(t) \end{aligned}$$

7)Now equate the 2nd line of Equation in 5) and 6), we get
dW(t) term:

$$\Delta(t) = \frac{\partial c(t, S(t))}{\partial S(t)}$$

dt term:

$$rc(t, S) = c_t(t, S(t)) + rS(t)\frac{\partial c(t, S(t))}{\partial S(t)} + \frac{1}{2}\sigma^2 S^(t)\frac{\partial c(t, S(t))}{\partial S^2(t)}$$

which is known as Black-Scholes-Merton partial differential equation.