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1 Basic of Fourier Transform

Fourier Series
If $x(t) = x(t+T)$ then $x(t)$ can be written as

$$x(t) = \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikt}{T}}$$

i is the imaginary unit, and k is an integer. The above expression is eligible because $e^{\frac{2\pi ikt}{T}}$ is a periodic function

$$e^{\frac{2\pi ikt}{T}} = e^{\frac{2\pi ik(t+T)}{T}}$$

Each basis $e^{\frac{2\pi i}{T}}$ represents a signal with frequency $f_k = \frac{k}{T}$. So the interval between each adjacent frequency $\Delta f = \frac{1}{T}$. Based on orthogonality, we can get c_k

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-i\frac{2\pi kt}{T}}dt$$

Fourier Series: Example

$$x(t) = cos(2\pi f_0 t) = \frac{1}{2}(e^{i2\pi f_0 t} + e^{-i2\pi f_0 t})$$

where $f_0 = \frac{1}{T}$

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T x(t)e^{-i\frac{2\pi kt}{T}}dt \\ &= \frac{1}{T}\int_0^T \frac{1}{2}(e^{\frac{2i\pi t}{T}} + e^{\frac{-2i\pi t}{T}})e^{-i\frac{2\pi kt}{T}}dt \end{aligned}$$

Only terms with k = +/-1 in the above expression can survive, so

$$c_1 = \frac{1}{T}\int_0^T \frac{1}{2}dt = \frac{1}{2}$$

Similarly, $c_{-1} = \frac{1}{2}$.

Fourier Transform
We can generalize the Fourier series to non-periodic functions. We define the Fourier transform as

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi ift}dt$$

With the inverse Fourier transform defined as

$$x(t) = \int_{-\infty}^{\infty} \mathcal{F}(f) e^{2\pi i f t} df$$

To see why the above makes sense, it is easy to prove the identity.

$$\begin{aligned}
x(t^{'}) &= \int_{-\infty}^{\infty} \mathcal{F}(f) e^{2\pi i f t^{'}} df \\
&= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt) e^{2\pi i f t^{'}} df \\
&= \int_{-\infty}^{\infty} x(t) (\int_{-\infty}^{\infty} e^{-2\pi i f t} e^{2\pi i f t^{'}}) df \\
&= \int_{-\infty}^{\infty} x(t) \delta(t - t^{'}) dt \\
&= x(t^{'})
\end{aligned}$$

Fourier Transform: Example
1. Constant Function

$$x(t) = 1$$

$$\begin{aligned}
\mathcal{F}(f) &= \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi i f t} dt \\
&= \lim_{a \to \infty} \int_{-a}^{a} e^{-2\pi i f t} dt \\
&= \lim_{a \to \infty} \frac{1}{-2\pi i f t} e^{-2\pi i f t} |_{-a}^{a} \\
&= \lim_{a \to \infty} \frac{1}{-2\pi i f a} (e^{-2\pi i f a} - e^{2\pi i f a}) \\
&= \lim_{a \to \infty} \frac{1}{2\pi i f a} (e^{2\pi i f a} - e^{-2\pi i f a}) \\
&= \lim_{a \to \infty} 2 \frac{sin(2\pi f a)}{2\pi f a} \\
&= 2 \lim_{a \to \infty} \frac{sin(2\pi f a)}{2\pi f a} \\
&= \delta(f)
\end{aligned}$$

2. Trigeometic Function Take the same x(t) as above in the discrete case

$$x(t) = cos(2\pi f_0 t) = \frac{1}{2}(e^{2i\pi f_0 t} + e^{-2i\pi f_0 t})$$

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} \frac{1}{2}(e^{2i\pi f_0 t} + e^{-2i\pi f_0 t})e^{-2i\pi f t}dt \quad = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Discrete Fourier Series
The above is the Fourier transform in continuous case, in discrete case If $x = n\Delta t$, where n = 1...N, and $T = N\Delta t$, then the Fourier series can be written as

$$\begin{aligned} f(n) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k n \Delta t}{N\Delta t}} \\ &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi i k n}{N}} \end{aligned}$$

$$c_k = \frac{1}{N\Delta t}\sum_{n=1}^{N} f(n\Delta t)e^{-i2\pi k \frac{1}{N\Delta t} n\Delta t} d(n\Delta t) = \frac{1}{N}\sum_{n=1}^{N} f(n)e^{-i2\pi k \frac{n}{N}}$$

This is the discrete Fourier series.

$$\Delta f = \frac{1}{T} = \frac{1}{N\Delta t}$$

Discrete Fourier Transform In the discrete case, suppose we sample a signal N times within time T. Then $t = n\Delta t$, where $\Delta t = N/T$, the integral in the Fourier transform becomes a summation. So we write the Fourier transform as

$$\mathcal{F}(f) = \sum_{0}^{N} x(n\delta t)e^{-2\pi i f n \Delta t}\frac{T}{N}$$

In frequency domain, the frequency also becomes discrete, and $\Delta f = \frac{1}{T}$. Let $f = k\Delta f$, then

$$\begin{aligned} \mathcal{F}(k\Delta f) &= \sum_{0}^{N} x(n\Delta t)e^{-2\pi i k \Delta f n \Delta t}\frac{T}{N} \\ &= \sum_{0}^{N} x(n\Delta t)e^{-2\pi i k n/N}\frac{T}{N} \end{aligned}$$

Then we work out the Fourier transform and inverse Fourier transform identity

$$\begin{aligned} x(n^{'}\Delta t) &= \sum_{k=0}^{N} \mathcal{F}(k\Delta f)e^{2\pi i k n^{'}/N}\Delta f \\ &= \sum_{k=0}^{N}(\sum_{n=0}^{N} x(n\Delta t)e^{-2\pi i k n/N}\frac{T}{N})e^{2\pi i k n^{'}/N}\Delta f \\ &\sum_{k=0}^{N}\frac{1}{N}(\sum_{n=0}^{N} x(n\Delta t)e^{-2\pi i k n/N})e^{2\pi i k n^{'}/N} \end{aligned}$$

So we define discrete Fourier transform

$$\mathcal{F}(k) = \sum_{n=0}^{N} x(n) e^{-2\pi ikn/N}$$

and the discrete inverse Fourier transform

$$x(n) = \frac{1}{N} \sum_{n=0}^{N} \mathcal{F}(k) e^{2\pi ikn/N}$$

Example
Let N = 4, and

$$f(n) = cos(2\pi \frac{n}{4}) = \frac{1}{2}(e^{i2\pi \frac{n}{4}} + e^{-i2\pi \frac{n}{4}})$$

$$c_k = \frac{1}{4} \sum_{n=1}^{4} \frac{1}{2}(e^{i2\pi \frac{n}{4}} + e^{-i2\pi \frac{n}{4}}) e^{-i\frac{2\pi kn}{4}}$$

Similary to the continuous case, only terms with k = +/-1 in the above expression can survive, when $k = 1$

$$\begin{aligned} c_1 &= \frac{1}{4} \sum_{n=1}^{4} \frac{1}{2} e^{i2\pi \frac{n}{4}} e^{-i\frac{2\pi n}{4}} \\ &= \frac{1}{4}\frac{1}{2}4 \\ &= \frac{1}{2} \end{aligned}$$

What about case for $k = -1$? We define k = 1, 2, 3, 4 so $k = -1$ is not defined. However, in discrete case we note $c_{-1} = c_3$ due to the periodicity. Similarly, we can calculate $c_3 = \frac{1}{2}$.
N is the total sample within time T.
Properties
1) To be eligible, f(x) has to be a period function with time T(with frequency $F = \frac{1}{T}$) in both continuous case and discrete case. The requirement in discrete case leads to uniform sampling theorem used in signal processing. The total sampling time $T_{sampling}$ has to be an integer multiple of T.

$$T_{sampling} = MT$$

while $T = \frac{N}{F_s}$ So

$$MT = N\Delta t$$

if we let $\Delta t = \frac{1}{F_s}$, where F_s is the sampling frequency, and $T = \frac{1}{F}$, we have

$$\frac{M}{F} = \frac{N}{F_s}$$

2) If $f(x)$ is real, which means $f(x) = f^*(x)$. We then substitute Fourier series for both $f(x)$ and $f*(x)$,

$$\sum_{-\infty}^{+\infty} c_k e^{2\pi i \frac{1}{T} kx} = \sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} \tag{1}$$

Since the summation on the right hand side is from $-\infty$ to ∞, it is eligible to replace k with k.

$$\sum_{-\infty}^{+\infty} c_k^* e^{-2\pi i \frac{1}{T} kx} = \sum_{\infty}^{-\infty} c_{-k}^* e^{2\pi i \frac{1}{T} kx} \tag{2}$$

Combine the above two equations 1 and 2, we can see $c_k = c_{-k}^*$. This means they are complex conjugate: their magnitude are equal, their phase are opposite. Namely $||c_k|| = ||c_{-k}||$, $\phi(c_k) = \phi(c_{-k})$.

3) Connection between complex representation and real representation.

We have shown that for real signal $c_k = c_{-k}^*$ and $c_k = |c_k|e^{j\theta_k}$, $c_{-k} = |c_k|e^{-j\theta_k}$. And in complex representation, we can combine the term with index k and -k,

$$c_k e^{j2\pi kF_0 t} + c_{-k} e^{-j2\pi kF_0 t} = 2|c_k|cos(2\pi kF_0 t + \theta_k)$$

$$\begin{aligned} f(x) &= \sum_{-\infty}^{+\infty} c_k e^{\frac{2\pi ikx}{T}} \\ &= c_0 + 2\sum_{k=1}^{\infty} |c_k|cos(2\pi kF_0 t + \theta_k) \\ &= a_0 + \sum_{k=1}^{\infty} (a_k cos(2\pi kF_0 t) - b_k sin(2\pi kF_0 t)) \end{aligned}$$

where $a_0 = c_0$, $a_k = 2|c_k|cos\theta_k$, $b_k = 2|c_k|sin\theta_k$. 4) $c_k = c_{k+N}$. So when a signal contains frequency component no larger than B, in other words, the bandwidth of the signal is $2B(-B(to)B)$, then in order to capture the whole bandwidth of the signal, $N\Delta f > 2B$. This leads to Nyquist sampling theorem $F_s > 2B(bandwidth)$.

5) Power density

$$\begin{aligned} P_x &= \frac{1}{T}\int |x(t)|^2 dt \\ &= \frac{1}{T}\int x(t) \sum_{-\infty}^{\infty} c_k^* e^{-j2\pi kF_0 t} \\ &= \sum_{-\infty}^{\infty} c_k^* [\frac{1}{T}\int x(t) e^{-j2\pi kF_0 t}] \\ &= \sum_{-\infty}^{\infty} |c_k|^2 \end{aligned}$$

When signal is real, then

$$
\begin{aligned}
P_x &= \sum_{-\infty}^{\infty} |c_k|^2 \\
&= a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)
\end{aligned}
$$

2 Fast Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

let

$$u_k = e^{-i2\pi k \frac{n}{N}}$$

then we have the basis orthogonality

$$u_{k1}^T u_{k2} = N\delta_{k_1,k_2}$$

We recognize we can write X_k with even index terms and odd index terms

$$
\begin{aligned}
X_k = &\text{ Even index parts } + \text{ Odd index parts} \\
&= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}2mk} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k} \\
&= \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2}mk}
\end{aligned}
$$

(We can view this as Fourier Transform of N/2 even indexed points, where k is 0,1N/2)

$$+\, e^{-\frac{2\pi i}{N}k}$$

$$\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2}mk}$$

(We can view this as Fourier Transform of N/2 odd indexed points, where k is 0,1N/2)

(Since each part is a Fourier transform of N/2 points, k has to be smaller than N/2)

$$= E_k + e^{-\frac{2\pi i}{N}k} O_k$$

As noted, the above derivation is for $k < N/2$, a very similar derivation for $N/2 <= k < N$ leads to

$$X_{k+N/2} = E_k - e^{-\frac{2\pi i}{N}k} O_k$$

Now we have divided the FFT of N points to two FFT with N/2 points Keep going till we reach the size to one, then combine together recursively.

3 Fourier Transform of Useful Functions

The Fourier Transform of Step Function
Let $u(t)$ be a step function: $u(t) = 1$ when $t \geq 0$, $u(t) = 0$ when $t < 0$. And its derivative is a delta function

$$\frac{d\,\mathrm{u}(t)}{dt} = \delta(t)$$

Taking Fourier transform on both sides yields

$$2\pi i f \mathcal{F}(f) = 1$$

So

$$\mathcal{F}(f) = \frac{1}{2\pi i f}|_{f \neq 0} + \mathcal{F}(f)|_{f=0}$$

Since any function with a different constant can have the same derivative, the Fourier transform of the original function has to have a constant, which corresponds to zero frequency component $F(0)$. The constant component of function $u(t)$ is its offset to zero, which is $1/2$. so

$$F(f) = \frac{1}{2\pi i f}|_{f \neq 0} + \frac{1}{2}\delta(f)$$

The Fourier Transform of a Shifted Step Function
Let $u(t)$ be a step function: $u(t-\tau) = 1$ when $t \geq \tau$, $u(t-\tau) = 0$ when $t < \tau$. Then

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} u(t-\tau) e^{-2\pi f t} dt$$

Let $t^{'} = t - \tau$, then

$$\mathcal{F}(f) = e^{-2\pi i f \tau} \int_{-\infty}^{\infty} u(t^{'}) e^{-2\pi f t^{'}} dt^{'}$$

So we see this is a factor times Fourier transform of step function, therefore

$$\begin{aligned}
\mathcal{F}(f) &= e^{-2\pi i f \tau}\left(\frac{1}{2\pi i f}|_{f \neq 0} + \frac{1}{2}\delta(f)\right) \\
&= e^{-2\pi i f \tau}\frac{1}{2\pi i f}|_{f \neq 0} + \frac{1}{2}\delta(f)
\end{aligned}$$

The Fourier Transform of Gaussian

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$$

$$\mathcal{F}(f) = e^{-2\pi^2\sigma^2 f^2}$$

So the Fourier transform of a Gaussian function is another Gaussian function but with different width.

The Fourier Transform of Dirac Comb

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

It is clearly that x(t) is periodic with period T. So we can expand that into Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t/T}$$

Where

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-i\frac{2\pi k t}{T}} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(0) e^{-i\frac{2\pi k t}{T}} \\ &= \frac{1}{T} \end{aligned}$$

So

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{2\pi i k t/T}$$

On the other hand, based on the formula of Fourier transform

$$\mathcal{F}(f) = \int \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-2\pi i f t} dt \quad = \sum_{n=-\infty}^{\infty} e^{-2\pi i n T f} = \sum_{n=-\infty}^{\infty} e^{-2\pi i n f/f_0}$$

Comparing the Fourier series of x(t) and the expression of F(f), they are the same except T being changed to f_0. Therefore we can conclude that F(f) itself is also a Dirac comb, which is

$$\mathcal{F}(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

The Fourier Transform of White Noise
Assuming noise we sample in time is $n[m]$, where m = 0,... M-1. $n[m]$ is a Gaussian random variable with zero mean and variance σ^2. The the FFT of $n[m]$ is

$$\begin{aligned}N[k] &= \frac{1}{M}\sum_{m=0}^{M-1} n[m]e^{-i2\pi mk/M}\\ &= \frac{1}{M}\sum_{m=0}^{M-1} n[m](cos(2\pi mk/M) - i\ n[m]sin(2\pi mk/M))\end{aligned}$$

The expected value is

$$\begin{aligned}E[N[k]] &= E[\frac{1}{M}\sum_{0}^{M-1} n[m]e^{-i2\pi mk/M}]\\ &= \frac{1}{M}\sum_{0}^{M-1} E[n[m]]e^{-i2\pi mk/M}]\\ &= 0(\text{because E[n[m]] = 0})\end{aligned}$$

The variance of the real part is

$$\begin{aligned}Var[R[N[k]]] &= E[(\frac{1}{M}\sum_{m=0}^{M-1} n[m](cos(2\pi mk/M)) * (\frac{1}{M}\sum_{p=0}^{M-1} n[p](cos(2\pi pk/M))]\\ &= \frac{1}{M^2}E[\sum_{m=0}^{M-1} n[m]n[p]\delta(n-p)cos(2\pi mk/M) * cos(2\pi pk/M)]\\ &= \frac{1}{M^2}\sum_{m=0}^{M-1} E[n[m]^2]cos^2(2\pi mk/M)\\ &= \frac{1}{M^2}\sigma^2(\sum_{m=0}^{M-1} cos^2(2\pi mk/M))\\ &= \frac{1}{M^2}\sigma^2(\frac{M}{2} + \frac{cos((M+1)2\pi k/M)sin(2\pi Mk/M)}{2sin(2\pi k/M)})\\ &= \frac{1}{M}\frac{\sigma^2}{2}\end{aligned}$$

The same derivation applies for the imaginary part. So the FFT is Gaussian noise with mean zero and variance σ^2.

4 Connection with Uncertainty Principle

Relationship between time length and frequency bandwidth
We consider a few examples

1) We consider a function g(t) which is infinitely long in time domain

$$g(t) = cos(2\pi f_0 t)$$

Its Fourier transform is

$$\begin{aligned}
F(f) &= \int \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2} e^{i2\pi f t} dt \\
&= \int \frac{1}{2} e^{i2\pi t(f_0+f)} dt + \int \frac{1}{2} e^{i2\pi t(f-f_0)} dt \\
&= \frac{1}{2}\delta(f+f_0) + \frac{1}{2}\delta(f-f_0)
\end{aligned}$$

The last line is based on $\int_{-\infty}^{\infty} e^{i2\pi f t} = \delta(f)$.
Since the delta function has width zero, so the the bandwidth in frequency domain is zero. We see a signal which is infinitely long in time domain has zero bandwidth in frequency domain.
2) We consider a function g(t) which has zero width in time, namely an impulse function.

$$g(t) = \delta(t)$$

Since this function is not a periodic function, we assume its period is infinity. Its Fourier transform is

$$F(f) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi f t} = 1$$

Now we see a signal which has zero width in time has infinitely long frequency bandwidth. This leads to the uncertainty principle.
Uncertainty Principle In quantum mechanics, if there is a particle with position x and momentum p, then uncertainty principle states

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Similar relationship holds for time t and Energy.

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

We can modify this expression to get the time and frequency relationship in our Fourier transform. Since $E = \hbar\omega$. Then

$$\Delta t \Delta \omega \geq \frac{1}{2}$$