1 ANOVA One-Way Model

One way ANOVA model states as following:

$$
\begin{aligned}
Y_{ij} &= \mu + \alpha_i + \epsilon_{ij} \\
i &\leq I; j \leq J
\end{aligned}
$$

with the assumption that $\sum_i \alpha_i = 0$, and ϵ_{ij} $N(0, \sigma^2)$.
We now calculate the expectation of the sum square error

$$
\begin{aligned}
E[SSE] &= E[\sum_{ij}(Y_{ij} - \bar{Y}_{i.})^2)] \\
&= E[\sum_{ij}(\mu + \alpha_i + \epsilon_{ij} - (\mu + \alpha_i + \frac{1}{J}\sum_j \epsilon_{ij}))^2] \\
&= E[\sum_{ij}(\epsilon_{ij} - \frac{1}{J}\sum_j \epsilon_{ij})^2] \\
&= \sum_{ij} E[(\epsilon_{ij} - \frac{1}{J}\sum_j \epsilon_{ij})^2] \\
&= \sum_{ij}(E[\epsilon_{ij}^2] - 2E[(\epsilon_{ij})(\frac{1}{J}\sum_j \epsilon_{ij}))] + E[(\frac{1}{J}\sum_j \epsilon_{ij})^2])
\end{aligned}
$$

So, $E[SSE]$ is

$$
\begin{aligned}
E[SSE] &= \sum_{ij}(\sigma^2 - \frac{2}{J}\sigma^2 + \frac{J}{J^2}\sigma^2) \\
&= IJ(\sigma^2 - \frac{2}{J}\sigma^2 + \frac{1}{J}\sigma^2) \\
&= IJ\sigma^2 - I\sigma^2 = I(J-1)\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
E[SS\alpha] &= E[\sum_{ij}(\bar{Y}_{i.} - \bar{Y}_{..})^2)] \\
&= E[\sum_{ij}(\mu + \alpha_i + \frac{\sum_j \epsilon_{ij}}{J} - (\mu + \sum_i \frac{\alpha_i}{I} + \frac{1}{IJ}\sum_{ij}\epsilon_{ij}))^2] \\
&= E[\sum_{ij}(\alpha_i + \sum_j \frac{\epsilon_{ij}}{J} - \sum_{ij}\frac{\epsilon_{ij}}{IJ})^2] \\
&= J\sum_i \alpha_i^2 + \sum_{ij}(E[(\frac{\sum_j \epsilon_{ij}}{J})^2] - 2E[(\frac{\sum_j \epsilon_{ij}}{J})(\frac{1}{IJ}\sum_{ij}\epsilon_{ij}))] + E[(\frac{1}{IJ}\sum_{ij}\epsilon_{ij})^2]) \\
&= J\sum_i \alpha_i^2 + \sum_{ij}(\frac{1}{J}\sigma^2 - \frac{2}{IJ}\sigma^2 + \frac{1}{IJ}\sigma^2) \\
&= J\sum_i \alpha_i^2 + (I-1)\sigma^2 \\
\frac{E(SS\alpha)}{I-1} &= J\frac{\sum_i \alpha^2}{I-1} + \sigma^2
\end{aligned}
$$

2 ANOVA Two-Way Model

$$
\begin{aligned}
Y_{ijk} &= \mu + \alpha_i + \beta_j + \epsilon_{ijk} \\
i &\leq I; j \leq J; k \leq K
\end{aligned}
$$

$$
\begin{aligned}
E[SSE] &= E[\sum_{ijk}(Y_{ijk} - \bar{Y}_{ij.})^2)] \\
&= E[\sum_{ij}(\mu + \alpha_i + \beta_j + \epsilon_{ijk} - (\mu + \alpha_i + \beta_j + \frac{1}{K}\sum_k \epsilon_{ijk}))^2] \\
&= E[\sum_{ij}(\epsilon_{ijk} - \frac{1}{K}\sum_k \epsilon_{ijk})^2]
\end{aligned}
$$

Based on the SSE we have derived in one-way model, we can easily see

$$E[SSE] = IJ(K-1)\sigma^2$$

$$
\begin{aligned}
E(SS\alpha) &= E[\sum_{ijk}(\bar{Y}_{i..} - \bar{Y}_{...})] \\
&= E[\sum_{ijk}(\mu + \alpha_i + \frac{\sum_j \beta_j}{J} + \frac{\sum_{jk}\epsilon_{ijk}}{JK} - \mu - \frac{\sum_i \alpha_i}{I} + \frac{\sum_j \beta_j}{J} + \frac{\sum_{ijk}\epsilon_{ijk}}{IJK}] \\
&= E[\sum_{ijk}(\alpha_i + \frac{\sum_{jk}\epsilon_{ijk}}{JK} - \frac{\sum_{ijk}\epsilon_{ijk}}{IJK})] \\
&= JK\sum_i \alpha_i^2 + (I-1)\sigma^2
\end{aligned}
$$

The last line is based on what we derived in one-way model.

3 ANOVA Two-Way nested Model

$Y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ijk}$

$$
\begin{aligned}
E[SSE] &= E[\sum_{ijk}(\mu + \alpha_i + \beta_j(i) + \epsilon_{ijk}) - (\mu + \alpha_i + \beta_j(i) + \frac{\sum_k \epsilon_{ijk}}{K})] \\
&= E[\sum_{ijk}(\epsilon_{ijk} - \frac{\sum_k \epsilon_{ijk}}{K})^2] \\
&= IJ(K-1)\sigma^2
\end{aligned}
$$

The last line is based on what we derived in one-way model.

$$
\begin{aligned}
E[SS\beta|\alpha] &= E[\sum_{ijk}(\mu + \alpha_i + \beta_{j(i)} + \frac{\epsilon_{ijk}}{K}) - \mu - \alpha_i - \frac{\sum_j \beta_j}{J} - \frac{\sum_{jk}\epsilon_{ijk}}{JK})^2] \\
&= K\sum_{ij}\beta_{j(i)}^2 + I(J-1)\sigma^2
\end{aligned}
$$

The last line is based on what we derived in one-way model.

$$
\begin{aligned}
E[SS\alpha] &= E[\sum_{ijk}(\mu + \alpha_i + \frac{\sum_j \beta_{j(i)}}{J} + \frac{\sum_{jk}\epsilon_{ijk}}{JK} - \mu - \frac{\sum_i \alpha_i}{I} - \frac{\beta_{j(i)}}{IJ} - \frac{\sum_{ijk}\epsilon_{ijk}}{IJK})^2] \\
&= E[\sum_{ijk}(\alpha_i + \frac{\sum_{jk}\epsilon_{ijk}}{JK} - \frac{\sum_{ijk}\epsilon_{ijk}}{IJK})^2] \\
&= JK\sum_i \alpha^2 + (I-1)\sigma^2
\end{aligned}
$$

The last line is based on what we derived in one-way model.