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# A Lyapunov-based small-gain theorem for interconnected switched systems[☆]

Guosong Yang*, Daniel Liberzon

*Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, United States*

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ABSTRACT

Stability of an interconnected system consisting of two switched systems is investigated in the scenario where in both switched systems there may exist some subsystems that are not input-to-state stable (non-ISS). We show that, providing the switching signals neither switch too frequently nor activate non-ISS subsystems for too long, a small-gain theorem can be used to conclude global asymptotic stability (GAS) of the interconnected system. For each switched system, with the constraints on the switching signal being modeled by an auxiliary timer, a correspondent hybrid system is defined to enable the construction of a hybrid ISS Lyapunov function. Apart from justifying the ISS property of their corresponding switched systems, these hybrid ISS Lyapunov functions are then combined to establish a Lyapunov-type small-gain condition which guarantees that the interconnected system is globally asymptotically stable.


## 1. Introduction

The study of interconnected systems plays a significant role in the development of stability theory of dynamic systems, as it allows one to investigate the stability property of a complex system by analyzing its less complicated components. In this context, the small-gain theorems have proved to be important tools in the analysis of feedback connections of multiple systems, which appear frequently in the control literature. A comprehensive summarization of classical small-gain theorems involving input–output gains of linear systems can be found in [1]. This technique was then generalized to nonlinear feedback systems in [2,3] within the input–output context. The notion of input-to-state stability (ISS) proposed by Sontag [4] was naturally adopted and extended in [5] to establish a general nonlinear small-gain theorem which guaranteed both external and internal stabilities. Instead of analyzing the behavior of solution trajectories, Jiang et al. [6] have developed a Lyapunov-type nonlinear small-gain theorem based on the construction of ISS Lyapunov functions. A variety of nonlinear small-gain theorems were summarized in [7, Section 10.6].

In this paper, we explore the stability property of interconnected nonlinear switched systems. The study of switched systems has attracted a lot of attention in recent years (see, e.g., [8] and references therein). It is well-known that, in general, a switched system does not necessarily inherit the stability properties of its subsystems. For example, in [8, Part II] it is shown that a switched system consisting of two asymptotically stable subsystems may not be stable. In the linear system context, it was proved in [9] that such a switched system can achieve asymptotic stability providing the switching signal satisfies a certain dwell-time condition. This approach was then generalized to the nonlinear system context and to the concept of average dwell-time condition in [10]. In [11] a similar result was developed for a linear switched system with both stable and unstable subsystems by restricting the fraction of time in which the unstable subsystems are active. The study of stability property inheritance in switched systems was extended to the ISS context by Vu et al. [12], and to the IOSS (input/output-to-state stability) context by Müller and Liberzon [13], both for nonlinear switched systems. Furthermore, in [13] the IOSS property of a nonlinear switched system was studied also for the general case where some of the subsystems are not input/output-to-state stable. In [14] a small-gain theorem was formulated to establish the ISS property of a switched interconnected nonlinear system under an average dwell-time condition, and the global stabilization of a switched nonlinear system in strict-feedback form with possibly non-ISS subsystems was investigated based on this small-gain theorem.

In this work, a sufficient condition is formulated to guarantee the global asymptotic stability (GAS) of an interconnected system consisting of two nonlinear switched systems. We have considered

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* Corresponding author.
*E-mail addresses:* yang150@illinois.edu (G. Yang), liberzon@illinois.edu (D. Liberzon).

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a very general scenario: in both switched systems there may exist some subsystems that are not input-to-state stable (non-ISS). It is proved that, providing the switching signals neither switch too frequently (average dwell-time constraint) nor activate non-ISS subsystems for too long (time-ratio constraint), a small-gain theorem can be established by introducing auxiliary timers and adopting hybrid system techniques. In particular, for each switched system, a hybrid system is defined such that their solutions are correspondent and the constraints on the switching signal are modeled by the auxiliary timer. For each hybrid system, an ISS Lyapunov function is then constructed to establish the ISS property for all complete solutions to the hybrid system, and therefore all solutions to the switched system. (Although the result that a switched system with not necessarily ISS subsystems is ISS under certain average dwell-time condition and time-ratio condition has already been proved in [13], the Lyapunov-type formulation in this paper exhibits an improvement: it not only generates an ISS Lyapunov function which is used later in the study of the interconnected system, but provides means for robustness analysis as well.) With these two ISS Lyapunov functions, a small-gain condition is then established to prove the GAS property of the interconnected switched system.

Hybrid systems are dynamic systems that possess both continuous-time and discrete-time features. Trajectory-based small-gain theorems for interconnected hybrid systems were first presented in [15,16], while Lyapunov-based formulations were introduced in [17]. The concept of ISS Lyapunov function was extended to hybrid systems in [18]. In our analysis of hybrid systems, we have adopted the modeling framework proposed by Goebel et al. [19], which proved to be general and natural from the viewpoint of Lyapunov stability theory. In the hybrid system context, a detailed study of small-gain theorems based on the construction of ISS Lyapunov functions using this modeling framework can be found in [20–22]. Comparing to [22], our result on modifying the ISS Lyapunov function to guarantee its decrease along solutions is more general in the sense that it applies to the situation where the original ISS Lyapunov functions are increasing both at the jumps and during some of the flows. Based on the idea of restricting non-ISS subsystems' total activation time proportion proposed in [11,13], an aforementioned auxiliary timer is introduced in the construction of the hybrid system to manage the non-ISS flows.

This paper is structured as follows. In Section 2, we introduce some mathematical preliminaries. Our main result – the small-gain theorem for interconnected switched systems with both ISS and non-ISS subsystems – is presented and interpreted in Section 3, followed by a corollary discussing relaxations in the assumptions to conclude GAS when all subsystems are ISS. A detailed proof, prefaced by an introduction to hybrid systems, is provided in Section 4. Section 5 concludes the paper with a short summary and an outlook on future research.

## 2. Preliminaries

Consider a family of dynamic systems

$$\dot{x} = f_p(x, u), \quad p \in \mathcal{P} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $\mathcal{P}$ is the *index set* (which can in principle be arbitrary). For all $p \in \mathcal{P}, f_p$ is locally Lipschitz and $f_p(0, 0) = 0$. Given the family (1), a *switched system*

$$\dot{x} = f_\sigma(x, u) \tag{2}$$

is generated by a *switching signal* $\sigma: \mathbb{R}_{\geq 0} \to \mathcal{P}$ which specifies the index of the active system at time $t$. The switching signal $\sigma$ is assumed to be piecewise constant and right-continuous. Let $\psi_k$ ($k \in \mathbb{Z}_{>0}$) denote the time when the $k$-th switch occurs and define $\Psi := \{\psi_k : k \in \mathbb{Z}_{>0}\}$ as the set of switching time instants, which is assumed to contain no accumulation points. (Thus the switched system (2) has at most one switch at any time instant and finitely many switches in any finite time interval.) A function $u$ is an admissible input to the switched system (2) if it is measurable and locally essentially bounded.

Following Morse [9], we say that a switching signal $\sigma$ satisfies the *dwell-time condition* if there exists a $\tau_d \in \mathbb{R}_{>0}$, called the *dwell-time*, such that for all consecutive switching time instants $\psi_k, \psi_{k+1} \in \Psi$,

$$\psi_{k+1} - \psi_k \geq \tau_d. \tag{3}$$

A generalized concept was introduced by Hespanha and Morse [10]: a switching signal $\sigma$ is said to satisfy the *average dwell-time condition* if there exists a $\tau_a \in \mathbb{R}_{>0}$, called the *average dwell-time*, and an $N_0 \in \mathbb{Z}_{\geq 0}$ such that

$$N(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_a} \quad \forall\, t_2 \geq t_1 \geq 0, \tag{4}$$

where $N(t_2, t_1)$ denotes the number of switchings in the time interval $(t_1, t_2]$. Note that the dwell-time condition can be interpreted as a special case of the average dwell-time condition with $N_0 = 1$ and $\tau_a = \tau_d$.

For two vectors $x_1, x_2$, $(x_1, x_2)$ is used to denote their concatenation, that is, $(x_1, x_2) := (x_1^\top, x_2^\top)^\top$. For a vector $x \in \mathbb{R}^n$, we use $|x|$ to denote its Euclidean norm. For a compact set $A \subset \mathbb{R}^n$, we use $|x|_A$ to denote the Euclidean distance from a vector $x$ to $A$. For a function $u: \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\|u\|_t$ is used to denote its essential supremum (Euclidean) norm on the interval $[0, t]$.

A function $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing and positive definite. A function $\gamma$ is of *class* $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\lim_{r\to\infty} \gamma(r) = \infty$. In particular, this implies that $\gamma$ is globally invertible. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for all fixed $t$, $\beta(r, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(r, t) = 0$ for all fixed $r$.

As introduced by Sontag [4], a dynamic system from family (1) is called *input-to-state stable* (ISS) if there exist functions $\gamma \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ such that for all initial states $x(0) \in \mathbb{R}^n$ and all inputs $u: \mathbb{R}_{\geq 0} \to \mathbb{R}^m$,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_t) \quad \forall\, t \in \mathbb{R}_{\geq 0}. \tag{5}$$

The definition of input-to-state stability (ISS) also applies to switched systems. Note that for an autonomous dynamic system (i.e. $u \equiv 0$), the ISS property (5) is equivalent to the notion of *global asymptotic stability* (GAS) [23, Proposition 2.5].

## 3. Main result

### 3.1. Interconnected switched system with both ISS and non-ISS subsystems

Consider two switched systems

$$\begin{aligned} \dot{x}_1 &= f_{1,\sigma_1}(x_1, u_1), \\ \dot{x}_2 &= f_{2,\sigma_2}(x_2, u_2), \end{aligned} \tag{6}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $\sigma_i \in \mathcal{P}_i$ for all $i \in \{1, 2\}$.[1] Suppose the two switched systems fulfill the same assumptions as those imposed on the switched system (2) in Section 2. If $m_1 = n_2$ and $m_2 = n_1$, an *interconnected switched system* with the state $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ can be constructed by letting $u_1 = x_2$ and $u_2 = x_1$:

[1] We use $f_{i,\sigma_i}$ instead of $f_{\sigma_i}$ to avoid confusion in case the two index sets $\mathcal{P}_1, \mathcal{P}_2$ contain common elements.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_{1,\sigma_1}(x_1,x_2)\\ f_{2,\sigma_2}(x_2,x_1)\end{bmatrix}. \tag{7}$$

Suppose that in the interconnected switched system (7), both switched systems may contain ISS as well as non-ISS subsystems. For all $i \in \{1,2\}$, let $\mathcal{P}_{s,i}$ and $\mathcal{P}_{u,i}$ denote the subsets of $\mathcal{P}_i$ containing the indices of ISS and non-ISS subsystems, respectively. Then $(\mathcal{P}_{s,i}, \mathcal{P}_{u,i})$ forms a partition of $\mathcal{P}_i$ (i.e., $\mathcal{P}_{s,i} \cup \mathcal{P}_{u,i} = \mathcal{P}_i$ and $\mathcal{P}_{s,i} \cap \mathcal{P}_{u,i} = \varnothing$). Following Müller and Liberzon [13], we define $T_{s,i}(t_2,t_1)$ as the total activation time of ISS subsystems (i.e., subsystems from $\mathcal{P}_{s,i}$) on the time interval $(t_1,t_2]$ and $T_{u,i}(t_2,t_1)$ that for non-ISS subsystems. Then $T_{s,i}(t_2,t_1) + T_{u,i}(t_2,t_1) = t_2 - t_1$.

We introduce three constraints in the following assumption; the first two are frequently used in the context of switched systems, while the last one (the Time-Ratio Constraint) is somewhat less standard. The idea of restricting the fraction of time during which non-ISS subsystems are active in the third constraint is essentially introduced by Zhai et al. [11] and Müller and Liberzon [13].

**Assumption 1.** For all $i, j \in \{1,2\}$ such that $i \neq j$, the following constraints are satisfied:

Uniform ISS Lyapunov-Type Constraint There exists a family of positive definite $\mathcal{C}^1$ functions $V_{i,p_i}\colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, $p_i \in \mathcal{P}_i$ such that the following conditions hold:

1. $\exists\, \alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$ such that for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i \in \mathcal{P}_i$,
$$\alpha_{1,i}(|x_i|) \leq V_{i,p_i}(x_i) \leq \alpha_{2,i}(|x_i|). \tag{8}$$
2. $\exists\, \phi_i \in \mathcal{K}_\infty, \lambda_{s,i}, \lambda_{u,i} \in \mathbb{R}_{>0}$ such that for all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$ and all $p_s \in \mathcal{P}_{s,i}$, $p_u \in \mathcal{P}_{u,i}$,
$$|x_i| \geq \phi_i(|x_j|) \Rightarrow \begin{cases} \dfrac{\partial V_{i,p_s}(x_i)}{\partial x_i} \cdot f_{i,p_s}(x_i,x_j) \leq -\lambda_{s,i} V_{i,p_s}(x_i), \\ \dfrac{\partial V_{i,p_u}(x_i)}{\partial x_i} \cdot f_{i,p_u}(x_i,x_j) \leq \lambda_{u,i} V_{i,p_u}(x_i). \end{cases} \tag{9}$$
3. $\exists\, \mu_i \in \mathbb{R}_{\geq 1}$ such that for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i, q_i \in \mathcal{P}_i$,
$$V_{i,p_i}(x_i) \leq \mu_i V_{i,q_i}(x_i). \tag{10}$$

Average Dwell-Time Constraint The switching signal $\sigma_i$ satisfies the average dwell-time condition (4) with constants $\tau_{a,i} \in \mathbb{R}_{>0}$ and $N_{0,i} \in \mathbb{Z}_{\geq 0}$.

Time-Ratio Constraint There exist $\rho_i \in [0,1)$ and $T_{0,i} \in \mathbb{R}_{\geq 0}$ such that the total activation time of non-ISS subsystems satisfies

$$T_{u,i}(t_2,t_1) \leq T_{0,i} + \rho_i(t_2 - t_1) \quad \forall\, t_2 \geq t_1 \geq 0. \tag{11}$$

The constraints in Assumption 1 apply to each switched system separately. Moreover, the Uniform ISS Lyapunov-Type Constraint is a constraint on the subsystems' dynamics, while the Average Dwell-Time Constraint and the Time-Ratio Constraint are constraints on the switching signals.

**Remark 1.** The Uniform ISS Lyapunov-Type Constraint in Assumption 1 is "Lyapunov-type" in the sense that it constrains not only the ISS subsystems, but the non-ISS subsystems as well. The existence of functions $V_{i,p_s}$ satisfying (9) for $p_s \in \mathcal{P}_{s,i}$ follows from the fact that these subsystems are ISS [24], while the existence of functions $V_{i,p_u}$ satisfying (9) for $p_u \in \mathcal{P}_{u,i}$ is equivalent to the forward completeness property of non-ISS subsystems [25].

**Remark 2.** The Uniform ISS Lyapunov-Type Constraint in Assumption 1 is "uniform" since for the $i$-th switched system, it is satisfied by ISS Lyapunov functions $V_{i,p_i}$ for all subsystems, with fixed class $\mathcal{K}_\infty$ functions $\alpha_{1,i}, \alpha_{2,i}, \phi_i$ and constants $\lambda_{s,i}, \lambda_{u,i}, \mu_i$. This uniformity can be concluded automatically for some particular types of index sets. For example, (8) is guaranteed if $\mathcal{P}_i$ is finite and all subsystems are ISS [12, Remark 1]. Besides, for positive definite functions $V_{i,p_i}$, the existence of the uniform ratio bound $\mu_i$ in (10) is a sufficient condition for the existence of the uniform comparison functions $\alpha_{1,i}, \alpha_{2,i}$ in (8).

Our main result is stated as the following theorem:

**Theorem 1.** *Consider the interconnected switched system (7). Suppose that Assumption 1 holds with the constants satisfying*

$$\lambda_{s,i} > \frac{\ln(\mu_i)}{\tau_{a,i}} + \rho_i(\lambda_{s,i} + \lambda_{u,i}) =: \gamma_i \quad \forall\, i \in \{1,2\}. \tag{12}$$

*For all $i, j \in \{1,2\}$ such that $i \neq j$, let*

$$\varGamma_i := N_{0,i}\ln(\mu_i) + T_{0,i}(\lambda_{s,i} + \lambda_{u,i}), \tag{13}$$

*and $\chi_i \in \mathcal{K}_\infty$ be defined as*

$$\chi_i(r) := \alpha_{2,i}(\phi_i(\alpha_{1,j}^{-1}(r)))\exp(\varGamma_i). \tag{14}$$

*Then the interconnected switched system is globally asymptotically stable if the following small-gain condition is satisfied:*

$$\chi_1(\chi_2(r)) < r \quad \forall\, r \in \mathbb{R}_{>0}. \tag{15}$$

**Remark 3.** The inequality in (12) can be rewritten as

$$(1-\rho_i)\lambda_{s,i} - \rho_i\lambda_{u,i} - \frac{\ln(\mu_i)}{\tau_{a,i}} > 0 \quad \forall\, i \in \{1,2\},$$

which helps provide a clearer interpretation of the condition. Here $(1-\rho_i)\lambda_{s,i}$ measures the average rate of exponential decay of the ISS Lyapunov functions due to the ISS subsystems, while $\rho_i\lambda_{u,i}$ measures their exponential growth due to the non-ISS subsystems, and $\ln(\mu_i)/\tau_{a,i}$ measures their exponential growth due to the switches. Thus this condition can be interpreted as saying that, for each switched system, the ISS Lyapunov functions are decreasing on average.

**Remark 4.** The inequality in (12) can also be rewritten as

$$\lambda_{s,i} > \frac{1}{1-\rho_i}\left(\frac{\ln(\mu_i)}{\tau_{a,i}} + \lambda_{u,i}\right) - \lambda_{u,i} \quad \forall\, i \in \{1,2\},$$

from which it is clear that by increasing $\lambda_{s,i}$ (with all other parameters fixed), we are able to accommodate a larger time-ratio $\rho_i$. On the other hand, from the definitions of $\varGamma_i$ (13) and $\chi_i$ (14), and the small-gain condition (15), for a fixed time-ratio $\rho_i$, we see that one should work with the smallest possible $\lambda_{s,i}$ satisfying (12) to have the least conservative gain estimate.

**Remark 5.** Consider the following interconnected switched system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_{1,\sigma_1}(x_1,x_2,v_1)\\ f_{2,\sigma_2}(x_2,x_1,v_2)\end{bmatrix}, \tag{16}$$

where, for each $i \in \{1,2\}$, $v_i \in \mathbb{R}^{k_i}$ is an external input. If the Uniform ISS Lyapunov-Type Constraint in Assumption 1 is modified so that the effects of the external inputs are included in (9) (e.g., change the left-hand-side to $|x_i| \geq \phi_i(|x_j|) + \phi_{u,i}(|v_i|)$), then Theorem 1 can be used to conclude the input-to-state stability of the interconnected switched system (16). The proof is essentially the same as that of Theorem 1.

### 3.2. Interconnected switched system with only ISS subsystems

In the stability analysis of the interconnected switched system (7), a less complicated scenario arises when all the subsystems are ISS for both of the switched systems (i.e., for all $i \in \{1,2\}$, $\mathcal{P}_{s,i} = \mathcal{P}_i$, $\mathcal{P}_{u,i} = \varnothing$). In this case, the global asymptotic stability of the interconnected switched system can be established under less restrictive assumptions, as stated in the following assumption and corollary (which closely correspond to the results in [22]).

**Assumption 2.** For all $i, j \in \{1, 2\}$ such that $i \neq j$, the following constraints are satisfied:

Uniform ISS Lyapunov Constraint There exists a family of positive definite $\mathcal{C}^1$ functions $V_{i,p_i}: \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, $p_i \in \mathcal{P}_i$ such that the following conditions hold:

1. $\exists\, \alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$ such that (8) holds for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i \in \mathcal{P}_i$.
2. $\exists\, \phi_i \in \mathcal{K}_\infty$, $\lambda_i \in \mathbb{R}_{>0}$ such that for all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$ and all $p_i \in \mathcal{P}_i$,
$$|x_i| \geq \phi_i(|x_j|) \Rightarrow \frac{\partial V_{i,p_i}(x_i)}{\partial x_i} \cdot f_{i,p_i}(x_i, x_j) \leq -\lambda_i V_{i,p_i}(x_i). \quad (17)$$
3. $\exists\, \mu_i \in \mathbb{R}_{\geq 1}$ such that (10) holds for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i, q_i \in \mathcal{P}_i$.

Average Dwell-Time Constraint The switching signal $\sigma_i$ satisfies the average dwell-time condition (4) with constants $\tau_{a,i} \in \mathbb{R}_{>0}$ and $N_{0,i} \in \mathbb{Z}_{\geq 0}$.

**Remark 6.** The existence of a positive definite $\mathcal{C}^1$ function $V_{i,p_i}$ satisfying (8) and (17) is equivalent to the fact that subsystem $p_i$ in the $i$-th switched system is ISS [24]. The set of possible ISS-Lyapunov functions is constrained by condition (10). While this condition might seem somehow restrictive (e.g., it does not hold if $V_{i,p}$ is quadratic and $V_{i,q}$ is quartic for some $p, q \in \mathcal{P}_i$), it is quite common in the study of switched systems under slow switching assumptions, and is a considerable relaxation to the condition requiring a common ISS Lyapunov condition (which is equivalent to the Uniform ISS Lyapunov Constraint in Assumption 2 with $\mu_i = 1$ [26]); cf. [12, Remark 1 and Subsection 4.1].

**Corollary 1.** *Consider the interconnected switched system (7). Suppose that Assumption 2 holds with constants satisfying*

$$\lambda_i > \frac{\ln(\mu_i)}{\tau_{a,i}} \quad \forall\, i \in \{1, 2\}. \quad (18)$$

*For all $i, j \in \{1, 2\}$ such that $i \neq j$, let $\chi_i \in \mathcal{K}_\infty$ be defined as*

$$\chi_i(r) := \alpha_{2,i}(\phi_i(\alpha_{1,j}^{-1}(r))) \exp(N_{0,i} \ln(\mu_i)). \quad (19)$$

*Then the interconnected switched system is globally asymptotically stable if the small-gain condition (15) is satisfied.*

**Remark 7.** For an interconnected switched system (7) in which only one of the two switched systems contains non-ISS subsystems, the global asymptotic stability can be established if Assumption 1 is satisfied by this switched system, while Assumption 2 is satisfied by the other, and the small-gain condition (15) is satisfied with the functions $\chi_1$, $\chi_2$ defined as in (14) and (19), accordingly.

**Remark 8.** Suppose that in one of the two switched systems, instead of the average dwell-time condition (4), the switching signal satisfies the dwell-time condition (3) with dwell-time $\tau_{d,i}$. Then the same result holds if the average dwell-time $\tau_{a,i}$ in (12) (or (18), in case this switched system consists of only ISS subsystems) is substituted by $\tau_{d,i}$ and the constant $N_{0,i}$ in (14) (or (19)) is equal to 1.

## 4. Proof of the main result

In this section, a detailed proof of Theorem 1 is presented. We start by introducing some preliminaries for hybrid systems in Section 4.1. In Section 4.2, a correspondent hybrid system is constructed for each switched system under Assumption 1 and the correspondence between solutions is proved. An ISS Lyapunov function for each hybrid system is defined and verified in Section 4.3. In Section 4.4, the ISS property of each hybrid system is established.[2] Section 4.5 concludes the proof of Theorem 1 by showing that any solution to the interconnected switched system is GAS.

### 4.1. Preliminaries for hybrid systems

Following Goebel et al. [19, Chapter 2], a *hybrid system* with inputs can be modeled as

$$\begin{cases} \dot{z} \in F(z, u), & z \in C, \\ z^+ \in G(z, u), & z \in D, \end{cases} \quad (20)$$

where $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $C \subset \mathbb{R}^n$ is the *flow set*, $D \subset \mathbb{R}^n$ is the *jump set*, $F: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the *flow map* and $G: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the *jump map*.[3] (In this model, if $z \in C$, the state can flow at a velocity $\dot{z} \in F(z, u)$; if $z \in D$, the state can jump to a point $z^+ \in G(z, u)$; if $z \in C \cap D$, the state can either flow or jump.) $\mathcal{H} = (C, F, D, G)$ is called the *data* of the hybrid system. The solutions to the hybrid system are defined on the so-called hybrid time domain. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if

$$E = \bigcup_{k=0}^{K} ([\theta_k, \theta_{k+1}], k) \quad (21)$$

for some finite sequence of time instants $0 = \theta_0 < \theta_1 < \cdots < \theta_{K+1}$. $E$ is a *hybrid time domain* if for all $(T, K) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, K\})$ is a compact hybrid time domain. A *hybrid arc* is a function $z: \operatorname{dom} z \to \mathbb{R}^n$ defined on a hybrid time domain such that for each fixed $k \in \mathbb{Z}_{\geq 0}$, $z(\cdot, k)$ is locally absolutely continuous on $\{t : (t, k) \in \operatorname{dom} z\} =: \Theta_k^z$. A hybrid arc is *complete* if its domain is unbounded. A *hybrid input* is a function $u: \operatorname{dom} u \to \mathbb{R}^m$ defined on a hybrid time domain such that for each fixed $k \in \mathbb{Z}_{\geq 0}$, $u(\cdot, k)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t, k) \in \operatorname{dom} u\} = \Theta_k^u$. A hybrid arc $z: \operatorname{dom} z \to \mathbb{R}^n$ is a solution to a hybrid system $\mathcal{H} = (C, F, D, G)$ with a hybrid input $u: \operatorname{dom} u \to \mathbb{R}^m$ if the following conditions hold:

1. $\operatorname{dom} z = \operatorname{dom} u$.
2. $z(t, k) \in C$ and $\dot{z}(t, k) \in F(z(t, k), u(t, k))$ for all $k \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \Theta_k^z$.[4]
3. $z(t, k) \in D$ and $z(t, k+1) \in G(z(t, k), u(t, k))$ for all $(t, k) \in \operatorname{dom} z$ such that $(t, k+1) \in \operatorname{dom} z$.

With proper assumptions on the data $\mathcal{H}$, one can establish the local existence of solutions to the hybrid system, which may not be necessarily unique (see, e.g., [19, Proposition 2.10]).

Following Cai and Teel [18], for a function defined on a hybrid time domain $z: \operatorname{dom} z \to \mathbb{R}^n$, the essential supremum (Euclidean) norm up to hybrid time $(t, k)$ is denoted by $\|z\|_{(t,k)}$ and defined as

$$\|z\|_{(t,k)} := \max \left\{ \operatorname*{ess\,sup}_{\substack{(s,l) \in \operatorname{dom} z \setminus J(z), \\ s \leq t, l \leq k}} |z(s, l)|, \sup_{\substack{(s,l) \in J(z), \\ s \leq t, l \leq k}} |z(s, l)| \right\},$$

where $J(z)$ is the set of all $(s, l) \in \operatorname{dom} z$ such that $(s, l+1) \in \operatorname{dom} z$. (Note that the set of measure 0 of hybrid time that can be ignored when computing this essential supremum norm cannot include any jump time instants.)

For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and vectors $x, v \in \mathbb{R}^n$, the *Clarke derivative* [28] of $f$ at $x$ in the direction of $v$ is defined as

$$f^\circ(x; v) := \lim_{h \to 0^+} \sup_{y \to x} \frac{f(y + hv) - f(y)}{h}. \quad (22)$$

[2] This subsection is not directly related to the proof of Theorem 1 but considered as an independent result. More details on this result can be found in [27].

[3] We use "$\rightrightarrows$" to denote a set-valued mapping.

[4] Here $z(t, k)$ represents the state of the system at time $t$ and after $k$ jumps.

### 4.2. A correspondent hybrid system

In this subsection, for each switched system, we construct a hybrid system with state consisting of variables representing the state of the corresponding switched system, the corresponding switching signal and an auxiliary timer $\tau_i$. The dynamics of each timer is specifically designed to not only incorporate the effect of the Average Dwell-Time Constraint and Time-Ratio Constraint in Assumption 1, but also enable us to construct an ISS Lyapunov function for the corresponding hybrid system in Section 4.3.

For each $i \in \{1,2\}$, consider the hybrid system with state $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i) \in \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Gamma_i] =: \mathcal{Z}_i$ and input $\widetilde{u}_i \in \mathbb{R}^{m_i}$ defined as follows:

$$\begin{cases} \dot{z}_i \in F_i(z_i, \widetilde{u}_i), & z_i \in C_i, \\ z_i^+ \in G_i(z_i), & z_i \in D_i, \end{cases} \tag{23}$$

where

$$F_i(z_i, \widetilde{u}_i) := \begin{cases} \begin{bmatrix} \{f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{u}_i)\} \\ \{0\} \\ [0, \gamma_i] \end{bmatrix}, & \text{if } \widetilde{\sigma}_i \in \mathcal{P}_{s,i}, \\ \begin{bmatrix} \{f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{u}_i)\} \\ \{0\} \\ \{\gamma_i - (\lambda_{s,i} + \lambda_{u,i})\} \end{bmatrix}, & \text{if } \widetilde{\sigma}_i \in \mathcal{P}_{u,i}, \end{cases} \tag{24}$$
$$C_i := \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Gamma_i],$$
$$G_i(z_i) := \{\widetilde{x}_i\} \times (\mathcal{P}_i \setminus \{\widetilde{\sigma}_i\}) \times \{\tau_i - \ln(\mu_i)\},$$
$$D_i := \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln(\mu_i), \Gamma_i],$$

and constants $\gamma_i$ and $\Gamma_i$ are defined in (12) and (13), respectively. We will show that the following proposition holds:

**Proposition 1.** *Consider a solution $x_i$ to the $i$-th switched system in (6) with an input $u_i$ and a switching signal $\sigma_i$. Suppose that Assumption 1 is satisfied. Then there is a complete solution $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)$ to the hybrid system (23) with a hybrid input $\widetilde{u}_i$ such that*

$$\begin{cases} \widetilde{u}_i(t,k) = u_i(t) \\ \widetilde{x}_i(t,k) = x_i(t) \end{cases} \quad \forall\, (t,k) \in \operatorname{dom} z_i.^{5} \tag{25}$$

**Proof.** Suppose $x_i$ is a solution to the $i$-th switched system in (6) with an input $u_i$ and a switching signal $\sigma_i$. We construct a hybrid arc $z_i$ and a hybrid input $\tilde{u}_i$ in a recursive manner. Define $\Psi_i := \{\psi_{i,k} : k \in \mathbb{Z}_{>0}\}$ as the set of the switching time instants of $\sigma_i$ and let $\psi_{i,0} = 0$. For all $T \in \mathbb{R}_{\geq 0}$, let $K_{i,T} := \max\{k \in \mathbb{Z}_{\geq 0} : \psi_{i,k} \leq T\}$ be the number of switches on $[0, T]$ and

$$E_{i,T} := \left( \bigcup_{k=0}^{K_{i,T}-1} ([\psi_{i,k}, \psi_{i,k+1}], k) \right) \cup ([\psi_{i,K_{i,T}}, T], K_{i,T}). \tag{26}$$

Then $E_{i,T}$ is a compact hybrid time domain. Consider the hybrid input $\widetilde{u}_i$ and the hybrid arc $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)$ defined so that for all $T \in \mathbb{R}_{\geq 0}$, the following conditions hold:

- The domains satisfy $\operatorname{dom} z_i \cap ([0, T] \times \{0, 1, \ldots, K_{i,T}\}) = E_{i,T}$ and $\operatorname{dom} \widetilde{u}_i = \operatorname{dom} z_i$;
- For all $(t,k) \in E_{i,T}$, $\widetilde{u}_i(t,k) = u_i(t)$, $\widetilde{x}_i(t,k) = x_i(t)$ and $\widetilde{\sigma}_i(t,k) = \sigma_i(\psi_{i,k})$;
- For all $(t,k) \in E_{i,T}$,

$$\tau_i(t,k) = \begin{cases} \Gamma_i, & \text{if } k = 0, \\ \min\{\Gamma_i, \bar{\tau}_{s,i}(t,k)\}, & \text{if } k > 0,\ \sigma_i(\psi_{i,k}) \in \mathcal{P}_{s,i}, \\ \bar{\tau}_{u,i}(t,k), & \text{if } k > 0,\ \sigma_i(\psi_{i,k}) \in \mathcal{P}_{u,i}, \end{cases} \tag{27}$$

where

$$\begin{aligned} \bar{\tau}_{s,i}(t,k) &:= \tau_i(\psi_{i,k}, k-1) - \ln(\mu_i) + \gamma_i(t - \psi_{i,k}), \\ \bar{\tau}_{u,i}(t,k) &:= \bar{\tau}_{s,i}(t,k) - (\lambda_{s,i} + \lambda_{u,i})(t - \psi_{i,k}). \end{aligned} \tag{28}$$

We will show that, if Assumption 1 is satisfied, the hybrid arc $z_i$ is a complete solution to the hybrid system (23) with the hybrid input $\widetilde{u}_i$.

Indeed, by construction, $z_i$ and $\widetilde{u}_i$ are defined on the same hybrid time domain and satisfy the dynamics of the hybrid system (23). Then it remains to prove that $z_i$ is complete and $z_i(t,k) \in C_i \cup D_i = \mathcal{Z}_i$, which amounts to showing the following properties:

1. By the Uniform ISS Lyapunov-Type Constraint in Assumption 1, the solution $x_i$ to the $i$-th switched system in (6) is forward complete and is thus defined for all $t \in \mathbb{R}_{\geq 0}$. Therefore $\operatorname{dom} z_i$ is unbounded in the $t$-direction and $\widetilde{x}_i(t,k) \in \mathbb{R}^{n_i}$ for all $(t,k) \in \operatorname{dom} z_i$.
2. As the range of the switching signal $\sigma_i$ is $\mathcal{P}_i$, $\widetilde{\sigma}_i(t,k) \in \mathcal{P}_i$ for all $(t,k) \in \operatorname{dom} z_i$.
3. From (27) and (28), it is clear that $\tau_i(t,k) \leq \Gamma_i$ for all $(t,k) \in \operatorname{dom} z_i$. On the other hand, for any $(t,k) \in \operatorname{dom} z_i$, let $(t_0, k_0) := \arg\max_{(s,l) \in \operatorname{dom} z_i} \{s + l \leq t + k : \tau_i(s,l) = \Gamma_i\}$. (Such $(t_0, k_0)$ always exists since $\tau_i(0,0) = \Gamma_i$.) Then according to the Time-Ratio Constraint and the Average Dwell-Time Constraint in Assumption 1 and the definitions of $\gamma_i$ in (12) and $\Gamma_i$ in (13), we have

$$\begin{aligned} \tau_i(t,k) &= \tau_i(t_0,k_0) - N(t,t_0)\ln(\mu_i) + T_{s,i}(t,t_0)\gamma_i \\ &\quad + T_{u,i}(t,t_0)(\gamma_i - (\lambda_{s,i} + \lambda_{u,i})) \\ &\geq \Gamma_i - (N_{0,i} + (t - t_0)/\tau_{a,i})\ln(\mu_i) + (t - t_0)\gamma_i \\ &\quad - (T_{0,i} + \rho_i(t - t_0))(\lambda_{s,i} + \lambda_{u,i}) \\ &= (\Gamma_i - N_{0,i}\ln(\mu_i) - T_{0,i}(\lambda_{s,i} + \lambda_{u,i})) \\ &\quad + (\gamma_i - \ln(\mu_i)/\tau_{a,i} - \rho_i(\lambda_{s,i} + \lambda_{u,i}))(t - t_0) \\ &= 0. \end{aligned}$$

Thus $\tau_i(t,k) \geq 0$ for all $(t,k) \in \operatorname{dom} z_i$.[6]

Therefore, the hybrid arc $z_i$ constructed above is a complete solution to the hybrid system (23) with the hybrid input $\widetilde{u}_i$.[7]

### 4.3. Hybrid ISS Lyapunov functions

Consider the hybrid system (23). Define a function $V_i \colon \mathcal{Z}_i \to \mathbb{R}_{\geq 0}$ as

$$V_i(z_i) := V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i), \tag{29}$$

where functions $V_{i,p_i}$, $p_i \in \mathcal{P}_i$ are the ISS Lyapunov functions in the Uniform ISS Lyapunov-Type Constraint in Assumption 1. For all $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i) \in \mathcal{Z}_i$, since $V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)$ is $\mathcal{C}^1$ with respect to $\widetilde{x}_i$, $V_i(z_i)$ is continuously differentiable with respect to $\widetilde{x}_i$ and $\tau_i$. We will show that the following uniform ISS Lyapunov conditions are satisfied:

[5] From the proof of Proposition 1, it will be clear that there exists a complete solution $z_i$ such that in addition to (25), we also have $\widetilde{\sigma}_i(t,k) = \sigma_i(t)$ for all $(t,k) \in \operatorname{dom} z_i$. But only (25) is required in the proof of Theorem 1.

[6] This property is equivalent to the fact that $\tau_i \geq \ln(\mu_i)$ whenever a jump occurs, since otherwise $\tau_i^+ < 0$.

[7] By Goebel et al. [19, Proposition 2.10], for a hybrid system with local existence of solutions, a solution is complete if it has no finite escape time and does not jump out of the union of the jump set and the closure of the flow set. Unfortunately, we cannot apply this result since in the hybrid system (23), the local existence of solutions is not satisfied everywhere. In particular, at $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, 0)$ where $\widetilde{\sigma}_i \in \mathcal{P}_{u,i}$, the condition (VC) in [19, Proposition 2.10] does not hold. However, the hybrid arcs we constructed will not arrive at such points.

**Proposition 2.** *$V_i$ satisfies the following conditions:*

1. *$\exists\, \underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that for all $z_i \in \mathcal{Z}_i$,*

$$\underline{\alpha}_i(|z_i|_{\mathcal{A}_i}) \le V_i(z_i) \le \overline{\alpha}_i(|z_i|_{\mathcal{A}_i}), \tag{30}$$

*where*

$$\mathcal{A}_i := 0^{n_i} \times \mathcal{P}_i \times [0, \Gamma_i]. \tag{31}$$

2. *$\exists\, \lambda_i \in \mathbb{R}_{>0}$ such that for all $z_i \in C_i$, all $\widetilde{u}_i \in \mathbb{R}^{m_i}$ and all $v_i \in F_i(z_i, \widetilde{u}_i)$,*

$$|z_i|_{\mathcal{A}_i} \ge \phi_i(|\widetilde{u}_i|) \Rightarrow V_i^\circ(z_i; v_i) \le -\lambda_i V_i(z_i). \tag{32}$$

3. *For all $z_i \in D_i$ and all $z_i^+ \in G_i(z_i)$,*

$$V_i(z_i^+) \le V_i(z_i). \tag{33}$$

**Proof.** Based on the Uniform ISS Lyapunov-Type Constraint in Assumption 1, we have:

1. Let

$$\begin{cases} \underline{\alpha}_i(r) := \alpha_{1,i}(r), \\ \overline{\alpha}_i(r) := \alpha_{2,i}(r)\exp(\Gamma_i), \end{cases} \tag{34}$$

then (30) is satisfied according to (8).

2. Let

$$\lambda_i := \lambda_{s,i} - \gamma_i, \tag{35}$$

then $\lambda_i > 0$ by (12). For all $z_i \in C_i$, all $\widetilde{u}_i \in \mathbb{R}^{m_i}$ and all $v_i \in F_i(z_i, \widetilde{u}_i)$, since $V_i(z_i)$ is continuously differentiable with respect to $\widetilde{x}_i$ and $\tau_i$, and $\dot{\widetilde{\sigma}}_i = 0$, the Clarke derivative in (32) is well-defined. (In fact, the standard directional derivative exists and is equal to the Clarke derivative, as one can see in the following proof.) According to (9), $|z_i|_{\mathcal{A}_i} \ge \phi_i(|\widetilde{u}_i|)$ implies the following:

i. If $\widetilde{\sigma}_i \in \mathcal{P}_{s,i}$,

$$\begin{aligned} V_i^\circ(z_i; v_i) &= \frac{\partial V_i(z_i)}{\partial z_i} \cdot v_i \\ &\le \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \widetilde{x}_i} \cdot f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{u}_i) + \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \tau_i} \cdot \gamma_i \\ &\le \frac{\partial V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)}{\partial \widetilde{x}_i} \cdot \exp(\tau_i) f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{u}_i) \\ &\quad + V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i)\gamma_i \\ &\le -(\lambda_{s,i} - \gamma_i) V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i) \\ &= -\lambda_i V_i(z_i). \end{aligned}$$

ii. If $\widetilde{\sigma}_i \in \mathcal{P}_{u,i}$,

$$\begin{aligned} V_i^\circ(z_i; v_i) &= \frac{\partial V_i(z_i)}{\partial z_i} \cdot v_i \\ &= \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \widetilde{x}_i} \cdot f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{u}_i) \\ &\quad + \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \tau_i} \cdot (\gamma_i - (\lambda_{s,i} + \lambda_{u,i})) \\ &= \frac{\partial V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)}{\partial \widetilde{x}_i} \cdot \exp(\tau_i) f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{u}_i) \\ &\quad + V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i)(\gamma_i - (\lambda_{s,i} + \lambda_{u,i})) \\ &\le (\lambda_{u,i} + \gamma_i - (\lambda_{s,i} + \lambda_{u,i})) V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i) \\ &= -\lambda_i V_i(z_i). \end{aligned}$$

Thus (32) is satisfied.

3. For all $z_i \in D_i$ and all $z_i^+ \in G_i(z_i)$, according to (10),

$$\begin{aligned} V_i(z_i^+) &= V_{i,\widetilde{\sigma}_i^+}(\widetilde{x}_i^+)\exp(\tau_i^+) \\ &\le \mu_i V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i - \ln(\mu_i)) \\ &= V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i) \\ &= V_i(z_i). \end{aligned}$$

Thus (33) is satisfied.

**Remark 9.** From the proof of Proposition 2, we see that the auxiliary timer $\tau_i$ is designed so that it compensates the increases in the concatenation of the ISS Lyapunov-type function of the active subsystem, $V_{i,\sigma_i}(x_i)$. Similar techniques can be found in [8] for switched systems, [29] for impulsive systems and [22] for hybrid systems. Our timer is more general in the sense that it is able to handle the undesired increases of $V_\sigma(x)$ both at the switches and when a non-ISS subsystem is active. In the latter case, our construction provides more decay in the auxiliary timer $\tau_i$ for compensation, as we can see clearly in the definition of the flow map $F_i$ (24).

### 4.4. Digression on ISS of the switched systems

In this subsection, we will show that, for all $i \in \{1, 2\}$, the hybrid ISS Lyapunov function $V_i$ defined in (29) can be conveniently used to prove the ISS property of the $i$-th switched system in (6). While not directly related to the proof of Theorem 1, this result is presented here to show the advantage of our method in comparison with that of [13]. More details on this result can be found in [27].

For all $i \in \{1, 2\}$, let $\alpha_i \in \mathcal{K}_\infty$, $\beta_i \in \mathcal{KL}$ be defined as

$$\begin{cases} \alpha_i(r) := \underline{\alpha}_i^{-1}(\overline{\alpha}_i(\phi_i(r))), \\ \beta_i(r,t) := \underline{\alpha}_i^{-1}(\overline{\alpha}_i(r)\exp(-\lambda_i t)), \end{cases}$$

then we have the following proposition, for which the proof is quite standard and omitted here:

**Proposition 3.** *Suppose $z_i$ is a complete solution to the hybrid system (23) with a hybrid input $\widetilde{u}_i$. Then for all $(t, k) \in \operatorname{dom} z_i$,*

$$|z_i(t,k)|_{\mathcal{A}_i} \le \beta_i(|z_i(0,0)|_{\mathcal{A}_i}, t) + \alpha_i(\|\widetilde{u}_i\|_{(t,k)}), \tag{36}$$

*where the set $\mathcal{A}_i$ is defined in (31).*

Combining Propositions 1 and 3 gives that, for each solution $x_i$ to the $i$-th switched system in (6) with an input $u_i$ and a switching signal $\sigma_i$, if Assumption 1 holds, then $x_i$ satisfies

$$|x_i(t)| \le \beta_i(|x_i(0)|, t) + \alpha_i(\|u_i\|_t) \quad \forall\, t \in \mathbb{R}_{\ge 0}. \tag{37}$$

Therefore, we have the following corollary, which has been first proved by Müller and Liberzon [13] using trajectory analysis:

**Corollary 2** (*[13, Theorem 2]*). *For all $i \in \{1, 2\}$, the $i$-th switched system in (6) is input-to-state stable if Assumption 1 holds with constants satisfying (12) for $n_j = m_i$ and $x_j = u_i$.*

### 4.5. GAS of the interconnected switched system

As shown by Jiang et al. [6], the small-gain condition (15) implies that there exists a $\mathcal{K}_\infty$ function $\delta$ such that

$$\chi_1^{-1}(r) > \delta(r) > \chi_2(r) \quad \forall\, r \in \mathbb{R}_{>0}, \tag{38}$$

and $\delta$ is $\mathcal{C}^1$ on $\mathbb{R}_{>0}$.

Let $z = (z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2 =: \mathcal{Z}$ and define a function $V\colon \mathcal{Z} \to \mathbb{R}_{\ge 0}$ as

$$V(z) := \max\{\delta(V_1(z_1)), V_2(z_2)\}, \tag{39}$$

where functions $V_1, V_2$ are defined in (29). Since for all $i \in \{1, 2\}$, $V_i$ is continuously differentiable with respect to $\widetilde{x}_i$ and $\tau_i$, and $\delta$ is $\mathcal{K}_\infty$ and $\mathcal{C}^1$ on $\mathbb{R}_{>0}$, $V$ is locally Lipschitz and thus absolutely continuous and almost everywhere differentiable away from its zero set (Rademacher's theorem [30]). We will show that, according to Proposition 2, $V$ satisfies the following Lyapunov conditions:

1. $\exists\, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|z|_{\mathcal{A}}) \le V(z) \le \overline{\alpha}(|z|_{\mathcal{A}}) \tag{40}$$

for all $z \in \mathcal{Z}$, where $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2$, and sets $\mathcal{A}_1, \mathcal{A}_2$ are defined in (31).

Indeed, let

$$\begin{cases}\underline{\alpha}(r) := \min\{\delta(\underline{\alpha}_1(r/\sqrt{2})), \underline{\alpha}_2(r/\sqrt{2})\},\\ \overline{\alpha}(r) := \max\{\delta(\overline{\alpha}_1(r)), \overline{\alpha}_2(r)\},\end{cases}$$

where functions $\underline{\alpha}_1, \underline{\alpha}_2, \overline{\alpha}_1, \overline{\alpha}_2$ are defined in (34), then (40) is satisfied according to (30). In particular, for all $z \in \mathcal{Z}$,

$$\begin{aligned}\underline{\alpha}(|z|_{\mathcal{A}}) &= \min\{\delta(\underline{\alpha}_1(|z|_{\mathcal{A}}/\sqrt{2})), \underline{\alpha}_2(|z|_{\mathcal{A}}/\sqrt{2})\}\\ &\le \min\left\{\max\{\delta(\underline{\alpha}_1(|z_1|_{\mathcal{A}_1})), \delta(\underline{\alpha}_1(|z_2|_{\mathcal{A}_2}))\},\right.\\ &\qquad \left.\max\{\underline{\alpha}_2(|z_1|_{\mathcal{A}_1}), \underline{\alpha}_2(|z_2|_{\mathcal{A}_2})\}\right\}\\ &\le \max\{\delta(\underline{\alpha}_1(|z_1|_{\mathcal{A}_1})), \underline{\alpha}_2(|z_2|_{\mathcal{A}_2})\}\\ &\le \max\{\delta(V_1(z_1)), V_2(z_2)\}\\ &= V(z).\end{aligned}$$

2. There exists a continuous and positive definite function $h\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$V^\circ(z; v) \le -h(V(z)) \tag{41}$$

for all $v \in F(z)$ and all $z \in C_1 \times C_2$, where

$$F(z) := \begin{bmatrix} F_1(z_1, \cancel{|z_2|_{\mathcal{A}_2}}) \\ F_2(z_2, \cancel{|z_1|_{\mathcal{A}_1}}) \end{bmatrix}. \tag{42}$$

Indeed, let $h$ be defined as

$$h(r) := \min\{\delta'(\delta^{-1}(r))\lambda_1\delta^{-1}(r), \lambda_2 r\},$$

where constants $\lambda_1, \lambda_2$ are defined in (35). As $\delta$ is $\mathcal{K}_\infty$ and $\mathcal{C}^1$ on $\mathbb{R}_{>0}$, $h$ is continuous and positive definite. For all $z \in C_1 \times C_2$ and all $v = (v_1, v_2)$ such that $v_1 \in F_1(z_1, \cancel{|z_2|_{\mathcal{A}_2}})$, $v_2 \in F_2(z_2, \cancel{|z_1|_{\mathcal{A}_1}})$, consider the following three cases:

i. $\delta(V_1(z_1)) > V_2(z_2)$. Then $V(z) = \delta(V_1(z_1))$ and, according to (38),

$$V_1(z_1) \ge \delta^{-1}(V_2(z_2)) > \chi_1(V_2(z_2)). \tag{43}$$

By the definitions of $\chi_1$ (14) and $\underline{\alpha}_i, \overline{\alpha}_i$ (34) and the inequality (30), $V_1(z_1) \ge \chi_1(V_2(z_2))$ implies $|z_1|_{\mathcal{A}_1} \ge \phi_1(|z_2|_{\mathcal{A}_2})$. Thus by (32),

$$\begin{aligned}V^\circ(z; v) &= \delta'(V_1(z_1))V_1^\circ(z_1; v_1)\\ &\le -\delta'(V_1(z_1))\lambda_1 V_1(z_1)\\ &= -\delta'(\delta^{-1}(V(z)))\lambda_1\delta^{-1}(V(z))\\ &\le -h(V(z)).\end{aligned}$$

ii. $\delta(V_1(z_1)) < V_2(z_2)$. Then $V(z) = V_2(z_2)$ and, according to (38),

$$V_2(z_2) \ge \delta(V_1(z_1)) > \chi_2(V_1(z_1)). \tag{44}$$

By the definition of $\chi_2$ in (14) and $\underline{\alpha}_i, \overline{\alpha}_i$ (34) and the inequality (30), $V_2(z_2) \ge \chi_2(V_1(z_1))$ implies $|z_2|_{\mathcal{A}_2} \ge \phi_2(|z_1|_{\mathcal{A}_1})$. Thus by (32),

$$\begin{aligned}V^\circ(z; v) &= V_2^\circ(z_2; v_2)\\ &\le -\lambda_2 V_2(z_2)\\ &= -\lambda_2 V(z)\\ &\le -h(V(z)).\end{aligned}$$

iii. $\delta(V_1(z_1)) = V_2(z_2)$. Then $V(z) = \delta(V_1(z_1)) = V_2(z_2)$, and (43) and (44) are both satisfied. By virtue of [22, Lemma II.1], which is a direct consequence of [28, Propositions 2.1.2 and 2.3.12], $V^\circ(z; v)$ is well-defined and satisfies

$$\begin{aligned}V^\circ(z; v) &\le \max\{\delta'(V_1(z_1))V_1^\circ(z_1; v_1), V_2^\circ(z_2; v_2)\}\\ &\le -h(V(z)),\end{aligned}$$

where the last inequality follows directly from the proof of the first two cases.

Thus (41) is satisfied.

3. For all $z^+ \in G(z)$ and all $z \in (\mathcal{Z}_1 \times D_2) \cup (D_1 \times \mathcal{Z}_2)$,

$$V(z^+) \le V(z), \tag{45}$$

where

$$G(z) := \begin{cases}\begin{bmatrix} G_1(z_1) \\ G_2(z_2)\end{bmatrix} & \text{if } z \in D_1 \times D_2,\\ \begin{bmatrix} \{z_1\} \\ G_2(z_2)\end{bmatrix} & \text{if } z \in \mathcal{Z}_1 \times D_2,\\ \begin{bmatrix} G_1(z_1) \\ \{z_2\}\end{bmatrix} & \text{if } z \in D_1 \times \mathcal{Z}_2.\end{cases} \tag{46}$$

Indeed, for all $z \in (\mathcal{Z}_1 \times D_2) \cup (D_1 \times \mathcal{Z}_2)$ and all $z^+ = (z_1^+, z_2^+) \in G(z)$, according to (33),

$$\begin{aligned}V(z^+) &\le \max\{\delta(V_1(z_1^+)), \delta(V_1(z_1)), V_2(z_2^+), V_2(z_2)\}\\ &\le \max\{\delta(V_1(z_1)), V_2(z_2)\}\\ &= V(z).\end{aligned}$$

Thus (45) is satisfied.

For all $i \in \{1, 2\}$, let $\overline{z}_i = (\overline{x}_i, \overline{\sigma}_i, \overline{\tau}_i)$ be a complete solution to the hybrid system (23). By definition, $\operatorname{dom}\overline{z}_i$ is a hybrid time domain. Define a hybrid arc $z = (z_1, z_2) : \operatorname{dom} z \to \mathcal{Z}_1 \times \mathcal{Z}_2$ as follows:

1. $\operatorname{dom} z$ is defined so that $z$ jumps if and only if at least one of $\overline{z}_1$ and $\overline{z}_2$ jumps;
2. For all $(t, k) \in \operatorname{dom} z$ such that $z$ does not jump at $(t, k)$, from the first condition, there exist unique $k_1, k_2$ such that $(t, k_1) \in \operatorname{dom}\overline{z}_1$, $(t, k_2) \in \operatorname{dom}\overline{z}_2$. Let $z_1(t, k) = \overline{z}_1(t, k_1)$ and $z_2(t, k) = \overline{z}_2(t, k_2)$.
3. For all $(t, k) \in \operatorname{dom} z$ such that $(t, k + 1) \in \operatorname{dom} z$, for all $i \in \{1, 2\}$, if there exists $k_i$ such that $(t, k_i), (t, k_i + 1) \in \operatorname{dom}\overline{z}_i$, let $z_i(t, k) = \overline{z}_i(t, k_i)$ and $z_i(t, k + 1) = \overline{z}_i(t, k_i + 1)$; otherwise, there exists a unique $k_i$ such that $(t, k_i) \in \operatorname{dom}\overline{z}_i$ and let $z_i(t, k) = z_i(t, k + 1) = \overline{z}_i(t, k_i)$.

**Remark 10.** In fact, $z$ is a complete solution to the following interconnected hybrid system:

$$\begin{cases}\dot{z} \in F(z), & z \in C_1 \times C_2,\\ z^+ \in G(z), & z \in (\mathcal{Z}_1 \times D_2) \cup (D_1 \times \mathcal{Z}_2),\end{cases}$$

where functions $F, G$ are defined in (42) and (46), respectively.

For all $i \in \{1, 2\}$, let $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)$. According to the definition of the jump maps in the system dynamics (23), one sees that the value of $\widetilde{x}_i$ does not change at jumps. Thus

$$\widetilde{x}_i(t, k) = \overline{x}_i(t, k_i) \tag{47}$$

for all $(t, k) \in \operatorname{dom} z$ and all $(t, k_i) \in \operatorname{dom}\overline{z}_i$.

By definition, for all $k \in \mathbb{Z}_{\ge 0}$, $z(t, k)$ is absolutely continuous in $t$ on $\Theta_k^z = \{t : (t, k) \in \operatorname{dom} z\}$. Since $V$ is almost everywhere differentiable and its Clarke derivative $V^\circ$ satisfies (41), following the argument in [31, p. 99], we conclude that, for all $k \in \mathbb{Z}_{\ge 0}$, $V(z(t, k))$ is absolutely continuous in $t$ on $\Theta_k^z$ and

$$\frac{\mathrm{d}V(z(t, k))}{\mathrm{d}t} \le -h(V(z(t, k))) \quad \text{a.e. on } \Theta_k^z. \tag{48}$$

More precisely, for all $k \in \mathbb{Z}_{\ge 0}$, since $z(t, k)$ is absolutely continuous in $t$ on $\Theta_k^z$ and $V$ is locally Lipschitz, by definition, for almost all $t \in \Theta_k^z$ and all $v = \dot{z}(t, k) \in F(z(t, k))$,

$$\begin{aligned}\frac{\mathrm{d}V(z(t, k))}{\mathrm{d}t} &= \lim_{h\to 0^+} \frac{V(z(t, k) + hv) - V(z(t, k))}{h}\\ &\le \lim_{h\to 0^+} \sup_{w\to z(t,k)} \frac{V(w + hv) - V(w)}{h}\\ &= V^\circ(z(t, k); v)\\ &\le -h(V(z(t, k))),\end{aligned}$$

where the last inequality follows from (41).

**Remark 11.** Consider the condition that either $V_1$ or $V_2$ is not constant in any open set in $\mathcal{Z}_1$ or $\mathcal{Z}_2$, respectively, which is quite common for ISS Lyapunov functions. Under this additional condition, the set $\{z = (z_1, z_2) : \delta(V_1(z_1)) = V_2(z_2)\}$ in case iii. in the proof of (41) has zero Lebesgue measure. (If not, consider an open ball contained in this set centered at $(z_1^*, z_2^*)$, then $\delta(V_1(z_1^*)) = V_2(z_2^*)$. Without loss of generality, assume $V_2$ is not constant in any open set in $\mathcal{Z}_2$. Then there exists another point $(z_1^*, z_2')$ in this open ball such that $z_2' \neq z_2^*$ and $V_2(z_2') \neq V_2(z_2^*) = \delta(V_1(z_1^*))$, which contradicts the assumption that this open ball is contained in the aforementioned set.) Then, by virtue of [29, Lemma 1], it is sufficient to conclude (48) from inequalities (43) and (44), and the analysis of case (c) becomes unnecessary.

Based on (48), we claim that there exists $\beta_V \in \mathcal{KL}$ such that

$$V(z(t,k)) \leq \beta_V(V(z(0,0)), t) \quad \forall (t,k) \in \operatorname{dom} z.$$

The proof essentially follows the proof of the comparison principle for hybrid systems proposed by Cai and Teel [18, Lemma C.1]. The only significant difference is that, at jump time instants, the function $V$ here satisfies (45), which is a weaker condition comparing to the second condition in [18, Lemma C.1], but it is sufficient for the existence of a class $\mathcal{KL}$ function. Indeed, if the hybrid arc $z$ jumps at $(\theta_{k+1}, k) \in \operatorname{dom} z$ (thus $(\theta_{k+1}, k+1) \in \operatorname{dom} z$), by (45),

$$\int_{V(z(\theta_{k+1},k))}^{V(z(\theta_{k+1},k+1))} \frac{\mathrm{d}s}{h(s)} \leq 0.$$

Following the proof of [18, Lemma C.1], it can be shown that

$$\int_{V(z(0,0))}^{V(z(t,k))} \frac{\mathrm{d}s}{h(s)} \leq -t \quad \forall (t,k) \in \operatorname{dom} z,$$

from which our claim follows exactly as in the proof of [23, Lemma 4.4].

Let $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ and define $\beta \in \mathcal{KL}$ as

$$\beta(r,t) := \underline{\alpha}^{-1}(\beta_V(\overline{\alpha}(r), t)).$$

Then, according to (40) and the definition of set $\mathcal{A}$,

$$|\widetilde{x}(t,k)| \leq \beta(|\widetilde{x}(0,0)|, t) \quad \forall (t,k) \in \operatorname{dom} z.$$

Finally, let $x = (x_1, x_2)$ be a solution to the interconnected switched system (7). Then, according to (47) and Proposition 1,

$$|x(t)| \leq \beta(|x(0)|, t) \quad \forall t \in \mathbb{R}_{\geq 0},$$

that is, the interconnected switched system (7) is globally asymptotically stable. This completes the proof of Theorem 1.

## 5. Conclusions

We have studied the stability property of an interconnected system consisting of two switched systems in the scenario where in both switched systems there may exist some subsystems that are not input-to-state stable. We have proved a small-gain theorem as a sufficient condition that guarantees the global asymptotic stability of the interconnected system via the hybrid system approach and the construction of appropriate ISS Lyapunov functions.

In this paper, for each switched system, we have categorized its subsystems by their ISS property (i.e., ISS or non-ISS). On the other hand, we have only assumed an upper-bound on the effect of the switches (i.e., Eq. (10)). This lack of symmetry has drawn our attention, and the case where switching between some of the subsystems produces a stabilizing effect could possibly become a future research topic.

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## References

[1] C. Desoer, M. Vidyasagar, Feedback Systems: Input–Output Properties, SIAM, 2009.
[2] D. Hill, A generalization of the small-gain theorem for nonlinear feedback systems, Automatica 27 (1991) 1043–1045.
[3] I. Mareels, D. Hill, Monotone stability of nonlinear feedback systems, J. Math. Syst. Estim. Control 2 (1992) 275–291.
[4] E. Sontag, Smooth stabilization implies coprime factorization, IEEE Trans. Automat. Control 34 (1989) 435–443.
[5] Z.-P. Jiang, A. Teel, L. Praly, Small-gain theorem for ISS systems and applications, Math. Control Signals Syst. 7 (1994) 95–120.
[6] Z.-P. Jiang, I. Mareels, Y. Wang, A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica 32 (1996) 1211–1215.
[7] A. Isidori, Nonlinear Control Systems II, Springer, 1999.
[8] D. Liberzon, Switching in Systems and Control, Birkhäuser, Boston, 2003.
[9] A. Morse, Supervisory control of families of linear set-point controllers—part I. Exact matching, IEEE Trans. Automat. Control 41 (1996) 1413–1431.
[10] J. Hespanha, A. Morse, Stability of switched systems with average dwell-time, in: 38th IEEE Conference on Decision and Control, Vol. 3, 1999, pp. 2655–2660.
[11] G. Zhai, B. Hu, K. Yasuda, A. Michel, Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach, in: 2000 American Control Conference, Vol. 1, 2000, pp. 200–204.
[12] L. Vu, D. Chatterjee, D. Liberzon, Input-to-state stability of switched systems and switching adaptive control, Automatica 43 (2007) 639–646.
[13] M. Müller, D. Liberzon, Input/output-to-state stability and state-norm estimators for switched nonlinear systems, Automatica 48 (2012) 2029–2039.
[14] L. Long, J. Zhao, A small-gain theorem for switched interconnected nonlinear systems and its applications, IEEE Trans. Automat. Control 59 (2014) 1082–1088.
[15] D. Nešić, D. Liberzon, A small-gain approach to stability analysis of hybrid systems, in: 44th IEEE Conference on Decision and Control, 2005, pp. 5409–5414.
[16] I. Karafyllis, Z.-P. Jiang, A small-gain theorem for a wide class of feedback systems with control applications, SIAM J. Control Optim. 46 (2007) 1483–1517.
[17] D. Liberzon, D. Nešić, Stability analysis of hybrid systems via small-gain theorems, in: J. Hespanha, A. Tiwari (Eds.), Hybrid Systems: Computation and Control, Springer, Berlin, Heidelberg, 2006, pp. 421–435.
[18] C. Cai, A. Teel, Characterizations of input-to-state stability for hybrid systems, Systems Control Lett. 58 (2009) 47–53.
[19] R. Goebel, R. Sanfelice, A. Teel, Hybrid Dynamical Systems: Modeling, Stability, and Robustness, Princeton University Press, 2012.
[20] D. Nešić, A. Teel, A Lyapunov-based small-gain theorem for hybrid ISS systems, in: 47th IEEE Conference on Decision and Control, 2008, pp. 3380–3385.
[21] D. Liberzon, D. Nešić, A. Teel, Small-gain theorems of LaSalle type for hybrid systems, in: 51st IEEE Conference on Decision and Control, 2012, pp. 6825–6830.
[22] D. Liberzon, D. Nešić, A. Teel, Lyapunov-based small-gain theorems for hybrid systems, IEEE Trans. Automat. Control 59 (2014) 1395–1410.
[23] Y. Lin, E. Sontag, Y. Wang, A smooth converse Lyapunov theorem for robust stability, SIAM J. Control Optim. 34 (1996) 124–160.
[24] L. Praly, Y. Wang, Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability, Math. Control Signals Syst. 9 (1996) 1–33.
[25] D. Angeli, E. Sontag, Forward completeness, unboundedness observability, and their Lyapunov characterizations, Systems Control Lett. 38 (1999) 209–217.
[26] J. Mancilla-Aquilar, R. Garcia, On the existence of common Lyapunov triples for ISS and iISS switched systems, in: 39th IEEE Conference on Decision and Control, Vol. 4, 2000, pp. 3507–3512.
[27] G. Yang, D. Liberzon, Input-to-state stability for switched systems with unstable subsystems: a hybrid Lyapunov construction, in: 53rd IEEE Conference on Decision and Control, 2014, pp. 6240–6245.
[28] F. Clarke, Optimization and Nonsmooth Analysis, SIAM, 1990.
[29] J. Hespanha, D. Liberzon, A. Teel, Lyapunov conditions for input-to-state stability of impulsive systems, Automatica 44 (2008) 2735–2744.
[30] L. Evans, R. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.
[31] A. Teel, L. Praly, On assigning the derivative of a disturbance attenuation control Lyapunov function, Math. Control Signals Syst. 13 (2000) 95–124.