Lyapunov small-gain theorems for networks of not necessarily ISS hybrid systems ⋆

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Abstract

We prove a novel Lyapunov-based small-gain theorem for networks of $n \geq 2$ hybrid systems which are not necessarily input-to-state stable. This result unifies and extends several small-gain theorems for hybrid and impulsive systems proposed in the last few years. We also show how average dwell-time (ADT) clocks and reverse ADT clocks can be used to modify the Lyapunov functions for subsystems and to enlarge the applicability of the derived small-gain theorems.

Key words: hybrid systems, input-to-state stability, small-gain theorems.

1 Introduction

The study of interconnected systems plays a significant role in the stability analysis of complex dynamics, as it allows one to establish stability of the interconnection based on properties of the less complex components. In this approach, small-gain theorems prove to be general and advantageous tools in the study of feedback interconnections, which appear frequently in the control literature. A comprehensive overview of classical small-gain theorems involving input-output gains of linear systems can be found in [14]. This technique was then generalized to nonlinear feedback systems in [20,28] within the input-output context. The next peak level in the stability analysis of interconnections was reached based on the input-to-state stability (ISS) framework proposed in [34], which unified the notions of internal and external stability. Nonlinear small-gain theorems for interconnections of two ISS systems were established in [22,21], and then generalized to arbitrary networks of n dynamic systems in [12,13]. A variety of nonlinear small-gain theorems were summarized in [6].

The results described above have been developed for systems of ordinary differential equations. However, in modeling real-world phenomena one often has to consider systems consisting of both continuous and discrete dynamics. A general framework for modeling such behaviors is the hybrid systems theory [17,15]. In this work we adopt the modeling framework proposed in [15], which proves to be general and natural from the viewpoint of Lyapunov stability theory [2,3].

During recent years great efforts have been devoted to the development of small-gain theorems for interconnected hybrid systems. Trajectory-based small-gain theorems for interconnections of two hybrid systems were derived in [32,24,7], while Lyapunov-based formulations were proposed in [26,33,27]. Some of these results were extended to interconnections of n hybrid ISS systems in [7].

A more challenging problem is the study of hybrid systems in which either the continuous or the discrete dynamics is non-ISS. In this case, input-to-state stability is usually achieved under restrictions on the density of jumps (discrete events) such as dwell-time [31], average dwell-time (ADT) [19] and reverse average dwell-time (RADT) [18]. For interconnections of such hybrid systems, the small-gain theorems derived in [7,27] cannot be directly applied. In [27] it was shown that one can modify the destabilizing dynamics of subsystems by adding aux-

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iliary clocks and constructing ISS Lyapunov functions for the augmented subsystems that decrease both during flows and at jumps. One advantage of this method is that it can be applied even if the destabilizing dynamics are of different types (i.e., if in some subsystems the continuous dynamics are non-ISS, while in some other ones the discrete dynamics are non-ISS). However, due to the modification the Lyapunov gains would increase exponentially w.r.t. constants in the ADT/RADT conditions, which restricts the applicability of this method as the enlarged gains may no longer satisfy the small-gain condition.

Another type of small-gain theorems has been proposed in [8,10] for ISS of interconnected impulsive systems with destabilizing continuous or discrete dynamics. The first step in this approach is to construct a candidate exponential ISS Lyapunov function for the interconnection. When the destabilizing dynamics of subsystems are of the same type (i.e., when the continuous dynamics of all subsystems or the discrete ones of all subsystems are ISS), the candidate exponential ISS Lyapunov function obtained above can be used to establish ISS of the interconnection under suitable ADT/RADT conditions. Compared with the previous method, this construction doesn't require modifications of subsystems, and hence preserves the Lyapunov gains and small-gain conditions. However, this method has been developed only for impulsive systems and requires candidate exponential ISS Lyapunov functions for subsystems. Moreover, it cannot be applied to interconnections of subsystems with different types of destabilizing dynamics.

In this paper we unify the two methods above. In Section 2 we introduce the modeling framework and main definitions, followed by a Lyapunov sufficient condition for ISS of hybrid systems. In Section 3 we first prove a general small-gain theorem for n interconnected hybrid systems based on the construction of a candidate ISS Lyapunov function for the interconnection, which generalizes the Lyapunov small-gain theorems from [33,8,7,10,27]. In the same section we also derive several implications of the general result, in particular, a small-gain theorem for interconnections of subsystems with the same type of destabilizing dynamics and with candidate exponential ISS Lyapunov functions with linear Lyapunov gains. In Section 4 we propose a version of the method of modifying ISS Lyapunov functions for subsystems from [27] in which a smaller number of subsystems are affected (and hence fewer Lyapunov gains are enlarged). In Section 5 we combine the results of this work into a unified method for establishing ISS of interconnected hybrid systems and conclude the paper with an outlook on future research.

A preliminary and shortened version of the paper has been presented at 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS 2014) [30].

2 Framework for hybrid systems

Let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For a vector x, let $|x|$ denote its Euclidean norm. For a set $\mathcal{A}$, its (Euclidean) distance to a vector x is defined as $|x|_\mathcal{A} := \inf_{y\in\mathcal{A}} |x - y|$. For n vectors $x_1, \ldots, x_n$, let $(x_1, \ldots, x_n) := (x_1^\top, \ldots, x_n^\top)^\top$ denote their concatenation. For two vectors x, y, we say $x \geq y$ and $x > y$ if the corresponding inequality holds in each dimension; and $x \not\geq y$ if there is at least one dimension i such that $x_i < y_i$. For a matrix $A \in \mathbb{R}^{n\times n}$, let $\rho(A)$ denote its spectral radius, that is,

$$\rho(A) := \max\{|\lambda_1|, \ldots, |\lambda_n|\}$$

with $\lambda_1, \ldots, \lambda_n$ being the eigenvalues of A.

Let id denote the identity function. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{P}$ if it is continuous and positive-definite (that is, $\alpha(r) = 0 \Leftrightarrow r = 0$); it is of class $\mathcal{K}$ if $\alpha \in \mathcal{P}$ and is strictly increasing; it is of class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{t\to\infty} \gamma(t) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t, and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed $r > 0$.

Motivated by [2], a hybrid system is modeled as a combination of continuous flows and discrete jumps of the form

$$\begin{aligned} \dot{x} &\in F(x, u), \qquad && (x, u) \in \mathcal{C}, \\ x^+ &\in G(x, u), \qquad && (x, u) \in \mathcal{D}, \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^N$ is the state, $u \in \mathcal{U} \in \mathbb{R}^M$ the input, $\mathcal{C} \subset \mathcal{X} \times \mathcal{U}$ the flow set, $\mathcal{D} \subset \mathcal{X} \times \mathcal{U}$ the jump set, $F : \mathcal{C} \rightrightarrows \mathbb{R}^N$ the flow map, and $G : \mathcal{D} \rightrightarrows \mathcal{X}$ the jump map. (In this model, the dynamics of (1) is continuous in $\mathcal{C}\backslash\mathcal{D}$, and discrete in $\mathcal{D}\backslash\mathcal{C}$. In $\mathcal{C} \cap \mathcal{D}$ it can be either continuous or discrete.) The hybrid system (1) is fully characterized by its *data* $\mathcal{H} := (\mathcal{C}, F, \mathcal{D}, G)$.

Solutions of (1) are defined on hybrid time domains. A set $E \subset \mathbb{R}_+ \times \mathbb{N}$ is called a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$. It is a *hybrid time domain* if $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in E$. On a hybrid time domain there is a natural ordering of points: $(s, k) \preceq (t, j)$ if $s + k \leq t + j$, and $(s, k) \prec (t, j)$ if $s + k < t + j$.

A *hybrid signal* is a function defined on a hybrid time domain. A hybrid signal $x : \operatorname{dom} x \to \mathcal{X}$ is called a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous for each j. A hybrid signal $u : \operatorname{dom} u \to \mathcal{U}$ is called a *hybrid input* if $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid arc $x : \operatorname{dom} x \to \mathcal{X}$ and a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ form a *solution pair* (x, u) to (1) if

- $\operatorname{dom} x = \operatorname{dom} u$;
- $(x(t,j), u(t,j)) \in \mathcal{C}$ and $\dot{x}(t,j) \in F(x(t,j), u(t,j))$ for all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in \operatorname{dom} x$; [1]
- $(x(t,j), u(t,j)) \in \mathcal{D}$ and $x(t,j+1) \in G(x(t,j), u(t,j))$ for all $(t,j) \in \operatorname{dom} x$ such that $(t,j+1) \in \operatorname{dom} x$.

With proper assumptions on the data $\mathcal{H}$, one can establish local existence of solutions, which are not necessarily unique (cf. [15, Proposition 2.10]). A solution pair (x,u) is *maximal* if it cannot be extended, and *complete* if $\operatorname{dom} x$ is unbounded. In this paper we only consider maximal solution pairs.

Following [2], the essential supremum norm of a hybrid signal u up to hybrid time (t,j) is defined as

$$\|u\|_{(t,j)} := \max\left\{ \operatorname*{ess\,sup}_{\substack{(s,k)\in\operatorname{dom} u,\\ (s,k)\preceq(t,j)}} |u(s,k)|, \sup_{\substack{(s,k)\in J(u),\\ (s,k)\preceq(t,j)}} |u(s,k)| \right\},$$

where $J(x) := \{(s,k) \in \operatorname{dom} u : (s,k+1) \in \operatorname{dom} u\}$ are the set of jump times. [2]

For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, its the *Dini derivative* at x in the direction $y \in \mathbb{R}^n$ is defined by

$$\dot{V}(x;y) = \overline{\lim_{h\searrow 0}} \frac{V(x+hy) - V(x)}{h}.$$

In this work, we study input-to-state stability (ISS) of the hybrid system (1) via ISS Lyapunov functions.

Definition 1. Following [27], a set of solution pairs $\mathcal{S}$ of (1) is *pre-input-to-state stable (pre-ISS) w.r.t.* $\mathcal{A} \subset \mathcal{X}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $(x,u) \in \mathcal{S}$,

$$|x(t,j)|_{\mathcal{A}} \le \max\{\beta(|x(0,0)|_{\mathcal{A}}, t+j), \gamma(\|u\|_{(t,j)})\} \quad (2)$$

for all $(t,j) \in \operatorname{dom} x$. If $\mathcal{S}$ contains all solution pairs (x,u) of (1), then we say that (1) is *pre-ISS w.r.t.* $\mathcal{A}$. If all solution pairs are in addition complete, then (1) is called *ISS w.r.t.* $\mathcal{A}$.

Remark 1. For the case without input (namely, $u \equiv 0$), pre-ISS is called *global pre-asymptotic stability (pre-GAS)*, and ISS is called *global asymptotic stability (GAS)* [27].

Remark 2. In [2], ISS of hybrid systems is defined in terms of class $\mathcal{KLL}$ functions and without requiring all solution pairs to be complete, which is equivalent to our definition of pre-ISS of hybrid systems [4, Lemma 6.1].

[1] Here $x(t,j)$ represents the state of the hybrid system at time t and after j jumps.

[2] Note that the set of hybrid times of measure 0 that can be ignored when computing the essential supremum norm cannot include any jump time.

Definition 2. A Lipschitz function $V : \mathcal{X} \to \mathbb{R}_+$ is a *candidate ISS Lyapunov function w.r.t.* $\mathcal{A} \subset X$ for (1) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|x|_{\mathcal{A}}) \le V(x) \le \psi_2(|x|_{\mathcal{A}}) \qquad \forall x \in \mathcal{X}, \quad (3)$$

and there exist $\chi \in \mathcal{K}, \alpha \in \mathcal{P}$ and a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that

$$V(x) \ge \chi(|u|) \Rightarrow \\ \begin{cases} \dot{V}(x;y) \le -\varphi(V(x)) & \forall y \in F(x,u), \forall (x,u) \in \mathcal{C}, \\ V(y) \le \alpha(V(x)) & \forall y \in G(x,u), \forall (x,u) \in \mathcal{D}. \end{cases} \quad (4)$$

In addition, if

$$\varphi(r) \equiv cr, \qquad \alpha(r) \equiv e^{-d}r \quad (5)$$

for some $c, d \in \mathbb{R}$ then V is called a *candidate exponential ISS Lyapunov function w.r.t.* $\mathcal{A}$ with *rate coefficients* c, d.

The following lemma gives an alternative description of the candidate ISS Lyapunov function, which will be useful for the formulation of small-gain theorems in Section 3.

Lemma 1. *A Lipschitz function $V : X \to \mathbb{R}_+$ is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for* (1) *if and only if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that* (3) *holds, and $\bar{\chi} \in \mathcal{K}, \alpha \in \mathcal{P}$ and a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that*

$$V(x) \ge \bar{\chi}(|u|) \Rightarrow \dot{V}(x;y) \le -\varphi(V(x)) \quad (6)$$

for all $(x,u) \in \mathcal{C}$ and $y \in F(x,u)$, and

$$V(y) \le \max\{\alpha(V(x)), \bar{\chi}(|u|)\} \quad (7)$$

for all $(x,u) \in \mathcal{D}$ and $y \in G(x,u)$.

Proof. The proof goes along the lines of the proof of [10, Proposition 1] for ISS Lyapunov functions of impulsive systems and is omitted here. ☐

Similar restatement can also be provided for exponential ISS Lyapunov functions. Note that the gain functions χ in Definition 2 and $\bar{\chi}$ in Lemma 1 are different in general.

In the definition of a candidate ISS Lyapunov function V it is not assumed that $\varphi \in \mathcal{P}$ or $\alpha < \mathrm{id}$. If both of these conditions are satisfied then V becomes an *ISS Lyapunov function*, which implies that (1) is pre-ISS [2, Proposition 2.7] (note that ISS in [2] means pre-ISS in this work; see Remark 2). Meanwhile, if only one of the two conditions is satisfied[3], then we can still establish ISS for a subset of solution pairs of (1) under additional restrictions on the density of jumps.

Proposition 1. *Let V be a candidate exponential ISS Lyapunov function for* (1) *w.r.t. $\mathcal{A}$ with rate coefficients c, d. For arbitrary constants $\eta, \lambda, \mu > 0$, let $\mathcal{S}[\eta, \lambda, \mu]$ denote the set of solution pairs (x, u) satisfying*

$$-(d-\eta)(j-k) - (c-\lambda)(t-s) \le \mu \tag{8}$$

for all $(s,k), (t,j) \in \operatorname{dom} x$ such that $(s,k) \preceq (t,j)$. Then $\mathcal{S}[\eta, \lambda, \mu]$ is pre-ISS w.r.t. $\mathcal{A}$.

Proof. The proof goes along the lines of the proof of [18, Theorem 1] for ISS of impulsive systems. Let $\chi \in \mathcal{K}$ be as in Definition 2. Consider arbitrary solution pair $(x,u) \in \mathcal{S}[\eta,\lambda,\mu]$ and $(t_0, j_0), (t_1, j_1) \in \operatorname{dom} x$ such that $(t_0, j_0) \preceq (t_1, j_1)$. If

$$V(x(s,k)) \ge \chi(\|u\|_{(s,k)}) \tag{9}$$

for all $(s,k) \in \operatorname{dom} x$ such that $(t_0, j_0) \preceq (s,k) \preceq (t_1, j_1)$, then (4), (5) and (8) imply that

$$\begin{aligned} V(x(t_1, j_1)) &\le e^{-d(j_1-j_0)-c(t_1-t_0)} V(x(t_0,j_0)) \\ &\le e^{-\eta(j_1-j_0)-\lambda(t_1-t_0)+\mu} V(x(t_0,j_0)). \end{aligned} \tag{10}$$

Consider an arbitrary $(t,j) \in \operatorname{dom} x$. If (9) holds for all $(s,k) \in \operatorname{dom} x$ such that $(s,k) \preceq (t,j)$, then (10) implies that

$$|x(t,j)|_{\mathcal{A}} \le \beta(|x(0,0)|_{\mathcal{A}}, t+j) \tag{11}$$

with $\beta \in \mathcal{KL}$ defined by

$$\beta(r,l) := \psi_1^{-1}(e^{-l\min\{\eta,\lambda\}+\mu} \psi_2(r)). \tag{12}$$

Otherwise, let

$$(t',j') = \operatorname*{argmax}_{\substack{(s,k)\in\operatorname{dom} x,\\ (s,k)\preceq(t,j)}} \{s+k : V(x(s,k)) \le \chi(\|u\|_{(s,k)})\}.$$

Then (9) holds for all $(s,k) \in \operatorname{dom} x$ such that $(t',j') \prec (s,k) \preceq (t,j)$, and hence (10) implies that

$$\begin{aligned} V(x(t,j)) &\le e^{-\eta(j-j')-\lambda(t-t')+\mu} e^{|d|} V(x(t',j')) \\ &\le e^{\mu+|d|} \chi(\|u\|_{(t',j')}) \\ &\le e^{\mu+|d|} \chi(\|u\|_{(t,j)}). \end{aligned}$$

(The term $e^{|d|}$ is needed when $(t', j'+1) \in \operatorname{dom} x$, and $V(x(t',j')) \le \chi(\|u\|_{t',j'})$ while $V(x(t',j'+1)) > \chi(\|u\|_{t',j'+1})$.) Hence

$$|x(t,j)|_{\mathcal{A}} \le \gamma(\|u\|_{(t,j)}) \tag{13}$$

with $\gamma \in \mathcal{K}$ defined by

$$\gamma(r) := \psi_1^{-1}\big(e^{\mu} \max\{1, e^{-d}\}\chi(r)\big).$$

Combining (11) and (13) shows that (2) holds for all $(x,u) \in \mathcal{S}[\eta,\lambda,\mu]$ and $(t,j) \in \operatorname{dom} x$. ☐

Remark 3. For a complete solution pair, the inequality (8) cannot hold with both $c, d < 0$ (as there is always a sufficiently large t or j such that $\eta j + \lambda t > \mu$); however, it can still hold for a maximal solution pair defined on a bounded hybrid time domain. Moreover, the claim of Proposition 1 also holds when $c > 0 > d$ and $\eta = 0$. The proof remains unchanged except that the last inequality in (10) becomes

$$\begin{aligned} & e^{-d(j_1-j_0)-c(t_1-t_0)} V(x(t_0,j_0)) \\ &\le e^{-\lambda(t_1-t_0)+\mu} V(x(t_0,j_0)) \\ &\le e^{(\lambda^2/c-\lambda)(t_1-t_0)-\lambda^2(t_1-t_0)/c+\mu} V(x(t_0,j_0)) \\ &\le e^{\lambda d(j_1-j_0)/c-\lambda^2(t_1-t_0)/c+(1+\lambda/c)\mu} V(x(t_0,j_0)), \end{aligned}$$

where the last inequality follows again from (8); and the exponent in (12) becomes $-l\min\{-\lambda d/c, \lambda^2/c\} + (1 + \lambda/c)\mu$. Analogously, the claim also holds when $d > 0 > c$ and $\lambda = 0$.

Remark 4. For $d < 0$, condition (8) can be transformed to the average dwell-time (ADT) condition [19] via division by $-(d-\eta)$; analogously, for $c < 0$, it can be transformed to the reverse average dwell-time (RADT) condition [18] via division by $-(c-\lambda)$.

Given a candidate exponential ISS Lyapunov function for (1) with rates $c > 0$ and/or $d > 0$, we can find the pre-ISS set of solution pairs via Proposition 1. In the following section we investigate the construction of such functions for interconnections of hybrid systems.

3 Interconnections and small-gain theorems

Consider the case where the hybrid system (1) is decomposed as

$$\begin{aligned} \dot{x}_i &\in F_i(x,u), \quad i = 1,\dots,n, \qquad (x,u) \in \mathcal{C}, \\ x_i^+ &\in G_i(x,u), \quad i = 1,\dots,n, \qquad (x,u) \in \mathcal{D}, \end{aligned} \tag{14}$$

where $x := (x_1,\dots,x_n) \in \mathcal{X} \subset \mathbb{R}^N$ with $x_i \in \mathcal{X}_i \subset \mathbb{R}^{N_i}$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^M$ the common (external) input, $\mathcal{C} := \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times \mathcal{C}_u$ and $\mathcal{D} := \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \times \mathcal{D}_u$ with $\mathcal{C}_i, \mathcal{D}_i \subset \mathcal{X}_i$ and $\mathcal{C}_u, \mathcal{D}_u \subset \mathcal{U}$, $F := (F_1,\dots,F_n)$ with $F_i : C \rightrightarrows \mathbb{R}^{N_i}$, and $G := (G_1,\dots,G_n)$ with $G_i : \mathcal{D} \rightrightarrows \mathcal{X}_i$. (Note that $N = N_1 + \cdots + N_n$ and $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$.)

[3] That is, either the continuous or the discrete dynamics taken alone is ISS, but not both; see [34] and [23] for the definitions of ISS for continuous and discrete dynamics, respectively.

We call the dynamics of x_i the i-th subsystem of (14) and denote it by Σ_i. The interconnection (14) is denoted by Σ. For each subsystem Σ_i, the states of all other subsystems are treated as (internal) inputs.

Remark 5. The flow set $\mathcal{C}$ and jump set $\mathcal{D}$ coincide for all subsystems as well as the interconnection Σ, which justifies the view of (14) as an interconnection of n hybrid subsystems.

Remark 6. Using Lemma 1 and standard considerations clarifying the influence of particular subsystems (e.g., [29, Lemma 2.4.1]) one can show that a Lipschitz function $V_i : \mathcal{X}_i \to \mathbb{R}_+$ is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}_i \subset \mathcal{X}_i$ for Σ_i when

(1) there exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that

$$\psi_{i1}(|x_i|_{\mathcal{A}_i}) \le V_i(x_i) \le \psi_{i2}(|x_i|_{\mathcal{A}_i}) \qquad \forall x_i \in \mathcal{X}_i; \tag{15}$$

(2) there exist *internal gains* $\chi_{ij} \in \mathcal{K}$ for $j \ne i$ and $\chi_{ii} \equiv 0$, an *external gain* $\chi_i \in \mathcal{K}$ and a continuous function $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi_i(0) = 0$ such that

$$V_i(x_i) \ge \max\left\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|)\right\} \tag{16}$$

implies

$$\dot{V}_i(x_i; y_i) \le -\varphi_i(V_i(x_i)) \tag{17}$$

for all $(x, u) \in \mathcal{C}$ and $y_i \in F_i(x, u)$;

(3) there exists an $\alpha_i \in \mathcal{P}$ such that

$$V_i(y_i) \le \max\left\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|)\right\} \tag{18}$$

for all $(x, u) \in \mathcal{D}$ and $y_i \in G_i(x, u)$.

In addition, if $\varphi_i(r) \equiv c_i r$ and $\alpha_i(r) \equiv e^{-d_i} r$ for some $c_i, d_i \in \mathbb{R}$ then V_i is called a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}_i$ with rate coefficients c_i, d_i.

Suppose that for each subsystem Σ_i, a candidate ISS Lyapunov function V_i is given. The question of whether the interconnection (14) is pre-ISS depends on properties of the *gain operator* $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(r_1, \dots, r_n) := \left(\max_{j=1}^{n} \chi_{1j}(r_j), \dots, \max_{j=1}^{n} \chi_{nj}(r_j)\right). \tag{19}$$

In order to construct a candidate ISS Lyapunov function for the interconnection (14), we adopt the notion of Ω-path [13].

Definition 3. Given an operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, a function $\sigma := (\sigma_1, \dots, \sigma_n)$ with $\sigma_1, \dots, \sigma_n \in \mathcal{K}_\infty$ is called an *Ω-path w.r.t.* Γ if

(1) σ_i^{-1} is locally Lipschitz on $(0, \infty)$;

(2) for each compact set $P \subset (0, \infty)$, there exist finite constants $K_2 > K_1 > 0$ such that

$$0 < K_1 \le (\sigma_i^{-1})'(r) \le K_2$$

for all points of differentiability of σ_i^{-1} in P;

(3) Γ is a contraction on $\sigma(\cdot)$, namely,

$$\Gamma(\sigma(r)) < \sigma(r) \qquad \forall r > 0. \tag{20}$$

We say that an operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies the *small-gain condition* if

$$\Gamma(v) \not\ge v \qquad \forall v \in \mathbb{R}^n_+ \backslash \{0\}, \tag{21}$$

or equivalently,

$$\Gamma(v) \ge v \Leftrightarrow v = 0.$$

As reported in [25, Proposition 2.7 and Remark 2.8] (see also [13, Theorem 5.2]), there exists an Ω-path σ w.r.t. the gain operator Γ defined in (19) provided that (21) holds. Furthermore, σ can be made smooth via standard mollification arguments [16, Appendix B.2].

Remark 7. We consider in this paper primarily Ω-paths w.r.t. the gain operator Γ defined by (19), due to the terms $\max_{j=1}^{n} \chi_{ij}(V_j(x_j))$ in (16) and (18) in the formulation of ISS Lyapunov functions for subsystems (which will be clear from the statement and proof of Theorem 2 below). However, there are other equivalent formulations of ISS Lyapunov functions for subsystems, which naturally lead to other types of gain operators (see, e.g., [12,13]). In particular, if the terms $\max_{j=1}^{n} \chi_{ij}(V_j(x_j))$ in (16) and (18) were replaced by $\sum_{j=1}^{n} \chi_{ij}(V_j(x_j))$, we would arrive at the gain operator $\Gamma_\Sigma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma_\Sigma(r_1, \dots, r_n) := \left(\sum_{j=1}^{n} \chi_{1j}(r_j), \dots, \sum_{j=1}^{n} \chi_{nj}(r_j)\right).$$

Since for the gain operator Γ in (19) we have $\Gamma(v) \le \Gamma_\Sigma(v)$ for all $v \in \mathbb{R}^n_+ \backslash \{0\}$, every Ω-path w.r.t. Γ_Σ is automatically an Ω-path w.r.t. Γ.

The following theorem shows that, if Γ satisfies the small-gain condition (21), a candidate ISS Lyapunov function for the interconnection (14) can be constructed based on the candidate ISS Lyapunov functions for subsystems and the corresponding Ω-path.

Theorem 2. *Consider the interconnection* (14). *Suppose that each subsystem Σ_i admits a candidate ISS Lyapunov function V_i w.r.t. $\mathcal{A}_i$ with internal gains χ_{ij} as in* (16), *and that the gain operator Γ defined by* (19) *satisfies the small-gain condition* (21). *Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be a smooth Ω-path w.r.t. Γ. Then the function $V : \mathcal{X} \to \mathbb{R}_+$ defined by*

$$V(x) := \max_{i=1}^{n} \sigma_i^{-1}(V_i(x_i)) \tag{22}$$

is a candidate ISS Lyapunov function w.r.t. $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *for* (14).

Proof. We will show that V defined by (22) satisfies the conditions of Lemma 1.

First, consider $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\begin{aligned}\psi_1(r) &:= \min_{i=1}^{n} \sigma_i^{-1}(\psi_{i1}(r/\sqrt{n})),\\ \psi_2(r) &:= \max_{i=1}^{n} \sigma_i^{-1}(\psi_{i2}(r)).\end{aligned}$$

Since all $\sigma_i, \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, it follows that $\psi_1, \psi_2 \in \mathcal{K}_\infty$. Then (3) follows from (15). In particular,

$$\begin{aligned}\psi_1(|x|_{\mathcal{A}}) &= \min_{i=1}^{n} \sigma_i^{-1}(\psi_{i1}(|x|_{\mathcal{A}}/\sqrt{n}))\\ &\le \min_{i=1}^{n} \sigma_i^{-1}\left(\psi_{i1}\left(\max_{j=1}^{n} |x_j|_{\mathcal{A}_j}\right)\right)\\ &\le \max_{j=1}^{n} \sigma_j^{-1}(\psi_{j1}(|x_j|_{\mathcal{A}_j}))\\ &\le \max_{j=1}^{n} \sigma_j^{-1}(V_j(x_j)) = V(x).\end{aligned}$$

Second, consider $\bar{\chi} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\bar{\chi}(r) := \max_{i=1}^{n} \sigma_i^{-1}(\chi_i(r)), \tag{23}$$

and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\varphi(r) := \min_{i=1}^{n} (\sigma_i^{-1})'(\sigma_i(r))\varphi_i(\sigma_i(r)). \tag{24}$$

Since $\sigma_i \in \mathcal{K}_\infty$ is smooth, $\chi_i \in \mathcal{K}$ and φ_i is continuous with $\varphi_i(0) = 0$ for each i, it follows that $\bar{\chi} \in \mathcal{K}$ and φ is continuous with $\varphi(0) = 0$. For each i, let a set M_i be defined by

$$M_i := \left\{ x \in \mathcal{X} : \sigma_i^{-1}(V_i(x_i)) > \max_{j \ne i} \sigma_j^{-1}(V_j(x_j)) \right\}.$$

The continuity of all V_i and σ_i^{-1} implies that all M_i are open, $M_i \cap M_j = \emptyset$ for all $i \ne j$, and

$$\mathcal{X} = \bigcup_{i=1}^{n} \overline{M}_i.$$

Hence for arbitrary $(x, u) \in \mathcal{C}$ and $y \in F(x, u)$, there are two possibilities:

(1) There exists a unique $i \in \{1, \ldots, n\}$ such that $x \in M_i$, namely,

$$V(x) = \sigma_i^{-1}(V_i(x_i)) > \max_{j \ne i} \sigma_j^{-1}(V_j(x_j)). \tag{25}$$

By combining (19), (20) and (25) we get

$$\begin{aligned}V_i(x_i) &= \sigma_i(V(x))\\ &> \max_{j=1}^{n} \chi_{ij}(\sigma_j(V(x)))\\ &\ge \max_{j=1}^{n} \chi_{ij}(V_j(x_j)).\end{aligned} \tag{26}$$

Suppose $V(x) \ge \bar{\chi}(|u|)$. By combining it with (23) and (25) we get

$$V_i(x_i) = \sigma_i(V(x)) \ge \sigma_i(\bar{\chi}(|u|)) \ge \chi_i(|u|). \tag{27}$$

Hence (16), and therefore (17), is satisfied. Substituting (17), (24) and (25) into the Dini derivative $\dot{V}(x; y)$ gives that

$$\begin{aligned}\dot{V}(x; y) &= \overline{\lim_{h \searrow 0}} \frac{V(x + hy) - V(x)}{h}\\ &= \overline{\lim_{h \searrow 0}} \frac{\sigma_i^{-1}(V_i(x_i + hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h}\\ &= (\sigma_i^{-1})'(V_i(x_i)) \overline{\lim_{h \searrow 0}} \frac{V_i(x_i + hy_i) - V_i(x_i)}{h}\\ &= (\sigma_i^{-1})'(V_i(x_i))\dot{V}_i(x_i; y_i)\\ &\le -(\sigma_i^{-1})'(\sigma_i(V(x)))\varphi_i(\sigma_i(V(x)))\\ &\le -\varphi(V(x)).\end{aligned}$$

(2) There exists a subset $I(x) \subset \{1, \ldots, n\}$ of indices with $|I(x)| \ge 2$ such that

$$x \in \bigcap_{i \in I(x)} \partial M_i,$$

where $\partial M_i := \overline{M}_i \backslash M_i$ is the boundary of M_i. Then for each $i \in I(x)$,

$$V(x) = \sigma_i^{-1}(V_i(x_i)) > \max_{j \notin I(x)} \sigma_j^{-1}(V_j(x_j)). \tag{28}$$

Following similar arguments as in the previous case while using (28) in place of (25), we see that (26), (27), and therefore (17), hold as well. Substituting (17), (24) and (28) into the Dini derivative $\dot{V}(x; y)$

gives that (cf. [9, proof of Theorem 4])

$$\begin{aligned}
\dot{V}(x;y) &= \overline{\lim_{h\searrow 0}} \frac{V(x+hy)-V(x)}{h} \\
&= \overline{\lim_{h\searrow 0}} \frac{1}{h}\left(\max_{i\in I(x)} \sigma_i^{-1}(V_i(x_i+hy_i)) - V(x)\right) \\
&= \overline{\lim_{h\searrow 0}} \max_{i\in I(x)} \frac{\sigma_i^{-1}(V_i(x_i+hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\
&= \max_{i\in I(x)} \overline{\lim_{h\searrow 0}} \frac{\sigma_i^{-1}(V_i(x_i+hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\
&= \max_{i\in I(x)} (\sigma_i^{-1})'(V_i(x_i))\dot{V}_i(x_i;y_i) \\
&\le \max_{i\in I(x)} -(\sigma_i^{-1})'(\sigma_i(V(x)))\varphi_i(\sigma_i(V(x))) \\
&\le -\varphi(V(x)),
\end{aligned}$$

where the second and fourth equalities follow partially from the continuity of all V_i and σ_i^{-1}.

Hence (6) holds for all $(x,u)\in\mathcal{C}$ and $y\in F(x,u)$.

Finally, consider $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ defined by

$$\alpha(r) := \max_{i=1}^{n}\left\{\sigma_i^{-1}(\alpha_i(\sigma_i(r))), \max_{j=1}^{n}\sigma_i^{-1}(\chi_{ij}(\sigma_j(r)))\right\}. \tag{29}$$

Since $\sigma_i\in\mathcal{K}_\infty$, $\chi_{ij}\in\mathcal{K}$, $\chi_{ii}\equiv 0$ and $\alpha_i\in\mathcal{P}$ for all i and $j\neq i$, it follows that $\alpha\in\mathcal{P}$. Consider arbitrary $(x,u)\in\mathcal{D}$ and $y\in G(x,u)$. By combining (22) and (29) we get

$$\alpha(V(x)) \ge \max_{i=1}^{n}\left\{\sigma_i^{-1}(\alpha_i(V_i(x_i))), \max_{j=1}^{n}\sigma_i^{-1}(\chi_{ij}(V_j(x_j)))\right\},$$

while from (23) we get

$$\bar{\chi}(|u|) \ge \max_{i=1}^{n}\sigma_i^{-1}(\chi_i(|u|)).$$

By combining the previous two inequalities with (18) and (22) we get

$$V(y) = \max_{i=1}^{n}\sigma_i^{-1}(V_i(y_i)) \le \max\{\alpha(V(x)), \bar{\chi}(|u|)\},$$

that is, (7) holds for all $(x,u)\in\mathcal{D}$ and $y\in G(x,u)$. Therefore, Lemma 1 shows that V is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for the interconnection (14). $\square$

Theorem 2 is a powerful tool in studying ISS of interconnections of hybrid systems. In the folowing we inspect some of its implications.

If each subsystem of (14) admits an ISS Lyapunov function, Theorem 2 implies the following result, which generalizes [27, Theorem III.1] and [7, Theorem 3.6].

Corollary 3. *Consider the interconnection* (14)*. Suppose that each subsystem Σ_i admits an ISS Lyapunov function V_i w.r.t. $\mathcal{A}_i$ with internal gains χ_{ij} as in* (16) *(i.e., $\varphi_i\in\mathcal{P}$ in* (17) *and $\alpha_i<\mathrm{id}$ in* (18)*), and that the gain operator Γ defined by* (19) *satisfies the small-gain condition* (21)*. Then* (14) *is pre-ISS w.r.t. $\mathcal{A}$.*

Proof. By Theorem 2, we know that the V defined by (22) is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (14). Since $\sigma_i\in\mathcal{K}_\infty$ is smooth and $\varphi_i\in\mathcal{P}$ for each i, from (24) we see that $\varphi\in\mathcal{P}$. Moreover, from (20) we see that $\sigma_i^{-1}\circ\chi_{ij}\circ\sigma_j<\mathrm{id}$ on $\mathbb{R}_+$ for all i,j, while since $\alpha_i<\mathrm{id}$ and $\sigma_i\in\mathcal{K}_\infty$ it follows that $\sigma_i^{-1}\circ\alpha_i\circ\sigma_i<\mathrm{id}$ on $\mathbb{R}_+$ for all i. Then from (29) we see that $\alpha<\mathrm{id}$. Hence V an ISS Lyapunov function, and (14) is pre-ISS w.r.t. $\mathcal{A}$ due to [2, Proposition 2.7]; cf. Remark 2. $\square$

As the assumptions in Corollary 3 are rather restrictive, we now investigate the case in which there may exist some i such that either $\varphi_i\notin\mathcal{P}$ or $\alpha_i(r)\ge r$ for some $r>0$ (cf. footnote 3). In this case, we cannot use Corollary 3 to prove pre-ISS of the interconnection (14), but are able to establish pre-ISS of the sets of solution pairs that jump neither too fast nor too slowly via Proposition 1. However, in general Theorem 2 is not sufficient to provide the candidate exponential ISS Lyapunov function needed in Proposition 1. In the following we will show that such a function can be constructed provided that each subsystem Σ_i admits a candidate exponential ISS Lyapunov function V_i with all internal gains χ_{ij} as in (16) being linear. In such cases, let the *gain matrix* $\Gamma_M\in\mathbb{R}^{n\times n}$ be defined by

$$\Gamma_M := (\chi_{ij})_{n\times n}. \tag{30}$$

If the spectral radius of the gain matrix satisfies

$$\rho(\Gamma_M) < 1 \tag{31}$$

then the small-gain condition (21) holds with the gain operator $v\mapsto\Gamma_M v$ [12, p. 110], and there exists a linear Ω-path [11, p. 78].

Theorem 4. *Consider the interconnection* (14)*. Suppose that each subsystem Σ_i admits a candidate exponential ISS Lyapunov function V_i w.r.t. $\mathcal{A}_i$ with rate coefficients c_i,d_i, that all internal gains χ_{ij} as in* (16) *are linear, and that the gain matrix Γ_M defined in* (30) *satisfies* (31)*. Let $\sigma: r\mapsto(s_1r,\ldots,s_nr)$ be a linear Ω-path w.r.t. the gain operator $v\mapsto\Gamma_M v$. Then the function $V:\mathcal{X}\to\mathbb{R}_+$ defined by*

$$V(x) := \max_{i=1}^{n}\frac{1}{s_i}V_i(x_i)$$

is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}$ for (14) *with rate coefficients* c, d *given by*

$$c := \min_{i=1}^{n} c_i, \qquad d := \min_{i,j:j\neq i} \left\{ d_i, -\ln\left(\frac{s_j}{s_i}\chi_{ij}\right)\right\}. \quad (32)$$

Proof. Let σ be as in the statement of the theorem. In view of Remark 7, it is also an Ω-path w.r.t. the gain operator defined by (19). From Remark 6 we see that $\varphi_i(r) \equiv c_i r$ and $\alpha_i(r) \equiv e^{-d_i} r$ for all i. Hence Theorem 2 implies that V is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (14) with

$$\varphi(r) \equiv \min_{i=1}^{n} c_i r, \quad \alpha(r) \equiv \max\left\{\max_{i=1}^{n} e^{-d_i}, \max_{i,j=1}^{n} \frac{s_j}{s_i}\chi_{ij}\right\} r$$

by (24) and (29), that is, a candidate exponential ISS Lyapunov function with rate coefficients c, d given by (32). ☐

Remark 8. For the more general case where the internal gains χ_{ij} are power functions instead of linear ones, a candidate exponential ISS Lyapunov function for (14) can be constructed in a similar manner; cf. [10, Theorem 9].

The following remark provides a simpler bound for the rate coefficient d in some important cases.

Remark 9. If the gain matrix Γ_M is irreducible then $\rho(\Gamma_M)$ is the Perron–Frobenius eigenvalue of Γ_M, and the corresponding eigenvector $\bar{s} = (s_1, \dots, s_n)$ satisfies that all $s_i > 0$ (Perron–Frobenius theorem [1, Theorem 2.1.3, p. 27]). Thus from (31) it follows that $\Gamma_M s = \rho(\Gamma_M) s < s$, that is, $\sigma : r \mapsto (s_1 r, \dots, s_n r)$ is a linear Ω-path w.r.t. the gain operator $v \mapsto \Gamma_M v$, and

$$\max_{j=1}^{n} \frac{s_j}{s_i}\chi_{ij} \leq \frac{1}{s_i}\sum_{j=1}^{n} s_j \chi_{ij} = \rho(\Gamma_M) \qquad \forall i \in \{1, \dots, n\}.$$

Then d in (32) satisfies $d \geq \min\{\min_i d_i, -\ln(\rho(\Gamma_M))\}$.

After applying Theorem 4, Proposition 1 can be used to establish pre-ISS of the sets of solution pairs satisfying suitable conditions on the density of jumps. However, if there exist $j, k \in \{1, \dots, n\}$ such that $c_j, d_k < 0$, then both $c, d < 0$ in (32) and hence Proposition 1 cannot be applied for complete solution pairs (see Remark 3). In the following section we handle such cases by modifying the candidate ISS Lyapunov functions for subsystems based on the concept of ADT and RADT clocks from [27].

4 Modifications of ISS Lyapunov functions for subsystems

In this subsection we construct new candidate exponential ISS Lyapunov functions for subsystems with rate coefficients $\tilde{c}_i, \tilde{d}_i$ such that either all $\tilde{c}_i > 0$ (i.e., all continuous dynamics are ISS) or all $\tilde{d}_i > 0$ (i.e., all discrete dynamics are ISS). To achieve this we first assign suitable conditions on the density of jumps, then augment the subsystems with auxiliary clocks to incorporate such conditions, and finally modify the corresponding candidate exponential ISS Lyapunov functions.

4.1 Making discrete dynamics ISS

In this subsection, we construct candidate exponential ISS Lyapunov functions such that all rate coefficients $\tilde{d}_i > 0$.

Let $I_d := \{i : d_i < 0\}$ denote the index set of subsystems with non-ISS discrete dynamics. Consider an arbitrary solution pair (x, u) of (14) such that for each $i \in I_d$, it satisfies the average dwell-time (ADT) condition [19]

$$j - k \leq \delta_i(t - s) + N_{0i} \quad (33)$$

for all $(s, k), (t, j) \in \operatorname{dom} x$ such that $(s, k) \preceq (t, j)$ with constants $\delta_i, N_{0i} > 0$. As shown in [5, Appendix], a hybrid time domain satisfies (33) if and only if it is the domain of a solution of the following hybrid system of the auxiliary clock τ_i:

$$\begin{aligned} \dot{\tau}_i &\in [0, \delta_i], & \tau_i &\in [0, N_{0i}], \\ \tau_i^+ &= \tau_i - 1, & \tau_i &\in [1, N_{0i}]. \end{aligned}$$

Let $z_i := x_i \in \mathcal{Z}_i := \mathcal{X}_i$ for $i \notin I_d$ and $z_i := (x_i, \tau_i) \in \mathcal{Z}_i := \mathcal{X}_i \times [0, N_{0i}]$ for $i \in I_d$. An augmented interconnection $\tilde{\Sigma}$ with state $z := (z_1, \dots, z_n)$ and input u is modeled as

$$\begin{aligned} \dot{z}_i &\in \tilde{F}_i(z, u), \quad i = 1, \dots, n, \qquad (z, u) \in \tilde{\mathcal{C}}, \\ z_i^+ &\in \tilde{G}_i(z, u), \quad i = 1, \dots, n, \qquad (z, u) \in \tilde{\mathcal{D}}, \end{aligned} \quad (34)$$

where $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_n \times \mathcal{C}_u$ and $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}_1 \times \cdots \times \tilde{\mathcal{D}}_n \times \mathcal{D}_u$ with $\tilde{\mathcal{C}}_i := \mathcal{C}_i$ and $\tilde{\mathcal{D}}_i := \mathcal{D}_i$ for $i \notin I_d$, and $\tilde{\mathcal{C}}_i := \mathcal{C}_i \times [0, N_{0i}]$ and $\tilde{\mathcal{D}}_i := \mathcal{D}_i \times [1, N_{0i}]$ for $i \in I_d$; and $\tilde{F}_i(z, u) := F_i(x, u)$ and $\tilde{G}_i(z, u) := G_i(x, u)$ for $i \notin I_d$, and $\tilde{F}_i(z, u) := F_i(x, u) \times [0, \delta_i]$ and $\tilde{G}_i(z, u) := G_i(x, u) \times \{\tau_i - 1\}$ for $i \in I_d$. Let $\tilde{F} := \tilde{F}_1 \times \cdots \times \tilde{F}_n$ and $\tilde{G} := \tilde{G}_1 \times \cdots \times \tilde{G}_n$. Then the augmented interconnection (34) is a hybrid system with the data $\tilde{\mathcal{H}} := (\tilde{\mathcal{C}}, \tilde{F}, \tilde{\mathcal{D}}, \tilde{G})$. The dynamics of z_i is called the i-th augmented subsystem $\tilde{\Sigma}_i$.

In the following proposition, we apply the modification technique in [27, Proposition IV.1] to construct a candidate exponential ISS Lyapunov function for each augmented subsystem $\tilde{\Sigma}_i$ based on the candidate exponential ISS Lyapunov functions for the subsystem Σ_i of the original interconnection (14) and the ADT clock τ_i.

Proposition 5. *Consider a subsystem Σ_i of the original interconnection* (14). *Suppose that it admits a candidate exponential ISS Lyapunov function V_i w.r.t. $\mathcal{A}_i$ with rate coefficients c_i, d_i. Then for a constant $L_i \in \mathbb{R}_+$, the function $W_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by*

$$W_i(z_i) := \begin{cases} V_i(x_i), & \text{if } i \notin I_d; \\ e^{L_i \tau_i} V_i(x_i), & \text{if } i \in I_d \end{cases}$$

is a candidate exponential ISS Lyapunov function w.r.t.

$$\tilde{\mathcal{A}}_i := \begin{cases} \mathcal{A}_i, & \text{if } i \notin I_d; \\ \mathcal{A}_i \times [0, N_{0i}], & \text{if } i \in I_d \end{cases}$$

for the augmented subsystem $\tilde{\Sigma}_i$ of (34) *with rate coefficients*

$$\begin{cases} \tilde{c}_i := c_i,\ \tilde{d}_i := d_i, & \text{if } i \notin I_d; \\ \tilde{c}_i := c_i - L_i\delta_i,\ \tilde{d}_i := d_i + L_i, & \text{if } i \in I_d. \end{cases}$$

More specifically,

(1) there exist $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that

$$\tilde{\psi}_{i1}(|z_i|_{\tilde{\mathcal{A}}_i}) \leq W_i(z_i) \leq \tilde{\psi}_{i2}(|z_i|_{\tilde{\mathcal{A}}_i}) \qquad \forall z_i \in \mathcal{Z}_i; \tag{35}$$

(2) there exist internal gains $\tilde{\chi}_{ij} \in \mathcal{K}$ for $j = 1, \dots, n$ defined by

$$\tilde{\chi}_{ij}(r) := \begin{cases} \chi_{ij}(r), & \text{if } i \notin I_d; \\ e^{L_i N_{0i}} \chi_{ij}(r), & \text{if } i \in I_d, \end{cases} \tag{36}$$

where χ_{ij} are as in (16),[4] *and an external gain $\tilde{\chi}_i \in \mathcal{K}$ such that*

$$W_i(z_i) \geq \max\left\{ \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)), \tilde{\chi}_i(|u|) \right\} \tag{37}$$

implies

$$\dot{W}_i(z_i; y_i) \leq -\tilde{c}_i W_i(z_i) \tag{38}$$

for all $(z,u) \in \tilde{\mathcal{C}}$ and $y_i \in \tilde{F}_i(z,u)$;

(3) it holds that

$$W_i(y_i) \leq \max\left\{ e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)), \tilde{\chi}_i(|u|) \right\} \tag{39}$$

for all $(z,u) \in \tilde{\mathcal{D}}$ and $y_i \in \tilde{G}_i(z,u)$.

[4] Note that if the original internal gain χ_{ij} is linear then so is $\tilde{\chi}_{ij}$.

Proof. For each $i \notin I_d$, it is straightforward to verify that the claim holds with all the functions and coefficients for $\tilde{\Sigma}_i$ being the same as those for Σ_i. Hence in the following proof we only consider $i \in I_d$. For each $i \in I_d$, let $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ be as in (15), and $\chi_i \in \mathcal{K}$ as in (16).

(1) Since $\tau_i \in [0, N_0^i]$, it follows that $|z_i|_{\tilde{\mathcal{A}}_i} = |x_i|_{\mathcal{A}_i}$. Then from (15) we see that (35) holds with

$$\tilde{\psi}_{i1}(r) := \psi_{i1}(r), \qquad \tilde{\psi}_{i2}(r) := e^{L_i N_{0i}} \psi_{i2}(r).$$

(2) For arbitrary $(z,u) \in \tilde{\mathcal{C}}$ and $y_i \in \tilde{F}_i(z,u)$, denote y_i by (y_{i1}, y_{i2}) with $y_{i1} \in F_i(x,u)$ and $y_{i2} \in [0, \delta_i]$. Suppose (37) holds with $\tilde{\chi}_{ij}$ defined by (36) and

$$\tilde{\chi}_i(r) := e^{L_i N_{0i}} \chi_i(r). \tag{40}$$

Then

$$\begin{aligned} V_i(x_i) &= e^{-L_i\tau_i} W_i(z_i) \\ &\geq e^{-L_i N_{0i}} \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)) \\ &= \max_{j=1}^{n} \chi_{ij}(e^{L_j\tau_j} V_j(x_j)) \\ &\geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \end{aligned}$$

and

$$V_i(x_i) = e^{-L_i\tau_i} W_i(z_i) \geq e^{-L_i N_{0i}} \tilde{\chi}_i(|u|) = \chi_i(|u|).$$

Hence (16), and therefore (17), is satisfied. By combining (17) and the assumption that $\varphi_i(r) \equiv c_i r$ we get

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{L_i\tau_i} \dot{V}_i(x_i; y_{i1}) + L_i e^{L_i\tau_i} V_i(x_i) y_{i2} \\ &\leq -c_i e^{L_i\tau_i} V_i(x_i) + L_i\delta_i e^{L_i\tau_i} V_i(x_i) \\ &= -\tilde{c}_i W_i(z_i). \end{aligned}$$

(3) For arbitrary $(z,u) \in \tilde{\mathcal{D}}$ and $y_i \in \tilde{G}_i(z,u)$, denote y_i by (y_{i1}, y_{i2}) with $y_{i1} \in G_i(x,u)$ and $y_{i2} = \tau_i - 1$. Then $W_i(y_i) = e^{L_i(\tau_i-1)} V_i(y_{i1})$. On the other hand,

$$e^{-\tilde{d}_i} W_i(z_i) = e^{-d_i - L_i} W_i(z_i) = e^{L_i(\tau_i-1)} e^{-d_i} V_i(x_i),$$

and by combining (36) and (40) we get

$$\begin{aligned} \tilde{\chi}_{ij}(W_j(z_j)) &= e^{L_i N_{0i}} \chi_{ij}(e^{L_j\tau_j} V_j(x_j)) \\ &\geq e^{L_i(\tau_i-1)} \chi_{ij}(V_j(x_j)) \end{aligned}$$

for all j, and

$$\tilde{\chi}_i(|u|) = e^{L_i N_{0i}} \chi_i(|u|) \geq e^{L_i(\tau_i-1)} \chi_i(|u|).$$

Hence (39) holds according to (18) and the assumption that $\alpha_i(r) \equiv e^{-d_i} r$.

Therefore, W_i is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_i$ for $\tilde{\Sigma}_i$ with rate coefficients $\tilde{c}_i, \tilde{d}_i$. $\square$

Proposition 5 shows that it is possible to make all $\tilde{d}_i > 0$ by choosing a large enough L_i for each $i \in I_d$ at the cost of decreasing the convergence rate of continuous dynamics (as $\tilde{c}_i = c_i - L_i\delta_i$) and increasing the internal gains (as $\tilde{\chi}_{ij}(\cdot) = e^{L_i N_{0i}}\chi_{ij}(\cdot)$). Thus for large enough N_{0i}, it is possible that the small-gain condition (21) holds for the gain operator Γ defined by (19), but not for $\tilde{\Gamma} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ define by

$$\tilde{\Gamma}(r_1, \dots, r_n) := \left(\max_{j=1}^{n} \tilde{\chi}_{1j}(r_j), \dots, \max_{j=1}^{n} \tilde{\chi}_{nj}(r_j) \right).$$

(However, if all the original internal gains χ_{ij} are linear, and the gain matrix Γ_M defined by (30) is of a triangular form, i.e., if the interconnection is a cascade interconnection, then (21) always holds since all the cycles of gains are equal to zero.)

To see the consequences of this fact clearer, consider for simplicity an interconnection of two subsystems Σ_1, Σ_2, with candidate exponential ISS Lyapunov functions V_1, V_2 with rate coefficients $c_1 > 0 > d_1$ and $c_2 < 0 < d_2$, and linear internal gains χ_{12}, χ_{21}. After we augment Σ_1 with an ADT clock, the gain matrix $\tilde{\Gamma}_M$ is given by

$$\tilde{\Gamma}_M = \begin{bmatrix} 0 & \tilde{\chi}_{12} \\ \tilde{\chi}_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{L_1 N_{01}}\chi_{12} \\ \chi_{21} & 0 \end{bmatrix},$$

and it satisfies $\rho(\tilde{\Gamma}_M) < 1$ if only if

$$\chi_{12}\chi_{21} < e^{-L_1 N_{01}}.$$

In order to have $\tilde{d}_1 = d_1 + L_1 > 0$, we need to choose $L_1 > -d_1$. Moreover, the chatter bound satisfies $N_{01} \geq 1$.[5] Hence we cannot apply Theorem 4 to the augmented interconnection unless the original internal gains χ_{12}, χ_{21} satisfy

$$\chi_{12}\chi_{21} \leq e^{d_1}.$$

The observations above hint us that we should add as small number of clocks as possible, and it may be better to make all $\tilde{c}_i > 0$ (instead of all $\tilde{d}_i > 0$ as we get in this subsection).[6]

[5] For $N_{01} = 1$, the ADT condition becomes the dwell-time condition [31]; while for $N_{01} < 1$, jumps are not allowed at all (this can be seen directly from (33) by taking $t - s$ small enough).

[6] For more comparisons on the two schemes, see [35].

4.2 Making continuous dynamics ISS

In this subsection, we construct candidate exponential ISS Lyapunov functions such that all rate coefficients $\tilde{c}_i > 0$.

Let $I_c := \{i : c_i < 0\}$ denote the index set of subsystems with non-ISS continuous dynamics. Consider an arbitrary solution pair (x, u) of (14) such that for each $i \in I_c$, it satisfies the reverse average dwell-time (RADT) condition [18]

$$t - s \leq \delta_i^*(j - k) + N_{0i}^*\delta_i^* \tag{41}$$

for all $(s, k), (t, j) \in \operatorname{dom} x$ such that $(s, k) \preceq (t, j)$ with constants $\delta_i^*, N_{0i}^* > 0$. As shown in [5, Appendix], a hybrid time domain satisfies (41) if and only if it is the domain of a solution of the following hybrid system of the auxiliary clock τ_i:

$$\begin{aligned} &\dot{\tau}_i = 1, && \tau_i \in [0, N_{0i}^*\delta_i^*], \\ &\tau_i^+ = \max\{0, \tau_i - \delta_i^*\}, && \tau_i \in [0, N_{0i}^*\delta_i^*]. \end{aligned}$$

Let $z_i := x_i \in \mathcal{Z}_i := \mathcal{X}_i$ for $i \notin I_c$, and $z_i := (x_i, \tau_i) \in \mathcal{Z}_i := \mathcal{X}_i \times [0, N_{0i}^*\delta_i^*]$ for $i \in I_c$. An augmented interconnection $\tilde{\Sigma}$ with state $z := (z_1, \dots, z_n)$ and input u is modeled as (34), where $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1 \times \dots \times \tilde{\mathcal{C}}_n \times \mathcal{C}_u$ and $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}_1 \times \dots \times \tilde{\mathcal{D}}_n \times \mathcal{D}_u$ with $\tilde{\mathcal{C}}_i = \mathcal{C}_i$ and $\tilde{\mathcal{D}}_i = \mathcal{D}_i$ for $i \notin I_c$, and $\tilde{\mathcal{C}}_i = \mathcal{C}_i \times [0, N_{0i}^*\delta_i^*]$ and $\tilde{\mathcal{D}}_i = \mathcal{D}_i \times [0, N_{0i}^*\delta_i^*]$ for $i \in I_c$; and $\tilde{F}_i(z, u) := F_i(x, u)$ and $\tilde{G}_i(z, u) := G_i(x, u)$ for $i \notin I_c$, and $\tilde{F}_i(z, u) := F_i(x, u) \times \{1\}$ and $\tilde{G}_i(z, u) := G_i(x, u) \times \max\{0, \tau_i - \delta_i^*\}$ for $i \in I_c$. Let $\tilde{F} := \tilde{F}_1 \times \dots \times \tilde{F}_n$ and $\tilde{G} := \tilde{G}_1 \times \dots \times \tilde{G}_n$. The the augmented interconnection (34) is a hybrid system with the data $\tilde{\mathcal{H}} := (\tilde{\mathcal{C}}, \tilde{F}, \tilde{\mathcal{D}}, \tilde{G})$. The dynamics of z_i is called the i-th augmented subsystem $\tilde{\Sigma}_i$.

In the following proposition, we apply the modification technique in [27, Proposition IV.4] to construct a candidate exponential ISS Lyapunov functions for each augmented subsystem $\tilde{\Sigma}_i$ based on the candidate exponential ISS Lyapunov function for the subsystems Σ_i of the original interconnection (14) and the RADT clock τ_i.

Proposition 6. *Consider a subsystem Σ_i of the original interconnection (14). Suppose that it admits an candidate exponential ISS Lyapunov function V_i w.r.t. $\mathcal{A}_i$ with rate coefficients c_i, d_i. Then for a constant $L_i \in \mathbb{R}_+$, the function $W_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by*

$$W_i(z_i) := \begin{cases} V_i(x_i), & \text{if } i \notin I_c; \\ e^{-L_i\tau_i}V_i(x_i), & \text{if } i \in I_c \end{cases} \tag{42}$$

is a candidate exponential ISS Lyapunov function w.r.t.

$$\tilde{\mathcal{A}}_i := \begin{cases} \mathcal{A}_i, & \text{if } i \notin I_c; \\ \mathcal{A}_i \times [0, N_{0i}^*\delta_i^*], & \text{if } i \in I_c \end{cases}$$

for the augmented subsystem $\tilde{\Sigma}_i$ with rate coefficients

$$\begin{cases} \tilde{c}_i := c_i,\ \tilde{d}_i := d_i, & \text{if } i \notin I_c; \\ \tilde{c}_i := c_i + L_i,\ \tilde{d}_i := d_i - L_i\delta_i^*, & \text{if } i \in I_c. \end{cases}$$

More specifically,

(1) there exist $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that (35) holds;
(2) there exist internal gains $\tilde{\chi}_{ij} \in \mathcal{K}$ for $j = 1, \ldots, n$ defined by

$$\tilde{\chi}_{ij}(r) := \begin{cases} \chi_{ij}(r), & \text{if } i \notin I_c; \\ \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} r), & \text{if } i \in I_c, \end{cases} \tag{43}$$

where χ_{ij} are as in (16) (see also footnote 4), and an external gain $\tilde{\chi}_i \in \mathcal{K}$ such that (37) implies (38) for all $(z, u) \in \tilde{\mathcal{C}}$ and $y_i \in \tilde{F}_i(z, u)$;
(3) the inequality (39) holds for all $(z, u) \in \tilde{\mathcal{D}}$ and $y_i \in \tilde{G}_i(z, u)$.

Proof. For all $i \notin I_c$, it is straightforward to verify that the claim holds with all the functions and constants for $\tilde{\Sigma}_i$ being the same as those for Σ_i. Hence in the following proof we only consider $i \in I_c$. For each $i \in I_c$, let $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ be as in (15), and $\chi_i \in \mathcal{K}$ as in (16).

(1) Since $\tau_i \in [0, N_{0i}^* \delta_i^*]$, it follows that $|z_i|_{\tilde{\mathcal{A}}_i} = |x_i|_{\mathcal{A}_i}$. Then from(15) we see that (35) holds with

$$\tilde{\psi}_{i1}(r) := e^{-L_i N_{0i}^* \delta_i^*} \psi_{i1}(r), \qquad \tilde{\psi}_{i2}(r) := \psi_{i2}(r).$$

(2) For arbitrary $(z, u) \in \tilde{\mathcal{C}}$ and $y_i \in \tilde{F}_i(z, u)$, denote y_i by (y_{i1}, y_{i2}) with $y_{i1} \in F_i(x, u)$ and $y_{i2} = 1$. Suppose (37) holds with $\tilde{\chi}_{ij}$ defined by (43) and

$$\tilde{\chi}_i(r) := \chi_i(r). \tag{44}$$

Then

$$\begin{aligned} V_i(x_i) &= e^{L_i \tau_i} W_i(z_i) \\ &\geq \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)) \\ &= \max_{j=1}^{n} \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} e^{-L_j \tau_j} V_j(x_j)) \\ &\geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)) \end{aligned}$$

and

$$V_i(x_i) = e^{L_i \tau_i} W_i(z_i) \geq \tilde{\chi}_i(|u|) \geq \chi_i(|u|).$$

Hence (16), and therefore (17), is satisfied. By combining (17) and the assumption that $\varphi_i(r) \equiv c_i r$ we get

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{-L_i \tau_i} \dot{V}_i(x_i; y_{i1}) - L_i e^{-L_i \tau_i} V_i(x_i) y_{i2} \\ &\leq -c_i e^{-L_i \tau_i} V_i(x_i) - L_i e^{-L_i \tau_i} V_i(x_i) \\ &= -\tilde{c}_i W_i(z_i). \end{aligned}$$

(3) For arbitrary $(z, u) \in \tilde{\mathcal{D}}$ and $y_i \in \tilde{G}_i(z, u)$, denote y_i by (y_{i1}, y_{i2}) with $y_{i1} \in G_i(x, u)$ and $y_{i2} = \max\{0, \tau_i - \delta_i^*\}$. Then

$$W_i(y_i) = \min\{1, e^{-L_i(\tau_i - \delta_i^*)}\} V_i(y_{i1}).$$

On the other hand,

$$e^{-\tilde{d}_i} W_i(z_i) = e^{-L_i(\tau_i - \delta_i^*)} e^{-d_i} V_i(x_i),$$

and by combining (43) and (44) we get

$$\begin{aligned} \tilde{\chi}_{ij}(W_j(z_j)) &= \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} e^{-L_j \tau_j} V_j(x_j)) \\ &\geq \chi_{ij}(V_j(x_j)) \end{aligned}$$

for all $j \in I_c$, while $\tilde{\chi}_{ij}(W_j(z_j)) = \chi_{ij}(V_j(x_j))$ for all $j \notin I_c$, and $\tilde{\chi}_i(|u|) = \chi_i(|u|)$. Hence (39) holds according to (18) and the assumption that $\alpha_i(r) \equiv e^{-d_i} r$.

Therefore, W_i is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_i$ for $\tilde{\Sigma}_i$ with rate coefficients $\tilde{c}_i, \tilde{d}_i$. □

4.3 Example

In this subsection, we demonstrate the method of modifying ISS Lyapunov functions in a case where we cannot directly apply Theorem 2 and Proposition 1 to establish stability.

Consider an interconnection of two hybrid subsystems with state $x = (x_1, x_2)$:

$$\begin{aligned} &\dot{x}_1 = x_1 + x_2^2, \quad \dot{x}_2 = -3x_2 + 0.1\sqrt{|x_1|}, && x \in \mathcal{C}, \\ &x_1^+ = e^{-1} x_1, \quad \dot{x}_2^+ = e x_2, && x \in \mathcal{D}, \end{aligned}$$

where $\mathcal{C} = \mathcal{D} = \mathbb{R}^2$.[7] The x_1-subsystem Σ_1 has stable discrete dynamics but non-ISS continuous dynamics, while the x_2-subsystem Σ_2 has ISS continuous dynamics but unstable discrete dynamics. Hence we cannot directly apply Theorem 2 and Proposition 1 to establish pre-GAS (as there is no external input) of the interconnection.

[7] Since $\mathcal{C} = \mathcal{D} = \mathbb{R}^2$, it follows that the system may flow or jump at any point in $\mathbb{R}^2$, and that all solutions are complete and hence the notions of pre-ISS and ISS, and those of pre-GAS and GAS coincide, respectively.

Consider functions $V_1, V_2 : \mathbb{R} \to \mathbb{R}_+$ defined by

$$V_1(x_1) := |x_1|, \qquad V_2(x_2) := |x_2|,$$

and $\chi_{12}, \chi_{21} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\chi_{12}(r) := \frac{1}{a} r^2, \qquad \chi_{21}(r) := \frac{1}{b}\sqrt{r}$$

with $a, b > 0$. Since

$$\begin{aligned} |x_1| \geq \chi_{12}(|x_2|) &\Rightarrow \dot{V}_1(x_1) \leq (a+1)V_1(x_1), \\ |x_2| \geq \chi_{21}(|x_1|) &\Rightarrow \dot{V}_2(x_2) \leq (0.1b - 3)V_2(x_2), \end{aligned}$$

and [8]

$$V_1(x_1^+) \leq e^{-1}V_1(x_1), \qquad V_2(x_2^+) \leq eV_2(x_2)$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, it follows that V_1, V_2 are candidate exponential ISS Lyapunov functions for the subsystems Σ_1, Σ_2 with internal gains χ_{12}, χ_{21}, respectively. As the discrete dynamics of the Σ_2 is unstable, we adopt the modification scheme in Section 4.1. Consider an arbitrary solution x satisfies the ADT condition

$$j - k \leq \delta_2(t - s) + N_{02} \tag{45}$$

for all $(s, k), (t, j) \in \operatorname{dom} x$ such that $(s, k) \preceq (t, j)$ with constants $\delta_2, N_{02} > 0$. Then the ADT clock τ_2 satisfies

$$\begin{aligned} &\dot{\tau}_2 \in [0, \delta_2], \qquad &\tau_2 \in [0, N_{02}], \\ &\tau_2^+ = \tau_2 - 1, \qquad &\tau_2 \in [1, N_{02}]. \end{aligned}$$

Let $z_1 := x_1$ and $z_2 := (x_2, \tau_2)$. Following Proposition 5, we see that the function $W_2 : \mathbb{R} \times [0, N_{02}] \to \mathbb{R}_+$ defined by

$$W_2(z_2) := e^{L_2\tau_2}V_2(x_2)$$

is a candidate exponential ISS Lyapunov function for the augmented subsystem $\tilde{\Sigma}_2$ with the internal gain $\tilde{\chi}_{21} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\tilde{\chi}_{21}(r) := e^{L_2N_{02}}\chi_{21}(r) = \frac{1}{b}e^{L_2N_{02}}\sqrt{r}$$

as

$$W_2(z_2) \geq \tilde{\chi}_{21}(V_1(z_1))$$

implies

$$\dot{W}_2(z_2; y_2) \leq (0.1b - 3 + L_2\delta_2)W_2(z_2)$$

for all $(z_1, z_2) \in \mathbb{R}^2 \times [0, N_{02}]$ and $y_2 \in \{-3x_2 + 0.1\sqrt{|x_1|}\} \times [0, \delta_2]$, and

$$W_2(y_2) \leq e^{1-L_2}W_2(z_2)$$

for all $z_2 \in \mathbb{R} \times [0, N_{02}]$ and $y_2 = (ex_2, \tau_2 - 1)$; see also footnote 8. Hence in order to ensure that the discrete dynamics of the z_2-subsystem is ISS, we set

$$L_2 > 1. \tag{46}$$

From (19) we see that the gain operator $\tilde{\Gamma} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ after modification is given by

$$\tilde{\Gamma}(r_1, r_2) = \begin{bmatrix} \chi_{12}(r_2) \\ \tilde{\chi}_{21}(r_1) \end{bmatrix},$$

and it satisfies the small-gain condition (21) if and only if

$$\chi_{12}(\tilde{\chi}_{21}(r)) = \frac{1}{ab^2}e^{2L_2N_{02}}r < r \qquad \forall r > 0,$$

or equivalently,

$$L_2 < \frac{\ln(ab^2)}{2N_{02}}. \tag{47}$$

Let $s > 0$ be such that

$$\frac{1}{b}e^{L_2N_{02}} < \frac{1}{s} < \sqrt{a}.$$

Then $\sigma := (\sigma_1, \sigma_1)$ with $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ defined by

$$\sigma_1(r) := r, \qquad \sigma_2(r) := \frac{1}{s}\sqrt{r}$$

is an Ω-path w.r.t. $\tilde{\Gamma}$. Following Theorem 2, we see that $W : \mathbb{R}^2 \times [0, N_{02}] \to \mathbb{R}_+$ defined by

$$\begin{aligned} W(z) &:= \max\{\sigma_1^{-1}(V_1(z_1)), \sigma_2^{-1}(W_2(z_2))\} \\ &= \max\{V_1(z_1), s^2W_2(z_2)^2\} \end{aligned}$$

is a candidate Lyapunov function w.r.t. $\tilde{\mathcal{A}} := \{0\} \times \{0\} \times [0, N_{02}]$ for the augmented interconnection with state $z := (z_1, z_2) \in \mathbb{R}^2 \times [0, N_{02}] =: \mathcal{Z}$, that is,

$$\dot{W}(z; y) \leq -cW(z)$$

for all $z \in \mathcal{Z}$ and $y \in \{-x_1 + x_2^2\} \times \{-3x_2 + 0.1\sqrt{|x_1|}\} \times [0, \delta_2]$ with

$$c := \min\{-(a+1), 2(3 - 0.1b - L_2\delta_2)\};$$

and

$$W(y) \leq e^{-d}W(z)$$

[8] Note that the discrete dynamics of both subsystems are autonomous, and hence we can ignore the terms corresponding to internal gains χ_{12}, χ_{21} in (7). Similar simplifications will be made when we apply Proposition 5 and Theorem 2.

for all $z \in \mathcal{Z}$ and $y = (e^{-1}x_1, ex_2, \tau_2 - 1)$ with [9]

$$d := \min\{1, 2(L_2 - 1)\}.$$

Hence W is a candidate exponential Lyapunov function for the augmented interconnection. Since L_2 satisfies (46), we get $c < 0 < d$. Following Proposition 1 and Remarks 3, 4, we see that a set of solutions is GAS provided that it satisfies the ADT condition (45) and the RADT condition

$$t - s \le \delta^*(j - k) + \delta^* N_0^* \tag{48}$$

for all $(s,k),(t,j) \in \operatorname{dom} x$ such that $(s,k) \preceq (t,j)$ with constant $\delta^*, N_0^* > 0$, and that

$$0 < \delta^* < \frac{d}{-c} = \frac{\min\{1, 2(L_2-1)\}}{\max\{a+1, 2(0.1b - 3 + L_2\delta_2)\}}$$

and (47) hold. For example, by setting $a = 1, b = 5$ and $L_2 = 1.5$ we see that the set of solutions satisfying the ADT condition (45) and the RADT condition (48) with $\delta_2 = 2.25, N_{02} = 1, \delta^* = 0.45$ and $N_0^* = 1$.

5 Conclusion and future research

In this paper we have proved several small-gain theorems for interconnected hybrid systems which provide candidate ISS Lyapunov functions for the interconnection. These results unify various Lyapunov-based small-gain theorems for hybrid [33,7,27] and impulsive systems [8,10], and pave the way to the following general scheme for establishing ISS of interconnected hybrid systems:

(1) Construct a candidate exponential ISS Lyapunov function V_i for each subsystem Σ_i with rate coefficients c_i, d_i and linear internal gains.
(2) Compute the non-ISS index sets I_d, I_c.
(3) Modify the subsystems Σ_i *either for all* $i \in I_d$ *or for all* $i \in I_c$.
(4) Apply Theorem 4 to construct a candidate exponential ISS Lyapunov function W for the augmented interconnection $\tilde{\Sigma}$ with rate coefficients c, d.
(5) Obtain the conditions for ISS of $\tilde{\Sigma}$ via Proposition 1.
(6) Summarize the conditions for ISS of the original interconnection Σ from those in Steps 3) and 5).

As we have seen from Section 4, the modification of candidate ISS Lyapunov functions in Step 3) leads to the substantial increase of internal gains. Therefore a considerable improvement of the scheme above lies in the fact that only the subsystems with indices in I_d or those with indices in I_c would be modified, instead of all the subsystems from $I_d \cup I_c$ as it was done in [27]. If either $I_d = \emptyset$ or $I_c = \emptyset$ then no subsystem needs to be modified at all. Moreover, the scheme above also applies to arbitrary interconnections of $n \ge 2$ subsystems.

In the scheme above it is assumed that all V_i are candidate exponential ISS Lyapunov functions with linear internal gains. However, the modification process works for candidate exponential Lyapunov functions with *nonlinear* internal gains as well, and Theorem 2 has been proved for arbitrary candidate ISS Lyapunov functions with nonlinear internal gains. Having generalized Proposition 1 to the case of non-exponential ISS Lyapunov functions, one can use the scheme above also for V_i with nonlinear internal gains. Such theorems have been proved in [10, Theorems 1,3] for impulsive systems, and we believe that they can be generalized to hybrid systems as well. This is one of the possible directions for future research.

More challenging are the questions whether one can establish ISS of an interconnection in the presence of destabilizing dynamics in subsystems without enlarging the internal gains or without modifying ISS Lyapunov functions at all. At the time these questions remain open.

[9] Note that the additional coefficients 2 in the second terms of the definitions of c and d are due to differentiating the second term $W_2(z_2)^2$ of the definition of W.

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