Adaptive learning in two-player Stackelberg games with continuous action sets

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Abstract

We study a two-player Stackelberg game in which the follower's best response depends on parameters that are unknown to the leader. An adaptive learning algorithm is designed to simultaneously estimate these unknown parameters and minimize the leader's cost, based on adaptive control techniques and hysteresis-type switching. The algorithm guarantees that the leader's estimated cost will be indistinguishable from its actual cost in finite time, up to an error no larger than a preselected, arbitrarily small threshold, and that the first-order necessary condition for optimality holds asymptotically for the estimated cost. Provided that an additional persistent of excitation condition holds, the difference between the parameter estimate and the actual parameter can also be bounded by a preselected, arbitrarily small threshold in finite time. The algorithm and the convergence results are illustrated via a simple simulation example in the domain of network security.

I. Introduction

A modern engineering system usually involves multiple self-interested parties whose actions have mutual consequences. Examples include communications sharing a network with limited capacity, and computer programs sharing a limited computational resource. Game theory provides a systematic framework for modeling cooperation and conflict between these so-called strategic players [1], and has been widely used in areas such as robust design, resource allocation, and network security [2]–[4].

In game theory, a fundamental question is whether the players could converge to a Nash equilibrium—a tuple of strategies for which no one has a unilateral incentive to change—if they play the game iteratively and adjust their strategies based on historical outcomes. A primary example of such a learning process is fictitious play [5], in which each player believes that its opponents are playing constant mixed strategies in agreement with the empirical distributions of their past actions, and plays the corresponding best response. Fictitious play has attracted significant research interests [6], [7], and has been used for prescribing behavior in distributed control [8]. Another well-known example is the gradient response method developed in [9].

In this paper, we propose an adaptive learning approach for a hierarchical game model proposed by Stackelberg [10]. In a two-player Stackelberg game, one player (called the *leader*) select its action first, and then the other player (called the *follower*), informed of the leader's choice, selects its own action. Therefore, a follower's strategy in Stackelberg game is a function that specifies a response to each leader's possible action.

Stackelberg games provide a natural framework for understanding systems with asymmetrical information structure, which is common in network routing [11] and scheduling [12]. This feature is especially useful for modeling security problems, where the defender (leader) is unaware of the attack objective ahead of time, whereas the attacker (follower) is able to observe the defense strategy and attack after careful planning. Stackelberg Security Games have been applied to various real-world security domains, and have lead to practical implementations such as the ARMOR program at the Los Angeles International Airport and the IRIS program used by the US Federal Air Marshals [13].

Asymmetrical information often leads to scenarios where a Nash equilibrium does not exist but a Stackelberg equilibrium does, as the sufficient conditions for existence of the former is much stronger than those of the latter [2, p. 181]. For these scenarios, learning options like fictitious play and gradient response cannot be applied, and novel approaches are needed to achieve convergence to a Stackelberg equilibrium. Existing results on learning in Stackelberg games are limited to linear and quadratic costs and finite action sets [14]–[16], which are too restrictive for many applications including network security.

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We study a Stackelberg game between two players, each with a continuous action set. We consider the scenario where the leader only has partial knowledge of the follower's cost function. Specifically, the follower's actual strategy (best-response function) admits a known functional form with unknown parameters. Our main contribution is an adaptive learning algorithm that simultaneously estimates these unknown parameters based on the follower's past actions and minimizes the leader's cost, designed based on adaptive control techniques and hysteresis-type switching. The algorithm guarantees that the leader's cost predicted using the parameter estimate will be indistinguishable from its actual cost in finite time, up to an error no larger than a preselected, arbitrarily small threshold, and that the first-order necessary condition for optimality holds asymptotically for the former. If an additional persistent of excitation condition holds, then the difference between the parameter estimate and the actual parameter can also be bounded by a preselected, arbitrarily small threshold in finite time. Furthermore, we consider the case where the function used by the leader to predict the follower's best response does not perfectly match the follower's actual strategy. In this case, our adaptive learning algorithm can be adjusted to guarantee the same convergence results with preselected error thresholds larger than the size of the mismatch. The algorithm and convergence results are illustrated via a simple simulation example motivated by link-flooding denial-of-service (DoS) attacks such as the Crossfire attack [17].

Notations: Let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. Let I_n be the identity matrix in $\mathbb{R}^{n\times n}$; the subscript is omitted when the dimension is implicit. Denote by $\|\cdot\|$ the Euclidean norm for vectors and the (induced) Euclidean norm for matrices. For a set $\mathcal{S} \subset \mathbb{R}^n$, denote by $\partial\mathcal{S}$, $\overline{\mathcal{S}}$, and $\overline{\text{conv}}\,\mathcal{S}$ its boundary, closure, and closed convex hull, respectively. A signal $u : [t_0, \infty) \to \mathbb{R}^n$ is of class $\mathcal{L}_\infty$ if $\sup_{t\geq t_0} \|u(t)\|$ is finite. A function is of class $\mathcal{C}^1$ if it is continuously differentiable. Denote by $\mathcal{B}_r(x)$ the closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$, that is, $\mathcal{B}_r(x) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$.

II. Problem formulation

Consider a two-player game where $\mathcal{U} \subset \mathbb{R}^{n_u}$ and $\mathcal{A} \subset \mathbb{R}^{n_a}$ are the *action sets* of the first and second players, respectively, and $J : \mathcal{U} \times \mathcal{A} \to \mathbb{R}$ and $H : \mathcal{A} \times \mathcal{U} \to \mathbb{R}$ are the corresponding *cost functions*. We are interested in a hierarchical game model proposed by Stackelberg [10], where the first player (called the *leader*) selects its action $u \in \mathcal{U}$ first, and then the second player (called the *follower*), informed of the leader's choice, selects its action $a \in \mathcal{A}$. The corresponding notion of equilibrium is defined as follows:

Definition 1 ([1], [2, Def. 4.6, p. 179])**.** An action $u^* \in \mathcal{U}$ is a *Stackelberg equilibrium action* for the *leader* if[1]

$$\sup_{a\in\beta_a(u^*)} J(u^*, a) = \inf_{u\in\mathcal{U}} \sup_{a\in\beta_a(u)} J(u, a),$$

where $\beta_a(u) \subset \mathcal{A}$ denotes the set of best responses against u, that is,

$$\beta_a(u) := \Big\{a \in \mathcal{A} : H(a, u) = \inf_{a'\in\mathcal{A}} H(a', u)\Big\}.$$

We consider games with perfect but incomplete information, where the leader only has partial knowledge of the follower's action set and cost function. Specifically, for a leader's choice $u \in \mathcal{U}$, the follower selects an action

$$a = f(\theta^*, u) \in \beta_a(u),$$

where the function $f : \Theta \times \mathcal{U} \to \mathbb{R}^{n_a}$ and the parameter set $\Theta \subset \mathbb{R}^{n_\theta}$ are known to the leader, but the specific value of the parameter $\theta^* \in \Theta$ is unknown.

In practice, assuming that the functional form of the follower's best-response is known introduces little loss of generality, as it can always be approximated on a compact set up to an arbitrary precision as a finite weighted sum of a preselected class of basis functions. An example of such an approximation is the *radial basis function (RBF)* model [18], in which the leader assumes

$$f(\theta^*, u) = \sum_{j=1}^{n_\theta} \theta_j^* F_j(u) = \sum_{j=1}^{n_\theta} \theta_j^* \phi(\|u - u_j^c\|),$$

[1]In particular, we let $\sup_{a\in\beta_a(u)} J(u, a) = \infty$ if $\beta_a(u) = \emptyset$.

where $\theta^* = (\theta_1^*, \ldots, \theta_{n_\theta}^*)$ is the unknown parameter, and each $F_j : \mathcal{U} \to \mathbb{R}^{n_a}$ is an RBF centered at u_j^c. Note that in the RBF model, the approximation is affine with respect to the unknown parameter, which is a common feature in many widely-used approximation models such as *orthogonal polynomials* and *multivariate splines* [18]. This motivates restricting our attention to affine maps $\theta \mapsto f(\theta, u)$. The following assumption captures this and additional regularity conditions that we use to guarantee existence of a Stackelberg equilibrium.

Assumption 1 (Regularity)**.** The leader's action set $\mathcal{U}$ and the parameter set Θ are convex and compact; the leader's cost function J is $\mathcal{C}^1$ with locally Lipschitz Jacobian; the function f is $\mathcal{C}^1$ with locally Lipschitz gradient; the map $\theta \mapsto f(\theta, u)$ is affine for each fixed $u \in \mathcal{U}$.

Under Assumption 1, existence of a Stackelberg equilibrium follows from standard results, e.g., [2, Th. 4.8, p. 180]. In particular, these condtions are much weaker than the sufficient conditions required for existence of a Nash equilibrium [2, p. 181], which is consistent with our interest in games where a Nash equilibrium does not exist but a Stackelberg equilibrium does; see also the example in Section VI.

We denote by $\nabla_u J(u, a)$ and $\nabla_a J(u, a)$ the gradients of the maps $u \mapsto J(u, a)$ and $a \mapsto J(u, a)$, respectively, and by $\nabla_\theta f(u)$ and $\nabla_u f(\theta, u)$ the Jacobian matrices of the maps $\theta \mapsto f(\theta, u)$ and $u \mapsto f(\theta, u)$, respectively. (To be consistent with the definition of Jacobian matrix, we consider gradients as row vectors.) In particular, the Jacobian matrix $\nabla_\theta f(u)$ is independent of θ following the affine condition in Assumption 1.

Our goal is to adjust the leader's action u to minimize its cost:

$$\begin{aligned} \text{minimize} \quad & J(u, f(\theta^*, u)), \\ \text{subject to} \quad & u \in \mathcal{U}, \end{aligned}$$

based on past observations of the follower's action $a = f(\theta^*, u)$ and the leader's cost $J(u, a)$, but without knowing the actual value of the unknown parameter θ^*. Our approach to solve this problem consists of two components:
1) Construct a *parameter estimate* θ that approaches the actual parameter θ^*.
2) Adjust the leader's action u based on a gradient descent method to minimize its *estimated cost*:

$$\begin{aligned} \text{minimize} \quad & \hat{J}(u, \theta) := J(u, f(\theta, u)), \\ \text{subject to} \quad & u \in \mathcal{U}. \end{aligned} \tag{1}$$

In this paper, our analysis and design are formulated using continuous-time dynamics, which is common in the literature of learning in game theory [6], [7].

III. Estimation and minimization

To specify the adaptive algorithm for estimating the unknown parameter θ^* that is known to lie in the compact convex set Θ and optimizing the leader's strategy u within the compact convex set $\mathcal{U}$, we recall the following notions and basic properties from convex analysis [19, Sec. 4.7].

For a convex set $\mathcal{S} \subset \mathbb{R}^n$, we denote by $N_{\mathcal{S}}(x)$ the *normal cone* at a point $x \in \mathcal{S}$, that is,

$$N_{\mathcal{S}}(x) := \{v \in \mathbb{R}^n : \forall\, y \in \mathcal{S},\ v^\top (y - x) \le 0\}, \tag{2}$$

and by $T_{\mathcal{S}}(x)$ the *tangent cone* at x, that is,[2]

$$T_{\mathcal{S}}(x) := \{w \in \mathbb{R}^n : \forall\, v \in N_{\mathcal{S}}(x),\ v^\top w \le 0\}. \tag{3}$$

In particular, $T_{\mathcal{S}}(x) = \mathbb{R}^n$ and $N_{\mathcal{S}}(x) = \{0\}$ for all interior points $x \in \mathcal{S} \backslash \partial\mathcal{S}$. For a function $g : \mathcal{S} \times \mathbb{R}^m \to \mathbb{R}^n$, denote by $[g(x, d)]_{T_{\mathcal{S}}}$ its *tangent projection* with respect to $\mathcal{S}$ at $(x, d) \in \mathcal{S} \times \mathbb{R}^m$, that is,

$$[g(x, d)]_{T_{\mathcal{S}}} := \operatorname*{arg\,min}_{v \in T_{\mathcal{S}}(x)} \|v - g(x, d)\|.$$

The convexity of $\mathcal{S}$ ensures that $[g(x, d)]_{T_{\mathcal{S}}}$ is unique for each $(x, d) \in \mathcal{S} \times \mathbb{R}^m$. Moreover, for all $x \in \mathcal{S}$ and $d \in \mathbb{R}^m$, the tangent projection satisfies

$$[g(x, d)]_{T_{\mathcal{S}}} \in T_{\mathcal{S}}(x), \quad g(x, d) - [g(x, d)]_{T_{\mathcal{S}}} \in N_{\mathcal{S}}(x), \tag{4}$$

and

$$\big(g(x, d) - [g(x, d)]_{T_{\mathcal{S}}}\big)^\top [g(x, d)]_{T_{\mathcal{S}}} = 0. \tag{5}$$

[2]As the set $\mathcal{S}$ is convex, the tangent cone $T_{\mathcal{S}}(x)$ can be defined using (3) [19, Prop. 4.6.3, p. 254].

A. Parameter estimation

We construct the parameter estimate θ by comparing past observation of the follower's action $a = f(\theta^*, u)$ and the leader's cost $J(u, a)$ with the corresponding estimated values $f(\theta, u)$ and $\hat{J}(u, \theta) = J(u, f(\theta, u))$. Their difference is defined as the following *observation error*

$$e_{\text{obs}} := \begin{bmatrix} f(\theta, u) - a \\ \hat{J}(u, \theta) - J(u, a) \end{bmatrix}. \tag{6}$$

In the following, we develop an estimation law in terms of the observation error e_{obs}, so that the norm of the *estimation error* $\|\theta - \theta^*\|$ is monotonically nonincreasing, regardless of how the leader's action u is adjusted.

First, we establish a relation between the observation error e_{obs} and the estimation error $\theta - \theta^*$.

Lemma 1. *The observation error e_{obs} satisfies*

$$e_{\text{obs}} = K(u, a, \theta)(\theta - \theta^*) \tag{7}$$

with the gain matrix

$$K(u, a, \theta) := \begin{bmatrix} I \\ \int_0^1 \nabla_a J(u, \lambda f(\theta, u) + (1 - \lambda)\, a)\, \mathrm{d}\lambda \end{bmatrix} \nabla_\theta f(u). \tag{8}$$

Proof. See Appendix II-A. □

Following Lemma 1, the observation error e_{obs} would be zero if the current estimate θ of the unknown parameter θ^* was correct. However, in most interesting scenarios, the dimension n_θ of the parameter vector θ^* is much larger than the dimension $n_a + 1$ of the observation error e_{obs}; thus the gain matrix $K(u, a, \theta)$ cannot be invertible, and a zero value for the observation error e_{obs} does not imply that the estimate θ is correct.

We propose the following estimation law to drive the parameter estimate θ towards the actual parameter θ^*:

$$\dot{\theta} = [-\lambda_e K(u, a, \theta)^\top e_{\text{obs}}]_{T_\Theta}, \tag{9}$$

with the gain matrix $K(u, a, \theta)$ defined by (8) and the *switching signal* $\lambda_e : \mathbb{R}_+ \to \{0,\, \lambda_1\}$ defined by

$$\lambda_e(t) := \begin{cases} \lambda_1 & \text{if } \|e_{\text{obs}}(t)\| \geq \varepsilon_{\text{obs}}; \\ \lim_{s \nearrow t} \lambda_e(s) & \text{if } \|e_{\text{obs}}(t)\| \in (\varepsilon_{\text{obs}}/2, \varepsilon_{\text{obs}}); \\ 0 & \text{if } \|e_{\text{obs}}(t)\| \leq \varepsilon_{\text{obs}}/2 \end{cases} \tag{10}$$

and $\lambda_e(0) := \lambda_1$ if $\|e_{\text{obs}}(0)\| \in (\varepsilon_{\text{obs}}/2, \varepsilon_{\text{obs}})$, where $\varepsilon_{\text{obs}}, \lambda_1 > 0$ are preselected constants. Several comments are in order: First, the gain matrix $K(u, a, \theta)$ depends on the estimate θ but not on the actual parameter θ^*, so (9) can be implemented without knowing θ^*. Second, the tangent projection $[\cdot]_{T_\Theta}$ is used to guarantee that the parameter estimate θ remains inside the compact convex set Θ. Finally, the right-continuous, piecewise constant switching signal λ_e is designed so that the adaption is on when $\|e_{\text{obs}}\| \geq \varepsilon_{\text{obs}}$ and off when $\|e_{\text{obs}}\| \leq \varepsilon_{\text{obs}}/2$, with a hysteresis-type switching rule that avoids chattering. The key feature of (9) is that the estimation error $\theta - \theta^*$ satisfies

$$\begin{aligned} \frac{\mathrm{d}\|\theta - \theta^*\|^2}{\mathrm{d}t} &= 2(\theta - \theta^*)^\top [-\lambda_e K(u, a, \theta)^\top e_{\text{obs}}]_{T_\Theta} \\ &\leq 2(\theta - \theta^*)^\top \big(-\lambda_e K(u, a, \theta)^\top e_{\text{obs}}\big) = -2\lambda_e \|e_{\text{obs}}\|^2, \end{aligned}$$

where the inequality follows from (4) and (2). More specifically, (4) implies

$$-\lambda_e K(u, a, \theta)^\top e_{\text{obs}} - [-\lambda_e K(u, a, \theta)^\top e_{\text{obs}}]_{T_\Theta} \in N_\Theta(\theta)$$

and thus (2) implies

$$\big(-\lambda_e K(u, a, \theta)^\top e_{\text{obs}} - [-\lambda_e K(u, a, \theta)^\top e_{\text{obs}}]_{T_\Theta}\big)^\top (\theta^* - \theta) \leq 0$$

as $\theta^* \in \Theta$. We thus conclude that the estimation law (9) with the switching signal (10) guarantees

$$\frac{\mathrm{d}\|\theta - \theta^*\|^2}{\mathrm{d}t} \leq -2\lambda_e \|e_{\text{obs}}\|^2 \leq 0, \tag{11}$$

which implies that $\|\theta - \theta^*\|$ is monotonically nonincreasing and will not stop approaching zero unless $\|e_{\text{obs}}\| < \varepsilon_{\text{obs}}$. In the convergence results in Section IV, we will show that the adaption of the estimate θ stops in finite time, and the observation error e_{obs} satisfies $\|e_{\text{obs}}\| < \varepsilon_{\text{obs}}$ afterwards.

B. Cost minimization

Several options are available to adjust the leader's strategy u, but in this paper our analysis will focus on a gradient descent method, which is fairly robust for a wide range of problems. Our ultimate goal is to minimize the leader's cost $J(u,a) = J(u, f(\theta^*, u))$. However, computing the gradient descent direction of the actual cost requires knowledge of the actual parameter θ^*. Therefore, we minimize instead the leader's estimated cost $\hat{J}(u,\theta) = J(u, f(\theta, u))$, which only depends on the estimate θ. This change in objective is justified by the fact that $\|\hat{J}(u,\theta) - J(u,a)\| \leq \|e_{\text{obs}}\| < \varepsilon_{\text{obs}}$ holds in finite time, which will be established in Section IV.

The time derivative of the estimated cost $\hat{J}(u,\theta)$ is

$$\dot{\hat{J}}(u,\theta) = \nabla_u \hat{J}(u,\theta)\,\dot{u} + \nabla_\theta \hat{J}(u,\theta)\,\dot{\theta}, \tag{12}$$

where

$$\begin{aligned}\nabla_u \hat{J}(u,\theta) &= \nabla_u J(u, f(\theta,u)) + \nabla_a J(u, f(\theta,u)) \nabla_u f(\theta,u),\\ \nabla_\theta \hat{J}(u,\theta) &= \nabla_a J(u, f(\theta,u)) \nabla_\theta f(u)\end{aligned}$$

are the gradients of the maps $u \mapsto \hat{J}(u,\theta)$ and $\theta \mapsto \hat{J}(u,\theta)$, respectively. (Note that here $\nabla_u J(u, f(\theta,u))$ denotes the gradient of the map $u \mapsto J(u,\hat{a})$ at $\hat{a} = f(\theta,u)$.) As we will establish that the adaption of the parameter estimate θ stops in finite time, we neglect the second term in (12) and focus exclusively in adjusting u along the gradient descent direction of $u \mapsto \hat{J}(u,\theta)$. This motivates the following minimization law to adjust the leader's action:

$$\dot{u} = \left[-\lambda_2 \nabla_u \hat{J}(u,\theta)^\top\right]_{T_{\mathcal{U}}} \tag{13}$$

where $\lambda_2 > 0$ is a preselected constant. The tangent projection $[\cdot]_{T_{\mathcal{U}}}$ is used to guarantee that the leader's action u remains inside the compact convex set $\mathcal{U}$. Consequently,

$$\begin{aligned}\dot{\hat{J}}(u,\theta) &= \nabla_u \hat{J}(u,\theta)\left[-\lambda_2 \nabla_u \hat{J}(u,\theta)^\top\right]_{T_{\mathcal{U}}} + \nabla_\theta \hat{J}(u,\theta)\,\dot{\theta}\\ &= -\left\|\left[-\lambda_2 \nabla_u \hat{J}(u,\theta)^\top\right]_{T_{\mathcal{U}}}\right\|^2/\lambda_2 + \nabla_\theta \hat{J}(u,\theta)\,\dot{\theta}\\ &= -\|\dot{u}\|^2/\lambda_2 + \nabla_\theta \hat{J}(u,\theta)\,\dot{\theta},\end{aligned}$$

where the second equality follows from (5). We thus conclude that the minimization law (13) guarantees

$$\dot{\theta} = 0 \implies \dot{\hat{J}}(u,\theta) \leq -\|\dot{u}\|^2/\lambda_2 \leq 0.$$

In the convergence results in Section IV, we will show that the leader's action u converges asymptotically to the set of points for which the first-order necessary condition for optimality holds for the optimization problem (1).

IV. Convergence analysis

We now state the main result of this paper:

Theorem 1. *Suppose that Assumption 1 holds. Given any threshold $\varepsilon_{\text{obs}} > 0$ in* (10)*, the estimation and minimization algorithm* (9) *and* (13) *with the switching signal* (10) *guarantees the following properties:*

1) There exists a time $T_1 \geq 0$ such that

$$\theta(t) = \theta(T_1) \qquad \forall\, t \geq T_1, \tag{14}$$

and

$$\|e_{\text{obs}}(t)\| < \varepsilon_{\text{obs}} \qquad \forall\, t \geq T_1. \tag{15}$$

2) The first-order necessary condition for optimality holds asymptotically for the optimization problem (1)*, that is,*

$$\lim_{t\to\infty} \left[-\nabla_u \hat{J}(u(t), \theta(T_1))^\top\right]_{T_{\mathcal{U}}} = 0. \tag{16}$$

Essentially, item 1) guarantees that the parameter estimate θ converges in finite time to a point which is indistinguishable from the actual parameter θ^* based on observations of the follower's action $a = f(\theta^*, u)$ and the leader's cost $J(u,a)$, up to an error no larger than the threshold ε_{obs}. Regarding item 2), we recall the first-order necessary condition for optimality from [19, Prop. 4.7.1, p. 255]:

Lemma 2. *If u^* is a local minimum point of the map $u \mapsto \hat{J}(u, \theta(T_1))$ on $\mathcal{U}$, then there exists a $\delta > 0$ such that for all $u \in \mathcal{U}$ satisfying $\|u - u^*\| < \delta$, it holds that*

$$\nabla_u \hat{J}(u^*, \theta(T_1))(u - u^*) \geq 0,$$

or equivalently,

$$\left[-\nabla_u \hat{J}(u^*, \theta(T_1))^\top\right]_{T_{\mathcal{U}}} = 0.$$

Proof of Theorem 1. As the right hand-sides of (9) and (13) are potentially discontinuous due to the tangent projections, the proof of Theorem 1 requires results from differential inclusions theory; see Appendix I for the necessary preliminaries.

First, we establish existence, boundedness and uniqueness of solutions for the system defined by (9) and (13).

Lemma 3. *For each $(\theta_0, u_0) \in \Theta \times \mathcal{U}$, there exists a unique Carathéodory solution to the system defined by* (9) *and* (13) *on $\mathbb{R}_+$ with $(\theta(0), u(0)) = (\theta_0, u_0)$, that is, there exist unique absolutely continuous functions $\theta : \mathbb{R}_+ \to \mathbb{R}^{n_\theta}$ and $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$ with $(\theta(0), u(0)) = (\theta_0, u_0)$ such that* (9) *and* (13) *hold almost everywhere on $\mathbb{R}_+$. Moreover, $(\theta(t), u(t)) \in \Theta \times \mathcal{U}$ for all $t \geq 0$, and*

$$\theta, \dot{\theta}, u, \dot{u}, e_{\text{obs}}, \dot{e}_{\text{obs}} \in \mathcal{L}_\infty. \tag{17}$$

Proof. Lemma 3 is established by combining the results on hysteresis switching from [20], the results on existence and boundedness of solutions for projected differential inclusions from [21], and the results on uniqueness of solutions for differential inclusions from [22]; see Appendix II-B for the complete proof. □

Second, we establish item 1) of Theorem 1 via arguments along the lines of the proof of Barbalat's lemma [23, Lemma 3.2.6, p. 76]. We cannot use Barbalat's lemma directly since the switching signal λ_e in (9) is not continuous but only piecewise continuous. Following (11), we see that $\|\theta - \theta^*\|^2$ is monotonically nonincreasing. Therefore $\lim_{t\to\infty} \|\theta(t) - \theta^*\|^2$, and thus

$$\lim_{t\to\infty} \int_0^t \lambda_e(s) \|e_{\text{obs}}(s)\|^2 \, \mathrm{d}s, \tag{18}$$

exists and is finite. On the other hand, (9) and (10) imply that (14) and (15) hold if there exists a time $T_1 \geq 0$ such that

$$\lambda_e(t) = 0 \qquad \forall\, t \geq T_1. \tag{19}$$

Assume (19) does not hold for any $T_1 \geq 0$. Then (10) implies that there exists an unbounded increasing sequence $(t_k)_{k\in\mathbb{N}}$ with $t_0 > 0$ such that

$$\lambda_e(t_k) = \lambda_1, \quad \|e_{\text{obs}}(t_k)\| > \varepsilon_{\text{obs}}/2 \qquad \forall\, k \in \mathbb{N}.$$

Now we show that there exists an unbounded sequence $(s_k)_{k\in\mathbb{N}}$ with $s_k \in [t_k - \delta, t_k]$ such that

$$\|e_{\text{obs}}(t)\| \geq \varepsilon_{\text{obs}}/2, \quad \lambda_e(t) = \lambda_1 \tag{20}$$

for all $k \in \mathbb{N}$ and $t \in [s_k, s_k + \delta]$ with the constant

$$\delta := \min\left\{t_0, \, \frac{\varepsilon_{\text{obs}}}{2 \sup_{s\geq 0} \|\dot{e}_{\text{obs}}(s)\|}\right\} > 0,$$

where $\delta > 0$ following $\dot{e}_{\text{obs}} \in \mathcal{L}_\infty$ in (17), $\varepsilon_{\text{obs}} > 0$, and $t_0 > 0$. Indeed, for each $k \in \mathbb{N}$, consider the following two possibilities:

1) If $\|e_{\text{obs}}(t)\| < \varepsilon_{\text{obs}}$ for all $t \in [t_k - \delta, t_k]$, then (10) and $\lambda_e(t_k) = \lambda_1$ imply that (20) holds with $s_k = t_k - \delta$.
2) Otherwise, there exists an $s_k \in [t_k - \delta, t_k]$ such that $\|e_{\text{obs}}(s_k)\| = \varepsilon_{\text{obs}}$, and (20) follows from the definition of δ and (10).

Moreover, $(s_k)_{k\in\mathbb{N}}$ is unbounded as $(t_k)_{k\in\mathbb{N}}$ is unbounded. Following (20), we see that

$$\int_{s_k}^{s_k+\delta} \lambda_e(s) \|e_{\text{obs}}(s)\|^2 \, \mathrm{d}s \geq \frac{\lambda_1 \varepsilon^2 \delta}{4} > 0$$

for the unbounded sequence $(s_k)_{k\in\mathbb{N}}$, which contradicts the property that (18) exists and is finite. Therefore, there exists a time $T_1 \geq 0$ such that (19), and thus (14) and (15), holds.

Finally, we establish item 2) of Theorem 1 based on an invariance principle for Filippov solutions [24, Th. 3.2]. After T_1, the system (13) becomes

$$\dot{u} = \left[-\lambda_2 \nabla_u \hat{J}(u, \theta(T_1))^\top\right]_{T_\mathcal{U}},$$

which can be modeled using the projected dynamical system (32) in Appendix I with the state $x := u$ and the set $\mathcal{S} := \mathcal{U}$. The corresponding function g in (32) is defined by

$$g(x) := -\lambda_2 \nabla_u \hat{J}(x, \theta(T_1))^\top.$$

Note that every Filippov solution to (32) is a Carathéodory solution to the differential inclusion (34). Then Proposition 5 in Appendix I implies that for each $u_0 \in \mathcal{U}$, there exists a unique Filippov solution u to (32) on $[T_1, \infty)$ with $u(T_1) = u_0$ which is also a Carathéodory solution, and $u(t) \in \mathcal{U}$ for all $t \geq T_1$. Consider the function $V : \mathcal{U} \to \mathbb{R}$ defined by

$$V(u) := \hat{J}(u, \theta(T_1)).$$

Then for all $u \in \mathcal{U}$ and $v \in G(u)$ with the corresponding set-valued function G defined by (33), it holds that

$$\begin{aligned} \nabla V(u)\, v &= \nabla_u \hat{J}(u, \theta(T_1))\, v \\ &= -g(u)^\top v/\lambda_2 \leq -\|[g(u)]_{T_\mathcal{U}}\|^2/\lambda_2 \leq 0, \end{aligned}$$

where the first inequality follows from (37). Consequently, the invariance principle [24, Th. 3.2] implies that all Filippov solutions, and therefore all Carathéodory solutions, to (32) converge to the largest invariant set in

$$\overline{\{u \in \mathcal{U} : \exists\, v \in G(u) \text{ s.t. } \nabla V(u)\, v = 0\}} \subset \{u \in \mathcal{U} : [g(u)]_{T_\mathcal{U}} = 0\}$$

Therefore, $\lim_{t\to\infty}[g(u(t))]_{T_\mathcal{U}} = 0$, that is, (16) holds. □

In Theorem 1, there is no claim that the parameter estimate θ necessarily converges to the actual parameter θ^*. However, this can be guaranteed if we assume that the following *persistent of excitation (PE)* condition holds.

Assumption 2 (Persistent of excitation)**.** There exist constants $\tau_0, \alpha_0 > 0$ such that the gain matrix $K(u, a, \theta)$ defined by (8) satisfies

$$\int_t^{t+\tau_0} K(u(s), a(s), \theta(s))^\top K(u(s), a(s), \theta(s))\, \mathrm{d}s \geq \alpha_0 I \tag{21}$$

for all $t \geq 0$.

Theorem 2. *Suppose that Assumptions 1 and 2 hold. Then by setting the threshold*

$$\varepsilon_{\text{obs}} := \varepsilon_\theta \sqrt{\alpha_0/\tau_0} \tag{22}$$

in (10) *for any given constant $\varepsilon_\theta > 0$, the estimation and minimization algorithm* (9) *and* (13) *with the switching signal* (10) *guarantees the following properties:*

1) There exists a time $T_1 \geq 0$ such that (14) *and* (15) *hold, and*

$$\|\theta(T_1) - \theta^*\| < \varepsilon_\theta. \tag{23}$$

2) The first-order necessary condition for optimality holds asymptotically for the optimization problem (1)*, that is,* (16) *holds.*

Proof. As (14), (15) and (16) are established in Theorems 1, it remains to prove (23).

To this effect, we note that (15) implies

$$\int_{T_1}^{T_1+\tau_0} \|e_{\text{obs}}(s)\|^2\, \mathrm{d}s < \varepsilon_{\text{obs}}^2 \tau_0 = \alpha_0 \varepsilon_\theta^2,$$

where the equality follows from (22). On the other hand, (7) and (14) imply

$$\begin{aligned}
&\int_{T_1}^{T_1+\tau_0} \|e_{\rm obs}(t)\|^2 \,\mathrm{d}t \\
&= \int_{T_1}^{T_1+\tau_0} \|K(u(t),a(t),\theta(T_1))(\theta(T_1)-\theta^*)\|^2 \,\mathrm{d}s \\
&= (\theta(T_1)-\theta^*)^\top \left(\int_{T_1}^{T_1+\tau_0} K(u(t),a(t),\theta(T_1))^\top K(u(t),a(t),\theta(T_1)) \,\mathrm{d}t \right) (\theta(T_1)-\theta^*) \\
&\geq \alpha_0 \|\theta(T_1)-\theta^*\|^2,
\end{aligned}$$

where the inequality follows from the PE condition (21). Combining the inequalities above yields (23). □

Remark 1. In view of (8), a sufficient condition for (21) is

$$\int_t^{t+\tau_0} \nabla_\theta f(u(s))^\top \nabla_\theta f(u(s)) \,\mathrm{d}s \geq \alpha_0 I \qquad \forall\, t \geq 0. \tag{24}$$

The PE condition (24) is more restrictive than (8), but has the advantage that it can be checked without knowing the estimate θ. Moreover, from the proof of Theorem 2, we see that (23) only requires (21) or (24) to hold at $t = T_1$. Consequently, in practice it suffices to enforce that (21) or (24) holds when λ_e in (10) has been set to zero.

V. Model mismatch

In this section, we study the effect of a bounded mismatch exists between the follower's actual best response a against a leader's action u and the leader's belief $f(\theta^*, u)$. Denoting by $f^* : \mathcal{U} \to \mathcal{A}$ the unknown follower's strategy (best-response function), that is,

$$a = f^*(u) \in \beta_a(u),$$

up to now we assumed that there was some unknown parameter $\theta^* \in \Theta$ such that

$$f^*(u) = f(\theta^*, u) \qquad \forall\, u \in \mathcal{U}$$

for the known function f. We now consider the case where such perfect matching between the follower's actual best response $f^*(u)$ and some $f(\theta^*, u)$ may not exist.

Assumption 3 (Mismatch)**.** The unknown follower's strategy f^* is locally Lipschitz and satisfies

$$\left\| \begin{bmatrix} I \\ \int_0^1 \nabla_a J(u, \lambda f(\theta,u) + (1-\lambda) f^*(u)) \,\mathrm{d}\lambda \end{bmatrix} (f(\theta^*,u) - f^*(u)) \right\| \leq \varepsilon_f \qquad \forall\, \theta \in \Theta, \forall\, u \in \mathcal{U}$$

for the known function f and some known constant $\varepsilon_f > 0$.

Similar arguments to those in the proof of Lemma 1 show that the observation error $e_{\rm obs}$ now satisfies

$$e_{\rm obs} = K(u,a,\theta)(\theta - \theta^*) + e_f \tag{25}$$

with the gain matrix $K(u,a,\theta)$ defined by (8) and the *mismatch error*

$$e_f := \begin{bmatrix} I \\ \int_0^1 \nabla_a J(u, \lambda f(\theta,u) + (1-\lambda)\, a) \,\mathrm{d}\lambda \end{bmatrix} (f(\theta^*,u) - a).$$

Then Assumption 3 implies

$$\|e_f(t)\| \leq \varepsilon_f \qquad \forall\, t \geq 0. \tag{26}$$

The effect of the mismatch error e_f in (25) can be mitigated by keeping the same estimation law (9), while adjusting the definition of the switching signal λ_e by

$$\lambda_e(t) := \begin{cases} \lambda_1 & \text{if } \|e_{\rm obs}(t)\| \geq \varepsilon_{\rm obs}; \\ \lim_{s \nearrow t} \lambda_e(s) & \text{if } \|e_{\rm obs}(t)\| \in ((\varepsilon_{\rm obs}+\varepsilon_f)/2, \varepsilon_{\rm obs}); \\ 0 & \text{if } \|e_{\rm obs}(t)\| \leq (\varepsilon_{\rm obs}+\varepsilon_f)/2 \end{cases} \tag{27}$$

and $\lambda_e(0) = \lambda_1$ if $\|e_{\rm obs}(0)\| \in ((\varepsilon_{\rm obs} + \varepsilon_f)/2, \varepsilon_{\rm obs})$, where $\varepsilon_{\rm obs} > \varepsilon_f$ and $\lambda_1 > 0$ are preselected constants. Under the estimation law (9), the estimation error $\theta - \theta^*$ now satisfies

$$\begin{aligned}\frac{\mathrm{d}\|\theta-\theta^*\|^2}{\mathrm{d}t} &= 2(\theta-\theta^*)^\top[-\lambda_e K(u,a,\theta)^\top e_{\rm obs}]_{T_\Theta}\\ &\le 2(\theta-\theta^*)^\top\big(-\lambda_e K(u,a,\theta)^\top e_{\rm obs}\big)\\ &= -2\lambda_e(e_{\rm obs}-e_f)^\top e_{\rm obs},\end{aligned}$$

where the inequality follows from (4) and (2). In the following, we prove that

$$\frac{\mathrm{d}\|\theta-\theta^*\|^2}{\mathrm{d}t} \le -2\lambda_e(e_{\rm obs}-e_f)^\top e_{\rm obs} \le 0, \tag{28}$$

where $\mathrm{d}\|\theta-\theta^*\|^2/\,\mathrm{d}t = 0$ if and only if $\lambda_e = 0$. Indeed, consider the following two possibilities:

1) If $\lambda_e = 0$, then $\mathrm{d}\|\theta-\theta^*\|^2/\,\mathrm{d}t = 0$.
2) Otherwise $\lambda_e = \lambda_1$, and thus $\|e_{\rm obs}\| > (\varepsilon_{\rm obs}+\varepsilon_f)/2 > \varepsilon_f \ge \|e_f\|$ following (26) and (27). Hence

$$\begin{aligned}\frac{\mathrm{d}\|\theta-\theta^*\|^2}{\mathrm{d}t} &\le -2\lambda_1(e_{\rm obs}-e_f)^\top e_{\rm obs}\\ &\le -\lambda_1(\|e_{\rm obs}\|^2 + \|e_{\rm obs}-e_f\| - \|e_f\|^2) < 0.\end{aligned}$$

The following two results extend Theorems 1 and 2 to the current case without perfect matching between the follower's actually best response $f^*(u)$ and some $f(\theta^*,u)$.

Theorem 3. *Suppose that Assumptions 1 and 3 hold. Given any threshold $\varepsilon_{\rm obs} > \varepsilon_f$ in* (27)*, the estimation and minimization algorithm* (9) *and* (13) *with the switching signal* (27) *guarantees the following properties:*

1) There exists a time $T_1 \ge 0$ such that (14) *and* (15) *hold.*
2) The first-order necessary condition for optimality holds asymptotically for the optimization problem (1)*, that is,* (16) *holds.*

Proof. The proof is along the lines of the proof of Theorem 1. First, Lemma 3 still holds as the function f^* is locally Lipschitz. Second, we establish item 1) of Theorem 3 via similar arguments to those for item 1) of Theorem 1. Following (28), we see that $\|\theta-\theta^*\|^2$ is monotonically nonincreasing. Therefore $\lim_{t\to\infty}\|\theta(t)-\theta^*\|^2$, and thus

$$\lim_{t\to\infty}\int_0^t \lambda_e(s)(e_{\rm obs}(s)-e_f(s))^\top e_{\rm obs}(s)\,\mathrm{d}s, \tag{29}$$

exists and is finite. On the other hand, (9) and (27) imply that (14) and (15) hold if there exists a time $T_1 \ge 0$ such that (19) holds. Assume (19) does not hold for any $T_1 \ge 0$. Then (27) implies that there exists an unbounded increasing sequence $(t_k)_{k\in\mathbb{N}}$ with $t_0 > 0$ such that

$$\lambda_e(t_k) = \lambda_1, \quad \|e_{\rm obs}(t_k)\| > (\varepsilon_{\rm obs}+\varepsilon_f)/2 \qquad \forall\, k \in \mathbb{N}.$$

Now we show that there exists an unbounded sequence $(s_k)_{k\in\mathbb{N}}$ with $s_k \in [t_k-\delta, t_k]$ such that

$$\|e_{\rm obs}(t)\| \ge (\varepsilon_{\rm obs}+\varepsilon_f)/2, \quad \lambda_e(t) = \lambda_1 \tag{30}$$

for all $k \in \mathbb{N}$ and $t \in [s_k, s_k+\delta]$ with the constant

$$\delta := \min\left\{t_0,\ \frac{\varepsilon_{\rm obs}-\varepsilon_f}{2\sup_{s\ge0}\|\dot e_{\rm obs}(s)\|}\right\} > 0,$$

where $\delta > 0$ following $\dot e_{\rm obs} \in \mathcal{L}_\infty$ in (17), $\varepsilon_{\rm obs} > \varepsilon_f$, and $t_0 > 0$. Indeed, for each $k \in \mathbb{N}$, consider the following two possibilities:

1) If $\|e_{\rm obs}(t)\| < \varepsilon_{\rm obs}$ for all $t \in [t_k-\delta, t_k]$, then (27) and $\lambda_e(t_k) = \lambda_1$ imply that (30) holds with $s_k = t_k - \delta$.
2) Otherwise, there exists an $s_k \in [t_k-\delta, t_k]$ such that $\|e_{\rm obs}(s_k)\| = \varepsilon$, and (30) follows from the definition of δ and (27).

Moreover, $(s_k)_{k\in\mathbb{N}}$ is unbounded as $(t_k)_{k\in\mathbb{N}}$ is unbounded. Following (30), we see that

$$\begin{aligned}&\int_{s_k}^{s_k+\delta} \lambda_e(s)(e_{\text{obs}}(s)-e_f(s))^\top e_{\text{obs}}(s)\,\mathrm{d}s\\ \geq\ &\lambda_1\int_{s_k}^{s_k+\delta}\|e_{\text{obs}}(s)\|(\|e_{\text{obs}}(s)\|-\varepsilon_f)\,\mathrm{d}s\\ \geq\ &\frac{\lambda_1(\varepsilon_{\text{obs}}^2-\varepsilon_f^2)\,\delta}{4}>0\end{aligned}$$

for the unbounded sequence $(s_k)_{k\in\mathbb{N}}$, which, combined with (28), contradicts the property that (29) exists and is finite. Therefore, there exists a time $T_1\geq 0$ such that (19), and thus (14) and (15), holds. Finally, item 2) of Theorem 3 is the same as item 2) of Theorem 1 as the minimization process is the same after the adaption of θ stops. □

Theorem 4. *Suppose that Assumptions 1–3 hold. Then by setting the threshold*

$$\varepsilon_{\text{obs}} := \sqrt{\alpha_0\varepsilon_\theta^2/(2\tau_0)-\varepsilon_f^2} \tag{31}$$

in (27) *for any given constant* $\varepsilon_\theta > 2\varepsilon_f\sqrt{\tau_0/\alpha_0}$*, the estimation and minimization algorithm* (9) *and* (13) *with the switching signal* (27) *guarantees the following properties:*

1) *There exists a time* $T_1\geq 0$ *such that* (14)*,* (15) *and* (23) *hold.*
2) *The first-order necessary condition for optimality holds asymptotically for the optimization problem* (1)*, that is,* (16) *holds.*

Proof. As (14), (15) and (16) are established in Theorems 3, it remains to prove (23).

To this effect, we note that (15) implies

$$\int_{T_1}^{T_1+\tau_0}\|e_{\text{obs}}(s)\|^2\,\mathrm{d}s<\varepsilon_{\text{obs}}^2\tau_0=\frac{\alpha_0\varepsilon_\theta^2}{2}-\varepsilon_f^2\tau_0,$$

where the equality follows from (31). On the other hand, (14) and (25) imply

$$\begin{aligned}&\int_{T_1}^{T_1+\tau_0}\|e_{\text{obs}}(t)\|^2\,\mathrm{d}t\\ =\ &\int_{T_1}^{T_1+\tau_0}\|K(u(t),a(t),\theta(T_1))(\theta(T_1)-\theta^*)+e_f(t)\|^2\,\mathrm{d}t\\ \geq\ &\int_{T_1}^{T_1+\tau_0}\left(\frac{1}{2}\|K(u(t),a(t),\theta(T_1))(\theta(T_1)-\theta^*)\|^2-\|e_f(t)\|^2\right)\mathrm{d}t\\ \geq\ &\frac{\alpha_0\|\theta(T_1)-\theta^*\|^2}{2}-\varepsilon_f^2\tau_0,\end{aligned}$$

where the first inequality follows from the triangle inequality, and the second one from the PE condition (21) and the mismatch bound (26). Combining the inequalities above yields (23). □

VI. Simulation example

We illustrate the estimation and minimization algorithm via a simple example motivated by link-flooding denial-of-service (DoS) attacks such as the Crossfire attack [17].

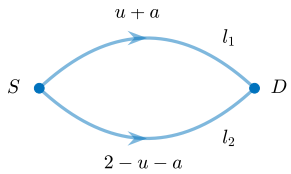


Fig. 1. A simple network with one source S, one destination D, and two links l_1 and l_2.

Consider the communication network in Fig. 1, and suppose there is a router (leader) that distributes 1 unit of legitimate traffic between the two links l_1 and l_2, and an attacker (follower) that disrupts communication by injecting 1 unit of malicious traffic on the two links. Denote by $u, a \in [0,1]$ the amounts of legitimate and malicious traffic on the link l_1, respectively. Then the total traffic on the links l_1 and l_2 are given by $f_1 = u + a$ and $f_2 = 2 - u - a$, respectively. We assume that a communication delay is incurred on each link, which is a quadratic function of the corresponding total traffic, that is, $p_1(f_1) := f_1^2$ and $p_2(f_2) := f_2^2$, and that the router aims to minimize the average delay for legitimate traffic, which corresponds to the cost function defined by

$$\begin{aligned} J(u,a) &:= up_1(f_1) + (1-u)p_2(f_2) \\ &= 5u^2 + 6ua + a^2 - 8u - 4a + 4, \end{aligned}$$

whereas the attacker aims to disrupt communication by maximizing the average delay for legitimate traffic, which corresponds to the cost function defined by

$$H(a,u) := -J(u,a).$$

Standard convex analysis shows that the attacker's best response against u is given by

$$\beta_a(u) := \begin{cases} 0 & u \le 1/2; \\ 1 & u > 1/2, \end{cases}$$

and the router's best response against a is given by

$$\beta_u(a) := (4 - 3a)/5.$$

Then it is straightforward to see that a Nash equilibrium does not exist for this game, but there is a Stackelberg equilibrium action $u^* = 1/2$ for the router.

If the router knew that the attacker's cost function was indeed H, it could use the Stackelberg equilibrium action $u^* = 1/2$. However, we consider a scenario where it does not and, instead, will use the approach proposed in this paper to construct its best action. To this effect, the router will assume the attacker's strategy (best-response function) to be of the form[3]

$$f(\theta, u) := \sum_{j=1}^{4} \theta_j F_j(u),$$

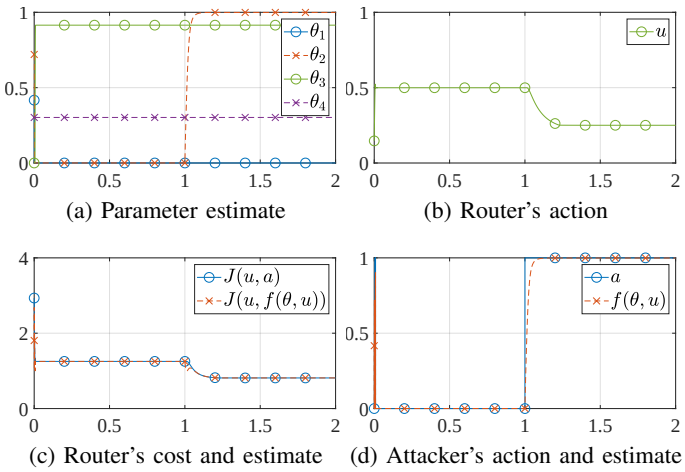
where the RBF $F_j : [0,1] \to \mathbb{R}$ are defined by

$$F_j(u) := \mathbb{1}_{(-1/8,1/8]}(\|u - (2j-1)/8\|), \qquad j = 1, \dots, 4,$$

and θ is the parameter estimate in the parameter set $\Theta := [0,1]^4$. For the specific cost function H, the attacker's best response is given by $f(\theta^*, u)$ with $\theta^* = (0,0,1,1)$.

In simulation shown in Fig. 2 and 3, the threshold ε_{obs} is set to 10^{-3}, and the initial values of the parameter estimate θ and the router's action u are randomly generated. For the case without enforcing PE in Fig. 2, in the first 10^4 iterations the router's action u converges to the optimum $u^* = 1/2$, despite that the parameter estimate θ does not converge to the actual parameter θ^*. In Fig. 3, we enforce PE by adding some random noise to u for a short interval when the observation error $\|e_{\text{obs}}\| < \varepsilon_{\text{obs}}$. In this case, in the first 10^4 iterations the router's action u converges to the optimum $u^* = 1/2$, and the parameter estimate θ converges to the actual parameter θ^*. In both cases, we also simulate the scenario where after 10^4 iterations, the attacker starts focusing more on disrupting the link l_1, so that the new value of the unknown parameter is $\theta^* = (0,1,1,1)$, and the new router's Stackelberg equilibrium action is $u^* = 1/4$. The corresponding simulation results show that our estimation and minimization algorithm is able to identify this switch in the attack, as the router's action converges to the new optimum $u^* = 1/4$ in both Fig. 2 and 3, and the parameter estimate θ converges to the new parameter θ^* in Fig. 3.

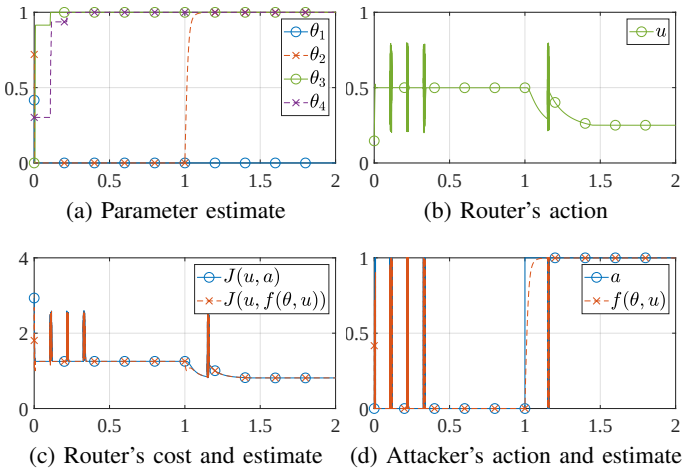
[3]The function f used here actually violates the regularity conditions in Assumption 1 as it is discontinuous in u. The continuity requirement of f in Assumption 1 is only needed so that the gradient descent is well-defined and does not lead to chattering. In simulation, these issues can be handled by using generalized subgradients at discontinuities [25] and setting $\dot{u} = 0$ when the right-hand side of (13) becomes very small.



(a) Parameter estimate (b) Router's action

(c) Router's cost and estimate (d) Attacker's action and estimate

Fig. 2. Simulation results w/o PE (horizontal axis unit: $\times 10^4$ iterations). In the first 10^4 iterations, the router's action u converges to the optimum $u^* = 1/2$; in the second 10^4 iterations, the attacker's cost function changes, and the router's action u converges to the new optimum $u^* = 1/4$; the parameter estimate θ does not converge to the actual parameter θ^* in either case.



(a) Parameter estimate (b) Router's action

(c) Router's cost and estimate (d) Attacker's action and estimate

Fig. 3. Simulation results w/ PE (horizontal axis unit: $\times 10^4$ iterations). In the first 10^4 iterations, the router's action u converges to the optimum $u^* = 1/2$; in the second 10^4 iterations, the attacker's cost function changes, and the router's action u converges to the new optimum $u^* = 1/4$; the parameter estimate θ converges to the actual parameter θ^* in both cases.

VII. Future Research Topics

An advantage of our estimation law (9) is that the norm of the estimation error $\|\theta - \theta^*\|$ will be monotonically nonincreasing and the observation error e_{obs} will be bounded by the preselected, arbitrarily small threshold ε_{obs} in finite time, regardless how the leader's strategy is adjusted. A future research topic is to integrate our estimation law with more efficient optimization methods for minimizing the leader's cost. Other future research topics include to relax the affine condition in Assumption 1, and to extend the current results to Stackelberg games on distributed networks.

Appendix I
Projected Dynamical Systems

Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact convex set, and $g : \mathcal{S} \to \mathbb{R}^n$ a locally Lipschitz function. In this section, we prove existence, boundedness and uniqueness of Carathéodory solutions for the *projected dynamical system*

$$\dot{x} = [g(x)]_{T_{\mathcal{S}}}. \tag{32}$$

The difficulty in analyzing (32) lies in that fact that its right-hand side is potentially discontinuous due to the tangent projection. Therefore, we extend (32) to a differential inclusion that satisfies a suitable notion of continuity. Consider the set-valued function $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$G(x) := \bigcap_{\varepsilon>0} \overline{\text{conv}}\{[g([y]_{\mathcal{S}})]_{T_{\mathcal{S}}} : y \in \mathcal{B}_\varepsilon(x)\}, \tag{33}$$

where

$$[y]_{\mathcal{S}} := \arg\min_{z\in\mathcal{S}} \|z - y\|$$

denotes the *projection* of a point $y \in \mathbb{R}^n$ onto $\mathcal{S}$. Existence, boundedness and uniqueness of Carathéodory solutions for (32) is established by examining the corresponding properties for the differential inclusion[4]

$$\dot{x} \in G(x). \tag{34}$$

Proposition 5. *For each $(t_0, x_0) \in \mathbb{R} \times \mathcal{S}$, there exists a unique Carathéodory solution to* (34) *on $[t_0, \infty)$ with $x(t_0) = x_0$, that is, there exists a unique absolutely continuous function $x : [t_0, \infty) \to \mathbb{R}^n$ with $x(t_0) = x_0$ such that* (34) *holds almost everywhere on $[t_0, \infty)$. Moreover, $x(t) \in \mathcal{S}$ for all $t \geq t_0$, and x is also the unique Carathéodory solution to* (32) *on $[t_0, \infty)$ with $x(t_0) = x_0$.*

Before proving Proposition 5, we present some useful properties of the functions g and G. As the set $\mathcal{S}$ is compact and the function g is locally Lipschitz, there exists a constant $\gamma \geq 0$ such that

$$\|g(x) - g(y)\| \leq \gamma\|x - y\| \qquad \forall\, x, y \in \mathcal{S}. \tag{35}$$

The following properties follow from the definition of tangent cone (2), the definition (33) of the function G, and the properties (3), (5) and (35).

Lemma 4. *The set-valued function G defined by* (33) *satisfies*

$$g(x) - v \in N_{\mathcal{S}}(x) \qquad \forall\, x \in \mathcal{S}, \forall\, v \in G(x) \tag{36}$$

and

$$g(x)^\top v \geq \|[g(x)]_{T_{\mathcal{S}}}\|^2 \qquad \forall\, x \in \mathcal{S}, \forall\, v \in G(x). \tag{37}$$

Proof. See Appendix II-C. □

Following (35) and (36), the set-valued function G satisfies the one-sided Lipschitz condition

$$(v - w)^\top (x - y) \leq \gamma\|x - y\|^2 \tag{38}$$

for all $x, y \in \mathcal{S}$, $v \in G(x)$ and $w \in G(y)$. Indeed,

$$\begin{aligned}(v - w)^\top (x - y) &= (v - g(x))^\top (x - y) \\ &\quad + (g(x) - g(y))^\top (x - y) + (g(y) - w)^\top (x - y),\end{aligned}$$

in which combining (36) and (2) implies

$$(v - g(x))^\top (x - y) \leq 0, \quad (g(y) - w)^\top (x - y) \leq 0,$$

and (35) implies

$$(g(x) - g(y))^\top (x - y) \leq \|g(x) - g(y)\|\|x - y\| \leq \gamma\|x - y\|^2.$$

Proof of Proposition 5. Consider an arbitrary $(t_0, x_0) \in \mathbb{R} \times \mathcal{S}$.

First, we establish existence of Carathéodory solutions for (34). The function G defined by (33) is upper semicontinuous on $\mathbb{R}^n$, and $G(x)$ is convex and compact for every $x \in \mathbb{R}^n$. Also, (35), together with the triangle inequality, implies

$$\|g(x)\| \leq \alpha(1 + \|x\|) \qquad \forall\, x \in \mathcal{S}$$

[4]Carathéodory solutions to (34) are called Krasovskii solutions to (32).

with the constant $\alpha := \max_{y\in\mathcal{S}}\{\|g(y)\| + \gamma\|y\|, \gamma\}$. Consider an arbitrary $T > t_0$. Then a standard result on existence of Carathéodory solutions for differential inclusions (see, e.g., [21, Appendix]) implies that there exists at least one absolutely continuous function $x : [t_0, T] \to \mathbb{R}^n$ with $x(t_0) = x_0$ such that (34) holds almost everywhere on $[t_0, T]$.

Next, we establish boundedness and uniqueness of Carathéodory solutions for (34). Consider an arbitrary Carathéodory solution x to (34) on $[t_0, T]$ with $x(t_0) = x_0$. Then [21, Lemma 1] implies $x(t) \in \mathcal{S}$ for all $t \in [t_0, T]$. As G satisfies the one-sided Lipschitz condition (38), [22, Cor. 2.4] implies that x is the unique Carathéodory solution to (34) on $[t_0, T]$ with $x(t_0) = x_0$.

Finally, [21, Lemma 2] implies that the solution x satisfies (32) almost everywhere on $[t_0, T]$. Therefore, x is also the unique Carathéodory solution to (32) on $[t_0, T]$ with $x(t_0) = x_0$. The proof is completed by noticing that $T > t_0$ is arbitrary. □

APPENDIX II
PROOF OF TECHNICAL LEMMAS

A. Proof of Lemma 1

As the map $\theta \mapsto f(\theta, u)$ is affine, its Jacobian matrix $\nabla_\theta f(u)$ is independent of θ. Thus for a fixed $u \in \mathcal{U}$, it holds that

$$f(\theta, u) - a = f(\theta, u) - f(\theta^*, u) = \nabla_\theta f(u)(\theta - \theta^*).$$

Next, consider the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(\lambda) := J(u, \lambda f(\theta, u) + (1 - \lambda)\, a),$$

which satisfies $g \in \mathcal{C}^1$ as $J \in \mathcal{C}^1$. Then

$$\hat{J}(u, \theta) - J(u, a) = g(1) - g(0) = \int_0^1 \frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}\,\mathrm{d}\lambda,$$

in which

$$\begin{aligned}\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda} &= \nabla_a J(u, \lambda f(\theta, u) + (1 - \lambda)\, a)(f(\theta, u) - a)\\ &= \nabla_a J(u, \lambda f(\theta, u) + (1 - \lambda)\, a)\nabla_\theta f(u)(\theta - \theta^*).\end{aligned}$$

Hence

$$\begin{aligned}&\hat{J}(u, \theta) - J(u, a)\\ &= \int_0^1 \nabla_a J(u, \lambda f(\theta, u) + (1 - \lambda)\, a)\nabla_\theta f(u)(\theta - \theta^*)\,\mathrm{d}\lambda\\ &= \left(\int_0^1 \nabla_a J(u, \lambda f(\theta, u) + (1 - \lambda)\, a)\,\mathrm{d}\lambda\right)\nabla_\theta f(u)(\hat{\theta} - \theta).\end{aligned}$$

B. Proof of Lemma 3

Consider an arbitrary $(\theta_0, u_0) \in \Theta \times \mathcal{U}$, and let $\lambda_e(0)$ be the corresponding value given by (6) and (10). Suppose $\lambda_e(0) = \lambda_1$, that is, $\|e_{\mathrm{obs}}(0)\| > \varepsilon_{\mathrm{obs}}/2$. In the following, we construct the unique Carathéodory solution to (9) and (13) on $\mathbb{R}_+$ with $(\theta(0), u(0)) = (\theta_0, u_0)$ recursively. The case where $\lambda_e(0) = 0$ can be proved based on the same arguments starting with the second step.

First, consider the system defined by (9) and (13) with $\lambda_e \equiv \lambda_1$, that is,

$$\begin{aligned}\dot{\theta} &= [-\lambda_1 K(u, f(\theta^*, u), \theta)^\top K(u, f(\theta^*, u), \theta)(\theta - \theta^*)]_{T_\Theta},\\ \dot{u} &= \left[-\lambda_2 \nabla_u \hat{J}(u, \theta)^\top\right]_{T_\mathcal{U}},\end{aligned} \tag{39}$$

which can be modeled using the projected dynamical system (32) in Appendix I with the state $x := (\theta, u)$ and the set $\mathcal{S} := \Theta \times \mathcal{U}$. Then Proposition 5 in Appendix I implies that there exists a unique Carathéodory solution (θ_1, u_1)

of (39) on $\mathbb{R}_+$ with $(\theta_1(0), u_1(0)) = (\theta_0, u_0)$, and $(\theta_1(t), u_1(t)) \in \Theta \times \mathcal{U}$ for all $t \geq 0$. Consider the corresponding observation error $e_{\mathrm{obs},1}$ and switching signal $\lambda_{e,1}$ given by (6) and (10), and let

$$t_1 := \inf\{t > 0 : \|e_{\mathrm{obs},1}(t)\| \leq \varepsilon_{\mathrm{obs}}/2\}.$$

Then (θ_1, u_1) is the unique Carathéodory solution to (9) and (13) on $[0, t_1)$ with $(\theta_1(0), u_1(0)) = (\theta_0, u_0)$. If $t_1 = \infty$ then the proof is complete. Otherwise, $e_{\mathrm{obs},1}(t_1) = \varepsilon_{\mathrm{obs}}/2$ and thus $\lambda_{e,1}(t_1) = 0$, and we continue with the second step below.

Second, consider the system defined by (9) and (13) with $\lambda_e \equiv 0$, that is,

$$\begin{aligned} \dot{\theta} &= 0, \\ \dot{u} &= \left[-\lambda_2 \nabla_u \hat{J}(u, \theta)^\top\right]_{T_{\mathcal{U}}}, \end{aligned} \tag{40}$$

which can also be modeled using the projected dynamical system (32) in Appendix I with the state $x := (\theta, u)$ and the set $\mathcal{S} := \Theta \times \mathcal{U}$. Then Proposition 5 in Appendix I implies that there exists a unique Carathéodory solution (θ_2, u_2) of (40) on $[t_1, \infty)$ with $(\theta_2(t_1), u_2(t_1)) = (\theta_1(t_1), u_1(t_1))$, and $(\theta_2(t), u_2(t)) \in \Theta \times \mathcal{U}$ for all $t \geq t_1$. Consider the corresponding observation error $e_{\mathrm{obs},2}$ and switching signal $\lambda_{e,2}$ given by (6) and (10), and let

$$t_2 := \inf\{t \geq t_1 : \|e_{\mathrm{obs},2}(t)\| \geq \varepsilon_{\mathrm{obs}}\}.$$

Then (θ_2, u_2) is the unique Carathéodory solution to (9) and (13) on $[t_1, t_2)$ with $(\theta_2(t_1), u_2(t_1)) = (\theta_1(t_1), u_1(t_1))$. If $t_2 = \infty$ then the proof is complete. Otherwise, $e_{\mathrm{obs},2}(t_2) = \varepsilon_{\mathrm{obs}}$ and thus $\lambda_{e,2}(t_2) = \lambda_1$, and we continue with the first step above.

In this way, we obtain an increasing sequence $(t_k)_{k\in\mathbb{N}}$ with $t_0 = 0$ and a corresponding sequence $(\theta_k, u_k)_{k\geq 1}$ of absolutely continuous functions $\theta_k : [t_{k-1}, \infty) \to \Theta$ and $u_k : [t_{k-1}, \infty) \to \mathcal{U}$. Moreover, following (6), (39), (40) and Assumption 1, as $\theta_k(t) \in \Theta$ and $u_k(t) \in \mathcal{U}$ for all $k \geq 1$ and $t \geq t_{k-1}$, there exists a constant $M \geq 0$ such that $\|\dot{e}_{\mathrm{obs},k}(t)\| \leq M$ for all $k \geq 1$ and $t \geq t_{k-1}$. Hence

$$t_k - t_{k-1} \geq \varepsilon_{\mathrm{obs}}/(2M) \qquad \forall\, k \geq 2,$$

and thus $\lim_{k\to\infty} t_k = \infty$. Therefore, (θ, u) with the absolutely continuous functions $\theta : \mathbb{R}_+ \to \Theta$ and $u : \mathbb{R}_+ \to \mathcal{U}$ defined by

$$\theta(t) := \theta_k(t), \quad u(t) := u_k(t), \qquad k \geq 1, t \in [t_{k-1}, t_k)$$

is the unique Carathéodory solution to the system defined by (9) and (13) on $\mathbb{R}_+$ with $(\theta(0), u(0)) = (\theta_0, u_0)$. The proof is completed by noticing that (17) follows from (6), (39), (40), Assumption 1, and the property that $(\theta(t), u(t)) \in \Theta \times \mathcal{U}$ for all $t \geq 0$.

C. Proof of Lemma 4

Consider arbitrary $x \in \mathcal{S}$ and $v \in G(t, x)$.

First, we establish (36). Consider an arbitrary $\varepsilon_1 > 0$, and let

$$\delta_1 := \min\left\{\frac{\varepsilon_1}{2\gamma \max_{y\in\mathcal{S}} \|y - x\|}, \frac{\varepsilon_1}{2 \max_{y\in\mathcal{S}} \|g(y)\|}\right\} > 0$$

with the Lipschitz constant γ in (35). Then for all $y \in \mathcal{S}$ and $z \in \mathcal{B}_{\delta_1}(x)$, it holds that

$$\begin{aligned} &\big(g(x) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top (y - x) \\ &= \big(g(x) - g([z]_{\mathcal{S}})\big)^\top (y - x) \\ &\quad + \big(g([z]_{\mathcal{S}}) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top (y - [z]_{\mathcal{S}}) \\ &\quad + \big(g([z]_{\mathcal{S}}) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top ([z]_{\mathcal{S}} - x), \end{aligned}$$

in which the first product satisfies

$$\begin{aligned} \big(g(x) - g([z]_{\mathcal{S}})\big)^\top (y - x) &\leq \|g(x) - g([z]_{\mathcal{S}})\| \|y - x\| \\ &\leq \gamma \|x - [z]_{\mathcal{S}}\| \|y - x\| \leq \gamma \delta_1 \|y - x\| \leq \varepsilon_1/2, \end{aligned}$$

where the second inequality follows from (35), the third one from $z \in \mathcal{B}_{\delta_1}(x)$, and the last one from the definition of δ_1; the second product satisfies

$$\big(g([z]_{\mathcal{S}}) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top (y - [z]_{\mathcal{S}}) \le 0$$

due to (4) and (2); the last product satisfies

$$\begin{aligned}
&\big(g([z]_{\mathcal{S}}) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top ([z]_{\mathcal{S}} - x) \\
&\le \|g([z]_{\mathcal{S}}) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\| \|[z]_{\mathcal{S}} - x\| \\
&\le \|g([z]_{\mathcal{S}})\| \delta_1 \le \varepsilon_1/2,
\end{aligned}$$

where the second inequality follows from (5) and $z \in \mathcal{B}_{\delta_1}(x)$, and the last one from the definition of δ_1. Therefore,

$$\big(g(x) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top (y - x) \le \varepsilon_1 \quad \forall\, y \in \mathcal{S}, \forall\, z \in \mathcal{B}_{\delta_1}(x).$$

Consequently, the definition (33) of G implies

$$(g(x) - v)^\top (y - x) \le \max_{z \in \mathcal{B}_{\delta_1}(x)} \big(g(x) - [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\big)^\top (y - x) \le \varepsilon_1$$

for all $y \in \mathcal{S}$. As $\varepsilon_1 > 0$ is arbitrary, it holds that

$$(g(x) - v)^\top (y - x) \le 0 \qquad \forall\, y \in \mathcal{S}.$$

Hence (36) follows from the definition of normal cone (2).

Second, we establish (37). Consider an arbitrary $\varepsilon_2 > 0$, and let

$$\delta_2 := \frac{\varepsilon_2}{\gamma \max_{y \in \mathcal{S}} \|g(y)\|} > 0$$

with the Lipschitz constant γ in (35). Then for all $z \in \mathcal{B}_{\delta_2}(x)$, it holds that

$$\begin{aligned}
&\big(2[g(x)]_{T_{\mathcal{S}}} - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \\
&= \|[g(x)]_{T_{\mathcal{S}}}\|^2 - \|[g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} - [g(x)]_{T_{\mathcal{S}}}\|^2 \\
&\quad + \big([g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \\
&\le \|[g(x)]_{T_{\mathcal{S}}}\|^2 + \big([g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \\
&= \|[g(x)]_{T_{\mathcal{S}}}\|^2 + \big([g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} - g([z]_{\mathcal{S}})\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \\
&\quad + \big(g([z]_{\mathcal{S}}) - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}},
\end{aligned}$$

in which the second product satisfies

$$\big([g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} - g([z]_{\mathcal{S}})\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} = 0$$

due to (5); the last product satisfies

$$\begin{aligned}
&\big(g([z]_{\mathcal{S}}) - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \\
&\le \big(\|g([z]_{\mathcal{S}}) - g(x)\| \|[g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}}\| \\
&\le \gamma \|[z]_{\mathcal{S}} - x\| \|g([z]_{\mathcal{S}})\| \le \gamma \delta_2 \|g([z]_{\mathcal{S}})\| \le \varepsilon_2,
\end{aligned}$$

where the second inequality follows from (35), the third one from $z \in \mathcal{B}_{\delta_2}(x)$, and the last one from the definition of δ_2. Therefore,

$$\big(2[g(x)]_{T_{\mathcal{S}}} - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \le \|[g(x)]_{T_{\mathcal{S}}}\|^2 + \varepsilon_2$$

for all $z \in \mathcal{B}_{\varepsilon_2}(x)$. Consequently, the definition (33) of G implies

$$\big(2[g(x)]_{T_{\mathcal{S}}} - g(x)\big)^\top v \le \max_{z \in \mathcal{B}_{\delta_2}(x)} \big(2[g(x)]_{T_{\mathcal{S}}} - g(x)\big)^\top [g([z]_{\mathcal{S}})]_{T_{\mathcal{S}}} \le \|[g(x)]_{T_{\mathcal{S}}}\|^2 + \varepsilon_2.$$

As $\varepsilon_2 > 0$ is arbitrary, it holds that

$$\big(2[g(x)]_{T_{\mathcal{S}}} - g(x)\big)^\top v \le \|[g(x)]_{T_{\mathcal{S}}}\|^2,$$

that is,

$$g(x)^\top v \geq 2[g(x)]_{T_S}^\top v - \|[g(x)]_{T_S}\|^2.$$

Combining (36) with (3) and (4) implies $(g(x) - v)^\top [g(x)]_{T_S} \leq 0$, that is,

$$[g(x)]_{T_S}^\top v \geq g(x)^\top [g(x)]_{T_S} = \|[g(x)]_{T_S}\|^2,$$

where the equality follows from (5). Therefore, (37) holds.

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