Newton and interior-point methods for (constrained) nonconvex-nonconcave minmax optimization with stability guarantees

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Abstract

We address the problem of finding a local solution to a nonconvex-nonconcave minmax optimization using Newton type methods, including interior-point ones. We modify the Hessian matrix of these methods such that, at each step, the modified Newton update direction can be seen as the solution to a quadratic program that locally approximates the minmax problem. Moreover, we show that by selecting the modification in an appropriate way, the only stable points of the algorithm's iterations are local minmax points. Using numerical examples, we show that the computation time of our algorithm scales roughly linearly with the number of nonzero elements in the Hessian. For minmax control problems with per-stage costs, this generally leads to computation times that scale linearly with the horizon length.

Keywords: minmax optimization, robust optimization, Newton method, interior-point method, local minmax

1 Introduction

In minmax optimization, one minimizes a cost function which is itself obtained from the maximization of a scalar function. Minmax optimization is a powerful modeling framework, generally used to guarantee robustness to an adversarial parameter such as accounting for disturbances in model predictive control [1, 2], security related problems [3, 4], or training neural networks to be robust to adversarial attacks [5]. It can also be used as a framework to model more general problem such as sampling from unknown distributions using generative adversarial networks [6], reformulating stochastic programming as minmax optimization [7–9], or producing robustness of a stochastic program with respect to the probability distribution [10]. Minmax optimization is also known as minimax or robust optimization.

Finding a global minmax point for nonconvex-nonconcave problems is generally difficult, and one has to settle for finding a local minmax point. Surprisingly, only recently a first definition of unconstrained local minmax was proposed in [11], and the definition of constrained local minmax in [12].

It is widely accepted that the application of Newton-like methods to the minimization of nonconvex functions requires the modification of the Hessian matrix through the addition of a matrix, typically a multiple of the identity. An initial contribution of our paper is the observation that by selecting this additive term so that a quadratic local approximation to the cost function becomes convex has two important consequences. First it guarantees progress towards a solution, in the sense that the function decreases with each Newton step – this result is well known [13, Chapter 3.4]. In addition, we show that the same additive term also guarantees that the set of locally asymptotically stable equilibrium points of the Newton iteration is precisely the set of strict local minimum of the optimization. This guarantees that convergence to a locally asymptotically stable equilibrium point necessarily implies convergence to a local minima. We also show that, in the case of constrained minimization, it is possible to analyze interior-point methods as a quadratic approximation which can also be appropriately modified. These results (presented in Section 2) directly motivate the design of novel Newton-type algorithms for minmax optimizations

The Newton-type algorithms proposed in this paper are motivated by a quadratic local approximation to the optimization criteria to which we add terms to make it have a finite minmax solution (without necessarily becoming convex-concave). Any additive terms that guarantee this are said to satisfy the Local Quadratic Approximation Condition (LQAC). We show that contrary to minimization, such modification does not lead to Newton-type iterations with desired stability: a local minmax can be unstable and an equilibrium point that is not a local minmax can be stable. Our first minmax result shows that additional conditions are needed to guarantee that every locally asymptotically stable equilibrium point of a Newton-type iteration is a local minmax. This additional condition is expressed in terms of the *inertia* of the modified Hessian matrix, i.e., in terms of the number of positive, negative, and zero eigenvalues.

To simplify the presentation, we first present this result in Section 3.1 for unconstrained minmax and then extend it in Section 3.2 to interior-point methods for constrained minmax.

The conditions described above to establish the equivalence between local minmax and local asymptotic stability of the equilibria to a Newton-type iteration are directly used to construct a numerical algorithm to find local minmax. By construction, when this algorithm converges to a locally asymptotically stable equilibrium point, its is guaranteed to obtain a local minmax. It is important to clarify that this result fall shy of guaranteeing *global* asymptotic convergence to a local minmax, as the algorithm could simply never converge.

Using numerical examples, we show that by using an appropriate implementation of the LDL decomposition, the numerical complexity increases roughly with the number of nonzero entries of the Hessian. This is important for problems with stage costs and constraints, such as robust Model Predictive Control, where the number of nonzero entries of the Hessian tend to increase linearly with the number of stages. The numerical results are implemented on MATLAB, using `TensCalc` as a backend for the symbolic computation [14]. A general solver for minmax optimization based on the results of this paper is being implemented to `TensCalc` to complement its other solvers.

Notation:

The set of real numbers is denoted by $\mathbb{R}$. Given a vector $v \in \mathbb{R}^n$, its transpose is denoted by v'. The operation $\text{diag}(v)$ creates a matrix with diagonal elements v and off-diagonal elements 0. The matrix I is the identity, $\mathbf{1}$ is the matrix of ones and $\mathbf{0}$ the matrix of zeros; their sizes will be provided as subscripts whenever it is not clear from context. If a matrix M only has real eigenvalues, we denote by $\lambda_{min}(M)$ and $\lambda_{max}(M)$ its smallest and largest eigenvalues.

Consider a differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^p$. The Jacobian (or gradient if $p = 1$) at a point $(\bar{x}, \bar{y})$ according to the x variable is a matrix of size $n \times p$ and is denoted by $\mathbf{\nabla}_x f(\bar{x}, \bar{y})$, and analogously for the variable y. When $p = 1$ and $f(\cdot)$ is twice differentiable, we use the notation $\mathbf{\nabla}_{yx} f(\bar{x}, \bar{y}) := \mathbf{\nabla}_y (\mathbf{\nabla}_x f)(\bar{x}, \bar{y})$ which has sizes $m \times n$. We use analogous definition for $\mathbf{\nabla}_{xy} f(\bar{x}, \bar{y})$, $\mathbf{\nabla}_{xx} f(\bar{x}, \bar{y})$ and $\mathbf{\nabla}_{yy} f(\bar{x}, \bar{y})$.

1.1 Literature Review

Traditionally, robust optimization focused on the convex-concave case, with two main methods. The first, robust reformulation, uses results from convex analysis to reformulate the minmax optimization as a counterpart minimization problem which has the same solution as the original problem [15–17]. The second, cutting-set methods, solves a sequence of minimization where the constraint of each minimization is based on subdividing the inner maximization [18]. The robust reformulation is problem specific, while the cutting-set approach requires solving many exact maximization which might not be feasible in large scale.

Motivated by some of the shortcomings of these methods and the necessities of machine learning, research on minmax optimization started to shift towards the study of methods based on variations of gradient descent-ascent. The results tend to focus on providing convergence complexity given different convexity/concavity assumptions on the target function. In multistep gradient descent ascent, also know as unrolled or GDmax, the minimizer is updated by a single gradient descent whereas the maximizer is updated by several gradient ascent steps that aim to approximately find the maximum [11, 19, 20]. In single step, the minimizer and maximizer are updated at each iteration, generally combined with some other features such as different step sizes, momentum or extra gradient [21–25]

In recent years, researchers have also started to work on algorithms that use second order derivatives to determine the directions. These algorithm, in their major part have not attracted as much attention as first order methods. In the Learning with Opponent Learning Awareness (LOLA), the minimizer anticipates the play of the maximizer using the Jacobian of the maximizer's gradient [26, 27]. In competitive gradient descent, both minimizer and maximizer use the cross derivative of the Hessian to compute their direction [28]. In follow the ridge, the gradient ascent step is corrected by a term that avoids a drift away from local maxima [29]. In the total gradient descent-ascent, similarly to LOLA, the descent direction is computed by taking to total derivative of a function which anticipates the maximizer's response to the minimizer [30]. Finally, the complete Newton borrows ideas from follow the ridge and total gradient to obtain a Newton method which prioritizes steps towards local minmax [31]. These three last algorithms are shown to only converge towards local minmax under some conditions, but in none of them it is addressed the issue of how to adjust the Hessian far away from a local minmax point.

Recently, some second order methods have been proposed for the nonconvex-strongly-concave case, where the Hessian is modified such that it is invertible and that the minimizer update is a descent direction of the objective function at its maximum. They either use cubic regularization [32, 33] or randomly perturb the Hessian [34]. Because of some of the assumptions these work make, most important the strong-concavity of the objective function with respect to the maximizer, they are able to establish complexity analysis and guarantee. It is also worth mention that these algorithms are all multistep based, meaning they (approximately) solve the maximization between each update of the minimizer, whereas our algorithm updates both the minimizer and the maximizer simultaneously.

2 Minimization

Let $f : \mathcal{X} \to \mathbb{R}$ a twice continuously differentiable cost function defined in a set $\mathcal{X} \subset \mathbb{R}^n$, and consider the minimization problem

$$\min_{x \in \mathcal{X}} f(x). \tag{1}$$

We recall that a point x^* is called a *strict local minimum of* $f(\cdot)$ if there exist $\delta > 0$ such that $\|x - x^*\| < \delta$ implies that $f(x^*) < f(x)$ for every $x \in \mathcal{X}$. If $f(\cdot)$ is twice continuously differentiable in a neighborhood of a point x which belongs to the interior of $\mathcal{X}$, $\boldsymbol{\nabla}_x f(x) = \mathbf{0}$ and $\boldsymbol{\nabla}_{xx} f(x) \succ 0$, then x is a strict local minimum of $f(\cdot)$ [13, Chapter 2].

Closely related to local minima, is the concept of descent direction. A vector $d_x \in \mathbb{R}^n$ is a *descent* direction of $f(\cdot)$ at a point x if $\exists \alpha_0 > 0$ such that $\forall \alpha \in (0, \alpha_0]$, $f(x + \alpha d_x) < f(x)$. It is straightforward to establish that there are no descent direction at a local minimum. For continuously differentiable functions, one can further show that a given vector $d_x \in \mathbb{R}^n$ is a descent direction if an only if $d_x' \boldsymbol{\nabla}_x f(x) < 0$ [13].

In a widely used modification of Newton's method for unconstrained (*i.e.*, $\mathcal{X} = \mathbb{R}^n$) nonconvex optimization, a descent direction d_x is obtained by solving the following local quadratic approximation to (1)

$$d_x = \arg\min_{\bar{d}_x} f(x) + \boldsymbol{\nabla}_x f(x)' \bar{d}_x + \frac{1}{2} \bar{d}_x (\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I) \bar{d}_x$$
$$= -(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1} \boldsymbol{\nabla}_x f(x) \qquad (2)$$

with $\epsilon_x(x) \geq 0$ chosen such that $(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)$ is positive definite. For twice differentiable strictly convex functions we can choose $\epsilon_x(x) = 0$ and this corresponds to the classical Newton's methods. However, when $f(\cdot)$ is not convex, the minimization in (2) is only well defined if $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I$ is positive definite and which requires selecting a strictly positive value for ϵ_x, leading to a perturbed Newton's method. Regardless of whether of not $f(\cdot)$ is convex, the positive definiteness of $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I$ guarantees that $d_x' \boldsymbol{\nabla}_x f(x) = -\boldsymbol{\nabla}_x f(x)(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I) \boldsymbol{\nabla}_x f(x) < 0$ and therefore d_x is a descent direction at x. The corresponding Newton iteration to obtain a local minimum is then given by

$$x^+ = x + d_x = x - (\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1} \boldsymbol{\nabla}_x f(x) \qquad (3)$$

where we use the notation x^+ to designate the value of x at the next iteration.

The following result establishes that the positive definiteness of $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I$ not only guarantees that d_x is a descent direction, but also that every locally asymptotically stable (LAS) equilibrium point of the Newton iteration (3) is a strict local minimum. This result has two important consequence: First, that if the Newton iteration starts sufficiently close to a strict local minimum, it will converge asymptotically fast to it (in fact exponentially fast). Second, regardless of where the iteration starts, if it converges to some asymptotically stable equilibrium point, then we can be sure that it converged to a strict local minimum.

Theorem 1 (Stability of modified Newton method for minimization) *Let x be an equilibrium point and assume that $\epsilon_x(\cdot)$ is differentiable, that $f(\cdot)$ is three times differentiable on a neighborhood around x and that $\nabla_{xx}f(x)$ and $(\nabla_{xx}f(x)+\epsilon_x(x)I)$ are invertible. Then x is a LAS equilibrium point of the Newton iteration* (3) *if and only if x is a strict local minimum.*

Proof The Jacobian of the expression $(\nabla_{xx}f(x) + \epsilon_x(x))^{-1}\nabla_x f(x)$ that appears in (3) at an equilibrium point x is given by

$$\nabla_x\left((\nabla_{xx}f(x) + \epsilon_x(x))^{-1}\nabla_x f(x)\right) = (\nabla_{xx}f(x) + \epsilon_x(x))^{-1}\nabla_{xx}f(x) +$$
$$\sum_{i=1}^{N}\nabla_x[(\nabla_{xx}f(x) + \epsilon_x(x))^{-1}]_i\nabla_x f(x)^{(i)}$$

where $\nabla_x f(x)^{(i)}$ is the ith element of $\nabla_x f(x)$ and $[(\nabla_{xx}f(x) + \epsilon_x(x))^{-1}]_i$ is the ith column of $(\nabla_{xx}f(x) + \epsilon_x(x))^{-1}$. Since $(\nabla_{xx}f(x) + \epsilon_x(x)I)$ is invertible, $\nabla_x[(\nabla_{xx}f(x) + \epsilon_x(x))^{-1}]_i$ is well defined and since x is an equilibrium point, $\nabla_x f(x)^{(i)} = 0$ for $i \in \{1 \dots N\}$ and therefore the Jacobian of right-hand side of (3) is given by

$$\nabla_x\left(x - (\nabla_{xx}f(x) + \epsilon_x(x))^{-1}\nabla_x f(x)\right) = I - (\nabla_{xx}f(x) + \epsilon_x(x))^{-1}\nabla_{xx}f(x). \quad (4)$$

The main argument of the proof is based on the following result. Let v be an eigenvector associated to an eigenvalue ρ of (4). Then

$$\left(I - (\nabla_{xx}f(x) + \epsilon_x(x)I)^{-1}\nabla_{xx}f(x)\right)v = \rho v$$
$$\Leftrightarrow (1 - \rho)v = (\nabla_{xx}f(x) + \epsilon_x(x)I)^{-1}\nabla_{xx}f(x)v$$
$$\Leftrightarrow \left(\rho\nabla_{xx}f(x) + (\rho - 1)\epsilon_x(x)I\right)v = \mathbf{0} \quad (5)$$

Therefore, ρ is an eigenvalue of (4) if and only if $\rho\nabla_{xx}f(x)+(\rho-1)\epsilon_x(x)I$ is singular.

We remind the reader that given a dynamical system, if the system's dynamic equation is continuously differentiable, a point is a LAS equilibrium point if all the eigenvalues of the linearized system are inside the unit circle. Conversely, if at least one of the eigenvalues of the linearized system is outside the unit circle, then the system is unstable [35, Chapter 8].

From (5), the case $\rho = 0$ happens if and only if $\epsilon_x(x) = 0$, which, by construction, can only happen if x is a local minmax, in which case x is a LAS equilibrium point of (3), as expected.

For $\rho \neq 0$, let us rewrite this expression as $\nabla_{xx}f(x) + \mu\epsilon_x(x)I$ with $\mu := 1 - 1/\rho$. We conclude that x is a LAS equilibrium point of (3) if $\nabla_{xx}f(x) + \mu\epsilon_x(x)$ is nonsingular $\forall \mu \in [0, 2]$. Conversely, x is an unstable equilibrium point of (3) if $\nabla_{xx}f(x) + \mu\epsilon_x(x)$ is singular for some $\mu \in [0, 2]$.

If x is a local minimum, then $\lambda_{min}(\nabla_{xx}f(x)) > 0$. As $\epsilon_x(x) > 0$, we conclude that $\lambda_{min}(\nabla_{xx}f(x) + \mu\epsilon_x(x)) > 0$ for every $\mu \geq 0$ and therefore x is a LAS equilibrium point of (3). Conversely, if x is not a local minimum then $\lambda_{min}(\nabla_{xx}f(x)) < 0$. By construction of $\epsilon_x(x)$, we have that $\lambda_{min}(\nabla_{xx}f(x) + \mu\epsilon_x(x)) > 0$, which, by continuity of the eigenvalue, implies $\exists \mu \in (0, 1)$ such that $\lambda_{min}(\nabla_{xx}f(x)+\mu\epsilon_x(x)) = 0$. Therefore x is an unstable equilibrium point of (3). $\square$

2.1 Constrained minimization and interior-point method

In order to make the presentation of the results of the rest of the paper more clear, it is useful to consider the case with more general constraint with the minimization set $\mathcal{X}$ involving equality and inequality constraints of the form

$$\mathcal{X} = \{x \in \mathbb{R}^n : G(x) = \mathbf{0}, F(x) \leq \mathbf{0}\}$$

where the functions $G : \mathbb{R}^n \rightarrow \mathbb{R}^l$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are all twice continuously differentiable. It will be convenient for the development of interior-point methods to use slack variables and rewrite (1) as

$$\min_{x,s:G(x)=\mathbf{0},F(x)+s=\mathbf{0},s\geq 0} f(x). \tag{6}$$

The Lagrangian of (6) is $L(z) = f(x) + \nu'G(x) + \lambda'(F(x) + s)$, where we use the shorthand notation $z := (x, s, \nu, \lambda)$. The Karush–Kuhn–Tucker (KKT) conditions [13, Chapter 12] for the optimization (6) are $g(z, 0) = \mathbf{0}$ and $\lambda, s \geq \mathbf{0}$ where

$$g(z, b) := \begin{bmatrix} \boldsymbol{\nabla}_x L(z) \\ \lambda \odot s - b\mathbf{1} \\ G_x(x) \\ F_x(x) + s_x \end{bmatrix} \tag{7}$$

with $\odot$ denoting the element wise Hadamard product of two vectors and $b \geq 0$ the barrier parameter (its role will be explained shortly).

Let $d_z := (d_x, d_s, d_\nu, d_\lambda)$ be the update direction for z, which will play an equivalent role to d_x in the unconstrained case. A basic interior-point method finds a candidate solution to (6) using the iterations

$$z^+ = z + \alpha d_z = z - \alpha \boldsymbol{\nabla}_z g(z, b)^{-1} g(z, b) \tag{8}$$

where the barrier parameter b is slowly decreased to 0, so that z converges to a root of $g(z, 0) = \mathbf{0}$ while $\alpha \in (0, 1]$ is chosen at each step such that the feasibility condition $\lambda, s > \mathbf{0}$ hold [13, Chapter 19]. This basic interior-point has similar limitation as a (non-modified) Newton method for unconstrained minimization: it might not converge towards a local minimum and $\boldsymbol{\nabla}_z g(z, b)$ might not be invertible. Similar to what we have done in the unconstrained case, we can modify this basic interior-point method such that the update direction d_z is obtained from a quadratic program that locally approximates (6).

Let us start with $\mathcal{X}$ described only by equality constraints (*i.e.*, no $F(x)$ and no s), in which case $L(z) = f(x) + \nu'G(x)$. If $\boldsymbol{\nabla}_x G(x)$ is full column rank, the update directions d_x can be obtained as the solution and d_ν as the associated Lagrange multiplier to the minimization

$$\min_{d_x:G(x)+\boldsymbol{\nabla}_x G(x)'d_x} L(z) + d_x' \boldsymbol{\nabla}_x L(z) + \frac{1}{2} d_x'(\boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z))d_x \tag{9}$$

where $\epsilon_x(x)$ is chosen such that the solution is unique. Notice that we use the second order linearization of the Lagrangian $L(z)$ as the cost function in (9), not the one of $f(x)$. The justification is that, if x^* is a local minimum of (6) with associated Lagrange multiplier ν^*, then x^* is also a local minimum of

$$\min_{x:G(x)=\mathbf{0}} f(x) + \nu^{*\,\prime}G(x).$$

Evidently, ν^* is not know in advance, so instead one uses the value of ν at the current iteration, which leads to the local approximation (9).

In the case where there are inequality constraints, using only $L(z)$ does not provide the desired behavior as it would not account for the constraint $s \geq 0$. To address this, a first step is to include the barrier function $-b\mathbf{1}'\log(s)$ (the $\log(\cdot)$ is element wise) which only accepts $s \geq 0$ and goes to $+\infty$ if $s \to 0$. The second order linearization of the barrier is

$$- b\mathbf{1}'\log(s) - d_s'b\mathbf{1} \oslash s + \frac{1}{2}d_s'b\operatorname{diag}(s)^{-2}d_s \tag{10}$$

where $\oslash$ designates the element wise division on $\mathbf{1}$ by s. We are going to modify this quadratic approximation, by using as as second order term $\operatorname{diag}(\lambda \oslash s)$ instead of $b\operatorname{diag}(s)^{-2}$. The justification is that, if we were at a point such that $g(z,b) = \mathbf{0}$, the two would be equivalent as $\lambda \odot s - b\mathbf{1} = \mathbf{0}$. This modified linearization tends to perform better because it provides directions d_s that also take into account the current value of λ in the quadratic form, which helps to get a direction d_z that does no violate the constraints $\lambda, s > 0$ [13, Chapter 19.3]. Putting this all together, we can modify the basic interior-point method such that (d_x, d_s) is obtained from the solution and (d_ν, d_λ) the associated Lagrange multiplier of the quadratic program

$$\min_{\substack{d_x,d_s: \\ G(x)+\boldsymbol{\nabla}_xG(x)'d_x, \\ F(x)+s+\boldsymbol{\nabla}_xF(x)'d_x+d_s}} L(z) - b\mathbf{1}'\log(s) + d_x'\boldsymbol{\nabla}_xL(z) + d_s'(\lambda - b\mathbf{1} \oslash s)$$

$$+ \frac{1}{2}d_x'(\boldsymbol{\nabla}_{xx}L(z) + \epsilon_x(z))d_x + \frac{1}{2}d_s'\operatorname{diag}(\lambda \oslash s)d_s \tag{11}$$

where $\epsilon_x(x)$ is chosen such that the solution is unique (the term $d_s'\lambda$ comes from the linearization term $d_s'\boldsymbol{\nabla}_sL(z)$). With some algebra, one could show that the dynamic equation of this modified interior-point is

$$z^+ = z + \alpha d_z = z - \alpha(\boldsymbol{\nabla}_zg(z,b) + E(x))^{-1}g(z,b), \tag{12}$$

where $E(x) := \operatorname{diag}(\epsilon_x\mathbf{1}_{n+m}, \mathbf{0}_{l+m})$. Conveniently, because we used $\operatorname{diag}(\lambda \oslash s)$ for the second order linearization of the barrier, when $\epsilon_x(x) = 0$ we recover the basic interior-point method from (8).

While it is beyond the scope of this paper, one could show that if some assumptions hold, the set of local minima of (6) coincides with the set of

asymptotically stable equilibrium points of (12). The proof uses very similar arguments as the proof of Theorem 1.

3 Minmax optimization

Consider the minmax optimization problem

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}(x)} f(x, y) \tag{13}$$

where $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ is a twice continuously differentiable objective function, $\mathcal{X} \subset \mathbb{R}^{n_x}$ is the feasible set for x and $\mathcal{Y} : \mathcal{X} \rightrightarrows \mathbb{R}^{n_y}$ is a set-valued map that defines an x dependent feasible set for y. A solution (x^*, y^*) to (13) is called a global minmax and satisfies

$$f(x^*, y) \le f(x^*, y^*) \le \max_{\tilde{y} \in \mathcal{Y}(x)} f(x, \tilde{y}) \qquad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}(x^*).$$

A point (x^*, y^*) is said to be a local minmax of (13) if there exist a constant $\delta_0 > 0$ and a positive function $h(\cdot)$ satisfying $h(\delta) \to 0$ as $\delta \to 0$, such that for every $\delta \in (0, \delta_0]$ and for every $(x, y) \in \{x \in \mathcal{X} : \|x - x^*\| \le \delta\} \times \{y \in \mathcal{Y}(x^*) : \|y - y^*\| \le h(\delta)\}$ we have

$$f(x^*, y) \le f(x^*, y^*) \le \max_{\tilde{y} \in \mathcal{Y}(x) : \|\tilde{y} - y^*\| \le h(\delta)} f(x, \tilde{y})$$

[11, 12]. Inspired by the properties of the modified Newton's method for minimization in Section 2, we want to develop a Newton-type iterative algorithm of the form

$$\begin{bmatrix} x^+ \\ y^+ \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}. \tag{14}$$

where d_x and d_y satisfy the following properties:

P1: At each time step, (d_x, d_y) is obtained from the solution of a quadratic program that locally approximates (13) and therefore (x^+, y^+) can be seen as an improvement over (x, y) .

P2: The set of asymptotically stable equilibrium points of (14) coincides with the set of local minmax of (15).

3.1 Unconstrained minmax

We start by considering the case where $\mathcal{X} = \mathbb{R}^{n_x}$ and $\mathcal{Y}(\cdot) = \mathbb{R}^{n_y}$ such that (13) simplifies to

$$\min_{x \in \mathbb{R}^{n_x}} \max_{y \in \mathbb{R}^{n_y}} f(x, y). \tag{15}$$

For this case, [11] establishes second order sufficient conditions to determine if a point (x, y) is a local minmax which can be stated in terms of the inertia

of the matrix

$$\boldsymbol{\nabla}_{zz} f(x,y) := \begin{bmatrix} \boldsymbol{\nabla}_{xx} f(x,y) & \boldsymbol{\nabla}_{xy} f(x,y) \\ \boldsymbol{\nabla}_{yx} f(x,y) & \boldsymbol{\nabla}_{yy} f(x,y) \end{bmatrix}.$$

We recall that the inertia $\text{In}(A)$ of a symmetric matrix A is a 3-tuple with the number of positive, negative and zero eigenvalues of A.

Proposition 1 (Unconstrained second order sufficient condition) *Let (x,y) be an equilibrium point in the sense that $\boldsymbol{\nabla}_x f(x,y) = 0$ and $\boldsymbol{\nabla}_y f(x,y) = 0$. If*

$$\text{In}(\boldsymbol{\nabla}_{yy} f(x,y)) = (0, n_y, 0) \ \text{and} \ \text{In}(\boldsymbol{\nabla}_{zz} f(x,y)) = (n_x, n_y, 0) \qquad (16)$$

then (x,y) is a local minmax.

The second order conditions in [11] are:

$$\text{In}(\boldsymbol{\nabla}_{yy} f(x,y)) = (0, n_y, 0) \ \text{and}$$
$$\text{In}(\boldsymbol{\nabla}_{xx} f(x,y) - \boldsymbol{\nabla}_{xy} f(x,y) \boldsymbol{\nabla}_{yy} f(x,y)^{-1} \boldsymbol{\nabla}_{yx} f(x,y)) = (n_x, 0, 0),$$

which turn out to be equivalent to the inertia conditions in Proposition 1 in view of Haynsworth inertia additivity formula [36, Theorem 1.6].

For the property P1 the Newton direction (d_x, d_y) for (14) should be obtained by solving the following local quadratic approximation to (15)

$$\min_{\bar{d}_x} \max_{\bar{d}_y} f(x,y) + \boldsymbol{\nabla}_x f(x,y)' \bar{d}_x + \boldsymbol{\nabla}_y f(x,y)' \bar{d}_y + \bar{d}_x' \boldsymbol{\nabla}_{xy} f(x,y) \bar{d}_y$$
$$+ \frac{1}{2} \bar{d}_x' \Big(\boldsymbol{\nabla}_{xx} f(x,y) + \epsilon_x(x,y) I \Big) \bar{d}_x + \frac{1}{2} \bar{d}_y' \Big(\boldsymbol{\nabla}_{yy} f(x,y) - \epsilon_y(x,y) I \Big) \bar{d}_y \quad (17)$$

with $\epsilon_x(\cdot)$ and $\epsilon_y(\cdot)$ chosen so that the minmax problem in (17) has a unique solution, which means that the inner (quadratic) maximization must be strictly concave and that the outer (quadratic) minimization of the maximized function must be strictly convex, which turns out to be precisely the second order sufficient conditions in Proposition 1, applied to the approximation in (17), which can be explicitly written as follows:

$$\text{In} \Big(\boldsymbol{\nabla}_{yy} f(x,y) - \epsilon_y(x,y) I \Big) = (0, n_y, 0) \ \text{and}$$
$$\text{In} \Big(\boldsymbol{\nabla}_{zz} f(x,y) + E(x,y) \Big) = (n_x, n_y, 0) \qquad \text{(LQAC)}$$

where $E(x,y) = \text{diag}(\epsilon_x(x,y) \mathbf{1}_{n_x}, -\epsilon_y(x,y) \mathbf{1}_{n_y})$. We call these condition the Local Quadratic Approximation Condition (LQAC). It is straightforward to show that the Newton iterations (14) with (d_x, d_y) obtained from the solution to (17) is given by

$$\begin{bmatrix} x^+ \\ y^+ \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \Big(\boldsymbol{\nabla}_{zz} f(x,y) + E(x,y) \Big)^{-1} \begin{bmatrix} \boldsymbol{\nabla}_x f(x,y) \\ \boldsymbol{\nabla}_y f(x,y) \end{bmatrix}. \quad (18)$$

For the property P2, we need all locally asymptotically stable equilibrium points of (17) to be local minmax of (15). For the minimization in Section 2, simply selecting $\epsilon_x(\cdot)$ such that the local quadratic approximation (2) has a well defined minimum suffices to guarantee that the only equilibrium points that are LAS for the Newton iterations (3) are strict local minima (Theorem 1). However, now the LQAC does not suffice to guarantee that P2 holds, as the two counter examples bellow show.

Example 1 Consider $f(x, y) = -1.5x^2 - 4xy + y^2$ for which the unique equilibrium point $x = y = 0$ is not a local minmax point. Take $\epsilon_y(0, 0) = 4$ and $\epsilon_x(0, 0) = 0$ which satisfy LQAC. The Jacobian of the dynamics is

$$I - \left(\begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix} \approx \begin{bmatrix} 0 & 0.72 \\ 0 & 0.54 \end{bmatrix}$$

which has eigenvalues approximately equal to $(0, 0.54)$. Therefore $(0, 0)$ is a LAS equilibrium point of (18) even though it is not a local minmax point.

Example 2 Consider $f(x, y) := -0.25x^2 + xy - 0.5y^2$, for which the unique equilibrium point $x = y = 0$ is a local minmax point. Take $\epsilon_y(0, 0) = 3$ and $\epsilon_x(0, 0) = 0.2$ which satisfy LQAC. The Jacobian of the dynamics is

$$I - \left(\begin{bmatrix} -0.5 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.5 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 15 \\ -1 & 4.5 \end{bmatrix},$$

for which the eigenvalues are 2 and -1.5. Therefore $(0, 0)$ is an unstable equilibrium point of (18) even though it is a local minmax point.

The main contribution of this section is a set of sufficient conditions that, in addition to LQAC, guarantee the P2 holds. In the results that follow, we use $\text{In}^+(\cdot)$ to designate the number of positive eigenvalues of a symmetric matrix and $\text{In}^-(\cdot)$ the number of negative ones.

Theorem 2 (Stability of modified Newton method for unconstrained minmax) *Let (x, y) be an equilibrium point of (18) for which $\boldsymbol{\nabla}_{zz} f(\cdot)$ and $E(\cdot)$ are differentiable on a neighborhood around (x, y) and LQAC holds. If (x, y) is such that (16) holds and either*

$$\epsilon_y(x, y) = 0 \ or \ \text{In}^- \left(\begin{bmatrix} \epsilon_x(x, y)I & \boldsymbol{\nabla}_{xy} f(x, y) \\ \boldsymbol{\nabla}_{yx} f(x, y) & \epsilon_y(x, y)^{-1} \boldsymbol{\nabla}_{yy} f(x, y)^2 \end{bmatrix} \right) = 0 \quad (19)$$

then (x, y) is a locally exponentially stable equilibrium point of (18). Conversely, suppose (x, y) is such that (16) does not hold. If there exist $\bar{\mu} \in (0, 1)$ for which $\boldsymbol{\nabla}_{yy} f(z) - \bar{\mu}\epsilon_y(x, y)$ and $\boldsymbol{\nabla}_{zz} f(x, y) + \bar{\mu}E(x, y)$ are nonsingular and either

- $\text{In}^+(\boldsymbol{\nabla}_{yy} f(z) - \bar{\mu}\epsilon_y(x, y)) = 0$ *and* $\text{In}^+(\boldsymbol{\nabla}_{zz} f(x, y) + \bar{\mu}E(x, y)) < n_x$ (20a)

- $n_x \geq \text{In}^+(\boldsymbol{\nabla}_{yy} f(z) - \bar{\mu}\epsilon_y(x, y)) > 0$ *and* $\text{In}^+(\boldsymbol{\nabla}_{zz} f(x, y) + \bar{\mu}E(x, y)) > n_x$
    (20b)

- $n_x < \text{In}^+(\boldsymbol{\nabla}_{yy} f(z) - \bar{\mu}\epsilon_y(x, y))$ (20c)

then (x, y) is an unstable equilibrium point of (18).

Essentially, Theorem 2 shows that if we select ϵ_x and ϵ_y so that both LQAC and either (19) or (20) hold, then (x, y) is a locally asymptotically stable equilibrium point of (18) if and only if this point satisfies the second order sufficient conditions in Proposition 1. The statement of the result is slightly more complicated than this because if the second order sufficient conditions do not hold, then the point is actually unstable (rather than just not exponentially stable). Conversely, if these conditions hold, we actually have local exponential stability.

Proof of Theorem 2 Using the same reasoning as in Theorem 1, we conclude that the Jacobian of the dynamical system (18) at (x, y) is

$$I - (\nabla_{zz} f(x, y) + E(x, y))^{-1} \nabla_{zz} f(x, y), \tag{21}$$

where we know that $(\nabla_{zz} f(x, y) + E(x, y))$ is nonsingular from the LQAC. Therefore, we can also use the same reasoning as in the proof of Theorem 1 to conclude that (x, y) is a LAS equilibrium point of (18) if $\nabla_{zz} f(x, y) + \mu E(x, y)$ is nonsingular $\forall \mu \in [0, 2]$. Conversely, (x, y) is an unstable equilibrium point of (18) if $\nabla_{zz} f(x, y) + \mu E(x, y)$ is singular for some $\mu \in [0, 2]$.

For the rest of the proof, it will be useful to have defined the function

$$S(\mu) = \nabla_{xx} f(x, y) - \nabla_{xy} f(x, y) (\nabla_{yy} f(x, y) - \mu \epsilon_y(x, y) I)^{-1} \nabla_{yx} f(x, y) + \mu \epsilon_x(x, y) I$$

and to drop the inputs (x, y) from the expressions in order to shorten them.

Let us start by proving that if (x, y) is such that (16) and (19) hold, then (x, y) is a LAS equilibrium point of (18). First, we want to show that condition (19) is equivalent to the condition

$$\epsilon_x(x, y) I - \epsilon_y(x, y) \nabla_{xy} f(x, y) \nabla_{yy} f(x, y)^{-2} \nabla_{yx} f(x, y) \succeq 0. \tag{22}$$

For $\epsilon_y(x, y) = 0$ the equivalence is true because $\epsilon_x(x, y) \geq 0$. For $\epsilon_y(x, y) > 0$, the equivalence comes from Haynsworth inertia additivity formula combined to the fact that $\nabla_{yy} f(x, y)^2 \succ 0$.

Let us show that this ensures that $\nabla_{zz} f + \mu E$ is nonsingular $\forall \, \mu > 0$, and therefore (x, y) is a LAS equilibrium point of (18). First, as $\nabla_{yy} f \prec 0$, $\mu \geq 0$, and $\epsilon_y \geq 0$, we have $\nabla_{yy} f - \mu \epsilon_y I \prec 0$ and is thus nonsingular. Second, let us show that the condition (22) implies that for any vector v

$$\min_{\mu \in [0, 2]} v' S(\mu) v = v' S(0) v. \tag{23}$$

Taking the derivative of $v' S(\mu) v$ with respect to μ we obtain

$$v' \Big(\epsilon_x I - \epsilon_y \nabla_{xy} f (\nabla_{yy} f - \mu \epsilon_y I)^{-2} \nabla_{yx} f \Big) v \succ v' \Big(\epsilon_x I - \epsilon_y \nabla_{xy} f \nabla_{yy} f^{-2} \nabla_{yx} f \Big) v$$

in which we use the the fact that $\nabla_{yy} f^{-2} \succeq (\nabla_{yy} f - \mu \epsilon_y I)^{-2}$ for all $\mu \geq 0$ as $\nabla_{yy} f \prec 0$, and $\epsilon_y \geq 0$. Therefore, if (22) holds, the derivative of $v' S(\mu) v$, thus the cost does not decrease with μ, which implies that the minimum is obtained for $\mu = 0$, which proves (23). Therefore if ϵ_x and ϵ_y are chosen to satisfy (19), then $\forall \mu \in [0, 2]$ it holds that $S(\mu) \succeq S(0) \succ 0$, where the second inequality comes from the second order sufficient conditions for unconstrained minmax (16). As neither $\nabla_{yy} f - \mu \epsilon_y I \prec 0$ nor $S(\mu)$ are singular for $\mu \in [0, 2]$, Haynsworth inertia additivity formula [36, Theorem 1.6] implies that $\nabla_{zz} f + \mu E$ is nonsingular $\forall \mu \in [0, 2]$, and therefore (x, y) is a LAS equilibrium point of (18).

(a) Illustration of (20a)   (b) Illustration of (20b)   (c) Illustration of (20c)
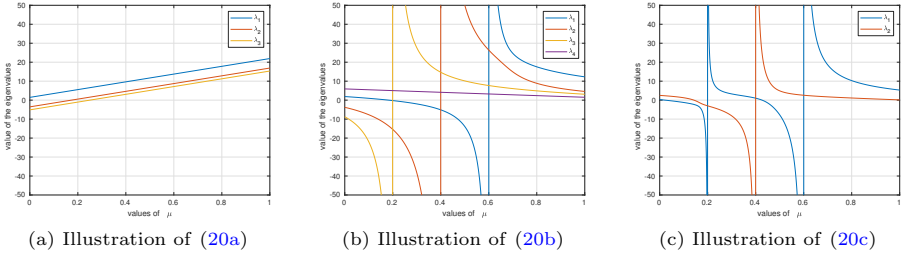
Fig. 1: Eigenvalues of $S(\mu)$ for the three differente scenarios in (20) illustrating the arguments for the proof of Theorem 2

Now the second part, let us prove that if (x, y) is not such that (16) and ϵ_x and ϵ_y are chosen to satisfy the LQAC and (20) then (x, y) is an unstable equilibrium point of (18). Let us start with the following general results. Using Haynsworth inertia additivity formula, if $\nabla_{yy} f - \mu \epsilon_y I$ is nonsingular then $\nabla_{zz} f + \mu E$ is singular if and only if $S(\mu)$ is singular. Let μ_k with $k = 1, 2, \ldots, \text{In}^+(\nabla_{yy} f(x, y))$ be the points such that $\nabla_{yy} f(x, y) - \mu_k \epsilon_y$ is singular with $\mu_1 > \cdots > \mu_{\text{In}^+(\nabla_{yy} f(x,y))}$. Moreover, the LQAC imply that $\nabla_{yy} f(x, y) - \mu \epsilon_y(x, y)I$ is negative definite for $\mu \in (\mu_1, 1)$.

Starting with (20a). First, notice that by construction of μ_1, we have that $\bar{\mu} > \mu_1$, and therefore $(\nabla_{yy} f(x, y) - \mu \epsilon_y(x, y)I) \prec 0 \ \forall \mu \geq \bar{\mu}$. As $\text{In}^+(\nabla_{zz} f(x, y)) < n_x$ implies that $\lambda_{min}(S(\bar{\mu})) < 0$ and the LQAC implies that $\lambda_{min}(S(1)) > 0$ (both by Haynsworth inertia additivity formula), from the continuity of the eigenvalues, $\exists \mu \in (\bar{\mu}, 1)$ such that $\lambda_{min}(S(\mu)) = 0$. We illustrate this reasoning in Figure 1a.

Now (20b). First, notice that, by construction of μ_1, $\bar{\mu} < \mu_1$. For each point μ_k, the limit of $\lambda_{max}(S(\mu))$ is $+\infty$ when μ tends to μ_k by the right and the limit of $\lambda_{min}(S(\mu))$ is $-\infty$ when μ tends to μ_k by the left. Let $\lambda_1(\mu)$ be the eigenvalue of $S(\mu)$ such that the limit of $\lambda_1(\mu)$ when μ tends to μ_1 by the left is $-\infty$. Between μ_2 and μ_1, $\lambda_1(\mu)$ is the smallest eigenvalue of $S(\mu)$. Between μ_3 and μ_2, $\lambda_1(\mu)$ is the second smallest eigenvalue of $S(\mu)$, as another eigenvalue has gone to $-\infty$ (it might be useful at this point to look at Figure 1b to have an illustration). If we show that (20b) is equivalent to $\lambda_1(\bar{\mu}) > 0$, then by continuity of the eigenvalue we know that there exist $\mu \in (\bar{\mu}, \mu_1)$ such that $\lambda_{min} S(\mu) = 0$. We now only need to show the equivalence of the conditions.

If $\bar{\mu} \in (\mu_2, \mu_1)$ then $\lambda_1(\bar{\mu}) > 0$ if $S(\bar{\mu})$ has n_x positive eigenvalues, as $\lambda_1(\mu)$ is the smallest eigenvalue of $S(\mu)$ for μ in (μ_2, μ_1). If $\bar{\mu} \in (\mu_3, \mu_2)$ then $\lambda_1(\bar{\mu}) > 0$ if $S(\bar{\mu})$ has at least $n_x - 1$ positive eigenvalues, as $\lambda_1(\mu)$ is the second smallest eigenvalue of $S(\mu)$ for μ in (μ_2, μ_1) (see discussion in previous paragraph). In the general case, if $\bar{\mu} \in (\mu_{k+1}, \mu_k)$ then $\lambda_1(\bar{\mu}) > 0$ if $S(\bar{\mu})$ has at least $n_x - k + 1$ positive eigenvalues as $\lambda_1(\mu)$ is the kth smallest eigenvalue of $S(\mu)$ for μ in (μ_{k+1}, μ_k). This is equivalent to say $\text{In}^+(S(\bar{\mu})) > n_x - k$. Using Haynsworth inertia additivity formula we obtain $\text{In}^+(\nabla_{zz} f + \bar{\mu} E) = \text{In}^+(S(\bar{\mu})) + \text{In}^+(\nabla_{yy} f(x, y) - \bar{\mu} \epsilon_y) > (n_x - k) + (k) = n_x$.

Finally, (20c). Continuing the reasoning from above, for $\mu \in (\mu_{n_x+1}, \mu_{n_x})$, $\lambda_1(\mu)$ is the largest eigenvalue of $S(\mu)$ (as all the $n_x - 1$ other eigenvalues have gone to $-\infty$). As $n_x < \text{In}^+(\nabla_{yy} f(z) - \bar{\mu} \epsilon_y(x, y))$, we know that at μ_{n_x+1}, the limit of $\lambda_{max}(S(\mu))$ (in this case, $\lambda_1(\mu)$) is $+\infty$ when μ tends to μ_{n_x+1} by the right, meaning that $\lambda_1(\mu)$ has crossed zero at some point between (μ_{n_x+1}, μ_1). We illustrate this reasoning in Figure 1c. $\square$

3.2 Constrained minmax

We now consider the case with more general constraint sets involving equality and inequality constraints of the form

$$\mathcal{X} = \{x \in \mathbb{R}^{n_x} : G_x(x) = \mathbf{0}, F_x(x) \leq \mathbf{0}\} \quad \text{and}$$
$$\mathcal{Y}(x) = \{y \in \mathbb{R}^{n_y} : G_y(x,y) = \mathbf{0}, F_y(x,y) \leq \mathbf{0}\} \tag{24}$$

where the functions $G_x : \mathbb{R}^{n_x} \to \mathbb{R}^{l_x}$, $F_x : \mathbb{R}^{n_x} \to \mathbb{R}^{m_x}$, $G_y : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{l_y}$ and $F_y : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{m_y}$ are all twice continuously differentiable. Similar to what we did in Section 2.1, it will be convenient for the development of the interior-point method to use slack variables and rewrite the constrained minmax (13) as

$$\min_{x,s_x : G_x(x)=\mathbf{0}, F_x(x)+s_x=\mathbf{0}, s_x \geq \mathbf{0}} \quad \max_{y,s_y : G_y(x,y)=\mathbf{0}, F_y(x,y)+s_y=\mathbf{0}, s_y \geq \mathbf{0}} f(x,y). \tag{25}$$

Similar to what we have done in the unconstrained case, we want to present second order conditions to determine if a point is a constrained local minmax. In order to do so, we need to extend some fundamental concepts of constrained minimization to constrained minmax optimization. The function

$$L(z) := f(x,y) + \nu_x' G_x(x) + \lambda_x'(F_x(x) + s_x) + \nu_y' G_y(x,y) - \lambda_y'(F_y(x,y) + s_y),$$

will play an equivalent role as the Lagrangian with $(\nu_x, \nu_y, \lambda_x, \lambda_y)$ as the equivalent of Lagrange multipliers; we use the shorthand notation $z = (x, s_x, y, s_y, \nu_y, \lambda_y, \nu_x, \lambda_x)$. Furthermore, the definition of linear independence constraint qualifications (LICQ) and the strict complementarity for minmax optimization are[1]:

Definition 1 (LICQ and strict complementarity for minmax) Let the sets of active inequality constraints for the minimization and maximization be defined, respectively, by

$$\mathcal{A}_x(x) = \{i : F_x^{(i)}(x) = 0, i = 1, \ldots, m_x\} \text{ and}$$
$$\mathcal{A}_y(x,y) = \{i : F_y^{(i)}(x,y) = 0, i = 1, \ldots, m_y\} \tag{26}$$

where $F_y^{(i)}(x,y)$ and $F_x^{(i)}(x)$ denote the ith element of $F_y(x,y)$ and $F_x(x)$. Then:

- The linear independence constraint qualification (LICQ) is said to hold at z if the vectors the sets

$$\{\boldsymbol{\nabla}_x G_x^{(i)}(x), i = 1, \ldots, l_x\} \bigcup \{\boldsymbol{\nabla}_x F_x^{(i)}(x), i \in \mathcal{A}_x(x)\} \text{ and}$$
$$\{\boldsymbol{\nabla}_y G_y^{(i)}(x,y), i = 1, \ldots, l_y\} \bigcup \{\boldsymbol{\nabla}_y F_y^{(i)}(x,y), i \in \mathcal{A}_y(x,y)\}$$

are linearly independent.

---

[1]Their definition for constrained minimization can be found in [13, Definitions 12.4 and 12.5].

- Strict complementarity is said to hold at z if $\lambda_y^{(i)} > 0 \ \forall i \in \mathcal{A}_y(x,y)$ and $\lambda_x^{(i)} > 0 \ \forall i \in \mathcal{A}_x(x)$

We have almost all the ingredients to present the second order condition. For the unconstrained minmax optimization, the second order condition in Proposition 1 required that gradients ($\boldsymbol{\nabla}_x f(x,y)$ and $\boldsymbol{\nabla}_y f(x,y)$) were equal to zero and that Hessians ($\boldsymbol{\nabla}_{zz} f(x,y)$ and $\boldsymbol{\nabla}_{yy} f(x,y)$) had a particular inertia. If it were not for the inequality constraints in (24), we would be able to state the second order conditions using gradients and Hessians of $L(z)$. Unfortunately, the inequality constraints make the statement a bit more complicated. The role of the gradient will be played by

$$
g(z,b) := \begin{bmatrix} \boldsymbol{\nabla}_x L(z) \\ \lambda_x \odot s_x - b\mathbf{1} \\ \boldsymbol{\nabla}_y L(z) \\ -\lambda_y \odot s_y + b\mathbf{1} \\ G_y(x,y) \\ -F_y(x,y) - s_y \\ G_x(x) \\ F_x(x) + s_x \end{bmatrix}
$$

where $\odot$ denotes the element wise Hadamard product of two vectors and $b \geq 0$ the barrier parameter, which is the extension to minmax of the function $g(\cdot)$ defined in (7) for the minimization. The role of $\boldsymbol{\nabla}_{yy} f(x,y)$ will be played by

$$
H_{yy} f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{yy} L(z) & \mathbf{0} & \boldsymbol{\nabla}_y G_y(x,y) & -\boldsymbol{\nabla}_y F_y(x,y) \\ \mathbf{0} & -\operatorname{diag}(\lambda_y) & \mathbf{0} & -\operatorname{diag}(\sqrt{s_y}) \\ \boldsymbol{\nabla}_y G_y(x,y)' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\nabla}_y F_y(x,y)' & -\operatorname{diag}(\sqrt{s_y}) & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (27a)
$$

while the role of $\boldsymbol{\nabla}_{zz} f(x,y)$ will be played by

$$
H_{zz} f(z) = \begin{bmatrix} H_{xx} f(z) & H_{xy} f(z) & H_{x\lambda} f(z) \\ H_{xy} f(z)' & H_{yy} f(z) & \mathbf{0} \\ H_{x\lambda} f(z)' & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (27b)
$$

with blocks defined by

$$
H_{xy} f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{xy} L(z) & \mathbf{0} & \boldsymbol{\nabla}_x G_y(x,y) & -\boldsymbol{\nabla}_x F_y(x,y) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$
$$
H_{xx} f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{xx} L(z) & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}(\lambda_x) \end{bmatrix} \quad H_{x\lambda} f(z) = \begin{bmatrix} \boldsymbol{\nabla}_x G_x(x) & \boldsymbol{\nabla}_x F_x(x) \\ \mathbf{0} & \operatorname{diag}(\sqrt{s_x}) \end{bmatrix} \quad (27c)
$$

Proposition 2 (Constrained second order sufficient conditions) *Let z be an equilibrium point in the sense that $g(z, 0) = \mathbf{0}$ with $\lambda_y, \lambda_x, s_y, s_x \geq \mathbf{0}$. If the LICQ and strict complementarity hold at z and*

$$\text{In}(H_{yy}f(z)) = (l_y + m_y, n_y + m_y, 0) \text{ and}$$
$$\text{In}(H_{zz}f(z)) = (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0) \tag{28}$$

then (x, y) is a local minmax of (13).

The conditions for Proposition 2 are slightly stricter than the ones in [12] as we require strict complementarity and LICQ both for the max and the min. However, our conditions allow us to verify whether a point is a local minmax using the inertia, instead of having to compute solution cones. We prove that given these stricter assumptions our conditions are equivalent to those in [12] in Appendix A.

Let $d_z = (d_x, d_{s_x}, d_y, d_{s_y}, d_{\nu_y}, d_{\lambda_y}, d_{\nu_x}, d_{\lambda_x})$ be a shorthand notation to designate the update direction of the variables $z = (x, s_x, y, s_y, \nu_y, \lambda_y, \nu_x, \lambda_x)$. Similar to the basic interior-point method introduced in Section 2.1, a basic interior-point method for minmax finds a candidate solution to (25) using the iterations

$$z^+ = z + \alpha d_z = z - \alpha \boldsymbol{\nabla}_z g(z, b)^{-1} g(z, b) \tag{29}$$

where the barrier parameter b is slowly decreased to 0, so that z converges to a root of $g(z, 0) = \mathbf{0}$ while $\alpha \in (0, 1]$ is chosen at each step such that the feasibility conditions $\lambda_y, \lambda_x, s_y, s_x > 0$ hold. We want to modify this basic interior-point so it satisfies the properties P1 and P2.

For the property P1, d_z needs to be obtained from the solution of a quadratic program that locally approximates (25). Using equivalent arguments as in the development of the quadratic program (11) for the constrained minimization in Section 2.1, we obtain the the objective function should be

$$K(d_x, d_{s_x}, d_y, d_{s_y}) = L(z) + \boldsymbol{\nabla}_x L(z)' d_x + (\lambda_x - b\mathbf{1} \oslash s_x)' d_{s_x} + \boldsymbol{\nabla}_y L(z)' d_y$$
$$- (\lambda_y - b\mathbf{1} \oslash s_y)' d_{s_y} + d_x' \boldsymbol{\nabla}_{xy} L(z) d_y + \frac{1}{2} d_x' (\boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z) I) d_x$$
$$+ \frac{1}{2} d_{s_x}' \operatorname{diag}(\lambda_x \oslash s_x) d_{s_x} + \frac{1}{2} d_y' (\boldsymbol{\nabla}_{yy} L(z) - \epsilon_y(z) I) d_y - \frac{1}{2} d_{s_y}' \operatorname{diag}(\lambda_y \oslash s_y) d_{s_y},$$

where $\epsilon_x(z) \geq 0$ and $\epsilon_y(z) \geq 0$ are scalar and $\oslash$ designates the element wise division of two vectors. The feasible sets $d\mathcal{X}$ for (d_x, d_{s_x}) and the set-valued map that defines a feasible set $d\mathcal{Y}(d_x)$ for (d_y, d_{s_y}) are obtained from the first order linearization of the functions in $\mathcal{X}$ and $\mathcal{Y}(d_y)$ and are given by

$$d\mathcal{X} = \{(d_x, d_{s_x}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{m_x} : G_x(x) + \boldsymbol{\nabla}_x G_x(x)' d_x = \mathbf{0},$$
$$F_x(x) + s_x + \boldsymbol{\nabla}_x F_x(x)' d_x + d_{s_x} = \mathbf{0}\}$$
$$d\mathcal{Y}(d_x) = \{(d_y, d_{s_y}) \in \mathbb{R}^{n_y} \times \mathbb{R}^{m_y} : G_y(x, y) + \boldsymbol{\nabla}_x G_y(x, y)' d_x + \boldsymbol{\nabla}_y G_y(x, y)' d_y$$

$$= \mathbf{0}, F_y(x, y) + s_y + \boldsymbol{\nabla}_x F_y(x, y)' d_x + \boldsymbol{\nabla}_y F_y(x, y)' d_y + d_{s_y} = \mathbf{0}\}.$$

If $\boldsymbol{\nabla}_x G_x(x)$ and $\boldsymbol{\nabla}_y G_y(x, y)$ have linearly independent columns, we propose to obtain $(d_x, d_{s_x}, d_y, d_{s_y})$ as the optimizers and $(d_{\nu_y}, d_{\lambda_y}, d_{\nu_x}, d_{\lambda_x})$ the associated Lagrange multipliers of the minmax optimization

$$\min_{\bar{d}_x, \bar{d}_{s_x} \in d\mathcal{X}} \quad \max_{\bar{d}_y, \bar{d}_{s_y} \in d\mathcal{Y}(\bar{d}_x)} K(\bar{d}_x, \bar{d}_{s_x}, \bar{d}_y, \bar{d}_{s_y}) \tag{30}$$

where $\epsilon_x(z)$ and $\epsilon_y(z)$ are chosen such that the solution to (30) is unique. We can apply to (30) the second order condition from Proposition 2 and obtain that $\epsilon_x(z)$ and $\epsilon_y(z)$ need to be chosen to satisfy

$$\mathrm{In}(J_{yy}f(z) - E_y(z)) = (l_y + m_y, n_y + m_y, 0) \text{ and}$$
$$\mathrm{In}(J_{zz}f(z) + E(z)) = (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0) \tag{ConsLQAC}$$

where $E_y(z)$:= $\mathrm{diag}(\epsilon_y(z)\mathbf{1}_{n_y}, \mathbf{0}_{l_y+2m_y})$ and $E(z)$:= $\mathrm{diag}(\epsilon_x(z)\mathbf{1}_{n_x}, \mathbf{0}_{m_x}, -\epsilon_y(z)\mathbf{1}_{n_y}, \mathbf{0}_{l_y+2m_y+l_x+m_x})$; $J_{zz}f(z)$ is the equivalent of the matrix defined in (27b) for the problem (30) and can be shown to be equal to

$$J_{zz}f(z) = S^{-1/2}H_{zz}f(z)S^{-1/2} = S^{-1}\boldsymbol{\nabla}_z g(z, b). \tag{31}$$

with $S = \mathrm{diag}(\mathbf{1}_{n_x}, s_x, \mathbf{1}_{n_y}, s_y, \mathbf{1}_{l_y+m_y+l_x+m_x})$; $J_{yy}(z)$ is the equivalent partition of $J_{zz}(z)$ as $H_{yy}(z)$ is of $H_{zz}(z)$. We will call these conditions the Constrained Local Quadratic Approximation Conditions (ConsLQAC). In this case, it is straightforward to show that modifying the basic interior-point iterations in (29) by taking d_z from the solution of (30) leads to the iterations

$$z^+ = z + \alpha d_z = z - \alpha(J_{zz}f(z) + E(z))^{-1}S^{-1}g(z, b). \tag{32}$$

For the property P2, one can deduce that analogously to the unconstrained case, choosing $\epsilon_x(z)$ and $\epsilon_y(z)$ such that the ConsLQAC hold is not sufficient to guarantee the desired stability/instability, and we need to develop equivalent conditions as those of Theorem 2. In order to state them, let us define the partitions, $J_{xx}f(z)$, $J_{yx}f(z)$, and $J_{x\lambda}f(z)$ of $J_{zz}f(z)$ analogously to the partitions $H_{xx}f(z)$, $H_{yx}f(z)$, and $H_{x\lambda}f(z)$ of $H_{zz}f(z)$. We also need to define the perturbation matrices $E_x(z) = \mathrm{diag}(\epsilon_x(z)\mathbf{1}_{n_x}, \mathbf{0}_{m_x})$ and $\tilde{E}_y(z) := \mathrm{diag}(\epsilon_y(z)\mathbf{1}_{n_y}, \xi\mathbf{1}_{l_y+2m_y})$ for an arbitrary choice of $\xi > 0$ (preferably small)

Theorem 3 (Stability of modified interior-point method for constrained minmax) *Let $\alpha = 1$ and (z, b), $b > 0$ be an equilibrium point of (32) (in the sense that $g(z, b) = \mathbf{0}$) for which $J_{zz}f(\cdot)$ and $E(\cdot)$ are differentiable on a neighborhood around*

z and [ConsLQAC]{.blue} and the LICQ hold. If *z* is such that (28) hold and either

$$\epsilon_y(z) = 0 \ or \ \text{In}^- \left(\begin{bmatrix} E_x(z) & J_{xy}f(z) & J_{x\lambda}f(z) \\ J_{xy}f(z)' & \tilde{E}_y(z)^{-1}J_{yy}f(z)^2 & 0 \\ J_{x\lambda}f(z)' & 0 & 0 \end{bmatrix} \right) = l_x + m_x \qquad (33)$$

then (z, b) *is a locally exponentially stable equilibrium point of* (32). *Conversely, suppose z is such that* (28) *does not hold. If there exist* $\bar{\mu} \in (0, 1)$ *for which* $H_{yy}f(z) - \bar{\mu}E_y(z)$ *and* $H_{zz}f(z) + \bar{\mu}E(z)$ *are nonsingular and either*

- $\text{In}^+(H_{yy}f(z) - \bar{\mu}E_y(z)) = l_y + m_y$

$$\text{and} \ \text{In}^+(H_{zz}f(z) + \bar{\mu}E(z)) < n_x + m_x + l_y + m_y \quad (34a)$$

- $n_x + m_x \geq \text{In}^+(H_{yy}f(z) - \bar{\mu}E_y(z)) - l_y - m_y > 0$

$$\text{and} \ \text{In}^+(H_{zz}f(z) + \bar{\mu}E(z)) > n_x + m_x + l_y + m_y \quad (34b)$$

- $n_x + m_x < \text{In}^+(H_{yy}f(z) - \bar{\mu}E_y(z)) - l_y - m_y \qquad (34c)$

then z is an unstable equilibrium point of (32).

Proof sketch First, using the same arguments as in the proof of Theorem 1, we conclude that the Jacobian of the dynamic system (32) around a point *z* such that $g(z, b) = \mathbf{0}$ is

$$I - \alpha \Big(J_{zz}f(z) + E(z) \Big)^{-1} S^{-1} \boldsymbol{\nabla}_z g(z, b) = I - \alpha \Big(J_{zz}f(z) + E(z) \Big)^{-1} J_{zz}f(z) \quad (35)$$

Second, from (31) we have that $\text{In}(H_{zz}f(z)) = \text{In}(S^{1/2}J_{zz}f(z)S^{1/2})$. Using Sylvester's law of inertia [36, Theorem 1.5], this simplifies to $\text{In}(H_{zz}f(z)) = \text{In}(J_{zz}f(z))$. If a point *z* is such that $g(z, b) = \mathbf{0}$, then one can check (28) using $J_{zz}f(z)$ and $J_{yy}f(z)$. The rest of the theorem's proof is analogous to the one of Theorem 2, but using the function

$$S(\mu) = Z_x(z)' \Big(J_{xx}f(z) - J_{xy}f(z)(J_{yy}f(z) - \mu E_y(z))^{-1} J_{yx}f(x, y) + \mu E_x(z) \Big) Z_x(z)$$

where $Z_x(z) \in \mathbb{R}^{n_x + m_x, n_x - l_x}$ is such that $J_{x\lambda}f(z)' Z_x(z) = 0$ and that $[J_{x\lambda}f(z), Z_x(z)]$ is full rank. Which comes from (A3), in the proof of Proposition 2.

First, the proof that if *z* is such that (28) and (33) hold then *z* is a LAS equilibrium point of (32). Similar to the unconstrained case, (33) implies that

$$Z_x(z)' \Big(E_x(z) - J_{xy}f(z)J_{yy}f(z)^{-1} \tilde{E}_y(z) J_{yy}f(z)^{-1} J_{yx}f(z) \Big) Z_x(z) \succeq 0$$

$$\implies Z_x(z)' \Big(E_x(z) - J_{xy}f(z)J_{yy}f(z)^{-1} E_y(z) J_{yy}f(z)^{-1} J_{yx}f(z) \Big) Z_x(z) \succeq 0 \quad (36)$$

The first implication comes from the arguments we used in the proof of Proposition 2, while the second implication comes from $\tilde{E}_y(z) \succ E_y(z)$.

The only extra argument needed is to show that condition (36) is always feasible for some $\epsilon_x(z)$ large enough. This is not evident as the matrix

$$R := H_{xy}f(z)H_{yy}f(z)^{-1}E_y(z)H_{yy}f(z)^{-1}H_{yx}f(z)$$

size is $(n_x + m_x) \times (n_x + m_x)$ while $E_x(z)$ has only n_x nonzero elements in the diagonal. However, because of the zero entries in $H_{xy}f(z)$ and $E_y(z)$, one can verify with some algebraic manipulation that $\text{rank}(R) := r \leq \min(n_x, n_y)$. Let Λ be the matrix with eigenvalues of *R* in decreasing order and *V* its associated eigenvectors such that $R = V\Lambda V'$. We can partition *V* into two diagonal block V_1 of size (r, r) associated to the nonzero eigenvalues of *R* and $V_2 = I_{n_x + m_x - r}$. Therefore $E_x(z) = V'E_x(z)V$, which concludes the proof sketch. $\qquad\square$

4 Algorithmic development and numerical examples

The following algorithm combines the result of the previous section to propose a method for selecting $\epsilon_x(z)$ and $\epsilon_y(z)$ that satisfies the ConsLQAC and guarantees the stability properties of Theorem 3. We only state the algorithm for the constrained case, its specialization to the unconstrained case is straightforward.

Algorithm 1 Interior-point method for minmax

Require: An initial point $z = (x, s_x, y, s_y, \nu_y, \lambda_y, \nu_x, \lambda_x)$, an initial barrier parameter value b, a barrier reduction factor $\sigma \in (0,1)$, a stopping accuracy δ_s, and a local stability switching criteria δ_l.

1: **while** $\|g(z,b)\|_\infty > \delta_s$ **do**
2:     **while** $\|g(z,b)\|_\infty > b$ **do**
3:         **if** $\|g(z,b)\|_\infty > \delta_\ell$ **then**
4:             Take $\epsilon_y(z)$ and $\epsilon_x(z)$ such that (ConsLQAC) is satisfied
5:         **else**
6:             **if** $J_{yy}(z)$ and $J_{zz}f(z)$ satisfy (28) **then**
7:                 Take $\epsilon_y(z)$ and $\epsilon_x(z)$ such that (33) is satisfied
8:             **else**
9:                 Take $\epsilon_y(z)$ and $\epsilon_x(z)$ such that (ConsLQAC) is satisfied.
10:                 Test conditions (34) for $\bar\mu = 0$
11:                 **if** conditions (34) do not hold for $\bar\mu = 0$ **then**
12:                     Pick candidates $\bar\mu_k > 0$, $k = 1, 2, \ldots$ with at least one of them satisfying $\mathrm{In}^+(J_{yy}f(z) - \bar\mu_k E_y(z)) - l_y - m_y > 0$
13:                     Increase ϵ_x until either condition (34a) or (34b) holds for one of the $\bar\mu_k$
14:                 **end if**
15:             **end if**
16:         **end if**
17:         Compute a new z using the equation

$$z \leftarrow z - \alpha \Big(J_{zz}f(z) + E(z) \Big)^{-1} S^{-1} g(z, b)$$

where $\alpha \in (0, 1]$ is selected such that the feasibility conditions $\lambda_y, \lambda_x, s_y, s_x > \mathbf{0}$ hold.
18:     **end while**
19:     $b \leftarrow \sigma b$
20: **end while**

The switching condition $\|g(z,b)\|_\infty > \delta_\ell$ between the two modes is used for computationally efficiency. As the stability results are only local, when

the system is far from an equilibrium point, generating local instability is not necessary, and $\epsilon_x(\cdot)$ and $\epsilon_y(\cdot)$ can be chosen simply to satisfy the constrained local quadratic approximation condition (ConsLQAC).

An option for line (7) would be to choose $\epsilon_y(z) = \epsilon_x(z) = 0$, which would guarantee that (28) always holds. However, it is often the case that the matrices $J_{yy}f(z)$ and $J_{zz}f(z)$ can be ill conditioned, meaning that one of their eigenvalues is close to zero. In this case, it can be interesting to choose non zero $\epsilon_x(z)$ and $\epsilon_y(z)$ which would generally make the computation more numerically stable.

Proposition 3 (Convergence to feasible local minmax points) *Suppose that Algorithm 1 generates an infinite sequence of iterates $\{(x_k, y_k)\}$ (i.e., $\delta_s = 0$) and that $\{b_k\} \to 0$. Then all limit points $(\hat{x}, \hat{y})$ of $\{(x_k, y_k)\}$ are feasible. Furthermore, if the LICQ and strict complementarity condition hold at a given limit point $(\hat{x}, \hat{y})$, then $g(\hat{z}, 0) = \mathbf{0}$.*

The proof is analogous to the proof of Theorem 19.1 in [13]. In addition to this result, because ϵ_x and ϵ_y are selected using the results from Theorem 3, which guarantees that the only stable equilibrium points of (32) are local minmax points, any given limit point $(\hat{x}, \hat{y})$ will also, in general, be a local minmax point.

Remark 1 (Strongly-concave case) Suppose that for all $x \in \mathcal{X}$ the function $f(x, y)$ is strongly concave and the set $\mathcal{Y}(x)$ is convex. Then one can set $\epsilon_y = 0$ as $\text{In}(J_{yy}f(z)) = (l_y + m_y, n_y + m_y, 0)$ and $J_{yy}f(z)$ will generally be well conditioned. Therefore, if $J_{zz}f(z)$ satisfies (28), the condition (33) will always hold, and if it does not satisfy (28), then choosing $\epsilon_x(z)$ such that the ConsLQAC is satisfied is enough to guarantee instability as (34a) will hold for $\bar{\mu} = 0$.   □

Remark 2 (Computing the inertia) It is not necessary to actually compute the eigenvalues of $J_{zz}f(z)$ in order to determine the inertia. A first option is to use the LBL decomposition [13, Appendix A], which decomposes $J_{zz}f(z)$ into the product LBL where L is a lower triangular matrix and B a block diagonal one, the inertia of B is the same as the inertia of $J_{zz}f(z)$.

Let $\Gamma = \text{diag}(\gamma\mathbf{1}_{n_x+m_x}, -\gamma\mathbf{1}_{n_y+m_y}, \gamma\mathbf{1}_{l_y+m_y}, -\gamma\mathbf{1}_{l_x+m_x})$, with γ a small positive number. A second approach is to use the LDL decomposition, to decompose $J_{zz}f(z) + \Gamma$ into the product LDL where L is a lower triangular matrix and D is a diagonal matrix; the inertia of D, which is given by the number of positive, negative and zero elements of the diagonal of D, gives the inertia of $J_{zz}f(z) + \Gamma$. The matrix Γ introduces a distortion in the inertia but it helps to stabilize the computation of the LDL decomposition, which tends to be faster than the LBL decomposition. This is the approach we use in our implementation; it has been studied in interior-point algorithms for minimization and the distortion introduced by Γ tends to be compensated by a better numerical algorithm [37, 38].   □

	Pure Newton			Alg 1, $\delta_\ell = 0$			Alg 1, $\delta_\ell = \infty$		
	Cnvg	Cnvg mm	Iter	Cnvg	Cnvg mm	Iter	Cnvg	Cnvg mm	Iter
f_1	1000	1000	4.1	1000	1000	4.0	937	937	5.7
f_2	1000	486	19	1000	997	14	890	890	13
f_3	700	49	6.0	1000	1000	5.0	979	979	5.0
f_4	1000	178	4.1	1000	322	4.9	320	320	4.9

Table 1: Comparing the performance of Algorithm 1 with two extreme parameters for δ_ℓ.

4.1 Benchmark example for unconstrained minmax

Consider the following functions

$$f_1(x, y) = 2x^2 - y^2 + 4xy + 4/3y^3 - 1/4y^4$$
$$f_2(x, y) = (4x^2 - (y - 3x + 0.05x^3)^2 - 0.1y^4)\exp\bigl(-0.01(x^2 + y^2)\bigr)$$
$$f_3(x, y) = (x - 0.5)(y - 0.5) + \exp\bigl(-(x - 0.25)^2 - (y - 0.75)^2\bigr)$$
$$f_4(x, y) = f_3(x, y) + 10x^2$$

which have been used as examples in [25, 29, 39] respectively, while $f_4(\cdot)$ we constructed ourselves. We have chosen these functions because, as we will show, they illustrate some interesting behaviors.

We want to investigate the performance of our algorithm for two extreme values of δ_ℓ, the switching parameter which chooses between when to guarantee stability, which will be $\delta_l = 0$ (*i.e.,* only verifying the LQAC without worrying about stability/instability) and $\delta_\ell = \infty$ (*i.e.,* always guarantee stability/instability even far away from an equilibrium point). As a baseline, we compare our algorithm with a "pure" Newton (PN) algorithm, using $\epsilon_x = \epsilon_y = 0$.

The pure Newton and Algorithm 1 with the different δ_ℓ are each initialized 1000 times, using the same initialization for the three of them each time. We compare their convergence properties according to three criteria: the number of times the algorithm converged (Cnvg), the number of times it converged to a local minmax point (Cnvg mm) and the average number of iterations to converge to a local minmax point (Iter). The algorithm is terminated when the infinity norm of the gradient is smaller than $\delta_s = 10^{-5}$ and we declare that they did not converge if it has not terminated in less than 100 iterations or (in the case of the pure Newton algorithm) the solver failed to invert the Hessian. The result of the comparison is displayed in Table 1. We can make the following observations from this comparison.

- The pure Newton algorithm is not suited to solve minmax problems. Except for f_1, it has substantially worst convergence towards minmax points compared to Algorithm 1. This suggests that by modifying the Newton method with ϵ_x and ϵ_y such that LQAC are satisfied, the directions (d_x, d_y) update (x, y) to the correct place.
- By selecting $\delta_\ell = \infty$ in Algorithm 1, we are able to guarantee that the Newton iterations only converge towards local minmax points, as predicted

by Theorem 2. However, if we compare these results to the case where we set $\delta_\ell = 0$, it seems that guaranteeing stability decreases the basin of attraction of equilibrium points, without increasing the basin of attraction of local minmax points. In $f_2(\cdot)$ the basin of attraction reduction impacts the convergence towards local minmax, with $\delta_\ell = 0$ converging to more local minmax than in the case of $\delta_\ell = 0$. In $f_4(\cdot)$, taking $\delta_\ell = \infty$ seems to only reduce the basin of attraction of equilibrium points that are not local minmax, but this does not translate to larger basin of attractions for local minmax points.

4.2 The homicidal chauffeur example for constrained minmax

In the homicidal chauffeur problem, a pursuer driving a car is trying to hit a pedestrian, who (understandably) is trying to evade it. The pursuer is modeled as a discrete time Dubins's vehicle with equations

$$x_p^+ = \begin{bmatrix} x_p^{(1)} + v \cos x_p^{(3)} \\ x_p^{(2)} + v \sin x_p^{(3)} \\ x_p^{(3)} + u \end{bmatrix} =: \phi_p(x_p, u)$$

where $x_p^{(i)}$ designates the i$^{\text{th}}$ element of the vector x_p, v is a constant forward speed and u is the steering, over which the driver has control. The pedestrian is modeled by the accumulator

$$x_e^+ = x_e + d =: \phi_e(x_e, d)$$

where d is the velocity vector. Given a time horizon T, and initial positions $x_e(t)$ and $x_p(t)$, we want to solve

$$\min_{U \in \mathcal{U}} \quad \max_{D \in \mathcal{D}} \sum_{i=0}^{T-1} \gamma_x \left\| x_p^{(1,2)}(t+i+1) - x_e(t+i+1) \right\|_2^2 + \gamma_u u(t+i)^2 - \gamma_d d(t+i)_2^2$$

$$(37)$$

where $x_p^{(1,2)}$ designates the first and second elements of the vector x_p; γ_x, γ_u and γ_d are positive weights; and $U, \mathcal{U}, D$ and $\mathcal{D}$ are defined for $i = 0, \dots, T-1$

$$\begin{aligned}
U &:= u(t+i), x_p(t+i+1) \\
\mathcal{U} &:= u(t+i), x_p(t+i+1) : \ |u(t+i)| \leq u_{max}, \\
&\qquad\qquad x_p(t+i+1) = \phi_p\big(x_p(t+i), u(t+i)\big) \\
D &:= d(t+i), x_e(t+i+1) \\
\mathcal{D} &:= \{d(t+i), x_e(t+i+1) : \ d(t+i)_2 \leq d_{max}, \\
&\qquad\qquad x_e(t+i+1) = \phi_e\big(x_e(t+i), d(t+i)\big)\}.
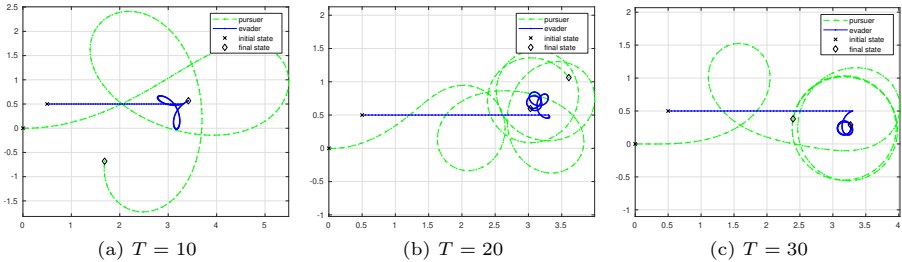\end{aligned}$$

Fig. 2: Trajectory for Homicidal Chauffeur problem with 200 time steps for different horizons T.

Instead of explicitly computing the solution of the trajectory of the pursuer and evaders, we are implicitly computing them by setting the dynamics as equality constraints. While this increases the number of variables to solve the optimization for, because the problem has stage costs and constraints, the number of nonzero entries of $J_{zz}f(z)$ grows in $O(T)$, instead of growing in $O(T^2)$ if we had chosen to solve the dynamical equations. As we will show shortly, this has important impact on the scalability of the algorithm.

Each player is controlled using Model Predictive Control, meaning that at each time step t we solve (37) obtaining controls $u(t)$ and $d(t)$, which are then used to control the system for the next time step. In Figure 2 we show the different trajectories over 200 time steps of the pursuer and evader for three different future horizon predictions.

We want to study the scalability of the algorithm by enlarging the horizon T. The number of variables and the number of constraints increase in $O(T)$, while the number of elements of $J_{zz}f(\cdot)$ increases in $O(T^2)$. However, as we can see from the sparsity pattern in Figure 3b, most of the entries in the Hessian are actually structurally zero (meaning they are always zero), and the number of nonzero elements in $J_{zz}f(\cdot)$ scales in $O(T)$. `TensCalc`'s implementation of the LDL factorization exploits sparsity patterns and scales roughly with the number of nonzero entries of the matrix, which makes it substantially more efficient than standard LDL decomposition, which scales in $O(T^3)$ [13, Appendix A]. At each step of Algorithm 1, most of the time is spent computing the LDL decomposition, either for adjusting ϵ_x and ϵ_y or to invert $H_{zz}f(z)$. As a consequence, we can see in Figure 3a that both the number of iterations necessary to solve the optimization as well as the time per iteration scale roughly linear, the first being multiplied by about 1.7 while the second by 3.5 while the horizon length T is multiplied by roughly 30.

Remark 3 (Minmax problems with shared dynamics) In the homicidal chauffeur, the control of the pursuer does not impact the *dynamics* of the evader, and vice versa. This is why in (37) the dynamics can be set as equality constraints independently for the min and for the max.
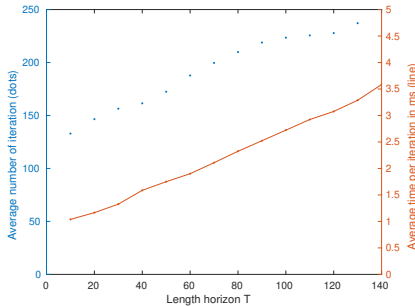
(a) Computational scaling for solving homicidal chauffeur per horizon length
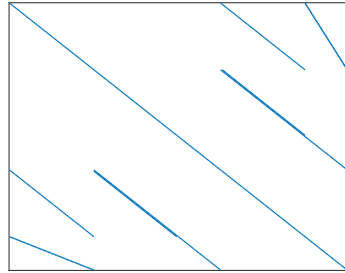
(b) Structural sparsity pattern of $J_{zz}f(z)$

Fig. 3: Scaling of homicidal chauffeur with horizon length and sparsity pattern of the Hessian

Now consider the problem

$$x^+ = f(x, u, d)$$

where u is the control and d is the disturbance and one wants to minimize a cost function $V(x(1), \ldots, x(T), u(0), \ldots, u(T-1))$ given the worst disturbance $d(1), \ldots, d(T)$. Because both the control and the disturbances influence the dynamics, we need to include the dynamics as equality constraints for the *maximization*, leading to the optimization problem

$$\min_{u(i) \in \mathcal{U}, i=0,\ldots,T-1} \max_{\substack{d(i) \in \mathcal{D}, x(i+1), i=0,\ldots,T-1: \\ x(i+1)=f(x(i),u(i),d(i))}} V\Big(x(1), \ldots, x(T), u(0), \ldots, u(T-1)\Big)$$

where $\mathcal{U}, \mathcal{D}$ are the feasible sets for the control and disturbances. It is important to notice that x just acts as a latent/dummy variable that allows us to avoid solving the trajectory equation. Setting it as a maximization variable does not changes the result as x is always exactly determined by the value of u and d. It does, however, improves the numerical efficiency of the algorithm as now the Hessian matrices are sparse and their LDL decomposition can be efficiently computed. $\qquad\square$

5 Conclusion

The main contribution of this article is the construction of Newton and interior-point algorithm for nonconvex-nonconcave minmax optimization. We show that by modifying the Newton/interior-point matrices such that the update steps can be seen as the solution of quadratic programs that locally approximate the minmax problem, we are able to make progress towards a solution even far away from it. While we show that these conditions are not enough to guarantee that the iterations are only stable at local minmax points, numerical results seem to indicate that guaranteeing the stability does not improve the convergence of the algorithm, and actually can even impair it.

The main future direction would be to obtain a method to generate instability that also improves convergence. A possible attempt would be to use trust region methods. Another future direction would be the development of merit functions for minmax problems. We used local quadratic approximation to obtain a proxy for what could be descent-ascent directions, but ideally what one would want to measure is how good of an improvement they are. Merit functions could also be used to establish some sort of Armijo rule which could be used to improve convergence.

Appendix A Proof of Proposition 2: Constrained second order sufficient conditions

First, $g(z, 0) = 0$ implies that $s_x = -F_x(x)$ and $s_y = -F_y(x, y)$, thus $\lambda_y \odot F_y(x, y) = \mathbf{0}$ and $\lambda_x \odot F_x(x) = \mathbf{0}$, and therefore $g(z, 0)$ is equivalent to the first order necessary condition in [12]. For the rest of the proof, let us substitute $s_x = -F_x(x)$ and $s_y = -F_y(x, y)$ back in the definitions of $H_{zz}f(z)$ and $H_{yy}f(z)$ as defined in (27).

Let $h_{xy}f(z)$ be the first block row of $H_{xy}f(z)$ (*i.e.,* without the block row of zeros). Because strict complementarity holds, the solution cones of the maximizer and minimizer are $\mathcal{C}_y(x, y) := \{d_y \in \mathbb{R}^{n_y}\backslash\{0\} : \boldsymbol{\nabla}_y G_y(x, y)d_y = 0, \boldsymbol{\nabla}_y f_y^{(i)}(x, y)d_y = 0, i = 1 \in \mathcal{A}_y(x, y)\}$ and $\mathcal{C}_x(x) := \{d_x \in \mathbb{R}^{n_x}\backslash\{0\} : \boldsymbol{\nabla}_x G_x(x)d_x = 0, \boldsymbol{\nabla}_x f_x^{(i)}(x)d_x = 0, i = 1 \in \mathcal{A}_x(x)\}$, respectively, where $\mathcal{A}_y(x, y)$ and $\mathcal{A}_x(x)$ are defined in (26). According to [12, Theorem 3.2], if $g(z, 0) = \mathbf{0}$ and

$$d_y' \boldsymbol{\nabla}_{yy} L(z) d_y < 0 \ \forall \ d_y \in \mathcal{C}_y(x, y) \tag{A1a}$$

and

$$d_x' \left(\boldsymbol{\nabla}_{xx} L(z) - h_{xy} f(z) H_{yy} f(z)^{-1} h_{xy} f(z)' \right) d_x > 0 \ \forall \ d_x \in \mathcal{C}_x(x) \tag{A1b}$$

then (x, y) is a local minmax of (13). We will show that the conditions in our proposition are equivalent to these. For the sake of notation convenience we will drop the dependency on the variable with the understanding the functions are being taken at $(x, y, \nu_x, \nu_y, \lambda_x, \lambda_y)$.

Let us start by the part on $\text{In}(H_{yy}f)$. First, reformulate the inner maximization using only equality constraints, *i.e.,*

$$\max_{y, w : G_y(x, y) = 0, F_y(x, y) + \frac{1}{2} w \odot w = 0} f(x, y).$$

Now all the constraints are active and for any solution $w = \sqrt{-F_y}$. Let

$$H_{y\lambda}f := \begin{bmatrix} \boldsymbol{\nabla}_y G_y & \boldsymbol{\nabla}_y F_y \\ \mathbf{0} & -\operatorname{diag}(\sqrt{-F_y}) \end{bmatrix}.$$

As the linear independence constraint qualification – which imply that the Lagrange multipliers are unique – and strict complementarity hold at z, we can use the so called projected Hessian form of the second order conditions [13, Chapter 12.5]. In this case, the sufficient condition (A1a) for the reformulated problem is equivalent to

$$Z'_y \begin{bmatrix} \boldsymbol{\nabla}_{yy}L & \mathbf{0} \\ \mathbf{0} & -\operatorname{diag}(\lambda_y) \end{bmatrix} Z_y \prec 0, \tag{A2}$$

where $Z_y \in \mathbb{R}^{n_y+m_y, n_y+m_y-m_y-l_y}$ is such that $H_{y\lambda}f' Z_y = 0$ and $[H_{y\lambda}f, Z_y]$ is full rank. The equivalence between (A1a) and (A2) comes from the constraints for the reformulated problem being all active, and therefore the solution cone $C_y(x,(y,w))$ is equal to the null space of $H_{y\lambda}f'$. Finally, according to [13, Theorem 16.3]

$$\operatorname{In}(H_{yy}f) = \operatorname{In}\left(Z'_y \begin{bmatrix} \boldsymbol{\nabla}_{yy}L(z) & \mathbf{0} \\ \mathbf{0} & -\operatorname{diag}(\lambda_y) \end{bmatrix} Z_y \right) + (l_y + m_y, l_y + m_y, 0).$$

Therefore (A2) holds if and only if $\operatorname{In}(H_{yy}f) = (l_y + m_y, n_y + m_y, 0)$. This finishes this part of the proof.

In an analogous way than for the inner maximization, reformulate the outer minimization using equality constraints and construct Z_x analogously to Z_y for $H_{x\lambda}f'$. Then the sufficient conditions (A1b) for the reformulated outer minimization is

$$Z'_x \left(H_{xx}f - H_{yx}f' H_{yy}f^{-1} H_{yx}f \right) Z_x \succ 0. \tag{A3}$$

We can verify that the matrix matrix

$$\bar{Z} := \begin{bmatrix} Z_x & \mathbf{0}_{n_x+m_x,n_y+m_y+l_y+m_y} \\ \mathbf{0}_{n_y+m_y+l_y+m_y,n_x-l_x} & I_{n_y+m_y+l_y+m_y,} \end{bmatrix}$$

is full column rank and its columns spans the null space of $[H_{x\lambda}f', \mathbf{0}_{l_x+m_x,n_y+m_y+l_y+m_y}]$ (the subscripts in $\mathbf{0}$ and I denote the sizes). Applying again [13, Theorem 16.3] but now to $H_{zz}f$, with the correct partitioning gives

$$\operatorname{In}(H_{zz}f)$$
$$= \operatorname{In}\left(\bar{Z}' \begin{bmatrix} H_{xx}f & H_{xy}f \\ H_{xy}f' & H_{yy}f \end{bmatrix} \bar{Z} \right) + (l_x + m_x, l_x + m_x, 0)$$
$$= \operatorname{In}\left(\begin{bmatrix} Z'_x H_{xx}f Z_x & Z'_x H_{xy}f \\ H_{xy}f' Z_x & H_{yy}f \end{bmatrix} \right) + (l_x + m_x, l_x + m_x, 0)$$
$$= \operatorname{In}\left(Z'_x \left(H_{xx}f - H_{xy}f H_{yy}f^{-1} H_{xy}f' \right) Z_x \right) + \operatorname{In}(H_{yy}f) + (l_x + m_x, l_x + m_x, 0)$$

where the last equality comes from Haynsworth inertia additivity formula [36, Theorem 1.6]. Therefore, if (A1a) holds, (A1b) is equivalent to

$$\text{In}(H_{zz}f) = (n_x - l_x, 0, 0) + (l_y + m_y, n_y + m_y, 0) + (l_x + m_x, l_x + m_x, 0)$$

which finishes the proof. $\square$

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