Lyapunov small-gain theorems for networks of not necessarily ISS hybrid systems ⋆

Andrii Mironchenko [a], Guosong Yang [b], Daniel Liberzon [b]

[a] *Faculty of Computer Science and Mathematics, University of Passau, Innstraße 33, 94032 Passau, Germany*

[b] *Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 W. Main St., Urbana, IL 61801 U.S.A.*

Abstract

We prove a novel Lyapunov-based small-gain theorem for networks composed of $n \geq 2$ hybrid systems which are not necessarily input-to-state stable. This result unifies and extends several small-gain theorems for hybrid and impulsive systems proposed in the last few years. We also show how average dwell-time (ADT) clocks and reverse ADT clocks can be used to modify the ISS Lyapunov functions for subsystems and to enlarge the applicability of the derived small-gain theorems.

Key words: hybrid systems, input-to-state stability, small-gain theorems.

1 Introduction

The study of interconnections plays a significant role in system theory, as it allows one to establish stability for a complex system based on properties of its less complex components. In this context, small-gain theorems prove to be useful and general in analyzing feedback interconnections, which are ubiquitous in the control literature. An overview of classical small-gain theorems involving input-output gains of linear systems can be found in [13]. In [19,29], the small-gain technique was extended to nonlinear feedback systems within the input-output context. The next peak in the stability analysis of interconnections was reached based on the input-to-state stability (ISS) framework proposed in [37], which unified the notions of internal and external stability. Nonlinear small-gain theorems for general feedback interconnections of two ISS systems were introduced in [21,20]. Their generalization to networks composed of $n \geq 2$ ISS systems were reported in [11,12], with several variations summarized in [5].

The results described above have been developed for continuous-time systems (i.e., ordinary differential equations). In the discrete-time context, small-gain theorems for general feedback interconnections of two ISS systems were established in [22,25], and their generalization to networks composed of $n \geq 2$ ISS system can be found in [28]. However, in modeling real-world phenomena one often has to consider interactions between continuous and discrete dynamics. A general framework for modeling such behaviors is the hybrid systems theory [16,14]. In this work we adopt the hybrid system model in [14], which proves to be natural and general from the viewpoint of Lyapunov stability theory [3,4]. The notions of input-to-state stability and ISS Lyapunov functions were extended for this class of hybrid systems in [2].

Due to their interactive nature, many hybrid systems can be inherently modeled as feedback interconnections [27, Section V]. During recent years great efforts have been devoted to the development of small-gain theorems for interconnected hybrid systems. Trajectory-based small-gain theorems for interconnections of two hybrid systems were reported in [34,23,6], while Lyapunov-based formulations were proposed in [26,35,27]. Some of these results were extended to networks composed of $n \geq 2$ ISS hybrid systems in [6].

A more challenging problem is the study of hybrid systems in which either the continuous or the discrete dynamics is destabilizing (non-ISS). In this case, input-

⋆ This paper was not presented at any IFAC meeting. Corresponding author A. Mironchenko. Tel. +49-851-509-3363.

Email addresses: `andrii.mironchenko@uni-passau.de` (Andrii Mironchenko), `yang150@uiuc.edu` (Guosong Yang), `liberzon@uiuc.edu` (Daniel Liberzon).

to-state stability is usually achieved under restrictions on the frequency of discrete events, such as dwell-time [33], average dwell-time (ADT) [18] and reverse average dwell-time (RADT) [17]. For interconnections of such hybrid systems, the small-gain theorems established in [6,27] cannot be applied directly. The results of [27] show that one can modify the non-ISS dynamics in subsystems by first adding auxiliary clocks and then constructing ISS Lyapunov functions for the augmented subsystems that decrease both during flow and at jumps. One advantage of this method is that it can be applied even if the non-ISS dynamics are of different types (i.e., if in some subsystems the continuous dynamics are non-ISS, and in some other ones the discrete dynamics are non-ISS). However, such modifications will lead to enlarged Lyapunov gains of subsystems, and hence make the small-gain condition more restrictive.

Another type of small-gain theorems was proposed in [7,9] for interconnected impulsive systems with continuous or discrete non-ISS dynamics. The first step in this method is to construct a candidate exponential ISS Lyapunov function for the interconnection. Provided that the non-ISS dynamics of subsystems are of the same type (i.e., when the continuous dynamics of all subsystems or the discrete dynamics of all subsystems are ISS), the aforementioned candidate exponential ISS Lyapunov function can be used to establish input-to-state stability of the interconnection under suitable ADT/RADT conditions. Compared with the previous approach, this one doesn't require modifications of subsystems, and hence preserves the Lyapunov gains and small-gain conditions. However, this method has been developed only for impulsive systems and requires candidate exponential ISS Lyapunov functions for subsystems. Moreover, it cannot be applied to interconnections of subsystems with different types of non-ISS dynamics.

In this paper we unify the two methods above. In Section 2, we introduce the modeling framework and main definitions, followed by a Lyapunov-based sufficient condition for ISS of hybrid systems with continuous or discrete non-ISS dynamics. In Section 3, we establish a general small-gain theorem for $n \geq 2$ interconnected hybrid systems by constructing a candidate ISS Lyapunov function for the interconnection, which generalizes the Lyapunov-based small-gain theorems from [35,7,6,9,27]. In the same section, we also derive several implications of the general result, in particular, a small-gain theorem for interconnections of subsystems with the same type of non-ISS dynamics and also candidate exponential ISS Lyapunov functions with linear Lyapunov gains. In Section 4 we propose a version of the method of modifying ISS Lyapunov functions for subsystems from [27], in which a smaller number of subsystems are affected (and hence fewer Lyapunov gains are enlarged). In Section 5, we summarize the results of this work as a unified method for establishing ISS of interconnected hybrid systems, and conclude the paper with an outlook on future research.

A preliminary and shortened version of the paper has been presented at the 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS 2014) [31].

2 Framework for hybrid systems

Let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. Given a vector $x \in \mathbb{R}^N$, denote by $|x|$ its Euclidean norm, and by $|x|_\mathcal{A}$ its Euclidean distance to a set $\mathcal{A} \subset \mathbb{R}^N$, namely, $|x|_\mathcal{A} := \inf_{y\in\mathcal{A}} |x - y|$. Given n vectors $x_1, \ldots, x_n$, denote by $(x_1, \ldots, x_n) := (x_1^\top, \ldots, x_n^\top)^\top$ their concatenation. Given two vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we say that $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, n$ and $x > y$ if $x_i > y_i$ for all $i = 1, \ldots, n$. We write $x \ngeq y$ if there is $i \in \{1, \ldots, n\}$ for which $x_i < y_i$. Given a set $\mathcal{A}$, denote by $\overline{\mathcal{A}}$ and $\operatorname{int}\mathcal{A}$ its closure and interior, respectively.

Denote by id the identity function. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{PD}$ if it is continuous and positive-definite (i.e., $\alpha(r) = 0 \Leftrightarrow r = 0$); it is of class $\mathcal{K}$ if $\alpha \in \mathcal{PD}$ and is strictly increasing; it is of class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{t\to\infty} \gamma(t) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t, and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed $r > 0$.

Motivated by [2], a hybrid system is modeled as the combination of a continuous flow and discrete jumps

$$
\begin{aligned}
\dot{x} &\in F(x, u), \qquad && (x, u) \in \mathcal{C}, \\
x^+ &\in G(x, u), \qquad && (x, u) \in \mathcal{D},
\end{aligned}
\tag{1}
$$

where $x \in \mathcal{X} \subset \mathbb{R}^N$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^M$ is the input, $\mathcal{C} \subset \mathcal{X} \times \mathcal{U}$ is the flow set, $\mathcal{D} \subset \mathcal{X} \times \mathcal{U}$ is the jump set, $F : \mathcal{C} \rightrightarrows \mathbb{R}^N$ is the flow map (here by $\rightrightarrows$ we mean that F is a set-valued function, which maps each element of $\mathcal{C}$ to a subset of $\mathbb{R}^N$), and $G : \mathcal{D} \rightrightarrows \mathcal{X}$ is the jump map. (In this model, the dynamics of (1) is continuous in $\mathcal{C}\backslash\mathcal{D}$, and discrete in $\mathcal{D}\backslash\mathcal{C}$. In $\mathcal{C} \cap \mathcal{D}$ it can be either continuous or discrete.) The hybrid system (1) is fully characterized by its *data* $\mathcal{H} := (F, G, \mathcal{C}, \mathcal{D}, \mathcal{X}, \mathcal{U})$.

Solutions of (1) are defined on hybrid time domains. A set $E \subset \mathbb{R}_+ \times \mathbb{N}$ is called a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$. It is a *hybrid time domain* if $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in E$. On a hybrid time domain there is a natural ordering of points, that is, $(s, k) \preceq (t, j)$ if $s + k \leq t + j$, and $(s, k) \prec (t, j)$ if $s + k < t + j$.

A function defined on a hybrid time domain is called a *hybrid signal*. A hybrid signal $x : \operatorname{dom} x \to \mathcal{X}$ (defined on the hybrid time domain $\operatorname{dom} x$) is called a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous for each j. A hybrid signal $u : \operatorname{dom} u \to \mathcal{U}$ is called a *hybrid input* if $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid arc $x : \operatorname{dom} x \to \mathcal{X}$ and a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ form a *solution pair* (x, u) of (1) if

- $\operatorname{dom} x = \operatorname{dom} u$ and $(x(0,0), u(0,0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$, where $x(t, j)$ represents the state of the hybrid system at time t and after j jumps (i.e. at hybrid time (t, j));
- for each $j \in \mathbb{N}$, it holds that $(x(t, j), u(t, j)) \in \mathcal{C}$ for all $t \in \operatorname{int} I_j$, and $\dot{x}(t, j) \in F(x(t, j), u(t, j))$ for almost all $t \in I_j$, where $I_j := \{t : (t, j) \in \operatorname{dom} x\}$;
- for each $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$, it holds that $(x(t, j), u(t, j)) \in \mathcal{D}$ and $x(t, j+1) \in G(x(t, j), u(t, j))$.

With proper assumptions on the data $\mathcal{H}$, one can establish local existence of solutions, which are not necessarily unique (see, e.g., [14, Proposition 2.10]). A solution pair (x, u) is *maximal* if it cannot be extended, and *complete* if $\operatorname{dom} x$ is unbounded. In this paper we only consider maximal solution pairs.

Following [2], the essential supremum norm of a hybrid signal u up to a hybrid time (t, j) is defined by

$$\|u\|_{(t,j)} := \max\left\{ \operatorname*{ess\,sup}_{\substack{(s,k)\in\operatorname{dom} u,\\ (s,k)\preceq(t,j)}} |u(s,k)|, \sup_{\substack{(s,k)\in J(u),\\ (s,k)\preceq(t,j)}} |u(s,k)| \right\},$$

where $J(x) := \{(s, k) \in \operatorname{dom} u : (s, k+1) \in \operatorname{dom} u\}$ is the set of jump times. In particular, the set of measure 0 of hybrid times that are ignored in computing the essential supremum norm cannot contain any jump time.

For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, its *Dini derivative* at a point $x \in \mathbb{R}^n$ in the direction $y \in \mathbb{R}^n$ is defined by

$$\dot{V}(x; y) = \overline{\lim_{h\searrow 0}} \frac{V(x + hy) - V(x)}{h},$$

where $\overline{\lim}$ denotes the limit superior.

In this work, we study input-to-state stability (ISS) properties of the hybrid system (1) via ISS Lyapunov functions. Let $\mathcal{A} \subset \mathcal{X}$ be a compact set.

Definition 1. Following [27], we say that a set of solution pairs $\mathcal{S}$ of (1) is *pre-input-to-state stable (pre-ISS) w.r.t.* $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{S}$,

$$|x(t, j)|_{\mathcal{A}} \leq \max\{\beta(|x(0, 0)|_{\mathcal{A}}, t + j),\ \gamma(\|u\|_{(t,j)})\} \quad (2)$$

for all $(t, j) \in \operatorname{dom} x$. If $\mathcal{S}$ contains all solution pairs of (1), then we say that (1) is *pre-ISS w.r.t.* $\mathcal{A}$. In addition, if all solution pairs are complete then we say that (1) is *ISS w.r.t.* $\mathcal{A}$.

Remark 1. If (2) holds with $\gamma \equiv 0$, then the set $\mathcal{S}$ is *globally pre-asymptotically stable (pre-GAS)*, which implies that all complete solution pairs in $\mathcal{S}$ converge to $\mathcal{A}$. In addition, if all solution pairs in $\mathcal{S}$ are complete then it is *globally asymptotically stable (GAS)* [27].

Remark 2. In [2], ISS of hybrid systems is defined in terms of class $\mathcal{KLL}$ functions and without requiring all solution pairs to be complete, which is equivalent to our definition of pre-ISS of hybrid systems with $\mathcal{KL}$ functions [3, Lemma 6.1].

Definition 2. For the hybrid system (1), a function $V : \mathcal{X} \to \mathbb{R}_+$ is a *candidate ISS Lyapunov function w.r.t.* $\mathcal{A}$ if it is locally Lipschitz outside $\mathcal{A}$,[1] and

1. there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that
$$\psi_1(|x|_{\mathcal{A}}) \leq V(x) \leq \psi_2(|x|_{\mathcal{A}}) \qquad \forall\, x \in \mathcal{X}; \quad (3)$$
2. there exist a gain function $\chi \in \mathcal{K}$ and a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that for all $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$,
$$V(x) \geq \chi(|u|) \Rightarrow \dot{V}(x; y) \leq -\varphi(V(x)) \qquad \forall\, y \in F(x, u); \quad (4)$$
3. there is a function $\alpha \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{D}$,[2]
$$V(x) \geq \chi(|u|) \Rightarrow V(y) \leq \alpha(V(x)) \qquad \forall\, y \in G(x, u). \quad (5)$$

In addition, if (4) and (5) hold with

$$\varphi(r) \equiv cr, \qquad \alpha(r) \equiv e^{-d} r \quad (6)$$

for some constants $c, d \in \mathbb{R}$, then V is a *candidate exponential ISS Lyapunov function w.r.t.* $\mathcal{A}$ with *rate coefficients* c, d.

The following lemma gives an alternative characterization of the candidate ISS Lyapunov function, which will be useful in formulating the small-gain theorems in Section 3.

Lemma 1. *A function $V : X \to \mathbb{R}_+$ is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for the hybrid system (1) if and only if it is locally Lipschitz outside $\mathcal{A}$, and*

[1] The Lipschitz condition here is used to ensure the existence of the Dini derivative in (4), and it can be relaxed to that the function V is locally Lipschitz on an open set containing all $x \notin \mathcal{A}$ such that $(x, u) \in \mathcal{C}$ for some $u \in \mathcal{U}$.

[2] There is no loss of generality in requiring $\alpha \in \mathcal{K}$ instead of $\alpha \in \mathcal{PD}$, as a class $\mathcal{PD}$ function can always be majorized by a class $\mathcal{K}$ one. Meanwhile, $\alpha \in \mathcal{K}$ is required in establishing the small-gain theorems below, as explained in footnote 4.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

1. *there exist functions* $\psi_1, \psi_2 \in \mathcal{K}_\infty$ *such that* (3) *holds;*
2. *there exist a gain function* $\bar{\chi} \in \mathcal{K}$ *and a continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *with* $\varphi(0) = 0$ *such that for all* $(x, u) \in \mathcal{C}$ *with* $x \notin \mathcal{A}$,
$$\begin{aligned} V(x) &\geq \bar{\chi}(|u|) \\ &\Rightarrow \dot{V}(x; y) \leq -\varphi(V(x)) \qquad \forall\, y \in F(x, u); \end{aligned} \quad (7)$$
3. *there exists a function* $\alpha \in \mathcal{K}$ *such that for all* $(x, u) \in \mathcal{D}$,
$$V(y) \leq \max\{\alpha(V(x)),\, \bar{\chi}(|u|)\} \quad \forall\, y \in G(x, u). \quad (8)$$

Proof. The proof is along the lines of the proof of [9, Proposition 1] for ISS Lyapunov functions for impulsive systems, and is omitted here. □

Exponential ISS Lyapunov functions can be characterized in a similar way. Note that the functions χ in Definition 2 and $\bar{\chi}$ in Lemma 1 are different in general.

The notion of candidate ISS Lyapunov function is defined to characterize the effects of destabilizing dynamics in a hybrid system. In Definition 2, it is not required that $\varphi \in \mathcal{PD}$ or $\alpha < \mathrm{id}$ on $(0, \infty)$ (i.e., V does not necessarily decrease along solutions of the hybrid system). If both of these conditions hold then V becomes an *ISS Lyapunov function*, and similar analysis to the proof of [2, Proposition 2.7] can be used to show that the hybrid system (1) is pre-ISS (note that ISS in [2] means pre-ISS in this paper; see Remark 2). Moreover, if only one of them holds,[3] we are still able to establish ISS for some sets of solution pairs of (1) satisfying suitable conditions on the density of jumps (i.e., the number of jumps per unit interval of continuous time).

Proposition 1. *Let* V *be a candidate exponential ISS Lyapunov function w.r.t.* $\mathcal{A}$ *for* (1) *with rate coefficients* c, d. *For arbitrary constants* $\eta, \lambda, \mu > 0$, *let* $\mathcal{S}[\eta, \lambda, \mu]$ *denote the set of solution pairs* (x, u) *for which all* $(s, k) \preceq (t, j)$ *in the hybrid time domain* $\operatorname{dom} x$ *satisfy*
$$-(d - \eta)(j - k) - (c - \lambda)(t - s) \leq \mu. \quad (9)$$
Then $\mathcal{S}[\eta, \lambda, \mu]$ *is pre-ISS w.r.t.* $\mathcal{A}$.

Proof. The proof is along the lines of the proof of [17, Theorem 1] for ISS of impulsive systems. Let the function χ be as in (4). Consider an arbitrary solution pair $(x, u) \in \mathcal{S}[\eta, \lambda, \mu]$. Let $(t_0, j_0), (t_1, j_1) \in \operatorname{dom} x$ be such that $(t_0, j_0) \preceq (t_1, j_1)$. If
$$V(x(s, k)) \geq \chi(\|u\|_{(s,k)}) \quad (10)$$
for all $(s, k) \in \operatorname{dom} x$ such that $(t_0, j_0) \preceq (s, k) \preceq (t_1, j_1)$, then (4)–(6) imply that
$$\begin{aligned} V(x(t_1, j_1)) &\leq e^{-d(j_1 - j_0) - c(t_1 - t_0)} V(x(t_0, j_0)) \\ &\leq e^{-\eta(j_1 - j_0) - \lambda(t_1 - t_0) + \mu} V(x(t_0, j_0)), \end{aligned} \quad (11)$$
where the last inequality follows from (9). Now consider an arbitrary $(t, j) \in \operatorname{dom} x$. If (10) holds for all $(s, k) \in \operatorname{dom} x$ such that $(s, k) \preceq (t, j)$, then (11) implies that
$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j) \quad (12)$$
with the function $\beta \in \mathcal{KL}$ defined by
$$\beta(r, l) := \psi_1^{-1}\big(e^{-l \min\{\eta,\, \lambda\} + \mu} \psi_2(r)\big). \quad (13)$$
Otherwise, let
$$(t', j') = \operatorname*{argmax}_{\substack{(s,k) \in \operatorname{dom} x, \\ (s,k) \preceq (t,j)}} \{s + k : V(x(s, k)) \leq \chi(\|u\|_{(s,k)})\}.$$
Then (10) holds for all $(s, k) \in \operatorname{dom} x$ such that $(t', j') \prec (s, k) \preceq (t, j)$, and hence (11) implies that
$$\begin{aligned} V(x(t, j)) &\leq e^{-\eta(j - j') - \lambda(t - t') + \mu} \max\{1, e^{-d}\} V(x(t', j')) \\ &\leq e^{\mu} \max\{1,\, e^{-d}\} \chi(\|u\|_{(t',j')}) \\ &\leq e^{\mu} \max\{1,\, e^{-d}\} \chi(\|u\|_{(t,j)}), \end{aligned}$$
where the term $\max\{1,\, e^{-d}\}$ is needed when $(t', j'+1) \in \operatorname{dom} x$ with $V(x(t', j')) < \chi(\|u\|_{(t',j')})$ and $V(x(t', j' + 1)) \geq \chi(\|u\|_{(t',j'+1)})$, and the second inequality is due to $\eta, \lambda > 0$. Hence
$$|x(t, j)|_{\mathcal{A}} \leq \gamma(\|u\|_{(t,j)}) \quad (14)$$
with the function $\gamma \in \mathcal{K}$ defined by
$$\gamma(r) := \psi_1^{-1}\big(e^{\mu} \max\{1,\, e^{-d}\} \chi(r)\big).$$
Combining (12) and (14), we obtain that (2) holds for all $(x, u) \in \mathcal{S}[\eta, \lambda, \mu]$ and all $(t, j) \in \operatorname{dom} x$. □

Remark 3. We observe that, for a complete solution pair (x, u), the inequality (9) cannot hold with both $c, d < 0$, since there is always a large enough t or j such that $\eta j + \lambda t > \mu$. However, it may hold with both $c, d < 0$ for a solution pair defined on a bounded hybrid time domain. Moreover, provided that $c > 0$, the claim of Proposition 1 also holds for $\eta = 0$. The proof is completely analogous. In particular, the last inequality in (11) becomes
$$\begin{aligned} &e^{-d(j_1 - j_0) - c(t_1 - t_0)} V(x(t_0, j_0)) \\ &\leq e^{-\lambda(t_1 - t_0) + \mu} V(x(t_0, j_0)) \\ &\leq e^{(\lambda^2/c - \lambda)(t_1 - t_0) - \lambda^2(t_1 - t_0)/c + \mu} V(x(t_0, j_0)) \\ &\leq e^{\lambda d(j_1 - j_0)/c - \lambda^2(t_1 - t_0)/c + (1 + \lambda/c)\mu} V(x(t_0, j_0)), \end{aligned}$$

[3] Namely, either the continuous or the discrete dynamics taken alone is ISS, but not both; see [37] and [22] for the definitions of ISS for continuous and discrete dynamics, respectively.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

where the first inequality follows from (9) with $\eta = 0$, and the last one comes from the estimate

$$e^{(\lambda^2/c-\lambda)(t_1-t_0)} = e^{(\lambda/c)(\lambda-c)(t_1-t_0)} \leq e^{(\lambda/c)(d(j_1-j_0)+\mu)}.$$

Hence (13) becomes

$$\beta(r,l) := \psi_1^{-1}\big(e^{-l\min\{-\lambda d/c,\,\lambda^2/c\}+(1+\lambda/c)\,\mu}\psi_2(r)\big).$$

Analogously, when $d > 0$, the claim of Proposition 1 also holds for $\lambda = 0$.

Remark 4. If $d < 0$, we can divide both sides of (9) by $-(d-\eta) > 0$ to transform it to the average dwell-time (ADT) condition [18]. Analogously, if $c < 0$, we can divide both sides of (9) by $-(c-\lambda) > 0$ to transform it to the reverse average dwell-time (RADT) condition [17].

Given a candidate exponential ISS Lyapunov function for (1) with rate coefficients $c > 0$ and/or $d > 0$, we can determine pre-ISS sets of solution pairs via Proposition 1. In the following section, we investigate the formulation of such functions for interconnections of hybrid systems.

3 Interconnections and small-gain theorems

We are interested in the case where the hybrid system (1) is decomposed as

$$\begin{aligned} \dot{x}_i &\in F_i(x,u), \quad i = 1,\dots,n, \qquad && (x,u) \in \mathcal{C}, \\ x_i^+ &\in G_i(x,u), \quad i = 1,\dots,n, \qquad && (x,u) \in \mathcal{D}, \end{aligned} \tag{15}$$

where $x := (x_1,\dots,x_n) \in \mathcal{X} \subset \mathbb{R}^N$ with $x_i \in \mathcal{X}_i \subset \mathbb{R}^{N_i}$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^M$ is the common (external) input, $\mathcal{C} := \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times \mathcal{C}_u$ with $\mathcal{C}_i \subset \mathcal{X}_i$ and $\mathcal{C}_u \subset \mathcal{U}$ is the flow set, $\mathcal{D} := \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \times \mathcal{D}_u$ with $\mathcal{D}_i \subset \mathcal{X}_i$ and $\mathcal{D}_u \subset \mathcal{U}$ is the jump set, $F := (F_1,\dots,F_n)$ with $F_i : \mathcal{C} \rightrightarrows \mathbb{R}^{N_i}$ is the flow map, and $G := (G_1,\dots,G_n)$ with $G_i : \mathcal{D} \rightrightarrows \mathcal{X}_i$ is the jump map. The dynamics of x_i is called the i-th subsystem of (15) and is denoted by Σ_i, and the interconnection (15) is denoted by Σ. For each subsystem Σ_i, the states of all other subsystems are treated as (internal) inputs.

Many systems with hybrid behaviors can be naturally tranformed into the form of (15). As demonstrated in [27, Section V], a networked control system can be treated as an interconnection of continuous states and hybrid errors due to network protocol, and a quantized control system can be modeled as an interconnection of continuous states and a discrete quantizer. Moreover, the "natural decomposition" of a hybrid system (1) as an interconnection of its continuous and discrete parts is often of interest as well.

Remark 5. All the subsystems, as well as the interconnection, share the same flow set $\mathcal{C}$ and the same jump set $\mathcal{D}$, which justifies the view of (15) as an interconnection composed of n hybrid subsystems.

Remark 6. Based on Lemma 1 and standard considerations clarifying the influence of particular subsystems (see, e.g., [30, Lemma 2.4.1]), one can show that a function $V_i : \mathcal{X}_i \to \mathbb{R}_+$ is a candidate ISS Lyapunov function w.r.t. a set $\mathcal{A}_i \subset \mathcal{X}_i$ for the subsystem Σ_i if and only if it is locally Lipschitz outside $\mathcal{A}_i$, and

1. there exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that
$$\psi_{i1}(|x_i|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \psi_{i2}(|x_i|_{\mathcal{A}_i}) \quad \forall\, x_i \in \mathcal{X}_i; \tag{16}$$
2. there exist *internal gains* $\chi_{ij} \in \mathcal{K}$ for $j \neq i$ and $\chi_{ii} \equiv 0$, an *external gain* $\chi_i \in \mathcal{K}$, and a continuous function $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi_i(0) = 0$ such that for all $(x,u) \in \mathcal{C}$ with $x_i \notin \mathcal{A}_i$,
$$V_i(x_i) \geq \max\Big\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)),\, \chi_i(|u|)\Big\} \tag{17}$$
implies that
$$\dot{V}_i(x_i; y_i) \leq -\varphi_i(V_i(x_i)) \qquad \forall\, y_i \in F_i(x,u); \tag{18}$$
3. there exists a function $\alpha_i \in \mathcal{K}$ such that for all $(x,u) \in \mathcal{D}$,
$$\begin{aligned} V_i(y_i) \leq \max\Big\{&\alpha_i(V_i(x_i)),\, \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \\ &\chi_i(|u|)\Big\} \qquad \forall\, y_i \in G_i(x,u). \end{aligned} \tag{19}$$

In addition, V_i is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}_i$ with rate coefficients c_i, d_i if and only if

$$\varphi_i(r) \equiv c_i r, \qquad \alpha_i(r) \equiv e^{-d_i} r. \tag{20}$$

Suppose that for each subsystem Σ_i, a candidate ISS Lyapunov function V_i is given (for discussions on the existence of candidate exponential ISS Lyapunov functions for hybrid systems, see [3, Theorem 8.1], [2, Section 2], and [38, Remark 3]). The question of whether the interconnection (15) is pre-ISS depends on properties of the *gain operator* $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by

$$\Gamma(r_1,\dots,r_n) := \Big(\max_{j=1}^{n} \chi_{1j}(r_j),\dots,\max_{j=1}^{n} \chi_{nj}(r_j)\Big). \tag{21}$$

In order to construct a candidate ISS Lyapunov function for the interconnection (15), we adopt the notion of Ω-path [12].

Definition 3. Given a function $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$, a function $\sigma := (\sigma_1,\dots,\sigma_n)$ with $\sigma_i \in \mathcal{K}_\infty$ for $i = 1,\dots,n$ is called an Ω*-path w.r.t.* Γ if

1. all σ_i^{-1} are locally Lipschitz on $(0,\infty)$;
2. for each compact set $P \subset (0,\infty)$, there exist finite constants $K_2 > K_1 > 0$ such that for all i,
$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2$$

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

for all points of differentiability of σ_i^{-1} in P;

3. Γ is a contraction on $\sigma(\cdot)$, namely,

$$\Gamma(\sigma(r)) < \sigma(r) \qquad \forall\, r > 0. \tag{22}$$

We say that a function $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ satisfies the *small-gain condition* if

$$\Gamma(v) \not\geq v \qquad \forall\, v \in \mathbb{R}_+^n \backslash \{0\}, \tag{23}$$

or equivalently,

$$\Gamma(v) \geq v \Leftrightarrow v = 0.$$

As reported in [24, Proposition 2.7 and Remark 2.8] (see also [12, Theorem 5.2]), there exists an Ω-path σ w.r.t. the gain operator Γ defined by (21) provided that (23) holds. Furthermore, σ can be made smooth on $(0, \infty)$ via standard mollification arguments [15, Appendix B.2].

Remark 7. In this paper, we consider primarily Ω-paths w.r.t. the gain operator Γ defined by (21), due to the terms $\max_{j=1}^n \chi_{ij}(V_j(x_j))$ in (17) and (19) in the formulation for candidate ISS Lyapunov functions for subsystems (which will be clear from the statement and proof of Theorem 2 below). However, there are other equivalent formulations for candidate ISS Lyapunov functions for subsystems, which will naturally lead to gain operators in different forms (see, e.g., [11,12]). In particular, if (17) and (19) were formulated using $\sum_{j=1}^n \chi_{ij}(V_j(x_j))$ instead of $\max_{j=1}^n \chi_{ij}(V_j(x_j))$, we would arrive at the alternative gain operator $\bar{\Gamma} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by

$$\bar{\Gamma}(r_1, \ldots, r_n) := \left(\sum_{j=1}^n \chi_{1j}(r_j), \ldots, \sum_{j=1}^n \chi_{nj}(r_j) \right).$$

Compared with the gain operator Γ defined by (21), it is clear that $\Gamma(v) \leq \bar{\Gamma}(v)$ for all $v \in \mathbb{R}_+^n \backslash \{0\}$, and hence every Ω-path w.r.t. $\bar{\Gamma}$ is an Ω-path w.r.t. Γ. This alternative construction will be useful in establishing Theorem 4 for the case of linear internal gains below.

Provided that the small-gain condition (23) holds for the gain operator Γ defined by (21), the following theorem provides a candidate ISS Lyapunov function for the interconnection (15) based on those for subsystems and the corresponding Ω-path.

Theorem 2. *Consider the interconnection* (15)*. Suppose that each subsystem Σ_i admits a candidate ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with the internal gains χ_{ij} as in* (17)*, and the small-gain condition* (23) *holds for the gain operator Γ defined by* (21)*. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an Ω-path w.r.t. Γ which is smooth on $(0, \infty)$. Then the function $V : \mathcal{X} \to \mathbb{R}_+$ defined by*

$$V(x) := \max_{i=1}^n \sigma_i^{-1}(V_i(x_i)) \tag{24}$$

is a candidate ISS Lyapunov function w.r.t. the set $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ for (15)*.*

Proof. As every $\sigma_i \in \mathcal{K}_\infty$ is smooth on $(0, \infty)$ and every V_i is locally Lipschitz outside $\mathcal{A}_i$, it follows that every $\sigma_i^{-1} \circ V_i$ is locally Lipschitz outside $\mathcal{A}_i$. Hence the function V defined by (24) is locally Lipschitz outside $\mathcal{A}$. In the following, we prove that it satisfies the conditions of Lemma 1, by combining and extending the analyses in proofs of [12, Theorem 5.3] and [27, Theorem III.1].

First, consider the functions $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\psi_1(r) := \min_{i=1}^n \sigma_i^{-1}(\psi_{i1}(r/\sqrt{n})),$$
$$\psi_2(r) := \max_{i=1}^n \sigma_i^{-1}(\psi_{i2}(r))$$

with ψ_{i1}, ψ_{i2} as in (16). As all $\sigma_i, \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, it follows that $\psi_1, \psi_2 \in \mathcal{K}_\infty$. Then (3) follows from (16). In particular,

$$\begin{aligned}\psi_1(|x|_\mathcal{A}) &\leq \min_{i=1}^n \sigma_i^{-1}\Big(\psi_{i1}\Big(\max_{j=1}^n |x_j|_{\mathcal{A}_j}\Big)\Big)\\ &\leq \max_{j=1}^n \sigma_j^{-1}(\psi_{j1}(|x_j|_{\mathcal{A}_j}))\\ &\leq \max_{j=1}^n \sigma_j^{-1}(V_j(x_j)) = V(x).\end{aligned}$$

Second, consider the function $\bar{\chi} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\bar{\chi}(r) := \max_{i=1}^n \sigma_i^{-1}(\chi_i(r)) \tag{25}$$

with χ_i as in (17), and the function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\varphi(r) := \min_{i=1}^n (\sigma_i^{-1})'(\sigma_i(r))\, \varphi_i(\sigma_i(r)) \tag{26}$$

with φ_i as in (18). As all $\sigma_i \in \mathcal{K}_\infty$ are smooth, $\chi_i \in \mathcal{K}$, and φ_i are continuous with $\varphi_i(0) = 0$, it follows that $\bar{\chi} \in \mathcal{K}$ and φ is continuous with $\varphi(0) = 0$. Consider the sets $\mathcal{M}_i \subset \mathcal{X}$, $i = 1, \ldots, n$ defined by

$$\mathcal{M}_i := \Big\{ x \in \mathcal{X} : \sigma_i^{-1}(V_i(x_i)) > \max_{j \neq i} \sigma_j^{-1}(V_j(x_j)) \Big\}.$$

As all V_i and σ_i^{-1} are continuous, it follows that all $\mathcal{M}_i$ are open, $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for $j \neq i$, and

$$\mathcal{X} = \bigcup_{i=1}^n \overline{\mathcal{M}_i}.$$

Thus for every $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$, there are two possibilities:

1) There is a unique $i \in \{1, \ldots, n\}$ such that $x \in \mathcal{M}_i$. Then $x_i \notin \mathcal{A}_i$ (since $V_i(x_i) > 0$ following the definition of $\mathcal{M}_i$), and

$$V(x) = \sigma_i^{-1}(V_i(x_i)). \tag{27}$$

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

Hence

$$\begin{aligned} V_i(x_i) &= \sigma_i(V(x)) \\ &\geq \max_{j=1}^{n} \chi_{ij}(\sigma_j(V(x))) \geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \end{aligned} \quad (28)$$

where the first inequality follows from (21) and (22), and the second one follows from (24). Meanwhile, suppose that $V(x) \geq \bar{\chi}(|u|)$. Then $V(x) \geq \max_{j=1}^{n} \sigma_j^{-1}(\chi_j(|u|))$ due to (25), and thus

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &\geq \sigma_i\Big(\max_{j=1}^{n} \sigma_j^{-1}(\chi_j(|u|))\Big) \\ &\geq \sigma_i(\sigma_i^{-1}(\chi_i(|u|))) = \chi_i(|u|). \end{aligned} \quad (29)$$

Hence (17), and therefore (18), holds. Given an arbitrary $y = (y_1, \ldots, y_n) \in F(x, u)$, as $\mathcal{M}_i$ is open, for all small enough $h > 0$ it holds that $x + hy \in \mathcal{M}_i$, that is, $V(x + hy) = \sigma_i^{-1}(V_i(x_i + hy_i))$. Hence

$$\begin{aligned} \dot{V}(x; y) &= \overline{\lim_{h \searrow 0}} \frac{V(x + hy) - V(x)}{h} \\ &= \overline{\lim_{h \searrow 0}} \frac{\sigma_i^{-1}(V_i(x_i + hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\ &= (\sigma_i^{-1})'(V_i(x_i)) \overline{\lim_{h \searrow 0}} \frac{V_i(x_i + hy_i) - V_i(x_i)}{h} \\ &= (\sigma_i^{-1})'(V_i(x_i)) \dot{V}_i(x_i; y_i) \\ &\leq -(\sigma_i^{-1})'(\sigma_i(V(x)))\, \varphi_i(\sigma_i(V(x))) \\ &\leq -\varphi(V(x)), \end{aligned}$$

where the first inequality follows from (18) and (27), and the last one follows from (26).
2) There is a subset $I(x) \subset \{1, \ldots, n\}$ of indices with the cardinality $|I(x)| \geq 2$ such that

$$x \in \bigcap_{i \in I(x)} \partial \mathcal{M}_i,$$

where $\partial \mathcal{M}_i := \overline{\mathcal{M}_i} \backslash \mathcal{M}_i$ denotes the boundary of $\mathcal{M}_i$. Then $x_i \notin \mathcal{A}_i$ and (27) holds for all $i \in I(x)$. Following similar arguments to those in the previous case, we see that if $V(x) \geq \bar{\chi}(|u|)$ holds then (28), (29), and therefore (18), hold for all $i \in I(x)$ as well. Also, given an arbitrary $y = (y_1, \ldots, y_n) \in F(x, u)$, as all $\mathcal{M}_i$ are open, for all small enough $h > 0$ it holds that $x + hy \in \cap_{i \in I(x)} \mathcal{M}_i$, that is, $V(x + hy) = \max_{i \in I(x)} \sigma_i^{-1}(V_i(x_i + hy_i))$. Hence

$$\begin{aligned} \dot{V}(x; y) &= \overline{\lim_{h \searrow 0}} \frac{V(x + hy) - V(x)}{h} \\ &= \overline{\lim_{h \searrow 0}} \frac{1}{h} \Big(\max_{i \in I(x)} \sigma_i^{-1}(V_i(x_i + hy_i)) - V(x) \Big) \\ &= \overline{\lim_{h \searrow 0}} \max_{i \in I(x)} \frac{\sigma_i^{-1}(V_i(x_i + hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\ &= \max_{i \in I(x)} \overline{\lim_{h \searrow 0}} \frac{\sigma_i^{-1}(V_i(x_i + hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\ &= \max_{i \in I(x)} (\sigma_i^{-1})'(V_i(x_i)) \dot{V}_i(x_i; y_i) \\ &\leq \max_{i \in I(x)} -(\sigma_i^{-1})'(\sigma_i(V(x)))\, \varphi_i(\sigma_i(V(x))) \\ &\leq -\varphi(V(x)), \end{aligned}$$

where the fourth equality follows partially from the continuity of all V_i and σ_i^{-1} (cf. [8, proof of Theorem 4]); the first inequality follows from (18) and (27) for all $i \in I(x)$, and the last one follows from (26).

Hence (7) holds for every $(x, u) \in \mathcal{C}$.

Finally, consider the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\alpha(r) := \max_{i,j=1}^{n} \Big\{ \sigma_i^{-1}(\alpha_i(\sigma_i(r))),\, \sigma_i^{-1}(\chi_{ij}(\sigma_j(r))) \Big\} \quad (30)$$

with α_i and χ_{ij} as in (19). As all $\sigma_i \in \mathcal{K}_\infty$, $\chi_{ij} \in \mathcal{K}$ for $j \neq i$, $\chi_{ii} \equiv 0$ and $\alpha_i \in \mathcal{K}$, it follows that $\alpha \in \mathcal{K}$. Consider an arbitrary $(x, u) \in \mathcal{D}$. From (24) and (30) it follows that [4]

$$\alpha(V(x)) \geq \max_{i,j=1}^{n} \Big\{ \sigma_i^{-1}(\alpha_i(V_i(x_i))),\, \sigma_i^{-1}(\chi_{ij}(V_j(x_j))) \Big\};$$

and (25) implies that

$$\bar{\chi}(|u|) = \max_{i=1}^{n} \sigma_i^{-1}(\chi_i(|u|)).$$

Consider an arbitrary $y = (y_1, \ldots, y_n) \in G(x, u)$. Combining the previous two inequalities with (19), we obtain

$$V(y) = \max_{i=1}^{n} \sigma_i^{-1}(V_i(y_i)) \leq \max\{\alpha(V(x)),\, \bar{\chi}(|u|)\}.$$

Hence (8) holds for every $(x, u) \in \mathcal{D}$.

Therefore, from Lemma 1 it follows that V is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (15). □

[4] Note that, if α_i is of class $\mathcal{PD}$ but not increasing, then it is possible that $\sigma_i(V(x)) > V_i(x_i)$ but $\alpha_i(\sigma_i(V(x))) < \alpha_i(V_i(x_i))$ for some i. Consequently, the inequality following this footnote may not hold. A similar issue arises in the proof of [27, Theroem III.1] where it was overlooked, but could be fixed by majorizing the class $\mathcal{PD}$ functions λ_1, λ_2 with class $\mathcal{K}$ ones.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

Theorem 2 is a powerful tool in establishing ISS of interconnections of hybrid systems. In the following we inspect some of its implications.

If every subsystem of (15) admits an ISS Lyapunov function, Theorem 2 implies the following result, which generalizes [27, Theorem III.1] and [6, Theorem 3.6].

Corollary 3. *Consider the interconnection* (15)*. Suppose that each subsystem* Σ_i *admits an ISS Lyapunov function* V_i *w.r.t. a set* $\mathcal{A}_i$ *(i.e.,* $\varphi_i \in \mathcal{PD}$ *and* $\alpha_i < \mathrm{id}$ *on* $(0,\infty)$ *in* (18) *and* (19)*, respectively) with the internal gains* χ_{ij} *as in* (17)*, and the small-gain condition* (23) *holds for the gain operator* Γ *defined by* (21)*. Then* (15) *is pre-ISS w.r.t.* $\mathcal{A}$*.*

Proof. Following Theorem 2, the function V defined by (24) is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (15). As all $\sigma_i \in \mathcal{K}_\infty$ are smooth and $\varphi_i \in \mathcal{PD}$, the function φ defined by (26) is of class $\mathcal{PD}$. Also, from (22) it follows that $\sigma_i^{-1} \circ \chi_{ij} \circ \sigma_j < \mathrm{id}$ on $(0,\infty)$ for all i, j; for all i, from $\sigma_i \in \mathcal{K}_\infty$ and $\alpha_i < \mathrm{id}$ it follows that $\sigma_i^{-1} \circ \alpha_i \circ \sigma_i < \mathrm{id}$ on $(0,\infty)$. Thus the function α defined by (30) satisfies that $\alpha < \mathrm{id}$ on $(0,\infty)$. Hence the function V is an ISS Lyapunov function, and (15) is pre-ISS w.r.t. $\mathcal{A}$ following similar analysis to the proof of [2, Proposition 2.7]; see also Remark 2. □

As the assumptions in Corollary 3 are quite restrictive, we now investigate the case where for some subsystems Σ_i, either $\varphi_i \notin \mathcal{PD}$ or $\alpha_i(r) \geq r$ for some $r > 0$ (cf. footnote 3). In this case, we cannot use Corollary 3 to prove pre-ISS for the interconnection (15) directly, but rather establish pre-ISS for the set of solution pairs that jump neither too fast nor too slowly via Proposition 1. However, in general Theorem 2 does not provide the candidate exponential ISS Lyapunov function needed in Proposition 1. We construct such a function in the next theorem, assuming that each subsystem Σ_i admits a candidate exponential ISS Lyapunov function V_i, and the internal gains χ_{ij} in (17) and (19) are all linear. With a slight abuse of notation, we let $\chi_{ij} \geq 0$ be scalars, and replace the terms $\chi_{ij}(V_j(x_j))$ in (17) and (19) with $\chi_{ij}V_j(x_j)$. Consider the *gain matrix* $\Gamma_M \in \mathbb{R}^{n\times n}$ defined by

$$\Gamma_M := (\chi_{ij})_{n\times n}. \tag{31}$$

Denote by $\rho(\Gamma_M)$ its spectral radius (i.e., the largest absolute value of its eigenvalues). Provided that

$$\rho(\Gamma_M) < 1, \tag{32}$$

the small-gain condition (23) holds for the function $v \mapsto \Gamma_M v$ on $\mathbb{R}^n_+$ [11, p. 110] (which corresponds to the alternative gain operator in Remark 7), and there exists a linear Ω-path w.r.t. it [10, p. 78] (for more results on Ω-paths, the reader may consult [36]).

Theorem 4. *Consider the interconnection* (15)*. Suppose that each subsystem* Σ_i *admits a candidate exponential ISS Lyapunov function* V_i *w.r.t. a set* $\mathcal{A}_i$ *with rate coefficients* c_i, d_i*. Assume also that the internal gains* χ_{ij} *in* (17) *and* (19) *are all linear, and* (32) *holds for the gain matrix* Γ_M *defined by* (31)*. Let* $\sigma : r \mapsto (s_1 r, \ldots, s_n r)$ *be a linear* Ω*-path w.r.t. the function* $v \mapsto \Gamma_M v$ *on* $\mathbb{R}^n_+$*. Then the function* $V : \mathcal{X} \to \mathbb{R}_+$ *defined by*

$$V(x) := \max_{i=1}^{n} \frac{1}{s_i} V_i(x_i) \tag{33}$$

is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}$ *for* (15) *with the rate coefficients*

$$c := \min_{i=1}^{n} c_i, \quad d := \min_{i,j:j\neq i} \left\{ d_i,\, -\ln\left(\frac{s_j}{s_i}\chi_{ij}\right)\right\}. \tag{34}$$

Proof. In view of Remark 7, σ is also an Ω-path w.r.t. the gain operator defined by (21) (with all $\chi_{ij}(r_j)$ replaced by $\chi_{ij} r_j$). Then from Theorem 2 it follows that V defined by (33) is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (15). Substituting (20) into (26) and (30), we obtain

$$\varphi(r) = \min_{i=1}^{n} c_i r, \quad \alpha(r) \equiv \max_{i,j=1}^{n} \left\{ e^{-d_i},\, \frac{s_j}{s_i}\chi_{ij}\right\} r.$$

Hence V is a candidate exponential ISS Lyapunov function with the rate coefficients c, d defined by (34). □

Remark 8. For the more general case where the internal gains χ_{ij} are power functions instead of linear ones, a candidate exponential ISS Lyapunov function for the interocnnection (15) can be constructed in a similar manner; cf. [9, Theorem 9].

The following remark provides a simpler bound for the rate coefficient d in some important cases.

Remark 9. If the gain matrix Γ_M defined by (31) is irreducible, then $\rho(\Gamma_M)$ is the Perron–Frobenius eigenvalue of Γ_M, and the corresponding eigenvector $v = (s_1, \ldots, s_n)$ satisfies that all $s_i > 0$ (Perron–Frobenius theorem [1, Theorem 2.1.3]). Hence if (32) holds then $\Gamma_M v = \rho(\Gamma_M)v < v$, and thus $\sigma : r \mapsto vr$ is a linear Ω-path as in Theorem 4. Moreover, it holds that

$$\max_{j=1}^{n} \frac{s_j}{s_i}\chi_{ij} \leq \frac{1}{s_i}\sum_{j=1}^{n} s_j \chi_{ij} = \rho(\Gamma_M)$$

for all $i \in \{1, \ldots, n\}$, and thus the rate coefficient d defined by (34) satisfies $d \geq \min\{\min_{i=1}^{n} d_i,\, -\ln(\rho(\Gamma_M))\}$.

Having applied Theorem 4, we may use Proposition 1 to establish pre-ISS for the set of solution pairs that jump neither too fast nor too slowly. However, if there are two subsystems Σ_k, Σ_l for which the rate coefficients $c_k, d_l < 0$, then c, d defined by (34) are negative as well,

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

and Proposition 1 cannot be applied to complete solution pairs (see Remark 3). In the following section, we handle such cases via the approach of modifying ISS Lyapunov functions for subsystems using ADT and RADT clocks from [27].

4 Modifying ISS Lyapunov functions for subsystems

Suppose that every subsystem Σ_i admits a candidate exponential ISS Lyapunov function with rate coefficients c_i, d_i, and there are Σ_k, Σ_l such that $c_k, d_l < 0 < c_l, d_k$. Our goal is to construct new candidate exponential ISS Lyapunov functions with rate coefficients $\tilde{c}_i, \tilde{d}_i$ so that either all $\tilde{c}_i > 0$ (i.e., all continuous dynamics are ISS) or all $\tilde{d}_i > 0$ (i.e., all discrete dynamics are ISS). To accomplish this, we first derive suitable conditions on the density of jumps, then augment the corresponding subsystems with auxiliary clocks to incorporate such conditions, and finally modify the corresponding candidate exponential ISS Lyapunov functions.

4.1 Making discrete dynamics ISS

In the following, we construct candidate exponential ISS Lyapunov functions so that all rate coefficients $\tilde{d}_i > 0$.

We say that a solution pair (x, u) of (15) admits an average dwell-time (ADT) [18] $\delta > 0$ if all $(s, k) \preceq (t, j)$ in the hybrid time domain $\operatorname{dom} x$ satisfy

$$j - k \le \delta(t - s) + N_0 \tag{35}$$

with an integer $N_0 \ge 1$.[5] Following [27, Section IV.A], a hybrid time domain satisfies (35) if and only if it is the domain of an ADT clock τ defined by

$$\begin{aligned} &\dot{\tau} \in [0, \delta], && \tau \in [0, N_0], \\ &\tau^+ = \tau - 1, && \tau_i \in [1, N_0]. \end{aligned} \tag{36}$$

Remark 10. This notion of ADT clock for hybrid systems first appeared in [4, Appendix], where the clock was defined by

$$\begin{cases} \dot{\tau} \in \eta_\delta(\tau) & \text{for } \tau \in C := [0, N_0] \\ \tau^+ = \tau - 1 & \text{for } \tau \in D := [1, N_0] \end{cases} \tag{37}$$

where

$$\eta_\delta(\tau) := \begin{cases} \delta & \text{for } \tau \in [0, N_0) \\ [0, \delta] & \text{for } \tau = N_0. \end{cases}$$

(See also [32] for a related earlier construction.) The ADT clocks defined by (36) and (37) are equivalent in the following sense. On the one hand, as $\tau \in [0, \delta]$, an ADT clock defined by (37) always satisfies (36). On the other hand, given an ADT clock defined by (36) that increases on $[0, N_0)$ with a speed less than δ, there always exists an ADT clock defined by (37) that increases on $[0, N_0)$ with the speed δ but stays longer at N_0 so that their hybrid time domains are the same.

Denote by $I_d := \{i : d_i < 0\}$ the index set of subsystems with non-ISS discrete dynamics. Let $z_i := x_i \in \mathcal{X}_i =: \mathcal{Z}_i$ for $i \notin I_d$ and $z_i := (x_i, \tau_i) \in \mathcal{X}_i \times [0, N_{0i}] =: \mathcal{Z}_i$ with a integer $N_{0i} \ge 1$ for $i \in I_d$. Consider the augmented interconnection $\tilde{\Sigma}$ with the state $z := (z_1, \dots, z_n) \in \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n =: \mathcal{Z}$, and the input $u \in \mathcal{U}$ modeled by

$$\begin{aligned} &\dot{z}_i \in \tilde{F}_i(z, u), \quad i = 1, \dots, n, && (z, u) \in \tilde{\mathcal{C}}, \\ &z_i^+ \in \tilde{G}_i(z, u), \quad i = 1, \dots, n, && (z, u) \in \tilde{\mathcal{D}}, \end{aligned} \tag{38}$$

where $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_n \times \mathcal{C}_u$ with $\tilde{\mathcal{C}}_i := \mathcal{C}_i$ for $i \notin I_d$ and $\tilde{\mathcal{C}}_i := \mathcal{C}_i \times [0, N_{0i}]$ for $i \in I_d$, $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}_1 \times \cdots \times \tilde{\mathcal{D}}_n \times \mathcal{D}_u$ with $\tilde{\mathcal{D}}_i := \mathcal{D}_i$ for $i \notin I_d$ and $\tilde{\mathcal{D}}_i := \mathcal{D}_i \times [1, N_{0i}]$ for $i \in I_d$, $\tilde{F} := (\tilde{F}_1, \dots, \tilde{F}_n)$ with $\tilde{F}_i(z, u) := F_i(x, u)$ for $i \notin I_d$ and $\tilde{F}_i(z, u) := F_i(x, u) \times [0, \delta_i]$ for $i \in I_d$, and $\tilde{G} := (\tilde{G}_1, \dots, \tilde{G}_n)$ with $\tilde{G}_i(z, u) := G_i(x, u)$ for $i \notin I_d$ and $\tilde{G}_i(z, u) := G_i(x, u) \times \{\tau_i - 1\}$ for $i \in I_d$. Then (38) is a hybrid system with the data $\tilde{\mathcal{H}} := (\tilde{F}, \tilde{G}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \mathcal{Z}, \mathcal{U})$. The dynamics of z_i is called the i-th augmented subsystem of (38) and is denoted by $\tilde{\Sigma}_i$.

In the following proposition, we apply the modification technique in [27, Proposition IV.1] to construct a candidate exponential ISS Lyapunov function for each augmented subsystem $\tilde{\Sigma}_i$ based on the one for the subsystem Σ_i of the original interconnection (15) and the ADT clock τ_i.

Proposition 5. *Consider a subsystem Σ_i of the original interconnection* (15)*. Suppose that it admits a candidate exponential ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with rate coefficients c_i, d_i. For a constant $L_i \ge 0$, the function $W_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by*

$$W_i(z_i) := \begin{cases} V_i(x_i), & \text{if } i \notin I_d; \\ e^{L_i \tau_i} V_i(x_i), & \text{if } i \in I_d \end{cases}$$

is a candidate exponential ISS Lyapunov function w.r.t. the set

$$\tilde{\mathcal{A}}_i := \begin{cases} \mathcal{A}_i, & \text{if } i \notin I_d; \\ \mathcal{A}_i \times [0, N_{0i}], & \text{if } i \in I_d \end{cases}$$

for the augmented subsystem $\tilde{\Sigma}_i$ of (38) *with the rate*

[5] If (35) holds with $N_0 = 1$ then the ADT condition becomes the dwell-time condition [33]; if it holds with $N_0 < 1$ then jumps are not allowed at all, which be seen directly from (35) by taking $t - s$ small enough.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

coefficients

$$\begin{cases} \tilde{c}_i := c_i, \quad \tilde{d}_i := d_i, & \text{if } i \notin I_d; \\ \tilde{c}_i := c_i - L_i\delta_i, \quad \tilde{d}_i := d_i + L_i, & \text{if } i \in I_d. \end{cases} \tag{39}$$

More specifically,

1. *there exist functions* $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ *such that*

$$\tilde{\psi}_{i1}(|z_i|_{\tilde{\mathcal{A}}_i}) \leq W_i(z_i) \leq \tilde{\psi}_{i2}(|z_i|_{\tilde{\mathcal{A}}_i}) \quad \forall\, z_i \in \mathcal{Z}_i; \tag{40}$$

2. *there exist internal gains* $\tilde{\chi}_{ij} \in \mathcal{K}$ *for* $j \neq i$ *defined by*

$$\tilde{\chi}_{ij}(r) := \begin{cases} \chi_{ij}(r), & \text{if } i \notin I_d; \\ e^{L_i N_{0i}}\chi_{ij}(r), & \text{if } i \in I_d \end{cases} \tag{41}$$

with χ_{ij} *as in* (17) *and* $\tilde{\chi}_{ii} = 0$*, and an external gain* $\tilde{\chi}_i \in \mathcal{K}$ *such that for all* $(z,u) \in \tilde{\mathcal{C}}$ *with* $z_i \notin \tilde{\mathcal{A}}_i$*,*

$$W_i(z_i) \geq \max\Big\{\max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)),\ \tilde{\chi}_i(|u|)\Big\} \tag{42}$$

implies that

$$\dot{W}_i(z_i; y_i) \leq -\tilde{c}_i W_i(z_i) \qquad \forall\, y_i \in \tilde{F}_i(z,u); \tag{43}$$

3. *for all* $(z,u) \in \tilde{\mathcal{D}}$*,*

$$W_i(y_i) \leq \max\Big\{e^{-\tilde{d}_i} W_i(z_i),\ \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)),\ \tilde{\chi}_i(|u|)\Big\} \qquad \forall\, y_i \in \tilde{G}_i(z,u). \tag{44}$$

Proof. If $i \notin I_d$ then the claim follows directly from the assumption that V_i is a candidate exponential ISS Lyapunov function with rate coefficients c_i, d_i. Therefore, in the following proof we only consider the case that $i \in I_d$. As V_i is locally Lipschitz outside $\mathcal{A}_i$ and the exponential function $\tau_i \mapsto e^{L_i\tau_i}$ is smooth, the function W_i is locally Lipschitz outside $\tilde{\mathcal{A}}_i$.

First, consider the functions $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ defined by

$$\tilde{\psi}_{i1}(r) := \psi_{i1}(r), \qquad \tilde{\psi}_{i2}(r) := e^{L_i N_{0i}}\psi_{i2}(r)$$

with ψ_{i1}, ψ_{i2} as in (16). Then (40) follows from (16).

Second, consider the function $\tilde{\chi}_i \in \mathcal{K}$ defined by

$$\tilde{\chi}_i(r) := e^{L_i N_{0i}}\chi_i(r) \tag{45}$$

with χ_i as in (17). For every $(z,u) \in \tilde{\mathcal{C}}$ with $z_i \notin \tilde{\mathcal{A}}_i$ such that (42) holds, it follows that

$$\begin{aligned} V_i(x_i) = e^{-L_i\tau_i} W_i(z_i) &\geq e^{-L_i N_{0i}} \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)) \\ &= \max_{j=1}^{n} \chi_{ij}(W_j(z_j)) \geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \end{aligned}$$

and $V_i(x_i) = e^{-L_i\tau_i}W_i(z_i) \geq e^{-L_i N_{0i}}\tilde{\chi}_i(|u|) = \chi_i(|u|)$. Hence (17), and therefore (18), holds. For all $y_i \in \tilde{F}_i(z,u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in F_i(x,u)$ and $y_{i2} \in [0, \delta_i]$. From (18), (20) and (39) it follows that

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{L_i\tau_i}\dot{V}_i(x_i; y_{i1}) + L_i e^{L_i\tau_i} V_i(x_i)\, y_{i2} \\ &\leq -c_i e^{L_i\tau_i} V_i(x_i) + L_i\delta_i e^{L_i\tau_i} V_i(x_i) = -\tilde{c}_i W_i(z_i). \end{aligned}$$

Finally, consider an arbitrary $(z,u) \in \tilde{\mathcal{D}}$. For all $y_i \in \tilde{G}_i(z,u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in G_i(x,u)$ and $y_{i2} = \tau_i - 1$. From (39) it follows that

$$e^{-\tilde{d}_i} W_i(z_i) = e^{-d_i - L_i + L_i\tau_i} V_i(x_i) = e^{L_i y_{i2} - d_i} V_i(x_i),$$

and from (41) and (45) it follows that

$$\begin{aligned} \tilde{\chi}_{ij}(W_j(z_j)) &= e^{L_i N_{0i}} \chi_{ij}(W_j(z_j)) \\ &\geq e^{L_i y_{i2}} \chi_{ij}(V_j(x_j)) \qquad \forall\, j \in \{1, \ldots, n\}, \end{aligned}$$

and $\tilde{\chi}_i(|u|) = e^{L_i N_{0i}}\chi_i(|u|) \geq e^{L_i y_{i2}}\chi_i(|u|)$, respectively. Substituting the previous equations into (19) gives (44).

Therefore, W_i is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_i$ for the augmented subsystem $\tilde{\Sigma}_i$ with the rate coefficients $\tilde{c}_i, \tilde{d}_i$ defined by (39). ☐

Proposition 5 shows that it is possible to make all $\tilde{d}_i > 0$ by choosing large enough scalars L_i for $i \in I_d$, at the cost of decreasing the convergence rates of continuous dynamics (as $\tilde{c}_i = c_i - L_i\delta_i$ in (39) above), and increasing the internal gains (as $\tilde{\chi}_{ij}(r) = e^{L_i N_{0i}}\chi_{ij}(r)$ in (41) above). Consequently, for large enough integers N_{0i}, it is possible that the small-gain condition (23) holds for the gain operator Γ defined by (21), but not for $\tilde{\Gamma} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by [6]

$$\tilde{\Gamma}(r_1, \ldots, r_n) := \Big(\max_{j=1}^{n} \tilde{\chi}_{1j}(r_j), \ldots, \max_{j=1}^{n} \tilde{\chi}_{nj}(r_j)\Big).$$

To see the consequence of this fact clearer, consider for simplicity an interconnection of two subsystems Σ_1, Σ_2,

[6] However, if all the original internal gains χ_{ij} are linear, and the gain matrix Γ_M defined by (31) is a triangular matrix (i.e., if (15) is a cascade interconnection), then (23) always holds for $\tilde{\Gamma}$, as all the cyclic gains equal zero.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

and their candidate exponential ISS Lyapunov functions V_1, V_2 with rate coefficients $c_1, d_2 > 0 > d_1, c_2$ and linear internal gains $\chi_{12}, \chi_{21} > 0$. After we augment Σ_1 with an ADT clock $\delta_1 \in [0, N_{01}]$, the gain matrix $\tilde{\Gamma}_M$ is given by

$$\tilde{\Gamma}_M = \begin{bmatrix} 0 & \tilde{\chi}_{12} \\ \tilde{\chi}_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{L_1 N_{01}} \chi_{12} \\ \chi_{21} & 0 \end{bmatrix},$$

and $\rho(\tilde{\Gamma}_M) < 1$ holds if only if $\chi_{12}\chi_{21} < e^{-L_1 N_{01}}$. In order to make the rate coefficient $\tilde{d}_1 = d_1 + L_1 > 0$, we need to choose a scalar $L_1 > -d_1$. Also, the integer $N_{01} \geq 1$. Hence we cannot apply Theorem 4 to the augmented interconnection unless the original internal gains χ_{12}, χ_{21} satisfy $\chi_{12}\chi_{21} \leq e^{d_1} < 1$.

The observation above hints that it may be better to make all $\tilde{c}_i > 0$ (instead of making all $\tilde{d}_i > 0$ as in this subsection). See [38] for a case-by-case study comparing the two schemes.

4.2 Making continuous dynamics ISS

In the following, we construct candidate exponential ISS Lyapunov functions so that all rate coefficients $\tilde{c}_i > 0$.

We say that a solution pair (x, u) of (15) admits a reverse average dwell-time (RADT) [17] $\delta^* > 0$ if all $(s, k) \preceq (t, j)$ in the hybrid time domain $\operatorname{dom} x$ satisfy

$$t - s \leq \delta^*(j - k) + N_0^* \delta^* \tag{46}$$

with an integer $N_0^* \geq 1$. Following [4, Appendix] and [27, Section IV.B], a hybrid time domain satisfies (46) if and only if it is the domain of an RADT clock τ defined by

$$\begin{aligned} &\dot{\tau} = 1, && \tau \in [0, N_0^* \delta^*], \\ &\tau^+ = \max\{0, \tau - \delta^*\}, && \tau \in [0, N_0^* \delta^*]. \end{aligned}$$

Denote by $I_c := \{i : c_i < 0\}$ the index set of subsystems with non-ISS continuous dynamics. Let $z_i := x_i \in \mathcal{X}_i =: \mathcal{Z}_i$ for $i \notin I_c$ and $z_i := (x_i, \tau_i) \in \mathcal{X}_i \times [0, N_{0i}^* \delta_i^*] =: \mathcal{Z}_i$ with an integer $N_{0i} \geq 1$ for $i \in I_c$. Consider the augmented interconnection $\tilde{\Sigma}$ with the state $z := (z_1, \dots, z_n) \in \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n =: \mathcal{Z}$ and the input $u \in \mathcal{U}$ modeled by (38), where $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_n \times \mathcal{C}_u$ with $\tilde{\mathcal{C}}_i = \mathcal{C}_i$ for $i \notin I_c$ and $\tilde{\mathcal{C}}_i = \mathcal{C}_i \times [0, N_{0i}^* \delta_i^*]$ for $i \in I_c$, $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}_1 \times \cdots \times \tilde{\mathcal{D}}_n \times \mathcal{D}_u$ with $\tilde{\mathcal{D}}_i = \mathcal{D}_i$ for $i \notin I_c$ and $\tilde{\mathcal{D}}_i = \mathcal{D}_i \times [0, N_{0i}^* \delta_i^*]$ for $i \in I_c$, $\tilde{F} := (\tilde{F}_1, \dots, \tilde{F}_n)$ with $\tilde{F}_i(z, u) := F_i(x, u)$ for $i \notin I_c$ and $\tilde{F}_i(z, u) := F_i(x, u) \times \{1\}$ for $i \in I_c$, and $\tilde{G} := (\tilde{G}_1, \dots, \tilde{G}_n)$ with $\tilde{G}_i(z, u) := G_i(x, u)$ for $i \notin I_c$ and $\tilde{G}_i(z, u) := G_i(x, u) \times \{\max\{0, \tau_i - \delta_i^*\}\}$ for $i \in I_c$. Then the interconnection modeled by (38) is a hybrid system with the data $\tilde{\mathcal{H}} := (\tilde{F}, \tilde{G}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \mathcal{Z}, \mathcal{U})$. The dynamics of z_i is called the i-th augmented subsystem of (38) and is denoted by $\tilde{\Sigma}_i$.

In the following proposition, we apply the modification technique in [27, Proposition IV.4] to construct a candidate exponential ISS Lyapunov functions for each augmented subsystem $\tilde{\Sigma}_i$ based on the one for the subsystem Σ_i of the original interconnection (15) and the RADT clock τ_i.

Proposition 6. *Consider a subsystem Σ_i of the original interconnection (15). Suppose that it admits an candidate exponential ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with rate coefficients c_i, d_i. For a constant $L_i \geq 0$, the function $W_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by*

$$W_i(z_i) := \begin{cases} V_i(x_i), & \text{if } i \notin I_c; \\ e^{-L_i \tau_i} V_i(x_i), & \text{if } i \in I_c \end{cases} \tag{47}$$

is a candidate exponential ISS Lyapunov function w.r.t. the set

$$\tilde{\mathcal{A}}_i := \begin{cases} \mathcal{A}_i, & \text{if } i \notin I_c; \\ \mathcal{A}_i \times [0, N_{0i}^* \delta_i^*], & \text{if } i \in I_c \end{cases}$$

for the augmented subsystem $\tilde{\Sigma}_i$ of (38) with the rate coefficients

$$\begin{cases} \tilde{c}_i := c_i, \quad \tilde{d}_i := d_i, & \text{if } i \notin I_c; \\ \tilde{c}_i := c_i + L_i, \quad \tilde{d}_i := d_i - L_i \delta_i^*, & \text{if } i \in I_c. \end{cases} \tag{48}$$

More specifically,

1. *there exist functions $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that (40) holds;*
2. *there exist internal gains $\tilde{\chi}_{ij} \in \mathcal{K}$ for $j \neq i$ defined by*

$$\tilde{\chi}_{ij}(r) := \begin{cases} \chi_{ij}(r), & j \notin I_c; \\ \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} r), & j \in I_c \end{cases} \tag{49}$$

with χ_{ij} as in (17) and $\tilde{\chi}_{ii} = 0$,[7] *and an external gain $\tilde{\chi}_i \in \mathcal{K}$ such that for all $(z, u) \in \tilde{\mathcal{C}}$ with $z_i \notin \tilde{\mathcal{A}}_i$, (42) implies (43);*

3. *for all $(z, u) \in \tilde{\mathcal{D}}$, (44) holds.*

Proof. If $i \notin I_c$ then the claim follows directly from the assumption that V_i is a candidate exponential ISS Lyapunov function with rate coefficients c_i, d_i. Therefore, in the following proof we only consider the case that $i \in I_c$. As V_i is locally Lipschitz outside $\mathcal{A}_i$ and the exponential function $\tau_i \mapsto e^{-L_i \tau_i}$ is smooth, the function W_i is locally Lipschitz outside $\tilde{\mathcal{A}}_i$.

First, consider the functions $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ defined by

$$\tilde{\psi}_{i1}(r) := e^{-L_i N_{0i}^* \delta_i^*} \psi_{i1}(r), \qquad \tilde{\psi}_{i2}(r) := \psi_{i2}(r)$$

[7] Note that the forms of the internal gains $\tilde{\chi}_{ij}$ defined by (41) dependent on whether $i \in I_d$, while the forms of these defined by (49) dependent on whether $j \in I_c$.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

with ψ_{i1}, ψ_{i2} as in (16). Then (40) follows from (16).

Second, consider the function $\tilde{\chi}_i \in \mathcal{K}$ defined by

$$\tilde{\chi}_i(r) := \chi_i(r) \tag{50}$$

with χ_i as in (17). For every $(z, u) \in \tilde{\mathcal{C}}$ with $z_i \notin \tilde{\mathcal{A}}_i$ such that (42) holds, it follows that

$$\begin{aligned} V_i(x_i) &= e^{L_i \tau_i} W_i(z_i) \geq W_i(z_i) \geq \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)) \\ &= \max_{j=1}^{n} \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} W_j(z_j)) \geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \end{aligned}$$

and $V_i(x_i) = e^{L_i \tau_i} W_i(z_i) \geq W_i(z_i) \geq \tilde{\chi}_i(|u|) \geq \chi_i(|u|)$. Hence (17), and therefore (18), holds. For all $y_i \in \tilde{F}_i(z, u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in F_i(x, u)$ and $y_{i2} = 1$. From (18), (20) and (48) it follows that

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{-L_i \tau_i} \dot{V}_i(x_i; y_{i1}) - L_i e^{-L_i \tau_i} V_i(x_i)\, y_{i2} \\ &\leq -c_i e^{-L_i \tau_i} V_i(x_i) - L_i e^{-L_i \tau_i} V_i(x_i) = -\tilde{c}_i W_i(z_i). \end{aligned}$$

Finally, consider an arbitrary $(z, u) \in \tilde{\mathcal{D}}$. For all $y_i \in \tilde{G}_i(z, u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in G_i(x, u)$ and $y_{i2} = \max\{0,\, \tau_i - \delta_i^*\}$. From (48) it follows that

$$e^{-\tilde{d}_i} W_i(z_i) = e^{-d_i + L_i \delta_i^* - L_i \tau_i} V_i(x_i) \geq e^{-L_i y_{i2} - d_i} V_i(x_i),$$

and from (49) and (50) it follows that

$$\begin{aligned} \tilde{\chi}_{ij}(W_j(z_j)) &= \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} W_j(z_j)) \geq \chi_{ij}(V_j(x_j)) \\ &\geq e^{-L_i y_{i2}} \chi_{ij}(V_j(x_j)) \qquad \forall\, j \in \{1, \ldots, n\}, \end{aligned}$$

and $\tilde{\chi}_i(|u|) = \chi_i(|u|) \geq e^{-L_i y_{i2}} \chi_i(|u|)$, respectively. Substituting the previous equations into (19) gives (44).

Therefore, W_i is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_i$ for the augmented subsystem $\tilde{\Sigma}_i$ with the rate coefficients $\tilde{c}_i, \tilde{d}_i$ defined by (48). $\square$

4.3 Example

We demonstrate the approach of modifying ISS Lyapunov functions in a case where we cannot apply Theorem 2 and Proposition 1 to establish stability directly.

Consider an interconnection of two hybrid subsystems with the state $x = (x_1, x_2)$ modeled by

$$\begin{aligned} &\dot{x}_1 = x_1 + x_2^2, \quad \dot{x}_2 = -3x_2 + 0.1\sqrt{|x_1|}, && x \in \mathcal{C}, \\ &x_1^+ = e^{-1} x_1, \quad x_2^+ = e x_2, && x \in \mathcal{D}, \end{aligned}$$

where $\mathcal{C} = \mathcal{D} = \mathbb{R}^2$. It can be represented in the form of the general interconnection (15) without the external input u by letting $n = 2$, $F_1(x) = x_1 + x_2^2$, $F_2(x) = -3x_2 + 0.1\sqrt{|x_1|}$, $G_1(x) = e^{-1} x_1$, and $G_2(x) = e x_2$. As $\mathcal{C} = \mathcal{D} = \mathbb{R}^2$, the system may flow or jump at any point in $\mathbb{R}^2$, and all solutions are complete. Hence the notions of pre-ISS and ISS coincide, and so do the notions of pre-GAS and GAS. The x_1-subsystem Σ_1 has stablizing discrete dynamics but non-ISS continuous dynamics, while the x_2-subsystem Σ_2 has ISS continuous dynamics but destablizing discrete dynamics. Thus we cannot apply Theorem 2 and Proposition 1 to establish pre-GAS of the interconnection directly.

Consider the functions $V_1, V_2 : \mathbb{R} \to \mathbb{R}_+$ defined by

$$V_1(x_1) := |x_1|, \qquad V_2(x_2) := |x_2|,$$

and the functions $\chi_{12}, \chi_{21} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\chi_{12}(r) := \frac{1}{a} r^2, \qquad \chi_{21}(r) := \frac{1}{b}\sqrt{r}$$

with some scalars $a, b > 0$. As

$$\begin{aligned} V_1(x_1) \geq \chi_{12}(V_2(x_2)) \quad &\Rightarrow \quad \dot{V}_1(x_1) \leq (a+1) V_1(x_1), \\ V_2(x_2) \geq \chi_{21}(V_1(x_1)) \quad &\Rightarrow \quad \dot{V}_2(x_2) \leq (0.1b - 3) V_2(x_2), \end{aligned}$$

and [8]

$$V_1(x_1^+) \leq e^{-1} V_1(x_1), \qquad V_2(x_2^+) \leq e V_2(x_2)$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, it follows that V_1, V_2 are candidate exponential ISS Lyapunov functions for the subsystems Σ_1, Σ_2 with the internal gains χ_{12}, χ_{21}, respectively. Since the discrete dynamics of the Σ_2 is destabilizing, we invoke the modification scheme in Section 4.1. Consider the solutions $x : \operatorname{dom} x \to \mathbb{R}^2$ admitting an ADT $\delta_2 > 0$, namely, all $(s, k) \preceq (t, j)$ in the hybrid time domains $\operatorname{dom} x$ satisfy

$$j - k \leq \delta_2 (t - s) + N_{02} \tag{51}$$

with an integer $N_{02} \geq 1$. The corresponding ADT clock τ_2 is defined by

$$\begin{aligned} &\dot{\tau}_2 \in [0, \delta_2], && \tau_2 \in [0, N_{02}], \\ &\tau_2^+ = \tau_2 - 1, && \tau_2 \in [1, N_{02}]. \end{aligned}$$

Let $z_1 := x_1$ and $z_2 := (x_2, \tau_2)$. Following Proposition 5, we see that the function $W_2 : \mathbb{R} \times [0, N_{02}] \to \mathbb{R}_+$ defined by

$$W_2(z_2) := e^{L_2 \tau_2} V_2(x_2)$$

[8] Note that the discrete dynamics of both subsystems are autonomous, and hence we can ignore the terms corresponding to internal gains χ_{12}, χ_{21} in (8). Similar simplifications will be made when we apply Proposition 5 and Theorem 2.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

is a candidate exponential ISS Lyapunov function for the augmented subsystem $\tilde{\Sigma}_2$ with the internal gain $\tilde{\chi}_{21} \in \mathcal{K}$ defined by

$$\tilde{\chi}_{21}(r) := e^{L_2 N_{02}} \chi_{21}(r) = \frac{1}{b} e^{L_2 N_{02}} \sqrt{r}.$$

More specifically, for all $(z_1, z_2) \in \mathbb{R}^2 \times [0, N_{02}]$, if

$$W_2(z_2) \geq \tilde{\chi}_{21}(V_1(z_1))$$

then

$$\begin{aligned} \dot{W}_2(z_2; y_2) &= e^{L_2 \tau_2} \dot{V}_2(x_2) + L_2 e^{L_2 \tau_2} V_2(x_2) \dot{\tau}_2 \\ &\leq (0.1b - 3) e^{L_2 \tau_2} V_2(x_2) + L_2 \delta_2 e^{L_2 \tau_2} V_2(x_2) \\ &= (0.1b - 3 + L_2 \delta_2) W_2(z_2) \end{aligned}$$

for all $y_2 \in \{-3x_2 + 0.1\sqrt{|x_1|}\} \times [0, \delta_2]$. Furthermore,

$$W_2(ex_2, \tau_2 - 1) = e^{L_2(\tau_2 - 1) + 1} V_2(x_2) \leq e^{1 - L_2} W_2(z_2)$$

(see also footnote 8). To make the discrete dynamics of the z_2-subsystem ISS, we set

$$L_2 > 1. \tag{52}$$

Following (21), the gain operator $\tilde{\Gamma} : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ after modification is defined by

$$\tilde{\Gamma}(r_1, r_2) = \begin{bmatrix} \chi_{12}(r_2) \\ \tilde{\chi}_{21}(r_1) \end{bmatrix},$$

and the small-gain condition (23) holds for $\tilde{\Gamma}$ if and only if

$$\chi_{12}(\tilde{\chi}_{21}(r)) < r \qquad \forall\, r > 0,$$

or equivalently,

$$L_2 < \frac{\ln(ab^2)}{2N_{02}}. \tag{53}$$

Let $s > 0$ be such that

$$\frac{1}{b} e^{L_2 N_{02}} < \frac{1}{s} < \sqrt{a}.$$

Then the function $\sigma := (\sigma_1, \sigma_1)$ with $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ defined by

$$\sigma_1(r) := r, \qquad \sigma_2(r) := \frac{1}{s}\sqrt{r}$$

is an Ω-path w.r.t. the gain operator $\tilde{\Gamma}$. Following Theorem 2, the function $W : \mathbb{R}^2 \times [0, N_{02}] \to \mathbb{R}_+$ defined by

$$\begin{aligned} W(z) &:= \max\{\sigma_1^{-1}(V_1(z_1)),\, \sigma_2^{-1}(W_2(z_2))\} \\ &= \max\{V_1(z_1),\, s^2 W_2(z_2)^2\} \end{aligned}$$

is a candidate Lyapunov function w.r.t. $\tilde{\mathcal{A}} := \{(0,0)\} \times [0, N_{02}]$ for the augmented interconnection with the state $z := (z_1, z_2) \in \mathbb{R}^2 \times [0, N_{02}] =: \mathcal{Z}$. More specifically, for all $z \in \mathcal{Z}$,

$$\dot{W}(z; y) \leq -cW(z)$$

for all $y \in \{-x_1 + x_2^2\} \times \{-3x_2 + 0.1\sqrt{|x_1|}\} \times [0, \delta_2]$ with

$$c := \min\{-(a+1),\, 2(3 - 0.1b - L_2 \delta_2)\} < 0,$$

where the inequality follows from $a > 0$. Furthermore,

$$W(e^{-1} x_1, e x_2, \tau_2 - 1) \leq e^{-d} W(z)$$

with

$$d := \min\{1,\, 2(L_2 - 1)\} > 0,$$

where the inequality follows from (52).

Thus W is a candidate exponential Lyapunov function for the augmented interconnection with the rate coefficients c, d. Consider the set of solutions $x : \operatorname{dom} x \to \mathbb{R}^2$ admitting the ADT $\delta_2 > 0$ and an RADT $\delta^* > 0$, namely, all $(s, l) \preceq (t, j)$ in the hybrid time domains $\operatorname{dom} x$ satisfy (51) and

$$t - s \leq \delta^*(j - k) + \delta^* N_0^* \tag{54}$$

with an integer $N_0^* \geq 1$. Following Proposition 1 and Remark 4, this set of solution pairs is GAS provided that

$$0 < \delta^* < \frac{d}{-c} = \frac{\min\{1,\, 2(L_2 - 1)\}}{\max\{a + 1,\, 2(0.1b - 3 + L_2 \delta_2)\}}$$

and (53) hold. For example, if $a = 1$, $b = 5$ and $L_2 = 1.5$, then the set of solutions satisfying the ADT condition (51) with $\delta_2 = 2.25$ and $N_{02} = 1$, and also the RADT condition (54) with $\delta^* = 0.45$ and $N_0^* = 1$ is GAS.

5 Conclusion and future research

In this paper, we proved several small-gain theorems for interconnected hybrid systems which yield candidate ISS Lyapunov functions for the interconnection. These results unify several Lyapunov-based small-gain theorems for hybrid systems [35,6,27] and impulsive systems [7,9], and pave the way to the following general scheme for establishing ISS of interconnected hybrid systems:

1. Construct a candidate exponential ISS Lyapunov function V_i for each subsystem Σ_i with rate coefficients c_i, d_i and linear internal gains.
2. Compute the index sets I_d, I_c of non-ISS dynamics.
3. Modify the candidate exponential ISS Lyapunov functions V_i *either for all* $i \in I_d$ via Proposition 5, *or for all* $i \in I_c$ via Proposition 6.
4. Invoke Theorem 4 to construct a candidate exponential ISS Lyapunov function W for the augmented interconnection $\tilde{\Sigma}$ with rate coefficients c, d.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST

5. Derive the conditions for ISS of $\tilde{\Sigma}$ via Proposition 1.
6. Summarize the conditions for ISS of the original interconnection Σ from those in Steps 3 and 5.

As we observed in Section 4, the modification of candidate ISS Lyapunov functions in Step 3 leads to the increase in internal gains. Therefore a considerable improvement of the scheme above lies in the fact that only the candidate ISS Lyapunov functions with indices from I_d or those with indices from I_c would be modified, instead of all those with indices from $I_d \cup I_c$ as it was done in [27]. If either $I_d = \emptyset$ or $I_c = \emptyset$, then no subsystem needs to be modified at all. Moreover, the scheme above also applies to arbitrary interconnections composed of $n \geq 2$ subsystems.

In the scheme above, it is assumed that all V_i are candidate exponential ISS Lyapunov functions with linear internal gains. However, the modification also works for candidate exponential Lyapunov functions with *nonlinear* internal gains, and Theorem 2 was proved for arbitrary candidate ISS Lyapunov functions with nonlinear internal gains. If Proposition 1 were extended to the case of non-exponential ISS Lyapunov functions, one could apply the scheme above for V_i with nonlinear internal gains as well. Such theorems have been proved in [9, Theorems 1,3] for impulsive systems, and we believe that they can be generalized to hybrid systems as well. This is one of the possible directions for future research.

The more challenging questions are whether one can establish ISS of an interconnection in the presence of destabilizing dynamics in subsystems without enlarging the internal gains, or without modifying ISS Lyapunov functions at all. At the time these questions remain open.

Acknowledgements

The work of A. Mironchenko was supported by the German Research Foundation (DFG) grant Wi 1458/13-1. The work of G. Yang and D. Liberzon was supported by the NSF grants CNS-1217811 and ECCS-1231196.

The authors thank Navid Naroozi for his comments on the proof of the main result. The authors are also grateful to the anonymous reviewers for their careful evaluation of the paper and valuable suggestions.

References

[1] Abraham Berman and Robert J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Society for Industrial and Applied Mathematics, 1994.

[2] Chaohong Cai and Andrew R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[3] Chaohong Cai, Andrew R. Teel, and Rafal Goebel. Smooth Lyapunov functions for hybrid systems—Part I: Existence is equivalent to robustness. *IEEE Transactions on Automatic Control*, 52(7):1264–1277, 2007.

[4] Chaohong Cai, Andrew R. Teel, and Rafal Goebel. Smooth Lyapunov functions for hybrid systems Part II: (Pre)Asymptotically stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734–748, 2008.

[5] Sergey Dashkovskiy, Denis V. Efimov, and Eduardo D. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72(8):1579–1614, 2011.

[6] Sergey Dashkovskiy and Michael Kosmykov. Input-to-state stability of interconnected hybrid systems. *Automatica*, 49(4):1068–1074, 2013.

[7] Sergey Dashkovskiy, Michael Kosmykov, Andrii Mironchenko, and Lars Naujok. Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems*, 6(3):899–915, 2012.

[8] Sergey Dashkovskiy and Andrii Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, 2013.

[9] Sergey Dashkovskiy and Andrii Mironchenko. Input-to-state stability of nonlinear impulsive systems. *SIAM Journal on Control and Optimization*, 51(3):1962–1987, 2013.

[10] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. On the construction of ISS Lyapunov functions for networks of ISS systems. In *17th International Symposium on Mathematical Theory of Networks and Systems*, pages 77–82, 2006.

[11] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

[12] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[13] Charles A. Desoer and Mathukumalli Vidyasagar. *Feedback Systems: Input-Output Properties.* Society for Industrial and Applied Mathematics, 2009.

[14] Rafal Goebel, Ricardo G. Sanfelice, and Andrew R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton University Press, 2012.

[15] Lars Grüne. *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization.* Springer Berlin Heidelberg, 2002.

[16] Wassim M. Haddad, VijaySekhar Chellaboina, and Sergey G. Nersesov. *Impulsive and Hybrid Dynamical Systems.* Princeton University Press, 2006.

[17] João P. Hespanha, Daniel Liberzon, and Andrew R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44(11):2735–2744, 2008.

[18] João P. Hespanha and A. Stephen Morse. Stability of switched systems with average dwell-time. In *38th IEEE Conference on Decision and Control*, volume 3, pages 2655–2660, 1999.

[19] David J. Hill. A generalization of the small-gain theorem for nonlinear feedback systems. *Automatica*, 27(6):1043–1045, 1991.

[20] Zhong-Ping Jiang, Iven M. Y. Mareels, and Yuan Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996.

[21] Zhong-Ping Jiang, Andrew R. Teel, and Laurent Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, 7(2):95–120, 1994.

[22] Zhong-Ping Jiang and Yuan Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37(6):857–869, 2001.

[23] Iasson Karafyllis and Zhong-Ping Jiang. A small-gain theorem for a wide class of feedback systems with control applications. *SIAM Journal on Control and Optimization*, 46(4):1483–1517, 2007.

[24] Iasson Karafyllis and Zhong-Ping Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28(3):309–344, 2011.

[25] Dina S. Laila and Dragan Nešić. Discrete-time Lyapunov-based small-gain theorem for parameterized interconnected ISS systems. *IEEE Transactions on Automatic Control*, 48(10):1783–1788, 2003.

[26] Daniel Liberzon and Dragan Nešić. Stability analysis of hybrid systems via small-gain theorems. In *Hybrid Systems: Computation and Control*, pages 421–435. Springer Berlin Heidelberg, 2006.

[27] Daniel Liberzon, Dragan Nešić, and Andrew R. Teel. Lyapunov-based small-gain theorems for hybrid systems. *IEEE Transactions on Automatic Control*, 59(6):1395–1410, 2014.

[28] Tengfei Liu, David J. Hill, and Zhong-Ping Jiang. Lyapunov formulation of the large-scale, ISS cyclic-small-gain theorem: The discrete-time case. *Systems & Control Letters*, 61(1):266–272, 2012.

[29] Iven M. Y. Mareels and David J. Hill. Monotone stability of nonlinear feedback systems. *Journal of Mathematical Systems, Estimation, and Control*, 2:275–291, 1992.

[30] Andrii Mironchenko. *Input-to-state stability of infinite-dimensional control systems*. Ph.d. dissertation, Universität Bremen, 2012.

[31] Andrii Mironchenko, Guosong Yang, and Daniel Liberzon. Lyapunov small-gain theorems for not necessarily ISS hybrid systems. In *21st International Symposium on Mathematical Theory of Networks and Systems*, pages 1001–1008, 2014.

[32] Sayan Mitra, Daniel Liberzon, and Nancy Lynch. Verifying average dwell time of hybrid systems. *ACM Transactions on Embedded Computing Systems*, 8(1):1–37, 2008.

[33] A. Stephen Morse. Supervisory control of families of linear set-point controllers—Part I. Exact matching. *IEEE Transactions on Automatic Control*, 41(10):1413–1431, 1996.

[34] Dragan Nešić and Daniel Liberzon. A small-gain approach to stability analysis of hybrid systems. In *44th IEEE Conference on Decision and Control*, pages 5409–5414, 2005.

[35] Dragan Nešić and Andrew R. Teel. A Lyapunov-based small-gain theorem for hybrid ISS systems. In *47th IEEE Conference on Decision and Control*, pages 3380–3385, 2008.

[36] Björn S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space. *Positivity*, 14(2):257–283, 2010.

[37] Eduardo D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[38] Guosong Yang, Daniel Liberzon, and Andrii Mironchenko. Analysis of different Lyapunov function constructions for interconnected hybrid systems. In *55th IEEE Conference on Decision and Control*, pages 465–470, 2016.

Preprint submitted to Automatica
Received April 22, 2017 13:42:57 PST