# Feedback stabilization of switched linear systems with unknown disturbances under data-rate constraints

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***Abstract*—We study the problem of stabilizing a switched linear system with a completely unknown disturbance using sampled and quantized state feedback. The switching is assumed to be slow enough in the sense of combined dwell-time and average dwell-time, each individual mode is assumed to be stabilizable, and the data rate is assumed to be large enough but finite. By extending the approach of reachable-set approximation and propagation from an earlier result on the disturbance-free case, we develop a communication and control strategy that achieves a variant of input-to-state stability with exponential decay. An estimate of the disturbance bound is introduced to counteract the unknown disturbance, and a novel algorithm is designed to adjust the estimate and recover the state when it escapes the range of quantization.**

## I. Introduction

Feedback control under data-rate constraints has been an active research area for years, as surveyed in [1], [2]. In many application-related scenarios, it is important to limit the information flow in the feedback loop due to cost concerns, physical restrictions, security considerations, etc. Besides these practical motivations, the question of how much information is needed to achieve a certain control objective is fundamental and intriguing from the theoretical viewpoint. In this work, a finite data transmission rate is achieved by generating the control input based on sampled and quantized state measurements, which is a standard modeling framework in the literature (see, e.g., [3], [4] and [5, Ch. 5]).

This paper studies the problem of feedback stabilization under data-rate constraints in the presence of external disturbances. In this context, [3], [4] assumed known bounds on the disturbances and addressed asymptotic stabilization with minimum data rates, while [6], [7] avoided such assumptions by alternating between "zooming-out" and "zooming-in" stages and achieved input-to-state stability (ISS) [8]. See also [9], [10] for related results in a stochastic setting.

The study of switched and hybrid systems has attracted a lot of attention lately (particularly relevant results are discussed in [5], [11], [12] and many references therein). In stability and stabilization of switched systems, it is a standard technique to impose suitable slow-switching conditions, especially in the sense of dwell-time [13] and average dwell-time (ADT) [14]. This approach also plays a crucial role in our analysis.

Towards switched systems with disturbances, [14] showed that ISS can be achieved under the same ADT condition as the one for stability in the disturbance-free case. Their result was made explicit only for the case of switched linear systems, and many similar results for switched nonlinear systems have been established since then (see, e.g., [15] for ISS with a dwell-time, [16] for ISS and integral-ISS with an ADT, and [17] for input/output-to-state stability with an ADT).

Early works on control under data-rate constraints in the context of switched systems were devoted to quantized control of Markov jump linear systems [18]–[20]. However, the discrete modes in the results above were always known to the controller, which would remove a major difficulty in our problem setup, making the control problem essentially the same as the one in the case without switching. The problem of asymptotically stabilizing a switched linear system (without disturbance) using sampled and quantized state feedback was studied in [21], which also serves as the basis for this work. In [21], the controller was assumed to have a partial knowledge of the switching, that is, the active mode was unknown except at sampling times, and the switching was subject to mild slow-switching conditions characterized by a dwell-time and an ADT. Assuming that the data rate was large enough but finite, asymptotic stability was achieved by propagating over-approximations of reachable sets of the state over sampling intervals. See [22] for a related result using output feedback.

This work generalizes the main result of [21] in the presence of a completely unknown disturbance. By extending the approach of reachable-set approximation and propagation from [21], we develop a communication and control strategy that achieves a variant of ISS with exponential decay. Due to the unknown disturbance, the state may be driven outside the approximation of reachable set at a sampling time after it has already been inside an earlier one (i.e., the state *escapes* the range of quantization). Consequently, the closed-loop system may alternate multiple times between stabilizing and searching stages. An estimate of the disturbance bound is introduced in approximating reachable sets so that the state cannot escape unless the disturbance is larger than the estimate. A novel algorithm is designed to adjust the estimate and recover the state after escapes, so that the total length of searching stages is finite and the system eventually stays in a stabilizing stage, provided that the disturbance is globally essentially bounded (by an unknown value).

To the best of our knowledge, this work is the first result that combines switching, disturbances, and data-rate constraints, with the exception of its preliminary version [23], and our earlier result [24] for the easier case of disturbances with known bounds. This paper improves [23] by formulating continuous gain functions in the main theorem, removing unnecessary conditions, substantiating the results with complete proofs and clarifying remarks, and providing a detailed simulation study.

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This paper is organized as follows: Section II introduces the system definition, the information structure, and the basic assumptions. Our main result is stated in Section III. Section IV explains the communication and control strategy, assuming that suitable approximations of reachable sets are available. Such approximations are constructed in Section V. Section VI details the stability analysis with major steps summarized as technical lemmas. The simulation study is provided in Section VII. Section VIII concludes the paper with a summary and an outlook on future research topics.

## II. Problem formulation

### A. System definition

We are interested in stabilizing a switched linear control system modeled by

$$\dot{x} = A_\sigma x + B_\sigma u + D_\sigma d, \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control, and $d \in \mathbb{R}^{n_d}$ is the external disturbance. The set $\{(A_p, B_p, D_p) : p \in \mathcal{P}\}$ denotes a collection of matrix triples defining the modes (subsystems), where $\mathcal{P}$ is a finite *index set*. The function $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ is a right-continuous, piecewise constant *switching signal* which specifies the active mode $\sigma(t)$ at each time $t$. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation (1) away from discontinuities of $\sigma$ (in particular, there is no state jump). An admissible disturbance $d(\cdot)$ is a Lebesgue measurable and locally essentially bounded function. The switching signal $\sigma$ is fixed but unknown to the sensor and the controller a priori. Discontinuities of $\sigma$ are called *switching times*, or simply *switches*. The number of switches on a time interval $(\tau, t]$ is denoted by $N_\sigma(t, \tau)$.

Our first basic assumption is that the switching is slow in the sense of combined dwell-time and average dwel-time.

**Assumption 1** (Switching)**.** The switching signal $\sigma$ admits

1) a *dwell-time* $\tau_d > 0$ such that $N_\sigma(t, \tau) \leq 1$ for all $\tau \geq 0$ and $t \in (\tau, \tau + \tau_d]$, and
2) an *average dwell-time (ADT)* $\tau_a > \tau_d$ such that

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a} \qquad \forall\, t > \tau \geq 0 \tag{2}$$

with a constant $N_0 \geq 1$.

The notions of dwell-time [13] and ADT [14] have become standard in the literature on switched systems. In Assumption 1, the dwell-time condition (item 1) can be written in the form of (2) with $\tau_a = \tau_d$ and $N_0 = 1$; meanwhile, the ADT condition (item 2) would be implied by the dwell-time condition if $\tau_a \leq \tau_d$. Switching signals satisfying Assumption 1 were referred to as "hybrid dwell-time" signals in [25].

Our second basic assumption is that every individual mode is stabilizable.

**Assumption 2** (Stabilizability)**.** For each $p \in \mathcal{P}$, the pair $(A_p, B_p)$ is stabilizable, that is, there exists a state feedback gain matrix $K_p$ such that $A_p + B_p K_p$ is Hurwitz.

In the following analysis, it is assumed that such a collection of stabilizing gain matrices $K_p$, $p \in \mathcal{P}$ has been selected and fixed. However, even in the disturbance-free case, and when all individual modes are stabilized via state feedback (or stable without feedback), stability of the switched system is not necessarily guaranteed (see, e.g., [5, p. 19]).

Throughout this work, $\|\cdot\|$ denotes the $\infty$-norm of a vector, or the (induced) $\infty$-norm of a matrix, that is,

$$\|v\| := \|v\|_\infty := \max_{1 \leq i \leq n} |v_i|$$

for $v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n$, and

$$\|M\| := \|M\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |M_{ij}|$$

for $M = (M_{ij}) \in \mathbb{R}^{n \times n}$. The left-sided limit of a piecewise absolutely continuous function $z$ approaching $t$ is denoted by $z(t^-) := \lim_{s \nearrow t} z(s)$.

We let $\delta_d$ denote the essential supremum $\infty$-norm of the disturbance $d$, that is,

$$\delta_d := \|d\|_\infty := \operatorname*{ess\,sup}_{s \geq 0} \|d(s)\| \leq \infty, \tag{3}$$

and call it the *disturbance bound*. In the following analysis, it is assumed that $\delta_d$ is finite (as the state bound (6) in our main result below holds trivially when $\delta_d = \infty$). However, its value is *unknown* to the sensor and the controller.

### B. Information structure

The feedback loop consists of a *sensor* and a *controller*. The sensor measures two sequences of data—quantized measurements (samples) of the state $x(t_k)$, and indices of the active modes $\sigma(t_k)$—and transmits them to the controller at *sampling times* $t_k = k\tau_s$, where $k$ is a nonnegative integer and $\tau_s > 0$ is the *sampling period*. Each sample is encoded by an integer $i_k$ from 0 to $N^{n_x}$, where $N$ is an odd integer (so that the equilibrium at the origin is preserved). The controller generates the control input $u(\cdot)$ to the switched linear system (1) based on the decoded data. As $\sigma(t_k) \in \mathcal{P}$ and $i_k \in \{0, 1, \dots, N^{n_x}\}$, the data transmission rate between the encoder and the decoder is given by

$$R = \frac{\log_2 |N^{n_x} + 1| + \log_2 |\mathcal{P}|}{\tau_s}$$

bits per unit of time, where $|\mathcal{P}|$ is the cardinality of the index set $\mathcal{P}$ (i.e., the number of modes). As illustrated in Fig. 1, this information structure allows us to separate the sensing and the control tasks in the following sense: the sensor does not have access to the exact control objective, and the controller does not have access to the exact state. The communication and control strategy is explained in detail in Section IV.

The sampling period $\tau_s$ is assumed to be no larger than the dwell-time $\tau_d$ in Assumption 1, that is,

$$\tau_s \leq \tau_d, \tag{4}$$

so that there is at most one switch on each *sampling interval* $(t_k, t_{k+1}]$. Due to the ADT $\tau_a > \tau_d$ in Assumption 1, switches actually occur less often than once per sampling period.

Our last basic assumption imposes a lower bound on the data rate $R$.

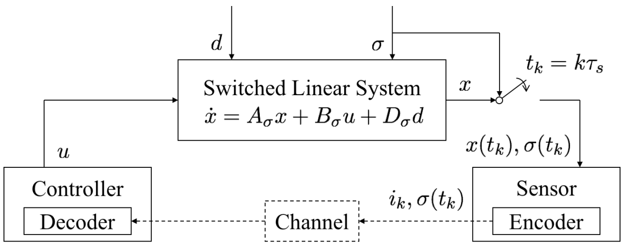


Fig. 1. Information structure

**Assumption 3** (Data rate)**.** The sampling period $\tau_s$ satisfies

$$\Lambda_p := \|e^{A_p \tau_s}\| < N \qquad \forall\, p \in \mathcal{P}. \tag{5}$$

The inequality in (5) can be interpreted as a lower bound on the data rate $R$ since it requires $\tau_s$ to be sufficiently small with respect to $N$. This bound is the same as the one for the disturbance-free case in [21, Assumption 3], and similar data-rate bounds appeared in [3], [4], [26] for stabilizing non-switched linear systems; see [7, Sec. V] and [21, Sec. 2.2] for more discussions on their relation.

## III. Main result

The control objective is to stabilize the system defined in Section II-A under the data-rate constraint described in Section II-B in a robust sense. More precisely, we intend to establish the following ISS-like property.

**Theorem 1** (Exponential decay)**.** *Consider the switched linear control system* (1)*. Suppose that Assumptions 1–3 and the inequality* (4) *hold. Then there is a communication and control strategy that yields the following property: provided that the average dwell-time $\tau_a$ is large enough, there exist a constant $\lambda > 0$ and gain functions $g, h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_d}$,*

$$\|x(t)\| \leq e^{-\lambda t} g(\|x_0\|) + h(\|d\|_\infty) \qquad \forall\, t \geq 0. \tag{6}$$

The communication and control strategy is described in Section IV. The lower bound on $\tau_a$ is given by (49) in Section VI-A3. The *exponential decay rate* $\lambda$ is given by (64), and the nonlinear gain functions $g$ and $h$ are given by (65), both in Section VI-C3. From the proof, it will be clear that both $g$ and $h$ can be made continuous and strictly increasing. However, $g(0) > 0$ due to the sampling and quantization, $h(0) > 0$ due to the unknown disturbance, and both $g(s)$ and $h(s)$ have superlinear growth rates as $s \to \infty$, which is consistent with [27, Cor. 2.3]. Consequently, the state bound (6) does not give the standard *input-to-state stability* (ISS) [8], but rather the *input-to-state practical stability* (ISpS) [28] with exponential decay, that is,

$$\|x(t)\| \leq e^{-\lambda t} \gamma_x(\|x_0\|) + \gamma_d(\|d\|_\infty) + C \qquad \forall\, t \geq 0$$

with the gain functions $\gamma_x, \gamma_d \in \mathcal{K}_\infty$ defined by[1]

$$\gamma_x(s) := g(s) - g(0), \quad \gamma_d(s) := h(s) - h(0) \tag{7}$$

[1] A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, positive definite, strictly increasing and unbounded.

and the constant

$$C := g(0) + h(0) > 0. \tag{8}$$

*Remark* 1. Along the lines of [29, Sec. VI], the state bound (6) can be restated as ISS with respect to a set. More specifically, (6) implies that the *uniform asymptotic gain* (UAG) property [29] holds for the set $\mathcal{A} := \{v \in \mathbb{R}^{n_x} : \|v\| \leq h(0)\}$, that is, for each pair of constants $\varepsilon, \delta > 0$, there exists a time $T_{\varepsilon,\delta} := \max\{\ln(g(h(0)+\delta)/\varepsilon)/\lambda,\, 0\}$ such that if $\|x_0\|_\mathcal{A} \leq \delta$, then $\|x(t)\|_\mathcal{A} \leq \gamma_d(\|d\|_\infty) + \varepsilon$ for all $t \geq T_{\varepsilon,\delta}$ with the gain function $\gamma_d \in \mathcal{K}_\infty$ defined in (7), where $\|v\|_\mathcal{A} := \inf_{v' \in \mathcal{A}} \|v - v'\|$ is the (Chebyshev) distance from a point $v$ to the set $\mathcal{A}$. In the context of non-switched systems, it has been shown that if UAG holds for $\mathcal{A}$, then the system is ISS with respect to the closure of the reachable set from $\mathcal{A}$ with $d \equiv 0$ [29, Lemma VI.2].

The state bound (6) implies the following stability property.

**Corollary 2** (Practical stability)**.** *In particular, the communication and control strategy in Theorem 1 yields the following property: provided that the average dwell-time $\tau_a$ is large enough, the system* (1) *is practically stable, that is, for each $\varepsilon > 0$, there exists a small enough $\delta > 0$ such that for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_d}$,*

$$\|x_0\|, \|d\|_\infty \leq \delta \implies \sup_{t \geq 0} \|x(t)\| \leq \varepsilon + C \tag{9}$$

*with the constant $C$ defined by* (8)*.*

Corollary 2 shows that, if the initial state and the disturbance are both small, then the solution is confined within a neighborhood of the hypercube of radius $C$ centered at the origin. In Section VI-D, we will establish practical stability with a smaller constant $C$ through a more direct approach.

## IV. Communication and control strategy

In this section, we describe the communication and control strategy in detail, assuming that suitable approximations of reachable sets of the state are available at all sampling times. (Such approximations are derived in the next section.)

The initial state $x_0$ is unknown. At $t_0 = 0$, the sensor and the controller are both provided with $x_0^* = 0$ and arbitrarily selected initial estimates $E_0 > 0$ and $\delta_0 > 0$ (for $\|x_0\|$ and the disturbance bound $\delta_d$ defined in (3), respectively). Starting from $t_0 = 0$, at each sampling time $t_k$, the sensor determines if the state $x(t_k)$ is inside the hypercube of radius $E_k$ centered at $x_k^*$ denoted by

$$\mathcal{S}_k := \{v \in \mathbb{R}^{n_x} : \|v - x_k^*\| \leq E_k\},$$

or equivalently, if

$$\|x(t_k) - x_k^*\| \leq E_k. \tag{10}$$

The hypercube $\mathcal{S}_k$ is the approximation of the reachable set at $t_k$, which is also used as the range of quantization. If (10) holds (i.e., if $x(t_k) \in \mathcal{S}_k$), we say that the state is *visible*, and the system is in a *stabilizing stage* described in Section IV-A. Otherwise the state is *lost*, and the system is in a *searching stage* described in Section IV-B.

To compensate for the unknown disturbance, we introduce an estimate $\delta_k$ of the disturbance bound $\delta_d$ in calculating $E_{k+1}$. Note that if $\delta_k < \delta_d$, then it is possible that $x(t_k) \in \mathcal{S}_k$ but $x(t_{k+1}) \notin \mathcal{S}_{k+1}$ (unlike in the disturbance-free case, where $x(t_k) \in \mathcal{S}_k$ implies that $x(t_l) \in \mathcal{S}_l$ for all $l \geq k$).

If the state is visible at $t_k$, then the system is in a stabilizing stage until the first sampling time $t_j > t_k$ such that $x(t_j) \notin \mathcal{S}_j$; in this case we say that the state *escapes* at $t_j$. Likewise, if the state is lost at $t_k$, then the system is in a searching stage until the first sampling time $t_i > t_k$ such that $x(t_i) \in \mathcal{S}_i$; in this case we say that the state *is recovered* at $t_i$. Due to the unknown disturbance, the system may alternate multiple times between stabilizing and searching stages. The rules for adjusting the estimate $\delta_k$ so that there are only a finite number of escapes are explained in Section IV-C.

## A. Stabilizing stage

At each sampling time $t_k$ in a stabilizing stage, the encoder divides the hypercube $\mathcal{S}_k$ into $N^{n_x}$ equal hypercubic boxes, $N$ per dimension, encodes each box by a unique integer index from 1 to $N^{n_x}$, and transmits the index $i_k$ of the box containing $x(t_k)$ to the decoder, along with the active mode $\sigma(t_k)$. The controller learns that (10) holds upon receiving $i_k \in \{1, \ldots, N^{n_x}\}$. The decoder follows the same pre-defined indexing protocol as the encoder, so that it is able to reconstruct the center $c_k$ of the hypercubic box containing $x(t_k)$ from $i_k$. Simple calculation shows that

$$\|x(t_k) - c_k\| \leq \frac{1}{N} E_k, \quad \|c_k - x_k^*\| \leq \frac{N-1}{N} E_k. \tag{11}$$

The controller then generates the control input $u(t) = K_{\sigma(t_k)} \hat{x}(t)$ for $t \in [t_k, t_{k+1})$, where $K_{\sigma(t_k)}$ is the state feedback gain matrix in Assumption 2, and $\hat{x}$ is the state of the auxiliary system

$$\dot{\hat{x}} = A_{\sigma(t_k)}\, \hat{x} + B_{\sigma(t_k)}\, u = (A_{\sigma(t_k)} + B_{\sigma(t_k)} K_{\sigma(t_k)})\, \hat{x} \tag{12}$$

with the boundary condition

$$\hat{x}(t_k) = c_k. \tag{13}$$

In particular, the auxiliary state $\hat{x}$ is reset to $c_k$ at each sampling time $t_k$ in a stabilizing stage. Both the sensor and the controller then use two functions $F$ and $G$ to calculate

$$\begin{aligned} x_{k+1}^* &:= F(\sigma(t_k), \sigma(t_{k+1}), c_k), \\ E_{k+1} &:= G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_k) \end{aligned} \tag{14}$$

for the next sampling time $t_{k+1}$ without further communication. The functions $F$ and $G$ are designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_d), \tag{15}$$

and $G$ is strictly increasing in the last argument, which is $\delta_k$ in (14) and $\delta_d$ in (15). Hence the state may escape at $t_{k+1}$ only if $\delta_k < \delta_d$. (However, $x(t_{k+1}) \in S_{k+1}$ does not imply that $\delta_k \geq \delta_d$.) The formulae for $F$ and $G$ are derived in Section V-A.

## B. Searching stage

At each sampling time $t_k$ in a searching stage, there is an unknown $\hat{D}_k$ such that

$$E_k < \|x(t_k) - x_k^*\| \leq \hat{D}_k. \tag{16}$$

For example, if the state escapes at $t_j$, then (15) implies that $\hat{D}_j = G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d)$; while if it is lost at $t_0 = 0$, then $\hat{D}_0 = \|x_0\|$. The encoder sends $i_k = 0$, the "overflow symbol", to the decoder. Upon receiving $i_k = 0$, the controller learns the state is lost, and sets the control input to be $u \equiv 0$ on $[t_k, t_{k+1})$. Both the sensor and the controller then use a function $\hat{G}$ to calculate

$$\begin{aligned} x_{k+1}^* &:= x_k^*, \\ E_{k+1} &:= \hat{G}(x_k^*, (1+\varepsilon_E) E_k, \delta_k) \end{aligned} \tag{17}$$

for the next sampling time $t_{k+1}$ without further communication, where $\varepsilon_E > 0$ is an arbitrary design parameter. The function $\hat{G}$ is designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq \hat{G}(x_k^*, \hat{D}_k, \delta_d), \tag{18}$$

and it is strictly increasing in the last two arguments. Note that the second argument of $\hat{G}$ in (18) is $\hat{D}_k$, whereas the one in (17) is $(1+\varepsilon_E)E_k$. With the additional coefficient $1+\varepsilon_E$, it is ensured that the growth rate of $E_k$ dominates that of $\hat{D}_k$; thus the state will be recovered in a finite time, as shown in Section V-B following the derivation of $\hat{G}$.

## C. Adjusting the estimate of the disturbance bound

When the state escapes at a sampling time $t_j$, the sensor and the controller learn that $\delta_{j-1} < \delta_d$, and adjust the estimate by enlarging it to $\delta_j = (1+\varepsilon_\delta)\delta_{j-1}$, where $\varepsilon_\delta > 0$ is an arbitrary design parameter. The estimate remains unchanged in all other cases; in particular, it is adjusted *once per searching stage*. Thus it is ensured that there is a finite number of searching stages in total, as the estimate becomes greater than or equal to the disturbance bound $\delta_d$ after finitely many adjustments, and the state cannot escape after that.

# V. Approximation of reachable sets

In this section, we derive the recursive formulas needed to implement the communication and control strategy. In Section V-A, we consider a stabilizing stage, and formulate the functions $F$ and $G$ in (14) so that (15) holds. In Section V-B, we consider a searching stage, formulate the function $\hat{G}$ in (17) so that (18) holds, and prove that the state will be recovered in a finite time.

## A. Stabilizing stage

Suppose that the state is visible at a sampling time $t_k$, that is, (10) holds.

*1) Sampling interval with no switch:* When

$$\sigma(t_k) = p = \sigma(t_{k+1}) \tag{19}$$

for some $p \in \mathcal{P}$, there is no switch on $(t_k, t_{k+1}]$ due to (4). Combining the switched linear system (1) and the auxiliary system (12), we obtain that

$$\begin{aligned} \dot{x} &= A_p x + B_p u + D_p d, \\ \dot{\hat{x}} &= A_p \hat{x} + B_p u. \end{aligned}$$

The error $e := x - \hat{x}$ satisfies that

$$\dot{e} = A_p e + D_p d, \qquad \|e(t_k)\| = \|x(t_k) - c_k\| \leq \frac{1}{N} E_k$$

on $[t_k, t_{k+1})$, where the boundary condition follows from (11) and (13). Hence

$$\begin{aligned} \|e(t_{k+1}^-)\| &= \left\| e^{A_p \tau_s} e(t_k) + \int_{t_k}^{t_{k+1}} e^{A_p(t_{k+1}-\tau)} D_p d(\tau) \mathrm{d}\tau \right\| \\ &\leq \|e^{A_p \tau_s}\| \|e(t_k)\| + \left( \int_0^{\tau_s} \|e^{A_p s} D_p\| \mathrm{d}s \right) \delta_d \\ &\leq \frac{\Lambda_p}{N} E_k + \Phi_p(\tau_s) \delta_d =: \hat{D}_{k+1} \end{aligned}$$

with the constant $\Lambda_p$ in (5) and the increasing function $\Phi_p : [0, \tau_s] \to \mathbb{R}$ defined by

$$\Phi_p(t) := \int_0^t \|e^{A_p s} D_p\| \mathrm{d}s. \tag{20}$$

Therefore, we set

$$E_{k+1} = G(p, p, x_k^*, E_k, \delta_k) := \frac{\Lambda_p}{N} E_k + \Phi_p(\tau_s) \delta_k. \tag{21}$$

As $x$ is continuous, (15) holds with $x_{k+1}^*$ set as the auxiliary state $\hat{x}$ approaching $t_{k+1}$, that is,

$$x_{k+1}^* = F(p, p, c_k) := \hat{x}(t_{k+1}^-) = S_p c_k \tag{22}$$

with the matrix $S_p := e^{(A_p + B_p K_p)\tau_s}$.

*2) Sampling interval with a switch:* When

$$\sigma(t_k) = p \neq q = \sigma(t_{k+1}) \tag{23}$$

for some $p, q \in \mathcal{P}$, there is exactly one switch on $(t_k, t_{k+1}]$ due to (4). Let $t_k + \bar{t}$ with $\bar{t} \in (0, \tau_s]$ denote the unknown switching time. Then

$$\sigma(t) = \begin{cases} p, & t \in [t_k, t_k + \bar{t}), \\ q, & t \in [t_k + \bar{t}, t_{k+1}]. \end{cases}$$

Before the switch, mode $p$ is active on $[t_k, t_k + \bar{t})$. Following essentially the calculations for the case with no switch, we see that the error $e = x - \hat{x}$ satisfies that

$$\|e(t_k + \bar{t})\| \leq \frac{\|e^{A_p \bar{t}}\|}{N} E_k + \Phi_p(\bar{t}) \delta_d$$

with the function $\Phi_p$ defined by (20). As $t_k + \bar{t}$ is unknown, we estimate $x(t_k + \bar{t})$ by comparing it with $\hat{x}(t_k + t') = e^{(A_p + B_p K_p)t'} c_k$ of the auxiliary system (12) at an arbitrarily selected time $t_k + t' \in [t_k, t_{k+1}]$ via the triangle inequality. First,

$$\begin{aligned} &\|\hat{x}(t_k + \bar{t}) - \hat{x}(t_k + t')\| \\ &\leq \|e^{(A_p + B_p K_p)\bar{t}} - e^{(A_p + B_p K_p)t'}\| \|c_k\| \\ &\leq \|e^{(A_p + B_p K_p)\bar{t}} - e^{(A_p + B_p K_p)t'}\| \left( \|x_k^*\| + \frac{N-1}{N} E_k \right), \end{aligned}$$

where the last inequality follows partially from (11). Then

$$\begin{aligned} &\|x(t_k + \bar{t}) - \hat{x}(t_k + t')\| \\ &\leq \|\hat{x}(t_k + \bar{t}) - \hat{x}(t_k + t')\| + \|e(t_k + \bar{t})\| \\ &\leq \|e^{(A_p + B_p K_p)\bar{t}} - e^{(A_p + B_p K_p)t'}\| \left( \|x_k^*\| + \frac{N-1}{N} E_k \right) \\ &\quad + \frac{\|e^{A_p \bar{t}}\|}{N} E_k + \Phi_p(\bar{t}) \delta_d \\ &=: \hat{D}'_{k+1}(t', \bar{t}, \delta_d). \end{aligned} \tag{24}$$

After the switch, mode $q$ is active on $[t_k + \bar{t}, t_{k+1}]$. Combining the switched linear system (1) and the auxiliary system (12) with $u = K_p \hat{x}$, we obtain that

$$\dot{z} = \bar{A}_{pq} z + \bar{D}_q d$$

for $z := (x^\top, \hat{x}^\top)^\top \in \mathbb{R}^{2n_x}$ with the matrices

$$\bar{A}_{pq} := \begin{bmatrix} A_q & B_q K_p \\ 0_{n_x \times n_x} & A_p + B_p K_p \end{bmatrix}, \quad \bar{D}_q = \begin{bmatrix} D_q \\ 0_{n_x \times n_d} \end{bmatrix}.$$

Combining it with a second auxiliary system

$$\dot{\hat{z}} = \bar{A}_{pq} \hat{z}, \quad \hat{z}(t_k + t') = (\hat{x}(t_k + t')^\top, \hat{x}(t_k + t')^\top)^\top, \tag{25}$$

we obtain that

$$\begin{aligned} \dot{z} &= \bar{A}_{pq} z + \bar{D}_q d, \\ \dot{\hat{z}} &= \bar{A}_{pq} \hat{z} \end{aligned}$$

with the boundary condition

$$\begin{aligned} &\|z(t_k + \bar{t}) - \hat{z}(t_k + t')\| \\ &= \max\{\|x(t_k + \bar{t}) - \hat{x}(t_k + t')\|, \|\hat{x}(t_k + \bar{t}) - \hat{x}(t_k + t')\|\} \\ &\leq \hat{D}'_{k+1}(t', \bar{t}, \delta_d), \end{aligned}$$

where the first inequality follows from the property that the $\infty$-norms of two vectors $v, w$ and their concatenation satisfy

$$\|(v^\top, w^\top)^\top\| = \max\{\|v\|, \|w\|\}. \tag{26}$$

Hence

$$\begin{aligned} &\|z(t_{k+1}^-) - \hat{z}(t_{k+1} - \bar{t} + t')\| \\ &= \left\| e^{\bar{A}_{pq}(\tau_s - \bar{t})} z(t_k + \bar{t}) + \int_{t_k + \bar{t}}^{t_{k+1}} e^{\bar{A}_{pq}(t_{k+1} - \tau)} \bar{D}_q d(\tau) \mathrm{d}\tau \right. \\ &\quad \left. - e^{\bar{A}_{pq}(\tau_s - \bar{t})} \hat{z}(t_k + t') \right\| \\ &\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \|z(t_k + \bar{t}) - \hat{z}(t_k + t')\| \\ &\quad + \left( \int_0^{\tau_s - \bar{t}} \|e^{\bar{A}_{pq} s} \bar{D}_q\| \mathrm{d}s \right) \delta_d \\ &\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \hat{D}'_{k+1}(t', \bar{t}, \delta_d) + \bar{\Phi}_{pq}(\tau_s - \bar{t}) \delta_d \end{aligned}$$

with the increasing function $\bar{\Phi}_{pq} : [0, \tau_s] \to \mathbb{R}$ defined by

$$\bar{\Phi}_{pq}(t) := \int_0^t \|e^{\bar{A}_{pq} s} \bar{D}_q\| \mathrm{d}s.$$

Again, we estimate $z(t_{k+1}^-)$ by comparing it with $\hat{z}(t_k+t'') = e^{\bar{A}_{pq}(t''-t')}\hat{z}(t_k+t')$ of the second auxiliary system (25) at an arbitrarily selected time $t_k + t'' \in [t_k, t_{k+1}]$ via the triangle inequality. First,

$$\begin{aligned}
&\|\hat{z}(t_{k+1} - \bar{t} + t') - \hat{z}(t_k + t'')\| \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t''-t')}\|\|\hat{z}(t_k + t')\| \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t''-t')}\|\|e^{(A_p+B_pK_p)t'}\| \\
&\quad \times \left(\|x_k^*\| + \frac{N-1}{N}E_k\right),
\end{aligned}$$

where the last inequality follows partially from (11) and (26). Then

$$\begin{aligned}
&\|z(t_{k+1}^-) - \hat{z}(t_k + t'')\| \\
&\leq \|z(t_{k+1}^-) - \hat{z}(t_{k+1} - \bar{t} + t')\| \\
&\quad + \|\hat{z}(t_{k+1} - \bar{t} + t') - \hat{z}(t_k + t'')\| \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\|\hat{D}'_{k+1}(t', \bar{t}, \delta_d) + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t''-t')}\| \\
&\quad \times \|e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta_d \\
&=: \hat{D}''_{k+1}(t', t'', \bar{t}, \delta_d). \qquad (27)
\end{aligned}$$

To remove the dependence of the unknown $\bar{t}$, we take the supremum over $\bar{t}$ (with fixed $t'$ and $t''$) and obtain that

$$\|z(t_{k+1}^-) - \hat{z}(t_k + t'')\| \leq \sup_{\bar{t}\in(0,\tau_s]} \hat{D}''_{k+1}(t', t'', \bar{t}, \delta_d) =: \hat{D}_{k+1}.$$

Therefore, we set $E_{k+1}$ by first replacing the disturbance bound $\delta_d$ in $\hat{D}''_{k+1}(t', t'', \bar{t}, \delta_d)$ with the estimate $\delta_k$, and then taking the maximum over $\bar{t}$, that is,

$$E_{k+1} = G(p, q, x_k^*, E_k, \delta_k) := \sup_{\bar{t}\in(0,\tau_s]} \hat{D}''_{k+1}(t', t'', \bar{t}, \delta_k). \qquad (28)$$

(Clearly, the design parameters $t'$ and $t''$ should be selected so that $E_{k+1}$ is minimized. However, their optimal values cannot be determined without imposing further constraints on the matrices $\{A_p, B_p, D_p, K_p : p \in \mathcal{P}\}$.) As $x$ is continuous, (15) holds with $x_{k+1}^*$ set as the projection of the second auxiliary state $\hat{z}$ approaching $t_k + t''$ onto the $x$-component, that is,

$$x_{k+1}^* = F(p, q, c_k) := (I_{n_x\times n_x}\ 0_{n_x\times n_x})\hat{z}(t_k + t'') = H_{pq}c_k \qquad (29)$$

with the matrix

$$H_{pq} := (I_{n_x\times n_x}\ 0_{n_x\times n_x})e^{\bar{A}_{pq}(t''-t')}\begin{pmatrix} I_{n_x\times n_x} \\ I_{n_x\times n_x}\end{pmatrix}e^{(A_p+B_pK_p)t'}.$$

In the remainder of this subsection, we derive a simpler but more conservative bound for $E_{k+1}$, which is more useful for computations. First, the norm of the difference of two matrix exponentials can be simplified via the following lemma.[2]

**Lemma 1.** *For all square matrices $X$ and $Y$,*

$$\|e^{X+Y} - e^X\| \leq e^{\|X\|+\|Y\|}\|Y\|.$$

*Proof:* See Appendix A. ■

[2]Using Lemma 1 instead of the inequality that $\|M - I\| \leq \|M\| + 1$ for all square matrices $M$ as in [21, eq. (20)] ensures that $\alpha_{pq} \to 0$ as $\tau_s \to 0$, a property we will use in the comparison to [14] in Remark 4. However, for a large enough $\tau_s$, it is possible that the bound in Lemma 1 is worse.

Based on Lemma 1 and the property that

$$\|e^{Ms}\| \leq e^{\|M\||s|} \qquad \forall\, M \in \mathbb{R}^{n\times n}, \forall\, s \in \mathbb{R},$$

from (28) it follows that

$$E_{k+1} \leq \alpha_{pq}\|x_k^*\| + \beta_{pq}E_k + \gamma_{pq}\delta_k \qquad (30)$$

with the constants

$$\begin{aligned}
\alpha_{pq} &:= e^{\|\bar{A}_{pq}\|\tau_s}e^{\|A_p+B_pK_p\|\max\{\tau_s,\, 2t'\}} \\
&\quad \times \|A_p + B_pK_p\|\max\{\tau_s - t',\, t'\} \\
&\quad + e^{\|\bar{A}_{pq}\|\max\{\tau_s,\, 2(t''-t'),\, \tau_s+2(t'-t'')\}}\|\bar{A}_{pq}\| \\
&\quad \times \max\{t'' - t',\, \tau_s + t' - t''\}\|e^{(A_p+B_pK_p)t'}\|, \qquad (31)\\
\beta_{pq} &:= \frac{N-1}{N}\alpha_{pq} + \frac{1}{N}e^{(\|\bar{A}_{pq}\|+\|A_p\|)\tau_s}, \\
\gamma_{pq} &:= e^{\|\bar{A}_{pq}\|\tau_s}\Phi_p(\tau_s) + \bar{\Phi}_{pq}(\tau_s).
\end{aligned}$$

*Remark* 2. If we set $t'' = t' = 0$, then (30) becomes

$$E_{k+1} \leq \alpha_{pq}^0\|x_k^*\| + \beta_{pq}^0E_k + \gamma_{pq}\delta_k$$

with the constants

$$\begin{aligned}
\alpha_{pq}^0 &:= e^{\|\bar{A}_{pq}\|\tau_s}(e^{\|A_p+B_pK_p\|\tau_s}\|A_p + B_pK_p\|\tau_s + \|\bar{A}_{pq}\|\tau_s, \\
\beta_{pq}^0 &:= \frac{N-1}{N}\alpha_{pq}^0 + \frac{1}{N}e^{(\|\bar{A}_{pq}\|+\|A_p\|)\tau_s}. \qquad (32)
\end{aligned}$$

Although this choice of $t'$ and $t''$ considerably simplifies the formula of the bound, it does not necessarily minimize $E_{k+1}$.

### *B. Searching stage*

Suppose that the state is lost at a sampling time $t_k$, that is, (16) holds.

*1) Reachable-set approximation:* Let $p = \sigma(t_k)$, and consider an arbitrary $t \in (t_k, t_{k+1}]$. If $\sigma(t) = p$, then there is no switch on $(t_k, t]$ due to (4); thus

$$\begin{aligned}
&\|x(t) - x_k^*\| \\
&= \left\|e^{A_p(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A_p(t-\tau)}D_pd(\tau)\mathrm{d}\tau - x_k^*\right\| \\
&\leq \|e^{A_p(t-t_k)} - I\|\|x_k^*\| + \|e^{A_p(t-t_k)}\|\|x(t_k) - x_k^*\| \\
&\quad + \left(\int_0^{t-t_k}\|e^{A_ps}D_p\|\mathrm{d}s\right)\delta_d \\
&\leq \bar{\Gamma}\|x_k^*\| + \bar{\Lambda}\hat{D}_k + \bar{\Phi}\delta_d
\end{aligned}$$

with the constants

$$\begin{aligned}
\bar{\Gamma} &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \|e^{A_pt} - I\|, \\
\bar{\Lambda} &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \|e^{A_pt}\| \geq 1, \qquad (33)\\
\bar{\Phi} &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \Phi_p(t) = \max_{p\in\mathcal{P}}\Phi_p(\tau_s).
\end{aligned}$$

If $\sigma(t) = q \neq p$, then there is exactly one switch on $(t_k, t]$ due to (4); thus

$$\begin{aligned}
&\|x(t) - x_k^*\| \\
&= \left\| e^{A_q(t-t_k-\bar{t})} x(t_k + \bar{t}) + \int_{t_k+\bar{t}}^{t} e^{A_q(t-\tau)} D_q d(\tau) \mathrm{d}\tau - x_k^* \right\| \\
&\leq \|e^{A_q(t-t_k-\bar{t})} - I\| \|x_k^*\| + \|e^{A_q(t-t_k-\bar{t})}\| \|x(t_k+\bar{t}) - x_k^*\| \\
&\quad + \left( \int_0^{t-t_k-\bar{t}} \|e^{A_q s} D_q\| \mathrm{d}s \right) \delta_d \\
&\leq \bar{\Gamma} \|x_k^*\| + \bar{\Lambda} \|x(t_k+\bar{t}) - x_k^*\| + \bar{\Phi} \delta_d \\
&\leq \bar{\Gamma} \|x_k^*\| + \bar{\Lambda}(\bar{\Gamma}\|x_k^*\| + \bar{\Lambda}\hat{D}_k + \bar{\Phi}\delta_d) + \bar{\Phi}\delta_d \\
&\leq (\bar{\Lambda}+1)\bar{\Gamma}\|x_k^*\| + \bar{\Lambda}^2 \hat{D}_k + (\bar{\Lambda}+1)\bar{\Phi}\delta_d,
\end{aligned}$$

where $t_k + \bar{t}$ denotes the unknown switching time. As $\bar{\Lambda} \geq 1$, the bound for the second case holds for both cases, that is,

$$\|x(t) - x_k^*\| \leq \bar{\alpha}\|x_k^*\| + \bar{\beta}\hat{D}_k + \bar{\gamma}\delta_d =: \hat{D}_{k+1} \tag{34}$$

for all $t \in (t_k, t_{k+1}]$ with the constants

$$\bar{\alpha} := (\bar{\Lambda}+1)\bar{\Gamma}, \quad \bar{\beta} := \bar{\Lambda}^2, \quad \bar{\gamma} := (\bar{\Lambda}+1)\bar{\Phi}.$$

From $\bar{\beta} = \bar{\Lambda}^2 \geq 1$, it follows that $\hat{D}_{k+1} \geq \hat{D}_k$. In order to dominate the growth rate of $\hat{D}_{k+1}$, we set

$$\begin{aligned}
E_{k+1} &= \hat{G}(x_k^*, (1+\varepsilon_E)E_k, \delta_k) \\
&:= \bar{\alpha}\|x_k^*\| + (1+\varepsilon_E)\bar{\beta}E_k + \bar{\gamma}\delta_k
\end{aligned} \tag{35}$$

with the arbitrary design parameter $\varepsilon_E > 0$.

*2) Recovery in a finite time:* Suppose that the state escapes at a sampling time $t_j$ (or it is lost at $t_j = t_0 = 0$), and remains lost at $t_{j+1}, \ldots, t_{k-1}$. Then the disturbance estimate satisfies that $\delta_{k-1} = \cdots = \delta_{j+1} = \delta_j = (1+\varepsilon_\delta)\delta_{j-1}$. From the recursive formulas (34) and (35), it follows that

$$\begin{aligned}
\hat{D}_k &= \bar{\beta}^{k-j}\hat{D}_j + \frac{\bar{\beta}^{k-j}-1}{\bar{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_d), \\
E_k &= \hat{\beta}^{k-j}E_j + \frac{\hat{\beta}^{k-j}-1}{\hat{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)
\end{aligned} \tag{36}$$

with the constant[3]

$$\hat{\beta} := (1+\varepsilon_E)\bar{\beta} > \bar{\beta}.$$

Let $c_\beta := (\hat{\beta}-1)/(\bar{\beta}-1)$, and consider the integer-valued functions $\eta_E, \eta_\delta : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$\begin{aligned}
\eta_E(s) &:= \begin{cases} \lceil \log_{1+\varepsilon_E} s \rceil, & s > 1; \\ 0, & 0 \leq s \leq 1, \end{cases} \\
\eta_\delta(s) &:= \begin{cases} \lceil \log_{1+\varepsilon_E}(c_\beta s) \rceil, & s > 1; \\ 0, & 0 \leq s \leq 1, \end{cases}
\end{aligned} \tag{37}$$

where $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function, that is, $\lceil s \rceil := \min\{m \in \mathbb{Z} : m \geq s\}$. Consider the integer

$$k' := j + \max\{\eta_E(\hat{D}_j/E_j),\ \eta_\delta(\delta_d/\delta_j)\}.$$

First, it holds that

$$\hat{\beta}^{k'-j}E_j \geq \bar{\beta}^{k'-j}(1+\varepsilon_E)^{\eta_E(\hat{D}_j/E_j)}E_j \geq \bar{\beta}^{k'-j}\hat{D}_j.$$

Second, if $\delta_d \leq \delta_j$ then

$$\frac{\hat{\beta}^{k'-j}-1}{\hat{\beta}-1}\delta_j \geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\delta_d$$

due to $\hat{\beta} > \bar{\beta}$ and $k' \geq j$; otherwise

$$\begin{aligned}
\frac{\hat{\beta}^{k'-j}-1}{\hat{\beta}-1}\delta_j &> \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}(1+\varepsilon_E)^{k'-j}\delta_j \\
&\geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\hat{\beta}-1}{\bar{\beta}-1}(1+\varepsilon_E)^{\eta_\delta(\delta_d/\delta_j)}\delta_j \geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\delta_d.
\end{aligned}$$

Hence $E_{k'} \geq \hat{D}_{k'}$, that is, the state is recovered no later than $t_{k'}$. Denote by $t_i$ the sampling time of recovery. Then[4]

$$i - j \leq \max\{\eta_E(\hat{D}_j/E_j),\ \eta_\delta(\delta_d/\delta_j)\}. \tag{38}$$

However, $\delta_d$ being unknown implies that neither the sensor nor the controller is able to predict how long it will take to recover the state.

## VI. Stability analysis

In this section, we show that the communication and control strategy described in Section IV fulfills the claims of Theorem 1. In Section VI-A, we formulate a Lyapunov-based bound with exponential decay for stabilizing stages. Then we derive its exponential growth for searching stages in Section VI-B. In Section VI-C, we calculate the maximum number of searching stages, and prove the ISS-like property in Theorem 1. A stronger version of Corollary 2 is established in Section VI-D.

### A. Stabilizing stage

*1) Sampling interval with no switch:* Consider a sampling interval $(t_k, t_{k+1}]$ such that (19) holds, as in Section V-A1. As $A_p + B_p K_p$ is Hurwitz, for $S_p$ in (22) there exist positive definite matrices $P_p, Q_p \in \mathbb{R}^{n_x \times n_x}$ such that

$$S_p^\top P_p S_p - P_p = -Q_p < 0. \tag{39}$$

Let $\overline{\lambda}(M)$ and $\underline{\lambda}(M)$ denote the largest and smallest eigenvalues of a matrix $M$, respectively, and define

$$\chi_p := \frac{2n_x^2\|S_p^\top P_p S_p\|^2}{\underline{\lambda}(Q_p)} + n_x\|S_p^\top P_p S_p\|. \tag{40}$$

Due to the inequality in (5), there exists a sufficiently small constant $\phi_1 > 0$ such that $(1+\phi_1)\Lambda_{p'}^2 < N^2$ for all $p' \in \mathcal{P}$. Then for each $p'$, there exists a sufficiently large constant $\rho_{p'} > 0$ such that

$$\frac{(N-1)^2}{N^2}\frac{\chi_{p'}}{\rho_{p'}} + \frac{(1+\phi_1)\Lambda_{p'}^2}{N^2} < 1. \tag{41}$$

[3]From (33) it follows that $\bar{\beta} = \bar{\Lambda}^2 \geq 1$, and $\bar{\Lambda} = 1$ only if all eigenvalues of all $A_p$ have nonpositive real parts. In the following analysis, we assume that $\bar{\beta} > 1$ (so that the first formula in (36) is well-defined), which can be achieved by letting $\bar{\beta} = \max\{\bar{\Lambda}^2, 1+\varepsilon\}$ for an arbitrary $\varepsilon > 0$ if necessary. The special case where $\bar{\beta} = 1$ can be treated using similar arguments, and is omitted here for brevity.

[4]The function $\eta_\delta$ is piecewise-defined since if $\delta_d \leq \delta_j$ in (38)—which is possible as the escape only implies $\delta_d > \delta_{j-1} = \delta_j/(1+\varepsilon_\delta)$—then the second term on the right-hand side of the second formula in (36) is larger than or equal to that of the first formula for all $k \geq j$. Similarly, the function $\eta_E$ is piecewise-defined since if $\hat{D}_0 = \|x_0\| \leq E_0$ in (57) below then there is no searching stage at the beginning.

Consider a family of positive definite functions $V_{p'} : \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$V_{p'}(x, E) := x^\top P_{p'} x + \rho_{p'} E^2, \qquad p' \in \mathcal{P}. \tag{42}$$

For the sampling interval $(t_k, t_{k+1}]$ with no switch, the following lemma provides a bound for $V_{\sigma(t_{k+1})}(x^*_{k+1}, E_{k+1})$ in terms of $V_{\sigma(t_k)}(x^*_k, E_k)$ and the disturbance estimate $\delta_k$.

**Lemma 2.** *Consider a sampling interval $(t_k, t_{k+1}]$ such that* (10) *and* (19) *hold. Then*

$$V_p(x^*_{k+1}, E_{k+1}) \leq \nu V_p(x^*_k, E_k) + \nu_d \delta_k^2 \tag{43}$$

*with the constants*[5]

$$\begin{aligned}
\nu &:= \max_{p \in \mathcal{P}} \nu_p, \\
\nu_p &:= \max \left\{ \frac{(N-1)^2}{N^2} \frac{\chi_p}{\rho_p} + \frac{(1+\phi_1)\Lambda_p^2}{N^2}, \, 1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)} \right\}, \\
\nu_d &:= \max_{p \in \mathcal{P}} \left(1 + \frac{1}{\phi_1}\right) \rho_p \Phi_p(\tau_s)^2.
\end{aligned} \tag{44}$$

*Proof:* See Appendix B. ■

*2) Sampling interval with a switch:* Consider a sampling interval $(t_k, t_{k+1}]$ such that (23) holds, as in Section V-A2. Let $h_{pq}$ be the largest singular value of $H_{pq}$ in (29), that is,

$$h_{pq} := \sqrt{\overline{\lambda}(H_{pq}^\top H_{pq})}.$$

Consider the functions $V_p$ and $V_q$ defined by (42). For the sampling interval $(t_k, t_{k+1}]$ with a switch, the following lemma provides a bound for $V_{\sigma(t_{k+1})}(x^*_{k+1}, E_{k+1})$ in terms of $V_{\sigma(t_k)}(x^*_k, E_k)$ and the disturbance estimate $\delta_k$.

**Lemma 3.** *Consider a sampling interval $(t_k, t_{k+1}]$ such that* (10) *and* (23) *hold. Then*

$$V_q(x^*_{k+1}, E_{k+1}) \leq \mu V_p(x^*_k, E_k) + \mu_d \delta_k^2 \tag{45}$$

*with the constants*

$$\begin{aligned}
\mu &:= \max_{p,q \in \mathcal{P}} \mu_{pq}, \\
\mu_{pq} &:= \max \left\{ \frac{2\overline{\lambda}(P_q) h_{pq}^2}{\underline{\lambda}(P_p)} + \frac{(2+\phi_2)\alpha_{pq}^2 \rho_q}{\underline{\lambda}(P_p)}, \right. \\
&\qquad \left. \frac{(N-1)^2}{N^2} \frac{2n_x \overline{\lambda}(P_q) h_{pq}^2}{\rho_p} + \frac{(2+\phi_2)\beta_{pq}^2 \rho_q}{\rho_p} \right\}, \\
\mu_d &:= \max_{p,q \in \mathcal{P}} \left(1 + \frac{2}{\phi_2}\right) \rho_q \gamma_{pq}^2,
\end{aligned} \tag{46}$$

*where $\phi_2 > 0$ is an arbitrary design parameter.*

*Proof:* See Appendix C. ■

*Remark* 3. From the definitions of $\nu$ in (44) and the inequality (41), it follows that $\nu < 1$. Meanwhile, if we set $t' = t'' = 0$ in (29), then $h_{pq} = 1$ for all $p, q \in \mathcal{P}$, and from the definition of $\mu$ in (46) it follows that $\mu \geq 1 > \nu$. While this may not hold for general $t', t'' \in [0, \tau_s]$, we are able to ensure

$$\mu > \nu \tag{47}$$

by letting $\mu = 1$ if all $\mu_{pq} < 1$. Meanwhile, the relation between $\mu_d$ and $\nu_d$ depends on the values of $\phi_1$ and $\phi_2$. Since (45) holds for all $\phi_2 > 0$, given an arbitrary $\phi_1$, a sufficiently small $\phi_2$ (e.g., $\phi_2 = \phi_1$) can be selected so that

$$\mu_d \geq \nu_d. \tag{48}$$

(Alternatively, we can simply replace $\mu_d$ with $\max\{\mu_d, \nu_d\}$ if necessary.) In the following analysis, we assume the inequalities (47) and (48) hold. Consequently, the bound in (45) holds for all sampling intervals in stabilizing stages, regardless of whether there is a switch.

*3) Combined bound at sampling times:* Combining the bounds (43) and (45), we derive a condition on the ADT $\tau_a$ in (2) that ensures a bound with exponential decay for $V_{\sigma(t_k)}(x^*_k, E_k)$ at sampling times $t_k$ in a stabilizing stage.

**Lemma 4.** *Consider a sequence of consecutive sampling times $t_i, \ldots, t_{k-1}$ in a stabilizing stage. Suppose that the average dwell-time $\tau_a$ satisfies*

$$\tau_a > \left(1 + \frac{\ln \mu}{\ln(1/\nu)}\right) \tau_s. \tag{49}$$

*There exists a sufficiently small constant $\phi_3 \in (0, 1)$ such that*

$$V_{\sigma(t_k)}(x^*_k, E_k) < \Theta^{N_0} \left(\theta^{k-i} V_{\sigma(t_i)}(x^*_i, E_i) + \Theta_d \delta_i^2\right) \tag{50}$$

*with the constants $N_0$ in* (2) *and*

$$\begin{aligned}
\theta &:= \frac{(\mu + \phi_3(1-\nu)\mu_d/\nu_d)^{\tau_s/\tau_a}}{(\nu + \phi_3(1-\nu))^{\tau_s/\tau_a - 1}} < 1, \\
\Theta &:= \frac{\mu + \phi_3(1-\nu)\mu_d/\nu_d}{\nu + \phi_3(1-\nu)} > 1, \\
\Theta_d &:= \frac{\mu}{\phi_3(1-\nu)} \nu_d + \mu_d.
\end{aligned} \tag{51}$$

*Proof:* See Appendix D. ■

*Remark* 4. In [14], the authors considered switched linear systems with inputs (disturbances) and derived a lower bound on the ADT that ensured a variant of ISS with exponential decay.[6] The lower bound (49) on the average dwell-time $\tau_a$ in Lemma 4, in the absence of sampling and quantization, is consistent with the one in [14, Th. 2]. More specifically, the case without sampling and quantization can be approximated by letting $\tau_s \to 0$ and $N \to \infty$. Consequently, $S_p \to I + (A_p + B_p K_p)\tau_s$ in (22), $H_{pq} \to I$ in (29), and $\alpha_{pq}, \beta_{pq} \to 0$ in (31); thus

$$\nu \to 1 - \min_{p \in \mathcal{P}} \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}, \quad \mu \to \max_{p,q \in \mathcal{P}} \frac{2\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)}$$

in (44) and (46) with large enough $\rho_p$ for all $p \in \mathcal{P}$. Moreover, the first order approximation in $\tau_s$ of the Lyapunov equation (39) is given by

$$\left((A_p + B_p K_p)^\top P_p + P_p(A_p + B_p K_p)\right)\tau_s = -Q_p.$$

[5] The denominator in the second term of the maximum in the definition of $\nu_p$ in (44) is reduced to $1/n_x$ of the corresponding term in [21, eq. (34)]. This improvement is due to the more suitable inequalities (67) and (68) from linear algebra. The first numerator in the first term of the maximum in the definition of $\mu_{pq}$ in (46) is reduced to $1/n_x$ of the corresponding term in [21, eq. (37)] for the same reason.

[6] More precisely, the result in [14] is stated in terms of "input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_\infty$-induced norm", which ensures an exponential decay rate.

As the index set $\mathcal{P}$ is finite, from Assumption 2 it follows that there exists a constant $\lambda_0 > 0$ such that all $A_p + B_pK_p + \lambda_0 I$ are Hurwitz; thus the (approximated) Lyapunov equation above holds with $P_p$ satisfying

$$(A_p + B_pK_p + \lambda_0 I)^\top P_p + P_p(A_p + B_pK_p + \lambda_0 I) = -I,$$

and $Q_p = (2\lambda_0 P_p + I)\tau_s$. Then (49) can be approximated by

$$\tau_a > \frac{\ln(2\mu^*)}{\min_{p\in\mathcal{P}}\left(\frac{\underline{\lambda}(P_p)}{2\overline{\lambda}(P_p)}2\lambda_0 + \frac{1}{2\overline{\lambda}(P_p)}\right)}$$

with

$$\mu^* := \max_{p,q\in\mathcal{P},\ p\neq q} \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)},$$

which is in a similar form as the lower bound

$$\tau_a \geq \tau_a^* > \frac{\ln \mu^*}{2\lambda_0}$$

in [14] (see [5, p. 59, eq. (3.10)] for an explicit bound on $\tau_a^*$). The additional terms result from the more complex Lyapunov functions (42) we used due to the sampling and quantization. In particular, the additional coefficients in the numerator and the first term of the denominator are generated when completing the squares. Meanwhile, we can made $\overline{\lambda}(P_p)$ arbitrarily large (and thus the second term of the denominator arbitrarily small) by selecting a sufficiently small $\lambda_0$.

### B. Searching stage

*1) Recovery:* Suppose that the state escapes at a sampling time $t_j$ and is recovered at $t_i$, as in Section V-B. Then $[t_j, t_i)$ is a searching stage, while the sampling period $[t_{j-1}, t_j)$ belongs to a stabilizing stage. At $t_j$, from (15) and (16) it follows that

$$E_j < \|x(t_j) - x_j^*\| \leq \hat{D}_j$$

with

$$\begin{aligned}\hat{D}_j &= G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d),\\ E_j &= G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_{j-1}).\end{aligned}$$

From the formulae (21) and (28) of $G$, it follows that

$$\hat{D}_j/E_j < \delta_d/\delta_{j-1} = (1+\varepsilon_\delta)\delta_d/\delta_j.$$

Let $c_\varepsilon := \max\{1+\varepsilon_\delta,\ (\hat{\beta}-1)/(\bar{\beta}-1)\}$ and define an integer-valued function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by

$$\begin{aligned}\eta(s) &:= \max\{\eta_E((1+\varepsilon_\delta)s),\ \eta_\delta(s)\}\\ &= \begin{cases}\lceil \log_{1+\varepsilon_E}(c_\varepsilon s)\rceil, & s > 1;\\ \lceil \log_{1+\varepsilon_E}((1+\varepsilon_\delta)s)\rceil, & (1+\varepsilon_\delta)^{-1} < s \leq 1;\\ 0, & 0 \leq s \leq (1+\varepsilon_\delta)^{-1}.\end{cases}\end{aligned} \tag{52}$$

From (38) it follows that

$$i - j \leq \eta(\delta_d/\delta_j), \tag{53}$$

which, combined with (36), implies that

$$\begin{aligned}E_i &= \hat{\beta}^{i-j}E_j + \frac{\hat{\beta}^{i-j}-1}{\hat{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)\\ &< \hat{\beta}^{i-j}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right)\\ &< \hat{\beta}^{\eta(\delta_d/\delta_j)}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right).\end{aligned} \tag{54}$$

For the searching stage $[t_j, t_i)$, the following lemma provides a bound for $V_{\sigma(t_i)}(x_i^*, E_i)$ at the recovery in terms of $V_{\sigma(t_j)}(x_j^*, E_j)$ at the escape and $\delta_d, \delta_j$.

**Lemma 5.** *Suppose that the state escapes at a sampling time $t_j$ and is recovered at $t_i$. Then*

$$V_{\sigma(t_i)}(x_i^*, E_i) \leq \hat{\beta}^{2\eta(\delta_d/\delta_j)}\big(\omega V_{\sigma(t_j)}(x_j^*, E_j) + \omega_d\delta_j^2\big) \tag{55}$$

*with*

$$\begin{aligned}\omega &:= \max_{p,q\in\mathcal{P}} \omega_{pq},\\ \omega_{pq} &:= \max\left\{\frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} + \frac{(2+\phi_4)\bar{\alpha}^2\rho_q}{(\hat{\beta}-1)^2\underline{\lambda}(P_p)},\ \frac{(2+\phi_4)\rho_q}{\rho_p}\right\},\\ \omega_d &:= \max_{q\in\mathcal{P}}\left(1+\frac{2}{\phi_4}\right)\frac{\bar{\gamma}^2\rho_q}{(\hat{\beta}-1)^2},\end{aligned} \tag{56}$$

*where $\phi_4 > 0$ is an arbitrary design parameter.*

*Proof:* See Appendix E. ■

*2) Initial capture:* The case where the state is lost at $t_0 = 0$ and is recovered at $t_{i_0}$ for the first time can be analyzed in a similar manner. From (38) with $j = 0$ and $\hat{D}_0 = \|x_0\|$, it follows that

$$i_0 \leq \eta_E(\|x_0\|/E_0) + \eta_\delta(\delta_d/\delta_0), \tag{57}$$

which, combined with (36) and $x_0^* = 0$, implies that

$$E_{i_0} < \hat{\beta}^{\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0)}\left(E_0 + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_0\right).$$

For the searching stage $[0, t_{i_0})$, the following lemma provides a bound for $V_{\sigma(t_{i_0})}(0, E_{i_0})$ at the first recovery in terms of $V_{\sigma(0)}(0, E_0) = \rho_{\sigma(0)}E_0^2$ at $t = 0$ and $\|x_0\|, E_0, \delta_d, \delta_0$.

**Lemma 6.** *Suppose that the state is lost at $t_0 = 0$ and is recovered at $t_{i_0}$. Then*

$$\begin{aligned}V_{\sigma(t_{i_0})}(0, E_{i_0}) &\leq \hat{\beta}^{2(\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0))}\\ &\quad\times\big(\omega_0 V_{\sigma(0)}(0, E_0) + \omega_d\delta_0^2\big)\end{aligned} \tag{58}$$

*with*

$$\omega_0 := \max_{q\in\mathcal{P}}\left(1+\frac{\phi_4}{2}\right)\frac{\rho_q}{\rho_{\sigma(0)}} \leq \frac{1}{2}\omega,$$

*where $\phi_4$ is the design parameter in* (56).

*Proof:* The proof is essentially the same as the one of Lemma 5 and is omitted here. ■

### C. Exponential decay

*1) Number of searching stages:* As explained in Section IV-C, the closed-loop system alternates between a finite number of searching and stabilizing stages, and eventually stays in a stabilizing stage. Let $0 = j_0 \leq i_0 < j_1 < i_1 < \cdots < j_{N_s} < i_{N_s}$ be such that $[t_{j_m}, t_{i_m})$ denotes a searching stage and $[t_{i_m}, t_{j_{m+1}})$ denotes a stabilizing stage for each $m \in \{0, \ldots, N_s\}$.[7] As the estimate is enlarged by a factor of $1 + \varepsilon_\delta$ every time the state escapes, it satisfies

$$\delta_k = (1+\varepsilon_\delta)^m \delta_0 \qquad \forall\, k \in \{j_m, \ldots, j_{m+1}-1\}.$$

Hence $N_s \leq N_d(\delta_d)$ with the integer-valued function $N_d : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$N_d(s) := \begin{cases} \lceil \log_{1+\varepsilon_\delta}(s/\delta_0) \rceil, & s > \delta_0; \\ 0, & 0 \leq s \leq \delta_0. \end{cases} \tag{59}$$

*2) Global bound at sampling times:* Combining the bound in Lemma 4 for stabilizing stages and the ones in Lemmas 5, 6 for searching stages, we establish a global bound for $V_{\sigma(t_k)}(x_k^*, E_k)$ in stabilizing stages in terms of $\rho_{\sigma(0)}$ at $t = 0$ and $\|x_0\|, E_0, \delta_d, \delta_0$.

**Lemma 7.** *Consider a sampling time $t_k$ such that* (10) *holds. Then*

$$\begin{aligned} V_{\sigma(t_k)}(x_k^*, E_k) \leq \Theta^{N_0} \Psi^{N_d(\delta_d)} \psi^{2L_d(\delta_d)} \big(&\theta^k \psi^{2L_x(\|x_0\|)} \\ &\times (\omega_0 \rho_{\sigma(0)} E_0^2 + \omega_d \delta_0^2) + C_d(\delta_d)\delta_0^2\big) \end{aligned}$$

*with the functions $L_x, L_d : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ and $C_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ defined by*[8]

$$\begin{aligned} L_x(s) &:= \eta_E(s/E_0), \\ L_d(s) &:= \eta_\delta(s/\delta_0) + \sum\nolimits_{l=1}^{N_d(s)} \eta((1+\varepsilon_\delta)^{-l} s/\delta_0), \\ C_d(s) &:= \Theta_d + (\Theta_d + \omega_d) \sum\nolimits_{l=1}^{N_d(s)} \psi_d^l, \end{aligned}$$

*and the constants*

$$\psi := \hat{\beta}\theta^{-1/2}, \quad \Psi := \omega\Theta^{N_0}, \quad \psi_d := (1+\varepsilon_\delta)^2/\Psi,$$

*where $\eta_E, \eta_\delta$ are defined by* (37)*, $\eta$ by* (52)*, and $N_d$ by* (59)*.*

*Proof:* See Appendix F. ■

*Remark* 5. The gain functions $N_d, L_x, L_d, C_d$ in Lemma 7 are piecewise constant, and satisfy that $L_x(s) = 0$ for all $0 \leq s \leq E_0$ and that $N_d(s) = L_d(s) = 0$ for all $0 \leq s \leq \delta_0$. A more conservative bound that depends continuously on $\|x_0\|$ and $\delta_d$ can be established by replacing them with continuous, strictly increasing gain functions. First, $N_d(s) \leq \bar{N}_d(s)$ for all $s \geq 0$ with $\bar{N}_d \in \mathcal{K}_\infty$ defined by

$$\bar{N}_d(s) := \begin{cases} 1 + \log_{1+\varepsilon_\delta}(s/\delta_0), & s > \delta_0; \\ s/\delta_0, & 0 \leq s \leq \delta_0. \end{cases}$$

Second, $L_x(s) \leq \bar{L}_x(s)$ for all $s \geq 0$ with $\bar{L}_x \in \mathcal{K}_\infty$ defined by

$$\bar{L}_x(s) := \begin{cases} 1 + \log_{1+\varepsilon_E}(s/E_0), & s > E_0; \\ s/E_0, & 0 \leq s \leq E_0. \end{cases}$$

Third, $L_d(s) \leq \bar{L}_d(s)$ for all $s \geq 0$ with $\bar{L}_d \in \mathcal{K}_\infty$ defined by

$$\begin{aligned} \bar{L}_d(s) := &\log_{1+\varepsilon_E}(c_\beta s/\delta_0) + (\bar{N}_d(s) - 1)\log_{1+\varepsilon_E}(c_\varepsilon s/\delta_0) \\ &+ \log_{1+\varepsilon_E}(s/\delta_0) + \bar{N}_d(s) + 1 \\ &- \big(\bar{N}_d(s)(\bar{N}_d(s)+1)/2 - 1\big)\log_{1+\varepsilon_E}(1+\varepsilon_\delta) \end{aligned}$$

for $s > \delta_0$; and

$$\bar{L}_d(s) := (2 + \log_{1+\varepsilon_E} c_\beta) s/\delta_0$$

for $0 \leq s \leq \delta_0$. Finally, $C_d(s) \leq \bar{C}_d(s)$ for all $s \geq 0$ with the continuous, strictly increasing function $\bar{C}_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ defined by[9]

$$\bar{C}_d(s) := \Theta_d + \frac{1 - \psi_d^{\bar{N}_d(s)}}{1 - \psi_d} \psi_d (\Theta_d + \omega_d).$$

*3) Intersample bound:* First, consider an arbitrary $t$ in a stabilizing stage, that is, $t \in [t_k, t_{k+1}]$ such that (10) holds. Following similar arguments as in Section V-A2 with $t' = t'' = 0$, we estimate $x(t)$ by comparing it with $c_k$ in (11), the center of the hypercubic box containing $x(t_k)$, via the triangle inequality. If there is no switch on $(t_k, t]$, then (24) holds with $t - t_k$ in place of $\bar{t}$; thus

$$\|x(t) - c_k\| = \|x(t) - \hat{x}(t_k)\| \leq \hat{D}'_{k+1}(0, t - t_k).$$

Otherwise, there is exactly one switch on $(t_k, t]$ due to (4), and (27) holds with $t$ in place of $t_{k+1}^-$ (and $\bar{t} \in (0, t - t_k]$ denoting the unknown switching time); thus

$$\|x(t) - c_k\| \leq \|z(t) - \hat{z}(t_k)\| \leq \hat{D}''_{k+1}(0, 0, t - t_k),$$

where the first inequality follows from (26). Comparing the corresponding coefficients in (24), (27), (31) and (32), we see that in both cases

$$\|x(t) - c_k\| \leq \alpha^0 \|x_k^*\| + \beta^0 E_k + \gamma\delta_d$$

with

$$\alpha^0 := \max_{p,q\in\mathcal{P}} \alpha_{pq}^0, \quad \beta^0 := \max_{p,q\in\mathcal{P}} \beta_{pq}^0, \quad \gamma := \max_{p,q\in\mathcal{P}} \gamma_{pq}. \tag{60}$$

Applying the triangle inequality, we obtain

$$\begin{aligned} \|x(t)\| &\leq \|c_k\| + \|x(t) - c_k\| \\ &\leq (\alpha^0 + 1)\|x_k^*\| + \left(\beta^0 + \frac{N-1}{N}\right) E_k + \gamma\delta_d, \end{aligned}$$

where the second inequality follows from (11). Let

$$\begin{aligned} \lambda_{\min}^P &:= \min_{p\in\mathcal{P}} \underline{\lambda}(P_p), & \lambda^P &:= \max_{p\in\mathcal{P}} \overline{\lambda}(P_p), \\ \rho_{\min} &:= \min_{p\in\mathcal{P}} \rho_p, & \rho &:= \max_{p\in\mathcal{P}} \rho_p, \end{aligned} \tag{61}$$

[7] There is a searching stage at the beginning (i.e., $i_0 > 0$) if and only if $\|x_0\| > E_0$; for the final stabilizing stage to be well-defined, we let $j_{N_s+1} := \infty$ and $t_{j_{N_s+1}} := \infty$.

[8] The sum $L_x(\|x_0\|) + L_d(\delta_d)$ gives a bound for the total length of all searching stages (in terms of sampling intervals).

[9] For $\bar{C}_d$ to be well-defined, the design parameter $\varepsilon_\delta$ should be selected so that $\psi_d \neq 1$. The special case where $\psi_d = 1$ can be treated via similar arguments, and is omitted here for brevity (cf. footnote 3).

and define

$$\xi := \frac{\lambda_{\min}^P}{\rho_{\min}} \frac{(\beta^0+1-1/N)^2}{(\alpha^0+1)^2},$$
$$\Xi := \sqrt{\frac{(\alpha^0+1)^2}{\lambda_{\min}^P} + \frac{(\beta^0+1-1/N)^2}{\rho_{\min}}}.$$

From Young's inequality with $\xi$, it follows that[10]

$$\begin{aligned}
&\Big((\alpha^0+1)\|x_k^*\| + \Big(\beta^0 + \frac{N-1}{N}\Big)E_k\Big)^2 \\
&\le (1+\xi)(\alpha^0+1)^2\|x_k^*\|^2 + \Big(1+\frac{1}{\xi}\Big)\Big(\beta^0+\frac{N-1}{N}\Big)^2 E_k^2 \\
&= \Xi^2(\lambda_{\min}^P\|x_k^*\|^2 + \rho_{\min}E_k^2) \\
&\le \Xi^2 V_{\sigma(t_k)}(x_k^*, E_k).
\end{aligned}$$

Hence

$$\|x(t)\| \le \Xi\sqrt{V_{\sigma(t_k)}(x_k^*, E_k)} + \gamma\delta_d$$

which, combined with Lemma 7 and Remark 5, implies that

$$\begin{aligned}
\|x(t)\| \le{}& \Xi\Theta^{N_0/2}\Psi^{\bar N_d(\delta_d)/2}\psi^{\bar L_d(\delta_d)}\Big(\theta^{k/2}\psi^{\bar L_x(\|x_0\|)} \\
&\times(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar C_d(\delta_d)}\delta_0\Big) + \gamma\delta_d,
\end{aligned}$$

where the last inequality follows partially from the property

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b} \qquad \forall\, a, b \ge 0. \tag{62}$$

Moreover, due to $t \in [t_k, t_{k+1}]$ and $\theta < 1$, it holds that

$$\theta^{k/2} \le \theta^{-1/2}\theta^{t/(2\tau_s)}.$$

Second, consider an arbitrary $t \in [t_{j_m}, t_{i_m})$. From (34) it follows that

$$\|x(t) - x_{j_m}^*\| \le \hat D_{i_m} \le E_{i_m}.$$

Following similar arguments as in the first case, we obtain

$$\begin{aligned}
\|x(t)\| &\le \|x_{j_m}^*\| + E_{i_m} \\
&\le (\alpha^0+1)\|x_{i_m}^*\| + \Big(\beta^0 + \frac{N-1}{N}\Big)E_{i_m} \\
&\le \Xi\sqrt{V_{\sigma(t_{i_m})}(x_{i_m}^*, E_{i_m})} \\
&\le \Xi\Theta^{N_0/2}\Psi^{\bar N_d(\delta_d)/2}\psi^{\bar L_d(\delta_d)}\Big(\theta^{i_m/2}\psi^{\bar L_x(\|x_0\|)} \\
&\quad\times(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar C_d(\delta_d)}\delta_0\Big) + \gamma\delta_d,
\end{aligned}$$

in which

$$\theta^{i_m/2} \le \theta^{t/(2\tau_s)} < \theta^{-1/2}\theta^{t/(2\tau_s)}.$$

Combining the results above, we obtain

$$\begin{aligned}
\|x(t)\| \le{}& \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\Psi^{\bar N_d(\delta_d)/2}\psi^{\bar L_d(\delta_d)}\Big(\theta^{t/(2\tau_s)}\psi^{\bar L_x(\|x_0\|)} \\
&\times(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar C_d(\delta_d)}\delta_0\Big) + \gamma\delta_d
\end{aligned} \tag{63}$$

[10]For an $\varepsilon > 0$, Young's inequality with $\varepsilon$ states that $ab \le \varepsilon a^2/2 + b^2/(2\varepsilon)$ for all $a, b \in \mathbb{R}$. When $\varepsilon = 1$, the term "with $\varepsilon$" is omitted for brevity.

for all $t \ge 0$. From Young's inequality with an arbitrary design parameter $\phi > 0$, it follows that

$$\begin{aligned}
\|x(t)\| \le{}& \Big(\frac{1}{2\phi}\psi^{2\bar L_x(\|x_0\|)} + \frac{\phi}{2}\Psi^{\bar N_d(\delta_d)}\psi^{2\bar L_d(\delta_d)}\Big)\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}} \\
&\times\theta^{t/(2\tau_s)}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) \\
&+ \Xi\Theta^{N_0/2}\Psi^{\bar N_d(\delta_d)/2}\psi^{\bar L_d(\delta_d)}\sqrt{\bar C_d(\delta_d)}\delta_0 + \gamma\delta_d \\
={}& \frac{1}{2\phi}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\psi^{2\bar L_x(\|x_0\|)}\theta^{t/(2\tau_s)} \\
&+ \frac{\phi}{2}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\Psi^{\bar N_d(\delta_d)}\psi^{2\bar L_d(\delta_d)} \\
&+ \Xi\Theta^{N_0/2}\sqrt{\bar C_d(\delta_d)}\delta_0\Psi^{\bar N_d(\delta_d)/2}\psi^{\bar L_d(\delta_d)} + \gamma\delta_d
\end{aligned}$$

for all $t \ge 0$.

*Remark* 6*.* Here Young's inequality is applied to restate the state bound (63) in the standard ISS form (6). However, besides increasing the value of the state bound, it also has the following consequence. If there is no disturbance and the sensor and the controller know that, then $\delta_d = \delta_0 = 0$; thus (63) becomes $\|x(t)\| \le \Xi\Theta^{N_0/2}\sqrt{\omega_0\rho/\theta}E_0\theta^{t/(2\tau_s)}\psi^{\bar L_x(\|x_0\|)}$, that is, it reduces to a similar form as the one for the disturbance-free case [21, eq. (5)]. Meanwhile, (6) cannot be reduced to the same form since $h(\delta_d) = \phi\Xi\Theta^{N_0/2}\sqrt{\omega_0\rho/\theta}E_0/2 > 0$ even if $\delta_d = \delta_0 = 0$.

Hence (6) holds with the exponential decay rate

$$\lambda := -\frac{\ln\theta}{2\tau_s} > 0, \tag{64}$$

and the gain functions $g, h : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ defined by

$$\begin{aligned}
g(s) &:= \frac{1}{2\phi}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\psi^{2\bar L_x(s)}, \\
h(s) &:= \frac{\phi}{2}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\Psi^{\bar N_d(s)}\psi^{2\bar L_d(s)} \\
&\quad + \Xi\Theta^{N_0/2}\sqrt{\bar C_d(s)}\delta_0\Psi^{\bar N_d(s)/2}\psi^{\bar L_d(s)} + \gamma s.
\end{aligned} \tag{65}$$

## D. Practical stability

Following essentially the calculations from [21, Sec. 5.5] and Sections V-A, VI-A3, and VI-C3, we establish a stronger version of Corollary 2 with a smaller constant $C$.

**Proposition 3** (Practical stability)**.** *Consider the switched linear control system* (1)*. Suppose that Assumptions 1–3 and the inequality* (4) *hold. Then the communication and control strategy in Theorem 1 yields the following property: provided that the average dwell-time $\tau_a$ satisfies* (49)*, for each $\varepsilon > 0$, there exists a small enough $\delta > 0$ such that* (9) *holds for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n_d}$ with the constant*

$$C = \Xi\Theta^{N_0/2}\sqrt{\Theta_d}\delta_0. \tag{66}$$

*Proof:* See Appendix G. ■

In particular, from

$$\begin{aligned}
&g(0) + h(0) \\
&= \Xi\Theta^{N_0/2}\Big(\Big(\frac{1}{2\phi} + \frac{\phi}{2}\Big)\frac{\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0}{\sqrt{\theta}} + \sqrt{\Theta_d}\delta_0\Big) \\
&> \Xi\Theta^{N_0/2}\sqrt{\Theta_d}\delta_0,
\end{aligned}$$

it follows that the constant $C$ in Proposition 3 is smaller than the one in Corollary 2.

Proposition 3 also improves the practical stability result in [23, Th. 1]. Moreover, from the proof it will be clear that the additional bound in [23, eq. (39)] on the average dwell-time $\tau_a$ is not necessary for establishing practical stability.

## VII. Simulation study

Our communication and control strategy is simulated with the following data: $\mathcal{P} = \{1, 2\}$,

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} -2 & 0 \end{bmatrix};$$

$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

and $\tau_s = 0.5$, $N = 5$, $\tau_d = 1.05$, $\tau_a = 7.55$, $N_0 = 5$ so that the basic Assumptions 1–3 hold. We set $t' = t'' = 0$ in (28), $\varepsilon_E = 0.8$ in (17) and $\varepsilon_\delta = 1$ in (59). The disturbance $d$ is kept $0$ most of the time and turned on for two sampling intervals with the constant value 10 when the state stays small (more specifically, when $\|x\| < 2$ for ten consecutive sampling intervals). The initial estimate is $\delta_0 = 2$. Figure 2 plots a typical behavior of the first component $x_1$ of the continuous state (in orange solid line) and the corresponding component $\hat{x}_1$ of the auxiliary state (in blue dash-dot line). Switching times are denoted by vertical gray dotted lines, and sampling times when the disturbance is turned on by vertical yellow dashed lines; captures are marked by red circles, and escapes by green crosses. Observe the searching stages at $t = 0$ (the state is lost due to $\|x_0\| > E_0$) and $t = 20.5$ and $31$ (the state escapes due to the disturbance), and the nonsmooth behavior of $x$ when $\hat{x}$ experiences a jump. The value of $\tau_a$ is empirically selected to be large enough to provide consistent convergence in simulations. For this example, the theoretical lower bound (49) on the average dwell-time $\tau_a$ is approximately 28.13, which is rather conservative. However, our result is significantly less conservative than the one in [21, Sec. 6] for the disturbance-free case, which generated a theoretical bound of $\tau_a \geq 85.5$ while consistent convergence was observed with $\tau_a = 7.55$ as well. The improvement is due to the more careful calculations in the stability analysis, such as the ones explained by footnotes 2 and 5.

Figure 3 exhibits the case where the unknown disturbance $d$ is transient or $d \equiv 0$, so that once the state is captured it will never escape. Due to the nonzero initial estimate $\delta_0$, the state $x$ will converge to the set $\mathcal{A} = \{v \in \mathbb{R}^{n_x} : \|v\| \leq h(0)\}$ (visualized by the shaded area) instead of the origin. Following essentially the idea of "zooming-in" from [6], we are able to make the state converge to the origin by halving the estimate $\delta_k$ every ten sampling intervals, as shown in Fig. 4. We conjecture that, for general disturbances, a similar modification to our communication and control strategy can be made to establish ISS with respect to the origin.

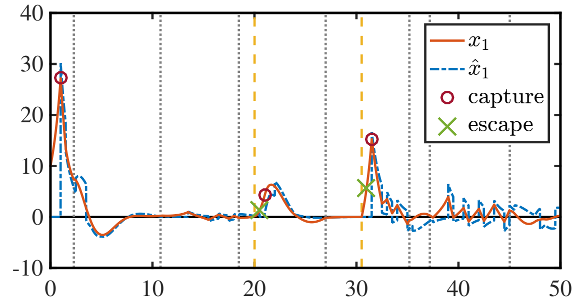


Fig. 2. Simulation example

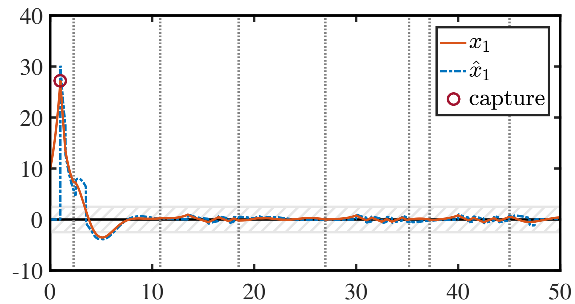


Fig. 3. With a constant estimate $\delta_k \equiv \delta_0$, the state $x \nrightarrow 0$ even if $d \equiv 0$.

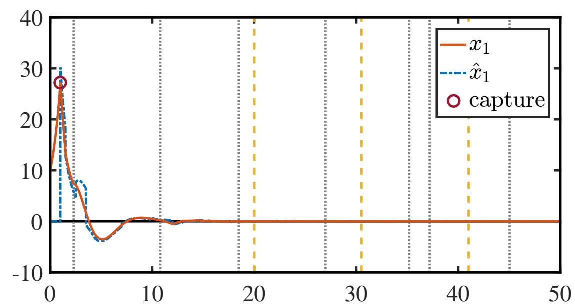


Fig. 4. With a converging estimate $\delta_k \to 0$, the state $x \to 0$ when $d \to 0$.

## VIII. Concluding remarks

In this paper, we studied the feedback stabilization of a switched linear system with a completely unknown disturbance under data-rate constraints. A finite data transmission rate was achieved via sampled and quantized state measurements. We extended the approach of reachable-set approximation and propagation from [21] by introducing an estimate of the disturbance bound to compensate for the disturbance. A communication and control strategy was designed to achieve a variant of input-to-state stability with exponential decay, based on a novel algorithm for adjusting the estimate and recovering the state when it escapes the range of quantization.

As discussed in Section VII, we intend to advance our result via the "zooming-in" technique from [6] to establish ISS with respect to the origin. However, reducing the estimate in stabilizing stages can lead to an unbounded number of searching stages, and further work is needed to establish convergence for the communication and control strategy.

For a non-switched linear control system, the minimum data rate for feedback stabilization equals the topological entropy [30] of the open-loop system [3], [4]. In the context of switched systems, neither the topological entropy nor the minimum data rate for feedback stabilization has been well-established. (See [31, Chap. 6] for some initial results on the topological entropy of switched linear systems.) These two notions and their relation could become intriguing topics for future research.

## Appendix A
## Proof of Lemma 1

For all square matrices $X$ and $Y$,

$$\begin{aligned}
&\|e^{X+Y} - e^X\| \\
&= \left\| \sum_{m=0}^{\infty} \frac{1}{m!}((X+Y)^m - X^m) \right\| \\
&= \left\| \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i=1}^{m} \binom{m}{i} X^{m-i} Y^i \right\| \\
&= \left\| \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{X^{m-1-j} Y^{j+1}}{j+1} \right\| \\
&\le \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \|X\|^{m-1-j} \|Y\|^{j+1} \\
&= e^{\|X\|+\|Y\|}\|Y\|.
\end{aligned}$$

## Appendix B
## Proof of Lemma 2

We first recall the following useful facts from linear algebra. From the definition of $\infty$-norm, it follows that

$$\|v\|^2 \le v^\top v, \quad v_1^\top v_2 \le n\|v_1\|\|v_2\| \tag{67}$$

for all vectors $v, v_1, v_2 \in \mathbb{R}^n$. Also,

$$\underline{\lambda}(S) \le \frac{v^\top S v}{v^\top v} \le \overline{\lambda}(S) \qquad \forall\, v \in \mathbb{R}^n \backslash \{0\} \tag{68}$$

for all symmetric matrices $S \in \mathbb{R}^{n\times n}$ (i.e., $S^\top = S$).

At $t_{k+1}$, from (19) it follows that

$$V_p(x_{k+1}^*, E_{k+1}) = (x_{k+1}^*)^\top P_p x_{k+1}^* + \rho_p E_{k+1}^2$$

with $x_{k+1}^*$ given by (22) and $E_{k+1}$ given by (21). First, (22) can be rewritten as

$$x_{k+1}^* = S_p c_k = S_p(x_k^* + \Delta_k)$$

with $\Delta_k := c_k - x_k^*$. Then

$$\begin{aligned}
&(x_{k+1}^*)^\top P_p x_{k+1}^* \\
&= (x_k^*)^\top S_p^\top P_p S_p x_k^* + 2(x_k^*)^\top S_p^\top P_p S_p \Delta_k + \Delta_k^\top S_p^\top P_p S_p \Delta_k \\
&\le (x_k^*)^\top (P_p - Q_p) x_k^* \\
&\quad + 2n_x \|x_k^*\| \|S_p^\top P_p S_p\| \|\Delta_k\| + n_x \|S_p^\top P_p S_p\| \|\Delta_k\|^2,
\end{aligned}$$

where the last inequality follows from (39) and (67). Moreover, (67) and (68) imply that

$$\begin{aligned}
&(x_k^*)^\top Q_p x_k^* \ge \underline{\lambda}(Q_p)(x_k^*)^\top x_k^* \ge \frac{\underline{\lambda}(Q_p)}{\overline{\lambda}(P_p)}(x_k^*)^\top P_p x_k^*, \\
&(x_k^*)^\top Q_p x_k^* \ge \underline{\lambda}(Q_p)(x_k^*)^\top x_k^* \ge \underline{\lambda}(Q_p)\|x_k^*\|^2.
\end{aligned}$$

Combining the inequalities above and completing the square, we obtain that

$$\begin{aligned}
&(x_{k+1}^*)^\top P_p x_{k+1}^* \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* - \frac{1}{2}\underline{\lambda}(Q_p)\|x_k^*\|^2 \\
&\quad + 2n_x\|x_k^*\|\|S_p^\top P_p S_p\|\|\Delta_k\| + n_x\|S_p^\top P_p S_p\|\|\Delta_k\|^2 \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \chi_p\|\Delta_k\|^2 \\
&\quad - \left(\sqrt{\frac{1}{2}\underline{\lambda}(Q_p)}\|x_k^*\| - \frac{\sqrt{2}n_x\|S_p^\top P_p S_p\|}{\sqrt{\underline{\lambda}(Q_p)}}\|\Delta_k\|\right)^2 \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \frac{(N-1)^2}{N^2}\chi_p E_k^2,
\end{aligned}$$

where the last inequality follows partially from (11). Second, from (21) and Young's inequality with $\phi_1$, it follows that

$$\begin{aligned}
E_{k+1}^2 &= \left(\frac{\Lambda_p}{N}E_k + \Phi_p(\tau_s)\delta_k\right)^2 \\
&\le \frac{(1+\phi_1)\Lambda_p^2}{N^2}E_k^2 + \left(1 + \frac{1}{\phi_1}\right)\Phi_p(\tau_s)^2\delta_k^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&V_p(x_{k+1}^*, E_{k+1}) \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \left(\frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p}\right. \\
&\quad \left. + \frac{(1+\phi_1)\Lambda_p^2}{N^2}\right)\rho_p E_k^2 + \left(1 + \frac{1}{\phi_1}\right)\rho_p\Phi_p(\tau_s)^2\delta_k^2,
\end{aligned}$$

which in turn implies (43).

## Appendix C
## Proof of Lemma 3

At $t_{k+1}$, from (23) it follows that

$$V_q(x_{k+1}^*, E_{k+1}) = (x_{k+1}^*)^\top P_q x_{k+1}^* + \rho_q E_{k+1}^2$$

with $x_{k+1}^*$ given by (29) and $E_{k+1}$ given by (28). First, (29) can be rewritten as

$$x_{k+1}^* = H_{pq} c_k = H_{pq}(x_k^* + \Delta_k)$$

with $\Delta_k = c_k - x_k^*$. Then

$$\begin{aligned}
&(x_{k+1}^*)^\top P_q x_{k+1}^* \\
&\le \overline{\lambda}(P_q) h_{pq}^2 (x_k^* + \Delta_k)^\top (x_k^* + \Delta_k) \\
&\le 2\overline{\lambda}(P_q) h_{pq}^2 (x_k^*)^\top x_k^* + 2n_x\overline{\lambda}(P_q) h_{pq}^2 \|\Delta_k\|^2 \\
&\le \frac{2\overline{\lambda}(P_q) h_{pq}^2}{\underline{\lambda}(P_p)}(x_k^*)^\top P_p x_k^* + \frac{(N-1)^2}{N^2} 2n_x\overline{\lambda}(P_q) h_{pq}^2 E_k^2,
\end{aligned}$$

where the inequalities follows from (11), (67), (68), and Young's inequality. Second, from (30) and Young's inequality with $\phi_2$, it follows that

$$\begin{aligned}
E_{k+1}^2 &\le (\alpha_{pq}\|x_k^*\| + \beta_{pq}E_k + \gamma_{pq}\delta_k)^2 \\
&\le (2+\phi_2)(\alpha_{pq}^2\|x_k^*\|^2 + \beta_{pq}^2 E_k^2) + \left(1 + \frac{2}{\phi_2}\right)\gamma_{pq}^2\delta_k^2
\end{aligned}$$

for every $\phi_2 > 0$, in which

$$\|x_k^*\|^2 \le (x_k^*)^\top x_k^* \le \frac{1}{\underline{\lambda}(P_p)}(x_k^*)^\top P_p x_k^*$$

due to (67) and (68). Therefore,

$$\begin{aligned}&V_q(x_{k+1}^*, E_{k+1})\\ &\le \left(\frac{2\overline{\lambda}(P_q)h_{pq}^2}{\underline{\lambda}(P_p)} + \frac{(2+\phi_2)\alpha_{pq}^2\rho_q}{\underline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^*\\ &\quad + \left(\frac{(N-1)^2}{N^2}\frac{2n_x\overline{\lambda}(P_q)h_{pq}^2}{\rho_p} + \frac{(2+\phi_2)\beta_{pq}^2\rho_q}{\rho_p}\right)\rho_p E_k^2\\ &\quad + \left(1+\frac{2}{\phi_2}\right)\rho_q\gamma_{pq}^2\delta_k^2,\end{aligned}$$

which in turn implies (45).

## Appendix D
## Proof of Lemma 4

First, consider the function $\zeta : [0,1) \to \mathbb{R}$ defined by

$$\zeta(s) = 1 + \frac{\ln(\mu + s(1-\nu)\mu_d/\nu_d)}{\ln(1/(\nu + s(1-\nu)))}.$$

From (47) and (48) it follows that $\Theta > 1$ in (51), and that $\zeta$ is continuous and increasing. Moreover, as

$$\zeta(0) = 1 + \frac{\ln\mu}{\ln(1/\nu)} < \frac{\tau_a}{\tau_s}$$

due to (49), there exists a sufficiently small constant $\phi_3 \in (0,1)$ such that $\zeta(\phi_3) < \tau_a/\tau_s$; thus $\theta < 1$ in (51).

The remaining proof follows in principle from the arguments in [17], [32]. If there is an integer $l \in \{i, \dots, k-1\}$ such that

$$V_{\sigma(t_l)}(x_l^*, E_l) > \frac{1}{\phi_3(1-\nu)}\nu_d\delta_l^2, \tag{69}$$

then (43) implies that

$$V_{\sigma(t_{l+1})}(x_{l+1}^*, E_{l+1}) < (\nu + \phi_3(1-\nu))V_{\sigma(t_l)}(x_l^*, E_l)$$

if $\sigma(t_{l+1}) = \sigma(t_l)$; whereas (45) implies that

$$V_{\sigma(t_{l+1})}(x_{l+1}^*, E_{l+1}) < (\mu + \phi_3(1-\nu)\mu_d/\nu_d)V_{\sigma(t_l)}(x_l^*, E_l)$$

if $\sigma(t_{l+1}) \neq \sigma(t_l)$. Hence for two integers $l', l''$ such that $i \le l' < l'' \le k$ and that (69) holds for all $l \in \{l', \dots, l''-1\}$,

$$\begin{aligned}&V_{\sigma(t_{l''})}(x_{l''}^*, E_{l''})\\ &< (\mu + \phi_3(1-\nu)\mu_d/\nu_d)^{N_\sigma(t_{l''}, t_{l'})}\\ &\quad \times (\nu + \phi_3(1-\nu))^{l''-l'-N_\sigma(t_{l''}, t_{l'})}V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'})\\ &= (\nu + \phi_3(1-\nu))^{l''-l'}\Theta^{N_\sigma(t_{l''}, t_{l'})}V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'}),\\ &< (\nu + \phi_3(1-\nu))^{l''-l'}\Theta^{N_0 + (l''-l')\tau_s/\tau_a}V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'})\\ &= \theta^{l''-l'}\Theta^{N_0}V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'}),\end{aligned}$$

where $N_\sigma(t_{l''}, t_{l'})$ denotes the number of switches on $(t_{l'}, t_{l''}]$, and the last inequality follows from $\Theta > 1$ and the ADT condition (2). Therefore, if (69) holds for all $l \in \{i, \dots, k-1\}$, then

$$V_{\sigma(t_k)}(x_k^*, E_k) < \theta^{k-i}\Theta^{N_0}V_{\sigma(t_i)}(x_i^*, E_i).$$

Otherwise, for

$$k' := \max\left\{l \le k-1 : V_{\sigma(t_l)}(x_l^*, E_l) \le \frac{1}{\phi_3(1-\nu)}\nu_d\delta_l^2\right\}$$

it holds that

$$\begin{aligned}V_{\sigma(t_{k'+1})}(x_{k'+1}^*, E_{k'+1}) &\le \mu V_{\sigma(t_{k'})}(x_{k'}^*, E_{k'}) + \mu_d\delta_{k'}^2\\ &\le \frac{\mu}{\phi_3(1-\nu)}\nu_d\delta_{k'}^2 + \mu_d\delta_{k'}^2 = \Theta_d\delta_{k'}^2\end{aligned}$$

(see also Remark 3); thus

$$\begin{aligned}V_{\sigma(t_k)}(x_k^*, E_k) &< \theta^{k-k'-1}\Theta^{N_0}V_{\sigma(t_{k'+1})}(x_{k'+1}^*, E_{k'+1})\\ &\le \Theta^{N_0}\Theta_d\delta_{k'}^2\end{aligned}$$

as (69) holds for all $l \in \{k'+1, \dots, k-1\}$. The proof of Lemma 4 is completed by combining the bounds for the two cases and noticing that $\delta_l = \delta_i$ for all $l \in \{i, \dots, k-1\}$.

## Appendix E
## Proof of Lemma 5

Let $p, q \in \mathcal{P}$ denote the active modes at sampling times $t_j, t_i$, respectively. At the sampling time $t_i$ of recovery,

$$V_q(x_i^*, E_i) = (x_i^*)^\top P_q x_i^* + \rho_q E_i^2$$

with $x_i^* = x_j^*$ and $E_i$ bounded by (54). First, from (68) it follows that

$$(x_i^*)^\top P_q x_i^* \le \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)}(x_j^*)^\top P_p x_j^*.$$

Second, following (54) and Young's inequality with $\phi_4$ we obtain

$$\begin{aligned}E_i^2 &\le \hat{\beta}^{2\eta(\delta_d/\delta_j)}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right)^2\\ &\le \hat{\beta}^{2\eta(\delta_d/\delta_j)}\Bigg((2+\phi_4)\left(\frac{\bar{\alpha}^2}{(\hat{\beta}-1)^2}\|x_j^*\|^2 + E_j^2\right)\\ &\quad + \left(1+\frac{2}{\phi_4}\right)\frac{\bar{\gamma}^2}{(\hat{\beta}-1)^2}\delta_j^2\Bigg)\end{aligned}$$

for every $\phi_4 > 0$, in which

$$\|x_j^*\|^2 \le (x_j^*)^\top x_j^* \le \frac{1}{\underline{\lambda}(P_p)}(x_j^*)^\top P_p x_j^*$$

due to (67) and (68). Therefore,

$$\begin{aligned}&V_q(x_i^*, E_i)\\ &\le \hat{\beta}^{2\eta(\delta_d/\delta_j)}\Bigg(\left(\frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} + \frac{(2+\phi_4)\bar{\alpha}^2\rho_q}{(\hat{\beta}-1)^2\underline{\lambda}(P_p)}\right)(x_j^*)^\top P_p x_j^*\\ &\quad + \frac{(2+\phi_4)\rho_q}{\rho_p}\rho_p E_k^2 + \left(1+\frac{2}{\phi_4}\right)\frac{\bar{\gamma}^2\rho_q}{(\hat{\beta}-1)^2}\delta_j^2\Bigg),\end{aligned}$$

which in turn implies (55).

## APPENDIX F
## PROOF OF LEMMA 7

Let $[t_{i_m}, t_{j_{m+1}})$ denote the stabilizing stage containing $t_k$, that is, $i_m \le k \le j_{m+1}-1$. Substituting (50) with $i = i_m$ and $k = j_{m+1}$ into (55) with $j = j_{m+1}$ and $i = i_{m+1}$, we obtain

$$\begin{aligned}
&V_{\sigma(t_{i_{m+1}})}(x^*_{i_{m+1}}, E_{i_{m+1}}) \\
&\le \hat{\beta}^{2\eta(\delta_d/\delta_{j_{m+1}})}\big(\omega V_{\sigma(t_{j_{m+1}})}(x^*_{j_{m+1}}, E_{j_{m+1}}) + \omega_d \delta^2_{j_{m+1}}\big) \\
&< \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})}\big(\omega \Theta^{N_0}\big(\theta^{j_{m+1}-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + \Theta_d(1+\varepsilon_\delta)^{2m}\delta_0^2\big) + \omega_d(1+\varepsilon_\delta)^{2(m+1)}\delta_0^2\big) \\
&= \Psi \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})}\big(\theta^{j_{m+1}-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\varepsilon_\delta)^{2m}\delta_0^2\big),
\end{aligned}$$

in which

$$\theta^{j_{m+1}-i_m} \le \theta^{i_{m+1}-i_m}\theta^{-\eta(\delta_d/\delta_{j_{m+1}})}$$

due to (53) and $\theta < 1$. Hence

$$\begin{aligned}
&V_{\sigma(t_{i_{m+1}})}(x^*_{i_{m+1}}, E_{i_{m+1}}) \\
&< \Psi \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})}\big(\theta^{i_{m+1}-i_m}\theta^{-\eta(\delta_d/\delta_{j_m})} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\varepsilon_\delta)^{2m}\delta_0^2\big) \\
&\le \Psi \psi^{2\eta(\delta_d/\delta_{i_{m+1}})}\big(\theta^{i_{m+1}-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\varepsilon_\delta)^{2m}\delta_0^2\big).
\end{aligned}$$

Based on this recursive bound, it is straightforward to derive that

$$\begin{aligned}
&V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&< \Psi \psi^{2\eta(\delta_d/\delta_{i_m})}\big(\theta^{i_m - i_{m-1}}\Psi\psi^{2\eta(\delta_d/\delta_{i_{m-1}})}\big(\theta^{i_{m-1}-i_{m-2}} \\
&\quad \times V_{\sigma(t_{i_{m-2}})}(x^*_{i_{m-2}}, E_{i_{m-2}}) + (\Theta_d + \psi_d\omega_d) \\
&\quad \times (1+\varepsilon_\delta)^{2(m-2)}\delta_0^2\big) + (\Theta_d + \psi_d\omega_d)(1+\varepsilon_\delta)^{2(m-1)}\delta_0^2\big) \\
&\le \Psi^2 \psi^{2(\eta(\delta_d/\delta_{i_m}) + \eta(\delta_d/\delta_{i_{m-1}}))} \\
&\quad \times \big(\theta^{i_m - i_{m-2}} V_{\sigma(t_{i_{m-2}})}(x^*_{i_{m-2}}, E_{i_{m-2}}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\psi_d)(1+\varepsilon_\delta)^{2(m-2)}\delta_0^2\big) \\
&< \cdots \\
&< \Psi^m \psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\big(\theta^{i_m - i_0} V_{\sigma(t_{i_0})}(0, E_{i_0}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1 + \psi_d + \cdots + \psi_d^{m-1})\delta_0^2\big), \\
&< \Psi^m \psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\Big(\theta^{i_m - i_0} V_{\sigma(t_{i_0})}(0, E_{i_0}) \\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big),
\end{aligned}$$

which, combined with (57) and (58), implies that

$$\begin{aligned}
&V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\le \Psi^m \psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\Big(\theta^{i_m - i_0}\hat{\beta}^{2(\eta_E(\|x_0\|/E_0) + \eta_\delta(\delta_d/\delta_0))} \\
&\quad \times \big(\omega_0 V_{\sigma(0)}(0, E_0) + \omega_d\delta_0^2\big) \\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big) \\
&\le \Psi^m \psi^{2(\eta_\delta(\delta_d/\delta_0) + \sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^{i_m}\psi^{2\eta_E(\|x_0\|/E_0)} \\
&\quad \times (\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big).
\end{aligned}$$

Finally, substituting the previous bound into (50) with $i = i_m$, we obtain

$$\begin{aligned}
&V_{\sigma(t_k)}(x^*_k, E_k) \\
&< \Theta^{N_0}\big(\theta^{k-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) + \Theta_d\delta^2_{i_m}\big) \\
&< \Theta^{N_0}\Big(\theta^{k-i_m}\Psi^m \psi^{2(\eta_\delta(\delta_d/\delta_0) + \sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^{i_m} \\
&\quad \times \psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) \\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big) + \Theta_d(1+\varepsilon_\delta)^{2m}\delta_0^2\Big) \\
&\le \Theta^{N_0}\Psi^m \psi^{2(\eta_\delta(\delta_d/\delta_0) + \sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))} \\
&\quad \times \Big(\theta^k \psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) \\
&\quad + \Big(\Theta_d \sum\nolimits_{l=0}^{m} \psi_d^l + \omega_d \sum\nolimits_{l=1}^{m} \psi_d^l\Big)\delta_0^2\Big).
\end{aligned}$$

The proof of Lemma 7 is completed by replacing $m$ with its upper bound $N_d(\delta_d)$.

## APPENDIX G
## PROOF OF PROPOSITION 3

First, suppose $\|x_0\| \le \delta \le E_0$ and $\delta_d \le \delta \le \delta_0$. Then the system is always in the stabilizing stage, and the estimate of the disturbance bound is always $\delta_0$. Suppose also that there is an integer $k_1 \ge 1$ such that $c_k = 0$ (i.e., the state $x(t_k)$ is inside the central hypercubic box) for all $k \le k_1 - 1$. Then similar arguments as in Sections V-A1 and V-A2 show that $u \equiv 0$ on $[0, t_{k_0})$ and $x^*_k = 0$ for all $k \in \{0, \ldots, k_1\}$; thus

$$E_{k+1} \ge \frac{\Lambda_{\min}}{N} E_k \qquad \forall\, k \in \{0, \ldots, k_1 - 1\} \tag{70}$$

with

$$\Lambda_{\min} := \min_{p \in \mathcal{P}} \Lambda_p$$

due to (21) and (28).

Second, following similar analysis on state bounds when $u \equiv 0$ as in Section V-B, for each $k \le k_1$,

$$\|x(t)\| \le \bar{\beta}^k \|x_0\| + \frac{\bar{\beta}^k - 1}{\bar{\beta} - 1}\bar{\gamma}\delta_d \qquad \forall\, t \le t_k. \tag{71}$$

Third, following similar arguments as in Section VI-C3, for each $k \ge k_1$,

$$\|x(t)\| \le \Xi\sqrt{V_{\sigma(t_k)}(x^*_k, E_k)} + \gamma\delta_d \qquad \forall\, t \in [t_k, t_{k+1}],$$

which, combined with (50) for $i = 0$ and (62), implies that

$$\|x(t)\| \le \Xi\Theta^{N_0/2}(\theta^{k_1}\sqrt{\rho}E_0 + \sqrt{\Theta_d}\delta_0) + \gamma\delta_d \qquad \forall\, t \ge t_{k_1} \tag{72}$$

with $\rho$ in (61).

Finally, the proof of Lemma 3 is completed via the following three steps. First, given an arbitrary $\varepsilon > 0$, from (66) and (72) it follows that if

$$\Xi\Theta^{N_0/2}\theta^{k_1}\sqrt{\rho}E_0 + \gamma\delta_d \le \varepsilon$$

then $\|x(t)\| \le \varepsilon + C$ for all $t \ge t_{k_1}$. Second, taking $E_0$ as fixed, calculate a sufficiently large $k_1$ such that

$$\Xi\Theta^{N_0/2}\theta^{k_1}\sqrt{\rho}E_0 \le \varepsilon/2.$$

Third, calculate a sufficiently small $\delta$ such that $\gamma\delta \leq \varepsilon/2$ and

$$\left(\bar{\beta}^{k_1} + \frac{\bar{\beta}^{k_1} - 1}{\bar{\beta} - 1}\bar{\gamma}\right)\delta \leq \varepsilon,$$

which, combined with (71), implies that $\|x(t)\| \leq \varepsilon$ for all $t \leq t_{k_1}$; and that $\delta \leq \delta_0$ and

$$\left(\bar{\beta}^{k_1-1} + \frac{\bar{\beta}^{k_1-1} - 1}{\bar{\beta} - 1}\bar{\gamma}\right)\delta \leq \left(\frac{\Lambda_{\min}}{N}\right)^{k_1-1}\frac{E_0}{N},$$

which, combined with (70) and (71), implies that $c_k = 0$ for all $k \leq k_1 - 1$ (and that the systems is always in the stabilizing stage), making the analysis above valid.

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