On topological entropy and stability of switched linear systems*

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ABSTRACT

This paper studies topological entropy and stability properties of switched linear systems. First, we show that the exponential growth rates of solutions of a switched linear system are essentially upper bounded by its topological entropy. Second, we estimate the topological entropy of a switched linear system by decomposing it into a part that is generated by scalar multiples of the identity matrix and a part that has zero entropy, and proving that the overall topological entropy is upper bounded by that of the former. Third, we prove that a switched linear system is globally exponentially stable if its topological entropy remains zero under a destabilizing perturbation. Finally, the entropy estimation via decomposition and the entropy-based stability condition are applied to three classes of switched linear systems to construct novel upper bounds for topological entropy and novel sufficient conditions for global exponential stability.

CCS CONCEPTS

• **Computing methodologies → Uncertainty quantification**; **Systems theory**; **Continuous models**; *Discrete-event simulation.*

KEYWORDS

Topological entropy; Switched systems; Stability

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1 INTRODUCTION

In systems theory, topological entropy describes the information accumulation needed to approximate trajectories with a finite precision, or the complexity growth of a system acting on sets with finite measure. The latter idea corresponds to Kolmogorov's original definition of entropy for dynamical systems [15], and shares a striking resemblance to Shannon's information entropy [27]. Adler first defined topological entropy as an extension of Kolmogorov's metric entropy, quantifying a map's expansion by the minimal cardinality of subcover refinements [1]. An alternative definition using the maximal number of trajectories separable with a finite precision was introduced by Bowen [3] and independently by Dinaburg [8]. An equivalence between the two definitions was established in [4]. Most results on topological entropy are for time-invariant systems, as time-varying dynamics introduce complexities which require new methods to understand [14, 16]. By studying the topological entropy of switched linear systems, we intend to clarify some of these complexities.

Entropy has played a prominent role in control theory, in which information flow appears between sensors and actuators for maintaining or inducing desired properties. Nair et al. first introduced topological feedback entropy for discrete-time systems [24], following the construction in [1]. Their definition extended the classical entropy notions, notably in allowing for non-compact state spaces, but still described the uncertainty growth with time. Colonius and Kawan later proposed a notion of invariance entropy for continuous-time systems [6], which is closer in spirit to the trajectory-counting formulation in [3, 8]. An equivalence between the two entropy notions was established in [7]. The results of [6] were extended from set invariance to exponential stabilization in [5]. Entropy has also been studied in the dual problem of state estimation in, e.g., [20, 26].

This paper studies the topological entropy of switched linear systems. Switched systems have become a popular topic in recent years (see, e.g., [18] and references therein). It is well known that, in general, a switched system does not inherit stability properties of its individual modes. One standard approach to establish stability for switched systems is to prove that there exists a common Lyapunov function, which ensures global asymptotic stability under arbitrary switching; see, e.g., [21] for the case of switched linear systems generated by Hurwitz triangular matrices. Another widely used technique is to impose suitable slow-switching conditions, especially in the sense of dwell time [22] or average dwell time (ADT) [10]. These results motivate us to examine the topological entropy of the corresponding classes of switched linear systems.

Our interest in studying the topological entropy of switched linear systems is strongly motivated by its relation to the data-rate requirements in control systems. For a linear time-invariant control system, it has been shown that the minimal data rate for stabilization equals the topological entropy in open-loop [11, 23, 29]. For switched systems, however, neither the minimal data rate nor the topological entropy are well-understood. Sufficient data rates for feedback stabilization of switched linear systems were established in [19, 32]. In [28], the notion of estimation entropy from [20] was extended to switched systems to formulate similar data-rate conditions. The paper [33] introduced a notion of topological entropy for switched systems, defined in terms of the minimal number of initial states needed to approximate all initial states with a finite precision, which is adopted in the current work. Formulae and bounds for the topological entropy of switched linear systems with diagonal, pairwise commuting, triangular, and general matrices were constructed in [30, 33].

The main contribution of this paper is the development of connections between topological entropy and stability properties for switched linear systems. In Section 2, we introduce the definition of topological entropy for switched systems and the stability notions, and define switching-related quantities such as active times of individual modes, which prove useful in calculating the topological entropy of switched linear systems. We also recall some basic results on the topological entropy, stability properties, and solutions of switched linear systems.

In Section 3, we formulate an entropy-based solution estimation for switched linear systems, which shows that the exponential growth rates of solutions are essentially upper bounded by the topological entropy. This result provides a fundamental connection between topological entropy and stability properties, which serves as the basis for the entropy estimation and the stability condition in later sections.

In Section 4, we design an approach for estimating the topological entropy of a switched linear system, by decomposing it into a part that is generated by scalar multiples of the identity matrix and a part that is "almost stable" (i.e., it has zero entropy), and proving that the overall topological entropy is upper bounded by that of the former. This result is obtained by establishing a similar but more general result based on the decomposition into a part that is generated by diagonal matrices and a second part that commutes with the first. The entropy estimation is then combined with standard stability results to construct novel upper bounds for the topological entropy of three classes of switched linear systems: one with general matrices and under arbitrary switching, one with triangular matrices and under arbitrary switching, and one with general matrices and under slow switching in the sense of an ADT.

In Section 5, we establish an entropy-based stability condition by proving that a switched linear system is globally exponentially stable (GES) if its topological entropy remains zero under a destabilizing perturbation. The stability condition is then combined with the upper bounds for the topological entropy of the three classes of switched linear systems from the previous section to construct novel sufficient conditions for GES, which extend stability results in the literature. Section 6 concludes the paper with a brief summary and an outlook on future research.

Notations: By default, all logarithms are natural logarithms. Let $\mathbb{R}_+ := [0, \infty)$. Denote by I_n the identity matrix in $\mathbb{R}^{n\times n}$; the subscript is omitted when the dimension is clear from context. For a complex scalar $a \in \mathbb{C}$, denote by $\mathrm{Re}(a)$ its real part. For a vector $v \in \mathbb{R}^n$, denote by v_i its i-th scalar component and write $v = (v_1, \dots, v_n)$. For a matrix $A \in \mathbb{R}^{n\times n}$, denote by $\mathrm{spec}(A)$ its spectrum, and by $\lambda_{\max}(A)$ the *spectral abscissa*—the largest real part of eigenvalues—of A, that is, $\lambda_{\max}(A) := \max_{\lambda\in\mathrm{spec}(A)} \mathrm{Re}(\lambda)$. For a set $E \subset \mathbb{R}^n$, denote by $|E|$ its cardinality. Denote by $\|v\|_\infty := \max_i |v_i|$ the ∞-norm of a vector v.

2 PRELIMINARIES

2.1 Entropy definitions

Consider a family of continuous-time dynamical systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P} \tag{1}$$

with the state $x \in \mathbb{R}^n$, in which each system is labeled with an index p from a finite *index set* $\mathcal{P}$, and all the functions $f_p : \mathbb{R}^n \to \mathbb{R}^n$ are locally Lipschitz. We are interested in the corresponding switched system defined by

$$\dot{x} = f_\sigma(x), \tag{2}$$

where $\sigma : \mathbb{R}_+ \to \mathcal{P}$ is a right-continuous, piecewise constant *switching signal*. The system with index p in (1) is called the p-th *mode*, or mode p, of the switched system (2), and $\sigma(t)$ is called the active mode at time t. Denote by $\xi_\sigma(x, t)$ the solution of (2) at time t with switching signal σ and initial state x. For fixed σ and x, the trajectory $\xi_\sigma(x, \cdot)$ is absolutely continuous and satisfies the differential equation (2) away from discontinuities of σ, which are called switching times, or simply *switches*. We assume that there is at most one switch at each time, and finitely many switches on each finite time interval (i.e., the set of switches contains no accumulation point). Denote by $N_\sigma(t, \tau)$ the number of switches on an interval $(\tau, t]$.

Denote by $\|\cdot\|$ some chosen norm on $\mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be a compact set of initial states with a nonempty interior, and σ a switching signal. Given a time horizon $T \geq 0$ and a radius $\varepsilon > 0$, we define the following open ball in K with center x:

$$B_{f_\sigma}(x, \varepsilon, T) := \left\{ x' \in K : \max_{t\in[0,T]} \|\xi_\sigma(x', t) - \xi_\sigma(x, t)\| < \varepsilon \right\}. \tag{3}$$

We say that a finite set $E \subset K$ is (T, ε)-*spanning* if

$$K = \bigcup_{\hat{x}\in E} B_{f_\sigma}(\hat{x}, \varepsilon, T),$$

or equivalently, for each $x \in K$, there is a point $\hat{x} \in E$ such that $\|\xi_\sigma(x, t) - \xi_\sigma(\hat{x}, t)\| < \varepsilon$ for all $t \in [0, T]$. Denote by $S(f_\sigma, \varepsilon, T, K)$ the minimal cardinality of a (T, ε)-spanning set, or equivalently, the cardinality of a minimal (T, ε)-spanning set. The *topological entropy* of the switched system (2) with initial set K and switching signal σ is defined in terms of the exponential growth rate of $S(f_\sigma, \varepsilon, T, K)$ by[1]

$$h(f_\sigma, K) := \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K). \tag{4}$$

The entropy $h(f_\sigma, K)$ is nonnegative as $S(f_\sigma, \varepsilon, T, K) \geq 1$ is nondecreasing in T and nonincreasing in ε.

[1] For brevity, we refer to $h(f_\sigma, K)$ simply as the (topological) entropy of the switched system (2) in the remainder of the paper.

Remark 2.1. In view of the equivalence of norms on a finite-dimensional vector space, the values of $h(f_\sigma, K)$ are the same for all norms $\|\cdot\|$ on $\mathbb{R}^n$; cf. [13, p. 109, Prop. 3.1.2]. For convenience and concreteness, $\|\cdot\|$ is taken to be the ∞-norm on $\mathbb{R}^n$ unless otherwise specified.

Next, we introduce an equivalent definition for the entropy of the switched system (2). With T and ε given as before, we say that a finite set $E \subset K$ is (T, ε)*-separated* if for each pair of distinct points $\hat{x}, \hat{x}' \in E$, it holds that

$$\hat{x}' \notin B_{f_\sigma}(\hat{x}, \varepsilon, T),$$

or equivalently, there is a time $t \in [0, T]$ such that $\|\xi_\sigma(\hat{x}', t) - \xi_\sigma(\hat{x}, t)\| \geq \varepsilon$. Denote by $N(f_\sigma, \varepsilon, T, K) \geq 1$ the maximal cardinality of a (T, ε)-separated set, or equivalently, the cardinality of a maximal (T, ε)-separated set, which is also nondecreasing in T and nonincreasing in ε. The entropy of (2) can be equivalently formulated in terms of the exponential growth rate of $N(f_\sigma, \varepsilon, T, K)$ as follows; the proof is along the lines of [13, p. 110] and thus omitted here.

Proposition 2.2. *The topological entropy of the switched system* (2) *satisfies*

$$h(f_\sigma, K) = \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log N(f_\sigma, \varepsilon, T, K).$$

2.2 Active time, active rates, and weighted averages

In this subsection, we define some switching-related quantities which will be useful in calculating the entropy of switched linear systems.

For a switching signal σ, we define the *active time* of each mode over $[0, t]$ by

$$\tau_p(t) := \int_0^t \mathbb{1}_p(\sigma(s))\, \mathrm{d}s, \qquad p \in \mathcal{P} \tag{5}$$

with the indicator function

$$\mathbb{1}_p(\sigma(s)) := \begin{cases} 1, & \sigma(s) = p, \\ 0, & \sigma(s) \neq p. \end{cases}$$

We also define the *active rate* of each mode over $[0, t]$ by

$$\rho_p(t) := \tau_p(t)/t, \qquad p \in \mathcal{P} \tag{6}$$

with $\rho_p(0) := \mathbb{1}_p(\sigma(0))$, and the *asymptotic active rate* of each mode by

$$\hat{\rho}_p := \limsup_{t \to \infty} \rho_p(t), \qquad p \in \mathcal{P}. \tag{7}$$

Clearly, the active times τ_p are nonnegative and nondecreasing, and satisfy $\sum_{p \in \mathcal{P}} \tau_p(t) = t$ for all $t \geq 0$; the active rates ρ_p take values in $[0, 1]$ and satisfy $\sum_{p \in \mathcal{P}} \rho_p(t) = 1$ for all $t \geq 0$. By contrast, due to the limit supremum in (7), it is possible that $\sum_{p \in \mathcal{P}} \hat{\rho}_p > 1$ for the asymptotic active rates $\hat{\rho}_p$; see [30, Example 1] for a numerical example.

For a family of scalars $\{\lambda_p \in \mathbb{R} : p \in \mathcal{P}\}$, we define the *asymptotic weighted average* by

$$\hat{\lambda} := \limsup_{t \to \infty} \sum_{p \in \mathcal{P}} \lambda_p \rho_p(t), \tag{8}$$

and the *maximal weighted average* over $[0, T]$ by

$$\bar{\lambda}(T) := \frac{1}{T} \max_{t \in [0,T]} \sum_{p \in \mathcal{P}} \lambda_p \tau_p(t). \tag{9}$$

The following lemma establishes a relation between these two notions.

Lemma 2.3 ([30, Lemma 1]). *The asymptotic weighted average $\hat{\lambda}$ defined by* (8) *and the maximal weighted average $\bar{\lambda}$ defined by* (9) *satisfy*

$$\limsup_{T \to \infty} \bar{\lambda}(T) = \max\{\hat{\lambda},\ 0\}.$$

2.3 Stability notions

When studying stability properties of the switched system (2), we assume that the origin is a common equilibrium for all modes, that is, $f_p(0) = 0$ for all $p \in \mathcal{P}$. The switched system (2) with a switching signal σ is *(Lyapunov) stable* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all initial states $x \in \mathbb{R}^n$ satisfying $\|x\| \leq \delta$, the corresponding solutions satisfy $\|\xi_\sigma(x, t)\| \leq \varepsilon$ for all $t \geq 0$; it is *globally exponentially stable (GES)* if there exist constants $c, \kappa > 0$ such that for all $x \in \mathbb{R}^n$,

$$\|\xi_\sigma(x, t)\| \leq c e^{-\kappa t} \|x\| \qquad \forall\, t \geq 0.$$

Clearly, stability implies $h(f_\sigma, K) = 0$ for a small enough initial set K, and GES implies $h(f_\sigma, K) = 0$ for all initial sets.

2.4 Switched linear systems

The main objective of this paper is to study the relations between topological entropy and stability properties for the switched linear system

$$\dot{x} = A_\sigma x \tag{10}$$

with a family of matrices $\{A_p \in \mathbb{R}^{n \times n} : p \in \mathcal{P}\}$. In this subsection, we recall some basic results on the entropy, stability properties, and solutions of (10), which will be useful in our analysis.

As the switching signal σ is piecewise constant, solutions of the switched linear system (10) can be formulated explicitly by

$$\xi_\sigma(x, t) = \Phi_\sigma(t, 0)\, x$$

with the state-transition matrix defined by

$$\Phi_\sigma(t, s) := \prod_{i=0}^{N_\sigma(t,s)} e^{A_{\sigma(t_i)}(t_{i+1} - t_i)}, \qquad t \geq s \geq 0,$$

where $t_1 < \cdots < t_{N_\sigma(t,s)}$ is the sequence of switches on $(s, t]$, and $t_0 := s$ and $t_{N_\sigma(t,s)+1} := t$.

Following [33, Prop. 2], the entropy of (10) is independent of the choice of initial set. Therefore, we omit the initial set and denote by $h(A_\sigma)$ the entropy of (10); here we think of matrices as linear operators. For convenience and concreteness, when we study entropy-related quantities of (10), the initial set K is taken to be the closed unit hypercube (recall that $\|\cdot\|$ is the ∞-norm) centered at the origin, that is, $K := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. By contrast, the initial states are arbitrary in $\mathbb{R}^n$ when we analyze stability properties of (10).

For the switched linear system (10), both stability and GES imply $h(A_\sigma) = 0$. However, it is possible that $h(A_\sigma) = 0$ while (10) is unstable; for example, consider the linear time-invariant system with a *shift matrix*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x.$$

3 AN ENTROPY-BASED SOLUTION ESTIMATION

In this section, we derive an entropy-based solution estimation for the switched linear system (10).

Theorem 3.1. *Consider the switched linear system* (10) *with a switching signal* σ. *For each* $\omega > 0$, *there exists a* $c_\omega > 0$ *such that for all initial states* $x \in \mathbb{R}^n$,

$$\|\xi_\sigma(x,t)\| \leq c_\omega e^{(h(A_\sigma)+\omega)t}\|x\| \qquad \forall\, t \geq 0. \tag{11}$$

Theorem 3.1 provides a fundamental connection between topological entropy and stability properties for the switched linear system (10). It indicates that the exponential growth rates of solutions are essentially upper bounded by the entropy $h(A_\sigma)$. In particular, if $h(A_\sigma) = 0$, then the solutions of (10) can only grow subexponentially.

The proof of Theorem 3.1 relies on the following technical lemma, which formulates a finite-horizon estimation for the solutions of (10) from the initial set K, in terms of the minimal cardinality of a spanning set.

Lemma 3.2. *Given a time horizon* $T \geq 0$ *and a radius* $\varepsilon > 0$, *the solutions of* (10) *with a switching signal* σ *satisfy that for all initial states* $x \in K$,

$$\max_{t\in[0,T]} \|\xi_\sigma(x,t)\| < 2\varepsilon S(A_\sigma, \varepsilon, T, K),$$

where $S(A_\sigma, \varepsilon, T, K)$ *is the minimal cardinality of a* (T, ε)*-spanning set.*

Proof. See Appendix A. □

Proof of Theorem 3.1. Consider an arbitrary initial state $x \in \mathbb{R}^n \backslash \{0\}$, as (11) holds trivially with $x = 0$.

First, define the unit vector $\bar{x} := x/\|x\|$. Then the corresponding solutions of (10) satisfy

$$\|\xi_\sigma(x,t)\| = \|x\|\|\xi_\sigma(\bar{x},t)\| \qquad \forall\, t \geq 0. \tag{12}$$

Second, recall that we take the initial set to be $K := \{x' \in \mathbb{R}^n : \|x'\| \leq 1\}$. As $\bar{x} \in K$, applying Lemma 3.2 yields

$$\max_{t\in[0,T]} \|\xi_\sigma(\bar{x},t)\| < 2\varepsilon S(A_\sigma, \varepsilon, T, K) \qquad \forall\, T \geq 0, \forall\, \varepsilon > 0. \tag{13}$$

Third, recall that $S(A_\sigma, \varepsilon, T, K)$ is the minimal cardinality of a (T, ε)-spanning set, which is nonincreasing in ε. Then the definition of topological entropy (4) implies

$$\limsup_{T\to\infty} \frac{1}{T} \log S(A_\sigma, \varepsilon, T, K) \leq h(A_\sigma) \qquad \forall\, \varepsilon > 0.$$

Given a radius $\varepsilon > 0$ and an $\omega > 0$, there exists a $T_{\varepsilon,\omega} \geq 0$ such that

$$S(A_\sigma, \varepsilon, T, K) < e^{(h(A_\sigma)+\omega)T} \qquad \forall\, T \geq T_{\varepsilon,\omega}. \tag{14}$$

Finally, combining (12)–(14) yields

$$\max_{t\in[0,T]} \|\xi_\sigma(x,t)\| \leq 2\varepsilon e^{(h(A_\sigma)+\omega)T}\|x\| \qquad \forall\, T \geq T_{\varepsilon,\omega}.$$

Let $c_\omega := 2\varepsilon e^{(h(A_\sigma)+\omega)T_{\varepsilon,\omega}}$. As $h(A_\sigma) + \omega > 0$, for $t \in [0, T_{\varepsilon,\omega}]$, it holds that

$$\|\xi_\sigma(x,t)\| \leq \max_{t\in[0,T_{\varepsilon,\omega}]} \|\xi_\sigma(x,t)\| \leq c_\omega\|x\| \leq c_\omega e^{(h(A_\sigma)+\omega)t}\|x\|,$$

and for $t > T_{\varepsilon,\omega}$, it holds that

$$\|\xi_\sigma(x,t)\| \leq 2\varepsilon e^{(h(A_\sigma)+\omega)t}\|x\| \leq c_\omega e^{(h(A_\sigma)+\omega)t}\|x\|. \quad \square$$

4 ENTROPY ESTIMATION VIA DECOMPOSITION

Based on Theorem 3.1, we design an approach for estimating the entropy of the switched linear system (10), by decomposing (10) into a part that is generated by scalar multiples of the identity matrix, and a part that is "almost stable" (i.e., it has zero entropy).

Theorem 4.1. *Consider the switched linear system* (10) *and a family of reference values* $\{\lambda_p \in \mathbb{R} : p \in \mathcal{P}\}$. *If the topological entropy of the residual switched linear system*

$$\dot{x} = \tilde{A}_\sigma x \tag{15}$$

defined by $\tilde{A}_p := A_p - \lambda_p I$ *for* $p \in \mathcal{P}$ *satisfies* $h(\tilde{A}_\sigma) = 0$, *then the topological entropy of* (10) *is upper bounded by*

$$h(A_\sigma) \leq \max\{n\hat{\lambda},\ 0\} \tag{16}$$

with the asymptotic weighted averages $\hat{\lambda}$ *defined by* (8).

Remark 4.2. Following [33, Th. 5], the upper bound in (16) is the entropy of the switched scalar system

$$\dot{x} = \lambda_\sigma x$$

defined by the reference values $\{\lambda_p : p \in \mathcal{P}\}$, multiplied by the dimension n, that is, (16) is equivalent to

$$h(A_\sigma) \leq nh(\lambda_\sigma).$$

Theorem 4.1 follows from Lemma 2.3 and the following general result, in which the switched linear system (10) is decomposed into a part that is generated by diagonal matrices, and a second part that commutes with the first.

Theorem 4.3. *Consider the switched linear system* (10). *Let*

$$\dot{x} = D_\sigma x \tag{17}$$

be a reference switched diagonal system with a family of diagonal matrices $\{D_p = \operatorname{diag}(a_p^1, \ldots, a_p^n) \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$ *such that each* D_p *commutes with every* A_q, *that is,*

$$D_p A_q = A_q D_p \qquad \forall\, p, q \in \mathcal{P}.$$

The topological entropy of (10) *is upper bounded by*

$$h(A_\sigma) \leq \limsup_{T\to\infty} \sum_{i=1}^{n} \bar{a}_i(T) + nh(\tilde{A}_\sigma) \tag{18}$$

with the component-wise maximal weighted averages over $[0, T]$ *defined by*

$$\bar{a}_i(T) := \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} a_p^i \tau_p(t), \qquad i = 1, \ldots, n,$$

where the active times τ_p *are defined by* (5), *and the topological entropy* $h(\tilde{A}_\sigma)$ *of the residual switched linear system* (15) *defined by* $\tilde{A}_p := A_p - D_p$ *for* $p \in \mathcal{P}$.

Remark 4.4. Following [33, Th. 7], the first term of the upper bound in (18) is the entropy of the reference switched diagonal system (17), that is, (18) is equivalent to

$$h(A_\sigma) \le h(D_\sigma) + nh(\tilde{A}_\sigma).$$

Proof of Theorem 4.3. Let $\xi_\sigma(x,t)$, $\hat{\xi}_\sigma(x,t)$, and $\tilde{\xi}_\sigma(x,t)$ be the solutions of (10), (17), and (15) with $\tilde{A}_p = A_p - D_p$ at time t with switching signal σ and initial state x, respectively. Note that all D_p commute with each other, and, as each D_p commutes with every A_q, it also commutes with every $\tilde{A}_q$. Hence for initial states $x, x' \in K$, the corresponding solutions satisfy

$$\begin{aligned}
\|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| &= \left\| \left(\prod_{i=0}^{N_\sigma(t,0)} e^{A_{\sigma(t_i)}(t_{i+1}-t_i)} \right)(x'-x) \right\| \\
&= \left\| \left(\prod_{i=0}^{N_\sigma(t,0)} e^{(D_{\sigma(t_i)}+\tilde{A}_{\sigma(t_i)})(t_{i+1}-t_i)} \right)(x'-x) \right\| \\
&= \left\| \left(\prod_{i=0}^{N_\sigma(t,0)} e^{\tilde{A}_{\sigma(t_i)}(t_{i+1}-t_i)} \right) e^{\sum_{p\in\mathcal{P}} D_p \tau_p(t)}(x'-x) \right\| \\
&= \|\tilde{\xi}_\sigma(\hat{\xi}_\sigma(x'-x,t),t)\|.
\end{aligned}$$

Given a radius $\varepsilon > 0$, applying Theorem 3.1 with $\omega = \varepsilon$ yields that there exists a $c_\varepsilon > 0$ such that (note that (11) holds for all $x \in \mathbb{R}^n$ in Theorem 3.1 above)

$$\|\tilde{\xi}_\sigma(\hat{\xi}_\sigma(x'-x,t),t)\| \le c_\varepsilon e^{(h(\tilde{A}_\sigma)+\varepsilon)t} \|\hat{\xi}_\sigma(x'-x,t)\|,$$

in which (recall that we take $\|\cdot\|$ to be the ∞-norm)

$$\begin{aligned}
\|\hat{\xi}_\sigma(x'-x,t)\| &= \left\| e^{\sum_{p\in\mathcal{P}} D_p \tau_p(t)}(x'-x) \right\| \\
&= \max_{i=1,\dots,n} e^{\sum_{p\in\mathcal{P}} a_p^i \tau_p(t)} |x'_i - x_i|.
\end{aligned}$$

Given a time horizon $T \ge 0$, let

$$\bar{\eta}_i(T) := \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} a_p^i \tau_p(t), \qquad i = 1,\dots,n.$$

Then

$$\begin{aligned}
&\max_{t\in[0,T]} \|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| \\
&\qquad \le c_\varepsilon e^{(h(\tilde{A}_\sigma)+\varepsilon)T} \max_{i=1,\dots,n} e^{\bar{\eta}_i(T)} |x'_i - x_i|. \qquad (19)
\end{aligned}$$

Consider the grid $G(\theta)$ defined by

$$G(\theta) := \{(k_1\theta_1, \dots, k_n\theta_n) \in K : k_1, \dots, k_n \in \mathbb{Z}\}$$

with the vector $\theta = (\theta_1, \dots, \theta_n)$ defined by

$$\theta_i := e^{-(h(\tilde{A}_\sigma)+\varepsilon)T - \bar{\eta}_i(T)} \varepsilon / c_\varepsilon, \qquad i = 1,\dots,n. \qquad (20)$$

Recall that we take the initial set to be $K := \{x \in \mathbb{R}^n : \|x\| \le 1\}$. Hence the cardinality of the grid $G(\theta)$ satisfies

$$|G(\theta)| = \prod_{i=1}^{n} (2\lfloor 1/\theta_i \rfloor + 1).$$

For a point $\hat{x} \in G(\theta)$, denote by $R(\hat{x})$ the open hyperrectangle in K with center $\hat{x}$ and sides $2\theta_1, \dots, 2\theta_n$, that is,

$$R(\hat{x}) := \{x \in K : |x_i - \hat{x}_i| < \theta_i \text{ for } i = 1,\dots,n\}. \qquad (21)$$

Then the union of all $R(\hat{x})$ covers the initial set K, that is,

$$K = \bigcup_{\hat{x}\in G(\theta)} R(\hat{x}).$$

By comparing (19)–(21) to (3), we see that $R(\hat{x}) \subset B_{A_\sigma}(\hat{x}, \varepsilon, T)$ for all $\hat{x} \in G(\theta)$. Hence the grid $G(\theta)$ is (T,ε)-spanning, and thus the minimal cardinality of a (T,ε)-spanning set satisfies

$$S(A_\sigma, \varepsilon, T, K) \le |G(\theta)| \le \prod_{i=1}^{n} (2/\theta_i + 1).$$

Then the definition (4) of the topological entropy implies

$$\begin{aligned}
h(A_\sigma) &\le \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{\log(2/\theta_i + 1)}{T} \\
&= \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{\log(1/\theta_i)}{T} + \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{\log(2+\theta_i)}{T} \\
&= \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{\bar{\eta}_i(T)}{T} + nh(\tilde{A}_\sigma) \\
&\quad + \lim_{\varepsilon\searrow 0} n\varepsilon + \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{n\log(c_\varepsilon/\varepsilon)}{T} \\
&= \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} a_p^i \tau_p(t) + nh(\tilde{A}_\sigma).
\end{aligned}$$

□

Clearly, the result of Theorem 4.1 holds if the residual switched linear system (15) is stable, which can always be achieved with large enough reference values λ_p. In the following subsections, this approach is combined with standard stability results to establish novel upper bounds for the entropy of three classes of switched linear systems. These upper bounds will also be combined with an entropy-based stability condition in the next section to extend stability results in the literature.

4.1 Switched linear systems under arbitrary switching

Based on Theorem 4.1, we construct the following general upper bound for the entropy of a switched linear system with an arbitrary switching signal.

Corollary 4.5. *The topological entropy of the switched linear system* (10) *is upper bounded by*

$$h(A_\sigma) \le \max\{n\hat{\lambda}_m^*, 0\} \qquad (22)$$

with

$$\hat{\lambda}_m^* := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \lambda_{\max}((A_p + A_p^\top)/2)\rho_p(t) \qquad (23)$$

where $\lambda_{\max}$ denotes the spectral abscissa, and the active rates ρ_p are defined by (6).

Proof. Consider an arbitrary $\delta > 0$, and let

$$\lambda_p := \lambda_{\max}((A_p + A_p^\top)/2) + \delta, \qquad p \in \mathcal{P}$$

in the residual switched linear system (15). Consider the quadratic function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by $V(x) := x^\top x/2$. Then for all

$p \in \mathcal{P}$,

$$\begin{aligned}\langle \nabla V(x), \tilde{A}_p x\rangle &= x^\top (A_p - \lambda_p I)\, x/2 + x^\top (A_p^\top - \lambda_p I)\, x/2 \\ &= x^\top (A_p + A_p^\top)\, x/2 - \lambda_{\max}((A_p + A_p^\top)/2)\, x^\top x - \delta x^\top x \\ &\le -\delta x^\top x\end{aligned}$$

for all $x \in \mathbb{R}^n$, where the inequality follows from the property that for all symmetric matrices $P \in \mathbb{R}^{n\times n}$,

$$\lambda_{\min}(P)\, x^\top x \le x^\top P x \le \lambda_{\max}(P)\, x^\top x \qquad \forall\, x \in \mathbb{R}^n, \tag{24}$$

where $\lambda_{\min}$ denotes the smallest eigenvalue, and $\lambda_{\max}$ is the largest eigenvalue as P is symmetric. Hence V is a common Lyapunov function, which implies that (15) is GES under arbitrary switching [18, p. 23], and thus $h(\tilde{A}_\sigma) = 0$. Then (22) follows from Theorem 4.1 and the fact that $\delta > 0$ is arbitrary. □

In [33, Th. 4], a general upper bound for the topological entropy $h(A_\sigma)$ of the switched linear system (10) was established using an arbitrary induced matrix norm $\|\cdot\|$:

$$h(A_\sigma) \le \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \|A_p\| \rho_p(t). \tag{25}$$

It is straightforward to verify that the upper bound (22) is better than (25) calculated with the Euclidean norm. For an arbitrary induced matrix norm, the relation between the upper bounds (22) and (25) is undetermined. However, $\|A_p\| \ge 0$ for all induced matrix norms, whereas it is possible for the spectral abscissae $\lambda_{\max}(A_p)$ to be negative, which will be needed for the stability results in the next section.

4.2 Switched triangular systems under arbitrary switching

In this subsection, we show that the general upper bound for entropy in Corollary 4.5 can be improved when the system matrices satisfy suitable commutation relations.

Consider the case of a switched linear system with simultaneously triangularizable matrices, that is, there exists a (possibly complex) change of basis under which the matrices A_p in (10) are all upper triangular.[2] Hence we assume, without loss of generality, that every A_p is upper triangular, and denote it by $U_p \in \mathbb{C}^{n\times n}$. Then (10) becomes the switched triangular system in $\mathbb{C}^n$ defined by[3]

$$\dot{x} = U_\sigma x. \tag{26}$$

Corollary 4.6. *The topological entropy of the switched triangular system* (26) *is upper bounded by*

$$h(U_\sigma) \le \max\{n\hat{\lambda}_m,\, 0\} \tag{27}$$

with

$$\hat{\lambda}_m := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \lambda_{\max}(U_p) \rho_p(t), \tag{28}$$

where $\lambda_{\max}$ denotes the spectral abscissa, and the active rates ρ_p are defined by (6).

As

$$\lambda_{\max}(A) \le \lambda_{\max}((A + A^\top)/2) \qquad \forall\, A \in \mathbb{R}^{n\times n} \tag{29}$$

(see, e.g., [2, p. 74, Prop. III.5.3]; the equality holds if and only if A is normal, i.e., $AA^\top = A^\top A$), it follows that $\hat{\lambda}_m \le \hat{\lambda}_m^*$ for the constants $\hat{\lambda}_m^*$ and $\hat{\lambda}_m$ defined by (23) and (28), respectively. Hence Corollary 4.6 provides a better upper bound for the entropy $h(U_\sigma)$ of the switched triangular system (26) than Corollary 4.5.

Proof of Corollary 4.6. Consider an arbitrary $\delta > 0$, and let

$$\lambda_p := \lambda_{\max}(U_p) + \delta, \qquad p \in \mathcal{P}.$$

Then the residual switched linear system (15) is defined by $\tilde{A}_p := U_p - (\lambda_{\max}(U_p) + \delta)\, I$ for $p \in \mathcal{P}$. As the matrices $\tilde{A}_p$ are all upper triangular, and their diagonal entries all have negative real parts, it follows that (15) is GES under arbitrary switching [18, p. 35, Prop. 2.9], and thus $h(\tilde{A}_\sigma) = 0$. Then (27) follows from Theorem 4.1 and the fact that $\delta > 0$ is arbitrary; see also footnote 3. □

In [33, Th. 11], an upper bound for the topological entropy $h(U_\sigma)$ of the switched triangular system (26) was established by deriving an explicit formula for the solutions:

$$h(U_\sigma) \le \limsup_{T\to\infty} \left(n\bar{a}_1(T) + \sum_{i=2}^{n} (n+1-i)\bar{d}_i(T) \right) \tag{30}$$

with

$$\bar{a}_1(T) := \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^1)\tau_p(t) \ge 0$$

and

$$\bar{d}_i(T) := \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i - a_p^{i-1})\tau_p(t) \ge 0, \qquad i = 2, \dots, n.$$

In general, the relation between the upper bounds (27) and (30) is undetermined and depends on the order of the diagonal entries of U_p. However, the upper bound (27) only depends on the active rates ρ_p and the spectral abscissae $\lambda_{\max}(U_p)$, and thus can be calculated for a switched linear system with simultaneously triangularizable matrices without the simultaneous triangularization.

4.3 Switched linear systems under slow switching

In stability analysis and stabilization of switched systems, it is a standard technique to impose suitable slow-switching conditions. In this section, we demonstrate how such a technique can also be used to formulate upper bounds for the entropy.

Following [10], we say the switching signal σ admits an *average dwell time (ADT)* $\tau_a > 0$ if there is a constant $N_0 \ge 1$ such that

$$N_\sigma(t_2, t_1) \le \frac{t_2 - t_1}{\tau_a} + N_0 \qquad \forall\, t_2 > t_1 \ge 0, \tag{31}$$

where $N_\sigma(t_2, t_1)$ is the number of switches on $(t_1, t_2]$. If $N_0 = 1$, then (31) becomes the *dwell-time* condition [22].

[2] Sufficient conditions for simultaneous triangularizability include that the matrices A_p commute pairwise [12, p. 81, Th. 2.3.3], or, more generally, that the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable (Lie's theorem, see, e.g., [25, p. 21, Th. A″]). See [17] and [12, p. 94, Th. 2.4.15] for more sufficient conditions, and [9] for a necessary and sufficient condition.

[3] It is straightforward to verify that all the results in this paper still hold if the state space is extended from $\mathbb{R}^n$ to $\mathbb{C}^n$ and the matrices $A_p \in \mathbb{C}^{n\times n}$. In particular, the reference values λ_p in Theorem 4.1 and the diagonal matrices D_p in Theorem 4.3 should still be selected from $\mathbb{R}$ and $\mathbb{R}^{n\times n}$, respectively.

Corollary 4.7. *Consider the switched linear system* (10). *Let*

$$\hat{\lambda}_m := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \lambda_{\max}(A_p)\rho_p(t), \tag{32}$$

where $\lambda_{\max}$ denotes the spectral abscissa, and the active rates ρ_p are defined by (6). *For each $\lambda > \hat{\lambda}_m$, there exists a finite $\tau_a^* \geq 0$ such that if the ADT condition* (31) *holds with $\tau_a > \tau_a^*$, then the topological entropy of* (10) *is upper bounded by*

$$h(A_\sigma) \leq \max\{n\lambda,\ 0\}. \tag{33}$$

The property (29) implies that Corollary 4.7 with a suitable ADT provides a better upper bound for the entropy $h(A_\sigma)$ than Corollary 4.5. The lower bound τ_a^* for ADT is established based on the solutions of continuous Lyapunov equations for modes of a residual switched linear system; an explicit formula is given by (34) in the proof below.

Proof of Corollary 4.7. Consider an arbitrary $\delta > 0$, define $\lambda_0 := \lambda - \hat{\lambda}_m > 0$, and let

$$\lambda_p := \lambda_{\max}(A_p) + \lambda_0 + \delta, \qquad p \in \mathcal{P}.$$

Then the residual switched linear system (15) is defined by $\tilde{A}_p := A_p - (\lambda_{\max}(A_p) + \lambda_0 + \delta)\, I$ for $p \in \mathcal{P}$. Stability of (15) is established along the lines of standard ADT results for stability of switched linear systems with Hurwitz matrices. More specifically, for each $p \in \mathcal{P}$, as $\tilde{A}_p + \lambda_0 I = A_p - (\lambda_{\max}(A_p) + \delta)\, I$ is Hurwitz, there exists a (symmetric) positive definite matrix $P_p \in \mathbb{R}^{n\times n}$ such that

$$P_p\,(\tilde{A}_p + \lambda_0 I) + (\tilde{A}_p^\top + \lambda_0 I)\, P_p = -I.$$

Then the family of quadratic functions $V_p : \mathbb{R}^n \to \mathbb{R}_+$ for $p \in \mathcal{P}$ defined by $V_p(x) := x^\top P_p x$ satisfy that for all $p \in \mathcal{P}$,

$$\langle \nabla V_p(x), \tilde{A}_p x\rangle = x^\top (P_p \tilde{A}_p + \tilde{A}_p^\top P_p)\, x < -2\lambda_0 V_p(x) \qquad \forall\, x \in \mathbb{R}^n.$$

Also, the property (24) implies that for all $p, q \in \mathcal{P}$,

$$V_p(x) \leq \mu V_q(x) \qquad \forall\, x \in \mathbb{R}^n$$

with the constant

$$\mu := \max_{p,q\in\mathcal{P}} \frac{\lambda_{\max}(P_p)}{\lambda_{\min}(P_q)} \geq 1,$$

where $\lambda_{\min}$ denotes the smallest eigenvalue, and $\lambda_{\max}$ is the largest eigenvalue as all P_p are symmetric. Hence if the ADT condition (31) holds with

$$\tau_a > \tau_a^* := \frac{\log \mu}{2\lambda_0} = \frac{\log \mu}{2(\lambda - \hat{\lambda}_m)} \geq 0, \tag{34}$$

then (15) is GES [10, Th. 2] (see [18, p. 59, Th. 3.2] for the explicit lower bound for ADT), and thus $h(\tilde{A}_\sigma) = 0$. Then (33) follows from Theorem 4.1 and the fact that $\delta > 0$ is arbitrary. In particular,

$$\begin{aligned}\limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \lambda_p \rho_p(t) &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} (\lambda_{\max}(A_p) + \lambda_0 + \delta)\rho_p(t) \\ &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \lambda_{\max}(A_p)\rho_p(t) + \lambda_0 + \delta \\ &= \hat{\lambda}_m + \lambda_0 + \delta = \lambda + \delta. \qquad \square\end{aligned}$$

5 ENTROPY-BASED STABILITY CONDITIONS

In this section, we establish an entropy-based stability condition for the switched linear system (10), based on the solution estimation in Theorem 3.1. The essential idea is that (10) is GES if its entropy remains zero under a destabilizing perturbation.

Theorem 5.1. *Consider the switched linear system* (10) *with a switching signal σ. If there exists a $\delta > 0$ such that the topological entropy of the auxiliary switched linear system*

$$\dot{x} = \bar{A}_\sigma x \tag{35}$$

defined by $\bar{A}_p := A_p + \delta I$ for $p \in \mathcal{P}$ satisfies $h(\bar{A}_\sigma) = 0$, then (10) *with σ is GES.*

Proof. Denote by $\xi_\sigma(x, t)$ and $\bar{\xi}_\sigma(x, t)$ the solutions of (10) and (35) at time t with switching signal σ and initial state x, respectively. Then

$$\begin{aligned}\|\bar{\xi}_\sigma(x, t)\| &= \left\| \left(\prod_{i=0}^{N_\sigma(t,0)} e^{(A_{\sigma(t_i)}+\delta I)(t_{i+1}-t_i)} \right) x \right\| \\ &= e^{\delta t} \left\| \left(\prod_{i=0}^{N_\sigma(t,0)} e^{A_{\sigma(t_i)}(t_{i+1}-t_i)} \right) x \right\| = e^{\delta t} \|\xi_\sigma(x, t)\|.\end{aligned}$$

As $h(\bar{A}_\sigma) = 0$, Applying Theorem 3.1 with $\omega = \delta/2$ implies that there exists a $c_{\delta/2} > 0$ such that for all $x \in \mathbb{R}^n$,

$$\|\bar{\xi}_\sigma(x, t)\| \leq c_{\delta/2} e^{\delta t/2} \|x\|.$$

Hence

$$\|\xi_\sigma(x, t)\| = e^{-\delta t} \|\bar{\xi}_\sigma(x, t)\| \leq c_{\delta/2} e^{-\delta t/2} \|x\|. \qquad \square$$

Theorem 5.1 can be readily combined with upper bounds for the entropy of switched linear systems to generate sufficient conditions for GES. In the following, we apply this result to formulate novel stability conditions for the three classes of switched linear systems in Subsections 4.2–4.3, which extend stability results in the literature.

First, combining Theorem 5.1 and Corollary 4.5 yields the following stability condition for the switched linear system (10) under arbitrary switching.

Corollary 5.2. *The switched linear system* (10) *with a switching signal σ is GES provided that $\hat{\lambda}_m^*$ defined by* (23) *satisfies $\hat{\lambda}_m^* < 0$.*

Second, combining Theorem 5.1 and Corollary 4.6 yields the following stability condition for the switched triangular system (26) under arbitrary switching.

Corollary 5.3. *The switched triangular system* (26) *with a switching signal σ is GES provided that $\hat{\lambda}_m$ defined by* (28) *satisfies $\hat{\lambda}_m < 0$.*

In particular, $\hat{\lambda}_m < 0$ if all $\lambda_{\max}(U_p) < 0$. Hence Corollary 5.3 extends the well known result that a switched linear system with Hurwitz triangular matrices is GES under arbitrary switching [18, p. 35, Prop. 2.9], in allowing for unstable modes that are not active for too long on average.

Finally, combining Theorem 5.1 and Corollary 4.7 yields the following stability condition for the switched linear system (10) under slow switching in the sense of an ADT.

Corollary 5.4. *Consider the switched linear system* (10) *with a switching signal σ. Provided that $\hat{\lambda}_m$ defined by* (32) *satisfies $\hat{\lambda}_m < 0$, there exists a finite $\tau_a^* \geq 0$ such that if the ADT condition* (31) *holds with $\tau_a > \tau_a^*$, then* (10) *with σ is GES.*

The lower bound τ_a^* in Corollary 5.4 can be set as the one in Corollary 4.7 with $\lambda = 0 > \hat{\lambda}_m$, that is,

$$\tau_a^* := -\frac{\log \mu}{2\hat{\lambda}_m} \geq 0$$

with

$$\mu := \inf_{\delta > 0} \max_{p,q \in \mathcal{P}} \frac{\lambda_{\max}(P_p)}{\lambda_{\min}(P_q)} \geq 1,$$

where P_p are the solutions of the continuous Lyapunov equations

$$P_p \left(A_p - (\lambda_{\max}(A_p) + \delta)\, I\right) + \left(A_p^\top - (\lambda_{\max}(A_p) + \delta)\, I\right) P_p = -I,$$

$\lambda_{\min}$ denotes the smallest eigenvalue, and each $\lambda_{\max}(P_p)$ is the largest eigenvalue of P_p as P_p is symmetric.

In particular, $\hat{\lambda}_m < 0$ if all $\lambda_{\max}(A_p) < 0$. Hence Corollary 5.4 extends the standard ADT result for stability of switched linear systems with Hurwitz matrices [10, Th. 2], in allowing for unstable modes that are not active for too long on average.

6 CONCLUSION

This paper studied the relations between topological entropy and stability properties for switched linear systems. We constructed novel upper bounds for topological entropy by designing a decomposition into a part that is generated by scalar multiples of the identity matrix and a part that is "almost stable" (i.e., it has zero entropy), and proving that the overall topological entropy is upper bounded by that of the former. We also formulated entropy-based sufficient conditions for global exponential stability, which, together with the upper bounds for entropy, extend stability results in the literature. In particular, our results demonstrate that the notions used to quantify switching in studying topological entropy (active times of individual modes) and stability and stabilization (ADT) can be combined to establish better estimates for topological entropy and more general stability conditions.

In [31, 34], a notion of *time-ratio constraint* was used to formulate sufficient conditions for stability properties of switched systems with unstable modes. The essential idea is to restrict the ratio of the time that unstable modes are active, which is similar to the stability conditions in Corollaries 5.2–5.4, and their comparison will be a topic for future research. Another potential direction is to extend the current results to switched nonlinear systems.

REFERENCES

[1] Roy L. Adler, Alan G. Konheim, and M. H. McAndrew. 1965. Topological entropy. *Trans. Amer. Math. Soc.* 114, 2 (Feb. 1965), 309–319. https://doi.org/10.2307/1994177
[2] Rajendra Bhatia. 1997. *Matrix Analysis*. Springer New York, New York, NY. https://doi.org/10.1007/978-1-4612-0653-8
[3] Rufus Bowen. 1971. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* 153 (Jan. 1971), 401–414. https://doi.org/10.1090/S0002-9947-1971-0274707-X
[4] Rufus Bowen. 1971. Periodic points and measures for axiom A diffeomorphisms. *Trans. Amer. Math. Soc.* 154 (Feb. 1971), 377–397. https://doi.org/10.2307/1995452
[5] Fritz Colonius. 2012. Minimal bit rates and entropy for exponential stabilization. *SIAM Journal on Control and Optimization* 50, 5 (Oct. 2012), 2988–3010. https://doi.org/10.1137/110829271
[6] Fritz Colonius and Christoph Kawan. 2009. Invariance entropy for control systems. *SIAM Journal on Control and Optimization* 48, 3 (May 2009), 1701–1721. https://doi.org/10.1137/080713902
[7] Fritz Colonius, Christoph Kawan, and Girish N. Nair. 2013. A note on topological feedback entropy and invariance entropy. *Systems & Control Letters* 62, 5 (May 2013), 377–381. https://doi.org/10.1016/j.sysconle.2013.01.008
[8] Efim I. Dinaburg. 1970. The relation between topological entropy and metric entropy. *Doklady Akademii Nauk SSSR* 190, 1 (1970), 19–22.
[9] Michael P. Drazin, James W. Dungey, and Karl W. Gruenberg. 1951. Some theorems on commutative matrices. *Journal of the London Mathematical Society* 26, 3 (July 1951), 221–228. https://doi.org/10.1112/jlms/s1-26.3.221
[10] João P. Hespanha and A. Stephen Morse. 1999. Stability of switched systems with average dwell-time. In *38th IEEE Conference on Decision and Control*, Vol. 3. IEEE, Phoenix, AZ, 2655–2660. https://doi.org/10.1109/CDC.1999.831330
[11] João P. Hespanha, Antonio Ortega, and Lavanya Vasudevan. 2002. Towards the control of linear systems with minimum bit-rate. In *15th International Symposium on Mathematical Theory of Networks and Systems*. Notre Dame, IN, 1–15. https://www3.nd.edu/~mtns/papers/13040_3.pdf
[12] Roger A. Horn and Charles R. Johnson. 1985. *Matrix Analysis*. Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/CBO9780511810817
[13] Anatole Katok and Boris Hasselblatt. 1995. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/CBO9780511809187
[14] Christoph Kawan and Yuri Latushkin. 2016. Some results on the entropy of non-autonomous dynamical systems. *Dynamical Systems* 31, 3 (July 2016), 251–279. https://doi.org/10.1080/14689367.2015.1111299
[15] Andrei N. Kolmogorov. 1958. A new metric invariant of transitive dynamical systems and automorphisms of Lebesgue spaces. *Doklady Akademii Nauk SSSR* 119, 5 (1958), 861–864.
[16] Sergiy Kolyada and L'ubomír Snoha. 1996. Topological entropy of nonautonomous dynamical systems. *Random & Computational Dynamics* 4, 2-3 (1996), 205–233.
[17] Thomas J. Laffey. 1978. Simultaneous triangularization of matrices—low rank cases and the nonderogatory case. *Linear and Multilinear Algebra* 6, 4 (Jan. 1978), 269–305. https://doi.org/10.1080/03081087808817249
[18] Daniel Liberzon. 2003. *Switching in Systems and Control*. Birkhäuser Boston, Boston, MA. https://doi.org/10.1007/978-1-4612-0017-8
[19] Daniel Liberzon. 2014. Finite data-rate feedback stabilization of switched and hybrid linear systems. *Automatica* 50, 2 (Feb. 2014), 409–420. https://doi.org/10.1016/j.automatica.2013.11.037
[20] Daniel Liberzon and Sayan Mitra. 2018. Entropy and minimal bit rates for state estimation and model detection. *IEEE Trans. Automat. Control* 63, 10 (Oct. 2018), 3330–3344. https://doi.org/10.1109/TAC.2017.2782478
[21] Yoshihiro Mori, Takehiro Mori, and Yasuaki Kuroe. 1997. A solution to the common Lyapunov function problem for continuous-time systems. In *36th IEEE Conference on Decision and Control*, Vol. 4. IEEE, San Diego, CA, 3530–3531. https://doi.org/10.1109/CDC.1997.652397
[22] A. Stephen Morse. 1996. Supervisory control of families of linear set-point controllers—Part I. Exact matching. *IEEE Trans. Automat. Control* 41, 10 (Oct. 1996), 1413–1431. https://doi.org/10.1109/9.539424
[23] Girish N. Nair and Robin J. Evans. 2003. Exponential stabilisability of finite-dimensional linear systems with limited data rates. *Automatica* 39, 4 (April 2003), 585–593. https://doi.org/10.1016/S0005-1098(02)00285-6
[24] Girish N. Nair, Robin J. Evans, Iven M. Y. Mareels, and William Moran. 2004. Topological feedback entropy and nonlinear stabilization. *IEEE Trans. Automat. Control* 49, 9 (Sept. 2004), 1585–1597. https://doi.org/10.1109/TAC.2004.834105
[25] Hans Samelson. 1990. *Notes on Lie Algebras*. Springer New York, New York, NY. https://doi.org/10.1007/978-1-4613-9014-5
[26] Andrey V. Savkin. 2006. Analysis and synthesis of networked control systems: topological entropy, observability, robustness and optimal control. *Automatica* 42, 1 (Jan. 2006), 51–62. https://doi.org/10.1016/j.automatica.2005.08.021
[27] Claude E. Shannon. 1948. A mathematical theory of communication. *The Bell System Technical Journal* 27, 3 (July 1948), 379–423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x
[28] Hussein Sibai and Sayan Mitra. 2017. Optimal data rate for state estimation of switched nonlinear systems. In *HSCC '17: 20th International Conference on Hybrid Systems: Computation and Control*. ACM, Pittsburgh, PA, 71–80. https://doi.org/10.1145/3049797.3049799
[29] Sekhar Tatikonda and Sanjoy Mitter. 2004. Control under communication constraints. *IEEE Trans. Automat. Control* 49, 7 (July 2004), 1056–1068. https://doi.org/10.1109/TAC.2004.831187
[30] Guosong Yang and João P. Hespanha. 2018. On topological entropy of switched linear systems with pairwise commuting matrices. In *56th Annual Allerton Conference on Communication, Control, and Computing*. Monticello, IL. To appear.
[31] Guosong Yang and Daniel Liberzon. 2014. Input-to-state stability for switched systems with unstable subsystems: A hybrid Lyapunov construction. In *53rd IEEE Conference on Decision and Control*. IEEE, Los Angeles, CA, 6240–6245. https://doi.org/10.1109/CDC.2014.7040367

[32] Guosong Yang and Daniel Liberzon. 2018. Feedback stabilization of a switched linear system with an unknown disturbance under data-rate constraints. *IEEE Trans. Automat. Control* 63, 7 (July 2018), 2107–2122. https://doi.org/10.1109/TAC.2017.2767822

[33] Guosong Yang, A. James Schmidt, and Daniel Liberzon. 2018. On topological entropy of switched linear systems with diagonal, triangular, and general matrices. In *57th IEEE Conference on Decision and Control*. IEEE, Miami Beach, FL, 5682–5687.

[34] Guisheng Zhai, Bo Hu, Kazunori Yasuda, and Anthony N. Michel. 2000. Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach. In *2000 American Control Conference*, Vol. 1. IEEE, Chicago, IL, 200–204. https://doi.org/10.1109/ACC.2000.878825

A PROOF OF LEMMA 3.2

Let E be a minimal (T, ε)-spanning set. Then its cardinality $|E| = S(A_\sigma, \varepsilon, T, K)$. Consider an arbitrary initial state $x \in K$.

Claim: In the initial set K, there exists a finite sequence of points $(v_k)_{k=0}^{2m}$ with $v_0 = x$ and $v_{2m} = 0$ such that $m \leq |E|$ and

$$\max_{t\in[0,T]} \|\xi_\sigma(v_k, t) - \xi_\sigma(v_{k-1}, t)\| \leq \varepsilon \qquad \forall\, k \in \{1, \ldots, 2m\}.$$

Reason: The essential idea is to iteratively cover the line segment from x to 0. Formally, we construct the sequence of points $(v_k)_{k=0}^{2m}$ as follows:

1. Let $v_0 := x$ and $\bar{K}_0 := \{\alpha x \in K : \alpha \in [0, 1]\}$, that is, $\bar{K}_0$ is the line segment from x to 0. Initialize $l = 0$.
2. Let a point $v_{2l+1} \in E \backslash \{v_1, v_3, \ldots, v_{2l-1}\}$ be such that $v_{2l} \in B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$; such a point exists as E is (T, ε)-spanning and $v_{2l} \notin \bigcup_{l'=0}^{l-1} B_{A_\sigma}(v_{2l'+1}, \varepsilon, T)$, which is trivial for $l = 0$ and will be shown iteratively below for $l \geq 1$. Then
$$\max_{t\in[0,T]} \|\xi_\sigma(v_{2l+1}, t) - \xi_\sigma(v_{2l}, t)\| < \varepsilon.$$
3. Check if $0 \in B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$.
   i. If $0 \in B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$, then let $m = l + 1$ and $v_{2m} = 0$, and the construction is complete.
   ii. Otherwise $0 \notin B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$, and thus it holds that $0 \notin \bigcup_{l'=0}^{l} B_{A_\sigma}(v_{2l'+1}, \varepsilon, T)$; else the construction would have been complete with a smaller l. Let
   $$v_{2l+2} := \Big(\inf_{v' \in \bar{K}_l \cap B_{A_\sigma}(v_{2l+1}, \varepsilon, T)} \|v'\|\Big) \frac{v_{2l}}{\|v_{2l}\|};$$
   such a point exists and is in $\bar{K}_l$ as $v_{2l} \in B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$. As $B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$ is an open set, v_{2l+2} is in the boundary of $B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$ but not in $B_{A_\sigma}(v_{2l+1}, \varepsilon, T)$. Hence
   $$\max_{t\in[0,T]} \|\xi_\sigma(v_{2l+2}, t) - \xi_\sigma(v_{2l+1}, t)\| = \varepsilon,$$
   and the line segment $\bar{K}_{l+1} := \{\alpha v_{2l+2} : \alpha \in [0, 1]\}$ satisfies $\bar{K}_{l+1} \cap B_{A_\sigma}(v_{2l+1}, \varepsilon, T) = \emptyset$, which also holds for all smaller l. Therefore, as $v_{2l+2} \in \bar{K}_{l+1} \subset \cdots \subset \bar{K}_1$ and $\bar{K}_{l'+1} \cap B_{A_\sigma}(v_{2l'+1}, \varepsilon, T) = \emptyset$ for all $l' \in \{0, \ldots, l\}$, it follows that
   $$v_{2l+2} \notin \bigcup_{l'=0}^{l} B_{A_\sigma}(v_{2l'+1}, \varepsilon, T).$$
   Update $l = l + 1$ and return to item 2).

As E is (T, ε)-spanning and all v_{2l+1} are distinct, the construction is complete with at most $|E|$ iterations; thus $m \leq |E|$.

Based on the claim and the triangle inequality, we obtain

$$\|\xi_\sigma(x, t)\| \leq \sum_{k=1}^{2m} \|\xi_\sigma(v_k, t) - \xi_\sigma(v_{k-1}, t)\| \leq 2\varepsilon S(A_\sigma, \varepsilon, T, K)$$

for all $t \in [0, T]$.