On Topological Entropy of Switched Linear Systems with Pairwise Commuting Matrices

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Motivation

Topological entropy in systems theory

- Originated from [Kolmogorov, 1958], defined by [Adler, Konheim, and McAndrew, 1965], [Bowen, 1971], and [Dinaburg, 1970].
- Essential idea:
  - The complexity growth of a system.
  - The information accumulation needed to approximate a trajectory.
- In control theory:
  - Topological feedback entropy [Nair, Evans, Mareels, and Moran, 2004]
  - Invariance entropy [Colonius and Kawan, 2009], exponential stabilization entropy [Colonius, 2012]
  - Estimation entropy [Savkin, 2006] and [Liberzon and Mitra]
- Minimal data rate for stabilizing linear time-invariant (LTI) system [Hespanha, Ortega, and Vasudevan, 2002], [Nair and Evans, 2003], and [Tatikonda and Mitter, 2004]

Motivation

Switched linear system with pairwise commuting matrices

- Switching is ubiquitous in realistic system models.
- Stability under arbitrary switching: pairwise commuting matrices [Narendra and Balakrishnan, 1994]
- Neither minimal data rate for stabilization nor topological entropy is well-understood:
  - Sufficient data rate [Liberzon, 2014], [Yang and Liberzon, 2018], and [Sibai and Mitra, 2017]
  - Topological entropy [Yang, Schmidt, and Liberzon]

Switched System

A finite family of continuous-time dynamical systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P}$$

with the state $x \in \mathbb{R}^n$ and an index set $\mathcal{P}$.

A switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K.$$

- Switching signal $\sigma : \mathbb{R}_+ \to \mathcal{P}$ is right-continuous and piecewise constant
- Initial set K is compact with a nonempty interior
- Modes $\{f_p : p \in \mathcal{P}\}$
- Denote by $\xi_\sigma(x, t)$ the solution at t with switching signal σ and initial state x

Entropy Definition

A switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K.$$

- Given a time horizon $T \geq 0$ and a radius $\varepsilon > 0$, define the open ball:

$$B_{f_\sigma}(x, \varepsilon, T) := \Big\{ x' \in K : \max_{t \in [0,T]} \|\xi_\sigma(x', t) - \xi_\sigma(x, t)\| < \varepsilon \Big\}.$$

- A finite set $E \subset K$ is (T, ε)-spanning if $K = \bigcup_{\hat{x} \in E} B_{f_\sigma}(\hat{x}, \varepsilon, T)$.
- Let $S(f_\sigma, \varepsilon, T, K)$ be the minimal cardinality of a (T, ε)-spanning set.
- The topological entropy with initial set K and switching signal σ is defined in terms of the exponential growth rate of $S(f_\sigma, \varepsilon, T, K)$ by

$$h(f_\sigma, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K).$$

Alternative Entropy Definition

A switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K.$$

- The topological entropy is defined by

$$h(f_\sigma, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K).$$

- A finite set of points $E \subset K$ is (T, ε)-separated if for all $\hat{x}, \hat{x}' \in E$,

$$\hat{x}' \notin B_{f_\sigma}(\hat{x}, \varepsilon, T) = \Big\{ x \in K : \max_{t \in [0,T]} \|\xi_\sigma(x,t) - \xi_\sigma(\hat{x},t)\| < \varepsilon \Big\}.$$

- Let $N(f_\sigma, \varepsilon, T, K)$ be the maximal cardinality of a (T, ε)-separated set.
- Proposition 1. The topological entropy satisfies

$$h(f_\sigma, K) = \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log N(f_\sigma, \varepsilon, T, K).$$

Proof. $N(f_\sigma, 2\varepsilon, T, K) \le S(f_\sigma, \varepsilon, T, K) \le N(f_\sigma, \varepsilon, T, K)$. $\square$

Active Time and Active Rates

For a switching signal σ, define the following quantities.

- The active time of each mode over an interval $[0, t]$ is

$$\tau_p(t) := \int_0^t \mathbb{1}_p(\sigma(s))\, \mathrm{d}s, \qquad p \in \mathcal{P}$$

  with the indicator function $\mathbb{1}$. Then $\sum_{p \in \mathcal{P}} \tau_p(t) = t$.

- The active rate of each mode over $[0, t]$ is

$$\rho_p(t) := \tau_p(t)/t, \qquad p \in \mathcal{P}$$

  with $\rho_p(0) := \mathbb{1}_p(\sigma(0))$. Then $\sum_{p \in \mathcal{P}} \rho_p(t) = 1$.

- The asymptotic active rate of each mode is

$$\hat{\rho}_p := \limsup_{t \to \infty} \rho_p(t), \qquad p \in \mathcal{P}.$$

  It is possible that $\sum_{p \in \mathcal{P}} \hat{\rho}_p > 1$.

Entropy of Switched Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K.$$

Results from [Yang, Schmidt, and Liberzon]:

- Proposition 2. The topological entropy of the switched linear system is independent of the choice of the initial set K.
- Proposition 3. The topological entropy of the switched linear system is upper bounded by

$$\limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(A_p)\rho_p(t) \le h(A_\sigma) \le \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} n\|A_p\|\rho_p(t)$$

with the active rates ρ_p.

Proof for the upper bound.

1. The solutions satisfy $\|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| \le e^{\sum_p \|A_p\|\tau_p(t)}\|x'-x\|$.
2. Construct a (T,ε)-spanning set using a grid. $\square$

Lack of "independence" between eigenspaces of different modes!

Switched Commuting Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K$$

with a commuting family $\{A_p : p \in \mathcal{P}\}$.

- If all A_p are diagonalizable, then there is a change of basis under which all A_p are diagonal.
- Every scalar component evolves independently (under the same switching signal).
- A formula for the entropy was established in [Yang, Schmidt, and Liberzon].

Switched Commuting Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K$$

with a commuting family $\{A_p : p \in \mathcal{P}\}$.

- A well-known result: in general, there is a change of basis under which all A_p are upper triangular.
- Each scalar component evolves in a "strict-feedback" fashion.
- An upper bound for the entropy was established in [Yang, Schmidt, and Liberzon].
- Being simultaneously triangularizable is weaker than being pairwise commuting!

Switched Commuting Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K$$

with a commuting family $\{A_p : p \in \mathcal{P}\}$.

- First goal: a suitable change of basis for pairwise commuting matrices.
- Jordan–Chevalley Decomposition [Humphreys, 1972]. For each matrix A, there exist polynomials f and g, without constant term, such that $f(A)$ is a diagonalizable matrix, $g(A)$ is a nilpotent matrix, and

$$A = f(A) + g(A).$$

- A polynomial of a matrix A commutes with all matrices that commute with A.

A Change of Basis

Proposition 5

For the commuting family $\{A_p : p \in \mathcal{P}\}$, there exists an invertible matrix $\Gamma \in \mathbb{C}^{n\times n}$ such that

$$\Gamma A_p \Gamma^{-1} = D_p + N_p \qquad \forall\, p \in \mathcal{P},$$

where all $D_p \in \mathbb{C}^{n\times n}$ are diagonal matrices, all $N_p \in \mathbb{C}^{n\times n}$ are nilpotent matrices, and $\{D_p, N_p : p \in \mathcal{P}\}$ is a commuting family.

Proof.

1. For each p, there are polynomials f_p and g_p such that $f_p(A_p)$ is diagonalizable, $g_p(A_p)$ is nilpotent, and $A_p = f_p(A_p) + g_p(A_p)$.
2. The set $\{f_p(A_p), g_p(A_p) : p \in \mathcal{P}\}$ is a commuting family.
3. There is an invertible $\Gamma \in \mathbb{C}^{n\times n}$ such that all $D_p := \Gamma f_p(A_p)\Gamma^{-1}$ are invertible.
4. All $N_p := \Gamma g_p(A_p)\Gamma^{-1}$ are nilpotent, and $\{D_p, N_p : p \in \mathcal{P}\}$ is a commuting family. $\square$

A Formula for Entropy

The switched commuting linear system becomes

$$\dot{x} = (D_\sigma + N_\sigma)x, \qquad x(0) \in K,$$

where $D_p = \operatorname{diag}(a_p^1, \dots, a_p^n)$ are diagonal, N_p are nilpotent, and $\{D_p, N_p : p \in \mathcal{P}\}$ is a commuting family.

Theorem 6

The topological entropy of the switched commuting system satisfies

$$h(D_\sigma + N_\sigma) = \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \operatorname{Re}(a_p^i)\tau_p(t)$$

with the active times τ_p.

- The entropy only depends on the diagonal part, i.e., $h(D_\sigma + N_\sigma) = h(D_\sigma)$.

A Formula for Entropy

Proof.

1. The solutions satisfy

$$\|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| = \left\| e^{\sum_{p\in\mathcal{P}} N_p\tau_p(t)} e^{\sum_{p\in\mathcal{P}} D_p\tau_p(t)}(x'-x)\right\|.$$

2. Lemma 2. Consider the commuting family of nilpotent matrices $\{N_p : p \in \mathcal{P}\}$. For each $\delta > 0$, there is a constant $c_\delta > 0$ such that for all $v \in \mathbb{C}^n$,

$$c_\delta^{-1} e^{-\delta t}\|v\| \le \left\| e^{\sum_{p\in\mathcal{P}} N_p\tau_p(t)} v\right\| \le c_\delta e^{\delta t}\|v\|$$

   for all $t \ge 0$ with the active times τ_p.

3. Given a radius $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that

$$\cdots \le \|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| \le c_\varepsilon e^{\varepsilon t} \max_{i=1,\dots,n} e^{\sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t)} |x_i' - x_i|.$$

4. For the upper/lower bound, construct a (T,ε)-spanning/separated set using a grid. □

The Non-Switched Case

The formula for entropy yields the following well-known result Bowen [1971]:

Corollary 7

The topological entropy of the linear time-invariant (LTI) system

$$\dot{x} = Ax, \qquad x(0) \in K$$

equals the sum of the positive real parts of the eigenvalues of A, that is,

$$h(A) = \sum_{\lambda \in \mathrm{spec}(A)} \max\{\mathrm{Re}(\lambda), 0\}.$$

Proof.

1. The spectrum $\mathrm{spec}(A) = \{a^1, \ldots, a^n\}$.
2. The entropy

$$h(A) = \limsup_{T \to \infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t \in [0,T]} \mathrm{Re}(a^i)t = \sum_{i=1}^{n} \max\{\mathrm{Re}(a^i), 0\}. \quad \square$$

More General Upper and Lower Bounds for Entropy

#	Formula/upper bounds	Sw	CoB
(1)	$= \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t)$	τ_p	Yes
(2)	$\leq \sum_{i=1}^{n} \max\left\{ \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\rho_p(t),\ 0 \right\}$	ρ_p	Yes
(3)	$\leq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} h(D_p)\rho_p(t)$	ρ_p	No
(4)	$\leq \sum_{p\in\mathcal{P}} h(D_p)\hat{\rho}_p$	$\hat{\rho}_p$	No
(5)	$\leq \max_{p\in\mathcal{P}} h(D_p)$	N/A	No
(6)	$\leq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} n\|A_p\|\rho_p(t)$	ρ_p	No
(7)	$\geq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(A_p)\rho_p(t)$	ρ_p	No

Numerical Example

Let $\mathcal{P} = \{1,\, 2\}$ and

$$D_1 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

	$(\hat{\rho}_1, \hat{\rho}_2)$	(1)	(2)	(3)	(4)	(5)	(6)	(7)
No switch	$(1, 0)$	2	2	2	2	3	4	1
Periodic switches	$(0.5, 0.5)$	2	2	2.5	2.5	3	5	2
Switches w/ set-points	$(0.9, 0.9)$	2.8	4.4	2.9	4.5	3	5.8	2.8

Conclusion

Contributions:

- Switched linear systems with pairwise commuting matrices.
- A change of basis under which each of the matrices can be decomposed into a diagonal part and a nilpotent part, and all the diagonal and nilpotent parts are pairwise commuting.
- A formula for the topological entropy, which only depends on the diagonal part.
- More general upper bounds for the entropy.

Future research:

- Reconcile the switching characterizations for entropy computation and for stability analysis and control design
  - Stability and stabilization: slow-switching conditions such as the average dwell-time
  - Entropy: the active time (rarely seen in the literature)

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