Analysis of different Lyapunov function constructions for interconnected hybrid systems

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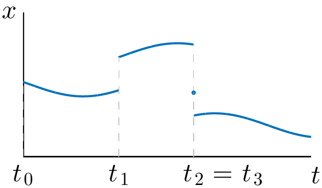
Introduction

- Hybrid systems: dynamical systems exhibiting both continuous and discrete behaviors



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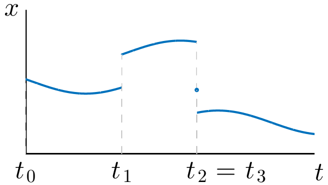

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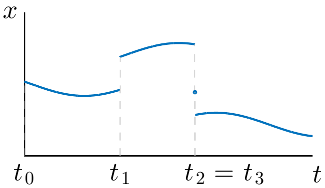

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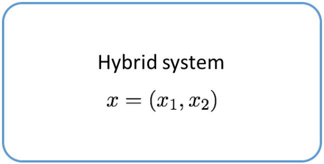

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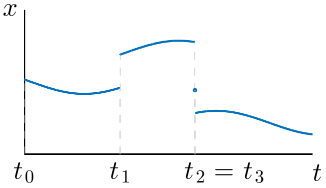

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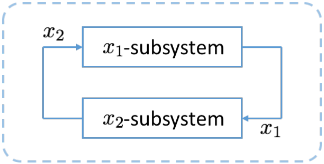

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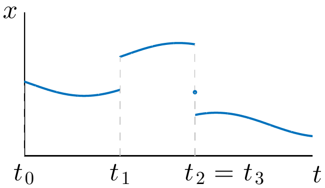

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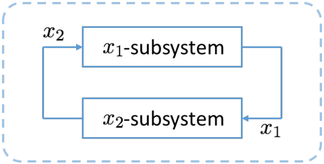

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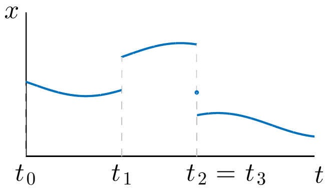

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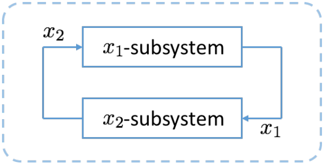

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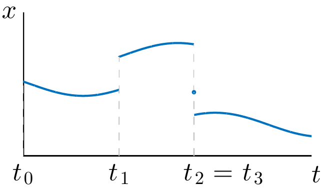

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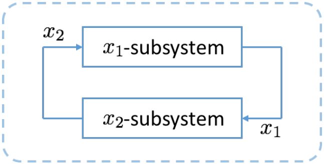

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$$\operatorname{dom} x = \bigcup_{j=0,1,\dots} [t_j, t_{j+1}] \times \{j\}$$

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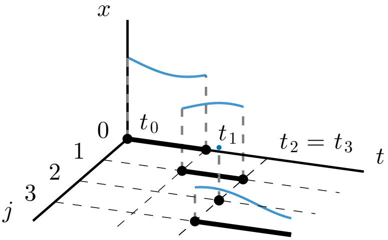

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Definition

A hybrid system is input-to-state stable (ISS) w.r.t. a set $\mathcal{A} \subset \mathcal{X}$ if there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that all solution pairs (x,u) satisfy

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma(\|u\|_{(t,j)}) \qquad \forall\,(t,j) \in \operatorname{dom} x.$$

[Son89] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, no. 4, pp. 435–443, 1989

Input-to-state stability

$$\dot{x} \in F(x,u), \qquad (x,u) \in \mathcal{C},$$
$$x^+ \in G(x,u), \qquad (x,u) \in \mathcal{D}.$$

- State $x \in \mathcal{X} \subset \mathbb{R}^n$, input $u \in \mathcal{U} \subset \mathbb{R}^m$

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- In the absence of inputs, ISS becomes global asymptotic stability (GAS)

[Son89] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, no. 4, pp. 435–443, 1989

Candidate ISS Lyapunov function

Definition 1

A locally Lipschitz function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a candidate ISS Lyapunov function w.r.t $\mathcal{A}$ if

1. $\exists$ bounds $\psi_1, \psi_2 \in \mathcal{K}_\infty$ s.t. $\psi_1(|x|_\mathcal{A}) \leq V(x) \leq \psi_2(|x|_\mathcal{A})$ for all $x \in \mathcal{X}$;

[CT09] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. & Control Lett.*, vol. 58, no. 1, pp. 47–53, 2009

[CT13] C. Cai and A. R. Teel, "Robust input-to-state stability for hybrid systems," *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 1651–1678, 2013

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It is an ISS Lyapunov function if $\phi(r) > 0$ and $\alpha(r) < r$ for all $r > 0$.

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Proposition 1 ([CT09, Prop. 2.7])

A hybrid system is ISS if it admits an ISS Lyapunov function.

[CT09] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. & Control Lett.*, vol. 58, no. 1, pp. 47–53, 2009

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Interconnection of two hybrid subsystems

- Hybrid system with state $x = (x_1, x_2)$

$$\dot{x}_1 = f_1(x), \dot{x}_2 = f_2(x), \quad x \in \mathcal{C},$$
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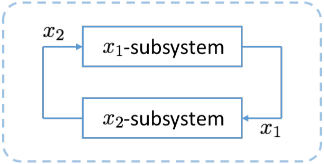

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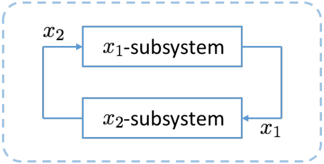

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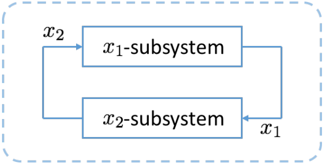

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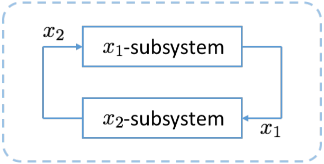

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- For all $x = (x_1, x_2) \in \mathcal{C}$,

$$V_1(x_1) \geq \gamma_1(V_2(x_2)) \quad \Rightarrow \quad \nabla_{f_1(x)} V_1(x_1) \leq -\phi_1(V_1(x_1)),$$
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with $\gamma_i(r) := \chi_i(\psi_{ij}^{-1}(r))$ for $i = 1, 2$

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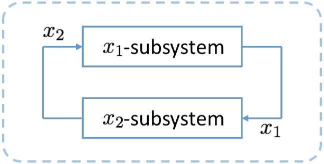

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- Small-gain condition (SG): the composition $\gamma_1 \circ \gamma_2 < \mathrm{Id}$

Small-gain theorem

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(SG1) The composition $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := \chi_i(\psi_{1j}^{-1}(r))$ for $i = 1, 2$.

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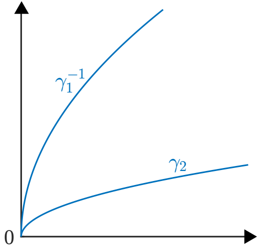
Lemma 1. ([JMW96, Lemma A.1])

Provided that

$$\gamma_1 \circ \gamma_2 < \mathrm{Id},$$



[JMW96] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996

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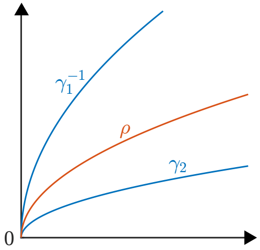
Lemma 1. ([JMW96, Lemma A.1])

Provided that

$$\gamma_1 \circ \gamma_2 < \mathrm{Id},$$

there exists a gain $\rho \in \mathcal{K}_\infty$ satisfying $\rho \in \mathcal{C}^1$ and $\rho' > 0$ on $\mathbb{R}_{>0}$ s.t.

$$\gamma_1^{-1}(r) > \rho(r) > \gamma_2(r) \qquad \forall\, r > 0.$$



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Proposition 2

Suppose Assumption 1 and (SG1) hold. Then $V(x) := \max\{\rho(V_1(x_1)), V_2(x_2)\}$ with ρ in Lemma 1 is a candidate Lyapunov function for the interconnection.

[LNT14] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based small-gain theorems for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 59, no. 6, pp. 1395–1410, 2014

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Proposition 3 ([LNT14, Th. III.1 and Cor. III.2])

Suppose Assumption 1 and (SG1) hold with ISS Lyapunov functions V_1, V_2. Then V defined in Proposition 2 is a Lyapunov function and ensures GAS.

[LNT14] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based small-gain theorems for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 59, no. 6, pp. 1395–1410, 2014

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Candidate exponential ISS Lyapunov functions

Definition 2

A candidate ISS Lyapunov function with rates ϕ, α satisfying

$$\phi(r) \equiv cr, \qquad \alpha(r) \equiv e^{-d}r$$

for some constants $c, d \in \mathbb{R}$ is a candidate exponential ISS Lyapunov function with rate coefficients c, d.
It is an exponential ISS Lyapunov function if $c, d > 0$.

Assumption 2

Each x_i-subsystem admits a candidate exponential ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ with bounds ψ_{1i}, ψ_{2i}, a gain χ_i, and rate coefficients c_i, d_i.

Non-ISS flows: reverse average dwell-time (RADT)

- Assumption 2 holds with V_1, V_2 and rate coefficients $c_1, c_2 \leq 0 < d_1, d_2$

[HLT08] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, 2008

[CTG08] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems Part II: (Pre)Asymptotically stable compact sets," *IEEE Trans. Automat. Contr.*, vol. 53, no. 3, pp. 734–748, 2008

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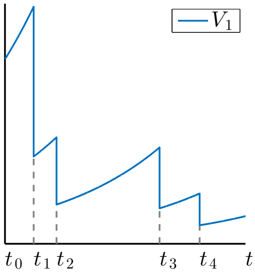

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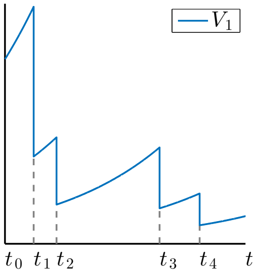

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- A solution x admits a reverse average dwell-time (RADT) $\tau_a^* > 0$ [HLT08] if

$$j - k \geq (t - s)/\tau_a^* - N_0^* \qquad \forall\, t \geq s$$

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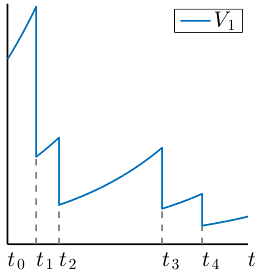

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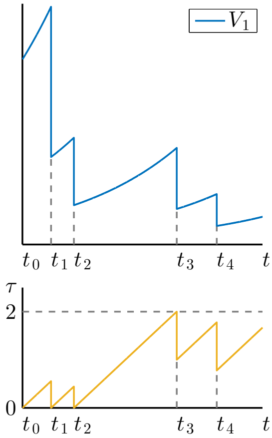
- [CTG08] Equivalently, $\operatorname{dom} x = \operatorname{dom} \tau$ for an RADT timer τ with

$$\dot{\tau} = 1/\tau_a^*, \qquad \tau \in [0, N_0^*],$$
$$\tau^+ = \max\{0, \tau - 1\}, \quad \tau \in [0, N_0^*].$$



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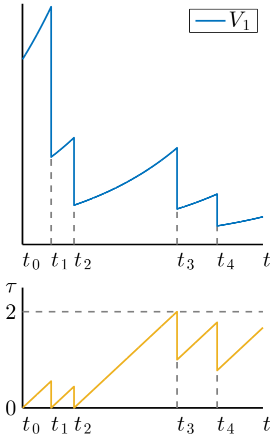
Non-ISS flows: RADT modification

- Assumption 2 holds with V_1, V_2 and rate coefficients $c_1, c_2 \leq 0 < d_1, d_2$
- Consider solutions that jump fast enough
- Consider the augmented interconnection with state (x_1, x_2, τ)



[MYL14] A. Mironchenko, G. Yang, and D. Liberzon, "Lyapunov small-gain theorems for not necessarily ISS hybrid systems," in *21st Int. Symp. Math. Theory Networks Syst.*, 2014, pp. 1001–1008

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- [MYL14, Prop. 6] Provided that $\tau_a^* < -d_i/c_i$, there exists an $L_i \in (-c_i\tau_a^*, d_i)$ s.t.

$$W_i(x_i, \tau) := e^{-L_i\tau} V_i(x_i)$$

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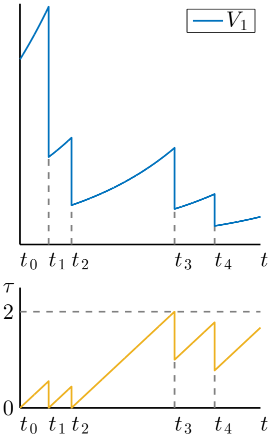

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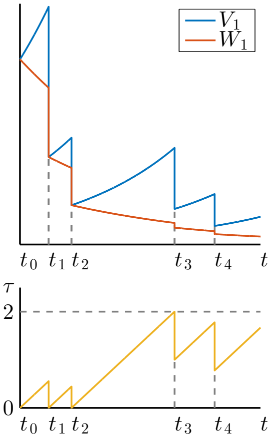

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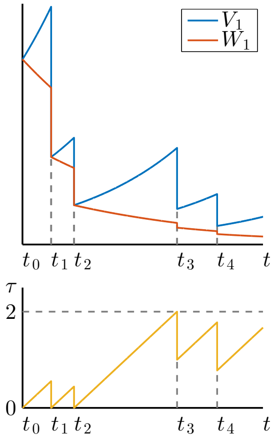

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[HM99] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *38th IEEE Conf. Decis. Control*, vol. 3, 1999, pp. 2655–2660

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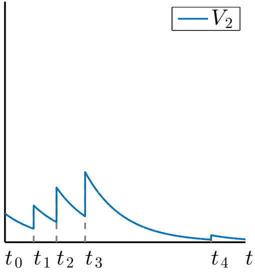

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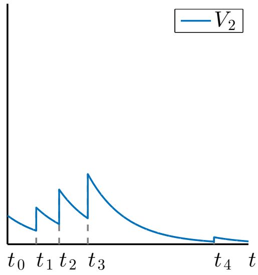

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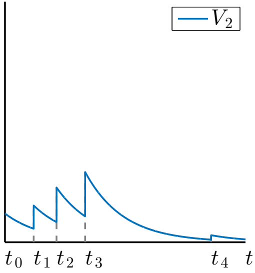

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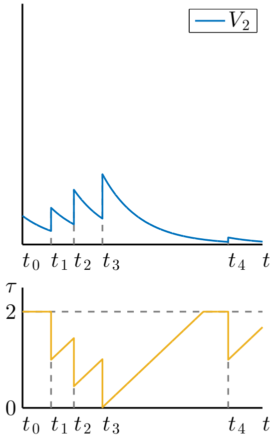
- [CTG08] Equivalently, $\operatorname{dom} x = \operatorname{dom} \tau$ for an ADT timer τ with

$$\dot{\tau} = [0, 1/\tau_a], \quad \tau \in [0, N_0],$$
$$\tau^+ = \tau - 1, \qquad \tau \in [1, N_0].$$



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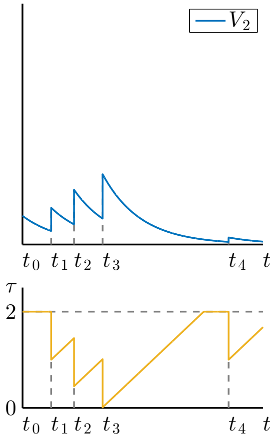

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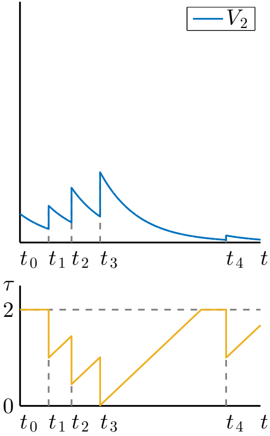

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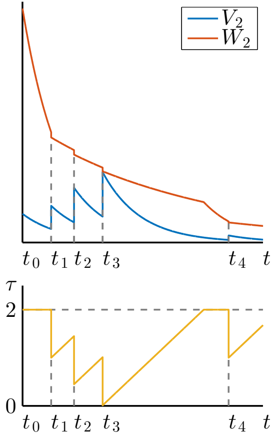

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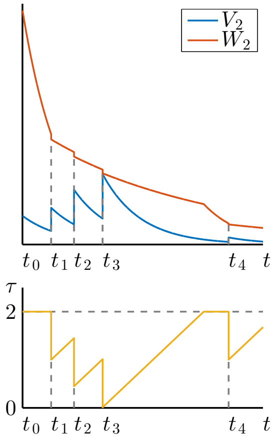

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Non-ISS jumps: an alternate construction

- Assumption 2 holds with rate coefficients $c_1, c_2 > 0 \geq d_1, d_2$
- Linear gains: $\gamma_1(r) \equiv \xi_1 r$ and $\gamma_2(r) \equiv \xi_2 r$ for some constant $\xi_1, \xi_2 > 0$
- Construct a candidate exponential Lyapunov function for the interconnection
- Establish GAS under ADT
- Advantage: it requires (SG1: $\xi_1 \xi_2 < 1$) instead of (SG3: $\xi_1 \xi_2 < e^{d_1 + d_2}$)
- Disadvantage: it requires linear gains

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Dynamics	Small-gain condition	Remark
ISS subsystems	(SG1)	
Non-ISS flows	(SG2)	Generic in (SG1)
Non-ISS jumps	(SG3)	Not generic in (SG1)
Non-ISS jumps	(SG1)	Linear gains
Non-ISS flow and jump	(SG4)	Not generic in (SG1)
Non-ISS flow and jump	(SG1)	Linear gains

Future research topics

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- Modifying non-exponential ISS Lyapunov functions