Question 1

We have the OGF for

$$\mathbb{N} = Seq(\{a\}) = z + z^2 + \cdots = \frac{z}{1-z}.$$

The OGF for

$$Seq(\mathbb{N}) = \frac{1}{1 - \frac{z}{1-z}} = \frac{1-z}{1-z-z} = \frac{1-z}{1-2z}.$$

Hence, we have

$$Seq(\mathbb{N}\backslash k) = \frac{1}{1 - (\frac{z}{1-z} - z^k)} = \frac{1-z}{(1-z) - z + z^k(1-z)} = \frac{1-z}{(1-z) - z + z^k - z^{k+1}} = \frac{1-z}{1 - 2z + z^k - z^{k+1}}.$$

Since the number of composition containing at least one k with weight n is the same as the number of composition with weight n minus the number of composition with no instance of k, we have

$$Seq(\mathbb{N}) - Seq(\mathbb{N}\backslash k) = \frac{1-z}{1-2z} - \frac{1-z}{1-2z+z^k-z^{k+1}}.$$

Question 2

Part a

By computing the first few term of the pentagonal number, we have

$$\begin{aligned} p_0 &= 3 - 2 = 1, \\ p_1 &= (3-2) + (6-2) = 5, \\ p_2 &= (3-2) + (6-2) + (9-2) = 12. \end{aligned}$$

We can draw the pentagonal number with dots and lines. Since $p_0 = 1$, we represent it to be a dot. For $p_1 = 5$, we draw a pentagon with 5 vertices and 5 edges. Now to draw the p_2 we can simply extend the two non-adjacent points in the pentagon and draw a bigger pentagon which includes the previous one with each edge consists one point. We can repeat this process to get p_3. We can repeat this process to construct all the remaining pentagonal numbers. The diagram below illustrates the above idea.

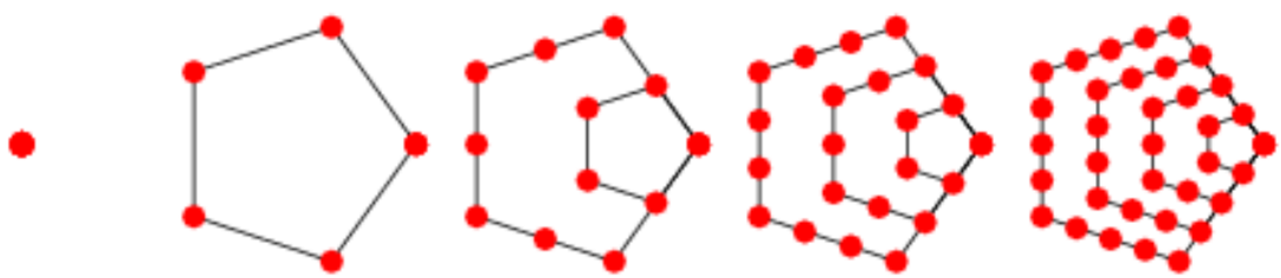


Figure 1: Pentagonal numbers

Part b

By inspecting the diagram, we have

$$p_n = 2p_{n-1} - p_{n-2} + 3, (n \geqslant 2) \tag{0.0.1}$$

where $n \geqslant 2$. We can also transfer this formula into the standard recurrence relation. Since $p_{n-1} - p_{n-2} = 3(n-1) - 2 + 3 = 3n - 2$, We have

$$p_n = 2p_{n-1} - p_{n-2} + 3 = p_{n-1} + (p_{n-1} - p_{n-2}) + 3 = p_{n-1} + 3n - 2.$$

Part c

Multiplying both side of (0.0.1) by z^n and summing over all n, we have

$$\sum_{n=2}^{\infty} p_n z^n = \sum_{n=2}^{\infty} 2p_n z^n - \sum_{n=2} (p_{n-2} - 3) z^n. \tag{0.0.2}$$

By shifting the summation index, we have

$$\text{LHS}(0.0.2) - 1 - 5z = z \sum_{n=2}^{\infty} 2p_{n-1} z^{n-1} - z^2 \sum_{n=2}^{\infty} (p_{n-2} - 3) z^{n-2} = z\Big(\sum_{n=0}^{\infty} 2p_n z^n - 2\Big) - z^2 \sum_{n=0}^{\infty} (p_n - 3) z^n. \tag{0.0.3}$$

Let $P(n) = \sum_{n=0}^{\infty} p_n z^n$. Then we have

$$P(n) - 1 - 5z = 2zP(n) - 2z - z^2 P(n) + 3z^2 \sum_{n=0}^{\infty} z^n. \tag{0.0.4}$$

By manipulating (0.0.4), we have

$$P(n)(1 - 2z + z^2) = 1 + 3z + \frac{3z^2}{1-z}.$$

Hence, we have

$$P(n) = \frac{1 + 3z + \frac{3z^2}{1-z}}{1 - 2z + z^2} = \frac{2z+1}{(1-z)^3}.$$

Recall the Taylor series

$$\frac{1}{(1-z)^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} z^{n-2}.$$

Hence, we have

$$P(n) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} (2z+1) z^{n-2}.$$

Rearranging the equation, we have

$$P(n) = \sum_{n=0}^{\infty} (n+2)(n+1) z^{n+1} + \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n = 1 \cdot z^0 + \sum_{n=1}^{\infty} \frac{(n+1)(3n+2)}{2} z^n. \tag{0.0.5}$$

Hence, we have

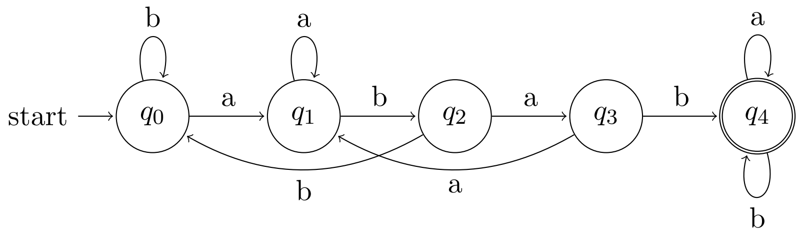
$$\text{RHS}(0.0.5) = \sum_{n=0}^{\infty} \frac{(n+1)(3n+2)}{2} z^n = \sum_{n=1}^{\infty} \frac{(n)(3n-1)}{2} z^{n-1}.$$

Hence, if n starts from 1. Then we have

$$p(n) = \sum_{k=1}^{n} (3k-2) = \frac{n(3n-1)}{2}.$$

Question 3

Part a



The staring state is q_0 and the final state is q_4. A word containing the block pattern *abab* must also pass the above state machine. Hence, the above machine checks wheather the word contains this parttern or not.

Part b

Now we can get

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Recall the Chomsky-Schutzenberger Thenorem, we have

$$W_G(z) = u(I - z \cdot T)^{-1} \cdot v,$$

where u is a row vector of length 5 with the first coordinate being 1 and others being 0 and v is a column vector with length 5 and the last coordinate being 1 and all the other being 0 because our final state only contains q_4. By wolfram, we get

$$W_G(z) = \frac{z^4}{(1-2z)(z^4 - 2z^3 + z^2 - 2z + 1)}.$$

Question 4

Part a

$P(A)$ is the set contains multiple instances of elements selected from the class A. $Q(B)$ is the set of all subsets of B.

Part b

First, we recall from the solution sheet 2 question 4 of the practical class that for any set $\mathcal{B}$ and we can express the ogf of the Mset in terms of Pset as follows

$$M(z) = \prod_{j=0}^{\infty} P(z^{2^j}).$$

Since we are taking $\mathcal{B}$ from subset of naturals, we realize that both counting sequence of $\mathcal{A}$ and $\mathcal{B}$ which we are constructing has the following form.

$$A_n = \begin{cases} 1 & \text{if} \quad n \in \mathcal{A}, \\ 0 & else \end{cases}$$

Similarly, we have

$$B_n = \begin{cases} 1 & \text{if} \quad n \in \mathcal{B}, \\ 0 & else \end{cases}$$

Now if there is only one element in B namely i. Then use the above formula, we get

$$\frac{1}{1-z^i} = \prod_{j=0}^{\infty}(1+z^{2^j i}) = \text{Pset}(A),$$

where A consists $2^j i$ for all $j \geqslant 0$. Now we consider any set B. We have

$$\text{Mset}(B) = \prod_{i \in B} \frac{1}{1-z^i} = \text{Pset}(A),$$

where A consists of all the the elements in B and also the elements $2^j i$ for all $j \geqslant 0$ and i an element of B. Since A is a set, there is no repeated element in A. Hence, for any $i \in \mathcal{B}$ we can not have $2^k i$ for $k \in \mathbb{Z}$ in there. Hence, now we have find all the necessary and sufficient condition.

Question 5

To the get word with the left most instance of a_i must happen to the left of the left most instance of a_{i+1}. We should focus on each of the a_is. Hence, we have the following

$$\text{Seq}_{k \geqslant 1}(\{a_1\})\,\text{Seq}_1(\{a_2\})\,\text{Seq}(\{a_1, a_2\}) \ldots \text{Seq}_1(a_{k-1})\,\text{Seq}(\{a_1, \ldots, a_{k-1}\})\,\text{Seq}_1(\{a_k\})\,\text{Seq}(\{a_1 \ldots a_k\}).$$

We denote $\text{Seq}_1(\{a_i\})$ to be sequence of length 1 containing 1 element namely a_i. This ogf certainly controls every left most a_i to be on the left of the left most a_{i+1}. Therefore, we have the ogf is equal to

$$\frac{z^k}{(1-z)(1-2z)\ldots(1-kz)}.$$

To work out this, we will use the partial fraction on

$$\frac{1}{(1-z)(1-2z)\ldots(1-kz)} = \sum_{j=1}^{k} \frac{a_j}{1-jz}.$$

Multiplying both side of the equation with $(1-rz)$ and substituting in $z = \frac{1}{r}$, we get

$$\begin{aligned} a_r &= \frac{1}{(1-\frac{1}{r})(1-\frac{2}{r})\ldots(1-\frac{(r-1)}{r})(1-\frac{(r+1)}{r})\ldots(1-\frac{k}{r})} \\ &= \frac{(-1)^{k-r} r^{k-1}}{(r-1)!(k-r)!} \\ &= \frac{(-1)^{k-r} r^k}{r!(k-r)!}. \end{aligned}$$

Now we will extract the coefficient of z^n from the equation

$$\begin{aligned} [z^n]\frac{z^k}{(1-z)(1-2z)\ldots(1-kz)} &= [z^n]\sum_{j=1}^{k} \frac{a_j}{1-jz} z^k \\ &= [z^{n-k}]\sum_{j=1}^{k} \frac{a_j}{1-jz} \\ &= \sum_{j=1}^{k} a_j [z^n] \frac{1}{1-jz} \end{aligned}$$

Since $\frac{1}{1-jz}$ can be expressed as a geometric series, we have

$$\sum_{j=1}^{k} a_j j^{n-k}.$$

Now we substitute the value of a_j, we have

$$\sum_{j=1}^{k} \frac{(-1)^{k-j} j^{k-1}}{(j-1)!(k-j)!} j^{n-k} = \sum_{j=1}^{k} \frac{j^n (-1)^{k-j}}{j!(k-j)!} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

Hence, this counts the number of elements required with weight n.

Question 6

We start by considering the cycle construction. Since this is the only thing which contains the totient function. Let $\mathcal{B} = Z$ be the atomic class. Hence we have $B(z) = z$ Then we have

$$\mathrm{Cyc}(\mathcal{B}) = \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log(\frac{1}{1-z^k}). \tag{0.0.6}$$

Using the Taylor series expansion of $\log(\frac{1}{1-z^k})$, we obtain

$$\mathrm{RHS}(0.0.6) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi(k)}{k} \frac{z^{nk}}{n} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi(k)}{nk} z^{nk}. \tag{0.0.7}$$

Now let $m = nk$. Now we basically looking at two ways of multiplying two numbers. Let m to be fixed. Then for any two integers n, k which $nk = m$, we must have m and k being a divisor of m. Hence summing over all the n and k is the same as summing over m and all it's divisors. Hence, we have the following result Hence, we have

$$\mathrm{RHS}(0.0.7) = \sum_{m=1}^{\infty} \sum_{k|m} \frac{\varphi(k)}{m} z^m.$$

Now since the $\mathrm{Cyc}(\{a\}) = \mathrm{Seq}(\{a\}) - \epsilon$ where ϵ is the neutral element. Therefore, we have

$$\sum_{m=1} z^m = \sum_{m=1}^{\infty} \sum_{k|m} \frac{\varphi(k)}{m} z^m.$$

Extracting the coefficient of z^m from both side, we obtain

$$\sum_{k|m} \varphi(k) = m.$$

Question 7

We consider the construction of a self-dual partition in two different ways. Firstly, we consider it as a power set of odd numbers. We can look at the self-dual partition as several layer of points i.e., the first layer(blue), the second layer(green) and the third layer(red) at the diagram below. Since the partition is self-dual, the points in each layer sum to an odd number. We also notice the number of points in each of the layer has to decrease otherwise we would not have a partition. Hence, we have the ogf of a powerset of odd numbers which is

$$\mathrm{Pset}(odds) = \mathrm{Pset}(z \times \mathrm{Seq}(\{z^2\})) = \prod_{k=1}^{\infty} (1 + z^{2k-1}).$$

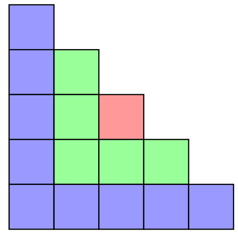


Figure 2: Pset representation

Now we consider the other construction of the self–dual partitions. Consider the partition being break into three parts, the first part(red) is a square with length k, the second part(blue) is a partition with number of parts at most k and the third part(blue) should be the reflection of the first part. Now there are two ways we can think about this. First, the number of ways of constructing the self–dual partition is the same as the number of ways constructing a square with length k and a partition with parts at most k. Therefore, we only require to construct a square with length k and a partition with parts at most k and then double the number of boxes of the partition to get the required self–dual partition. Second, we can construct both blue part at the same time as a Cartesian product. Hence, we get the following ogf

$$\bigoplus_{k\geqslant 0} Z^{k^2} \times \mathrm{Mset}_{\leqslant k}(\mathrm{Seq}(\{z^2\})) = 1 + \sum_{k=1}^{\infty} \frac{z^{k^2}}{(1-z^2)(1-z^4)\ldots(1-z^{2k})}$$

as required.

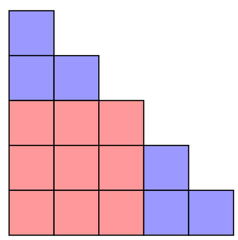


Figure 3: Mset representation