Quantum group and Crystal bases

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Gradation

Let M be an abelian group and let V be a vector space. A M-gradation is a vector space decomposition

$$V = \bigoplus_{\alpha \in M} V_\alpha,$$

where V_α are subspaces of V. Elements of V_α are called **homogeneous** degree of α. A subspace $U \subseteq V$ is called graded if

$$U = \bigoplus_{\alpha \in M} (U \cap V_\alpha).$$

The following proposition tells us that the gradation gets preserved from a module to it's submodules

Proposition 0.1. *Let $\mathfrak{h}$ be a commmutative Lie algebra, V a diagonalizable $\mathfrak{h}$-module, i.e.,*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \text{ where } v_\lambda = \{v \in V \mid h(v) = \lambda(h)v \quad \text{for all } h \in \mathfrak{h}.$$

Then any submodule U of V is graded with gradation given by

$$U = \bigoplus_{\lambda \in \mathfrak{h}^*} (V_\lambda \cap U).$$

Is formal topology completion important (?????????????) for what the purpose I am doing(??????????) Why is it in this spot of Kacs? Now for the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is also a vector space, there is also an M-gradtion defined on the Lie algebra $\mathfrak{g}$.

Definition. Let M be an abelian group. An M-gradation of a Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} = \bigoplus_{\alpha \in M} \mathfrak{g}_\alpha,$$

such that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

Now we can look at a classical example of the $\mathfrak{g}$ in terms of a Q-gradation which is the root lattice.

Example 0.1. *Let*

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}$$
$$\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$$

be the root basis and the coroot basis. Elements of Π and $\Pi^\vee$ are called the simple roots and the simple coroots respectively.

The Example $A_2^{(2)}$

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and let

$$\sigma : \mathfrak{g} \to \mathfrak{g} \quad \text{with} \quad \sigma^m = 1$$

for some positive integer m. In some sense

$$x^m = 1, \quad \text{is the minimal polynomial of } \sigma.$$

So the eigenvalues are roots of unities

$$\varepsilon = \exp\left(\frac{2\pi i}{m}\right) \quad \text{and} \quad \varepsilon^j \text{ is an eigenvalue of } \sigma \text{ for each } j \in \mathbb{Z}/m\mathbb{Z}.$$

Now if σ is an $m \times m$ matrix and because each eigenvalues are distinct, we have σ is diagonalizable. Hence, we have the following eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j,$$

where

$$\mathfrak{g}_j \quad \text{is the eigenspace of } \sigma \text{ for the eigenvalue } \varepsilon^j.$$

Let

$$\mathfrak{g} = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C}E'_\alpha\right)$$

be a simple finite-dimensional Lie algebra of type X_N, with the normalized invariant form $(. \mid .)$. Let

$$\overline{\mu} \quad \text{be an automorphism of Dynkin diagram of } \mathfrak{g} \text{ of order } r.$$

Let

$$\mu \text{ be the corresponding diagram automorphism of } \mathfrak{g}.$$

If the order of the automorphism equal to 2, then

$$\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}, \quad \text{and} \quad \mathfrak{h}' = \mathfrak{h}_{\overline{0}} \oplus \mathfrak{h}_{\overline{1}},$$

where $\overline{s} \in \mathbb{Z}/2\mathbb{Z}$ is the residue of s in mod 2. Let

$$\Pi' = \{\alpha'_1, \ldots, \alpha'_N\} \in \mathfrak{h} \quad \text{be the set of simple roots of } \mathfrak{g},$$

where

$$\alpha' = \sum_{i \in R} (k_i - m_i)\alpha_i, \quad \text{with} \quad \alpha = \sum_{i \in R} k_i \alpha_i$$

and

$$\beta = \sum_i m_i \alpha_i \quad \text{is the highest root in } \Delta.$$

To compute the root system of A_2^2, we look at the root system of $A_2 = \mathfrak{sl}_3$. Recall the root system for $\mathfrak{sl}_n$, let $\mathbb{R}^n$ be the n-dimensional real Euclidean space with standard basis $e_1, \ldots, e_n$ and the bilinear form

$$(e_i \mid e_j) = \delta_{ij}.$$

Define

$$\begin{aligned}
Q = Q^\vee &= \left\{ \sum_i k_i v_i \in \mathbb{R}^{n+1} \mid k_i \in \mathbb{Z}, \sum_i k_i = 0 \right\} \\
\Delta &= \{e_i - e_j \mid i \neq j\}, \\
\Pi &= \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}, \quad \text{where} \quad \alpha_i = e_i - e_{i+1}, \\
\theta &= e_1 - e_{\ell+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell, \\
W &= \{\text{all permutations of the } e_i\} = S_{n+1}.
\end{aligned}$$

Example 0.2. *For $\ell = 2$, we have $n = \ell + 1 = 3$. So the longest root is*

$$\theta = \alpha_1 + \alpha_2 + \alpha_3$$

and the set

$$\Pi' = \{\alpha_1', \alpha_2', \alpha_3'\},$$

where

$$\begin{aligned}\alpha_1' &= -\alpha_2 - \alpha_3 \\ \alpha_2' &= -\alpha_1 - \alpha_3 \\ \alpha_3' &= -\alpha_1 - \alpha_2\end{aligned}$$

Now recall the elements for the twisted Kac-Moody algebra A_2^2 with the order of the automorphism being 2 and

$$\begin{aligned}&\theta^0 = \alpha_1' + \cdots + \alpha_{2\ell}' \\ &H_i = H_i' + H_{2\ell-i+1}', \quad \text{for } i \in \{1, \ldots, \ell-1\}, \quad H_0 = 2(H_\ell' + H_{\ell+1}'), \quad H_\ell = -\theta^0 \\ &E_i = E_i' + E_{2\ell-i+1}' \quad \text{for } i \in \{1, \ldots, \ell-1\}, \quad E_0 = \sqrt{2}(E_\ell' + E_{\ell+1}')m, \quad E_\ell = E_{-\theta^0}' \\ &F_i = F_i' + F_{2\ell-i+1}' \quad \text{for } i \in \{1, \ldots, \ell-1\}, \quad F_0 = \sqrt{2}(F_\ell' + F_{\ell+1}'), \quad F_\ell = -E_{\theta^0}'.\end{aligned}$$

where

$$\varepsilon = \ell \quad \text{and} \quad I = \{0, 1, \ldots, \ell\} \backslash \{\varepsilon\}.$$

For the $A_2^{(2)}$ model, we will have

$$\varepsilon = 1, \quad \text{and} \quad I = \{0, 1\} \backslash \{1\} = \{0\}.$$

Let I be a finite index set. A square matrix $A = (a_{ij})_{i,j \in I}$ with $a_{ij} \in \mathbb{Z}$ is called a **generalized Cartan matrix** if it satisfies

$$\begin{aligned}a_{ii} &= 2 \quad \text{for all } i \in I, \\ a_{ij} &\leq 0 \quad \text{if } i \neq j, \\ a_{ij} &= 0 \quad \text{if and only if} \quad a_{ij} = 0.\end{aligned}$$

Example 0.3. *For the purpose of this writing, we are only focusing on one particular generalzied Cartan matrix which is for the twisted affine Kac-Moody algebra $A_2^{(2)}$, we have*

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

and define

$$a_0 = 2, \quad \textit{and} \quad a_1 = 1.$$

A matrix A is symmetrizable, if there exists a matrix $D = (s_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that

$$(DA)^T = DA.$$

Using the Chevalley presentation of $U_q(\mathfrak{g})$ taking from Michael and Paul, Weston and Ikhlef's paper for rank 1 affine Lie algebra $\mathfrak{g}$, we have the generators are

$$E_0, E_1, F_0, F_1, T_0, T_1,$$

satisfying the relations

$$\begin{aligned}
T_iT_i^{-1} = T_i^{-1}T_i = 1, &\quad T_iT_j = T_jT_i \quad \text{for } i \in \{0,1\} \\
T_iE_jT_i^{-1} = q^{s_ia_{ij}}E_j, &\quad T_iF_jT_i^{-1} = q^{-d_ia_{ij}}F_j \\
E_iF_j - F_jE_i &= \delta_{ij}\frac{T_i - T_i^{-1}}{q^{d_i} - q^{-d_i}} \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} & (E_i)^{1-a_{ij}-k}E_j(E_i)^k = 0, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} & (F_i)^{1-a_{ij}-k}F_j(F_i)^k = 0.
\end{aligned}$$

Example 0.4. *Let*

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

where $d_0 = 1$ and $d_1 = 4$. Then

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad DA = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$$

which is a symmetric matrix. Now the quantum group $U_q(A_2^{(2)})$ is generated by

$$E_0, E_1, F_0, F_1, T_0, T_1$$

satisfying the relations

$$\begin{aligned}
&T_iT_i^{-1} = T_i^{-1}T_i = 1, \quad T_iT_j = T_jT_i \quad \text{for } i \in \{0,1\} \\
&T_0E_0T_0^{-1} = q^2E_0, \quad T_0E_1T_0^{-1} = q^{-4}E_1, \quad T_1E_0T_1^{-1} = q^{-4}E_0, \quad T_1E_1T_1^{-1} = q^8E_1, \\
&T_0F_0T_0^{-1} = q^{-2}F_0, \quad T_0F_1T_0^{-1} = q^4F_1, \quad T_1F_0T_1^{-1} = q^4F_0, \quad T_1F_1T_1^{-1} = q^{-8}F_1, \\
&E_0F_0 - F_0E_0 = \frac{T_0 - T_0^{-1}}{q - q^{-1}}, \quad E_0F_1 - F_1E_0 = 0, \quad E_1F_0 - F_0E_1 = 0, \quad E_1F_1 - F_1E_1 = \frac{T_1 - T_1^{-1}}{q^4 - q^{-4}} \\
&E_0^{-4}E_1 - \frac{q^5 - q^{-5}}{q - q^{-1}}E_0^{-5}E_1E_0 + \frac{(q^5 - q^{-5})(q^4 - q^{-4})}{(q^2 - q^{-2})(q - q^{-1})}E_0^{-6}E_1E_0^2 - \frac{(q^5 - q^{-5})(q^4 - q^{-4})}{(q^2 - q^{-2})(q - q^{-1})}E_0^{-7}E_1E_0^3 \\
&\qquad + \frac{q^5 - q^{-5}}{q - q^{-1}}E_0^{-8}E_1E_0^4 - E_0^{-9}E_1E_0^5 = 0 \\
&E_1^2E_0 - \frac{q^8 - q^{-8}}{q^4 - q^{-4}}E_1E_0E_1 + E_1E_0^2 = 0 \\
&F_0^{-4}F_1 - \frac{q^5 - q^{-5}}{q - q^{-1}}F_0^{-5}F_1F_0 + \frac{(q^5 - q^{-5})(q^4 - q^{-4})}{(q^2 - q^{-2})(q - q^{-1})}F_0^{-6}F_1F_0^2 - \frac{(q^5 - q^{-5})(q^4 - q^{-4})}{(q^2 - q^{-2})(q - q^{-1})}F_0^{-7}F_1F_0^3 \\
&\qquad + \frac{q^5 - q^{-5}}{q - q^{-1}}F_0^{-8}F_1F_0^4 - F_0^{-9}F_1F_0^5 = 0 \\
&F_1^2F_0 - \frac{q^8 - q^{-8}}{q^4 - q^{-4}}F_1F_0F_1 + F_1F_0^2 = 0
\end{aligned}$$

Recall from Michael and Paul's paper. The coproduct of these generators is given by

$$\Delta(E_i) = E_i \otimes \mathrm{Id} + T_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes T_i^{-1} + \mathrm{Id} \otimes F_i, \quad \Delta(T_i) = T_i \otimes T_i.$$

We do the following computation

$$\sum_{k=0}^{5}(-1)^k \begin{bmatrix}5\\k\end{bmatrix}_q (E_0)^{1-5-k}E_1E_0^k$$
$$= \frac{[5]!}{[0]![5]!}E_0^4E_1 - \frac{[5]!}{[4]![1]!}E_0^{-5}E_1E_0 + \frac{[5]!}{[3]![2]!}E_0^{-6}E_1E_0^2 - \frac{[5]!}{[2]![3]!}(E_0)^{-7}E_1E_0^3$$
$$+ \frac{[5]!}{[1]![4]!}E_0^{-8}E_1E_0^4 - \frac{[5]!}{[5]![0]!}(E_0)^{-9}E_1E_0^5$$
$$= E_0^{-4}E_1 - \frac{q^5-q^{-5}}{q-q^{-1}}E_0^{-5}E_1E_0 + \frac{(q^5-q^{-5})(q^4-q^{-4})}{(q^2-q^{-2})(q-q^{-1})}E_0^{-6}E_1E_0^2 - \frac{(q^5-q^{-5})(q^4-q^{-4})}{(q^2-q^{-2})(q-q^{-1})}E_0^{-7}E_1E_0^3$$
$$+ \frac{q^5-q^{-5}}{q-q^{-1}}E_0^{-8}E_1E_0^4 - E_0^{-9}E_1E_0^5$$

and

$$\begin{aligned}\sum_{k=0}^{2}(-1)^k \begin{bmatrix}2\\k\end{bmatrix}_{q^4} E_1^{2-k}E_0E_1^k &= \begin{bmatrix}2\\0\end{bmatrix}_{q^4} E_1^2E_0 - \begin{bmatrix}2\\1\end{bmatrix}_{q^4} E_1E_0E_1 + \begin{bmatrix}2\\2\end{bmatrix}_{q^4} E_1E_0^2\\ &= \frac{[2]!}{[2]!}E_1^2E_0 - \frac{[2]!}{[1]![1]!}E_1E_0E_1 + E_1E_0^2\\ &= E_1^2E_0 - \frac{q^8-q^{-8}}{q^4-q^{-4}}E_1E_0E_1 + E_1E_0^2\end{aligned}$$

We need one more information, how does the counit defined for $A_2^{(2)}$???? Similarly, the computation for F can be down exactly the same as way as above but with E replaced with F. Now the root system of $A_2^{(2)}$ is given by

$$\begin{aligned}\Phi^{re} &= \{(2n\pm 1)\alpha_0 + n\alpha_1, 4n\alpha_0 + (2n\pm 1)\alpha_1 \mid n \in \mathbb{Z}\}\\ \Phi^{im} &= \{2n\alpha_0 + n\alpha_1 \mid n \in \mathbb{Z}, n \neq 0\},\end{aligned}$$

where $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Phi$. This is the same in Kac's book exercises 6.6. Now the denominator identity for any given root system is given by

$$\prod_{\alpha\in\Phi_+}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{w\in W}(-1)^{\ell(w)}e^{w\rho-\rho}.$$

From this we should be able to derive the quintuple product identity for $A_2^{(2)}$ which is

$$\prod_{n=1}^{\infty}(1-p^{2n}q^n)(1-p^{2n-1}q^{n-1})(1-p^{2n-1}q^n)(1-p^{4n-4}q^{2n-1})(1-p^{4n}q^{2n-1})$$
$$= \sum_{k\in\mathbb{Z}}\left(p^{3k^2-2k}q^{\frac{3k^2+k}{2}} - p^{3k^2-4k+1}q^{\frac{3k^2-k}{2}}\right),$$

Proof. THIS SHOULD BE DOABLE BECAUSE IT IS AN EXCERSIE IN KANG AND HONG. □

The F-matrix

It is now the time to discuss a bit about the F-matrix. From Maillet, Santos's paper,

Definition. Let $\mathcal{A}$ be a qusi-triangular Hopf algebra. Then a twist is an element

$$F = \sum_i a_i \otimes b_i \in \mathcal{A}$$

satisfying the following conditions:

1. $(\varepsilon \otimes \mathrm{Id})F = (\mathrm{Id} \otimes \varepsilon)F = \mathbf{1}$
2. $(F \otimes \mathbf{1})(\Delta \otimes \mathrm{Id})F = (\mathbf{1} \otimes F)(\mathrm{Id} \otimes \Delta)F.$

Kashiwara operator

Let $n \in \mathbb{Z}$ and let x be an indeterminate. Define

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}$$

with

$$[0]_x! = 1 \quad \text{and} \quad [n]_x! = [n]_x[n-1]_x \cdots [1]_x \text{ for } n \in \mathbb{Z}_{>0}.$$

Example 0.5. *Let $n = 5$, and let $x = q$. Then*

$$[5]_q = \frac{q^5 - q^{-5}}{q - q^{-1}}.$$

Moreover,

$$[5]_q! = [5]_q[4]_q[3]_q[2]_q[1]_q = \frac{q^5 - q^{-5}}{q - q^{-1}} \frac{q^4 - q^{-4}}{q - q^{-1}} \frac{q^3 - q^{-3}}{q - q^{-1}} \frac{q^2 - q^{-2}}{q - q^{-1}}.$$

Let $m \geq n \geq 0$ be two non-negative integers. Define the x-factorial binomial coefficients to be

$$\begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x!}{[n]_x![m-n]_x!}.$$

Fix $q \in \mathbb{C}$. Then the q-**integers** and q-**binomial coefficients** are

$$[n]_q \quad \text{and} \quad \begin{bmatrix} m \\ n \end{bmatrix}_q$$

It can be show that these q-integers and q-binomial coefficients are elements of $\mathbb{Z}[q, q^{-1}]$.

Lemma 0.2. *Let $q \in \mathbb{C}$. Then*

$$[n]_q \quad \textit{and} \quad \begin{bmatrix} m \\ n \end{bmatrix}_q \textit{ are in } \mathbb{Z}[q, q^{-1}].$$

Note that the q-integers and q-binomials are just generalization of the integers and the binomial coefficients

$$[n]_q \to n \quad \text{and} \quad \begin{bmatrix} m \\ n \end{bmatrix}_q \to \binom{m}{n} \quad \text{as } q \to 1.$$

Let $\lambda \in P$. Then

$$D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}.$$

Let L be a Lie algebra and V be an L-module. Then $x \in L$ acts **locally nilpotently** on V

$$\text{if for any } v \in V \text{ there exists } N \in \mathbb{Z}_{>0} \text{ such that } x^N \cdot v = 0.$$

A weight module V over a Kac-Moody algebra $\mathfrak{g}$ is **called** integrable

$$\text{if all } e_i \text{ and } f_i \text{ are locally nilpotent on } V.$$

The category $\mathcal{O}^q_{int}$ consists of $U_q(\mathfrak{g})$-modules V^q satisfies the following conditions:

- The module V^q has a weight space decomposition

  $$V^q = \bigoplus_{\lambda \in P} V^q,$$

  where

  $$V^q_\lambda = \{v \in V^q \mid q^h v = q^{\lambda(h)} v \quad \text{for all } h \in P^\vee\}$$

  and $\dim(V^q_\lambda) < \infty$ for all $\lambda \in P$,
- there exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

  $$\mathrm{wt}(V^q) \subseteq D(\lambda_1) \cup \cdots \cup D(\lambda_s),$$
- all e_i and f_i for $i \in I$ are locally nilpotent on V^q.

Quantum group

Let $U(L)$ be a Hopf algebra. Define

1. Δ to be the coproduct,
2. ε to be the counit,
3. S to be the antipode.

Assume that both antipode S and skew antipode S^{-1} exists. Let $a \in U(L)$ and define

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$$
$$\Delta^{op}(a) = \sum_a a_{(2)} \otimes a_{(1)}$$

If V and W are $L(U)$ modules, then there is a natural action of $L(U)$ on the tensor product $V \otimes W$ by

$$a(v \otimes w) = \Delta(a)(v \otimes w) = \sum_a a_{(1)}v \otimes a_{(2)}w,$$

where $a \in L(U)$, $v \in V$ and $w \in W$. Hence, if V is a $L(U)$ module and W is also an $L(U)$ module, then $V \otimes W$ is also an $L(U)$ module via the action defined above.

Definition. A **quasitriangular** Hopf algebra is a pair $(U, \mathcal{R})$ consisting of a Hopf algebra $U(L)$ and an invertible element $\mathcal{R} \in U(L) \otimes U(L)$(Note this is the universal R matrix) such that

$$\begin{aligned} \mathcal{R}\Delta(a)\mathcal{R}^{-1} &= \Delta^{op}(a), \quad \text{for all} \quad a \in U \\ (\Delta \otimes \mathrm{Id})(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23}, \\ (\mathrm{Id} \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12}, \end{aligned}$$

where, if $\mathcal{R} = \sum_i a_i \otimes b_i$, then

$$\mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad \mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad \mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i.$$

Let $(U(L), \mathcal{R})$ be a quasitriangular Hopf algebra, let $\mathcal{R} = \sum_i a_i \otimes b_i \in U(L) \otimes U(L)$ and let $\mathcal{R}_{21} = \sum_i b_i \otimes a_i$, and define

$$u = \sum_i S(b_i)a_i \in U(L) \quad \text{and} \quad z = uS(u).$$

Next we going to list the following facts:

$$\begin{aligned} (S \otimes \mathrm{Id})(\mathcal{R}) &= \mathcal{R}^{-1}, \\ (S \otimes S)(\mathcal{R}) &= \mathcal{R}, \\ u^{-1} &= \sum_j S^{-1}(d_j)c_j, \quad \text{where } \mathcal{R}^{-1} = \sum_j c_j \otimes d_j, \\ uau^{-1} &= S^2(a), \quad \text{for all } a \in U(L), \\ \Delta(u) &= (\mathcal{R}_{21}\mathcal{R})^{-1}(u \otimes u), \end{aligned}$$

z is an invertible central element of $U(L)$,

$\Delta(a) = (\mathcal{R}_{21}\mathcal{R})^{-2}(z \otimes z)$

A **ribbon Hopf algebra** is a triple $(L(U), \mathcal{R}, v)$ consisting of a quasitriangular Hopf algebra $(L(U), \mathcal{R})$, and an invertible element v in the center of $U(L)$, such that

$$v^2 = uS(u), \quad S(v) = v, \quad \varepsilon(v) = 1$$
$$\Delta(v) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(v \otimes v).$$

Let $\mathbb{C}[[h]]$ be the ring of formal power series with indeterminate h. Define

$$e^x = \sum_{k\geq 0} \frac{x^k}{k!},$$

and define $q = e^{\frac{h}{2}}$. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Let

$$\alpha_i \in \mathfrak{h}^* \quad \text{and} \quad H_i = \alpha_i^\vee \in \mathfrak{h}$$

be the simple roots and simple coroots respectively so that

$$(\langle \alpha_i, \alpha_j^\vee \rangle) = (a_{ij}) = A.$$

Let $U_h(L)$ be the associative algebra with 1 over $\mathbb{C}[[h]]$ generated by $\mathfrak{h}$ and the elements

$$X_1, \dots, X_r, Y_1, \dots, Y_r$$

with relations

$$\begin{aligned}
[a_1, a_2] &= 0, \quad \text{for all } a_1, a_2 \in \mathfrak{h}, \\
[a, X_j] = \langle \alpha_j, a\rangle X_j, \quad [a, Y_j] &= \langle -\alpha_j, a\rangle Y_j, \quad \text{for all } a \in \mathfrak{h}, \\
X_i Y_j - Y_j X_i &= \delta_{ij} \frac{e^{(\frac{h}{2})H_i} - e^{-\frac{h}{2}H_i}}{h}, \\
\sum_{s+t=1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix} X_i^s X_j X_i^t &= 0, \quad i \neq j, \\
\sum_{s+t=1-a_{ji}} (-1)^t \begin{bmatrix} 1-a_{ji} \\ s \end{bmatrix} Y_i^s Y_j Y_i^t &= 0, \quad i \neq j.
\end{aligned}$$

The Hopf algebra structure on $L(U)$ is given by

$$\begin{aligned}
\Delta(X_i) &= X_i \otimes e^{\frac{h}{4}H_i} + e^{-(\frac{h}{4})H_i} \otimes X_i, \\
\Delta(Y_i) &= Y_i \otimes e^{\frac{h}{4}H_i} + e^{-(\frac{h}{4})H_i} \otimes Y_i, \\
\varepsilon(X_i) &= \varepsilon(Y_i) = \varepsilon(a) = 0, \qquad \text{for all } a \in \mathfrak{h} \\
S(X_i) &= -e^{\frac{h}{2}} X_i, \quad S(Y_i) = -e^{-\frac{h}{2}} Y_i, \quad S(a) = -a, \qquad \text{for all } a \in \mathfrak{h}.
\end{aligned}$$

The **rational form of the Drinfel'd-Jimbo quantum group** $U_q\mathfrak{g}$ corresponding to $\mathfrak{g}$ is the algebra generated by

$$F_1, F_2, \dots, F_r, \quad K_1, K_2, \dots, K_r, \quad K_1^{-1}, K_2^{-1}, \dots K_r^{-1}, E_1, E_2, \dots, E_r,$$

with relations

$$\begin{aligned}
K_i K_j &= K_j K_i, \qquad \text{for } 1 \leq i, j \leq r, \\
K_i K_i^{-1} &= K_i^{-1} K_i = 1, \qquad \text{for all } 1 \leq i \leq r, \\
K_i E_j K_i^{-1} &= q^{d_i \alpha_j(H_i)} E_j, \qquad \text{for all } 1 \leq i, j \leq r, \\
K_i F_j K_i^{-1} &= q^{-d_i \alpha_j(H_i)} F_j, \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \qquad 1 \leq i, j \leq r, \\
\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{q^{d_i}} E_i^s E_j E_i^t &= 0, \quad \text{for } i \neq j, \\
\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{q^{d_i}} F_i^s F_j F_i^t &= 0, \quad \text{for } i \neq j.
\end{aligned}$$

and the Hopf algebra structure is given by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$S(K_i) = K_i^{-1}, S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i,$$
$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0.$$

There is a $\mathbb{Z}$ grading on the algebra $U_h(\mathfrak{g})$ is determined by

$$\deg(h) = 0, \qquad \text{for all } h \in \mathfrak{h},$$
$$\deg(E_i) = 1, \quad \deg(F_i) = -1, \quad \text{for all } 1 \leq i \leq r.$$

Define

$$U_h(\mathfrak{g})^{\geq 0} = \langle X_1, X_2, \ldots, X_r, \mathfrak{h} \rangle$$
$$U_h(\mathfrak{g})^{\leq 0} = \langle Y_1, Y_2, \ldots, Y_r, \mathfrak{h} \rangle$$

Let

$$\tilde{H}_1, \ldots, \tilde{H}_r \quad \text{be an orthonormal basis of } \mathfrak{h} \text{ and let } t_0 = \sum_{i=1}^{r} \tilde{H}_i \otimes \tilde{H}_i.$$

The algebra $U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra and the universal R-matrix of $U_h(\mathfrak{g})$ is written in the form

$$\mathcal{R} = \exp\left(\frac{h}{2} t_0\right) + \sum_i a_i^+ \otimes b_i^-,$$

where the elements $a_i^+ \in U_h(\mathfrak{g})^{\geq 0}$, $b_i^- \in U_h(\mathfrak{g})^{\leq 0}$.

Definition. Let the symmetrizable generalized Cartan matrix be

$$A = (a_{ij})_{i,j \in I} \text{ with a symmetrizing matrix } D = (s_i \in \mathbb{Z}_{>0} \mid i \in I).$$

The **quantum group** $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbb{F}(q)$ with 1 and generators

$$e_i, f_i \quad \text{and} \quad q^h \text{ for } h \in P^\vee$$

with the relations

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h \in P^\vee,$$
$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad \text{for } h \in P^\vee,$$
$$q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P^\vee,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i - 1}{q_i - q_i^{-1}} \quad \text{for } i, j \in I,$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{for } i \neq j$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad \text{for } i \neq j,$$

where

$$q_i = q^{s_i} \quad \text{and} \quad K_i = q^{s_i h_i}.$$

Let $\alpha = \sum_i n_i \alpha_i \in Q$ be in the root lattice. Define

$$K_\alpha = \prod_i K_i^{n_i}.$$

Set

$$\deg(f_i) = -\alpha_i, \quad \deg(q^h) = 0, \quad \text{and } \deg(e_i) = \alpha_i.$$

With the above grading, we see that the set of relations defined above are all homogeneous. Since the defining relations of the quantum group is homogeneous, we have a root space decomposition(this can referenced to Kac-Moody algebra)

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} (U_q)_\alpha,$$

where

$$(U_q)_\alpha = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u \quad \text{for all } h \in P^\vee\}$$

Representation theory of Quantum group

A $U_q(\mathfrak{g})$-module V^q is called a **weight module** if it admits a **weight space decomposition**

$$V^q = \bigoplus_{\mu \in P} V^q_\mu, \quad \text{where } V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \quad \text{for all } h \in P^\vee\}.$$

A vector of $v \in V^q_\mu$ is called a **weight vector** of weight μ. A vector $v \in V^q$ is called **maximal** if

$$e_i v = 0 \text{ for all } i \in I.$$

If $V^q_\mu \neq 0$, then μ is called a **weight** of V^q and V^q_μ is the weight space with weight $\mu \in P$. Define

Evaluation module and vertex model

In this section, we introduce the formal way of thinking about the lattice model via the evaluation modules. We will follow the same procedure as in Kang and Hong to produce the $R^\vee$ matrix for the six vertex model. Moreover, we will first consider the model $U_q(\widehat{\mathfrak{sl}_2})$. Then we will generalize to the model for $U_q(\widehat{\mathfrak{A}^{(2)}_2})$. Recall for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$, the generalized Cartan is given by

$$A = (a_{ij})_{i,j \in \{0,1\}} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The set of **simple roots** and **simple coroots** of $U_q(\widehat{\mathfrak{sl}_2})$ is

$$\Pi = \{\alpha_0, \alpha_1\} \quad \text{and} \quad \Pi^\vee = \{h_0, h_1\}.$$

The **dual weight lattice** is the free abelian group

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}d,$$

where d is the grading element satisfying

$$\alpha_0(d) = 1, \qquad \alpha_1(d) = 0.$$

Define the **fundamental weights** to be

$$\Lambda_1, \Lambda_2 \in \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee \quad \text{such that} \quad \Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \text{ for } i, j \in \{0,1\}.$$

Define the imagary root to be

$$\delta = \alpha_0 + \alpha_1.$$

Then the **weight lattice** is

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta.$$

Definition. Let q be nonzero complex number which is not a root of unity. The **quantum affine algebra** $U_q(\widehat{\mathfrak{sl}_2})$ is the quantum group assiociated with the

Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ be a two dimensional vector space and tensor the space V with a Laurent polynomial, we obtain

$$V^{\text{aff}} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} V,$$

where z is an indeterminate(this is often called spectral parameter in the vertex model bussiness). Let ζ be a non-zero complex number. Define

$$\mathbf{J}_\zeta = (z - \zeta) \quad \text{to be a maximal ideal of } \mathbb{C}[z, z^{-1}].$$

The ideal $\mathbf{J}_\zeta$ is maximal because

$$\mathbb{C}[z, z^{-1}]/\mathbf{J}_\zeta \cong \mathbb{C} \qquad \text{which is a field.}$$

Define the evaluation space

$$V(z = \zeta) := \mathbb{C} \otimes_{\mathbb{C}[z,z^{-1}]} V^{\text{aff}}.$$

Now the following are a series of isomorphism illustrates how one should understand this

$$V_\zeta = \mathbb{C} \otimes_{\mathbb{C}[z,z^{-1}]} V^{\text{aff}} \cong \mathbb{C}[z, z^{-1}]/\mathbf{J}_\zeta \otimes V^{\text{aff}} \cong V^{\text{aff}}/\mathbf{J}_\zeta V^{\text{aff}}.$$

Notice here that all the isomophism are $\mathbb{C}[z, z^{-1}]$ module isomorphism and we should understand V^{aff} as a $\mathbb{C}[z, z^{-1}]$-module. Define

$$\zeta_1, \zeta_2 \quad \text{be complex numbers that are not roots of unity.}$$

Remark 0.3. Roots of unity plays a very important role here. If these parameters are roots of unity than the R matrix will be different.