TYPE $\mathfrak{gl}_n$ MACDONALD POLYNOMIALS

By

Weiying Guo

Supervised by
Arun Ram

Submitted In Partial Fulfillment
of the Requirements for the Degree
Master of Science in the
School of Mathematics and Statistics

The University of Melbourne

October 2020

ABSTRACT

Macdonald polynomials are known to form an orthogonal basis of the ring of symmetric functions. It is well-known that Haglund, Haiman and Loehr [HHL06] provided a combinatorial formula for the non-symmetric Macdonald polynomial on the root system of $\mathfrak{gl}_n$ using non-attacking fillings. Soon after, Ram and Yip [RY08] constructed a general formula of the non symmetric Macdonald polynomials on any root system using alcove walks. In this thesis, we are going to discuss some connections between those two different formulas and those combinatorial objects which have been used.

ACKNOWLEDGEMENTS

Firstly, I would like to thank my supervisor Arun Ram for endless support, encouragement and most important being patient during my master project. I am also very thankful for many helpful conversations with David Ridout and Michael Wheeler on the affine root systems and Macdonald polynomials. Without their helps this thesis is never going to be done. I would also like to thank Chengjing Zhang, Spencer Wong, William Meadow and Qiaolin Mu who read my thesis and provided helpful suggestions.

Contents

Chapter 1

Introduction

Macdonald polynomials play central roles in the ring of symmetric functions. The symmetric Macdonald polynomials can be generalised to a non-symmetric version and both of them have strong connections to the root systems of Lie groups and Lie algebras. The connection between Macdonald polynomials and the root systems of Lie algebras categorises the Macdonald polynomials according to their root systems. In the combinatorial aspect, the root system of the Lie algebra $\mathfrak{gl}_n$ is known to be the simplest among all the classical Lie algebras. Therefore, understanding the Macdonald polynomials in type $\mathfrak{gl}_n$ is a stepping stone to the general theory of Macdonald polynomials.

In 2008, Haglund, Haiman and Loehr [HHL07] established a combinatorial formula for both symmetric and non-symmetric Macdonald polynomials. In this paper, the combinatorial object non-attacking filling was introduced and it was realised to have connections with the affine permutations and the affine root systems. In 2008, Ram and Yip [RY08] have generalised the combinatorial formula for both symmetric and non-symmetric Macdonald polynomials to all root systems. The combinatorial object alcove walks were introduced and it is also closely related to the affine root systems. More recently, Borodin and Wheeler[BW19] constructed a bijection between the μ-legal lattice configurations and the non-attacking fillings. It converts the Haglund-Haiman-Loehr formula into a more symmetric form involving descents and ascents of the non-attacking fillings. This allows us to convert the Ram-Yip formula to type $\mathfrak{gl}_n$ and make comparisons between the different combinatorial statistics occurred in the Haglund-Haiman-Loehr formula and the Ram-Yip formula.

The aim is to convert the Ram-Yip formula to the type $\mathfrak{gl}_n$ root system and make comparisons between the interwiner construction and the unwinding μ construction of the non-symmetric Macdonald polynomials. The connect between these inductive constructions reveal a non-trivial relationship between the alcove walks, paths, non-attacking fillings and affine root systems. This thesis is organised in six chapters:

- In Chapter 2, the affine Weyl group of type $\mathfrak{gl}_n$ and the combinatorics in relation to the statistics defined in Haglund-Haiman-Loehr formula will be discussed.
- In Chapter 3, a brief introduction about the type $\mathfrak{gl}_n$ double affine Hecke algebra(DAHA) and interwiner construction and the unwinding μ construction for non-symmetric Macdonald polynomials will be discussed.
- In Chapter 4, a brief introduction of the Haglund-Haiman-Loehr formula and its proof will

be discussed.

- In Chapter 5, the Borodin-Wheeler formula and the combinatorial statistics ordered triples will be discussed. Furthermore, a comparison between Haglund-Haiman-Loehr formula, Borodin-Wheeler formula and type $\mathfrak{gl}_n$ Ram-Yip formula will be discussed.
- In Chapter 6, we present a picture of the alcoves in type $\mathfrak{gl}_2$ and some computations of the non-symmetric Macdonald polynomials using the unwinding μ construction.

After converting the Ram-Yip formula to the root system of type $\mathfrak{gl}_n$, it will be interesting to understand what lies in the core of these different formulas. We have established two "different" inductive constructions of non-symmetric Macdonald polynomials. One comes from the Haglund-Haiman-Loehr formula which is called the unwinding μ construction, and the other comes from the Ram-Yip formula which is called the interwiner construction. We realised that the unwinding μ construction leads to a more "compressed" formula than the interwiner construction. One of the remaining questions is to understand the compression occurred from the Ram-Yip formula to Haglund-Haiman-Loehr formula, and some discussions on these topics can be found in [Len08].

Although the conversion of the Ram-Yip formula to type $\mathfrak{gl}_n$ has been obtained, the connections between the affine root systems, non-attacking fillings, alcove walks and their statistics are still not clear. Once the connections between these statistics have been worked out thoroughly, we believe that the proof of the Haglund-Haiman-Loehr formula can be simplified using the affine root systems.

Chapter 2

The Affine Weyl group and boxes

The main focus of this section is to introduce some properties of the root system of $\mathfrak{gl}_n$ and the Weyl group associated with it. In Section 2.1–2.2, we discuss the root system of $\mathfrak{gl}_n$, the Weyl group and the affine Weyl group associated with it. In Section 2.3–2.4, we introduce two combinatorial statistics, Harm and Hleg, associated with a box. Their relationship with the statistics arm and leg for partitions will be discussed at the end of Section 2.4.

2.1 The Lie algebra $\mathfrak{gl}_n$

The Lie algebra $\mathfrak{gl}_n$ is the set of $n \times n$ matrices with entries in $\mathbb{C}$. Let $A, B \in \mathfrak{gl}_n$. The multiplication on $\mathfrak{gl}_n$ is defined by

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{gl}_n \times \mathfrak{gl}_n &\to \mathfrak{gl}_n \\ (A, B) &\mapsto AB - BA. \end{aligned} \tag{2.1.1}$$

For example,

$$\mathfrak{gl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

A $\mathbb{C}$-basis of $\mathfrak{gl}_n$ is

$$\{E_{ij} \mid i, j \in \{1, \dots, n\}\},$$

where E_{ij} is the $n \times n$ matrix with the (i, j) entry 1 and all the other entries 0. For example, one of the bases of $\mathfrak{gl}_2$ is

$$\{E_{11}, E_{12}, E_{21}, E_{22}\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The Cartan subalgebra of $\mathfrak{gl}_n$ is

$$\mathfrak{a}_{\mathfrak{gl}_n} = \mathrm{span}_{\mathbb{C}}\{E_{ii} \mid i \in \{1, \dots, n\}\}. \tag{2.1.2}$$

For example,

$$\mathfrak{a}_{\mathfrak{gl}_2} = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mid a_1, a_2 \in \mathbb{C} \right\}.$$

Let $\varepsilon_i = (0, \dots, 1, \dots, 0)$ be the canonical basis of $\mathbb{R}^n$ with i-th entry 1 and all the other entries zero. The **weight lattice** of $\mathfrak{a}_{\mathfrak{gl}_n}$ is

$$\mathfrak{a}_{\mathbb{Z}}^* = \mathrm{span}_{\mathbb{Z}}\{\varepsilon_1, \dots, \varepsilon_n\}. \tag{2.1.3}$$

The **co-weight lattice** of $\mathfrak{a}_{\mathfrak{gl}_n}$ is the dual space $\mathfrak{a}_{\mathbb{Z}}$ of $\mathfrak{a}_{\mathbb{Z}}^*$, i.e.,

$$\mathfrak{a}_{\mathbb{Z}} = \mathrm{span}_{\mathbb{Z}}\{\varepsilon_1^\vee, \dots, \varepsilon_n^\vee\}, \qquad \text{where} \qquad \varepsilon_i(\varepsilon_j^\vee) = \delta_{ij}$$

and δ_{ij} is the Kronecker symbol. Let K, Λ_0 be formal variables. The **affine weight lattice** and the **affine co-weight lattice** are

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathfrak{a}_{\mathbb{Z}}^* \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta \quad \text{and} \quad \mathfrak{h}_{\mathbb{Z}} = \mathfrak{a}_{\mathbb{Z}} \oplus \mathbb{Z}K \oplus \mathbb{Z}d \quad \text{respectively.} \tag{2.1.4}$$

Extend $\mathbb{Z}$-modules $\mathfrak{a}_{\mathbb{Z}}$, $\mathfrak{a}_{\mathbb{Z}}^*$, $\mathfrak{h}_{\mathbb{Z}}$ and $\mathfrak{h}_{\mathbb{Z}}^*$ to $\mathbb{R}$-modules, i.e.,

$$\mathfrak{a}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{a}_{\mathbb{Z}}, \quad \mathfrak{a}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{a}_{\mathbb{Z}}^*, \quad \mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}} \quad \text{and} \quad \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*.$$

The **simple roots** are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in \{1, \dots, n-1\}$. Define the **simple affine roots** to be

$$\alpha_1, \alpha_2, \dots, \alpha_{n-1} \quad \text{and} \quad \alpha_0 = \delta - (\varepsilon_1 - \varepsilon_n). \tag{2.1.5}$$

The set of **positive roots** and **positive co-roots** are

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\} \quad \text{and} \quad (R^\vee)^+ = \{\varepsilon_i^\vee - \varepsilon_j^\vee \mid i < j\} \quad \text{respectively.}$$

(CHCK THE delta)The set of **positive affine roots** and **positive affine co-roots** are

$$\begin{aligned} \widehat{R}^+ &= \{\varepsilon_i - \varepsilon_j + k\Lambda_0 \mid i < j \text{ and } k \in \mathbb{Z}_{\geq 0}\} \cup \{\varepsilon_j - \varepsilon_i + k\Lambda_0 \mid i < j \text{ and } k \in \mathbb{Z}_{>0}\} \quad \text{and} \\ (\widehat{R}^\vee)^+ &= \{\varepsilon_i^\vee - \varepsilon_j^\vee + mK \mid i < j \text{ and } m \in \mathbb{Z}_{\geq 0}\} \cup \{\varepsilon_j^\vee - \varepsilon_i^\vee + m\Lambda_0 \mid i < j \text{ and } m \in \mathbb{Z}_{\geq 0}\} \end{aligned}$$

respectively.
Let $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R}^n = \mathbb{R}\varepsilon_1 + \cdots + \mathbb{R}\varepsilon_n$. There are **hyperplanes**

$$\mathfrak{a}^{\varepsilon_i^\vee - \varepsilon_j^\vee + aK} = \{(\mu_1, \dots, \mu_n) \in \mathbb{R}^n \mid \mu_i - \mu_j = -a\} \quad \text{and} \quad \mathfrak{a}^{\varepsilon_k^\vee + aK} = \{(\mu_1, \dots, \mu_n) \in \mathbb{R}^n \mid \mu_k = -a\}$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\}$.
Let $S = \{\varepsilon_1^\vee, \varepsilon_2^\vee, \dots, \varepsilon_n^\vee\}$ and $M = (R^\vee)^+ \cup S$. The **alcoves** of $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{a}_{\mathbb{Z}}^*$ are the connected components of

$$\mathfrak{a}_{\mathbb{R}}^* \backslash \Bigg(\bigcup_{x^\vee \in M, m \in \mathbb{Z}} \mathfrak{a}^{x^\vee + mK} \Bigg). \tag{2.1.6}$$

Informally, the alcoves are regions in $\mathfrak{a}_{\mathbb{R}}^*$ which are bounded by the hyperplanes(walls). The picture describing the alcoves of $\mathfrak{a}_{\mathbb{Z}} = \mathbb{R}^2$ are presented in Appendix 6.1. In comparisons with the alcoves defined in [RY08, pg 4, (2.14)], the alcoves defined in (2.1.6) are bounded by more hyperplanes than the ones in [RY08, pg 4, (2.14)].
Define the **fundamental alcove** to be

$$c_0 = \{a_1\varepsilon_1 + \cdots + a_n\varepsilon_n \mid 0 \leq a_i \leq 1\}. \tag{2.1.7}$$

2.2 The Weyl group and the affine Weyl group

The symmetric group S_n is the set of bijections $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. The **finite Weyl group** of $\mathfrak{gl}_n$ is

$$S_n \quad \text{with function composition as the multiplication.}$$

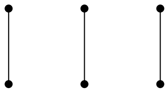
The **simple transpositions** are $s_1, \dots, s_{n-1}$ given by

$$s_i(i) = i+1, \quad s_i(i+1) = i, \quad s_i(j) = j \qquad \begin{matrix} \text{for } i \in \{1, \dots, n-1\} \\ \text{and } j \notin \{i, i-1\}. \end{matrix} \tag{2.2.1}$$

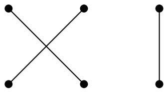
The finite Weyl group S_n can be presented by generators $s_1, \cdots, s_{n-1}$ and relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \qquad \begin{matrix} \text{for } i \in \{1, \dots, n-1\} \\ \text{and } j \notin \{i+1, i-1\}. \end{matrix}$$

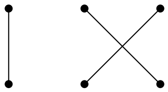
For example, $S_3 = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$, where the elements can be presented pictorially as



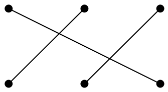
e



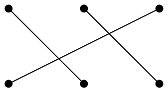
s_1



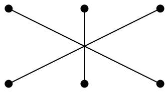
s_2



$s_1 s_2$



$s_2 s_1$



$s_1 s_2 s_1$

The finite Weyl group S_n acts on the lattice $\mathfrak{a}_{\mathbb{Z}}$ and $\mathfrak{a}_{\mathbb{Z}}^*$ by

$$s_i(\varepsilon_j) = \varepsilon_{s_i(j)} \quad \text{and} \quad s_i(\varepsilon_j^\vee) = \varepsilon_{s_i(j)}^\vee \qquad \begin{matrix} \text{for } i \in \{1, \dots, n-1\}, \\ \text{and } j \in \{1, \dots, n\}, \end{matrix} \tag{2.2.2}$$

and by linearity S_n acts on $\mathfrak{a}_{\mathbb{Z}}$ and $\mathfrak{a}_{\mathbb{Z}}^*$.
Let $\{s_1, \dots, s_{n-1}\}$ be the set of generators of S_n. A **reduced word** of $w \in S_n$ is a product

$$w = s_{i_1} \cdots s_{i_m} \quad \text{with } m \text{ minimal},$$

where m is the length of w and we denote the **length** of w to be $\ell(w)$.
An **inversion** of $w \in S_n$ is a pair

$$(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}, \quad \text{such that} \quad i < j \text{ and } w(i) > w(j),$$

the set of inversions of w is denoted as $\mathrm{FInv}(w)$. Then

$$\ell(w) = \# \mathrm{FInv}(w).$$

For $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$, we say μ is

$$\text{dominant if } \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \quad \text{or} \quad \text{anti-dominant if } \mu_1 \leq \mu_2 \cdots \leq \mu_n. \tag{2.2.3}$$

For a positive integer n, an n-**periodic permutation** is a bijection

$$w : \mathbb{Z} \to \mathbb{Z}, \quad \text{such that} \quad w(i+n) = w(i) + n. \tag{2.2.4}$$

We notice that the permutation extends both directions in $\mathbb{Z}$ via the relation in (2.2.4). Any $\sigma \in S_n$ can be extended uniquely to an n-periodic permutation $\hat{\sigma}$ via the relation $\hat{\sigma}(i+n) = \hat{\sigma}(i) + n$. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}. \quad \text{Then} \quad \begin{matrix} \hat{\sigma}(6) = 1+5, & \hat{\sigma}(7) = 4+5, & \hat{\sigma}(10) = 1+5, \\ \hat{\sigma}(8) = 3+5, & \hat{\sigma}(9) = 2+5, & \textit{etc.} \end{matrix} \tag{2.2.5}$$

We remark that not every n-periodic permutation is an extension of the symmetric group. For example, the two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 10 & 20 & 3 & 5 & -6 \end{pmatrix}$$

defines a 5-periodic permutation, but it is not an extension of an element in S_n.

The **affine Weyl group** $W_{\mathfrak{gl}_n}$ is the group of n-periodic permutations. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n = \mathbb{Z}\varepsilon_1^\vee + \mathbb{Z}\varepsilon_2^\vee + \cdots + \mathbb{Z}\varepsilon_n^\vee$, define the translation $t_\mu \in W_{\mathfrak{gl}_n}$ to be

$$t_\mu(1) = 1 + n\mu_1, \quad t_\mu(2) = 2 + n\mu_2, \quad \dots, \quad t_\mu(n) = n + n\mu_n. \tag{2.2.6}$$

Define $\pi \in W_{\mathfrak{gl}_n}$ by

$$\pi(1) = 2, \quad \pi(2) = 3, \quad \dots, \quad \pi(n) = n + 1. \tag{2.2.7}$$

By abuse of notation, set $s_i = \widehat{s_i}$. The element $s_0 \in W_{\mathfrak{gl}_n}$ is an n-periodic permutation given by

$$s_0(n) = n + 1, \quad s_0(n+1) = n, \quad s_0(j) = j, \quad \text{with} \quad j \notin \{n, n+1\}. \tag{2.2.8}$$

Lemma 2.2.1. *Let $s_0, s_1, \dots, s_{n-1}, \pi$ and t_μ be elements in $W_{\mathfrak{gl}_n}$ defined in* (2.2.6)*,* (2.2.7) *and* (2.2.8)*. Then*

$$s_0 = t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}, \quad t_{\varepsilon_1^\vee} = \pi s_{n-1} \cdots s_2 s_1 \tag{2.2.9}$$

and

$$t_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i, \quad \pi s_i \pi^{-1} = s_{i+1} \quad \textit{for } i \in \{1, \dots, n-2\}. \tag{2.2.10}$$

Proof. We check the first relation in (2.2.9) and the remaining relations can be checked by similar computations. Using (2.2.6) and (2.2.8), we obtain

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(n) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(1) = 1 + n$$

and

$$\begin{aligned} t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(1+n) &= t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(1) + n \\ &= t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(n) + n = n - n + n = n. \end{aligned}$$

For $j \in \{2, \dots, n-1\}$,

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(j) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(j) = j. \qquad \square$$

The affine Weyl group $W_{\mathfrak{gl}_n}$ can also be presented as a semi-direct product [Bou08, pg 186, Prop1]

$$W_{\mathfrak{gl}_n} = \mathbb{Z}^n \rtimes S_n = \{t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n\}, \tag{2.2.11}$$

where the conjugation action is provided by

$$v t_\mu = t_{v\mu} v. \tag{2.2.12}$$

Proposition 2.2.2. *Let π and s_i be defined in* (2.2.7)*,* (2.2.8)*. The affine Weyl group $W_{\mathfrak{gl}_n}$ has an alternative presentation with generators*

$$\pi, s_0, s_1, \dots, s_{n-1}, \tag{2.2.13}$$

and relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \quad s_i^2 = 1, \qquad \begin{array}{c} \textit{for } i \in \{0, 1, \dots, n-1\}, \\ \textit{and } j \notin \{i+1, i-1\}, \end{array} \tag{2.2.14}$$

$$\pi s_{n-1} \pi^{-1} = s_0 \quad \textit{and} \quad \pi s_i \pi^{-1} = s_{i+1}, \qquad \textit{for } i \in \{0, 1, \dots, n-2\}. \tag{2.2.15}$$

Proof. Using the semidirect product property of $W_{\mathfrak{gl}_n}$ in (2.2.11), we obtain the following set of generators

$$A = \{t_{\varepsilon_1^\vee}, \cdots t_{\varepsilon_n^\vee}, s_1, \cdots s_{n-1}\}$$

and the relations of s_i's defined in (2.2.14). In addition we also have the following relations

$$t_{\varepsilon_i^\vee} t_{\varepsilon_j^\vee} = t_{\varepsilon_i^\vee + \varepsilon_j^\vee}, \quad s_j t_{\varepsilon_i^\vee} = t_{\varepsilon_{s_j(i)}^\vee} s_j \quad \text{for } i, j \in \{1, \ldots, n\}. \tag{2.2.16}$$

We want to show that the following set

$$B = \{\pi, s_0, s_1, \cdots, s_{n-1}\}$$

with relations defined in (2.2.14) and (2.2.15) is another presentation of $W_{\mathfrak{gl}_n}$.
Writing generators in set A in terms of generators in set B, we obtain

$$t_{\varepsilon_i^\vee} = s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i. \tag{2.2.17}$$

Using the definition of t_μ defined in (2.2.6),

$$t_{\varepsilon_i^\vee} t_{\varepsilon_j^\vee} = t_{\varepsilon_i^\vee + \varepsilon_j^\vee}.$$

To show the second relation in (2.2.16): Consider the following four cases.
Case 1: Assume $j = i$. Using (2.2.17) to expand $s_j t_{\varepsilon_i^\vee}$ gives

$$\begin{aligned} s_j t_{\varepsilon_i^\vee} &= s_i(s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i) = (s_i s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_{i+1}) s_i \\ &= t_{\varepsilon_{i+1}^\vee} s_i = t_{\varepsilon_{s_j(i)}^\vee} s_j. \end{aligned}$$

Case 2: Assume $j = i - 1$, then

$$\begin{aligned} s_j t_{\varepsilon_i^\vee} &= s_{i-1}(s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i) = s_{i-2} \cdots s_1 \pi s_{n-1} \cdots s_i \\ &= (s_{i-2} \cdots s_1 \pi s_{n-1} \cdots s_i s_{i-1}) s_{i-1} = t_{\varepsilon_{i-1}^\vee} s_{i-1} = t_{\varepsilon_{s_j(i)}^\vee} s_j. \end{aligned}$$

Case 3: Assume $j \in \{1, \ldots, i-2\}$. Then

$$\begin{aligned} s_j t_{\varepsilon_i^\vee} &= s_{i-1} \cdots (s_j s_{j+1} s_j) \cdots s_1 \pi s_{n-1} \cdots s_i = s_{i-1} \cdots (s_{j+1} s_j s_{j+1}) s_{j-1} \cdots s_1 \pi s_{n-1} \cdots s_i \\ &= s_{i-1} \cdots s_1 (s_{j+1} \pi) s_{n-1} \cdots s_i = s_{i-1} \cdots s_1 (\pi s_j) s_{n-1} \cdots s_i = (s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i) s_j \\ &= t_{\varepsilon_i^\vee} s_j = t_{\varepsilon_{s_j(i)}^\vee} s_j. \end{aligned}$$

Case 4: Assume $j \in \{i+1, \ldots, n-1\}$. Then

$$\begin{aligned} s_j t_{\varepsilon_i^\vee} &= s_{i-1} \cdots s_1 (s_j \pi) s_{n-1} \cdots s_i = s_{i-1} \cdots s_1 (\pi s_{j-1}) s_{n-1} \cdots s_i \\ &= s_{i-1} \cdots s_1 \pi s_{n-1} \cdots (s_{j-1} s_j s_{j-1}) \cdots s_i = s_{i-1} \cdots s_1 \pi s_{n-1} \cdots (s_j s_{j-1} s_j) s_{j-2} \cdots s_i \\ &= s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i s_j = t_{\varepsilon_{s_j(i)}^\vee} s_j. \end{aligned}$$

Hence,

$$s_j t_{\varepsilon_i^\vee} = t_{\varepsilon_{s_j(i)}^\vee} s_j \quad \text{for } j \in \{1, \ldots, n-1\}.$$

Writing generators in set B in terms of generators in set A gives

$$\pi = t_{\varepsilon_1^\vee} s_1 s_2 \cdots s_{n-1}, \quad \text{and} \quad s_0 = t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}. \tag{2.2.18}$$

The relations defined in (2.2.14) are the same as the ones in (2.2.11).
To show the second relation in (2.2.15): Using relations in (2.2.18) give

$$\begin{aligned}\pi s_0 &= (t_{\varepsilon_1^\vee} s_1 s_2 \cdots s_{n-1})(t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}) = t_{\varepsilon_1^\vee} t_{-\varepsilon_1^\vee + \varepsilon_2^\vee} s_2 \cdots s_{n-1} \\ &= t_{\varepsilon_2^\vee} s_2 \cdots s_{n-1} = t_{\varepsilon_2^\vee}(s_1 s_1) s_2 \cdots s_{n-1} = s_1 t_{\varepsilon_1^\vee} s_1 s_2 \cdots s_{n-1} = s_1 \pi,\end{aligned}$$

and

$$\begin{aligned}\pi s_i &= (t_{\varepsilon_1^\vee} s_1 s_2 \cdots s_{n-1}) s_i = t_{\varepsilon_1^\vee} s_1 s_2 \cdots (s_i s_{i+1} s_i) s_{i+2} \cdots s_{n-1} \\ &= t_{\varepsilon_1^\vee} s_1 s_2 \cdots (s_{i+1} s_i s_{i+1}) s_{i+2} \cdots s_{n-1} = s_{i+1} \pi,\end{aligned}$$

for $i \in \{1, \ldots, n-1\}$. One checks the first relation in (2.2.15) by similar computation. □

Remark 2.2.3. We note that since $t_{\varepsilon_1^\vee} = \pi s_{n-1} \cdots s_2 s_1$, the element s_0 can be written in terms of products of π and s_i's for $i \in \{1, \ldots, n-1\}$. Then $\{s_1, \ldots, s_{n-1}, \pi\}$ forms a set of generators of $W_{\mathfrak{gl}_n}$ with relations defined in (2.2.14) and (2.2.15).

Set $s_\pi = \pi$. Then $\{s_1, \ldots, s_{n-1}, s_\pi\}$ is a set of generators of $W_{\mathfrak{gl}_n}$ by Remark 2.2.3. A **reduced word** of $w \in W_{\mathfrak{gl}_n}$ is a product

$$w = s_{i_1} \cdots s_{i_m} \quad \text{with } m \text{ minimal.}$$

The set of **affine inversions** of an element s in $W_{\mathfrak{gl}_n}$ is

$$\mathrm{AInv}(s) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i < j \text{ and } s(i) > s(j)\}. \tag{2.2.19}$$

If $(i,j) \in \mathrm{AInv}(s)$, then $(i,j) + (n,n)\mathbb{Z} \subseteq \mathrm{AInv}(s)$. Therefore, for each $s \in W_{\mathfrak{gl}_n}$, there exists a finite set of representatives

$$\mathrm{AInv}_R(s) = \{(i_1, j_1), \ldots, (i_k, j_k) \mid i_n \in \{1, \ldots, n\}, j \in \mathbb{Z}\} \quad \text{such that}$$

$$\mathrm{AInv}(s) = \bigcup_{(x,y) \in \mathrm{AInv}_R(s)} \{(x,y) + (n,n)\mathbb{Z}\}.$$

The **length** of a reduced word $s = s_{i_1} \cdots s_{i_r}$ in $W_{\mathfrak{gl}_n}$ is given by

$$\ell(s) = \#\{\text{transpositions } s_i \text{ appearing in the reduced word of } s\}. \tag{2.2.20}$$

Note that $\ell(s_\pi) = \ell(\pi) = 0$.

Theorem 2.2.4. *Let $s = s_{i_1} \cdots s_{i_r}$ be a reduced word in $W_{\mathfrak{gl}_n}$. Then there are bijections between*

$$\begin{aligned}&\{\text{transpositions } s_i \text{ appearing in the reduced word of } s\} \leftrightarrow \mathrm{AInv}_R(s) \\ &\qquad \leftrightarrow \left\{\text{hyperplanes separates the alcove } c_0 \text{ and } s^{-1}c_0\right\}.\end{aligned}$$

Proof. The first bijection is provided in [Bou08, Charpter 6, Section 1, Cor 2]. By the bijection between alcoves and the elements of $W_{\mathfrak{gl}_n}$ in Lemma 6.1.1 and each transposition s_i(except s_π) is a crossing of a hyperplane, it follows that there is a bijection between the transpositions in the reduced word and the hyperplanes between c_0 and $s^{-1}c_0$. For more details of the second bijection, readers may refer to [Mac03, pg19, (2.2.1)]. □

Recall that the set $S = \{\varepsilon_1^\vee, \varepsilon_2^\vee, \dots, \varepsilon_n^\vee\}$ and $M = (R^\vee)^+ \cup S$, and the alcoves of $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*$ are the connected components of

$$\mathfrak{h}_{\mathbb{R}}^* \backslash \Big(\bigcup_{x^\vee \in M, m \in \mathbb{Z}} \mathfrak{h}^{x^\vee + mK} \Big).$$

By Theorem 2.2.3, the length of an element s of the affine Weyl group

$$\begin{aligned} \ell(s) = \#\,\mathrm{AInv}_R(s) &= \#\{\text{hyperplanes separates the fundamental alcoves } c_0 \text{ and } s^{-1}c_0\} \\ &= \#\{\text{hyperplanes crossed by the alcove walk from } c_0 \text{ to } s^{-1}c_0\}. \end{aligned}$$

2.3 The cylindrical wrapping and fillings

A **box** is an element of $\{1, \dots, n\} \times \mathbb{Z}_{\geq 0}$ and the set of boxes is denoted by $\{\text{boxes}\}$. The **basement** is the set

$$\{(i, 0) \in \{\text{boxes}\} \mid i \in \{1, \dots, n\}\}.$$

The **cylindrical wrapping** is the bijection

$$\mathrm{cyl}_n : \mathbb{Z}_{>0} \to \{\text{boxes}\} \quad \text{defined by} \quad \mathrm{cyl}_n(i + nj) = (i, j). \tag{2.3.1}$$

The **cylindrical coordinate** for a box (i, j) is the integer $\mathrm{cyl}_n^{-1}((i, j))$. For example, for $n = 5$, the cylindrical wrapping is pictorially expressed by

$$\mathrm{cyl}_5 \left(\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{21} & \boxed{22} & \boxed{23} & \boxed{24} & \boxed{25} \\ \boxed{16} & \boxed{17} & \boxed{18} & \boxed{19} & \boxed{20} \\ \boxed{11} & \boxed{12} & \boxed{13} & \boxed{14} & \boxed{15} \\ \boxed{6} & \boxed{7} & \boxed{8} & \boxed{9} & \boxed{10} \\ \hline \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \end{array} \right) = \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{(1,4)} & \boxed{(2,4)} & \boxed{(3,4)} & \boxed{(4,4)} & \boxed{(5,4)} \\ \boxed{(1,3)} & \boxed{(2,3)} & \boxed{(3,3)} & \boxed{(4,3)} & \boxed{(5,3)} \\ \boxed{(1,2)} & \boxed{(2,2)} & \boxed{(3,2)} & \boxed{(4,2)} & \boxed{(5,2)} \\ \boxed{(1,1)} & \boxed{(2,1)} & \boxed{(3,1)} & \boxed{(4,1)} & \boxed{(5,1)} \\ \hline \boxed{(1,0)} & \boxed{(2,0)} & \boxed{(3,0)} & \boxed{(4,0)} & \boxed{(5,0)} \end{array}$$

Given $\mu \in \mathbb{Z}_{\geq 0}^n$, the **column diagram** $\mathrm{dg}(\mu)$ is

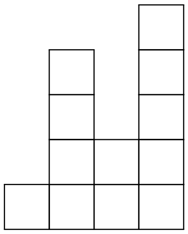
$$\mathrm{dg}(\mu) = \{(i, j) \in \{\text{boxes}\} \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, \mu_i\}\},$$

and the **extended column diagram** $\widehat{\mathrm{dg}}(\mu)$ is

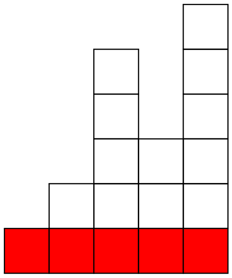
$$\widehat{\mathrm{dg}}(\mu) = \{(i, j) \in \{\text{boxes}\} \mid i \in \{1, \dots, n\} \text{ and } j \in \{0, \dots, \mu_i\}\}. \tag{2.3.2}$$

Example 2.3.1. For example, if $\mu = (0, 1, 4, 2, 5) \in \mathbb{Z}_{\geq 0}^5$, then the column diagram and the extended column diagram are

$\mathrm{dg}(\mu) =$



$\widehat{\mathrm{dg}}(\mu) =$



where the part highlighted by red in $\widehat{\mathrm{dg}}(\mu)$ is the basement.

A **filling** of $\mathrm{dg}(\mu)$ is a function

$$\sigma_\mu : \mathrm{dg}(\mu) \to \{1, \dots, n\}.$$

An **augmented filling** of $\widehat{\mathrm{dg}}(\mu)$ is a function

$$\widehat{\sigma_\mu} : \widehat{\mathrm{dg}}(\mu) \to \{1, \dots, n\}. \tag{2.3.3}$$

2.4 Arms and legs of boxes

A **partition** is a tuple

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n_{\geq 0} \quad \text{such that} \quad \lambda \text{ is dominant.}$$

If λ is a partition, then λ' denotes the conjugate partition of λ. Let $s = (i, j)$ be a box in $\mathrm{dg}(\lambda)$. Then, the statistics arm and leg [Mac95, pg337, (6.14)] are defined by

$$\mathrm{arm}_\lambda(s) = \lambda_i - j \quad \text{and} \quad \mathrm{leg}_\lambda(s) = \lambda'_j - i. \tag{2.4.1}$$

Let $\mu \in \mathbb{Z}^n_{\geq 0}$, and let $u = (i, j)$ be a box in $\mathrm{dg}(\mu)$. Identify the cylindrical coordinates (2.3.1) with the boxes, define attacking boxes, Harm and Hleg functions to be

$$\mathrm{attack}_\mu(u) = \{u - 1, \dots, u - n + 1\} \cap \widehat{\mathrm{dg}}(\mu), \quad \mathrm{Hleg}_\mu(u) = (u + n\mathbb{Z}_{>0}) \cap \widehat{\mathrm{dg}}(\mu), \tag{2.4.2}$$

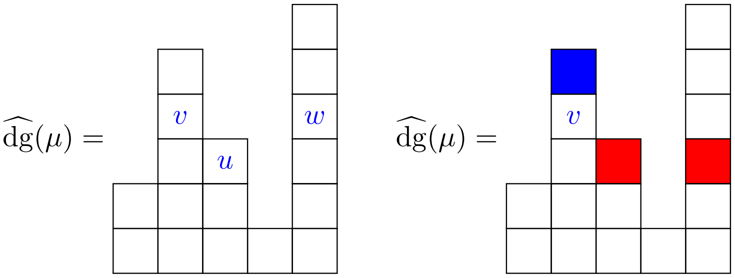
$$\text{and} \quad \mathrm{Harm}_\mu(u) = \{v \in \mathrm{attack}_\mu(u) \mid \mathrm{Hleg}_\mu(v) \leq \mathrm{Hleg}_\mu(u)\}. \tag{2.4.3}$$

Then cardinalities of Hleg_μ and Harm_μ are

$$\begin{aligned} \#\,\mathrm{Harm}_\mu(i, j) &= \#\{(r', j) \in \mathrm{dg}(\mu) \mid r' < i \text{ and } \mu_{r'} \leq \mu_i\} \\ &\quad + \#\{(r', j - 1) \in \widehat{\mathrm{dg}}(\mu) \mid i < r' \text{ and } \mu_{r'} < \mu_i\}, \\ \#\,\mathrm{Hleg}_\mu(i, j) &= \#\{(i, j') \in \mathrm{dg}(\mu) \mid j' > j\} = \mu_i - j. \end{aligned}$$

Informally, $\mathrm{Hleg}_\mu(u)$ is the set of boxes above u in the same column and $\mathrm{Harm}_\mu(u)$ is the set of boxes v in $\mathrm{attack}_\mu(u)$ such that the number of boxes above v is less or equal to the number of boxes above u.

Example 2.4.1. Let $\mu = (1, 4, 2, 0, 5)$, and let $u = (3, 2), v = (2, 3)$ and $w = (5, 3)$. Then



The statistics of the boxes u, v, w in the left diagram are listed in a table as follows:

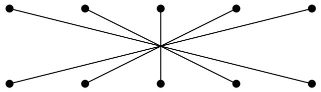
Boxes	attack_μ	Hleg_μ	Harm_μ	$\#\,\mathrm{Hleg}_\mu$	$\#\,\mathrm{Harm}_\mu$
u	$\{(2,2),(5,1)\}$	$\emptyset$	$\emptyset$	0	0
v	$\{(5,2),u\}$	$\{(2,4)\}$	$\{u\}$	1	1
w	$\{v\}$	$\{(5,4),(5,5)\}$	$\{v\}$	2	1

The red boxes in the right diagram indicate the set $\mathrm{attack}_\mu(v)$ and the blue box indicates the set $\mathrm{Hleg}_\mu(v)$.

Let

$$w_0 \text{ be the maximal length element of } S_n. \tag{2.4.4}$$

Example 2.4.2. For S_5, the maximal length element is $w_0 = (s_4s_3s_2s_1)(s_4s_3s_2)(s_4s_3)s_4$ and pictorially,

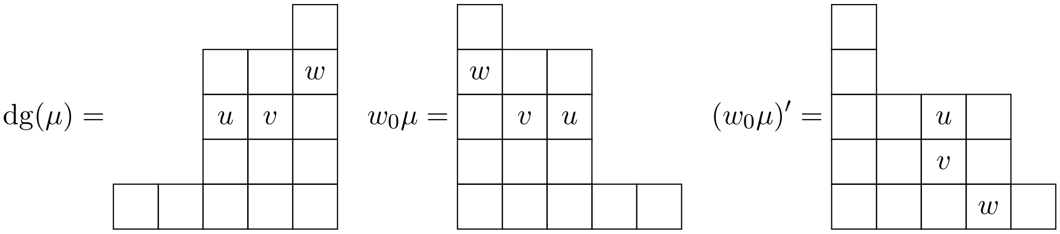


w_0 in S_5

Let $\mu = (\mu_1, \dots, \mu_n)$ be anti-dominant. Then $w_0\mu$ is a partition and the statistics $\#\,\mathrm{Harm}_\mu$, $\#\,\mathrm{Hleg}_\mu$ and $\mathrm{leg}_{w_0\mu}$, $\mathrm{arm}_{w_0\mu}$ defined in (2.4.1) are connected in the following way:

$$\begin{aligned}
\#\,\mathrm{Harm}_\mu(i,j) &= \#\{(r',j) \in \mathrm{dg}(\mu) \mid r' < i \text{ and } \mu_{r'} \le \mu_i\} \\
&= (w_0\mu)'_j - (n-i+1) = \mathrm{leg}_{w_0\mu}(n-i+1,j), \qquad \text{and} \\
\#\,\mathrm{Hleg}_\mu(i,j) &= \mu_i - j = (w_0\mu)_{n-i+1} - j = \mathrm{arm}_{w_0\mu}(n-i+1,j).
\end{aligned}$$

Example 2.4.3. Let $\mu = (1,1,4,4,5)$ and let $u = (3,3), v = (4,3), w = (5,4)$. Then $w_0\mu = (5,4,4,1,1)$, $(w_0\mu)' = (5,3,3,3,1)$ and



The statistics are

Boxes	$\#\,\mathrm{Harm}_\mu$	$\mathrm{leg}_{w_0\mu}$
u	0	0
v	1	1
w	2	2

Lemma 2.4.1 makes connection between the cardinality of Harm and the element w_μ.

Lemma 2.4.1. *Let* $\mu \in \mathbb{Z}^n_{\ge 1}$. *Then*

$$\#\,\mathrm{Harm}(i,1) = w_\mu^{-1}(i) - 1.$$

Proof. Since $\mu \in \mathbb{Z}^n_{\ge 1}$,

$$\begin{aligned}
w_\mu^{-1}(i) - 1 &= \#\{j \in \{1,\dots,n\} \mid j < i \text{ and } \mu_j \le \mu_i\} + \#\{j \in \{1,\dots,n\} \mid j > i \text{ and } \mu_j < \mu_i\} \\
&= \#\{(r',1) \in \mathrm{dg}(\mu) \mid r' < i \text{ and } \mu_{r'} \le \mu_i\} + \#\{(r',0) \in \widehat{\mathrm{dg}}(\mu) \mid i < r' \text{ and } \mu_{r'} < \mu_i\} \\
&= \#\,\mathrm{Harm}(i,1).
\end{aligned}$$

□

Chapter 3

Type $\mathfrak{gl}_n$ DAArt, DAHA and their polynomial representations

The double affine Hecke algebra is a quotient of the group algebra of the double affine Artin group. Macdonald polynomials can be naturally realised as the simultaneous eigenvectors of the $Y_i's$ elements of double affine Hecke algebra(DAHA) acting on the polynomial representations. One of the methods to understand Macdonald polynomials is to study the polynomial representation of DAHA. This chapter aims to study the type $\mathfrak{gl}_n$ double affine Hecke algebra, and its polynomial representation. In Section 3.1–3.2, we are going to introduce two presentations of the type $\mathfrak{gl}_n$ double affine Artin group. A duality between the two different presentations will also be given in Section 3.1 along with proofs. In Section 3.2, the type $\mathfrak{gl}_n$ double affine Hecke algebra will be introduced. The intertwiners $\tau_\pi^\vee, \tau_i^\vee$, which will be used to generate Macdonald polynomials, are introduced in Section 3.3 and their commutation relations with Y_i's will be given in Proposition 3.3.2. In Section 3.4, we will discuss some polynomial representations of the double affine Hecke algebra and introduce the interwiner construction for Macdonald polynomials. Finally, Proposition 3.5.3 states that Macdonald polynomials are simultaneous eigenvectors of the elements Y_i's.

3.1 Type $\mathfrak{gl}_n$ double affine Artin group

Let $q, t \in \mathbb{C}$. The double affine Artin group $\widetilde{B}_{\mathfrak{gl}_n}$ is generated by [Mac03, pg38, (3.1.6)]

$$q, \pi^\vee, T_0^\vee, \pi, T_0, T_1, \ldots, T_{n-1} \tag{3.1.1}$$

with relations

$$T_iT_jT_i = T_jT_iT_j, \quad T_iT_j = T_jT_i \quad \text{for } j \notin \{i-1, i+1\} \text{ and } i \in \{1, \ldots, n-1\}, \tag{3.1.2}$$

$$\pi T_0 \pi^{-1} = T_1, \quad \pi T_i \pi^{-1} = T_{i+1}, \quad \pi T_{n-1} \pi^{-1} = T_0, \tag{3.1.3}$$

$$\pi^\vee T_0^\vee (\pi^\vee)^{-1} = T_1, \quad \pi^\vee T_i (\pi^\vee)^{-1} = T_{i+1}, \quad \pi^\vee T_{n-1} (\pi^\vee)^{-1} = T_0^\vee, \tag{3.1.4}$$

for $i \in \{1, \ldots, n-1\}$, and

$$\pi^\vee \pi T_{n-1} = T_1^{-1} \pi \pi^\vee, \quad T_{n-1}^{-1} \cdots T_1^{-1} \pi (\pi^\vee)^{-1} = q (\pi^\vee)^{-1} \pi T_{n-1} \cdots T_1. \tag{3.1.5}$$

Define

$$Y^{\varepsilon_1^\vee} = \pi T_{n-1} \cdots T_1, \quad \text{and} \quad X^{\varepsilon_1} = \pi^\vee T_{n-1}^{-1} \cdots T_1^{-1}. \tag{3.1.6}$$

Theorem 3.1.1. *There is an involution* $\iota : \widetilde{B}_{\mathfrak{gl}_n} \to \widetilde{B}_{\mathfrak{gl}_n}$ *with*

$$\iota(q) = q^{-1}, \quad \iota(T_i) = T_i^{-1}, \text{ for } i \in \{1, \dots, n-1\},$$
$$\iota(T_0^\vee) = T_0^{-1}, \quad \iota(\pi) = \pi^\vee.$$

Proof. It is enough to check that ι preserves relation (3.1.2), (3.1.3), (3.1.4), (3.1.5). From the definition of ι, it preserves the relations (3.1.2), (3.1.3), (3.1.4). To show ι preserves (3.1.5), we note that ι is a homomorphism. Then

$$\iota(T_1\pi^\vee\pi = \pi\pi^\vee T_{n-1}^{-1}) \iff T_1^{-1}\pi^\vee\pi = \pi^\vee\pi T_{n-1}$$

and

$$\begin{aligned}\iota(T_{n-1}^{-1}\cdots T_1^{-1}\pi(\pi^\vee)^{-1} = q(\pi^\vee)^{-1}\pi T_{n-1}\cdots T_1) \iff& \\ T_{n-1}\cdots T_1\pi^\vee\pi^{-1} = q^{-1}\pi^{-1}\pi^\vee T_{n-1}^{-1}\cdots T_1^{-1}&.\end{aligned} \tag{3.1.7}$$

Hence, ι is an involution by definition. □

The second equality in Lemma 3.1.2 is the remark in [JV17, pg 17, Section 4.3] with π replaced with $\pi^\vee$. The third and fourth equality in Lemma 3.1.2 are the $\mathfrak{gl}_n$ case of second equality and third equality in [RY08, pg 5, (2.27)].

Lemma 3.1.2. *Let* $\varphi = \varepsilon_1 - \varepsilon_n$*, and let* $\varphi^\vee = \varepsilon_1^\vee - \varepsilon_n^\vee$*. Then*

$$\pi^n = Y^{\varepsilon_1^\vee + \cdots + \varepsilon_n^\vee} \quad \text{and} \quad (\pi^\vee)^n = X^{\varepsilon_1 + \cdots + \varepsilon_n}, \tag{3.1.8}$$
$$T_0T_{s_\varphi} = Y^{\varepsilon_1^\vee - \varepsilon_n^\vee} \quad \text{and} \quad (T_0^\vee)^{-1}T_{s_\varphi}^{-1} = X^{\varepsilon_1 - \varepsilon_n}, \tag{3.1.9}$$

where $T_{s_\varphi} = T_{n-1}\cdots T_1\cdots T_{n-1}$*.*

Proof. By the involution ι in Theorem 3.1.1, we only need to check one of the relations in (3.1.8) and (3.1.9). For the second relation in (3.1.8), it is better to illustrate an example rather than give a proof. For a complete proof one can change all the 5's in the following computation and obtain a proof in general case. Let $n = 5$. Using the first relation in (3.1.16) and the second relation in (3.1.4) obtain

$$\begin{aligned}X_1X_2X_3X_4X_5 &= \pi^\vee T_4^{-1}T_3^{-1}T_2^{-1}(T_1^{-1}T_1)\pi^\vee T_4^{-1}T_3^{-1}(T_2^{-1}T_2)T_1 \\ &\qquad \pi^\vee T_4^{-1}(T_3^{-1}T_3)T_2T_1\pi^\vee(T_4^{-1}T_4)T_3T_2T_1\pi^\vee \\ &= \pi^\vee T_4^{-1}T_3^{-1}T_2^{-1}\pi^\vee T_4^{-1}T_3^{-1}T_1\pi^\vee T_4^{-1}T_2T_1\pi^\vee T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^2T_3^{-1}T_2^{-1}T_1^{-1}T_4^{-1}T_3^{-1}T_1\pi^\vee T_4^{-1}T_2T_1\pi^\vee T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^2T_3^{-1}T_4^{-1}T_2^{-1}T_3^{-1}(T_1^{-1}T_1)\pi^\vee T_4^{-1}T_2T_1\pi^\vee T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^2T_3^{-1}T_4^{-1}T_2^{-1}T_3^{-1}\pi^\vee T_4^{-1}T_2T_1\pi^\vee T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^3T_2^{-1}T_3^{-1}T_1^{-1}T_2^{-1}T_4^{-1}T_2T_1\pi^\vee T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^3T_2^{-1}T_3^{-1}T_4^{-1}(T_1^{-1}T_2^{-1}T_2T_1)\pi^\vee T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^4T_1^{-1}T_2^{-1}T_3^{-1}T_3T_2T_1\pi^\vee \\ &= (\pi^\vee)^5.\end{aligned}$$

This gives the second relation in (3.1.8). To complete the proof, we remain to show the first relation in (3.1.9). Since the following computation

$$\begin{aligned} T_0^\vee = \pi T_{n-1}\pi^{-1} &= \pi T_{n-1}T_{n-2}\cdots T_1T_1^{-1}\cdots T_{n-2}^{-1}\pi^{-1} = Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-2}^{-1}\pi^{-1} \\ &= Y^{\varepsilon_1^\vee}\pi^{-1}T_2^{-1}\cdots T_{n-1}^{-1} = Y^{\varepsilon_1^\vee}\pi^{-1}T_1\cdots T_{n-1}T_{n-1}^{-1}\cdots T_1^{-1}T_2^{-1}\cdots T_{n-1}^{-1} \\ &= Y^{\varepsilon_1^\vee}Y^{-\varepsilon_n^\vee}T_{n-1}^{-1}\cdots T_1^{-1}\cdots T_{n-1}^{-1} = Y^{\varepsilon_1^\vee-\varepsilon_n^\vee}T_{n-1}^{-1}\cdots T_1^{-1}\cdots T_{n-1}^{-1}, \end{aligned}$$

gives

$$T_0^\vee = Y^{\varepsilon_1^\vee-\varepsilon_n^\vee}T_{n-1}^{-1}\cdots T_1^{-1}\cdots T_{n-1}^{-1} = Y^{\varepsilon_1^\vee-\varepsilon_n^\vee}(T_{s_\varphi})^{-1}. \qquad \square$$

Theorem 3.1.2 is discovered by Cherednik [Che95, Thm2.2], which is proved in [Mac03, 3.5-3.7] and in [Hai06, 4.13-4.18]. We give a proof of the duality theorem in type $\mathfrak{gl}_n$.

Theorem 3.1.3 (Duality)**.**

1. *The double affine Artin group $\widetilde{B}_{\mathfrak{gl}_n}$ can be generated by $q, \pi^\vee, T_0^\vee, T_1, \ldots, T_{n-1}$ and $Y^{\varepsilon_1^\vee}, \ldots, Y^{\varepsilon_n^\vee}$, and relations*

$$\pi^\vee T_0^\vee(\pi^\vee)^{-1} = T_1, \quad \pi^\vee T_i(\pi^\vee)^{-1} = T_{i+1}, \quad \pi^\vee T_{n-1}(\pi^\vee)^{-1} = T_0^\vee \ \textit{for } i \in \{1, \ldots, n-2\}, \tag{3.1.10}$$

$$q \in Z(\widetilde{B}_{\mathfrak{gl}_n}), \quad Y^{\varepsilon_k^\vee}Y^{\varepsilon_j^\vee} = Y^{\varepsilon_j^\vee}Y^{\varepsilon_k^\vee} \quad \textit{for } k, j \in \{1, \ldots, n\}, \tag{3.1.11}$$

$$Y^{\varepsilon_{i+1}^\vee} = T_i^{-1}Y^{\varepsilon_i^\vee}T_i^{-1}, \quad T_iY^{\varepsilon_j^\vee} = Y^{\varepsilon_j^\vee}T_i \quad \begin{array}{c}\textit{for } i \in \{1, \ldots, n-1\} \\ \textit{and } j \in \{1, \ldots, n\}\backslash\{i, i+1\},\end{array} \tag{3.1.12}$$

$$\pi^\vee Y^{\varepsilon_i^\vee}(\pi^\vee)^{-1} = Y^{\varepsilon_{i+1}^\vee} \ \textit{for } i \in \{1, \ldots, n-1\} \quad \textit{and} \quad \pi^\vee Y^{\varepsilon_n^\vee}(\pi^\vee)^{-1} = qY^{\varepsilon_1^\vee}. \tag{3.1.13}$$

2. *The group $\widetilde{B}_{\mathfrak{gl}_n}$ is also presented by generators $q, \pi, T_0, T_1, \ldots, T_{n-1}$ and $X^{\varepsilon_1}, \ldots, X^{\varepsilon_n}$ and relations*

$$\pi T_0(\pi)^{-1} = T_1, \quad \pi T_i\pi^{-1} = T_{i+1}, \quad \pi T_{n-1}\pi^{-1} = T_0 \ \textit{for } i \in \{1, \ldots, n-1\}, \tag{3.1.14}$$

$$q \in Z(\widetilde{B}_{\mathfrak{gl}_n}), \quad X^{\varepsilon_k}X^{\varepsilon_j} = X^{\varepsilon_j}X^{\varepsilon_k} \quad \textit{for } k, j \in \{1, \ldots, n\}, \tag{3.1.15}$$

$$X^{\varepsilon_i+1} = T_iX^{\varepsilon_i}T_i, \quad \textit{and} \quad T_iX^{\varepsilon_j} = X^{\varepsilon_j}T_i, \quad \begin{array}{c}\textit{for } i \in \{1, \ldots, n-2\} \\ \textit{and } j \neq i, i+1,\end{array} \tag{3.1.16}$$

$$\pi X^{\varepsilon_i}\pi^{-1} = X^{\varepsilon_{i+1}}, \quad \textit{for } i \in \{1, 2, \ldots, n-1\} \quad \textit{and} \quad \pi X^{\varepsilon_n}\pi^{-1} = q^{-1}X^{\varepsilon_1}. \tag{3.1.17}$$

Proof. Proof of part (1), relations in (3.1.10) is the same as in (3.1.4), and relations in (3.1.12) is the same as (3.1.6). It remains to show (3.1.11) and (3.1.13). The proof of relations in (3.1.13) is completed by the following computation. Using relation $\pi^\vee\pi T_{n-1} = T_1^{-1}\pi\pi^\vee$ and (3.1.6),

$$\begin{aligned} \pi^\vee Y^{\varepsilon_1^\vee} &= \pi^\vee(\pi T_{n-1}\cdots T_1) = T_1^{-1}\pi(\pi^\vee T_{n-2}\cdots T_1) = T_1^{-1}\pi T_{n-1}\cdots T_2\pi^\vee \\ &= T_1^{-1}(\pi T_{n-1}\cdots T_2T_1)T_1^{-1}\pi^\vee = (T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1})\pi^\vee = Y^{\varepsilon_{1+1}^\vee}\pi^\vee = Y^{\varepsilon_2^\vee}\pi^\vee. \end{aligned}$$

Using

$$\begin{aligned} Y^{\varepsilon_2^\vee} &= T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1}, \\ Y^{\varepsilon_3^\vee} &= T_2^{-1}T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1}T_2^{-1}, \\ Y^{\varepsilon_i^\vee} &= T_i^{-1}\cdots T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1}, \end{aligned} \tag{3.1.18}$$

and (3.1.4) gives

$$\begin{aligned}\pi^\vee Y^{\varepsilon_i^\vee}(\pi^\vee)^{-1} &= \pi^\vee(T_{i-1}^{-1}\cdots T_1^\vee Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{i-1}^{-1})(\pi^\vee)^{-1} = T_i^{-1}\cdots T_2^{-1}(\pi^\vee Y^{\varepsilon_1^\vee}(\pi^\vee)^{-1})T_2^{-1}\cdots T_i^{-1}\\ &= T_i^{-1}\cdots T_2^{-1}Y^{\varepsilon_2^\vee}T_2^{-1}\cdots T_i^{-1} = Y^{\varepsilon_{i+1}^\vee}\end{aligned}$$

for $i \in \{1, \ldots, n-1\}$.
Using (3.1.6), (3.1.18) and (3.1.5),

$$\begin{aligned}\pi^\vee Y^{\varepsilon_n^\vee}(\pi^\vee)^{-1} &= \pi^\vee(T_{n-1}^{-1}\cdots T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-1}^{-1})(\pi^\vee)^{-1}\\ &= \pi^\vee T_{n-1}^{-1}\cdots T_1^{-1}(\pi T_{n-1}\cdots T_1)T_1^{-1}\cdots T_{n-1}^{-1}(\pi^\vee)^{-1}\\ &= \pi^\vee(T_{n-1}^{-1}\cdots T_1^{-1}\pi(\pi^\vee)^{-1}) = \pi^\vee q(\pi^\vee)^{-1}\pi T_{n-1}\cdots T_1 = q\pi T_{n-1}\cdots T_1 = qY^{\varepsilon_1^\vee}.\end{aligned}$$

Since (3.1.18) and (3.1.6), $Y^{\varepsilon_i^\vee} = T_{i-1}^{-1}\cdots T_1^{-1}\pi T_{n-1}\cdots T_i$ and

$$\begin{aligned}Y^{\varepsilon_n^\vee} &= T_{n-1}^{-1}\cdots T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-1}^{-1} = T_{n-1}^{-1}\cdots T_1^{-1}(\pi T_{n-1}\cdots T_1)T_1^{-1}\cdots T_{n-1}^{-1}\\ &= T_{n-1}^{-1}\cdots T_1^{-1}\pi.\end{aligned}\tag{3.1.19}$$

Now, to show (1) $\implies$ (3.1.2), (3.1.3) and (3.1.4):
To show the first relation in (3.1.3), use (3.1.9),

$$\begin{aligned}\pi T_0\pi^{-1} &= Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-1}^{-1}(Y^{\varepsilon_1^\vee-\varepsilon_n^\vee}T_{n-1}^{-1}\cdots T_1^{-1}\cdots T_{n-1}^{-1})T_{n-1}\cdots T_1Y^{-\varepsilon_1^\vee}\\ &= Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-1}^{-1}Y^{-\varepsilon_n^\vee}Y^{\varepsilon_1^\vee}T_{n-1}^{-1}\cdots T_2^{-1}Y^{-\varepsilon_1^\vee}\\ &= Y^{\varepsilon_1^\vee}T_1^{-1}\cdots(T_{n-1}^{-1}T_{n-1}\cdots T_1Y^{-\varepsilon_1^\vee}T_1\cdots T_{n-1})Y^{\varepsilon_1^\vee}Y^{-\varepsilon_1^\vee}T_{n-1}^{-1}\cdots T_2^{-1}\\ &= Y^{\varepsilon_1^\vee}Y^{-\varepsilon_1^\vee}T_1 = T_1;\end{aligned}$$

To show the second relation in (3.1.3), use the definition (3.1.6) and the relation (3.1.1)

$$\begin{aligned}\pi T_i\pi^{-1} &= (Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-1}^{-1})T_i(T_{n-1}\cdots T_1Y^{-\varepsilon_1^\vee}) = Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{i+1}^{-1}T_iT_{i+1}\cdots T_1Y^{-\varepsilon_1^\vee}\\ &= Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{i+1}^{-1}T_iT_{i+1}T_i\cdots T_1Y^{-\varepsilon_1^\vee}T_{i+1}^{-1}T_{i+1}\\ &= Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{i-1}^{-1}T_{i+1}T_{i-1}\cdots T_1Y^{-\varepsilon_1^\vee}T_{i+1}^{-1}T_{i+1}\\ &= Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{i-1}^{-1}T_{i-1}\cdots T_1Y^{-\varepsilon_1^\vee}T_{i+1} = T_{i+1};\end{aligned}$$

To show the third relation in (3.1.3), use the second relation in (3.1.3) and (3.1.12),

$$\begin{aligned}\pi T_{n-1}\pi^{-1} &= \pi T_{n-1}(T_{n-2}\cdots T_1T_1^{-1}\cdots T_{n-2}^{-1})\pi = Y^{\varepsilon_1^\vee}(T_1^{-1}\cdots T_{n-2}^{-1}T_{n-2}\cdots T_1)T_1^{-1}\cdots T_{n-2}^{-1}\pi^{-1}\\ &= Y^{\varepsilon_1^\vee}(T_1^{-1}\cdots T_{n-2}^{-1}\pi^{-1}) = Y^{-\varepsilon_1^\vee}\pi^{-1}T_2^{-1}\cdots T_{n-1}^{-1} = Y^{\varepsilon_1^\vee}(T_{n-1}T_{n-2}\cdots T_1Y^{-\varepsilon_1^\vee})T_2^\vee\cdots T_{n-1}^{-1}\\ &= Y^{\varepsilon_1^\vee}\pi^{-1}T_1T_2\cdots T_{n-1}T_{n-1}^{-1}\cdots T_1^{-1}T_2^{-1}\cdots T_{n-1}^{-1}\\ &= Y^{\varepsilon_1^\vee}(Y^{-\varepsilon_n^\vee}T_{n-1}^{-1}\cdots T_1^{-1})T_1T_2\cdots T_{n-1}T_{n-1}^{-1}\cdots T_1^{-1}T_2^{-1}\cdots T_{n-1}^{-1}\\ &= Y^{\varepsilon_1^\vee-\varepsilon_n^\vee}T_{n-1}^{-1}\cdots T_1^{-1}T_2^{-1}\cdots T_{n-1}^{-1} = T_0;\end{aligned}$$

To show the first relation in (3.1.5), use (3.1.6), (3.1.10), (3.1.12) and (3.1.13)

$$\begin{aligned}T_1\pi^\vee\pi &= T_1\pi^\vee Y^{\varepsilon_1^\vee}(T_1^{-1}\cdots T_{n-1}^{-1} = T_1Y^{\varepsilon_2^\vee}\pi^\vee T_1^{-1}\cdots T_{n-1}^{-1})\\ &= T_1(T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1})\pi^\vee T_1^{-1}\cdots T_{n-1}^{-1} = Y^{\varepsilon_1^\vee}T_1^{-1}\pi^\vee T_1^{-1}\cdots T_{n-1}^{-1}\\ &= (\pi T_{n-1}\cdots T_1)T_1^{-1}\pi^\vee T_1^{-1}\cdots T_{n-1}^{-1} = \pi(T_{n-1}\cdots T_2\pi^\vee)T_1^{-1}\cdots T_{n-1}^{-1}\\ &= \pi\pi^\vee(T_{n-2}\cdots T_1T_1^{-1}\cdots T_{n-1}^{-1}) = \pi\pi^\vee T_{n-1}^{-1};\end{aligned}$$

To show the second relation in (3.1.5), use (3.1.6), (3.1.13)

$$\begin{aligned}T_{n-1}^{-1}\cdots T_1^{-1}\pi(\pi^\vee)^{-1} &= T_{n-1}^{-1}\cdots T_1^{-1}\pi(T_{n-1}\cdots T_1T_1^{-1}\cdots T_{n-1}^{-1})(\pi^\vee)^{-1}\\ &= (T_{n-1}^{-1}\cdots T_1^{-1}Y^{\varepsilon_1^\vee}T_1^{-1}\cdots T_{n-1}^{-1})(\pi^\vee)^{-1} = Y^{\varepsilon_n^\vee}(\pi^\vee)^{-1}\\ &= (\pi^\vee)^{-1}(\pi^\vee Y^{\varepsilon_n^\vee})(\pi^\vee)^{-1} = (\pi^\vee)^{-1}qY^{\varepsilon_1^\vee} = q(\pi^\vee)^{-1}Y^{\varepsilon_1^\vee}\\ &= q(\pi^\vee)^{-1}\pi T_{n-1}\cdots T_1.\end{aligned}$$

The proof of part (2) follows from the involution ι defined in Theorem 5.1. ☐

It remains to complete the relation (3.1.9), this will be formulated into the following theorem.

Theorem 3.1.4. *Let $n \in \mathbb{Z}_{\geq 2}$, $k \in \{1, \ldots, n-1\}$, and let $\ell \in \{1, \ldots, n\}\backslash\{k, k+1\}$. Then $X_kX_{k+1} = X_{k+1}X_k$ and $X_kX_\ell = X_\ell X_k$.*

Proof. This can be obtained by either a direct computation using the relations stated in part 2 of Theorem 3.1.3 or the topological properties of the double affine Artin groups. For more details, readers may refer to either [LR97, pg 9, Cor1.6] or [JV17, pg 17, remarks after Def4.4]. ☐

The two different presentations in Theorem 3.1.3 are stated and proved for arbitrary affine root systems in [Mac03, pg 38, (3.1.6)] and [Mac03, pg42, (3.3.1)]. The proof in [Mac03,pg 38,(3.1.6)] is different from the proof provided in Theorem 3.1.3 and the proof here only uses the definition of double affine Artin group provided in (3.1.1).

3.2 Type $\mathfrak{gl}_n$ double affine Hecke algebra(DAHA)

The double affine Hecke algebra $\widetilde{H}_{\mathfrak{gl}_n}$ over $\mathbb{C}$ is the quotient of the group algebra of $\widetilde{B}_{\mathfrak{gl}_n}$ by the relations

$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0, \quad \text{for } i \in \{1, \ldots, n-1\}. \tag{3.2.1}$$

We remark here that there are two other ways of defining the T_i relation:
In [JV17, pg18, Def 4.6], the relations of T_i are

$$(T_i - t)(T_i + t^{-1}) = 0 \quad \text{for } i \in \{1, \ldots, n-1\}; \tag{3.2.2}$$

In [HHL06, pg 3, (4)] and [BW19, pg5, (2.1)], the relations of T_i are

$$(T_i - t)(T_i + 1) = 0 \quad \text{for } i \in \{1, \ldots, n-1\}. \tag{3.2.3}$$

Let $w = s_{i_1}\cdots s_{i_k} \in W_{\mathfrak{gl}_n}$ be a reduced word such that $i_1, \ldots, i_k \in \{\pi, 1, \ldots, n-1\}$ and $s_\pi = \pi$. Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ and $\beta = (b_1, \ldots, b_n) \in \mathbb{Z}^n = \mathbb{Z}\varepsilon_1^\vee + \cdots \mathbb{Z}\varepsilon_n^\vee$. Then

$$T_w = T_{i_1}\cdots T_{i_k}, \quad X^\alpha = X_1^{a_1}\cdots X_n^{a_n}, \quad \text{and} \quad Y^\beta = Y_1^{b_1}\cdots Y_n^{b_n}, \tag{3.2.4}$$

where $T_\pi = \pi^\vee$. The **affine Hecke algebra** [Ch04, pg53, (8.3)]

$$\widetilde{H}_{\text{aff}} = \langle T_0, T_1, \ldots, T_n, \pi^{\pm 1}\rangle$$

is the subalgebra of $\widetilde{H}_{\mathfrak{gl}_n}$ generated by $T_0, T_1, \ldots, T_n, \pi^{\pm 1}$. The double affine Hecke algebra also has two distinct subalgebras

$$H_1 = \langle T_1, \ldots, T_{n-1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}\rangle, \quad H_2 = \langle T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}\rangle.$$

Proposition 3.2.1. *Let $Y^d = q^{-1}$. Then the set $\{T_\sigma \mid \sigma \in W_{\mathfrak{gl}_n}\}$ is a $\mathbb{C}$-basis of the affine Hecke algebra $\widetilde{H}_{\mathrm{aff}}$ and the set $\{T_\sigma Y^\beta \mid \sigma \in W_{\mathfrak{gl}_n}, \beta \in \mathbb{Z}^n\}$ forms a $\mathbb{C}$-basis of the double affine Hecke algebra $\widetilde{H}_{\mathfrak{gl}_n}$.*

Proof. The proof that the set $\{T_w \mid w \in W_{\mathfrak{gl}_n}\}$ forms a $\mathbb{C}$-basis of $\widetilde{H}_{\mathrm{aff}}$ is provided in [Mac03, pg 56, (4.1.3)] and the proof that the set $\{T_w Y^\beta \mid w \in W_{\mathfrak{gl}_n}, \beta \in \mathbb{Z}^n\}$ forms a $\mathbb{C}$-basis of $\widetilde{H}_{\mathfrak{d}_{<n}}$ is provided in [Mac03, (4.2.7)]. $\square$

For $\mu \in \mathbb{Z}^n$, $w \in S_n$ and $w' \in W_{\mathfrak{gl}_n}$ such that $w' = t_\mu w \in W_{\mathfrak{gl}_n}$ define

$$X^{t_\mu w} = X^\mu T_w, \quad \text{where } T_w = T_{i_1} \cdots T_{i_\ell} \quad \text{if } w = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word.}$$

Then

$$X^{\mu} T_{ws_i} = \begin{cases} X^\mu T_w T_i, & \text{if } w < ws_i, \\ X^\mu T_w T_i^{-1}, & \text{if } ws_i < w, \end{cases} \tag{3.2.5}$$

where the order is determined by the number of inversions in an affine permutation. Moreover,

$$X^\mu T_w \pi^\vee = X^\mu X_{w^{-1}(1)} T_{ws_1 \cdots s_{n-1}}. \tag{3.2.6}$$

3.3 Intertwiners of $\widetilde{H}_{\mathfrak{gl}_n}$

Define the intertwiners $\tau_i^\vee$ and $\tau_\pi^\vee$ of double affine Hecke algebra $\widetilde{H}_{\mathfrak{gl}_n}$ to be

$$\tau_\pi^\vee = \pi^\vee \quad \text{and} \quad \tau_i^\vee = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}}, \quad \text{for } i \in \{1, \ldots, n-1\}. \tag{3.3.1}$$

Lemma 3.3.1. *Let τ_i be defined in* (3.3.1). *Then*

$$\tau_i^\vee = T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)(Y_i^{-1} Y_{i+1})}{1 - Y_i^{-1} Y_{i+1}}. \tag{3.3.2}$$

Proof. By (3.2.1), we obtain

$$T_i = T_i^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}). \tag{3.3.3}$$

By (3.3.1) and (3.3.3), we have

$$\begin{aligned} \tau_i^\vee &= \left(T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}}\right) = \left(T_i^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}}\right) \\ &= T_i^{-1} + \frac{t^{-\frac{1}{2}}(t-1)(1 - Y_i^{-1} Y_{i+1}) + t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}} = T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)(Y_i^{-1} Y_{i+1} - 1 + 1)}{1 - Y_i^{-1} Y_{i+1}} \\ &= T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)(Y_i^{-1} Y_{i+1})}{1 - Y_i^{-1} Y_{i+1}}. \end{aligned}$$

$\square$

Proposition 3.3.2 explains the interwining property of $\tau_i^\vee$ and $\tau_\pi^\vee$.

Proposition 3.3.2. *Let $\tau_i^\vee$, $\tau_\pi^\vee$ be defined as in* (3.3.1)*, and let $Y_i = Y^{\varepsilon_i^\vee}$ for $i \in \{1, \dots, n\}$. Then*

$$Y_i\tau_\pi^\vee = \tau_\pi^\vee Y_{i-1}, \quad Y_{i+1}\tau_i^\vee = \tau_i^\vee Y_i, \quad Y_i\tau_i^\vee = \tau_i^\vee Y_{i+1}, \text{ and } \quad Y_k\tau_i^\vee = \tau_i^\vee Y_k, \tag{3.3.4}$$

for $i \in \{2, \dots, n-1\}$ and $k \in \{1, \dots, n\}\backslash\{i, i+1\}$.

Proof. The identity $Y_i\tau_\pi^\vee = \tau_\pi^\vee Y_{i-1}$ follows from $\tau_\pi^\vee = \pi^\vee$ defined in (3.3.1) and the first relation in (3.1.13).

To show the second identity in (3.3.4): Using the first relation in (3.1.12) and (3.3.2),

$$\begin{aligned}\tau_i^\vee Y_i &= \left(T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)(Y_i^{-1}Y_{i+1})}{1-Y_i^{-1}Y_{i+1}}\right)Y_i = T_i^{-1}Y_i + \frac{t^{-\frac{1}{2}}(1-t)(Y_i^{-1}Y_{i+1})}{1-Y_i^{-1}Y_{i+1}}Y_i \\ &= Y_{i+1}T_i + \frac{t^{-\frac{1}{2}}(1-t)Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} = Y_{i+1}\left(T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right) = Y_{i+1}\tau_i^\vee.\end{aligned}$$

To show the third identity in (3.3.4): Using the first relation in (3.1.12) and (3.3.2),

$$\begin{aligned}\tau_i^\vee Y_{i+1} &= \left(T_i^\vee + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right)Y_{i+1} = T_i^\vee Y_{i+1} + \frac{t^{-\frac{1}{2}}(1-t)Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} \\ &= Y_i(T_i^\vee)^{-1} + \frac{t^{-\frac{1}{2}}(1-t)Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} = Y_i\left((T_i^\vee)^{-1} + \frac{t^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}\right) = Y_i\tau_i^\vee.\end{aligned}$$

To show the fourth equality in (3.3.4): Let $k \in \{1, \dots, n\}\backslash\{i, i+1\}$. Using the second relation in (3.1.12) and (3.1.11), we have

$$T_iY_k = Y_kT_i, \quad Y_iY_k = Y_kY_i \quad \text{and} \quad Y_{i+1}Y_k = Y_kY_{i+1}.$$

Then

$$\tau_i^\vee Y_k = \left(T_i^\vee + \frac{(1-t)}{1-Y_i^{-1}Y_{i+1}}\right)Y_k = Y_k\left(T_i^\vee + \frac{(1-t)}{1-Y_i^{-1}Y_{i+1}}\right) = Y_k\tau_i^\vee. \qquad \square$$

3.4 Type $\mathfrak{gl}_n$ polynomial representation of DAHA

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Define

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

The Laurent polynomial ring $\mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ and the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ has $\mathbb{C}$-basis $\{x^\mu \mid \mu \in \mathbb{Z}^n\}$ and $\mathbb{C}$-basis $\{x^\mu \mid \mu \in \mathbb{Z}^n_{\geq 0}\}$ respectively. The actions of $\sigma \in S_n$ on $\mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ and $\mathbb{C}[x_1, \dots, x_n]$ are defined by

$$\begin{aligned}\sigma : \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}] &\to \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}] \\ x_i^{\pm1} &\mapsto x_{\sigma(i)}^{\pm1},\end{aligned}$$

and

$$\begin{aligned}\sigma : \mathbb{C}[x_1, \dots, x_n] &\to \mathbb{C}[x_1, \dots, x_n] \\ x_i &\mapsto x_{\sigma(i)},\end{aligned}$$

for $i \in \{1, \dots, n\}$. Recall from (2.2.2) the action of S_n on the lattice $\mathfrak{a}_{\mathbb{Z}} = \mathbb{Z}^n$. Then

$$wx^\mu = x^{w\mu} \quad \text{for} \quad w \in S_n \text{ and } \mu \in \mathbb{Z}^n.$$

Define the action of the **Demazure-Lusztig operators** [Che95, pg 63, (9.11)] $T_1, \dots, T_{n-1}$ on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by

$$T_i = t^{-\frac{1}{2}}\left(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\right), \quad \text{for } i \in \{1, \dots, n\}. \tag{3.4.1}$$

Define the action of the length zero elements π and $\pi^\vee$ on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by

$$\pi = s_1 s_2 \cdots s_{n-1} T_{q^{-1}, x_n}, \quad \pi^\vee = x_1 T_1 \cdots T_{n-1}.$$

The **difference-trigonometric Dunkl** operators [Che95, pg 38, After (5.1)] are

$$Y_1 = \pi T_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T^{-1}, \quad Y_3 = T_2^{-1} Y_2 T_2^{-1}, \dots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_n^{-1}.$$

The polynomial representation of the double affine Hecke algebra $\widetilde{H}_{\mathfrak{gl}_n}$ is

$$\mathrm{Ind}_{\widetilde{H}_{\mathrm{aff}}}^{\widetilde{H}_{\mathfrak{gl}_n}}(\mathbf{1}) = \mathbb{C}[q^{\pm 1}, t^{\pm \frac{1}{2}}][x^{\pm \varepsilon_1}, \dots, x^{\pm \varepsilon_n}]$$

with actions

$$T_i \mathbf{1} = t^{\frac{1}{2}} \mathbf{1} \quad \text{and} \quad \pi \mathbf{1} = \mathbf{1} \quad \text{for } i \in \{1, \dots, n-1\}. \tag{3.4.2}$$

The action of Demazure-Lusztig operators T_i's on the polynomial representation defined in [HHL06, pg3, (4)] and [JV17, pg17, (4.3)] is

$$T_i \mathbf{1} = t\mathbf{1} \quad \text{for } i \in \{1, \dots, n-1\},$$

which differ from (3.4.2) by a factor of $t^{\frac{1}{2}}$. More importantly, the coefficient of the leading term X^μ in the non-symmetric Macdonald polynomials E_μ^{RY} defined in [RY08, pg8, (3.3)] differed from E_μ^{HHL} defined in [HHL06, pg 4, first sentence] by a factor of $t^{-\frac{1}{2}\ell(w_\mu)}$, where w_μ is defined in (3.5.3).

Define T_{q^{-1}, x_n} to be the shifting operator [Mac03, Ch. VI(3.1)] given by

$$T_{q^{-1}, x_n} h(x_1, \dots, x_n) = h(x_1, \dots, q^{-1} x_n), \tag{3.4.3}$$

where $h(x_1, \dots, x_n)$ is a polynomial in $\mathbb{C}[q^{\pm 1}, t^{\pm \frac{1}{2}}][x^{\pm \varepsilon_1}, \dots, x^{\pm \varepsilon_n}]$.

The operators $Y^{\varepsilon_i^\vee}$ act on $\mathbf{1}$ by

$$Y^{\varepsilon_i^\vee} \mathbf{1} = T_{i-1}^{-1} \cdots T_1^{-1} Y^{\varepsilon_1^\vee} T_1^{-1} \cdots T_{i-1}^{-1} \mathbf{1} = t^{\frac{1}{2}(n-1)} t^{-\frac{1}{2}(i-1)} t^{-\frac{1}{2}(i-1)} \mathbf{1} = t^{\frac{1}{2}(n-1)-i+1} \mathbf{1}, \tag{3.4.4}$$

for $i \in \{1, \dots, n\}$. Moreover, we set $X_i = X^{\varepsilon_i}$ and define

$$X_i \mathbf{1} = x_i, \quad \text{for } i \in \{1, \dots, n\}.$$

Proposition 3.4.1. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, and let $X_i = X^{\varepsilon_i}$ for $i \in \{1, \dots, n\}$. Define $X^\mu = X^{\mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n}$. Then,*

$$\pi X^\mu = q^{-\mu_n} X^{\gamma\mu} \pi,$$

where γ in cyclic notation is $(12 \cdots n) = s_{n-1} \cdots s_1$, and

$$T_i X^\mu = (s_i X^\mu) T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_i X_{i+1}^{-1}}(1 - s_i) X^\mu, \text{ for } i \in \{1, \dots, n-1\}. \tag{3.4.5}$$

Proof. Since $T_iX_iT_i = X_{i+1}$ and $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$,

$$\begin{aligned}
T_iX_i = X_{i+1}T_i^{-1} = X_{i+1}(T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})) = X^{s_i\varepsilon_i}T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{X_i - X_{i+1}}{1 - X_iX_{i+1}^{-1}} \\
= X^{s_i\varepsilon_i}T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X_i
\end{aligned}$$

and

$$\begin{aligned}
T_iX_{i+1} &= T_i^2X_iT_i = ((t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1)X_iT_i \\
&= X_iT_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_iX_iT_i = X_iT_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_{i+1} \\
&= X^{s_i\varepsilon_{i+1}}T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{X_{i+1} - X_i}{1 - X_iX_{i+1}^{-1}} = (s_iX_{i+1})T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{X_{i+1} - X_i}{1 - X_iX_{i+1}^{-1}} \\
&= (s_iX_{i+1})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X_{i+1}.
\end{aligned}$$

For $j \notin \{i, i+1\}$,

$$T_iX_j = X_jT_i = (s_iX_j)T_i + 0 = (s_iX_j)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X_j.$$

Let

$$T_iX^\mu = (s_iX^\mu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^\mu \quad \text{and} \quad T_iX^\nu = (s_iX^\nu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^\nu.$$

Since

$$\begin{aligned}
T_iX^{\mu+\nu} &= T_iX^\mu X^\nu = ((s_iX^\mu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^\mu)X^\nu \\
&= ((s_iX^\mu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^\mu - X^{s_i\mu}))X^\nu \\
&= (s_iX^\mu)T_iX^\nu + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^\mu X^{nu} - X^{s_i\mu}X^\nu) \\
&= (s_iX^\mu)\left((s_iX^\nu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^\nu\right) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^\mu X^\nu - X^{s_i\mu}X^\nu) \\
&= X^{s_i\mu+s_i\nu}T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(s_iX^\mu)(1 - s_i)X^\nu + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^\mu X^\nu - X^{s_i\mu}X^\nu) \\
&= X^{s_i(\mu+\nu)} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^{s_i\mu}X^\nu - X^{s_i\mu}X^{s_i\nu}) + X^\mu X^\nu - X^{s_i\mu}X^\nu) \\
&= s_iX^{\mu+\nu} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^\mu X^\nu - X^{s_i\mu+s_i\nu}) = s_iX^{\mu+\nu} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(X^{\mu+\nu} - X^{s_i(\mu+\nu)}) \\
&= (s_iX^{\mu+\nu})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^{\mu+\nu},
\end{aligned}$$

then

$$T_iX^{\mu+\nu} = (s_iX^{\mu+\nu})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^{\mu+\nu}.$$

Since $\mathbb{Z}^n$ is a free $\mathbb{Z}$-module with basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$, this shows that

$$T_iX^{\mu} = (s_iX^{\mu})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^{\mu}, \qquad \text{for} \qquad \mu \in \mathbb{Z}^n.$$

Since $\pi X_n \pi^{-1} = q^{-1}X_1$ and $\pi X_i \pi^{-1} = X_{i+1}$ then

$$\begin{aligned} \pi X^{(\mu_1,\ldots,\mu_n)} &= \pi X_1^{\mu_1} \cdots X_n^{\mu_n} = X_2^{\mu_1} \cdots X_n^{\mu_{n-1}} \pi X_n^{\mu_n} = q^{-\mu_n} X_2^{\mu_1} \cdots X_n^{\mu_{n-1}} X_1^{\mu_n} \pi \\ &= q^{-\mu_n}(X_1^{\mu_n} X_2^{\mu_1} \cdots X_n^{\mu_n - 1})\pi = q^{-\mu_n}(X^{(\mu_n,\mu_1,\ldots,\mu_{n-1})})\pi = q^{-\mu_n} X^{\gamma\mu}\pi. \qquad \square \end{aligned}$$

We note that in [HHL06, pg 3, (7)], the action of T_i on the polynomial representation is defined by

$$T_iX^{\mu} = (s_iX^{\mu})T_i + \frac{t - 1}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^{\mu} \tag{3.4.6}$$

with action of T_i on the trivial representation being

$$T_i\mathbf{1} = t\mathbf{1}, \qquad \text{for } i \in \{1, \ldots, n-1\}. \tag{3.4.7}$$

Proposition 3.4.2. *The operators π and T_i for $i \in \{1, \ldots, n-1\}$ on the polynomial representation are*

$$\pi = \gamma T_{q^{-1},X_n} \qquad \text{and} \qquad T_i = t^{-\frac{1}{2}}\left(t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1 - s_i)\right),$$

where γ in cyclic notation is $(12\cdots n) = s_{n-1}\cdots s_1$.

Proof. Using (3.4.5) to expand the term T_iX^{μ} gives

$$\begin{aligned} T_iX^{\mu}\mathbf{1} &= (s_iX^{\mu})t^{\frac{1}{2}} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^{\mu}\mathbf{1} \\ &= \left(t^{\frac{1}{2}} - t^{\frac{1}{2}}(1 - s_i) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)\right)X^{\mu}\mathbf{1} \\ &= \left(t^{\frac{1}{2}} + \frac{1}{1 - X_iX_{i+1}^{-1}}(t^{\frac{1}{2}}(1 - s_i)(X_iX_{i+1}^{-1} - 1) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(1 - s_i))\right)X^{\mu}\mathbf{1} \\ &= \left(t^{\frac{1}{2}} + \frac{1}{1 - X_iX_{i+1}^{-1}}((1 - s_i)(t^{\frac{1}{2}}X_iX_{i+1}^{-1} - t^{-\frac{1}{2}})\right)X^{\mu}\mathbf{1} \\ &= \left(t^{\frac{1}{2}} - \frac{1}{X_{i+1} - X_i}(X_{i+1}t^{-\frac{1}{2}} - t^{\frac{1}{2}}X_i)(1 - s_i)\right)X^{\mu}\mathbf{1} \\ &= t^{-\frac{1}{2}}\left(t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1 - s_i)\right)X^{\mu}\mathbf{1} \end{aligned}$$

Because $T_{q^{-1},x_n}X^{\mu} = q^{-\mu_n}X^{\mu}$, $\pi X^{\mu} = (\gamma T_{q^{-1},x_n}X^{\mu})\mathbf{1}$. $\square$

Remark 3.4.3. Repeat the computation in Proposition 3.4.2, with action defined in (3.4.7). The Demazure-Lusztig operators defined in (3.4.6) acts on the polynomial representation by

$$T_i = \left(t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1 - s_i)\right). \tag{3.4.8}$$

3.5 Inductive constructions

In this section, we discuss two important inductive constructions which will be used to generate the Haglund-Hamain-Loehr formula in (4.3.1) and the type $\mathfrak{gl}_n$ Ram-Yip formula in (5.4.4). We prove these two inductive constructions for type $\mathfrak{gl}_n$ Macdonald polynomials are equivalent.

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$. The elements π and s_i act on μ by

$$\pi(\mu_1, \dots, \mu_n) = (\mu_n + 1, \mu_1, \dots, \mu_{n-1}) \quad \text{and} \tag{3.5.1}$$

$$s_i(\mu_1, \dots, \mu_n) = (\mu_1, \dots, \mu_{i+1}, \mu_i, \dots, \mu_n), \quad \text{for } i \in \{1, \dots, n-1\}. \tag{3.5.2}$$

Define

$$w_\mu \text{ to be the minimal length element of } S_n \text{ such that } w_\mu^{-1}\mu \text{ is anti-dominant.} \tag{3.5.3}$$

Example 3.5.1. If $\mu = (1,0,0)$, then $s_2s_1(1,0,0) = s_2(0,1,0) = (0,0,1)$. So $w_{(1,0,0)}^{-1} = s_2s_1$.

Lemma 3.5.1. *The element* $w_\mu^{-1} : \{1, \dots, n\} \to \{1, \dots, n\}$ *can be determined by*

$$\begin{aligned} w_\mu^{-1}(i) = 1 &+ \#\{j \in \{1, \dots, n\} \mid j < i \text{ and } \mu_j \le \mu_i\} \\ &+ \#\{j \in \{1, \dots, n\} \mid j > i \text{ and } \mu_j < \mu_i\}, \end{aligned}$$

for $i \in \{1, \dots, n\}$.

Proof. Assume $w_\mu^{-1}\mu$ is anti-dominant i.e.,

$$\text{if } \mu = (\mu_1, \dots, \mu_n), \text{ then } w_\mu^{-1}\mu = (\mu_{w_\mu^{-1}(1)}, \mu_{w_\mu^{-1}(2)}, \dots, \mu_{w_\mu^{-1}(n)})$$

with $\mu_{w_\mu^{-1}(1)} \le \mu_{w_\mu^{-1}(2)} \le \cdots \mu_{w_\mu^{-1}(n)}$. Note that if $j < i$ and $\mu_j > \mu_i$, then

$$w_\mu^{-1}(j) > w_\mu^{-1}(i).$$

Moreover, if $j > i$ and $\mu_i \le \mu_j$, then

$$w_\mu^{-1}(j) > w_\mu^{-1}(i).$$

So

$$\begin{aligned} w_\mu^{-1}(i) &= n - (\#\{j < i \mid \mu_j > \mu_i\} + \#\{j > i \mid \mu_i \le \mu_j\}) \\ &= n - \#\{j < i \mid \mu_j > \mu_i\} - \#\{j > i \mid \mu_i \le \mu_j\} \\ &= \#\{j < i \mid \mu_j \le \mu_i\} + \#\{j > i \mid \mu_j < \mu_i\} + 1. \end{aligned}$$

□

Recall from (2.2.9) and (2.2.10) that

$$t_{\varepsilon_1^\vee} = \pi s_{n-1} \cdots s_2 s_1, \quad \text{and} \quad t_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i. \tag{3.5.4}$$

Let $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$, and let $t_\nu = t_{\nu_1\varepsilon_1^\vee} t_{\nu_2\varepsilon_2^\vee} \cdots t_{\nu_n\varepsilon_n^\vee}$. By (3.5.4), the action of t_ν on $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ is given by

$$t_\nu(\mu_1, \dots, \mu_n) = (\mu_1 + \nu_1, \dots, \mu_n + \nu_n). \tag{3.5.5}$$

Let

$$m_\mu \text{ be the minimal length element of the coset } t_\mu S_n \text{ in } W_{\mathfrak{gl}_n}. \tag{3.5.6}$$

From [Mac03, pg 23, (2.4.3)] we know that

$$m_\mu = t_\mu w_\mu, \quad \text{where } w_\mu \text{ is defined in (3.5.3).} \tag{3.5.7}$$

Example 3.5.2. For example, let $\mu = (0,4,1,5,4)$. Then

$$w_\mu^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} \qquad \text{and} \qquad w_\mu^{-1}\mu = (0,1,4,4,5),$$

which gives

$$m_\mu = t_\mu w_\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1+5\cdot 0 & 3+5\cdot 4 & 2+5\cdot 1 & 5+5\cdot 5 & 4+5\cdot 4 \end{pmatrix}.$$

Let $\mu \in \mathbb{Z}^n$. Using actions of π and s_i defined in (3.5.1) and (3.5.2), write

$$\mu \xrightarrow{\pi} \nu \quad \text{if} \quad \mu = \pi\nu \quad \text{and}$$
$$\mu \xrightarrow{s_i} \nu \quad \text{if} \quad \mu = s_i\nu.$$

Lemma 3.5.2. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$. One way to obtain the element m_μ is to* ***unwind*** *μ in the following steps:*

1. *If $\mu_1 > 0$, then write $\mu \xrightarrow{\pi} \pi^{-1}\mu$;*
2. *If $\mu_1 = 0, \mu_2 = 0, \ldots, \mu_i = 0$ and $\mu_{i+1} \neq 0$, then write $\mu \xrightarrow{s_i} s_i\mu$.*

repeat item (1) and (2) until $\pi^{-1}\mu = (0,0,\ldots,0)$. Record the arrow and the action on the arrow at each step, we obtain the following sequence

$$\mu \xrightarrow{w_1} \mu' \xrightarrow{w_2} \cdots \xrightarrow{w_m} (0,0,\ldots,0),$$

where $w_i \in \{\pi, s_1, \ldots, s_{n-1}\}$. Then

$$m_\mu = w_1 w_2 \cdots w_m.$$

Proof. Recall from (3.5.6) that m_μ is the minimal length element in the coset $t_\mu S_n$. We need to show two conditions: Firstly the sequence $w_1 w_2, \cdots w_m$ obtained by unwinding μ is in the left coset $t_\mu S_n$. Secondly, this sequence is minimal in $t_\mu S_n$.
For the first part, fix $\mu \in \mathbb{Z}^n_{\geq 0}$ and let

$$w_1 w_2 \cdots w_m (0,0,\cdots,0) = \mu, \tag{3.5.8}$$

where $w_i \in W_{\mathfrak{gl}_n}$ for $i \in \{1,\ldots,m\}$. Since $w_1 w_2 \cdots w_m \in W_{\mathfrak{gl}_n}$, by (2.2.11) there exists $\mu' \in \mathbb{Z}^n_{\geq 0}$ such that $w_1 w_2 \cdots w_m$ can be written uniquely as

$$w_1 w_2 \cdots w_m = t_{\mu'} w, \qquad \text{for some } w \in S_n.$$

By (3.5.8) and the action defined in (3.5.4),

$$\mu = w_1 w_2 \cdots w_m (0,0,\ldots,0) = t_{\mu'} w(0,0,\ldots,0) = \mu'.$$

Hence, $w_1 w_2 \ldots w_m = t_\mu w \in t_\mu S_n$.
For the second part, let $w' \in S_n$ such that

$$t_\mu w' \neq w_1 w_2 \cdots w_m.$$

Since $t_\mu w'$ and $w_1 w_2 \cdots w_m$ are both in the same left coset $t_\mu S_n$, there exists $w'' \in S_n$ such that

$$t_\mu w' = w_1 w_2 \cdots w_m w''.$$

By construction, the word $w_1 w_2 \cdots w_m$ is reduced and therefore

$$\ell(w_1 w_2 \cdots w_m w'') \geq \ell(w_1 w_2 \cdots w_m).$$

Hence, $w_1 w_2 \cdots w_m$ is minimal. □

Example 3.5.3. For $\mu = (2,1,0)$, using item 1 and 2 in Lemma 3.5.2 to unwind μ, we obtain

$$(2,1,0) \xrightarrow{\pi} (1,0,1) \xrightarrow{\pi} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi} (0,0,0),$$

which gives

$$m_{(2,1,0)} = \pi\pi s_1 \pi.$$

Using (2.2.9) and (2.2.12),

$$\begin{aligned} m_{(2,1,0)} &= \pi\pi s_1\pi = (t_{\varepsilon_1^\vee} s_1 s_2)(t_{\varepsilon_1^\vee} s_1 s_2) s_1 (t_{\varepsilon_1^\vee} s_1 s_2) = t_{\varepsilon_1^\vee}(s_1 t_{\varepsilon_1^\vee}) s_2 s_1 s_2 s_1 t_{\varepsilon_1^\vee} s_1 s_2 \\ &= t_{\varepsilon_1^\vee} t_{\varepsilon_2^\vee}(s_1 s_2 s_1) s_2 s_1 t_{\varepsilon_1^\vee} s_1 s_2 = t_{\varepsilon_1^\vee+\varepsilon_2^\vee} s_2 s_1 (s_2 s_2) s_1 t_{\varepsilon_1^\vee} s_1 s_2 = t_{\varepsilon_1^\vee+\varepsilon_2^\vee}(s_2 t_{\varepsilon_1^\vee}) s_1 s_2 \\ &= t_{2\varepsilon_1^\vee+\varepsilon_2^\vee} s_2 s_1 s_2. \end{aligned}$$

Since the action of s_i in (3.5.2), we obtain

$$s_2 s_1 s_2 (2,1,0) = s_2 s_1 (2,0,1) = s_2(0,2,1) = (0,1,2).$$

We checked that $w_{(2,1,0)} = w_{(2,1,0)}^{-1} = s_2 s_1 s_2$.

Example 3.5.4. For $s_1\mu = s_1(2,1,0) = (1,2,0)$, unwind $(1,2,0)$, we obtain

$$(1,2,0) \xrightarrow{\pi} (2,0,0) \xrightarrow{\pi} (0,0,1) \xrightarrow{s_2} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi} (0,0,0),$$

which gives

$$m_{(1,2,0)} = \pi\pi s_2 s_1 \pi.$$

Using (2.2.9) and (2.2.12),

$$\begin{aligned} m_{(1,2,0)} &= \pi\pi s_2 s_1 \pi = (t_{\varepsilon_1} s_1 s_2)(t_{\varepsilon_1^\vee} s_1 s_2) s_2 s_1 (t_{\varepsilon_1^\vee} s_1 s_2) = t_{\varepsilon_1^\vee+\varepsilon_2^\vee} s_1 s_2 (s_1 s_2 s_2 s_1) t_{\varepsilon_1^\vee} s_1 s_2 \\ &= t_{\varepsilon_1^\vee+2\varepsilon_2^\vee} s_1 (s_2 s_1 s_2) = t_{\varepsilon_1^\vee+2\varepsilon_2^\vee}(s_1 s_1) s_2 s_1 = t_{\varepsilon_1^\vee+2\varepsilon_2^\vee} s_2 s_1. \end{aligned}$$

Since

$$s_1 s_2 (1,2,0) = s_1 s_2 (1,2,0) = s_1(1,0,2) = (0,1,2),$$

we checked $w_{(1,2,0)} = s_2 s_1$.

Recall the intertwiners $\tau_i^\vee$ and $\tau_\pi^\vee$ of $\widetilde{H}_{\mathfrak{gl}_n}$ are defined by

$$\tau_\pi^\vee = \pi^\vee \quad \text{and} \quad \tau_i^\vee = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1}Y_{i+1}} = T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1}}{1 - Y_i^{-1}Y_{i+1}}, \tag{3.5.9}$$

for $i \in \{1, \ldots, n-1\}$.

Intertwiner construction: Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$, and m_μ be the reduced word in $W_{\mathfrak{gl}_n}$ defined in (3.5.6). **The Knop-Sahi intertwiners** $\tau_{m_\mu}^\vee$ is the product of elements in $\{\tau_\pi^\vee, \tau_1^\vee, \ldots, \tau_{n-1}^\vee\}$ defined inductively by the condition $m_{(0,\ldots,0)} = 1$,

$$\tau^\vee_{m_{(0,\ldots,0,\mu_{i+1},0,\mu_{i+2},\ldots,\mu_n)}} = \tau_i^\vee \tau^\vee_{m_{(0,\ldots,0,0,\mu_{i+1},\ldots,\mu_n)}} \qquad \text{and} \tag{3.5.10}$$

$$\tau^\vee_{m_{(\mu_1,\ldots,\mu_n)}} = \tau_\pi^\vee \tau^\vee_{m_{(\mu_2,\ldots,\mu_n,\mu_1-1)}} \qquad \text{if } \mu_1 \neq 0. \tag{3.5.11}$$

For $\mu \in \mathbb{Z}_{\geq 0}^n$, the **non-symmetric Macdonald polynomial** E_μ is defined by

$$E_\mu = t^{-\frac{1}{2}\ell(w_\mu)} \tau^\vee_{m_\mu} \mathbf{1}. \tag{3.5.12}$$

Example 3.5.5. Using (3.5.10), (3.5.11) and (3.5.12), the non-symmetric Macdonald polynomial $E_{(1,2,0)}$ can be inductively constructed by the following

$$\begin{aligned}
E_{(0,0,0)} &= 1\\
E_{(1,0,0)} &= t^{\frac{1}{2}(\ell(w_{(0,0,0)})-\ell(w_{(1,0,0)}))}\tau_\pi^\vee E_{(0,0,0)} = t^{-\frac{2}{2}}X_1T_1T_2\mathbf{1} = X_1\mathbf{1} = x_1\\
E_{(0,1,0)} &= t^{\frac{1}{2}(\ell(w_{(1,0,0)})-\ell(w_{(0,1,0)})}\tau_1^\vee E_{(1,0,0)} = t^{\frac{1}{2}(2-1)}\tau_1^\vee E_{(1,0,0)}\\
&= t^{\frac{1}{2}}\left(T_1+\frac{t^{-\frac{1}{2}}(1-t)}{1-qt}\right)X_1\mathbf{1} = t^{\frac{1}{2}}X_2T_1^{-1}\mathbf{1}+\frac{1-t}{1-qt}X_1\mathbf{1} = x_2+\frac{1-t}{1-qt^2}x_1\\
E_{(0,0,1)} &= t^{\frac{1}{2}(\ell(w_{(0,1,0)})-\ell(w_{(0,0,1)}))}\tau_2^\vee E_{(0,1,0)} = t^{\frac{1}{2}(1-0)}\tau_2^\vee\left(X_2\mathbf{1}+\frac{1-t}{1-qt^2}X_1\mathbf{1}\right)\\
&= t^{\frac{1}{2}}\left(T_2+\frac{t^{-\frac{1}{2}}(1-t)}{1-qt}\right)\left(X_2\mathbf{1}+\frac{1-t}{1-qt^2}X_1\mathbf{1}\right)\\
&= t^{\frac{1}{2}}X_3T_2^{-1}\mathbf{1}+\frac{1-t}{1-qt}X_2\mathbf{1}+\frac{t^{\frac{1}{2}}(1-t)}{1-qt^2}X_1T_2\mathbf{1}+\frac{1-t}{1-qt}\frac{1-t}{1-qt^2}X_2\mathbf{1}\\
&= x_3+\frac{1-t}{1-qt}x_2+t\frac{1-t}{1-qt^2}x_1+\frac{1-t}{1-qt}\frac{1-t}{1-qt^2}x_1\\
E_{(2,0,0)} &= t^{\frac{1}{2}(\ell(w_{(0,1,0)})-\ell(w_{(2,0,0)}))}\tau_\pi^\vee E_{(0,0,1)} = t^{\frac{1}{2}(1-2)}\tau_\pi^\vee E_{(0,0,1)}\\
&= x_1^2+\frac{1-t}{1-qt}x_1x_3+qt\frac{1-t}{1-qt^2}x_1x_2+q\frac{1-t}{1-qt}\frac{1-t}{1-qt^2}x_1x_2\\
E_{(1,2,0)} &= t^{\frac{1}{2}(\ell(w_{(2,0,0)})-\ell(w_{(1,2,0)}))}\tau_\pi^\vee E_{(2,0,0)} = t^{\frac{1}{2}(2-2)}\tau_\pi^\vee E_{(2,0,0)}\\
&= x_1x_2^2+\frac{1-t}{1-qt}x_1^2x_2+qt\frac{1-t}{1-qt^2}x_1x_2x_3+q\frac{1-t}{1-qt}\frac{1-t}{1-qt^2}x_1x_2x_3
\end{aligned}$$

Proposition 3.5.3 is not new. It is stated in [BW19, pg6, (2.9)] and in [CMW19, pg 8, (9)] for μ being a partition.

Proposition 3.5.3. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$, and let $m_\mu = t_\mu w_\mu$ be the minimal length element in the coset $t_\mu S_n$. Recall from* (3.5.12) *the non-symmetric Macdonald polynomial is defined by*

$$E_\mu = t^{-\frac{1}{2}\ell(w_\mu)}\tau_{m_\mu}^\vee\mathbf{1}. \qquad \textit{Then} \quad Y_iE_\mu = q^{-\mu_i}t^{\frac{1}{2}(n-1)-(w_\mu^{-1}(i)-1)}E_\mu, \tag{3.5.13}$$

for $i \in \{1,\ldots,n\}$, and the coefficient of x^μ in E_μ is 1.

Proof. Since m_μ is a reduce word in $W_{\mathfrak{gl}_n}$, $m_\mu = s_{i_1}\cdots s_{i_k}$ for some $j_1,\ldots,j_k \in \{\pi,1,2,\ldots,n-1\}$. Then $\tau_{m_\mu}^\vee = \tau_{i_1}\ldots\tau_{i_k}$. Since

$$t_\mu w_\mu t_{-w_\mu^{-1}(\mu)}w_\mu^{-1} = t_{\mu-\mu}w_\mu w_\mu^{-1} = 1, \qquad \text{then} \qquad (t_\mu w_\mu)^{-1} = t_{-w_\mu^{-1}(\mu)}w_\mu^{-1}.$$

Using (3.3.1) and (3.4.4),

$$\begin{aligned}
Y_iE_\mu &= t^{-\frac{1}{2}\ell(w_\mu)}Y_i\tau_{m_\mu}^\vee\mathbf{1} = Y_it^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee\mathbf{1}\\
&= t^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee Y_{s_{i_k}\cdots s_{i_1}\pi^{-1}(i)}\mathbf{1} = t^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee Y_{(t_\mu w_\mu)^{-1}(i)}\mathbf{1}\\
&= t^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee Y_{t_{-w_\mu^{-1}(\mu)}w_\mu^{-1}(i)}\mathbf{1} = t^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee Y_{w_\mu^{-1}(i)-n(w_\mu^{-1}(\mu))_{w_\mu^{-1}(i)}}\mathbf{1}\\
&= t^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee Y_{-n\mu_i}Y_{w_\mu^{-1}(i)}\mathbf{1} = t^{-\frac{1}{2}\ell(w_\mu)}\tau_\pi^\vee\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee t^{\frac{1}{2}(n-1)-w_\mu^{-1}(i)+1}Y_{-n\mu_i}\mathbf{1}\\
&= t^{\frac{1}{2}(n-1)-w_\mu^{-1}(i)+1}q^{-\mu_i}t^{-\frac{1}{2}\ell(w_\mu)}\tau_{i_1}^\vee\cdots\tau_{i_k}^\vee\mathbf{1} = t^{\frac{1}{2}(n-1)-w_\mu^{-1}(i)+1}q^{-\mu_i}E_\mu.
\end{aligned}$$

□

Example 3.5.6. In this example, we compute the non-symmetric Macdonald polynomial $E_{(1,2,0)}$ directly from the definition (3.5.12). Let $w, w' \in S_n$. Then write

$$\begin{aligned} X^{\mu}T_w &\xrightarrow{\pi^\vee} X^{\mu'}T_{w'} \quad \text{if} \quad X^{\mu}T_w\pi^\vee = X^{\mu'}T_{w'}, \\ X^{\mu}T_w &\xrightarrow{T_i^{\pm 1}} X^{\mu}T_{w'} \quad \text{if} \quad X^{\mu}T_wT_i = X^{\mu}T_{w'} \text{ or } X^{\mu}T_wT_i^{-1} = X^{\mu}T_{w'}. \end{aligned}$$

Firstly, recall from Example 3.5.3 and 3.5.4, unwind $\mu = (1,2,0)$ and $\nu = (2,1,0)$, we have the following sequences

$$\begin{aligned} (1,2,0) \xrightarrow{\pi} (2,0,0) \xrightarrow{\pi} (0,0,1) \xrightarrow{s_2} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi} (0,0,0), \\ (2,1,0) \xrightarrow{\pi} (1,0,1) \xrightarrow{s_1} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi} (0,0,0), \end{aligned}$$

which give

$$\begin{aligned} m_{(1,2,0)} &= \pi\pi s_2 s_1 \pi, \quad \text{and} \quad \tau^\vee_{m_{(1,2,0)}} = \tau^\vee_\pi \tau^\vee_\pi \tau^\vee_2 \tau^\vee_1 \tau^\vee_\pi, \\ m_{(2,1,0)} &= \pi\pi s_1 \pi, \quad \text{and} \quad \tau^\vee_{m_{(2,1,0)}} = \tau^\vee_\pi \tau^\vee_\pi \tau^\vee_1 \tau^\vee_\pi, \end{aligned}$$

Using (3.2.5) and (3.2.6) to multiply out the leading terms of $E_{(1,2,0)}$ and $E_{(2,1,0)}$ gives

$$T_{(1,2,3)} \xrightarrow{\pi^\vee} X_1T_{(3,1,2)} \xrightarrow{\pi^\vee} X_1X_2T_{(2,3,1)} \xrightarrow{T_2} X_1X_2T_{(3,2,1)} \xrightarrow{T_1^{-1}} X_1X_2T_{(3,1,2)} \xrightarrow{\pi^\vee} X_1X_2X_2T_{(2,3,1)}, \tag{3.5.14}$$

$$T_{(1,2,3)} \xrightarrow{\pi^\vee} X_1T_{(3,1,2)} \xrightarrow{\pi^\vee} X_1X_2T_{(2,3,1)} \xrightarrow{T_1^{-1}} X_1X_2T_{(1,3,2)} \xrightarrow{\pi^\vee} X_1X_2X_1T_{(3,2,1)}. \tag{3.5.15}$$

Then leading terms of $E_{(1,2,0)}$ and $E_{(2,1,0)}$ are

$$X_1X_2X_2T_2T_1 \quad \text{and} \quad X_1X_2X_1T_1T_2T_1 \quad \text{respectively.}$$

Since $w_{(1,2,0)} = s_1s_2$, $\ell(w_{(1,2,0)}) = 2$. Using (3.5.14)(highlighted with red) and (3.5.15)(highlighted with red) to expand $\tau_i^\vee$ via (3.5.9) accordingly,

$$\begin{aligned} E_{(1,2,0)} &= t^{-\frac{2}{2}}\tau^\vee_\pi\tau^\vee_\pi\tau^\vee_2\tau^\vee_1\tau^\vee_\pi\mathbf{1} = t^{-\frac{2}{2}}\tau^\vee_\pi\tau^\vee_\pi\left(T_2 + \frac{t^{-\frac{1}{2}}(1-t)}{1-qt}\right)\tau^\vee_1\tau^\vee_\pi\mathbf{1} \\ &= t^{-\frac{2}{2}}\left(\tau^\vee_\pi\tau^\vee_\pi T_2\tau^\vee_1\tau^\vee_\pi + \tau^\vee_\pi\tau^\vee_\pi\frac{t^{-\frac{1}{2}}(1-t)}{1-qt}\tau^\vee_1\tau^\vee_\pi\right)\mathbf{1} \\ &= t^{-\frac{2}{2}}\left(\tau^\vee_\pi\tau^\vee_\pi T_2\left(T_1^{-1} + \frac{t^{-\frac{1}{2}}(1-t)qt^2}{1-qt^2}\right)\tau^\vee_\pi + \tau^\vee_\pi\tau^\vee_\pi\frac{t^{-\frac{1}{2}}(1-t)}{1-qt}\left(T_1^{-1} + \frac{t^{-\frac{1}{2}}(1-t)qt^2}{1-qt^2}\right)\tau^\vee_\pi\right)\mathbf{1} \\ &= t^{-\frac{2}{2}}\left(X_1X_2X_2T_2T_1\mathbf{1} + \frac{t^{-\frac{1}{2}}(1-t)qt^2}{1-qt^2}X_1X_2X_3T_1\mathbf{1} + \frac{t^{-\frac{1}{2}}(1-t)}{1-qt}X_1X_2X_1T_1T_2T_1\right. \\ &\qquad \left. + \frac{t^{-\frac{1}{2}}(1-t)}{1-qt}\frac{t^{-\frac{1}{2}}(1-t)qt^2}{1-qt^2}X_1X_2X_3\right)\mathbf{1} \\ &= x_1x_2 + \frac{1-t}{1-qt}x_1^2x_2 + \left(qt\frac{1-t}{1-qt} + q\frac{1-t}{1-qt}\frac{1-t}{1-qt^2}\right)x_1x_2x_3. \end{aligned}$$

Example 3.5.7. Recall from Example 3.5.6 that for $\mu = (2,1,0)$ and $s_1\mu = (1,2,0)$ we have

$$m_{(2,1,0)} = \pi\pi s_1\pi \quad \text{and} \quad m_{(1,2,0)} = \pi\pi s_2 s_1 \pi$$

also

$$w_{(2,1,0)} = s_2 s_1 s_2 \quad \text{and} \quad w_{(1,2,0)} = s_2 s_1. \tag{3.5.16}$$

We note that by (2.2.15)

$$s_1 m_{(2,1,0)} = s_1\pi\pi s_1\pi = \pi s_0 \pi s_1 \pi = \pi\pi s_2 s_1 \pi = m_{(1,2,0)}$$

which implies

$$\tau_1^\vee \tau_{m_{(2,1,0)}}^\vee = \tau_{m_{(1,2,0)}}^\vee. \tag{3.5.17}$$

Using properties of $\tau_i^\vee$ and $\tau_\pi^\vee$ in (3.3.4), (3.5.13) and (3.4.4), we obtain

$$\begin{aligned} Y_1 E_{(2,1,0)} &= t^{-\frac{3}{2}} Y_1 \tau_\pi^\vee \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \mathbf{1} = t^{-\frac{3}{2}} \tau_\pi^\vee q^{-1} Y_3 \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \mathbf{1} = t^{-\frac{3}{2}} \tau_\pi^\vee q^{-1} Y_3 \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \mathbf{1} \\ &= t^{-\frac{3}{2}} \tau_\pi^\vee q^{-1} \tau_\pi^\vee Y_2 \tau_1^\vee \tau_\pi^\vee \mathbf{1} = t^{-\frac{3}{2}} q^{-1} \tau_\pi^\vee \tau_\pi^\vee \tau_1^\vee Y_1 \tau_\pi^\vee \mathbf{1} = t^{-\frac{3}{2}} q^{-2} \tau_\pi^\vee \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee Y_3 \mathbf{1} \\ &= t^{-\frac{3}{2}} q^{-2} \tau_\pi^\vee \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee t^{\frac{1}{2}(3-1)-(3-1)} \mathbf{1} = q^{-2} t^{\frac{1}{2}(3-1)-(3-1)} E_{(2,1,0)}. \end{aligned} \tag{3.5.18}$$

Similarly,

$$Y_2 E_{(2,1,0)} = t^{-\frac{3}{2}} Y_2 \tau_\pi^\vee \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \mathbf{1} = q^{-1} t^{\frac{1}{2}(3-1)-(2-1)} E_{(2,1,0)}, \tag{3.5.19}$$

and

$$Y_3 E_{(2,1,0)} = Y_3 \tau_\pi^\vee \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \mathbf{1} = t^{\frac{1}{2}(3-1)-(1-1)} E_{(2,1,0)}. \tag{3.5.20}$$

Using (3.5.18), (3.5.19), (3.5.20) and (3.5.20), we determine the action of Y_1, Y_2, Y_3 inductively on $E_{(1,2,0)}$,

$$\begin{aligned} Y_1 E_{(1,2,0)} &= Y_1 t^{-\frac{1}{2}\ell(w_{(1,2,0)})} \tau_{m_{(1,2,0)}}^\vee \mathbf{1} = Y_1 t^{\frac{1}{2}(\ell(w_{(2,1,0)})-\ell(w_{(1,2,0)})} \tau_1^\vee E_{(2,1,0)} \\ &= t^{\frac{1}{2}\ell(w_{(2,1,0)})-\ell(w_{(1,2,0)})} Y_1 \tau_1^\vee E_{(2,1,0)} = t^{\frac{1}{2}\ell(w_{(2,1,0)})-\ell(w_{(1,2,0)})} \tau_1^\vee Y_2 E_{(2,1,0)} \\ &= q^{-1} t^{\frac{1}{2}(3-1)-(2-1)} t^{\frac{1}{2}\ell(w_{(2,1,0)})-\ell(w_{(1,2,0)})} \tau_1^\vee E_{(2,1,0)} = q^{-1} t^{\frac{1}{2}(3-1)-(2-1)} E_{(1,2,0)}. \end{aligned} \tag{3.5.21}$$

Similarly,

$$Y_2 E_{(1,2,0)} = Y_2 t^{\frac{1}{2}(\ell(w_{(2,1,0)})-\ell(w_{(1,2,0)})} \tau_1^\vee E_{(2,1,0)} = q^{-2} t^{\frac{1}{2}(3-1)-(3-1)} E_{(1,2,0)} \tag{3.5.22}$$

and

$$Y_3 E_{(1,2,0)} = Y_3 t^{\frac{1}{2}(\ell(w_{(2,1,0)})-\ell(w_{(1,2,0)})} \tau_1^\vee E_{(2,1,0)} = q^{-0} t^{\frac{1}{2}(3-1)-(1-1)} E_{(1,2,0)}. \tag{3.5.23}$$

By (3.5.16),

$$w_{(1,2,0)} = s_2 s_1 \quad \text{and} \quad w_{(1,2,0)}^{-1} = s_1 s_2.$$

Since

$$\begin{aligned} &s_2 s_1(1) = 3, \quad s_2 s_1(2) = 1, \quad s_2 s_1(3) = 2, \\ &s_1 s_2(1) = 2, \quad s_1 s_2(2) = 3, \quad s_1 s_2(3) = 1, \end{aligned}$$

equation (3.5.21), (3.5.22), (3.5.23) can be written as

$$\begin{aligned} Y_1 E_{(1,2,0)} &= q^{-1} t^{\frac{1}{2}(3-1)-(2-1)} E_{(1,2,0)} = q^{-1} t^{\frac{1}{2}(3-1)-(w_{1,2,0}^{-1}(1)-1)} E_{(1,2,0)}, \\ Y_2 E_{(1,2,0)} &= q^{-2} t^{\frac{1}{2}(3-1)-(3-1)} E_{(1,2,0)} = q^{-2} t^{\frac{1}{2}(3-1)-(w_{(1,2,0)}^{-1}(2)-1)} E_{(1,2,0)}, \\ Y_3 E_{(1,2,0)} &= q^{-0} t^{\frac{1}{2}(3-1)-(1-1)} E_{(1,2,0)} = q^{-0} t^{\frac{1}{2}(3-1)-(w_{(1,2,0)}^{-1}(3)-1)} E_{(1,2,0)} \end{aligned}$$

which in line with (3.5.13).

Set $\mathbf{0} = (0, 0, \ldots, 0)$. The **unwinding μ construction** of a non-symmetric Macdonald polynomial E_μ with $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ can be described by the following steps:

1. Using Lemma 3.5.2 to unwind μ and obtain the element m_μ,

2. Constructing E_μ from $E_{\mathbf{0}} = 1$ by applying m_μ on $\mathbf{0}$ and using the following two equalities [HHL06, pg 11,15, Cor 4.1.3, (62)]

$$E_\mu = E_{\pi\nu}(x_1, \ldots, x_n) = q^{\nu_n} x_1 E_\nu(x_2, \ldots, x_n, q^{-1}x_1), \tag{3.5.24}$$

$$E_\mu = E_{s_i\nu} = \left(T_i + \frac{1-t}{1-q^{\nu_i}t^{a(i,1)}}\right)E_\nu. \tag{3.5.25}$$

Example 3.5.8. For example, the non-symmetric Macdonald polynomial $E_{(2,0,1)}$ can be constructed inductively from unwinding

$$(2,0,1) \xrightarrow{\pi} (0,1,1) \xrightarrow{s_1} (1,0,1) \xrightarrow{\pi} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi} (0,0,0) \tag{3.5.26}$$

such that

$$m_\mu = \pi s_1 \pi s_1 \pi. \tag{3.5.27}$$

By (3.5.27), (4.2.2) and (4.2.1), the non-symmetric Macdonald polynomial is inductively constructed by

$$\begin{aligned}
E_{(0,0,0)} &= 1, \\
E_{(1,0,0)} &= E_{\pi(0,0,0)} = x_1, \\
E_{(0,1,0)} &= E_{s_1(1,0,0)} = \left(T_1 + \frac{1-t}{1-qt^2}\right)E_{(1,0,0)} = T_1 x_1 + \frac{1-t}{1-qt^2}x_1 = x_2 + \frac{1-t}{1-qt^2}x_1, \\
E_{(1,0,1)} &= E_{\pi(0,1,0)} = x_1\left(q^{-1}x_1 + \frac{1-t}{1-qt^2}x_2\right) = x_1x_3 + \frac{1-t}{1-qt^2}x_1x_2, \\
E_{(0,1,1)} &= E_{s_1(1,0,1)} = \left(T_1 + \frac{1-t}{1-qt}\right)E_{(1,0,1)}, \\
&= \left(\frac{1-t}{1-qt}\right)x_1x_3 + x_2x_3 + t\left(\frac{1-t}{1-qt^2}\right)x_1x_2 + \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)x_1x_2, \\
E_{(2,0,1)} &= E_{\pi(0,1,1)} = qx_1E_{\pi(0,1,1)} \\
&= qx_1\left(q^{-1}\left(\frac{1-t}{1-qt}\right)x_2x_1 + q^{-1}x_3x_1 + t\left(\frac{1-t}{1-qt^2}\right)x_2x_3 + \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)x_2x_3\right), \\
&= \left(\frac{1-t}{1-qt}\right)x_1^2x_2 + x_1^2x_3 + qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 + q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3.
\end{aligned}$$

Proposition 3.5.4. *Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$. With E_μ defined in* (3.5.13)*, the non-symmetric Macdonald polynomials satisfy the following:*

1. $E_\mu = E_{\pi^{-1}\mu}(x_1, \ldots, x_n) = q^{\nu_n}x_1E_\nu(x_2, \ldots, x_n, q^{-1}x_1)$*;*

2. *If* $\mu_1 = 0, \mu_2 = 0, \ldots, \mu_i = 0,$ *and* $\mu_{i+1} \neq 0$*, then* $\nu = s_i\mu$ *and*

$$E_\mu = E_{s_i\nu} = \left(T_i + \frac{1-t}{1-q^{\nu_i}t^{a(i,1)}}\right)E_\nu.$$

Proof. The equality (3.5.10) becomes (3.5.25) by multiplying both side of (3.5.10) by $\mathbf{1}$ and using the definition of $\tau_i^\vee$. It remains to show that (3.5.25) is equivalent to (3.5.10). Using $m_{\pi\mu} = \pi m_\mu$ and Proposition 3.5.3,

$$\begin{aligned} E_{\pi\mu} &= t^{-\frac{1}{2}\ell(w_{\pi\mu})}\tau^\vee_{m_{\pi\mu}}\mathbf{1} = t^{-\frac{1}{2}\ell(w_{\pi\mu})}t^{\frac{1}{2}\ell(w_\mu)}\tau^\vee_\pi t^{-\frac{1}{2}\ell(w_\mu)}\tau^\vee_{m_\mu}\mathbf{1} \\ &= t^{\frac{1}{2}(\ell(w_\mu)-\ell(w_{\pi\mu}))}\tau^\vee_\pi E_\mu = t^{\frac{1}{2}(\ell(w_\mu)-\ell(w_{\pi\mu}))}\pi^\vee E_\mu. \end{aligned}$$

Using

$$\ell(w_\mu) = \ell(w_\mu^{-1}) = \{i < j \mid \mu_i > \mu_j\},$$

we obtain

$$\ell(w_{\pi\mu}) = \ell(w_\mu) - \#\{i < n \mid \mu_i > \mu_n\} + \#\{i < n \mid \mu_n + 1 > \mu_i\}.$$

Then

$$\begin{aligned} \ell(w_\mu) - \ell(w_{\pi\mu}) &= \#\{i \in \{1, \ldots, n-1\} \mid \mu_i > \mu_n\} - \#\{i \in \{1, \ldots, n-1\} \mid \mu_n + 1 > \mu_i\} \\ &= \#\{i \in \{1, \ldots, n-1\} \mid \mu_i > \mu_n\} - \#\{i \in \{1, \ldots, n-1\} \mid \mu_n \geq \mu_i\} \\ &= (n-1) - 2(\#\{i \in \{1, \ldots, n-1\} \mid \mu_n \geq \mu_i\}) && (3.5.28) \\ &= (n-1) - 2(1 + \#\{i < n \mid \mu_n \geq \mu_i\} + \#\{i > n \mid \mu_n > \mu_i\} - 1) \\ &= (n-1) - 2(w_\mu^{-1}(n) - 1). && (3.5.29) \end{aligned}$$

Using Proposition 3.5.3, the right hand side of (3.5.24) is

$$\begin{aligned} q^{\mu_n}x_1E_\mu(x_2, \ldots, x_n, q^{-1}x_1) &= X_1q^{\mu_n}s_1s_2\cdots s_{n-1}E_\mu(x_1, \ldots, q^{-1}x_n) \quad \text{by (3.4.3)} \\ &= x_1q^{\mu_n}s_1s_2\cdots s_{n-1}T_{q^{-1},x_n}E_\mu \quad \text{by (3.4.1)} \\ &= q^{\mu_n}x_1\pi E_\mu = q^{\mu_n}x_1T_1\cdots T_{n-1}Y^{\varepsilon_n^\vee}E_\mu \quad \text{by (3.1.19)} \\ &= q^{\mu_n}x_1T_1\cdots T_{n-1}q^{-\mu_n}t^{\frac{1}{2}(n-1)-(w_\mu^{-1}(n)-1)}E_\mu \quad \text{by (3.4.4)} \\ &= t^{\frac{1}{2}(n-1)-(w_\mu^{-1}(n)-1)}\pi^\vee E_\mu = t^{\frac{1}{2}(\ell(w_\mu)-\ell(w_{\pi\mu}))}\pi^\vee E_\mu \text{ by (3.5.29).} \end{aligned}$$

Hence,

$$E_{\pi\mu}(x_1, \ldots, x_n) = t^{\frac{1}{2}(\ell(w_\mu)-\ell(w_{\pi\mu}))}\pi^\vee E_\mu = q^{\mu_n}x_1E_\mu(x_2, \ldots, x_n, q^{-1}x_1). \tag{3.5.30}$$

□

Example 3.5.9. We illustrate the second equality of (3.5.30) by the following example. Recall from Appendix Section 6.2 that

$$E_{(0,1,1)} = \frac{1-t}{1-qt}X_1X_3 + \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)X_1X_2 + X_2X_3 + \frac{t(1-t)}{1-qt^2}X_1X_2.$$

Then

$$\begin{aligned}
E_{(2,0,1)} &= E_{\pi(0,1,1)} = t^{-1}\pi^{\vee}E_{(0,1,1)} \\
&= t^{-1}\biggl(X_1T_1T_2X_2X_3 + \frac{t(1-t)}{1-qt^2}X_1T_1T_2X_1X_2 + \frac{1-t}{1-qt}X_1T_1T_2X_1X_3 \\
&\qquad + \frac{1-t}{1-qt}\frac{1-t}{1-qt^2}X_1T_1T_2X_1X_2\biggr)\mathbf{1} \\
&= t^{-1}\biggl(t^{\frac{2}{2}}X_1^2X_3\mathbf{1} - (1-t)X_1X_2X_3\mathbf{1} + t^{-1}\left(\frac{t(1-t)}{1-qt^2}\right)X_1X_2X_3\mathbf{1} \\
&\qquad t\left(\frac{1-t}{1-qt}\right)X_1^2X_2\mathbf{1} - (t^{-1}-1)\frac{1-t}{1-qt}X_1X_2X_3\mathbf{1} + t^{-1}\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)X_1X_2X_3\mathbf{1}\biggr) \\
&= t^{-1}\biggl(tX_1^2X_3\mathbf{1} + t\left(\frac{1-t}{1-qt}\right)X_1^2X_2\mathbf{1} + \left(\left(\frac{1-t}{1-qt^2}\right) - (1-t)\right)X_1X_2X_3\mathbf{1} \\
&\qquad + \left(\frac{(1-t)(1-t^{-1})}{1-qt} + \frac{t^{-1}(1-t)^2}{(1-qt)(1-qt^2)}\right)X_1X_2X_3\mathbf{1}\biggr) \\
&= t^{-1}\biggl(tX_1^2X_3\mathbf{1} + t\left(\frac{1-t}{1-qt}\right)X_1^2X_2\mathbf{1} + qt^2\frac{1-t}{1-qt^2}X_1X_2X_3\mathbf{1} + qt\frac{1-t}{1-qt^2}\frac{1-t}{1-qt}X_1X_2X_3\mathbf{1}\biggr) \\
&= x_1^2x_3 + \left(\frac{1-t}{1-qt}\right)x_1^2x_2 + qt\frac{1-t}{1-qt^2}x_1x_2x_3 + q\frac{1-t}{1-qt^2}\frac{1-t}{1-qt}x_1x_2x_3.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E_{(2,0,1)} &= q^1x_1E_{(0,1,1)}(x_2,x_3,q^{-1}x_1) \\
&= qx_1\left(q^{-1}\frac{1-t}{1-qt}x_2x_1 + \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)x_2x_3 + q^{-1}x_3x_1 + \frac{t(1-t)}{1-qt^2}x_2x_3\right) \\
&= x_1^2x_3 + \left(\frac{1-t}{1-qt}\right)x_1^2x_2 + qt\frac{1-t}{1-qt^2}x_1x_2x_3 + q\frac{1-t}{1-qt^2}\frac{1-t}{1-qt}x_1x_2x_3.
\end{aligned}$$

Chapter 4

The Haglund-Haiman-Loehr(HHL) formula

The main focus of this section is to study the statistics inversion, coinversion defined in [HHL06, pg7, (22)(23)(24)] and give a proof of the HHL formula [HHL06, Theorem 3.5.1]. Following the proof, we will discuss examples of Macdonald polynomials generated using the HHL combinatorial formula defined in (4.3.1). These examples will support the proof of Theorem 4.3.1.

4.1 The non-attacking fillings and statistics inv, coinv

The set of **ascents** and **descents** are denoted by

$$\mathrm{Aes}(\widehat{\sigma_\mu}) = \{(i,j) \in \mathrm{dg}(\mu) \mid \sigma_\mu(i,j) < \widehat{\sigma_\mu}(i,j-1)\}, \tag{4.1.1}$$

$$\mathrm{Des}(\widehat{\sigma_\mu}) = \{(i,j) \in \mathrm{dg}(\mu) \mid \sigma_\mu(i,j) > \widehat{\sigma_\mu}(i,j-1)\}. \tag{4.1.2}$$

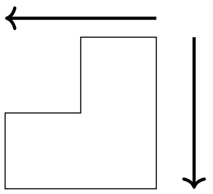
An ascent and a descent of a box $(i,j) \in \mathrm{dg}(\mu)$ with $\widehat{\sigma_\mu}(i,j) = b$ and $\widehat{\sigma_\mu}(i,j-1) = a$ can be viewed pictorially as

$$\begin{array}{|c|}\hline b \\ \hline a \\ \hline \end{array} \quad b > a \quad \text{and} \quad \begin{array}{|c|}\hline b \\ \hline a \\ \hline \end{array} \quad b < a \quad \text{respectively.}$$

Let $u, v \in \mathrm{dg}(\mu)$. The reading order of $\mathrm{dg}(\mu)$ is defined to be

$$u < v \quad \text{if} \quad \mathrm{cyl}_n^{-1}(u) > \mathrm{cyl}_n^{-1}(v). \tag{4.1.3}$$

Informally, it is the ordering from **top to bottom** and from **right to left**. Pictorially,



A **non-attacking filling** is a filling $\widehat{\sigma_\mu} : \mathrm{dg}(\mu) \to \{1, \dots, n\}$ such that

1. $\widehat{\sigma_\mu}(i,0) = i$;

2. if $k \in \mathbb{Z}_{>0}$, $j \in \{1, \dots, n-1\}$, $\mathrm{cyl}_n(k) \in \widehat{\mathrm{dg}}(\mu)$ and $\mathrm{cyl}_n(k-j) \in \widehat{\mathrm{dg}}(\mu)$, then
$$\widehat{\sigma_\mu}(\mathrm{cyl}_n(k)) \neq \widehat{\sigma_\mu}(\mathrm{cyl}_n(k-j)).$$

The first condition says the basement is filled with $1, 2, \dots, n$ and the second condition is says if $k \in \mathbb{Z}_{>0}$ such that $u = \mathrm{cyl}_n(k)$ is a box in $\mathrm{dg}(\mu)$, then the entries in the boxes $\mathrm{cyl}_n(\{k, k-1, \cdots, k-n+1\})$ are all distinct. The non-attacking filling defined here is equivalent to the non-attacking filling in [HHL06, pg6, 3.1].

Example 4.1.1. For example, one non-attacking filling $\widehat{\sigma_\mu}$ of $\mu = (3, 1, 2, 4, 3, 0, 4)$ is

$$\widehat{\sigma_\mu} = \begin{array}{ccccccc} & & & 3 & & & 5 \\ 1 & & & 5 & 4 & & 2 \\ 7 & & 2 & 3 & 4 & & 6 \\ 1 & 2 & 3 & 4 & 5 & & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

A pair of boxes $(u, v) \in \widehat{\mathrm{dg}}(\mu) \times \widehat{\mathrm{dg}}(\mu)$ is **attacking** if either
$$u \in \mathrm{attack}_\mu(v) \quad \text{or} \quad v \in \mathrm{attack}_\mu(u).$$

The set of inversions is defined as
$$\mathrm{HInv}(\widehat{\sigma_\mu}) = \left\{ (u, v) \in \widehat{\mathrm{dg}}(\mu) \times \widehat{\mathrm{dg}}(\mu) \;\middle|\; \begin{array}{c} (u, v) \text{ is attacking and} \\ u < v \text{ and } \widehat{\sigma_\mu}(u) > \widehat{\sigma_\mu}(v) \end{array} \right\}. \tag{4.1.4}$$

The statistics major, inversion and coinversion for a non-attacking filling $\widehat{\sigma_\mu}$ are
$$\mathrm{maj}(\widehat{\sigma_\mu}) = \sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} (\#\,\mathrm{Hleg}(u) + 1), \tag{4.1.5}$$

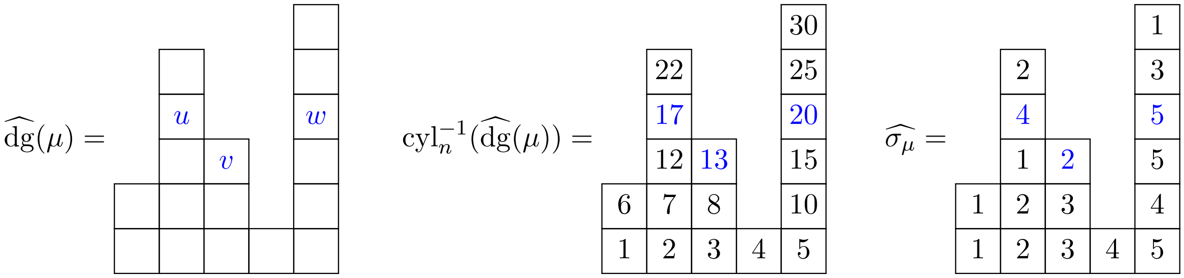
$$\mathrm{inv}(\widehat{\sigma_\mu}) = \#\,\mathrm{HInv}(\widehat{\sigma_\mu}) - \#\{i < j \mid \mu_i \leq \mu_j\} - \sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \#\,\mathrm{Harm}(u), \tag{4.1.6}$$

$$\mathrm{coinv}(\widehat{\sigma_\mu}) = \left(\sum_{u \in \mathrm{dg}(\mu)} \#\,\mathrm{Harm}(u) \right) - \mathrm{inv}(\widehat{\sigma_\mu}). \tag{4.1.7}$$

Let $\widehat{\sigma_\mu}$ be a non-attacking filling, and let $u = (i, j)$ be a box in $\mathrm{dg}(\mu)$. Define the **weight** of $\widehat{\sigma_\mu}$ by
$$\mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}) = x^{\widehat{\sigma_\mu}} q^{\mathrm{maj}(\widehat{\sigma_\mu})} t^{\mathrm{coinv}(\widehat{\sigma_\mu})} \prod_{\substack{u \in \mathrm{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)}} \frac{1-t}{1 - q^{\#\,\mathrm{Hleg}(u)+1} t^{\#\,\mathrm{Harm}(u)+1}}, \tag{4.1.8}$$

where $x^{\widehat{\sigma_\mu}} = x_1^{\#1 \text{ in } \widehat{\sigma_\mu}} \cdots x_n^{\#n \text{ in } \widehat{\sigma_\mu}}$.

Example 4.1.2. Let $\mu = (1,4,2,0,5)$. Then the extended column diagram, cylindrical coordinates and a non-attacking filling of μ are given by



Let $u = (2,3), v = (3,2)$ and $w = (5,3)$. Then $\mathrm{cyl}_5^{-1}(u) = 17, \mathrm{cyl}_5^{-1}(v) = 13$ and $\mathrm{cyl}_5^{-1}(w) = 20$. Moreover, $u \in \mathrm{attack}_\mu(w)$ and $v \in \mathrm{attack}_\mu(u)$. The statistics $\#\,\mathrm{Hleg}$ and $\#\,\mathrm{Harm}$ of u, v, w are

Boxes	# Hleg	# Harm
u	1	1
v	0	0
w	2	1

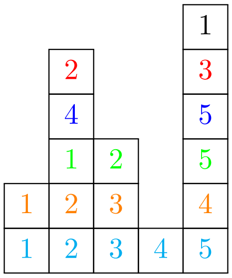
Counting from top to bottom and right to left, the inversions of $\widehat{\sigma_\mu}$ are pairs

$$(3,2), (5,4), (5,2), (5,1), (2,1), (4,3), (4,2), (4,1), (3,2), (3,1), (2,1)$$

and

$$(5,4), (5,3), (5,2), (5,1), (4,3), (4,2), (4,1), (3,2), (3,1), (2,1),$$

pictorially



The cardinality of the set of inversions is

$$\#\,\mathrm{HInv}(\widehat{\sigma_\mu}) = 1 + 1 + 3 + \binom{4}{2} + \binom{5}{2} = 21.$$

The statistics descent, major, inversion and coinversion are

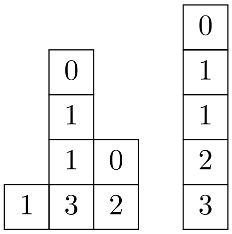
$$\mathrm{Des}(\widehat{\sigma_\mu}) = \{(2,3),(5,2)\}, \quad \mathrm{maj}(\widehat{\sigma_\mu}) = (3+1) + (2+1),$$

$$\mathrm{inv}(\widehat{\sigma_\mu}) = 21 - 6 - 3 = 12, \quad \mathrm{coinv}(\widehat{\sigma_\mu}) = 15 - 12 = 3.$$

The table of $\#\,\mathrm{Harm}$ of each box in $\mathrm{dg}(\mu)$ and the column diagram $\mathrm{dg}(\mu)$ filled with $\#\,\mathrm{Harm}$ are

(1, 1)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(3, 1)
1	3	1	1	0	2
(3, 2)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)
0	3	2	1	1	0

and



Now we reveal [HHL06, pg 8, (3.6)] the combinatorial natural of the statistics inversion and coinversion using triples defined in (4.1.12). Recall from (4.1.4) the inversion statistics associated to a filling $\widehat{\sigma_\mu}$ is

$$\mathrm{HInv}(\widehat{\sigma_\mu}) = \left\{ (u,v) \in \widehat{\mathrm{dg}}(\mu) \times \widehat{\mathrm{dg}}(\mu) \;\middle|\; \begin{array}{c} (u,v) \text{ is attacking and} \\ u < v \text{ and } \widehat{\sigma_\mu}(u) > \widehat{\sigma_\mu}(v) \end{array} \right\}. \tag{4.1.9}$$

Let $S = \{((i,0),(i',0)) \in \widehat{\mathrm{dg}}(\mu) \times \widehat{\mathrm{dg}}(\mu) \mid i < i' \text{ and } \mu_i \leq \mu_{i'}\}$. Define

$$\mathrm{Inv}^S(\widehat{\sigma_\mu}) = \mathrm{HInv}(\widehat{\sigma_\mu}) \backslash S.$$

Then

$$\# \mathrm{Inv}^S(\widehat{\sigma_\mu}) = \# \mathrm{HInv}(\widehat{\sigma_\mu}) - \#\{i < i' \mid \mu_i \leq \mu_{i'}\} \tag{4.1.10}$$

and

$$\mathrm{inv}(\widehat{\sigma_\mu}) = \# \mathrm{Inv}^S(\widehat{\sigma_\mu}) - \sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \# \mathrm{Harm}(u) \tag{4.1.11}$$

Define a **triple** of a box $u \in \widehat{\mathrm{dg}}(\mu)$ to be

$$\mathrm{trip}(u) = \{v \in \mathrm{dg}(\mu) \mid \mathrm{Harm}_\mu(u)\}. \tag{4.1.12}$$

Pictorially, a **triple** is

$$\begin{array}{cc} \boxed{u} & \\ \boxed{w} & \boxed{v} \end{array} \quad \text{or} \quad \begin{array}{cc} \boxed{v} & \boxed{u} \\ & \boxed{w} \end{array} \quad \text{with } v \in \mathrm{Harm}_\mu(u),$$

we denote the triples on the left to be **Type A** and on the right to be **Type B**. The set of triples of $\widehat{\sigma_\mu}$ is

$$\mathrm{trip}(\widehat{\sigma_\mu}) = \bigcup_{u \in \mathrm{dg}(\mu)} \mathrm{trip}(u) \quad \text{so that} \quad \# \mathrm{trip}(\widehat{\sigma_\mu}) = \sum_{u \in \mathrm{dg}(\mu)} \# \mathrm{Harm}_\mu(u). \tag{4.1.13}$$

Equivalently, $\mathrm{trip}(\widehat{\sigma_\mu})$ is the set of pairs

$$\mathrm{trip}(\widehat{\sigma_\mu}) = \{(u,v) \in \widehat{\sigma_\mu} \times \widehat{\sigma_\mu} \mid v \in \mathrm{Harm}_\mu(u)\}.$$

For $(u,v) \in \mathrm{trip}(\widehat{\sigma_\mu})$ define

$$\begin{aligned} \chi_{uv}(\widehat{\sigma_\mu}) &= \delta_{(u,v) \in \mathrm{HInv}(\widehat{\sigma_\mu})} = \delta_{\widehat{\sigma_\mu}(u) > \widehat{\sigma_\mu}(v)}, \\ \chi_{vw}(\widehat{\sigma_\mu}) &= \delta_{(v,w) \in \mathrm{HInv}(\widehat{\sigma_\mu})} = \delta_{\widehat{\sigma_\mu}(v) > \widehat{\sigma_\mu}(w)}, \qquad \text{so that} \qquad \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}) - \chi_{uw}(\widehat{\sigma_\mu}) \in \{0,1\}. \\ \chi_{uw}(\widehat{\sigma_\mu}) &= \delta_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} = \delta_{\widehat{\sigma_\mu}(u) > \widehat{\sigma_\mu}(w)}, \end{aligned}$$

Then the **inversion triples** and **coinversion triples** associated to the box u are defined by

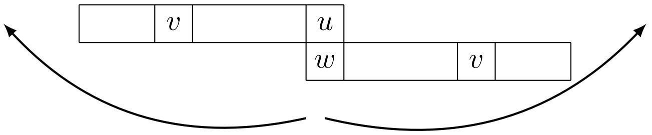
$$\mathrm{invtrip}(u) = \{v \in \mathrm{Harm}_\mu(u) \mid \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}) - \chi_{uw}(\widehat{\sigma_\mu}) = 1\}, \tag{4.1.14}$$

$$\mathrm{coinvtrip}(u) = \{v \in \mathrm{Harm}_\mu(u) \mid \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}) - \chi_{uw}(\widehat{\sigma_\mu}) = 0\}. \tag{4.1.15}$$

The set of inversion triples and the coinversion triples are

$$\begin{aligned} \mathrm{invtrip}(\widehat{\sigma_\mu}) &= \bigcup_{u \in \mathrm{dg}(\mu)} \mathrm{invtrip}(u) \quad \text{so that} \quad \# \mathrm{invtrip}(\widehat{\sigma_\mu}) = \sum_{u \in \mathrm{dg}(\mu)} \# \mathrm{invtrip}(u), \\ \mathrm{coinvtrip}(\widehat{\sigma_\mu}) &= \bigcup_{u \in \mathrm{dg}(\mu)} \mathrm{coinvtrip}(u) \quad \text{so that} \quad \# \mathrm{coinvtrip}(\widehat{\sigma_\mu}) = \sum_{u \in \mathrm{dg}(\mu)} \# \mathrm{coinvtrip}(u). \end{aligned}$$

The **coinversion triples** can be represented pictorially



The arrows indicate the order of decreasing elements. For example, if v is on the left of u, then $\widehat{\sigma_\mu}(w) < \widehat{\sigma_\mu}(v) < \widehat{\sigma_\mu}(u)$. If v is on the right of u, then $\widehat{\sigma_\mu}(u) < \widehat{\sigma_\mu}(w) < \widehat{\sigma_\mu}(v)$. The relationship between cardinality of the triples, inversion triples and coinversion triples is

$$\#\,\mathrm{trip}(\widehat{\sigma_\mu}) = \#\,\mathrm{invtrip}(\widehat{\sigma_\mu}) + \#\,\mathrm{coinvtrip}(\widehat{\sigma_\mu}). \tag{4.1.16}$$

Lemma 4.1.1. *Let $\widehat{\sigma_\mu}$ be a filling, and let x, y be two attacking boxes in $\widehat{\sigma_\mu}$ with $x, y \notin \{(i,0),(i',0)\}$ for $i < i'$ and $\mu_i \leq \mu'_i$. Then x, y lies in a unique triple,*

Proof. Let x and y be two attacking boxes in σ_μ. Then either $y \in \mathrm{attack}_\mu(x)$ or $x \in \mathrm{attack}_\mu(y)$. Hence, there are four cases.

Case 1: If $x = (i,j)$ and $y = (i',j)$ with $i < i'$ and $\mu_i \leq \mu_{i'}$, then $x \in \mathrm{Harm}_\mu(y)$, $y \notin \mathrm{Harm}_\mu(u)$ and

$$\begin{array}{|c|}\hline u \\ \hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline \end{array} \text{ is not a triple} \qquad \text{and} \qquad \begin{array}{|c|}\hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline u \\ \hline \end{array} \text{ is a triple}$$

Case 2: If $x = (i,j)$ and $y = (i',j)$ with $i < i'$ and $\mu_i > \mu_{i'}$ then $y \in \mathrm{Harm}_\mu(u)$, $x \notin \mathrm{Harm}_\mu(y)$ and

$$\begin{array}{|c|}\hline u \\ \hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline \end{array} \text{ is a triple} \qquad \text{and} \qquad \begin{array}{|c|}\hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline u \\ \hline \end{array} \text{ is not a triple}$$

Case 3: If $x = (i,j)$ and $y = (i',j-1)$ with $i < i'$ and $\mu_i \leq \mu_{i'}$, then $x \in \mathrm{Harm}_\mu(u)$, $y \notin \mathrm{Harm}_\mu(x)$ and

$$\begin{array}{|c|}\hline x \\ \hline u \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline \end{array} \text{ is not a triple} \qquad \text{and} \qquad \begin{array}{|c|}\hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline u \\ \hline y \\ \hline \end{array} \text{ is a triple}$$

Case 4: If $x = (i,j)$ and $y = (i',j-1)$ with $i < i'$ and $\mu_i > \mu_{i'}$ then $y \in \mathrm{Harm}_\mu(x)$, $x \notin \mathrm{Harm}_\mu(u)$ and

$$\begin{array}{|c|}\hline x \\ \hline w \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline \end{array} \text{ is a triple} \qquad \text{and} \qquad \begin{array}{|c|}\hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline u \\ \hline y \\ \hline \end{array} \text{ is not a triple.}$$

Hence, in all four cases the attacking boxes x, y in $\widehat{\sigma_\mu}$ with $x \neq (i,0)$ and $y \neq (i',0)$, for $i < i'$ and $\mu_i \leq \mu_{i'}$ is in a unique triple. Moreover if $x = (i,0)$, $y = (i'.0)$ for $i < i'$ and $\mu_i \leq \mu'_i$, then $y \notin \mathrm{Harm}_\mu(u)$, $w \notin \widehat{\mathrm{dg}}(\mu)$ and

$$\begin{array}{|c|}\hline u \\ \hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline \end{array} \text{ is not a triple} \qquad \text{and} \qquad \begin{array}{|c|}\hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline y \\ \hline w \\ \hline \end{array} \text{ is not a triple.} \qquad \square$$

For a box $u \in \widehat{\mathrm{dg}}(\mu)$, set $\#\,\mathrm{Harm}_\mu(u) = a(u)$. Recall from (4.1.6) and (4.1.7) the statistics inversion and coinversion associated with a filling $\widehat{\sigma_\mu}$ are

$$\mathrm{inv}(\widehat{\sigma_\mu}) = \#\,\mathrm{HInv}(\widehat{\sigma_\mu}) - \#\{i < j \mid \mu_i \leq \mu_j\} - \sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} a(u),$$

$$\operatorname{coinv}(\widehat{\sigma_\mu}) = \Bigg(\sum_{u \in \operatorname{dg}(\mu)} a(u) \Bigg) - \operatorname{inv}(\widehat{\sigma_\mu}). \tag{4.1.17}$$

Lemma 4.1.1 tells us that each pair of attacking boxes, which is not in S, is covered by exactly one triple. So [HHL06, pg 9, Prop3.6.2] tells us that

$$\# \operatorname{Inv}^S(\widehat{\sigma_\mu}) = \sum_{(u,v) \in \operatorname{trip}(\widehat{\sigma_\mu})} \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}) \quad \text{and} \quad - \sum_{u \in \operatorname{Des}(\widehat{\sigma_\mu})} a(u) = \sum_{(u,v) \in \operatorname{trip}(\widehat{\sigma_\mu})} -\chi_{uw}. \tag{4.1.18}$$

The combinatorial nature of the inversion and coinversion statistics are revealed in Proposition 4.1.2.

Proposition 4.1.2. *Let* $\widehat{\sigma_\mu}$ *be a non-attacking filling of* μ*. Then*

$$\operatorname{inv}(\widehat{\sigma_\mu}) = \# \operatorname{invtrip}(\widehat{\sigma_\mu}), \qquad \textit{and} \qquad \operatorname{coinv}(\widehat{\sigma_\mu}) = \# \operatorname{coinvtrip}(\widehat{\sigma_\mu}).$$

Proof. Since Lemma 5.1.1, each attacking pair $(u, v) \in \widehat{\sigma_\mu} \backslash S$ is covered by exactly one triple and these are all the triples of $\widehat{\sigma_\mu}$. Hence,

$$\# \operatorname{Inv}^S(\widehat{\sigma_\mu}) = \sum_{(u,v) \in \operatorname{trip}(\widehat{\sigma_\mu})} \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}).$$

By (4.1.10),

$$\# \operatorname{HInv}(\widehat{\sigma_\mu}) - \#\{i < i' \mid \mu_i \leq \mu'_i\} = \sum_{(u,v) \in \operatorname{trip}(\widehat{\sigma_\mu})} \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}). \tag{4.1.19}$$

Combining the second equality in (4.1.18) with (4.1.19) gives

$$\sum_{(u,v) \in \operatorname{trip}(\widehat{\sigma_\mu})} \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}) - \chi_{uw}(\widehat{\sigma_\mu}) \tag{4.1.20}$$

$$= \# \operatorname{Inv}(\widehat{\sigma_\mu}) - \#\{i < i' \mid \mu_i \leq \mu'_i\} - \sum_{u \in \operatorname{Des}(\widehat{\sigma_\mu})} a(u) = \operatorname{inv}(\widehat{\sigma_\mu}). \tag{4.1.21}$$

By (4.1.14) and (4.1.15), the left hand side of (4.1.21) is

$$\sum_{(u,v) \in \operatorname{trip}(\widehat{\sigma_\mu})} \chi_{uv}(\widehat{\sigma_\mu}) + \chi_{vw}(\widehat{\sigma_\mu}) - \chi_{uw}(\widehat{\sigma_\mu}) = \# \operatorname{invtrip}(\sigma_\mu).$$

Then

$$\operatorname{inv}(\widehat{\sigma_\mu}) = \# \operatorname{invtrip}(\widehat{\sigma_\mu}).$$

Using (4.1.16), (4.1.17) and (4.1.13), we obtain

$$\operatorname{coinv}(\widehat{\sigma_\mu}) = \# \operatorname{trip}(\widehat{\sigma_\mu}) - \# \operatorname{invtrip}(\widehat{\sigma_\mu}) = \# \operatorname{coinvtrip}(\widehat{\sigma_\mu}).$$ □

Example 4.1.3. Let $\mu = (0, 3, 0, 2)$ and let

$$\heartsuit = \begin{array}{cccc} & \boxed{1} & & \\ & \boxed{3} & & \boxed{2} \\ & \boxed{1} & & \boxed{4} \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \end{array}$$

The statistics of the filling $\widehat{\sigma_\mu}$ are listed in the following table

Fillings	$\mathrm{Inv}(\widehat{\sigma_\mu})$	$\#\{i<j \mid \mu_i \le \mu_j\}$	$\sum_{u\in \mathrm{Des}(\widehat{\sigma_\mu})} a(u)$	$\sum_{u\in \mathrm{dg}(\mu)} a(u)$	coinv
♡	7	4	1	4	2

The set of coinversion triples (u,v,w) is

$$\mathrm{coinvtrip}(\widehat{\sigma_\mu}) = \left\{ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline\end{array}, \quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline\end{array} \right\}.$$

4.2 Properties of the staistics major, coinversion

In this section, we provide a brief survey of the proof in [HHL06, pg7, (3.5)] and the inductive construction used in [HHL06, pg4, Lemma2.1.2] for generating non-symmetric Macdonald polynomials. Through out the section, we denote $\#\,\mathrm{Harm}_\mu(u) = a(u)$ and $\#\,\mathrm{Hleg}_\mu(u) = \ell(u)$, where u is a box in $\mathrm{dg}(\mu)$.
Recall from (3.5.1) and (3.5.2) the actions of π, s_i on $\mathbb{R}^n$ are given by

$$\begin{aligned} \pi(\mu_1,\dots,\mu_n) &= (\mu_n+1,\mu_1,\dots,\mu_n), && \text{and} \\ s_i(\mu_1,\dots,\mu_n) &= (\mu_1,\dots,\mu_{i+1},\mu_i,\dots,\mu_n), && \text{for } i\in\{1,\dots,n-1\}. \end{aligned}$$

Let $\mu,\nu \in \mathbb{R}^n$, and let $w \in W_{\mathfrak{gl}_n}$. Write

$$\mu \xrightarrow{w} \nu \quad \text{if} \quad \mu = w\nu.$$

We recall the unwinding μ construction from Section 3.5 again for convenience. The **unwinding μ construction** of a non-symmetric Macdonald polynomial E_μ with $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{Z}_{\ge 0}$ can be described by the following steps:

1. Starting from $\mu = (\mu_1,\dots,\mu_n)$. Then unwind μ to obtain

$$\nu = \begin{cases} \pi^{-1}\mu & \text{if } \mu_1 > 0, \\ s_i\mu & \text{if } \mu_1 = 0 \text{ and } \mu_1 = 0, \mu_2 = 0, \dots, \mu_i = 0, \mu_{i+1} \neq 0. \end{cases}$$

2. If $\nu \neq \mathbf{0} = (0,0,\dots,0)$, then replace μ with ν and repeat step 1 until $\nu = \mathbf{0}$. Set $s_\pi = \pi$ and write

$$m_\mu = s_{i_1}\cdots s_{i_m}, \quad \text{for the resulting sequence.}$$

3. Constructing E_μ from $E_{\mathbf{0}} = 1$ by applying m_μ on $\mathbf{0}$ and using the following two equalities

$$E_\mu = E_{\pi\nu}(x_1,\dots,x_n) = q^{\nu_n} x_1 E_\nu(x_2,\dots,x_n,q^{-1}x_1), \tag{4.2.1}$$

$$E_\mu = E_{s_i\nu} = \left(T_i + \frac{1-t}{1-q^{\nu_i}t^{a(i,1)}}\right)E_\nu. \tag{4.2.2}$$

Lemma 4.2.1. *Let $\mu = s_i\nu$ for some positive integer i, then define*

$$G_0^\nu = \sum_{\substack{\widehat{\sigma_\nu} \text{ type } \nu \\ \widehat{\sigma_\nu}(i,1)\neq i}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu}), \quad G_1^\nu = \sum_{\substack{\widehat{\sigma_\nu} \text{ type } \nu \\ \widehat{\sigma_\nu}(i,1)= i}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu}) \quad \textit{so that} \quad E_\nu = G_0^\nu + G_1^\nu, \tag{4.2.3}$$

and

$$G_2 = \sum_{\substack{\widehat{\sigma_\mu} \textit{ type } \mu \\ \widehat{\sigma_\mu}(i+1,1)=i+1}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}). \tag{4.2.4}$$

Then

1. *G_0^ν is symmetric in x_i, x_{i+1};*
2. *$G_1^\nu + G_2$ is symmetric in x_i, x_{i+1};*
3. *$tx_{i+1}G_1^\nu + x_iG_2$ is symmetric in x_i, x_{i+1}.*

Proof. Reader may refer to [HHL06, pg 14-16]. ☐

Remark 4.2.2. We call those non-attacking fillings $\widehat{\sigma_\mu}$ in Lemma 4.2.1 satisfying $\widehat{\sigma_\mu}(i,1) \neq i$ to be Type G_0^μ and those non-attacking fillings $\widehat{\sigma_\mu}$ satisfying $\widehat{\sigma_\mu}(i,1) = i$ to be Type G_1^μ.

Lemma 4.2.3. *Let $f_1, f_2 \in \mathbb{Q}(q,t)X$, and $i \in \{1, \ldots, n-1\}$. Then the following are equivalent*

1. *$f_2 = T_if_1$;*
2. *$f_1 + f_2$ and $tx_{i+1}f_1 + x_if_2$ are symmetric in x_i, x_{i+1}.*

Proof. Reader may refer to [HHL06, pg15, Lemma 4.3.1] ☐

Corollary 4.2.4. *Let G_0^ν be as in (4.2.3). Then $T_iG_0^\nu = tG_0^\nu$.*

Proof. Replacing the action in Proposition 3.4.2 with HHL action $T_i\mathbf{1} = t\mathbf{1}$ on the polynomial representation gives

$$T_iX^\mu = \left(t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1 - s_i)\right)X^\mu. \tag{4.2.5}$$

Then by Lemma 4.2.1(1),

$$T_iG_0^\nu = \left(t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1 - s_i)\right)G_0^\nu = tG_0^\nu. \qquad \square$$

Recall from Section 2.4, Lemma 2.4.1, for $\mu \in \mathbb{Z}^n_{\geq 1}$

$$\#\,\mathrm{Harm}(i,1) = w_\mu^{-1}(i) - 1.$$

This gives us a simple way to enumerate $\#\,\mathrm{Harm}(i,1)$.

Lemma 4.2.5. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$ such that $\mu_1 = 0$ and $\mu_1 = 0, \mu_2 = 0, \ldots, \mu_i = 0, \mu_{i+1} \neq 0$, and let $a(u) = \#\,\mathrm{Harm}_\mu(u)$, $\ell(u) = \#\,\mathrm{Hleg}_\mu(u)$. Then*

$$E_{s_i\mu} = \left(T_i + \frac{1-t}{1 - q^{\mu_i}t^{a(i,1)}}\right)E_\mu = \left(\frac{1 - q^{\mu_i}t^{a(i,1)+1}}{1 - q^{\mu_i}t^{a(i,1)}}\right)G_0^\mu + \left(\frac{1-t}{1 - q^{\mu_i}t^{a(i,1)}}\right)G_1^\mu + G_2.$$

Proof. Item (2), (3) of Lemma 4.2.1 and Lemma 4.2.3 implies

$$G_2 = T_iG_1^\mu \quad \text{for } i \in \{1, \ldots, n-1\}. \tag{4.2.6}$$

By Corollary 4.2.4 and (4.2.6),

$$\begin{aligned}\left(T_i+\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)E_\mu &= \left(T_i+\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)(G_0^\mu+G_1^\mu)\\ &= T_iG_0^\mu+\left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\mu+T_iG_1^\mu+\left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\mu\\ &= tG_0^\mu+\left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\mu+G_2+\left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\mu\\ &= \left(\frac{t-q^{\mu_i}t^{a(i,1)+1}+1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\mu+\left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\mu+G_2\\ &= \left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\mu+\left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\mu+G_2. \qquad \square\end{aligned}$$

Remark 4.2.6. Lemma 4.2.5 simplifies the inductive construction of Macdonald polynomials with $E_\mu = E_{s_i\nu}$. For example, if $\mu = (0,2,1)$, then $\nu = s_1\mu = (2,0,1)$ and the non-attacking fillings $\widehat{\sigma_\nu}$ and their weights are

$$\clubsuit = \begin{array}{|c|c|c|} \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\cline{3-3} 1 & & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \heartsuit = \begin{array}{|c|c|c|} \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\cline{3-3} 1 & & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \diamondsuit = \begin{array}{|c|c|c|} \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\cline{3-3} 1 & & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \spadesuit = \begin{array}{|c|c|c|} \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\cline{3-3} 1 & & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\tfrac{1-t}{1-qt}x_1^2x_2 \qquad x_1^2x_3 \qquad qt\big(\tfrac{1-t}{1-qt^2}\big)x_1x_2x_3 \qquad q\big(\tfrac{1-t}{1-qt^2}\big)\big(\tfrac{1-t}{1-qt}\big)x_1x_2x_3.$$

Fillings	Des	# Inv	inv	maj	coinv
♣	$\emptyset$	4	3	0	0
♡	$\emptyset$	4	3	0	0
♢	$\{(1,2)\}$	4	2	1	1
♠	$\{(1,2)\}$	5	3	1	0

Moreover, the non-symmetric Macdonald polynomial

$$E_\nu = E_{(2,0,1)} = \left(\frac{1-t}{1-qt}\right)x_1^2x_2+x_1^2x_3+qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3+q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3.$$

The corresponding G_1^ν is

$$G_1^\nu = \left(\frac{1-t}{1-qt}\right)x_1^2x_2+x_1^2x_3+qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3+q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3. \tag{4.2.7}$$

The non-attacking fillings $\widehat{\sigma_\mu}$ and their weights are

$$\clubsuit = \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \heartsuit = \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 1 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \diamondsuit = \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \spadesuit = \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 1 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\tfrac{1-t}{1-qt^2}x_1x_2x_3 \qquad \tfrac{1-t}{1-qt}x_1x_2^2 \qquad x_2^2x_3 \qquad qt\big(\tfrac{1-t}{1-qt^2}\big)\big(\tfrac{1-t}{1-qt}\big)x_1x_2x_3$$

Fillings	Des	maj	# Inv	inv	coinv
♣	$\emptyset$	0	4	2	0
♡	$\emptyset$	0	4	2	0
♢	$\emptyset$	0	4	2	0
♠	$\{(2,2)\}$	1	4	1	1

Thus,

$$G_2 = \left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 + \frac{1-t}{1-qt}x_1x_2^2 + x_2^2x_3 + qt\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3.$$

Using $G_0^\nu = 0$, then (4.2.7) and Lemma(4.2.5) imply

$$\begin{aligned}
E_\mu = E_{s_1(2,0,1)} &= \left(\frac{1-q^{\nu_i}t^{a(i,1)+1}}{1-q^{\nu_i}t^{a(i,1)}}\right)G_0^\nu + \left(\frac{1-t}{1-q^{\nu_i}t^{a(i,1)}}\right)G_1^\nu + G_2 \\
&= \left(\frac{1-t}{1-q^2t^2}\right)\Bigg(\left(\frac{1-t}{1-qt}\right)x_1^2x_2 + qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 \qquad \text{(4.2.8)} \\
&\quad +q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3 + x_1^2x_3\Bigg) + \left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 \\
&\quad +\frac{1-t}{1-qt}x_1x_2^2 + x_2^2x_3 + qt\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3 \\
&= \left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)x_1^2x_2 + qt\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 \\
&\quad + q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3 + \left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 \\
&\quad + \frac{1-t}{1-qt}x_1x_2^2 + x_2^2x_3 + qt\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3 \\
&\quad + \left(\frac{1-t}{1-q^2t^2}\right)x_1^2x_3,
\end{aligned}$$

where in (4.2.8) the part highlighted with blue corresponding to G_1^ν and highlighted with red corresponding to G_2.

Lemma 4.2.7. *Let $i \in \{1,\dots,n-1\}$ and $\mu = s_i\nu$ with $\nu_i \neq 0$ and $\nu_{i+1} = 0$. Then*

1. $\operatorname{maj}(\widehat{\sigma_{s_i\nu}}) = \operatorname{maj}(\widehat{\sigma_\nu})$;
2. $\operatorname{coinv}(\widehat{\sigma_{s_i\nu}}) = \operatorname{coinv}(\widehat{\sigma_\nu})$;
3.

$$\operatorname{wt}(\widehat{\sigma_\mu}) = \operatorname{wt}(\widehat{\sigma_{s_i\nu}}) = \begin{cases} \left(\dfrac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)\operatorname{wt}(\widehat{\sigma_\nu}), & \textit{for } \widehat{\sigma_\nu}(i,1) \neq i, \\ \left(\dfrac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)\operatorname{wt}(\widehat{\sigma_\nu}), & \textit{for } \widehat{\sigma_\nu}(i,1) = i. \end{cases}$$

Proof. The equation

$$\#\operatorname{Des}(\widehat{\sigma_{s_i\nu}}) = \#\{(i,j) \in \operatorname{dg}(s_i\nu) \mid \widehat{\sigma_{s_i\nu}}(i,j) > \widehat{\sigma_{s_i\nu}}(i,j-1)\} = \#\operatorname{Des}(\widehat{\sigma_\nu}) \tag{4.2.9}$$

gives

$$\operatorname{maj}(\widehat{\sigma_{s_i\nu}}) = \sum_{u\in\operatorname{Des}(\widehat{\sigma_{s_i\nu}})} \ell(u)+1 = \sum_{u\in\operatorname{Des}(\widehat{\sigma_\nu})} \ell(u)+1 = \operatorname{maj}(\widehat{\sigma_\nu}).$$

As $\nu_i \neq 0$ and $\nu_{i+1} = 0$,

$$\#\{i<j \mid \mu_i \le \mu_j\} = \#\{i<j \mid \nu_i \le \nu_j\} + 1.$$

Then $\#\operatorname{Inv}(\widehat{\sigma_{s_i\nu}}) = \#\operatorname{Inv}(\widehat{\sigma_\nu})$ and (4.2.9) give

$$\begin{aligned}
\operatorname{inv}(\widehat{\sigma_{s_i\nu}}) &= \#\operatorname{Inv}(\widehat{\sigma_{s_i\nu}}) - \#\{i<j \mid \mu_i \leq \mu_j\} - \sum_{u\in \operatorname{Des}(\widehat{\sigma_{s_i\nu}})} a(u) \\
&= \#\operatorname{Inv}(\widehat{\sigma_\nu}) - (\#\{i<j \mid \nu_i \leq \nu_j\} + 1) - \sum_{u\in \operatorname{Des}(\widehat{\sigma_\nu})} a(u) \\
&= \#\operatorname{Inv}(\widehat{\sigma_\nu}) - \#\{i<j \mid \nu_i \leq \nu_j\} - 1 - \sum_{u\in \operatorname{Des}(\widehat{\sigma_\nu})} a(u) \\
&= \operatorname{inv}(\widehat{\sigma_\nu}) - 1.
\end{aligned}$$

Since $\sum_{u\in \operatorname{dg}(\widehat{s_i\nu})} a(u) = \big(\sum_{u\in \operatorname{dg}(\widehat{\nu})} a(u)\big) - 1$, it follows that

$$\operatorname{coinv}(\widehat{\sigma_{s_i\nu}}) = \sum_{u\in \widehat{\operatorname{dg}}(s_i\nu)} a(u) - \operatorname{inv}(\widehat{\sigma_{s_i\nu}}) = \Bigg(\sum_{u\in \operatorname{dg}(\nu)} a(u)\Bigg) - 1 - \operatorname{inv}(\widehat{\sigma_\nu}) + 1 = \operatorname{coinv}(\widehat{\sigma_\nu}).$$

If $\widehat{\sigma_\nu}(i,1) \neq i$, then $\#\operatorname{Hleg}(i,1) + 1 = \mu_i - 1 + 1 = \mu_i$ and

$$\begin{aligned}
\operatorname{wt}(\widehat{\sigma_{s_i\nu}}) &= X^{\widehat{\sigma_{s_i\nu}}} q^{\operatorname{maj}(\widehat{\sigma_{s_i\nu}})} t^{\operatorname{coinv}(\widehat{\sigma_{s_i\nu}})} \prod_{\substack{u\in \operatorname{dg}(s_i\nu) \\ \widehat{\sigma_{s_i\nu}}(u-n)\neq \widehat{\sigma_{s_i\nu}}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}} \\
&= X^{\widehat{\sigma_\nu}} q^{\operatorname{maj}(\widehat{\sigma_\nu})} t^{\operatorname{coinv}(\widehat{\sigma_\nu})} \prod_{\substack{u\in \operatorname{dg}(\nu) \\ \widehat{\sigma_\nu}(u-n)\neq \widehat{\sigma_\nu}(u) \\ u\neq (i,1)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)}} \frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}.
\end{aligned}$$

Since

$$\begin{aligned}
\left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)\operatorname{wt}(\widehat{\sigma_\nu}) &= \left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right) X^{\widehat{\sigma_\nu}} q^{\operatorname{maj}(\widehat{\sigma_\nu})} t^{\operatorname{coinv}(\widehat{\sigma_\nu})} \\
&\qquad \prod_{\substack{u\in \operatorname{dg}(\nu) \\ \widehat{\sigma_\nu}(u-n)\neq \widehat{\sigma_\nu}(u) \\ u\neq (i,1)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}} \frac{1-t}{1-q^{\mu_i}t^{a(i,1)}+1} \\
&= q^{\operatorname{maj}(\widehat{\sigma_\nu})} t^{\operatorname{coinv}(\widehat{\sigma_\nu})} \\
&\qquad \prod_{\substack{u\in \operatorname{dg}(\nu) \\ \widehat{\sigma_\nu}(u-n)\neq \widehat{\sigma_\nu}(u) \\ u\neq (i,1)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)}} \frac{1-t}{1-q^{\mu_i}t^{a(i,1)}},
\end{aligned}$$

then

$$\operatorname{wt}(\widehat{\sigma_{s_i\nu}}) = \left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)\operatorname{wt}(\widehat{\sigma_\nu}) \quad \text{for} \quad \widehat{\sigma_\nu}(i,1) \neq i.$$

Consider the case where $\widehat{\sigma_\nu}(i,1) = i$, then $\widehat{\sigma_{s_i\nu}}(i+1,1) = i \neq i+1$. Hence,

$$\operatorname{wt}(\widehat{\sigma_{s_i\nu}}) = \left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)\operatorname{wt}(\widehat{\sigma_\nu}). \qquad \square$$

Example 4.2.1. For $\nu = (0,2,0)$, the non-attacking fillings of Type G_0^ν, G_1^ν and their weights are

$$\diamondsuit = \begin{ytableau} \none & 1 \\ \none & 1 \\ 1 & 2 & 3 \end{ytableau} \qquad \heartsuit = \begin{ytableau} \none & 2 \\ \none & 1 \\ 1 & 2 & 3 \end{ytableau} \qquad \clubsuit = \begin{ytableau} \none & 3 \\ \none & 1 \\ 1 & 2 & 3 \end{ytableau}$$

$$\left(\tfrac{1-t}{1-q^2t^2}\right)X_1^2 \qquad q\left(\tfrac{1-t}{1-q^2t^2}\right)\left(\tfrac{1-t}{1-qt}\right)X_1X_2 \qquad q\left(\tfrac{1-t}{1-q^2t^2}\right)\left(\tfrac{1-t}{1-qt}\right)X_1X_3$$

$$\spadesuit = \begin{ytableau} \none & 1 \\ \none & 2 \\ 1 & 2 & 3 \end{ytableau} \qquad \blacktriangledown\!\!\!\heartsuit = \begin{ytableau} \none & 2 \\ \none & 2 \\ 1 & 2 & 3 \end{ytableau} \qquad \blacklozenge = \begin{ytableau} \none & 3 \\ \none & 2 \\ 1 & 2 & 3 \end{ytableau}$$

$$\left(\tfrac{1-t}{1-qt}\right)X_1X_2 \qquad X_2^2 \qquad q\left(\tfrac{1-t}{1-qt}\right)X_2X_3$$

Fillings	Des	# Inv	inv	maj	coinv
♢	$\emptyset$	3	1	0	0
♡	$\{(2,2)\}$	3	1	1	0
♣	$\{(2,2)\}$	3	1	1	0

Fillings	Des	# Inv	inv	maj	coinv
♠	$\emptyset$	3	1	0	0
♥	$\emptyset$	3	1	0	0
♦	$\{(2,2)\}$	3	1	1	0

The corresponding G_0^ν and G_1^ν are

$$G_0^\nu = \left(\frac{1-t}{1-q^2t^2}\right)X_1^2 + q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)X_1X_2 + q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)X_1X_3,$$
$$G_1^\nu = \left(\frac{1-t}{1-qt}\right)X_1X_2 + X_2^2 + q\left(\frac{1-t}{1-qt}\right)X_2X_3.$$

For $\mu = (0,0,2)$, the non-attacking fillings of type G_0^ν, G_1^ν(defined in Remark 4.2.2) and their weights are

$$\clubsuit = \begin{ytableau} \none & \none & 1 \\ \none & \none & 1 \\ 1 & 2 & 3 \end{ytableau} \qquad \diamondsuit = \begin{ytableau} \none & \none & 2 \\ \none & \none & 1 \\ 1 & 2 & 3 \end{ytableau} \qquad \spadesuit = \begin{ytableau} \none & \none & 3 \\ \none & \none & 1 \\ 1 & 2 & 3 \end{ytableau}$$

$$\left(\tfrac{1-t}{1-q^2t}\right)X_1^2 \qquad q\left(\tfrac{1-t}{1-qt}\right)\left(\tfrac{1-t}{1-q^2t}\right)X_1X_2 \qquad q\left(\tfrac{1-t}{1-qt}\right)\left(\tfrac{1-t}{1-q^2t}\right)X_1X_3$$

$$\blacklozenge = \begin{ytableau} \none & \none & 1 \\ \none & \none & 2 \\ 1 & 2 & 3 \end{ytableau} \qquad \blacktriangledown\!\!\!\heartsuit = \begin{ytableau} \none & \none & 2 \\ \none & \none & 2 \\ 1 & 2 & 3 \end{ytableau} \qquad \heartsuit = \begin{ytableau} \none & \none & 3 \\ \none & \none & 2 \\ 1 & 2 & 3 \end{ytableau}$$

$$\left(\tfrac{1-t}{1-qt}\right)\left(\tfrac{1-t}{1-q^2t}\right)X_1X_2 \qquad \left(\tfrac{1-t}{1-q^2t}\right)X_2^2 \qquad q\left(\tfrac{1-t}{1-qt}\right)\left(\tfrac{1-t}{1-q^2t}\right)X_2X_3$$

Fillings	Des	# Inv	inv	maj	coinv
♣	$\emptyset$	3	0	0	0
♢	$\{(3,2)\}$	3	0	1	0
♠	$\{(3,2)\}$	3	0	1	0
♦	$\emptyset$	3	0	0	0
♥	$\emptyset$	3	0	0	0
♡	$\{(3,2)\}$	3	0	1	0

$$G_0^\mu = \left(\frac{1-t}{1-q^2t}\right)X_1^2 + q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)X_1X_2 + q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)X_1X_3$$
$$+\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)X_1X_2 + \left(\frac{1-t}{1-q^2t}\right)X_2^2 + q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)X_2X_3.$$

The non-attacking fillings of Type G_0^μ separate into the non-attacking fillings of G_0^ν(highlighted with red) and G_1^ν(highlighted with blue). The major, coinversion and the corresponding weights match the result in item 1, 2, 3 of Lemma 4.2.7. We remark that when $\widehat{\sigma_\mu}(i,1) \neq i$, the cancellation occurs in $\mathrm{wt}(\widehat{\sigma_{s_i\mu}})$, and when $\widehat{\sigma_\mu}(i,1) = i$ we obtain a factor of fraction in the weight.

Lemma 4.2.8. *Let $\mu, \nu \in \mathbb{Z}^n_{\geq 0}$ such that $\mu = \pi\nu$ and let $\mathcal{N}_\nu$ and $\mathcal{N}_\mu$ be the set of non-attacking fillings of ν and μ respectively. Then there is a bijection*

$$\pi : \mathcal{N}_\nu \to \mathcal{N}_\mu,$$
$$\widehat{\sigma_\nu} \mapsto \widehat{\sigma_\mu},$$

where

$$\widehat{\sigma_\mu}(i,j) = \begin{cases} \widehat{\sigma_\nu}(i-1,j)+1, & \text{for } i \in \{2,\ldots,n\} \text{ and } j \in \{1,\ldots,n\} \text{ and } \widehat{\sigma_\nu}(i-1,j) \neq n, \\ \widehat{\sigma_\nu}(n,j-1)+1, & \text{for } i = 1 \text{ and } j \in \{2,\ldots,n\} \text{ and } \widehat{\sigma_\nu}(i-1,j) \neq n, \\ 1, & \text{otherwise }. \end{cases}$$

Proof. Since for $\widehat{\sigma_\nu}(i-1,j) = n$, $\widehat{\sigma_\mu}(i,j) = 1$. Hence, the map π is well-defined. It remains to show that π is injective and there exists a inverse of π.
Let $\widehat{\sigma_\nu}, \widehat{\sigma_\nu}'$ be two fillings of ν such that $\pi(\widehat{\sigma_\nu}) = \pi(\widehat{\sigma_\nu}')$.
Then

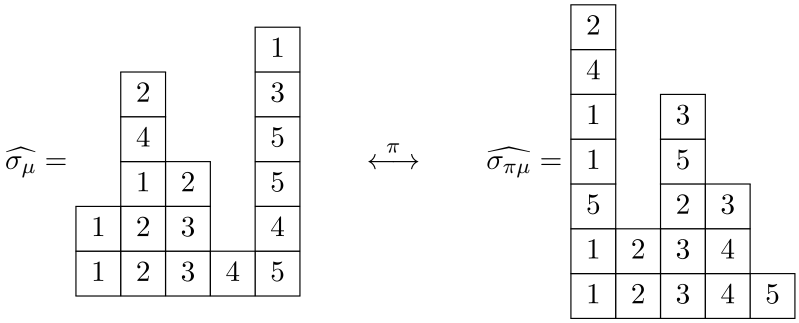
$$\widehat{\sigma_\nu}(i-1,j)+1 = \widehat{\sigma_\nu}'(i-1,j)+1 \quad \widehat{\sigma_\nu}(i-1,j) = \widehat{\sigma_\nu}'(i-1,j)$$

for $i \in \{2,\ldots,n\}$ and $j \in \{1,\ldots,n\}$ and

$$\widehat{\sigma_\nu}(n,j-1)+1 = \widehat{\sigma_\nu}'(n,j-1)+1$$

for $i = 1$. Hence, $\widehat{\sigma_\nu} = \widehat{\sigma_\nu}'$ and π is injective. The inverse is not difficult to define, I will illustrate with an example below. ☐

Example 4.2.2. For $\mu = (1,4,2,0,5)$, the map π can be described as



We state the following observation of [HHL06, Lemma4.1.1] in Lemma 4.2.9.

Lemma 4.2.9. *Let $u \in \mathrm{dg}(\mu)$, and $v \in \widehat{\mathrm{dg}}(\mu)$. Then*

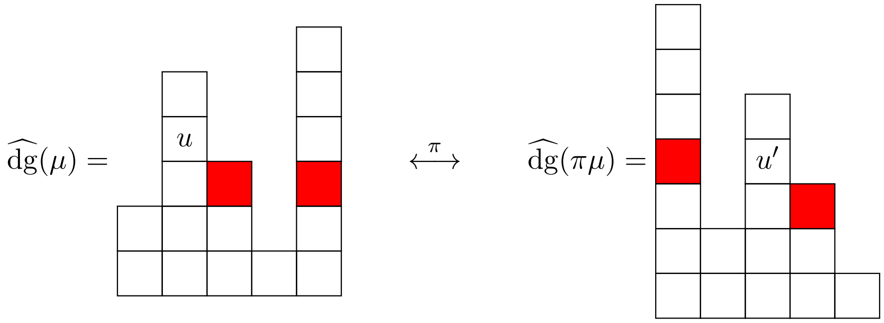
$$v \in \mathrm{Harm}_\mu(u) \text{ if and only if } \pi(v) \in \mathrm{Harm}_\mu(\pi(u)).$$

Proof. Let $v \in \mathrm{Harm}_\mu(u)$. By (2.4.3), $v \in \mathrm{attack}_\mu(u)$ and $\#\,\mathrm{Hleg}(v) \leq \#\,\mathrm{Hleg}(u)$. Identify each box with it's cylindrical coordinates, then $v \in \{u-1, \dots, u-n+1\} \cap \widehat{\mathrm{dg}}(\mu)$.
Since $\pi(v) = v+1$ and $\pi(u) = u+1$, we have

$$\pi(v) \in \{u, \dots, u-n+2\} = \mathrm{attack}_\mu(\pi(u)).$$

It remains to show that $\#\,\mathrm{Hleg}(\pi(v)) \leq \#\,\mathrm{Hleg}(\pi(u))$. Since $\#\,\mathrm{Hleg}(\pi(v)) = \#\,\mathrm{Hleg}(v)$ and $\#\,\mathrm{Hleg}(\pi(u)) = \#\,\mathrm{Hleg}(u)$, $\#\,\mathrm{Hleg}(\pi(v)) \leq \#\,\mathrm{Hleg}(\pi(u))$.
For the converse, let $\pi(v) \in \mathrm{Harm}_\mu(\pi(u))$. Then $\pi(v) \in \mathrm{attack}_\mu(\pi(u))$. By the same augment as before, $v \in \mathrm{attack}_\mu(u)$. Similarly, $\#\,\mathrm{Hleg}(\pi(v)) \leq \#\,\mathrm{Hleg}(\pi(u))$. Hence, $v \in \mathrm{Harm}_\mu(u)$. ☐

Example 4.2.3. Take $\mu = (1,4,2,0,5)$ as in Example 4.2.2, then $\pi\mu = (6,1,4,2,0)$. Let $u = (2,2)$ be a box in $\widehat{\mathrm{dg}}(\mu)$ and $u' = \pi(u) = (3,2)$. Then the column diagrams are



The set of boxes in $\mathrm{attack}_\mu(u)$ on the left diagram and the set of boxes in $\mathrm{attack}_\mu(\pi(u))$ are highlighted by red in both diagrams. We observe that attacking boxes wrap around so are the boxes in $\mathrm{Harm}_\mu(u)$.

Proposition 4.2.10. *(HHL06, pg 11, Prop4.1.2) Let $\widehat{\sigma_\mu}$ be a non-attacking filling, and let $\mu, \nu \in \mathbb{Z}_{\geq 0}$ such that $\mu = \pi\nu$. Then*

$$\#\,\mathrm{coinv}(\widehat{\sigma_\mu}) = \#\,\mathrm{coinv}(\widehat{\sigma_\nu}).$$

Proof. By Proposition 4.1.2, $\mathrm{coinv}(\widehat{\sigma_\mu}) = \#\,\mathrm{coinvtrip}(\sigma_\mu)$. It is enough to show the map

$$\begin{aligned} \pi_{\mathrm{trip}} : \mathrm{coinvtrip}(\widehat{\sigma_\nu}) &\to \mathrm{coinvtrip}(\widehat{\sigma_\mu}) \\ (u,v) &\mapsto (\pi(u), \pi(v)) \end{aligned}$$

is a bijection, where π is defined in Lemma 4.2.8. Let $(u,v) \in \mathrm{trip}(\widehat{\sigma_\nu})$. Then $v \in \mathrm{Harm}_\nu(u)$. By Lemma 4.2.9, $\pi(v) \in \mathrm{Harm}_\nu(\pi(u))$. Hence, $(\pi(u), \pi(v)) \in \mathrm{trip}(\widehat{\sigma_{\pi\nu}}) = \mathrm{trip}(\widehat{\sigma_\mu})$. Now we need to show that if $(u,v) \in \mathrm{coinvtrip}(\widehat{\sigma_\mu})$, then $(\pi(u), \pi(v)) \in \mathrm{coinvtrip}(\widehat{\sigma_\mu})$. Define

$$\begin{aligned} S &= \{(u,v) \in \mathrm{coinvtrip}(\widehat{\sigma_\nu}) \mid n \notin \{\widehat{\sigma_\mu}(u), \widehat{\sigma_\mu}(v), \widehat{\sigma_\mu}(w)\}\} \quad \text{and} \\ S' &= \{(u,v) \in \mathrm{coinvtrip}(\widehat{\sigma_\nu}) \mid n \in \{\widehat{\sigma_\mu}(u), \widehat{\sigma_\mu}(v), \widehat{\sigma_\mu}(w)\}\} \quad \text{and} \\ E &= \{(u,v) \in \mathrm{coinvtrip}(\widehat{\sigma_\nu}) \mid (n,i) \notin \{u,v,w\} \text{ for } i \in \{1, \dots, \mu_n\}\}. \end{aligned}$$

Then $\operatorname{coinvtrip}(\widehat{\sigma_\mu}) = S \cup S'$. Assume $(u, v) \in S \backslash E$. Then $(\pi(u), \pi(v))$ is a triple with the same shape as (u, v) i.e.,

$$\begin{array}{ccccc} \boxed{u} & & & \boxed{u'} & \\ \boxed{w} & \boxed{v} & \xrightarrow{\pi} & \boxed{w'} & \boxed{v'} \end{array}$$

$$\begin{array}{ccccc} \boxed{v} & \boxed{u} & \xrightarrow{\pi} & \boxed{v'} & \boxed{u'} \\ & \boxed{w} & & & \boxed{w'} \end{array}$$

where $u' = \pi(u)$, $v' = \pi(v)$ and $w' = \pi(w)$. Since fillings of the coinversion triples increase in either a clockwise order or a counterclockwise order, adding one on the filling in each box does not change the order. Hence, for triples $(u, v) \in S \cap E$, $(\pi(u), \pi(v)) \in \operatorname{coinvtrip}(\widehat{\sigma_\mu})$. Consider the triples $(u, v) \in S \backslash E$ of Type A. Then there are three cases:
Case 1: if $\widehat{\sigma_\nu}(u) < \widehat{\sigma_\nu}(w) < \widehat{\sigma_\nu}(v)$, then $\widehat{\sigma_\mu}(u') < \widehat{\sigma_\mu}(\pi(w)) < \widehat{\sigma_\mu}(\pi(v))$.
Case 2: if $\widehat{\sigma_\nu}(w) < \widehat{\sigma_\nu}(v) < \widehat{\sigma_\nu}(u)$, then $\widehat{\sigma_\mu}(\pi(w)) < \widehat{\sigma_\mu}(\pi(v)) < \widehat{\sigma_\mu}(\pi(u))$.
Case 3: if $\widehat{\sigma_\nu}(v) < \widehat{\sigma_\nu}(u) < \widehat{\sigma_\nu}(w)$, then $\widehat{\sigma_\mu}(\pi(v)) < \widehat{\sigma_\mu}(\pi(u)) < \widehat{\sigma_\mu}(\pi(w))$.
The following diagram illustrates how the shape changes

$$\begin{array}{ccccc} \boxed{u} & & & \boxed{v'} & \boxed{u'} \\ \boxed{w} & \boxed{v} & \xrightarrow{\pi} & & \boxed{w'} \end{array}$$

where $u' = \pi(u), v' = \pi(v)$ and $w' = \pi(w)$. The triples $(u, v) \in S \backslash E$ of Type B can be analysed in the same way and the shape after applying π is

$$\begin{array}{ccccc} \boxed{v} & \boxed{u} & \xrightarrow{\pi} & \boxed{u'} & \\ & \boxed{w} & & \boxed{w'} & \boxed{v'} \end{array}$$

Hence, for triples $(u, v) \in S \backslash E$, $(\pi(u), \pi(v)) \in \operatorname{coinvtrip}(\widehat{\sigma_\mu})$. For triples $(u, v) \in S' \cap E$, there are three cases for each Type A and Type B triples according to the position of n. We are only going to illustrate the coinversion triples of type B. Type A triples can be done in the same way.
Case 1: If $u = n$ and $\widehat{\sigma_\nu}(w) < \widehat{\sigma_\nu}(v) < \widehat{\sigma_\nu}(u)$, then $\widehat{\sigma_\mu}(\pi(u)) < \widehat{\sigma_\mu}(\pi(w)) < \widehat{\sigma_\mu}(\pi(v))$ and

$$\begin{array}{ccccc} \boxed{v} & \boxed{n} & \xrightarrow{\pi} & \boxed{v'} & \boxed{1} \\ & \boxed{w} & & & \boxed{w'} \end{array}$$

Case 2: If $w = n$ and $\widehat{\sigma_\nu}(v) < \widehat{\sigma_\nu}(u) < \widehat{\sigma_\nu}(w)$, then $\widehat{\sigma_\mu}(\pi(w)) < \widehat{\sigma_\mu}(\pi(v)) < \widehat{\sigma_\mu}(\pi(u))$ and

$$\begin{array}{ccccc} \boxed{v} & \boxed{u} & \xrightarrow{\pi} & \boxed{v'} & \boxed{u'} \\ & \boxed{n} & & & \boxed{1} \end{array}$$

Case 3: If $v = n$ and $\widehat{\sigma_\nu}(u) < \widehat{\sigma_\nu}(w) < \widehat{\sigma_\nu}(v)$, then $\widehat{\sigma_\mu}(\pi(v)) < \widehat{\sigma_\mu}(\pi(u)) < \widehat{\sigma_\mu}(\pi(w))$ and

$$\begin{array}{ccccc} \boxed{v} & \boxed{n} & \xrightarrow{\pi} & \boxed{v'} & \boxed{1} \\ & \boxed{w} & & & \boxed{w'} \end{array}$$

Consider the triples $(u, v) \in S' \backslash E$. Then there are only two cases we need to consider.
Case 1: For type A triples, if $v = n$ and $\widehat{\sigma_\nu}(u) < \widehat{\sigma_\nu}(w) < \widehat{\sigma_\nu}(v)$, then $\widehat{\sigma_\mu}(\pi(v)) < \widehat{\sigma_\mu}(\pi(u)) < \widehat{\sigma_\mu}(\pi(w))$ and

$$\begin{array}{ccccc} \boxed{u} & & & \boxed{1} & \boxed{u'} \\ \boxed{w} & \boxed{n} & \xrightarrow{\pi} & & \boxed{w'} \end{array}$$

Case 2: For type B, if $u = n$ and $\widehat{\sigma_\nu}(w) < \widehat{\sigma_\nu}(v) < \widehat{\sigma_\nu}(u)$, then $\widehat{\sigma_\mu}(\pi(u)) < \widehat{\sigma_\mu}(\pi(w)) < \widehat{\sigma_\mu}(\pi(v))$ and

$$\begin{array}{|c|}\hline v\\ \hline \end{array} \qquad \begin{array}{|c|}\hline n\\ \hline w\\ \hline \end{array} \xrightarrow{\pi} \begin{array}{|c|}\hline 1\\ \hline w'\\ \hline \end{array} \qquad \begin{array}{|c|}\hline v'\\ \hline \end{array}$$

Similarly argument can be done for $w = n$. Hence, for a coinversion triple in $\widehat{\sigma_\nu}$ the map π_{trip} sends it to another coinversion triple in $\widehat{\sigma_\mu}$. Thus, the map π_{trip} is well-defined. Since π is a bijection, if $(\pi(u), \pi(v)) = (\pi(x), \pi(y))$, then $u = x$ and $v = y$. Hence, π_{trip} is injective. It remains to show that π_{trip} is surjective. It has been shown from the above argument that all the triples in $\widehat{\sigma_\mu}$ which do not include the box $(1, 0)$ are in the image of π_{trip}. It remains to consider these triples containing the box $(1, 0)$. Since the filling $\widehat{\sigma_\mu}$ is a non-attacking filling, $\widehat{\sigma_\mu}(1, 0) = \widehat{\sigma_\mu}(1, 1) = 1$. Hence, those triples containing the box $(1, 0)$ can not be a coinversion triple. Therefore, the map π_{trip} is surjective. ☐

Lemma 4.2.11. *(HHL06, pg 11, Prop4.1.2) Let* $\mu, \nu \in \mathbb{Z}^n_{\geq 0}$ *such that* $\mu = \pi\nu$, *and let* $\widehat{\sigma_\nu}$ *be a non-attacking filling containing* r *numbers of* n *and* $\widehat{\sigma_\mu}$ *be the corresponding non-attacking filling. Then*

1. $\operatorname{maj}(\widehat{\sigma_\mu}) = \nu_n - r + \operatorname{maj}(\widehat{\sigma_\nu})$;
2. $\operatorname{coinv}(\widehat{\sigma_\mu}) = \operatorname{coinv}(\widehat{\sigma_\nu})$;
3. $X^{\widehat{\sigma_\mu}} = X_1^{(\# n \text{ in } \widehat{\sigma_\nu})+1} X_2^{\# 1 \text{ in } \widehat{\sigma_\nu}} \cdots X_n^{\#(n-1) \text{ in } \widehat{\sigma_\nu}}$.

Proof. To show (1): Let $u = (i, j) \in \operatorname{dg}(\mu)$ and $d(u) = (i, j-1) \in \widehat{\operatorname{dg}}(\mu)$. Consider the following two sets

$$S = \{u \in \operatorname{dg}(\mu) \mid \widehat{\sigma_\nu}(u) = n\},$$
$$S' = \{u \in \operatorname{dg}(\mu) \mid \widehat{\sigma_\nu}(d(u)) = n\}.$$

Then $\pi(u) \in \operatorname{Des}(\widehat{\sigma_{\pi\nu}})$ if and only if $u \in \operatorname{Des}(\widehat{\sigma_\nu}) \backslash S$ or $u \in S' \backslash S$. If $u \in \operatorname{dg}(\mu)$, then $\ell(u) = \ell(\pi(u))$. Since $\ell(\pi(u)) = \ell(u)$, it gives

$$\operatorname{maj}(\widehat{\sigma_{\pi\nu}}) = \sum_{u \in \operatorname{Des}(\widehat{\sigma_{\pi\nu}})} \ell(u) + 1 = \sum_{u \in \operatorname{Des}(\widehat{\sigma_\nu}) \backslash S} (\ell(u) + 1) + \sum_{u \in S' \backslash S} (\ell(u) + 1).$$

Since $S \cap \operatorname{Des}(\widehat{\sigma_\nu}) = S \backslash S'$,

$$\begin{aligned}
\operatorname{maj}(\widehat{\sigma_{\pi\nu}}) &= \sum_{u \in \operatorname{Des}(\widehat{\sigma_\nu}) \backslash S} (\ell(u) + 1) + \sum_{u \in S' \backslash S} (\ell(u) + 1) \\
&= \sum_{u \in \operatorname{Des}(\widehat{\sigma_\nu})} (\ell(u) + 1) - \sum_{u \in S \backslash S'} (\ell(u) + 1) + \sum_{u \in S' \backslash S} (\ell(u) + 1) \\
&= \sum_{u \in \operatorname{Des}(\widehat{\sigma_\nu})} (\ell(u) + 1) - \sum_{u \in S} (\ell(u) + 1) + \sum_{u \in S \cap S'} (\ell(u) + 1) + \sum_{u \in S'} (\ell(u) + 1) \\
&\qquad - \sum_{u \in S \cap S'} (\ell(u) + 1) \\
&= \sum_{u \in \operatorname{Des}(\widehat{\sigma_\nu})} (\ell(u) + 1) - \sum_{u \in S} (\ell(u) + 1) + \sum_{u \in S'} (\ell(u) + 1).
\end{aligned}$$

Define the map

$$\begin{aligned} d' : S'\backslash\{v\} &\to S \cap \{u \in \mathrm{dg}(\nu) \mid \ell(u) \neq 0\} \\ u &\mapsto d(u). \end{aligned} \tag{4.2.10}$$

Since $u \in S'\backslash\{v\}$, $d(u)$ is a box in S with $\ell(d(u)) \neq 0$. Hence, the map d' is well-defined. Let $u = (i, j) \in S \cap \{u \in \mathrm{dg}(\mu) \mid \ell(u) \neq 0\}$. Then $u' = (i, j-1)$ is the box above u such that $u' \in S'\backslash v$. Hence, the map d' is a bijection. If $\nu_n \geq 0$ and $v = (n, 1)$, then $v \in S'$ and $\ell(v) + 1 = \nu_n - 1 + 1 = \nu_n$. Hence,

$$\sum_{u \in S'} (\ell(u) + 1) = \Bigg(\sum_{\substack{u \in S' \\ u \neq (n,1)}} \ell(u) + 1 \Bigg) + \nu_n = \Bigg(\sum_{\substack{u \in S \\ \ell(u) \neq 0}} \ell(u) \Bigg) + \nu_n = \sum_{u \in S} \ell(u) + \nu_n. \tag{4.2.11}$$

By the bijection defined in (4.2.10) and (4.2.11),

$$\begin{aligned} \mathrm{maj}(\widehat{\sigma_{\pi\nu}}) &= \sum_{u \in \mathrm{Des}(\widehat{\sigma_\nu})} (\ell(u) + 1) - \sum_{u \in S} (\ell(u) + 1) + \sum_{u \in S'} (\ell(u) + 1) \\ &= \sum_{u \in \mathrm{Des}(\widehat{\sigma_\nu})} (\ell(u) + 1) + \sum_{u \in S} \ell(u) + \nu_n - \sum_{u \in S} (\ell(u) + 1) \\ &= \mathrm{maj}(\widehat{\sigma_\nu}) + \nu_n - \#S = \mathrm{maj}(\widehat{\sigma_\nu}) + \nu_n - r. \end{aligned}$$

The proof of item (2) is done in Proposition 4.2.10. For item (3), it follows directly from the definitions of $\widehat{\sigma_\mu}$ and $\widehat{\sigma_\nu}$. □

Example 4.2.4. Let $\mu = (2, 0, 1) = \pi\nu$ with $\nu = (0, 1, 1)$. The non-attacking fillings of ν are

$$\diamondsuit = \begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \quad \clubsuit = \begin{array}{ccc} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \quad \spadesuit = \begin{array}{ccc} & \boxed{2} & \boxed{1} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \quad \heartsuit = \begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array}$$

$$\left(\tfrac{1-t}{1-qt}\right) x_1 x_3 \qquad x_2 x_3 \qquad t\left(\tfrac{1-t}{1-qt^2}\right) x_1 x_2 \qquad \left(\tfrac{1-t}{1-qt}\right)\left(\tfrac{1-t}{1-qt^2}\right) x_1 x_2$$

Fillings	Des	# Inv	inv	maj	coinv
♢	∅	4	1	0	0
♣	∅	4	1	0	0
♠	∅	3	0	0	1
♥	∅	4	1	0	0

The associated non-attacking fillings of $\mu = (2, 0, 1)$ are

$$\diamondsuit = \begin{array}{ccc} \boxed{1} & & \\ \boxed{1} & & \boxed{2} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \quad \spadesuit = \begin{array}{ccc} \boxed{1} & & \\ \boxed{1} & & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \quad \clubsuit = \begin{array}{ccc} \boxed{2} & & \\ \boxed{1} & & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \quad \heartsuit = \begin{array}{ccc} \boxed{3} & & \\ \boxed{1} & & \boxed{2} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array}$$

$$\left(\tfrac{1-t}{1-qt}\right) x_1^2 x_2 \qquad x_1^2 x_3 \qquad qt\left(\tfrac{1-t}{1-qt^2}\right) x_1 x_2 x_3 \qquad q\left(\tfrac{1-t}{1-qt^2}\right)\left(\tfrac{1-t}{1-qt}\right) x_1 x_2 x_3$$

Fillings	Des	# Inv	inv	maj	coinv
♢	∅	4	3	0	0
♠	∅	4	3	0	0
♣	{(1, 2)}	4	2	1	1
♥	{(1, 2)}	5	3	1	0

We observe from this example that the major statistics of $\widehat{\sigma_\mu}$ depends on the number of 3's appeared in the filling $\widehat{\sigma_\nu}$ and its major statistics. Moreover, let $v = (3,1) \in \widehat{\sigma_\nu}$, we observe that if $\widehat{\sigma_\nu}(v) \neq 3$, then the box v contributes a descent in $\widehat{\sigma_\mu}$. Note that the statistics $\#\,\mathrm{Hleg}_\mu(v)+1 = \nu_3$. This example illustrates that the coinversion statistics in $\widehat{\sigma_\mu}$ and $\widehat{\sigma_\nu}$ match.

4.3 The Haglund-Haiman-Loehr formula

Now we rewrite the proof of the Haglund-Haiman-Loehr formula in [HHL06, pg7, Thm3.5.1].

Theorem 4.3.1 (Haglund-Haiman-Loehr formula)**.** *Let $u = (i,j) \in \mathrm{dg}(\mu)$, $\ell(u) = \#\,\mathrm{Hleg}(u)$, $a(u) = \#\,\mathrm{Harm}(u)$ and let $\mu \in \mathbb{Z}^n_{\geq 0}$. Non-symmetric Macdonald polynomials E_μ on the root system $\mathfrak{gl}_n$ are given by*

$$E_\mu(x;q,t) = \sum_{\substack{\widehat{\sigma_\mu} \\ \text{non-attacking}}} x^{\widehat{\sigma_\mu}} q^{\mathrm{maj}(\widehat{\sigma_\mu})} t^{\mathrm{coinv}(\widehat{\sigma_\mu})} \prod_{\substack{u \in \mathrm{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}}, \tag{4.3.1}$$

where $x^{\widehat{\sigma_\mu}} = x_1^{\#1 \text{ in } \widehat{\sigma_\mu}} \cdots x_n^{\#n \text{ in } \widehat{\sigma_\mu}}$.

Proof. The proof is by induction on non-symmetric Macdonald polynomials E_μ. Starting with $E_{\mathbf{0}} = 1$ and using the unwinding μ construction defined in (4.2.2) and (4.2.1), the non-symmetric Macdonald polynomial E_μ can be expressed as $E_\mu = G_0^\mu + G_1^\mu$. It is enough to consider two cases.
Case 1: If $\mu = s_i\nu$ where $\nu_i \neq 0$ and $\nu_{i+1} = 0$, then $E_\mu = G_0^\mu + G_1^\mu$.
Assume $\mu = s_i\nu$ where $\nu_i \neq 0$ and $\nu_{i+1} = 0$.
By Lemma (4.2.5),

$$E_\mu = E_{s_i\nu} = \left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\nu + \left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\nu + G_2.$$

From Lemma 4.2.7 item 3, we know

$$\mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}) = \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_{s_i\nu}}) = \begin{cases} \left(\dfrac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)\mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu}) & \text{for } \widehat{\sigma_\nu}(i,1) \neq i, \\ \left(\dfrac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)\mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu}) & \text{for } \widehat{\sigma_\nu}(i,1) = i. \end{cases} \tag{4.3.2}$$

Then by the definitions of G_0^μ, G_0^ν and G_1^ν

$$G_0^\mu = \sum_{\substack{\widehat{\sigma_{s_i\nu}} \text{ type } s_i\nu \\ \widehat{\sigma_{s_i\nu}}(i+1,1)\neq i+1}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_{s_i\nu}}), \quad G_0^\nu = \sum_{\substack{\widehat{\sigma_\nu} \text{ type } \nu \\ \widehat{\sigma_\nu}(i,1)\neq i}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu}), \quad G_1^\nu = \sum_{\substack{\widehat{\sigma_\nu} \text{ type } \nu \\ \widehat{\sigma_\nu}(i,1)=i}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu})$$

and (4.3.2), we obtain

$$G_0^\mu = \left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\nu + \left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\nu. \tag{4.3.3}$$

Since

$$G_2 = \sum_{\substack{\sigma_{s_i\nu} \text{ type } s_i\nu \\ \widehat{\sigma_{s_i\nu}}(i+1,1)=i+1}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_{s_i\nu}}) \quad \text{and} \quad G_1^\mu = \sum_{\substack{\widehat{\sigma_{s_i\nu}} \text{ type } s_i\nu \\ \sigma_{s_i\nu}(i+1,1)=i+1}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_{s_i\nu}}),$$

we obtain

$$G_1^\mu = G_2.$$

Hence,

$$\begin{aligned} E_\mu = E_{s_i\nu} &= \left(\frac{1-q^{\mu_i}t^{a(i,1)+1}}{1-q^{\mu_i}t^{a(i,1)}}\right)G_0^\nu + \left(\frac{1-t}{1-q^{\mu_i}t^{a(i,1)}}\right)G_1^\nu + G_2 \\ &= G_0^\mu + G_1^\mu. \end{aligned}$$

Case 2: Consider the case where $\mu = \pi\nu$ for some $\nu \in \mathbb{Z}_{\geq 0}^n$. Then by the inductive hypothesis, E_ν satisfies equality (4.3.1). By (4.2.1),

$$E_\mu = E_{\pi\nu}(x_1, \ldots, x_n) = q^{\nu_n} x_1 E_\nu(x_2, \ldots, x_n, q^{-1}x_1). \tag{4.3.4}$$

Assume $\widehat{\sigma_\nu}$ has r number of n. Then by item $(1), (2), (3)$ in Lemma 4.2.8,

$$\begin{aligned} q^{\nu_n} x_1 \mathrm{wt}(\widehat{\sigma_\nu}) &= q^{\nu_n} x_1 q^{-r} q^{\mathrm{maj}(\widehat{\sigma_\nu})} t^{\mathrm{coinv}\,\widehat{\sigma_\nu}} x_1^{\widehat{\sigma_\nu}(n)} x_2^{\widehat{\sigma_\nu}(1)} \cdots x_n^{\widehat{\sigma_\nu}(n-1)} \prod_{\substack{u \in \widehat{\sigma_\nu} \\ \widehat{\sigma_\nu}(u-n) \neq \widehat{\sigma_\nu}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}} \\ &= q^{\nu_n - r + \mathrm{maj}(\widehat{\sigma_\nu})} t^{\mathrm{coinv}(\widehat{\sigma_\nu})} x_1^{\widehat{\sigma_\nu}(n)+1} x_2^{\widehat{\sigma_\nu}(1)} \cdots x_n^{\widehat{\sigma_\nu}(n-1)} \prod_{\substack{u \in \widehat{\sigma_\nu} \\ \widehat{\sigma_\nu}(u-n) \neq \widehat{\sigma_\nu}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}} \\ &= q^{\nu_n - r + \mathrm{maj}(\widehat{\sigma_\nu})} t^{\mathrm{coinv}(\widehat{\sigma_\nu})} x_1^{\widehat{\sigma_\nu}(n)+1} x_2^{\widehat{\sigma_\nu}(1)} \cdots x_n^{\widehat{\sigma_\nu}(n-1)} \prod_{\substack{u \in \widehat{\sigma_\nu} \\ \widehat{\sigma_\nu}(u-n) \neq \widehat{\sigma_\nu}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}} \\ &= q^{\mathrm{maj}(\widehat{\sigma_{\pi\nu}})} t^{\mathrm{coinv}(\widehat{\sigma_{\pi\nu}})} x^{\widehat{\sigma_{\pi\nu}}} \prod_{\substack{u \in \widehat{\sigma_{\pi\nu}} \\ \widehat{\sigma_{\pi\nu}}(u-n) \neq \widehat{\sigma_{\pi\nu}}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}} \\ &= \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_{\pi\nu}}) = \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}). \end{aligned}$$

Since the map π defined in Lemma 4.2.8 is a bijection between the set of non-attacking fillings of ν and μ,

$$E_\mu = q^{\nu_n} x_1 E_\nu(x_2, \ldots, x_n, q^{-1}x_1) = q^{\nu_n} x_1 \sum_{\substack{\widehat{\sigma_\nu} \\ \text{non-attacking}}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\nu}) = \sum_{\substack{\widehat{\sigma_\mu} \\ \text{non-attacking}}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}). \square$$

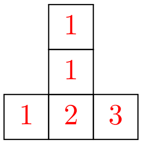
Example 4.3.1. We examine closely at terms G_0^μ, G_0^ν and G_1^ν of $E_{(0,0,2)}$. Recall from Example 4.2.1, the G_0^μ, G_0^ν and G_1^ν of $E_{(0,0,2)}$ are

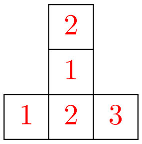
$$\begin{aligned} G_0^\nu &= \left(\frac{1-t}{1-q^2t^2}\right)x_1^2 + q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2 + q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_3, \\ G_1^\nu &= \left(\frac{1-t}{1-qt}\right)x_1x_2 + x_2^2 + q\left(\frac{1-t}{1-qt}\right)x_2x_3 \end{aligned}$$

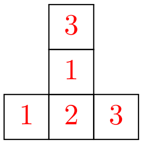
and

$$\begin{aligned} G_0^\mu = &\left(\frac{1-t}{1-q^2t}\right)x_1^2 + q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_1x_2 + q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_1x_3 \\ &+ \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_1x_2 + \left(\frac{1-t}{1-q^2t}\right)x_2^2 + q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_2x_3. \end{aligned}$$

The non-attacking fillings corresponding to the weights in G_0^ν are

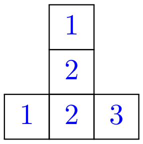


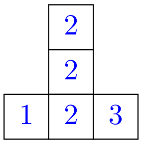


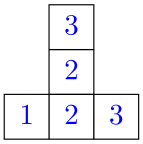


$\left(\frac{1-t}{1-q^2t^2}\right)x_1^2$ $\quad q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2$ $\quad q\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_3$

The non-attacking fillings corresponding to the weights of G_1^ν are

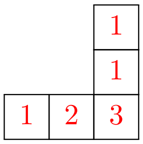


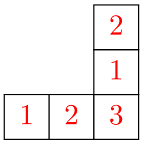


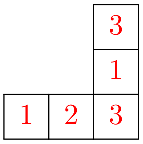


$\left(\frac{1-t}{1-qt}\right)x_1x_2$ $\quad x_2^2$ $\quad q\left(\frac{1-t}{1-qt}\right)x_2x_3$

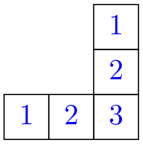
The non-attacking fillings of G_0^ν convert to the part of non-attacking fillings of G_1^μ highlighted by red and the non-attacking fillings of G_1^ν convert to the part of non-attacking fillings of G_1^μ highlighted by blue, i.e.,

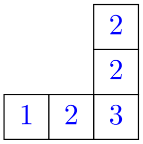


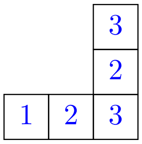




$\left(\frac{1-t}{1-q^2t}\right)x_1^2$ $\quad q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_1x_2$ $\quad q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_1x_3$



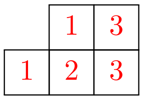


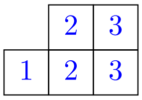


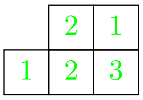
$\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_1x_2$ $\quad \left(\frac{1-t}{1-q^2t}\right)x_2^2$ $\quad q\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-q^2t}\right)x_2x_3$

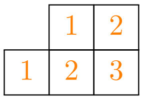
The weights of G_0^ν multiplied by a factor of $\frac{1-q^2t^2}{1-q^2t}$ become the weights of G_0^μ and the weights of G_1^ν multiplied by a factor of $\frac{1-t}{1-q^2t}$ become the weights of G_0^μ. This illustrates the equality in (4.3.3).

Example 4.3.2. Recall from Example 4.2.3, the non-attacking fillings of ν are



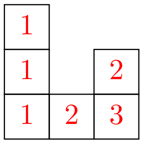


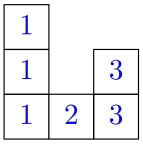


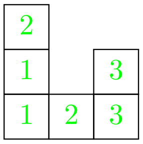


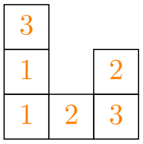
$\left(\frac{1-t}{1-qt}\right)x_1x_3$ $\quad x_2x_3$ $\quad t\left(\frac{1-t}{1-qt^2}\right)x_1x_2$ $\quad \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)x_1x_2$

The non-attacking fillings of μ are









$\left(\frac{1-t}{1-qt}\right)x_1^2x_2$ $\quad x_1^2x_3$ $\quad qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3$ $\quad q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3$

The bijection π sends the non-attacking fillings of μ to the non-attacking fillings of ν indicated by the colors. The non-symmetric Macdonald polynomial of $\nu = (0, 1, 1)$ is

$$E_\nu = E_{(0,1,1)} = \left(\frac{1-t}{1-qt}\right)x_1x_3 + x_2x_3 + t\left(\frac{1-t}{1-qt^2}\right)x_1x_2 + \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right)x_1x_2.$$

The weight in $E_{(2,0,1)}$ changes accordingly

$$\begin{aligned}E_\mu = E_{(2,0,1)} &= E_{\pi(0,1,1)} = q^1x_1E_{(0,1,1)}(x_2, \ldots, x_n, q^{-1}x_1)\\ &= qx_1\left(q^{-1}\left(\frac{1-t}{1-qt}\right)x_2x_1\right) + qx_1\left(q^{-1}x_3x_1\right) + qx_1\left(t\left(\frac{1-t}{1-qt^2}\right)x_2x_3\right)\\ &\qquad + qx_1\left(\left(\frac{1-t}{1-qt^2}\right)x_2x_3\right)\\ &= \left(\frac{1-t}{1-qt}\right)x_1^2x_2 + x_1^2x_3 + qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 + q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3.\end{aligned}$$

Chapter 5

The Borodin-Wheeler(BW) formula

This section includes a general overview of the statistics provided in [BW19] and introduces the Borodin-Wheeler formula for type $\mathfrak{gl}_n$ Macdonald polynomials [BW19, pg30, Prop 5.8]. The Borodin-Wheeler formula serves as a bridge between the Haglund-Hamian-Loehr formula and the Ram-Yip formula in type $\mathfrak{gl}_n$. In this section, we describe the connection between the ordered triples(statistics in Borodin-Wheeler) and the inversion(coinversion) triples(statistics in Haglund-Hamian-Loehr). Finally, we make comparisons between Haglund-Haiman-Loehr formula, Borodin-Wheeler formula and type $\mathfrak{gl}_n$ Ram-Yip formula.

5.1 Ordered triples, inversion triples and coinversion triples

Let $\widehat{\sigma_\mu}$ be a non-attacking filling. An **ordered triple** is a collection of three boxes (u,v,w) such that

$$u = (i, j) \in \mathrm{dg}(\mu), \quad w = (i', j-1) \in \widehat{\mathrm{dg}}(\mu) \quad \text{and} \quad v = (i', j) \in \mathrm{dg}(\mu) \text{ with } i < i'$$

and satisfies one of the following

$$\widehat{\sigma_\mu}(i', j) > \widehat{\sigma_\mu}(i, j) > \widehat{\sigma_\mu}(i', j-1), \tag{5.1.1}$$

$$\widehat{\sigma_\mu}(i', j) < \widehat{\sigma_\mu}(i, j) < \widehat{\sigma_\mu}(i', j-1), \tag{5.1.2}$$

where $\widehat{\sigma_\mu}(i', j) = \infty$ if $(i', j) \notin \mathrm{dg}(\mu)$. Denote the set of ordered triples of a filling $\widehat{\sigma_\mu}$ to be $\mathrm{Otrip}(\widehat{\sigma_\mu})$.

The set of **positive ordered triples** and the set of **negative ordered triples** are

$$\mathrm{ord}_+(\widehat{\sigma_\mu}) = \{(u, v, w) \in \mathrm{Otrip}(\widehat{\sigma_\mu}) \mid \widehat{\sigma_\mu}(i', j) > \widehat{\sigma_\mu}(i, j) > \widehat{\sigma_\mu}(i', j-1)\} \quad \text{and}$$
$$\mathrm{ord}_-(\widehat{\sigma_\mu}) = \{(u, v, w) \in \mathrm{Otrip}(\widehat{\sigma_\mu} \mid \widehat{\sigma_\mu}(i', j) < \widehat{\sigma_\mu}(i, j) < \widehat{\sigma_\mu}(i', j-1)\} \quad \text{respectively.}$$

Then

$$\mathrm{Otrip}(\widehat{\sigma_\mu}) = \mathrm{ord}_+(\widehat{\sigma_\mu}) \cup \mathrm{ord}_-(\widehat{\sigma_\mu}).$$

Define

$$\Delta(\widehat{\sigma_\mu}) = \#\,\mathrm{ord}_+(\widehat{\sigma_\mu}) - \#\,\mathrm{ord}_-(\widehat{\sigma_\mu}). \tag{5.1.3}$$

Lemma 5.1.1 makes a connection between the coinversion statistics of a non-attacking filling defined in (4.1.7) and the difference of ordered triples defined in (5.1.3).

Lemma 5.1.1. *Let* $\widehat{\sigma_\mu}$ *be a non-attacking filling of* μ*,* $\Delta(\widehat{\sigma_\mu})$ *be defined in* (5.1.3) *and* $\mathrm{coinv}(\widehat{\sigma_\mu})$ *be defined in* (4.1.7) *and* $\mathrm{Des}(\widehat{\sigma_\mu})$ *be defined in* (4.1.2)*. Set* $\#\,\mathrm{Harm}_\mu(u) = a(u)$*. Then*

$$\mathrm{coinv}(\widehat{\sigma_\mu}) + \Delta(\widehat{\sigma_\mu}) = \sum_{u\in\mathrm{Des}(\widehat{\sigma_\mu})} a(u)$$

Proof. The ordered triples can be understood as a Type A triple with an additional condition that these ordered triples also included the boxes that are not in $\widehat{\mathrm{dg}}(\mu)$ and their fillings are counted as infinity. For more details, reader may refer to [BW19, pg28, Section 5]. $\square$

Example 5.1.1. For example, the non-attacking fillings of $\mu = (2, 0, 1)$ are

$$\diamondsuit = \begin{array}{|c|c|c|}\hline 1 & & \\ \hline 1 & & 2 \\ \hline 1 & 2 & 3 \\ \hline\end{array} \quad \clubsuit = \begin{array}{|c|c|c|}\hline 1 & & \\ \hline 1 & & 3 \\ \hline 1 & 2 & 3 \\ \hline\end{array} \quad \heartsuit = \begin{array}{|c|c|c|}\hline 2 & & \\ \hline 1 & & 3 \\ \hline 1 & 2 & 3 \\ \hline\end{array} \quad \spadesuit = \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 1 & & 2 \\ \hline 1 & 2 & 3 \\ \hline\end{array}$$

and the statistics coinv, Δ and $\sum_{u\in\mathrm{Des}(\widehat{\sigma_\mu})}$ are

Fillings	coinv	Δ	$\sum_{u\in\mathrm{Des}(\widehat{\sigma_\mu})} a(u)$
♢	0	0	0
♣	0	0	0
♡	1	0	1
♠	0	1	1

The statistics coinv and Δ of $\widehat{\sigma_\mu}$ sum up to $\sum_{u\in\mathrm{Des}(\widehat{\sigma_\mu})} a(u)$ which illustrated the result in Lemma 5.1.1.

5.2 Bijection between lattice configurations and non-attacking fillings

In this section, we discuss the bijection between μ-legal configurations [BW19, pg30, (5.9)] and non-attacking fillings. A **lattice configuration**

$$P : \{1, \ldots, n\} \times \mathbb{Z}_{\geq 0} \to \{0, 1, \ldots, n\}$$

is a filling of boxes.
A μ-legal lattice configuration $P_{\mu\text{legal}}$ [BW19, pg30, after (5.9)] is a configuration satisfies the following three conditions:

1. Let $S_i = \{(a, b) \in \{1, \ldots, n\} \times \mathbb{Z}_{>0} \mid P(a, b) = i\}$. Then

   $$\#S_i = \mu_i \text{ and there exists exactly one } (a, j) \in S_i \text{ for each } j \in \{1, \ldots, \mu_i\};$$

2. For entries $(i, 0) \in \{1, \ldots, n\} \times \{0\}$,

   $$P((i, 0)) = i;$$

3. For $k \in \mathbb{Z}_{>0}$ with $P(\mathrm{cyl}_n(k)) \neq 0$,

   $$P(\mathrm{cyl}_n(k)) \geq P(\mathrm{cyl}_n(k - n)),$$

where cyl_n is the cylindrical wrapping defined in (2.3.1).

Lemma 5.2.1. *The bijection between the set of non-attacking fillings and set of μ-legal configurations is given by ([BW19, pg31 , (5.10)] and [CMW18, pg23, Def A.6])*

$$P_{\mu\text{legal}}(k,j)=i \quad \text{if and only if} \quad \widehat{\sigma_\mu}(i,j)=k. \tag{5.2.1}$$

Example 5.2.1. For example, for $\mu=(2,0,1)$, there are four non-attacking fillings

$$\clubsuit=\begin{array}{ccc} 1 & & \\ 1 & & 2 \\ 1 & 2 & 3 \end{array} \quad \heartsuit=\begin{array}{ccc} 1 & & \\ 1 & & 3 \\ 1 & 2 & 3 \end{array} \quad \diamondsuit=\begin{array}{ccc} 2 & & \\ 1 & & 3 \\ 1 & 2 & 3 \end{array} \quad \spadesuit=\begin{array}{ccc} 3 & & \\ 1 & & 2 \\ 1 & 2 & 3 \end{array}$$

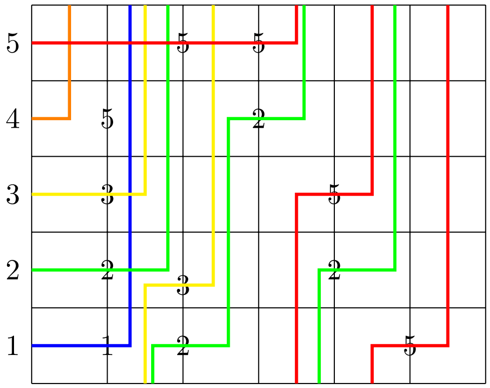
The conversions to pipe dreams are

$$\clubsuit=\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\\hline 1 & 3 & 0 \\\hline 1 & 2 & 3 \\\hline \end{array} \quad \heartsuit=\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\\hline 1 & 0 & 3 \\\hline 1 & 2 & 3 \\\hline \end{array} \quad \diamondsuit=\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\\hline 1 & 0 & 3 \\\hline 1 & 2 & 3 \\\hline \end{array} \quad \spadesuit=\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\\hline 1 & 3 & 0 \\\hline 1 & 2 & 3 \\\hline \end{array}$$

Example 5.2.2. For $\mu=(1,4,2,0,5)$, the non-attacking filling

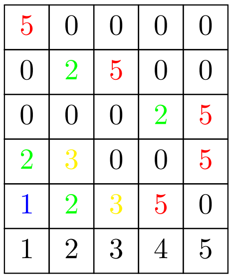
$$\widehat{\sigma_\mu}=\begin{array}{ccccc} & & & & 1 \\ & 2 & & & 3 \\ & 4 & & & 5 \\ & 1 & 2 & & 5 \\ 1 & 2 & 3 & & 4 \\ 1 & 2 & 3 & 4 & 5 \end{array} \quad \text{with the corresponding configuration} \quad \begin{array}{|c|c|c|c|c|}\hline 5 & 0 & 0 & 0 & 0 \\\hline 0 & 2 & 5 & 0 & 0 \\\hline 0 & 0 & 0 & 2 & 5 \\\hline 2 & 3 & 0 & 0 & 5 \\\hline 1 & 2 & 3 & 5 & 0 \\\hline 1 & 2 & 3 & 4 & 5 \\\hline \end{array} \tag{5.2.2}$$

Pictorially, we label the horizontal lines with $1,2,3,4,5$ bottom to top and call them the **height**. For the horizontal label of height i, we put entries in the i-th column of the μ-legal configuration, where we omit the label zero in the diagram. For the label i, we draw lines going across the labels with different colors as below



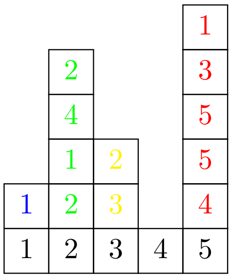
The conversion from a μ-legal configuration to a non-attacking filling is more straight forward.

For example, with the configuration



5	0	0	0	0
0	2	5	0	0
0	0	0	2	5
2	3	0	0	5
1	2	3	5	0
1	2	3	4	5

the non-attacking filling is $\widehat{\sigma_\mu} =$



				1
	2			3
	4			5
	1	2		5
1	2	3		4
1	2	3	4	5

where the fillings j in the i-th column and height k of the non-attacking fillings are those entries in the μ-legal configuration with fillings i, column j and height k.

5.3 Borodin-Wheeler formula

Recall from (4.1.1) and (4.1.2) that the set of ascents and descents associated with a non-attacking filling $\widehat{\sigma_\mu}$ are given by

$$\operatorname{Aes}(\widehat{\sigma_\mu}) = \{(i,j) \in \operatorname{dg}(\mu) \mid \sigma_\mu(i,j) < \widehat{\sigma_\mu}(i,j-1)\}, \tag{5.3.1}$$

$$\operatorname{Des}(\widehat{\sigma_\mu}) = \{(i,j) \in \operatorname{dg}(\mu) \mid \sigma_\mu(i,j) > \widehat{\sigma_\mu}(i,j-1)\}. \tag{5.3.2}$$

Theorem 5.3.1. *(Borodin-Wheeler[BW19, pg30, Prop 5.8]) Let $\mu \in \mathbb{Z}^n_{\geq 0}$, and let Δ be defined in (5.1.3). Fix $a(u) = \#\operatorname{Harm}_\mu(u)$ and $\ell(u) = \#\operatorname{Hleg}_\mu(u)$ for $u \in \operatorname{dg}(\mu)$. The non-symmetric Macdonald polynomials are given by the following formula:*

$$E^{\mathrm{BW}}_\mu(x;q,t) = \sum_{\substack{\widehat{\sigma_\mu} \\ \text{non-attacking}}} x^{\widehat{\sigma_\mu}} t^{\Delta(\widehat{\sigma_\mu})} \prod_{u \in \operatorname{Des}(\widehat{\sigma_\mu})} \frac{1-t}{1-q^{\ell(u)+1}t^{a(s)+1}} \prod_{u \in \operatorname{Aes}(\widehat{\sigma_\mu})} \frac{q^{\ell(u)+1}t^{a(u)}(1-t)}{1-q^{\ell(u)+1}t^{a(u)+1}}, \tag{5.3.3}$$

where $x^{\widehat{\sigma_\mu}} = x_1^{\#1 \text{ in } \widehat{\sigma_\mu}} \cdots x_n^{\#n \text{ in } \widehat{\sigma_\mu}}$.

Proof. The proof of this formula is given in [BW19, pg 30, Prop 5.8]. The key step of proving this formula is to use the bijection between non-attacking fillings and μ-legal configurations defined in (5.2.1). By assigning a weight to each lattice configuration and matching the resulting terms piece-by-piece, it gives the formula of non-symmetric Macdonald polynomials in (5.3.3). ☐

With notation given in Theorem 5.3.1. Define the BW-weight of $\widehat{\sigma_\mu}$ by

$$\mathrm{wt}^{\mathrm{BW}}(\widehat{\sigma_\mu}) = x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u \in \operatorname{Des}(\widehat{\sigma_\mu})} \frac{1-t^{-1}}{1-q^{-(\ell(u)+1)}t^{-(a(u)+1)}} \prod_{u \in \operatorname{Aes}(\widehat{\sigma_\mu})} \frac{q^{-(\ell(u)+1)}t^{-a(u)}(1-t^{-1})}{1-q^{-(\ell(u)+1)}t^{-(a(u)+1)}}.$$

Recall from (4.1.8) that the HHL-weight of $\widehat{\sigma_\mu}$ is given by

$$\mathrm{wt}^{\mathrm{HHL}} = x^{\widehat{\sigma_\mu}} q^{\operatorname{maj}(\widehat{\sigma_\mu})} t^{\operatorname{coinv}(\widehat{\sigma_\mu})} \prod_{\substack{u \in \operatorname{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)}} \frac{1-t}{1-q^{\#\operatorname{Hleg}(u)+1}t^{\#\operatorname{Harm}(u)+1}}.$$

Theorem 5.3.2 makes connection between the Haglund-Hamian-Loehr formula in (4.3.1) and the Borodin-Wheeler formula in (5.3.3). Moreover, it separates the product over boxes u with $\widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)$ to a box either in $\operatorname{Des}(\widehat{\sigma_\mu})$ or in $\operatorname{Aes}(\widehat{\sigma_\mu})$. This is similar to the two products occurred in [RY08, pg9, Thm 3.1].

Theorem 5.3.2. *Let $\mu \in \mathbb{Z}_{\geq 0}$ and $\widehat{\sigma_\mu}$ be a filling of μ, and let $\Delta(\widehat{\sigma_\mu}), \mathrm{Des}(\widehat{\sigma_\mu})$ and $\mathrm{Aes}(\widehat{\sigma_\mu})$ be defined as in (5.1.3), (5.3.1) and (5.3.2). Fix $a(u) = \#\,\mathrm{Harm}_\mu(u)$ and $\ell(u) = \#\,\mathrm{Hleg}_\mu(u)$ for $u \in \mathrm{dg}(\mu)$. Then*

$$E_\mu^{\mathrm{BW}}(x, q^{-1}, t^{-1}) = \sum_{\substack{\widehat{\sigma_\mu} \\ \textit{non-attacking}}} \mathrm{wt}^{\mathrm{BW}}(\widehat{\sigma_\mu}) = \sum_{\substack{\widehat{\sigma_\mu} \\ \textit{non-attacking}}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}) = E_\mu^{\mathrm{HHL}}(x; q, t).$$

Proof. Using $\mathrm{maj}(\widehat{\sigma_\mu}) = \sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \ell(u) + 1$ and Lemma 5.1.1,

$$\begin{aligned}
\mathrm{wt}^{\mathrm{BW}}(\widehat{\sigma_\mu}) &= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{1 - t^{-1}}{1 - q^{-(\ell(u)+1)} t^{-(a(u)+1)}} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{q^{-(\ell(u)+1)} t^{-a(u)} (1 - t^{-1})}{1 - q^{-(\ell(u)+1)} t^{-(a(u)+1)}} \\
&= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{(1 - t^{-1})(q^{\ell(u)+1} t^{a(u)+1})}{q^{\ell(u)+1} t^{a(u)+1} - 1} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{t(1 - t^{-1})}{q^{\ell(u)+1} t^{a(u)+1} - 1} \qquad (5.3.4) \\
&= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{q^{\ell(u)+1} t^{a(u)} (1 - t)}{1 - q^{\ell(u)+1} t^{a(u)+1}} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}} \qquad (5.3.5) \\
&= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} q^{\sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \ell(u)+1} t^{\sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} a(u)} \prod_{\substack{u \in \mathrm{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)}} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}} \\
&= x^{\widehat{\sigma_\mu}} q^{\mathrm{maj}(\widehat{\sigma_\mu})} t^{-\Delta(\widehat{\sigma_\mu}) + \sum_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} a(u)} \prod_{\substack{u \in \mathrm{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)}} \frac{1 - t}{1 - q^{\ell(s)+1} t^{a(s)+1}} \qquad \text{by Lemma 5.1.1} \\
&= x^{\widehat{\sigma_\mu}} q^{\mathrm{maj}(\widehat{\sigma_\mu})} t^{\mathrm{coinv}(\widehat{\sigma_\mu})} \prod_{\substack{u \in \mathrm{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n) \neq \widehat{\sigma_\mu}(u)}} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}} = \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}).
\end{aligned}$$

☐

Corollary 5.3.3. *With the same statistics defined in Theorem 5.3.2, the non-symmetric Macdonald polynomials are*

$$E_\mu(x; q, t) = \sum_{\substack{\widehat{\sigma_\mu} \\ \textit{non-attacking}}} x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu}) - \#\,\mathrm{Des}(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{(1 - t) q^{\ell(u)+1} t^{a(u)+1}}{1 - q^{\ell(u)+1} t^{a(u)+1}}. \qquad (5.3.6)$$

Proof. By (5.3.5),

$$\begin{aligned}
\mathrm{wt}^{\mathrm{BW}} &= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{q^{\ell(u)+1} t^{a(u)} (1 - t)}{1 - q^{\ell(u)+1} t^{a(u)+1}} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}} \\
&= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{t^{-1} q^{\ell(u)+1} t^{a(u)+1} (1 - t)}{1 - q^{\ell(u)+1} t^{a(u)+1}} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}} \\
&= x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} t^{-\#\,\mathrm{Des}(\widehat{\sigma_\mu})} \prod_{u \in \mathrm{Des}(\widehat{\sigma_\mu})} \frac{q^{\ell(u)+1} t^{a(u)+1} (1 - t)}{1 - q^{\ell(u)+1} t^{a(u)+1}} \prod_{u \in \mathrm{Aes}(\widehat{\sigma_\mu})} \frac{1 - t}{1 - q^{\ell(u)+1} t^{a(u)+1}}.
\end{aligned}$$

Summing over non-attacking fillings of μ, we obtain the equality in (5.3.6). ☐

Example 5.3.1. For $E_{(2,1,0)}$, the non-attacking fillings and their weights are

$$\heartsuit = \begin{array}{ccc}\cline{1-1} \multicolumn{1}{|c|}{1} & & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \diamondsuit = \begin{array}{ccc}\cline{1-1} \multicolumn{1}{|c|}{3} & & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$x_1^2x_2 \qquad t^{-1}\tfrac{1-t^{-1}}{1-q^{-1}t^{-2}}x_1x_2x_3$$

The statistics are

Fillings	ord_+	ord_-	Δ	Des	Aes
$\heartsuit$	0	0	0	$\emptyset$	$\emptyset$
$\diamondsuit$	1	0	1	$\{(1,2)\}$	$\emptyset$

.

Since

$$\begin{aligned} t^{-1}x_1x_2x_3\frac{1-t^{-1}}{1-q^{-1}t^{-2}} &= x_1x_2x_3\frac{t^{-1}-t^{-2}}{1-q^{-1}t^{-2}} = qx_1x_2x_3\frac{t^{-1}-t^{-2}}{1-q^{-1}t^{-2}}\frac{1}{q} && (5.3.7)\\ &= qx_1x_2x_3\frac{t^{-1}-t^{-2}}{q-t^{-2}} = qx_1x_2x_3\frac{t^{-1}-t^{-2}}{q-t^{-2}}\frac{t^2}{t^2} && (5.3.8)\\ &= qx_1x_2x_3\frac{t-1}{qt^2-1} = qx_1x_2x_3\frac{1-t}{1-qt^2}, \end{aligned}$$

the BW-weights of the non-attacking fillings $\heartsuit$ and $\diamondsuit$ are

$$\mathrm{wt}^{\mathrm{BW}}(\heartsuit) = x_1^2x_2, \qquad \mathrm{wt}^{\mathrm{BW}}(\diamondsuit) = t^{-1}\frac{1-t^{-1}}{1-q^{-1}t^{-2}}x_1x_2x_3 = q\frac{1-t}{1-qt^2}x_1x_2x_3.$$

Using Theorem 5.3.2, we obtain

$$E_{(2,1,0)} = x_1^2x_2 + q\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3.$$

This way of generating the Macdonald polynomial $E_{(2,1,0)}$ matches exactly the one generated by the Haglund-Haiman-Loehr formula defined in (4.3.1). See Appendix 6.2. The step performed in (5.3.7) illustrates the translation from (5.3.4) to (5.3.5).

Example 5.3.2. For $E_{(2,0,1)}$, the non-attacking fillings and their weights are

$$\heartsuit = \begin{array}{ccc}\cline{1-1} \multicolumn{1}{|c|}{1} & & \\ \cline{1-1}\cline{3-3} \multicolumn{1}{|c|}{1} & & \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \diamondsuit = \begin{array}{ccc}\cline{1-1} \multicolumn{1}{|c|}{1} & & \\ \cline{1-1}\cline{3-3} \multicolumn{1}{|c|}{1} & & \multicolumn{1}{|c|}{3} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \spadesuit = \begin{array}{ccc}\cline{1-1} \multicolumn{1}{|c|}{2} & & \\ \cline{1-1}\cline{3-3} \multicolumn{1}{|c|}{1} & & \multicolumn{1}{|c|}{3} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \clubsuit = \begin{array}{ccc}\cline{1-1} \multicolumn{1}{|c|}{3} & & \\ \cline{1-1}\cline{3-3} \multicolumn{1}{|c|}{1} & & \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$\left(\tfrac{q^{-1}(1-t)}{1-q^{-1}t^{-1}}\right)x_1^2x_2 \quad x_1^2x_3 \quad \left(\tfrac{1-t^{-1}}{1-q^{-1}t^{-2}}\right)x_1x_2x_3 \quad t^{-1}\left(\tfrac{1-t^{-1}}{1-q^{-1}t^{-2}}\right)\left(\tfrac{1-t^{-1}q^{-1}}{1-q^{-1}t^{-1}}\right)x_1x_2x_3$$

The statistics are

Fillings	ord_+	ord_-	Δ	Des	Aes
$\heartsuit$	0	0	0	$\emptyset$	$\{(3,1)\}$
$\diamondsuit$	0	0	0	$\emptyset$	$\emptyset$
$\spadesuit$	0	0	0	$\{(1,2)\}$	$\emptyset$
$\clubsuit$	1	0	1	$\{(1,2)\}$	$\{(3,1)\}$

(5.3.9)

Fillings	Des	maj	# Inv	inv	coinv
♡	$\emptyset$	0	4	3	0
♢	$\emptyset$	0	4	3	0
♠	$\{(1,2)\}$	1	4	2	1
♣	$\{(1,2)\}$	1	5	3	0

(5.3.10)

The conversion of $E_{(2,0,1)}$ from the Borodin-Wheeler form to the Haglund-Haiman-Loehr form is

$$\begin{aligned} E_{(2,0,1)} &= \left(\frac{t(1-t^{-1})}{qt-1}\right)x_1^2x_2 + x_1^2x_3 + qt^2\left(\frac{1-t^{-1}}{qt^2-1}\right)x_1x_2x_3 \\ &\quad + \left(\frac{t^{-1}-t^{-2}}{1-q^{-1}t^{-2}}\right)\left(\frac{(1-t^{-1})q^{-1}}{1-q^{-1}t^{-1}}\right)x_1x_2x_3 \\ &= \left(\frac{1-t}{1-qt}\right)x_1^2x_2 + x_1^2x_3 + qt\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3 + q\left(\frac{1-t}{1-qt^2}\right)\left(\frac{1-t}{1-qt}\right)x_1x_2x_3. \end{aligned}$$

5.4 Comparisons of formulas

In this section, we make comparisons between the Haglund-Haiman-Loehr formula defined in (4.3.1), the Borodin-Wheeler formula in (5.3.3) and the type $\mathfrak{gl}_n$ Ram-Yip formula in (5.4.1).

Theorem 5.4.1. *Let $\mu \in \mathfrak{a}^*_{\mathbb{Z}}$ and let m_μ be the minimal length element in the coset $t_\mu S_n$. Fix a reduced word $m_\mu = s_{i_1}\cdots s_{i_\ell}$, let*

$$\beta_1^\vee = s_{i_\ell}\cdots s_{i_2}\alpha_{i_1}^\vee, \quad \beta_2^\vee = s_{i_\ell}\cdots s_{i_3}\alpha_{i_2}^\vee, \quad \ldots, \quad \beta_\ell^\vee = \alpha_{i_\ell}^\vee,$$

with notations defined in [RY08, pg 6, 2.35] and let $\mathcal{B}(1, m_\mu) = m_\mu\omega_0$ be the set of alcove walks of type m_μ, where ω_0 is the empty alcove. Then

$$E_\mu^{\mathfrak{gl}_n}(x;q,t) = \sum_{p\in\mathcal{B}(1,m_\mu)} X^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(w_\mu)-\#f(p))} \left(\prod_{k\in f^+(p)} \frac{(1-t)}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right) \left(\prod_{k\in f^-(p)} \frac{(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right), \tag{5.4.1}$$

where $X^{\mathrm{wt}(p)}$, $\varphi(p)$ and $f(p)$ are defined in [RY08,pg 7,(2.36)(2.37)(3.5)(3.6)], and $\mathrm{sh}(\beta_k^\vee)$, $\mathrm{ht}(\beta_k^\vee)$ defined in [Yip12, pg9, (2.3.9)].

Proof. Recall from [MRY19, pg 15, Thm1.1], the formula for Ram-Yip on arbitrary root system is

$$E_\mu^R = \sum_{p\in\mathcal{B}(1,m_\mu)} X^{\mathrm{wt}(p)} t_{\varphi(p)}^{\frac{1}{2}} \left(\prod_{k\in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right)\left(\prod_{k\in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right).$$

In the root system of type $\mathfrak{gl}_n$, $t_i = t$ for all $i \in \{1,\ldots,n\}$. Since (3.5.12),

$$E_\mu^{\mathfrak{gl}_n} = t^{-\frac{1}{2}\ell(w_\mu)}E_\mu^R.$$

Hence,

$$\begin{aligned}E_\mu^{\mathfrak{gl}_n} &= \sum_{p\in\mathcal{B}(1,m_\mu)} X^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p)-\ell(w_\mu))} \left(\prod_{k\in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-q^{\mathrm{sh}\,(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right) \\ &\qquad \left(\prod_{k\in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right) \\ &= \sum_{p\in\mathcal{B}(1,m_\mu)} X^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(w_\mu)-\#f^+(p)-\#f^-(p))} \left(\prod_{k\in f^+(p)} \frac{(1-t)}{1-q^{\mathrm{sh}\,(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right) \\ &\qquad \left(\prod_{k\in f^-(p)} \frac{(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right) \\ &= \sum_{p\in\mathcal{B}(1,m_\mu)} X^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(w_\mu)-\#f(p))} \left(\prod_{k\in f^+(p)} \frac{(1-t)}{1-q^{\mathrm{sh}\,(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right) \\ &\qquad \left(\prod_{k\in f^-(p)} \frac{(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right). \qquad \square\end{aligned}$$

In Chapter 4, we have seen that the Haglund-Haiman-Loehr formula for type $\mathfrak{gl}_n$ non-symmetric Macdonald polynomial has the form

$$E_\mu(x;q,t) = \sum_{\substack{\widehat{\sigma_\mu} \\ \text{non-attacking}}} \mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}), \tag{5.4.2}$$

where

$$\mathrm{wt}^{\mathrm{HHL}}(\widehat{\sigma_\mu}) = x^{\widehat{\sigma_\mu}} q^{\mathrm{maj}(\widehat{\sigma_\mu})} t^{\mathrm{coinv}(\widehat{\sigma_\mu})} \prod_{\substack{u\in\mathrm{dg}(\mu) \\ \widehat{\sigma_\mu}(u-n)\neq\widehat{\sigma_\mu}(u)}} \frac{1-t}{1-q^{\ell(u)+1}t^{a(u)+1}}.$$

In Section 5.3, we have seen that the Borodin-Wheeler formula for type $\mathfrak{gl}_n$ non-symmetric Macdonald polynomial has the form

$$E_\mu^{\mathrm{BW}}(x;q,t) = \sum_{\substack{\widehat{\sigma_\mu} \\ \text{non-attacking}}} \mathrm{wt}^{\mathrm{BW}}(\widehat{\sigma_\mu}), \tag{5.4.3}$$

where

$$\mathrm{wt}^{\mathrm{BW}}(\widehat{\sigma_\mu}) = x^{\widehat{\sigma_\mu}} t^{-\Delta(\widehat{\sigma_\mu})} \prod_{u\in\mathrm{Des}(\widehat{\sigma_\mu})} \frac{1-t^{-1}}{1-q^{-(\ell(u)+1)}t^{-(a(u)+1)}} \prod_{u\in\mathrm{Aes}(\widehat{\sigma_\mu})} \frac{q^{-(\ell(u)+1)}t^{-a(u)}(1-t^{-1})}{1-q^{-(\ell(u)+1)}t^{-(a(u)+1)}}.$$

In Theorem 5.4.1, we have seen that the Ram-Yip formula for type $\mathfrak{gl}_n$ has the form

$$E_\mu^{\mathrm{RY}}(x;q,t) = \sum_{p\in\mathcal{B}(1,m_\mu)} \mathrm{wt}^{\mathrm{RY}}(p), \tag{5.4.4}$$

where

$$\mathrm{wt}^{\mathrm{RY}}(p) = x^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(w_\mu)-\#f(p))} \left(\prod_{k\in f^+(p)} \frac{(1-t)}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right)\left(\prod_{k\in f^-(p)} \frac{(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}\right).$$

Observe that even though these formulas are summed over different combinatorial objects, they are all in a very similar form i.e., summing over some weights. To understand the relationship between these different formulas, one might want to work on the connections between their weights. From Chapter 4, we understood that the unwinding μ construction is the core which has been used to generate non-symmetric Macdonald polynomials in [HHL06]. Similarly, the interwiner construction, which can also be generalised to generate non-symmetric Macdonald polynomials on any root system [RY08, pg7, Thm2.2], is equivalent to the unwinding μ construction in type $\mathfrak{gl}_n$.

So the algebraic side of the conversion is mostly completed. But the interesting part of this story has just begin. We have seen from Theorem 5.3.2 or more precisely from [BW19, pg 28, section 5] that the Borodin-Wheeler formula and the Haglund-Haiman-Loehr formula are the same. We have known from [BW19, pg31, (5.10)] there is a bijection between the non-attacking fillings(HHL) and the μ-legal configurations(BW). We observed one connection between the statistics coinversion(HHL) and the statistics Δ(BW) in Lemma 5.1.1. However, the connection between the two statistics is not clear i.e., the connection between coinversion triples and the ordered triples is not obvious.

In Theorem 5.4.1, we have established the type $\mathfrak{gl}_n$ Ram-Yip formula with the scalar adjusted so that the leading coefficient of E_μ is 1 and matches the Haglund-Haiman-Loehr formula. We observed that the Haglund-Haiman-Loehr formula uses the combinatorial object non-attacking fillings, but the type $\mathfrak{gl}_n$ Ram-Yip formula uses alcove walks. The relations between these two combinatorial objects are not trivial. One can start by first matching the non-attacking fillings of μ with the affine roots associated with the minimal length element m_μ in the coset $t_\mu S_n$.

Chapter 6

Appendix

6.1 Alcoves of $\mathfrak{gl}_2$

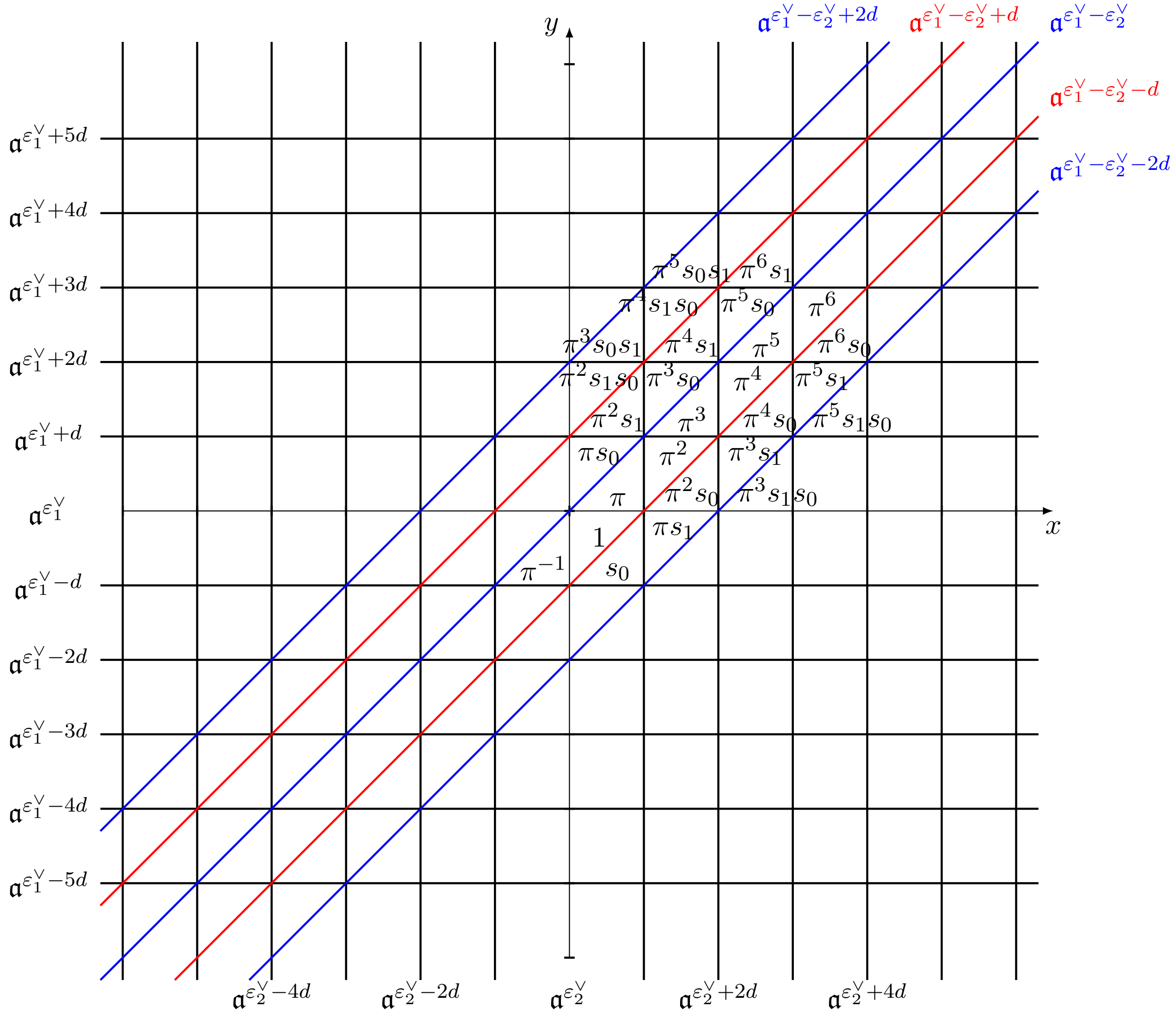
Lemma 6.1.1. *There is a bijection between the affine Weyl group* $W_{\mathfrak{gl}_n}$ *and alcoves of* $\mathfrak{gl}_n$ *given by*

$$\begin{aligned} f : W_{\mathfrak{gl}_n} &\to \{\text{alcoves}\} \\ w &\mapsto wc_0, \end{aligned} \tag{6.1.1}$$

where c_0 *is the fundamental alcove.*

Proof. By Proposition 11.5 of [Kan01], $W_{\mathfrak{gl}_n}$ acts freely transitively on the alcoves so the map f is a bijection. □

For example, if $n = 2$ the picture of $\mathfrak{h}^*_{\mathbb{R}} \cong \mathbb{R}^2$ with hyperplanes $\mathfrak{a}^{\varepsilon_1^\vee - \varepsilon_2^\vee + ad}$, $\mathfrak{a}^{\varepsilon_1^\vee + ad}$ and $\mathfrak{a}^{\varepsilon_2^\vee + ad}$ are



6.2 Example in type $\mathfrak{gl}_3$(HHL)

Using (4.2.1), (4.2.2) to expand the first few Macdonald polynomials gives

$$
\begin{aligned}
E_{(1,0,0)} &= E_{\pi(0,0,0)} = x_1 E_{(0,0,0)} = x_1, \\
E_{(0,1,0)} &= E_{s_1(1,0,0)} = \left(T_1 + \frac{1-t}{1-qt^2}\right) E_{(1,0,0)} \\
&= \left(T_1 + \frac{1-t}{1-qt^2}\right) E_{(1,0,0)} = T_1 x_1 + \frac{1-t}{1-qt^2} x_1 = x_2 + \frac{1-t}{1-qt^2} x_1, \\
E_{(0,0,1)} &= E_{s_2(0,1,0)} = \left(T_2 + \frac{1-t}{1-qt}\right) E_{(0,1,0)} = \left(\frac{1-qt^2}{1-qt}\frac{1-t}{1-qt^2}\right) x_1 + \left(\frac{1-t}{1-qt}\right) x_2 + x_3 \\
&= \left(\frac{1-t}{1-qt}\right) x_1 + \frac{1-t}{1-qt} x_2 + x_3 = \frac{1-t}{1-qt}(x_1 + x_2) + x_3, \\
E_{(1,1,0)} &= E_{\pi(1,0,0)} = q^0 x_1 E_{(1,0,0)} = x_1 x_2, \\
E_{(0,1,1)} &= E_{s_1(1,0,1)} = \left(T_1 + \frac{1-t}{1-qt}\right) E_{(1,0,1)} \\
&= \left(\frac{1-t}{1-qt}\right)\left(x_1 x_3 + \frac{1-t}{1-qt^2} x_1 x_2\right) + x_2 x_3 + \frac{t(1-t)}{1-qt^2} x_1 x_2 \\
&= \frac{1-t}{1-qt} x_1 x_3 + \left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt^2}\right) x_1 x_2 + x_2 x_3 + \frac{t(1-t)}{1-qt^2} x_1 x_2 \\
&= \frac{1-t}{1-qt^2} x_1 x_3 + \frac{1-t}{1-qt^2}\left(t + \frac{1-t}{1-qt}\right) x_1 x_2 + x_2 x_3 \\
&= \frac{1-t}{1-qt}(x_1 x_2 + x_1 x_3) + x_2 x_3. \\
E_{(1,0,1)} &= E_{\pi(0,1,0)} = x_1\left(q^{-1} x_1 + \frac{1-t}{1-qt^2} x_2\right) = x_1 x_3 + \frac{1-t}{1-qt^2} x_1 x_2. \\
E_{(2,0,0)} &= E_{\pi(0,0,1)} = q x_1 E_{(0,0,1)} = q x_1\left(q^{-1} x_1 + \frac{(1-t)}{1-qt}(x_2 + x_2)\right) \\
&= x_1^2 + \frac{q(1-t)}{1-qt}(x_1 x_2 + x_1 x_3). \\
E_{(2,1,0)} &= q x_1 E_{(1,0,1)} = q x_1\left(x_2 q^{-1} x_1 + \frac{1-t}{1-qt^2} x_2 x_3\right) = x_1^2 x_2 + \frac{q(1-t)}{1-qt^2} x_1 x_2 x_3. \\
E_{(1,2,0)} &= E_{\pi(2,0,0)} = x_1 E_{(2,0,0)} = x_1\left(x_2^2 + \frac{q(1-t)}{1-qt}(x_2 x_3 + x_2 q^{-1} x_1)\right) \\
&= x_1 x_2^2 + \frac{q(1-t)}{1-qt} x_1 x_2 x_3 + \frac{1-t}{1-qt} x_1^2 x_2 = x_1 x_2^2 + \frac{(1-t)}{1-qt} x_1^2 x_2 + \frac{q(1-t)}{1-qt} x_1 x_2 x_3, \\
E_{(1,0,2)} &= E_{\pi(0,2,0)} = x_1 E_{(0,2,0)} \\
&= \left(\frac{1-t}{1-q^2t^2}\right) x_1^3 + \frac{q(1-t)^2}{(1-qt)(1-q^2t^2)} x_1^2 x_2 + \frac{q(1-t)^2}{(1-qt)(1-q^2t^2)} x_1^2 x_3 \\
&\qquad + x_1 x_2^2 + \frac{q(1-t)}{1-qt} x_1 x_2 x_3 + \left(\frac{1-t}{1-qt}\right) x_1^2 x_2.
\end{aligned}
$$

Bibliography

[Ale16] Per Alexandersson. Non-symmetric Macdonald polynomials and Demazure-Lusztig operators. *Séminaire Lotharingien de Combinatoire*, B76d:27, 2019. *arXiv:1602.05153*, 2016.

[BW19] Alexei Borodin and Michael Wheeler. Nonsymmetric Macdonald polynomials via integrable vertex models. *arXiv:1904.06804*, April 2019.

[Bou08] Nicolas Bourbaki, *Lie groups and Lie algebras: Chapters 7-9*, volume 3, Springer Science and Business Media, 2008.

[CMW19] Sylive Corteel, Olya Mandelshtam, and Lauren Williams. From multiline queues to Macdonald polynomials via the exclusion process. *arXiv:1811.01024*, 2018.

[CHMMW20] Sylvie Corteel, James Haglund, Olya Mandelshtam, Sarah Mason and Lauren Williams. Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials. *arXiv2004.11907*, 2020.

[Che95] Ivan Cherednik. Double affine Hecke algebras and Macdonald's conjectures. *Annals of mathematics*, 141(1):191–216, 1995.

[Hag04] James Haglund. A combinatorial model for the Macdonald polynomials. *Proccedings of the National Acdaemy of Science*, 101(46):16127–16131. *arXiv:math/0409538*, November 2004.

[Hai06] Mark Haiman. Cherednik algebras, Macdonald polynomials and combinatorics. In *International Congress of Mathematicians*, 3:843–872, 2006. *http://sporadic.stanford.edu/macdo/papers/Haiman-ICM.pdf.*

[HHL06] James Haglund, Mark Haiman, and Nicholas Loehr. A combinatorial formula for nonsymmetric Macdonald polynomials. *American Journal of Mathematics.* 130(2):359–383. *arXiv:math/0601693*, January 2006.

[JV17] David Jordan and Monica Vazirani. The rectangular representation of the double affine Hecke algebra via elliptic Schur-Weyl duality. *International Mathematics Research Notices.* *arXiv:1708.06024*, August 2017.

[Kan01] Richard Kane. *Reflection groups and invariant theory.* Springer Science and Business Media, 2001.

[Len08] Cristian Lenart. On combinatorial formulas for Macdonald polynomials. *Advances in Mathematics*, 220(1):324–340. *arXiv:0804.4716*, April 2008.

[LR97] Robert Leduc and Arun Ram. A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: The Brauer, Birman-Wenzl, and Type A Iwahori–Hecke algebras. *Advances in Mathematics*, 125(1):1-94, 1997.

[Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 1998.

[Mac03] Ian Grant Macdonald. *Affine Hecke algebras and orthogonal polynomials*, volume 157. Cambridge University Press, 2003.

[MRY19] Finn McGlade, Arun Ram, and Yaping Yang. Positive level, negative level and level zero. *To appear in the special volume in conjunction with the conference International Festival in Schubert Calculus, Sun-Yat Sen University, Guangzhou China, 5-11. arXiv:1907.11796*, 2019.

[Ram06] Arun Ram. Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux. *Pure and Applied Mathematics*, 2(4):963-1013. *arXiv:0601.343*, January 2006.

[RY06] Arun Ram and Martha Yip. A combinatorial formula for Macdonald polynomials. *Advances in Mathematics*, 226(1):309–331. *arXiv:0807.3602*, 2011.

[Sah96] Siddhartha Sahi. Interpolation, integrality, and a generalization of Macdonald's polynomials. *International Mathematics Research Notices*, 1996(10):457–471, 1996.

[Yip12] Martha Yip. A Littlewood–Richardson rule for Macdonald polynomials. *Mathematische Zeitschrift*, 272(3-4):1259-1290. *arXiv:1010.0722*, 2012.