Review: vector space

Vector space

- A set V where the following two operations are defined on the elements
- **Addition**: for any two elements $\bar{x}, \bar{y}$ belonging to V (i.e., $\bar{x} \in V$ and $\bar{y} \in V$),
$$\bar{x} + \bar{y} = \bar{z} \in V$$
- **Multiplication by scalar:** $\alpha\, \bar{x} = \bar{y} \in V$
  - A scalar is generally a real number (→ real vector space) or complex (→ complex vector space) but can be an element of a more general set (a Field)
- V contains an **identity element** $\bar{\phi}$ such that
$$\bar{x} + \bar{\phi} = \bar{x}$$

from which it follows that (prove)
$$\alpha\, \bar{\phi} = \bar{\phi}$$

- The elements of a vector space are called **vectors**

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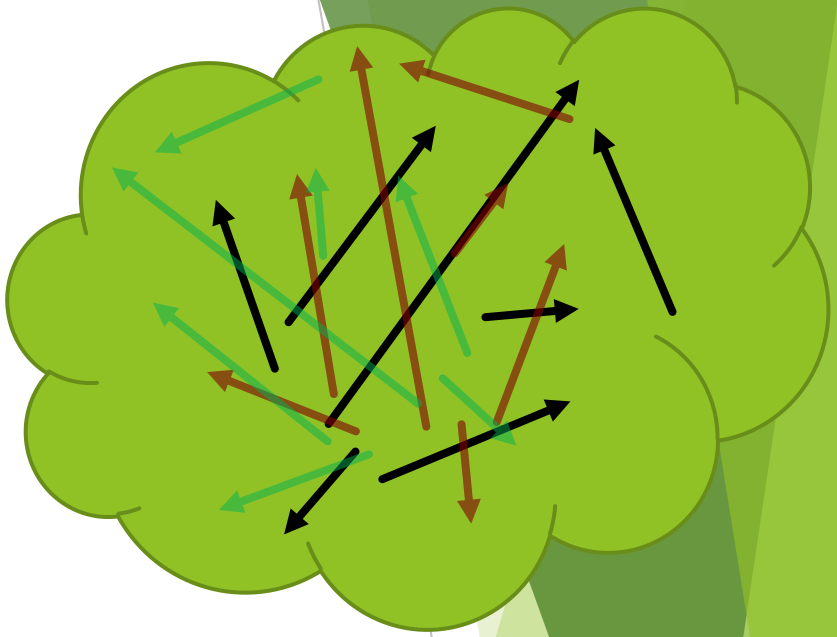
- The elements of a vector space are called **vectors**
- Note: Look up a textbook for a more rigorous definition of vector spaces. There are additional axioms involved.

Vector space

- A familiar example of a vector space is that of displacements
- A displacement is a rule that tells us how far to go and in what direction
- It is standard to represent a displacement as an arrow whose length is proportional to the distance and that points along the direction of displacement
- The set of these displacement rules along with rules for addition (e.g., parallelogram law) and multiplication by real numbers gives us a vector space



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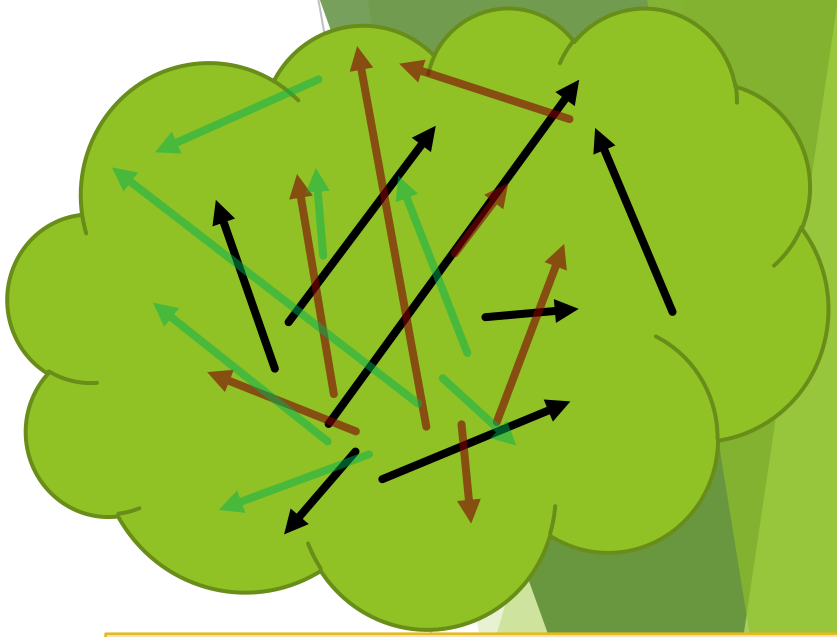
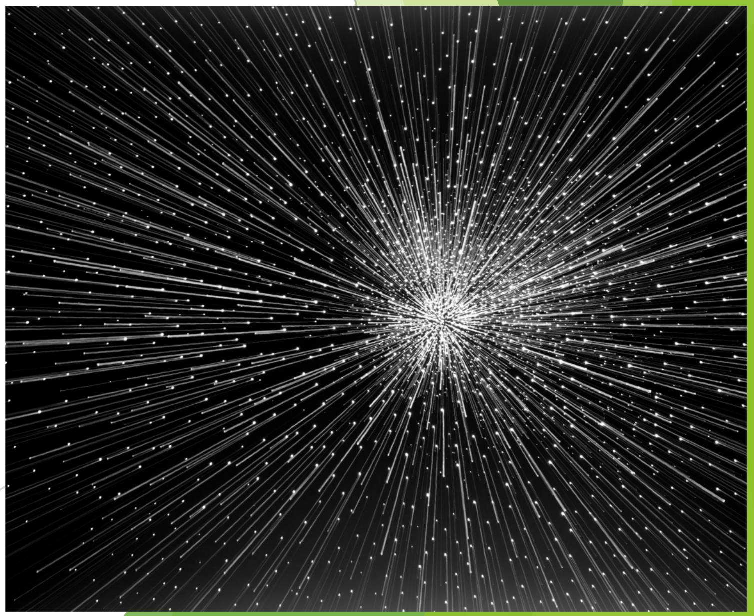


Different visualizations of the set of all displacements



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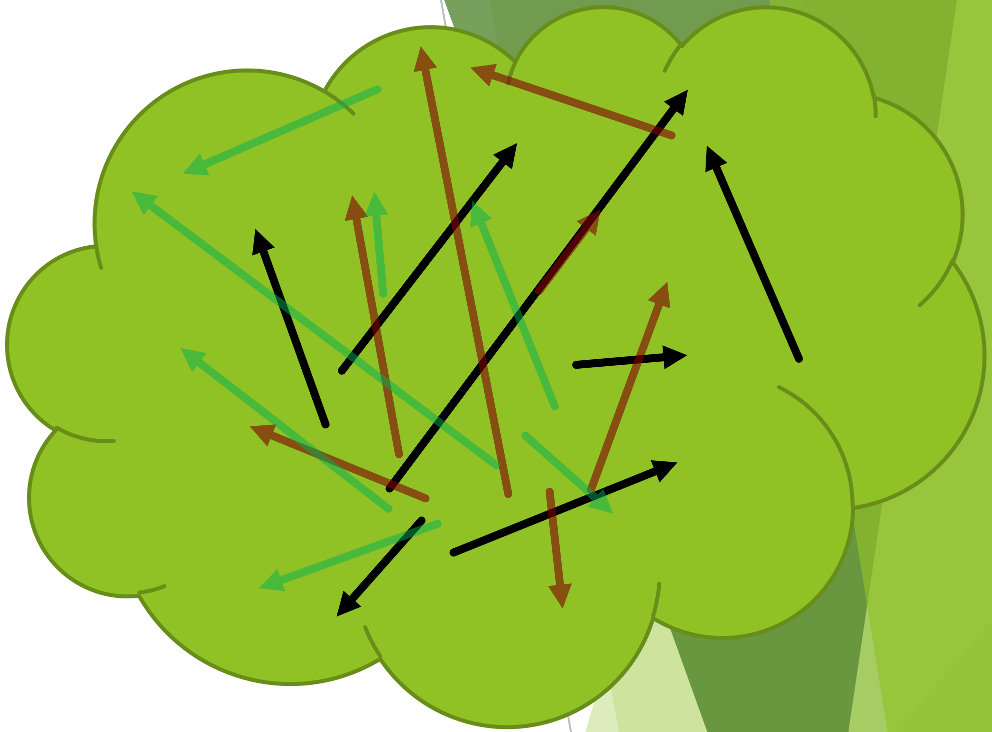
- But we cannot do much with these abstract displacement rules, we need numbers …
- … This is where the concept of **linear independence** comes in handy



Vector space

- **Linear combination:** a combination of vectors of the form $\sum_{i=1}^{m} \alpha_i \, \bar{x}_i$
- A subset B of a vector space V is said to be **linearly independent** if

$$\sum_{i=1}^{m} \alpha_i \, \bar{x}_i = 0 \Rightarrow \alpha_i = 0 \; \forall i \in \{1,2, \dots, m\}$$

- (The symbol $\Rightarrow$ means "implies" and $\forall$ means "for all")
- In other words,

$$\alpha_j \, \bar{x}_j = -\sum_{\substack{i=1 \\ i \neq j}}^{m} \alpha_i \, \bar{x}_i \quad \text{OR} \quad \bar{x}_j = \sum_{\substack{i=1 \\ i \neq j}}^{m} \beta_i \, \bar{x}_i$$

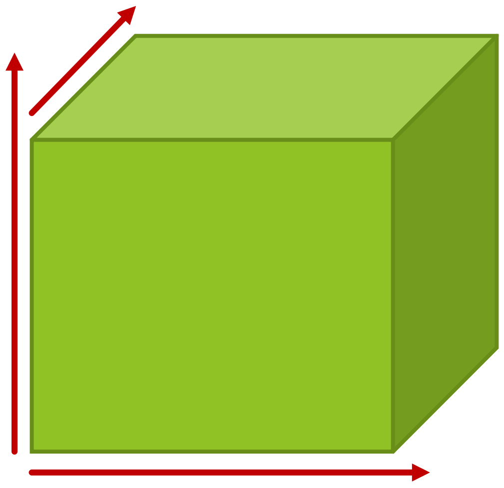
is not possible, or equivalently, an element of B cannot be obtained by linear combination of the other elements of B

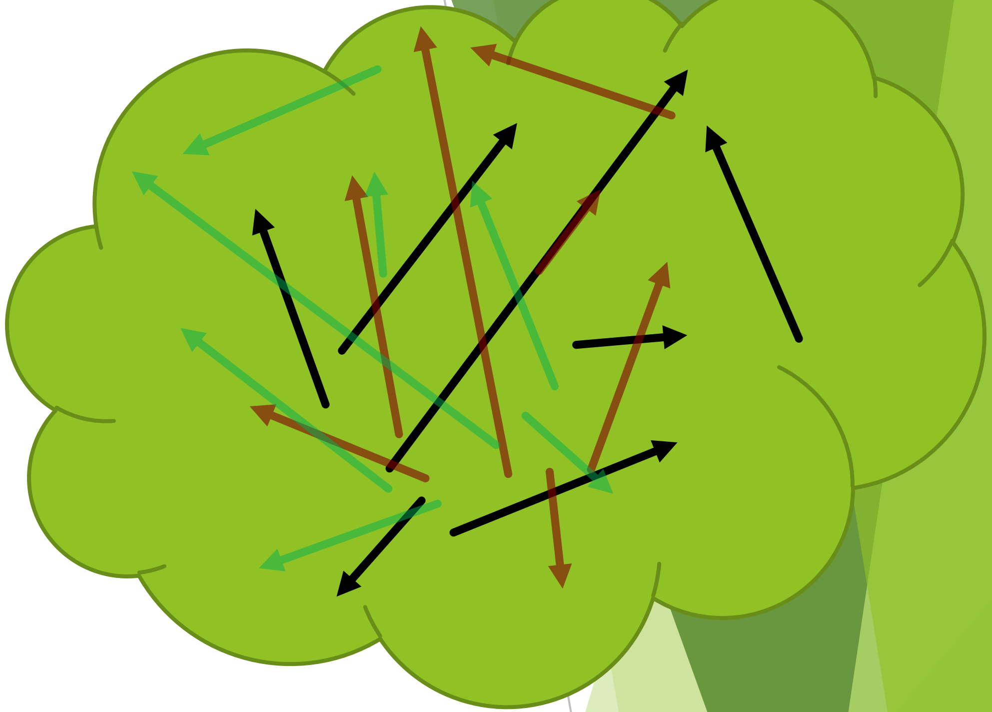
Vector space

- The largest subset of V that is linearly independent is called a **Basis set** (or basis) of V
- The elements of a basis set B are called **basis vectors**

Vector space

- For the set of displacement rules in 3-dimensional space, there are 3 basis vectors
- We typically pick these to be along the X, Y, Z directions



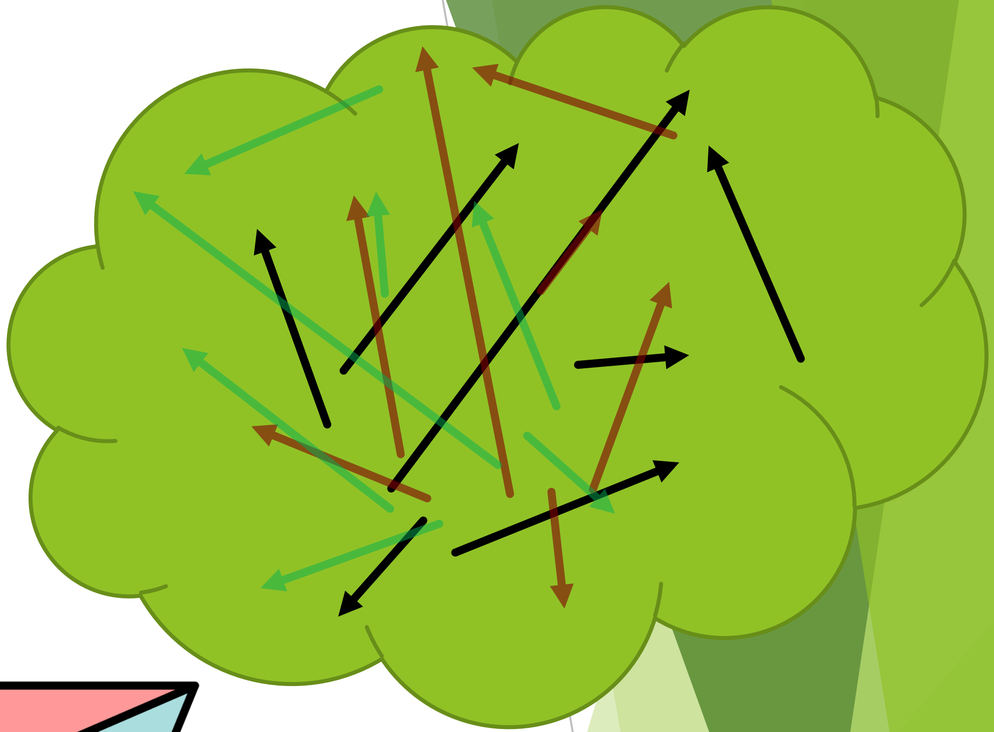


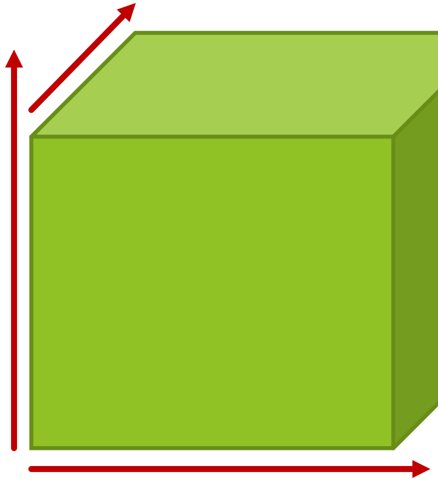
Vector space

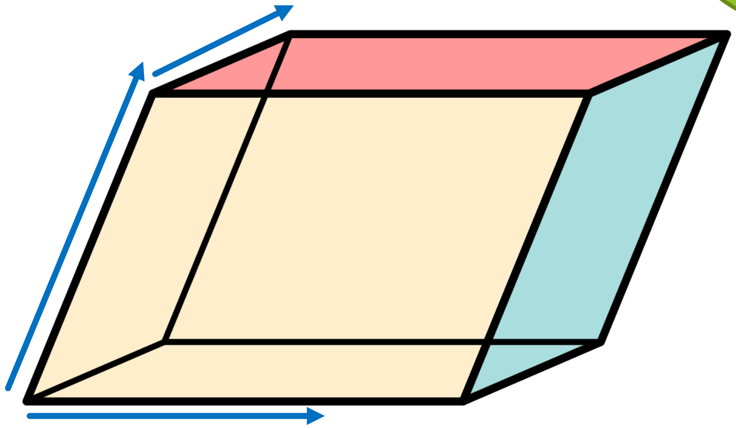
- *Basis sets are not unique*: any subset of linearly independent vectors can form a basis set

Vector space

- For the set of displacement rules in 3-dimensional space, any 3 directions that do not all lie in a plane can be used to form a basis







Vector space

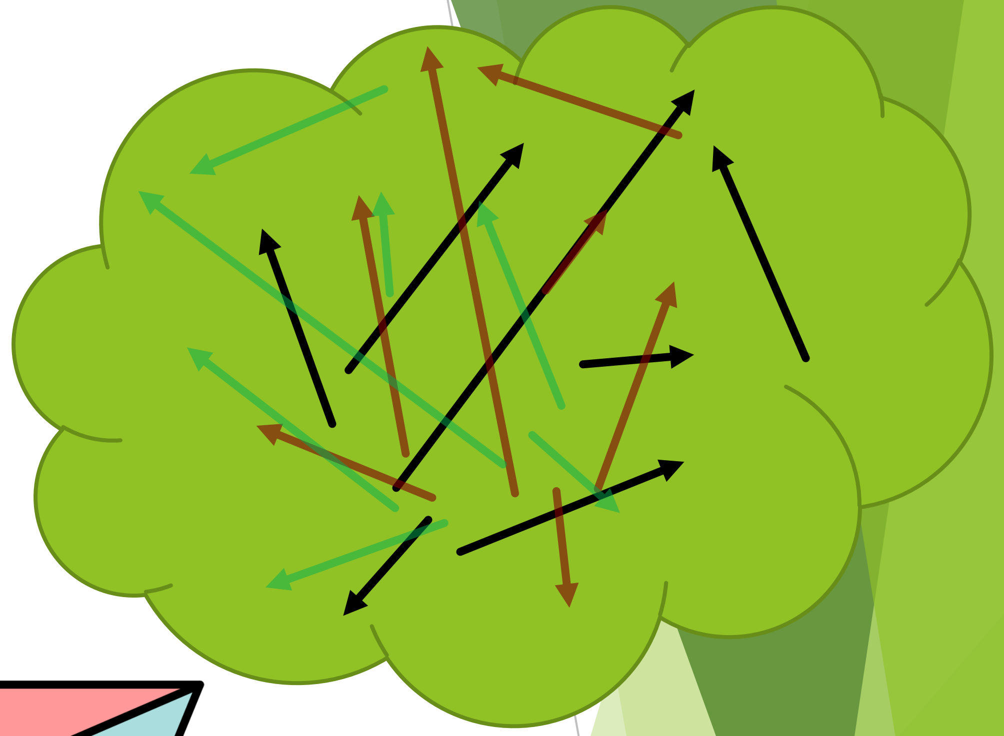
- By definition of a basis, any vector outside a basis set must be expressible as a linear combination of the basis vectors
- This means that all basis sets have the same number of basis vectors: If not, then a larger basis will have elements expressible in terms of a smaller basis and, hence, cannot be a basis set by itself
- The number of basis vectors is a characteristic property of V: its **Dimensionality**
  - We will denote the dimensionality of a vector space in general by N

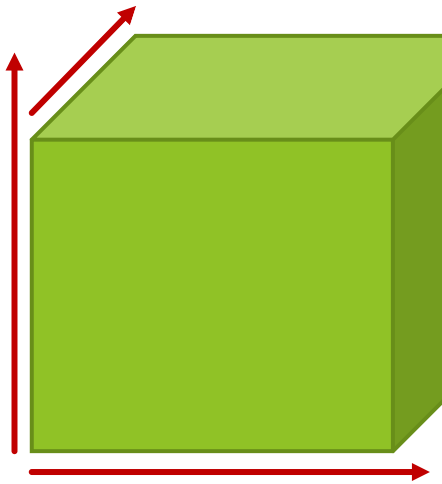
Vector space

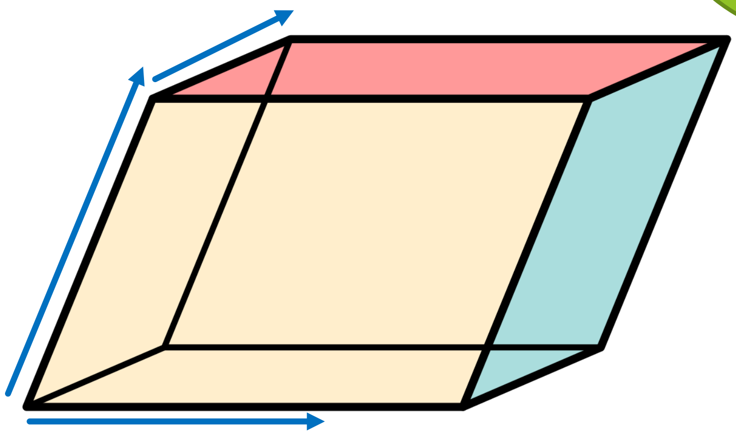
- Taking all possible linear combinations of vectors in a linearly independent set generates a vector space that is a subset, called a **subspace**, of the full vector space
  - The subspace is the full vector space if the linearly independent set used to generate it is a basis set
  - The subspace has a dimensionality (= number of linear independent vectors used) that is $\leq N$
- The subspace generated by a given set of linearly independent vectors is called its **span**

Vector space

- Taking the linear combination of any two of the three basis vectors below will generate a 2D plane which is a subspace of the full 3D space
- Similarly, any 2D plane in 3D space is spanned by two linearly independent vectors







Vector space

- For a given basis, the coefficients of a vector expressed as a linear combination of its basis vectors are unique

$$\sum_{i=1}^{N} \alpha_i \, \bar{x}_i = \bar{z} \Rightarrow \alpha_i \text{ is unique}$$

- Proof: Assume the contrary,

$$\sum_{i=1}^{N} \alpha_i \, \bar{x}_i = \bar{z} = \sum_{i=1}^{N} \beta_i \, \bar{x}_i \text{, where } \beta_i \neq \alpha_i$$

Then,

$$\sum_{i=1}^{N} \alpha_i \, \bar{x}_i - \sum_{i=1}^{N} \beta_i \, \bar{x}_i = \bar{z} - \bar{z} = 0$$

$$\sum_{i=1}^{N} (\alpha_i - \beta_i) \, \bar{x}_i = \sum_{i=1}^{N} \gamma_i \, \bar{x}_i = 0$$

But this cannot happen for a basis set by definition unless $\gamma_i = 0 \Rightarrow \alpha_i = \beta_i$

Vector space

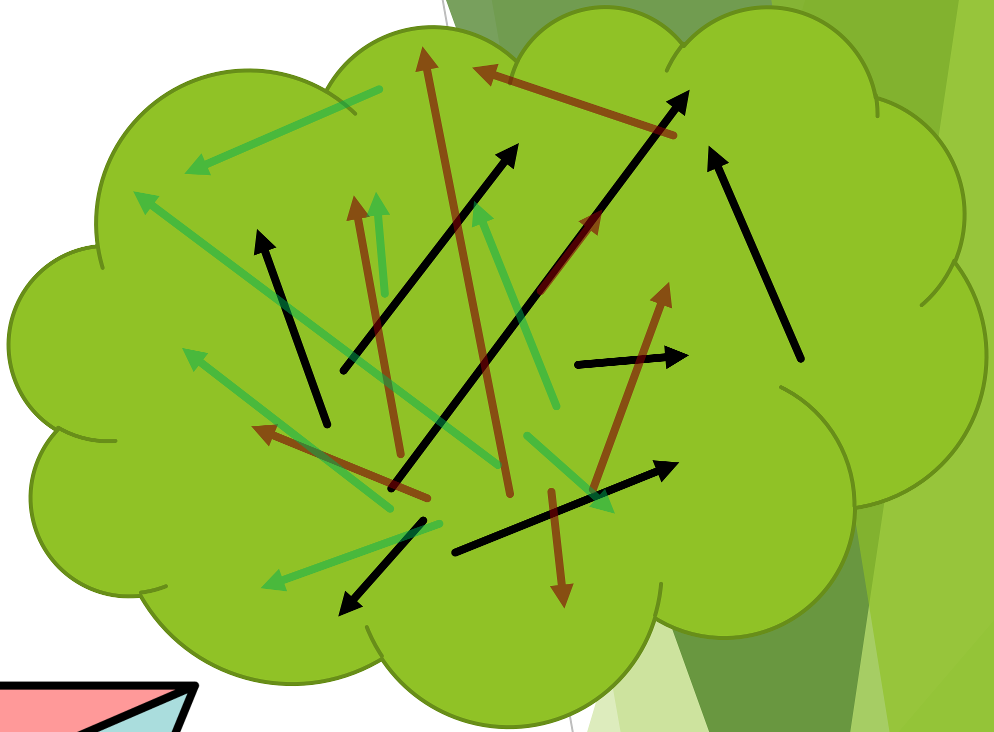
- For a given basis, the coefficients of a vector expressed as a linear combination of its basis vectors are unique

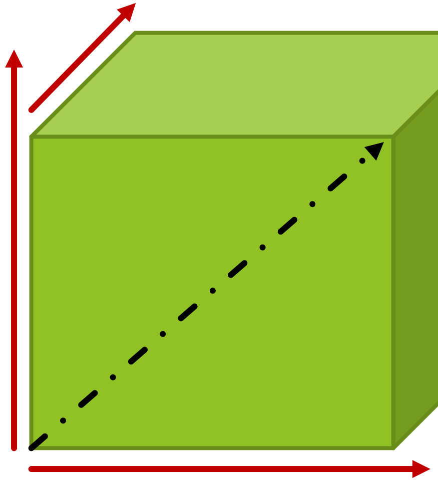
$$\sum_{i=1}^{N} \alpha_i \, \bar{x}_i = \bar{z} \Rightarrow \alpha_i \text{ is unique}$$

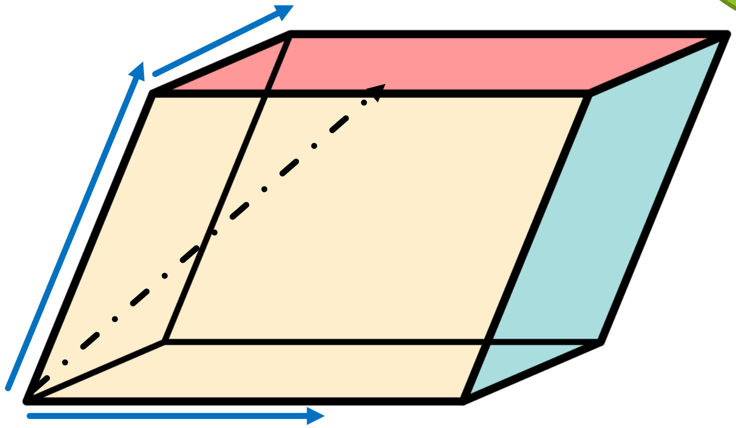
- This means that, for a given choice of basis, every vector in V can be uniquely represented as a sequence of N numbers: $(\alpha_1, \alpha_2, \dots, \alpha_N)$
  - These linear combination coefficients are called the **components** of a vector in a given basis
- The same vector has different components in different bases

Vector space

- The same vector (dot-dashed arrow) has different components in different bases







Vector space: Inner product

- An additional useful operation on a vector space is an inner product
$$\langle \bar{x}, \bar{y} \rangle = \alpha$$

Where α is a scalar

  - **Linearity** property: $\langle \bar{x} + \bar{z}, \bar{y} \rangle = \langle \bar{x}, \bar{y} \rangle + \langle \bar{z}, \bar{y} \rangle$
  - See a textbook for more requirements on an inner product
- **Orthogonal vectors:** $\bar{x}$ and $\bar{y}$ are orthogonal if $\langle \bar{x}, \bar{y} \rangle = 0$
- **Norm** (induced by the inner product): $||\bar{x}||^2 = \langle \bar{x}, \bar{x} \rangle \geq 0$
  - Only the norm of the identity vector is zero

Vector space

- Orthonormal basis: $\{\bar{e}_0, \bar{e}_1, \dots, \bar{e}_{N-1}\}$

$$\langle \bar{e}_i, \bar{e}_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- Given an orthornormal basis, and components of vectors in that basis, the inner product of vectors can be evaluated in term of the components alone

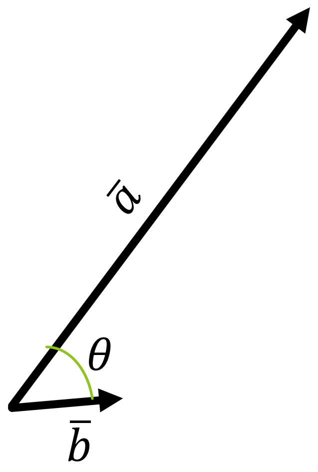
$$\bar{x} = \sum_{i=0}^{N-1} x_i\, \bar{e}_i\,, \qquad \bar{y} = \sum_{j=0}^{N-1} y_j\, \bar{e}_j$$

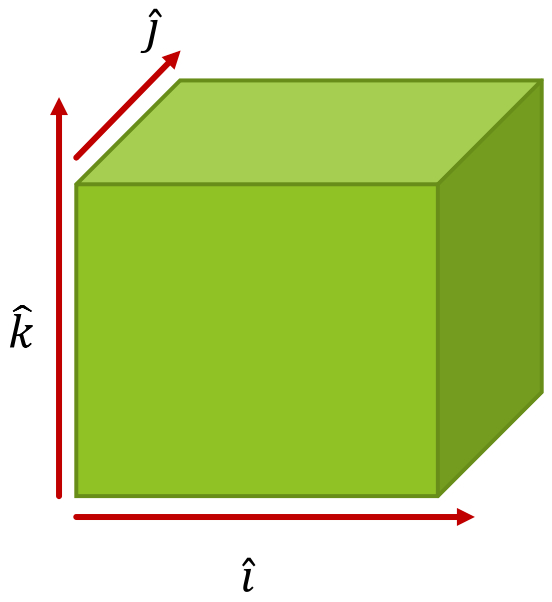
$$\langle \bar{x}, \bar{y} \rangle = \left\langle \sum_{i=0}^{N-1} x_i\, \bar{e}_i\,, \bar{y} \right\rangle = \sum_{i=0}^{N-1} x_i \langle \bar{e}_i, \bar{y} \rangle = \sum_{i=0}^{N-1} x_i \langle \bar{e}_i, \sum_{j=0}^{N-1} y_j\, \bar{e}_j \rangle = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} x_i y_j \langle \bar{e}_i, \bar{e}_j \rangle = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} x_i y_j \delta_{ij} = \sum_{i=0}^{N-1} x_i y_i$$

$$\langle \bar{x}, \bar{y} \rangle = \sum_{i=0}^{N-1} x_i y_i$$

Vector space

- Familiar dot product (Euclidean inner product):
$$\langle \bar{a}, \bar{b} \rangle = |\bar{a}||\bar{b}| \cos\theta$$
- Using the orthornormal basis shown,
$$\langle \bar{a}, \bar{b} \rangle = a_x b_x + a_y b_y + a_z b_z$$





Signals and vector spaces

- The set of all sequences $\{(x_0, x_1, \dots, x_{N-1})\}$ with N real numbers, called N-tuples, is a vector space
  - Usual symbol for this vector space: $\mathbb{R}^N$
  - The addition and multiplication rules are the usual ones for real numbers
- Any vector in $\mathbb{R}^N$ can be expressed as a linear combination of N other independent vectors: **Basis set**

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- **Standard basis:**

$$
\begin{aligned}
\bar{e}_0 &= (1,0,0,\dots,0) \\
\bar{e}_1 &= (0,1,0,\dots,0) \\
&\vdots \\
\bar{e}_{N-1} &= (0,0,0,\dots,1)
\end{aligned}
$$

- The elements of $\bar{x} \in \mathbb{R}^N$are the components of $\bar{x}$ in the standard basis:

$$\bar{x} = x_0 \times (1,0,0,\dots,0) + x_1 \times (0,1,0,\dots,0) + \cdots x_{N-1}(0,0,0,\dots,1)$$

Example

- Consider the vector space $\mathbb{R}^3$
- Given a vector wtith components (x_1, x_2, x_3) in the standard basis, what are its components in the orthonormal basis $\{(-1,0,0), (0,-1,0), (0,0,-1)\}$?

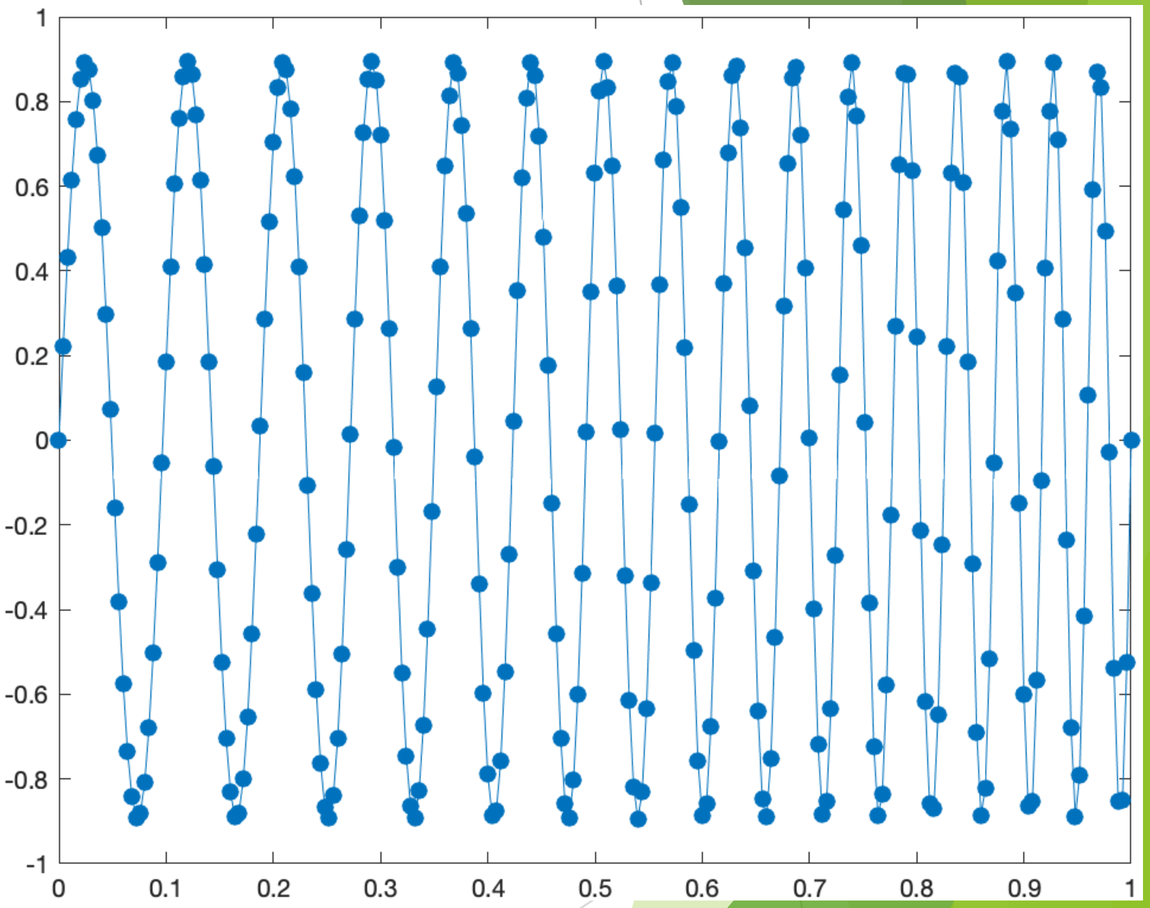
$$(x_1, x_2, x_3) = (-x_1)\times(-1,0,0) + (-x_2)\times(0,-1,0) + (-x_3)\times(0,0,-1)$$

- Hence, the components are $(-x_1, -x_2, -x_3)$
  - Note that the same components in the standard basis correspond to a completely different vector
- Coefficients of a given vector in different basis sets are related by a transformation matrix:

$$\underbrace{\begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}}_{\substack{Coefficients \\ in\ another \\ basis}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\substack{Coeficients \\ in\ Standard\ basis}}$$

Signals and vector spaces

- A finite length discrete time signal with N samples, $(s_0, s_1, \dots, s_{N-1})$ is an element of $\mathbb{R}^N$
- We can represent the same signal in different basis sets
- Different basis sets emphasize different information in the signal



Signals and vector spaces

- A finite length discrete time signal with N samples, $(s_0, s_1, \dots, s_{N-1})$ is an element of $\mathbb{R}^N$
- We can carry over the same kind of visualization aids provided we understand their limitations to 3 dimensions