Lab Topic 2

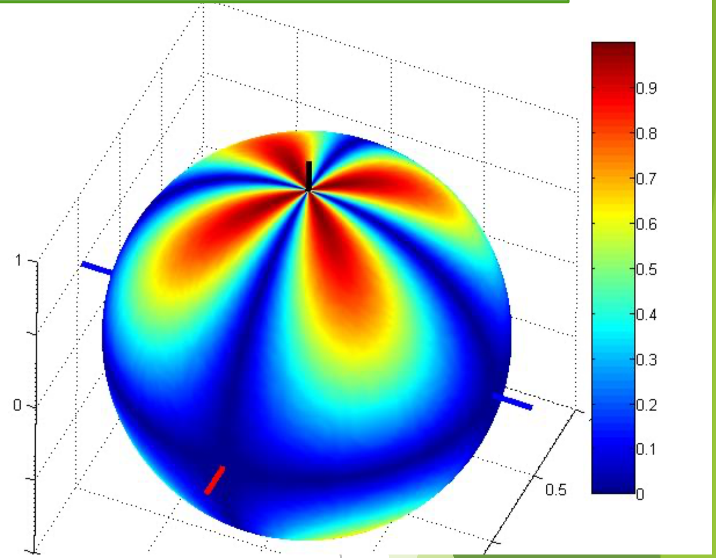
Learning objectives

- Learn how to calculate the response of a GW detector to a plane GW
  - Response calculations for both LIGO and LISA
  - Long wavelength approximation
  - Detector rotation and motion included for LISA

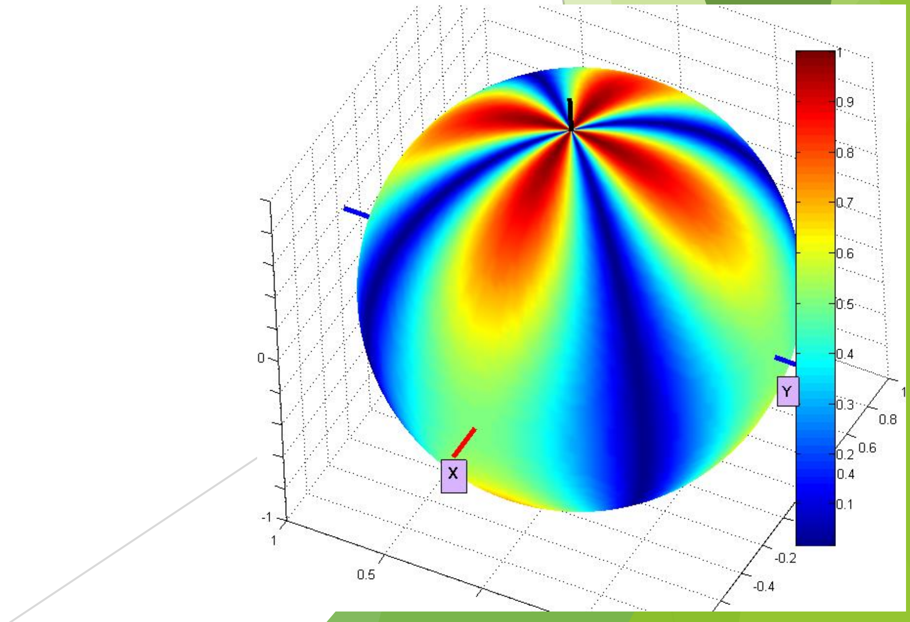
Antenna Patterns: Local frame← Analytical forms

- Long wavelength and static detector approximation throughout this Lab
- Write a code to calculate F_+ and $F_\times$ in an L-shaped interferometer's local frame from their analytical formulae
  - Source direction: (θ, ϕ) in detector frame
  - Plot them on a sphere using the GWSC/GW/skyplot.m function
  - F_+ Demo code: GWSC/GW/formulafp.m, GWSC/GW/testskyplot.m
- Plots should agree with the pictures in the lecture slides

$$F_\times(\theta, \phi) = \cos\theta \sin 2\phi$$

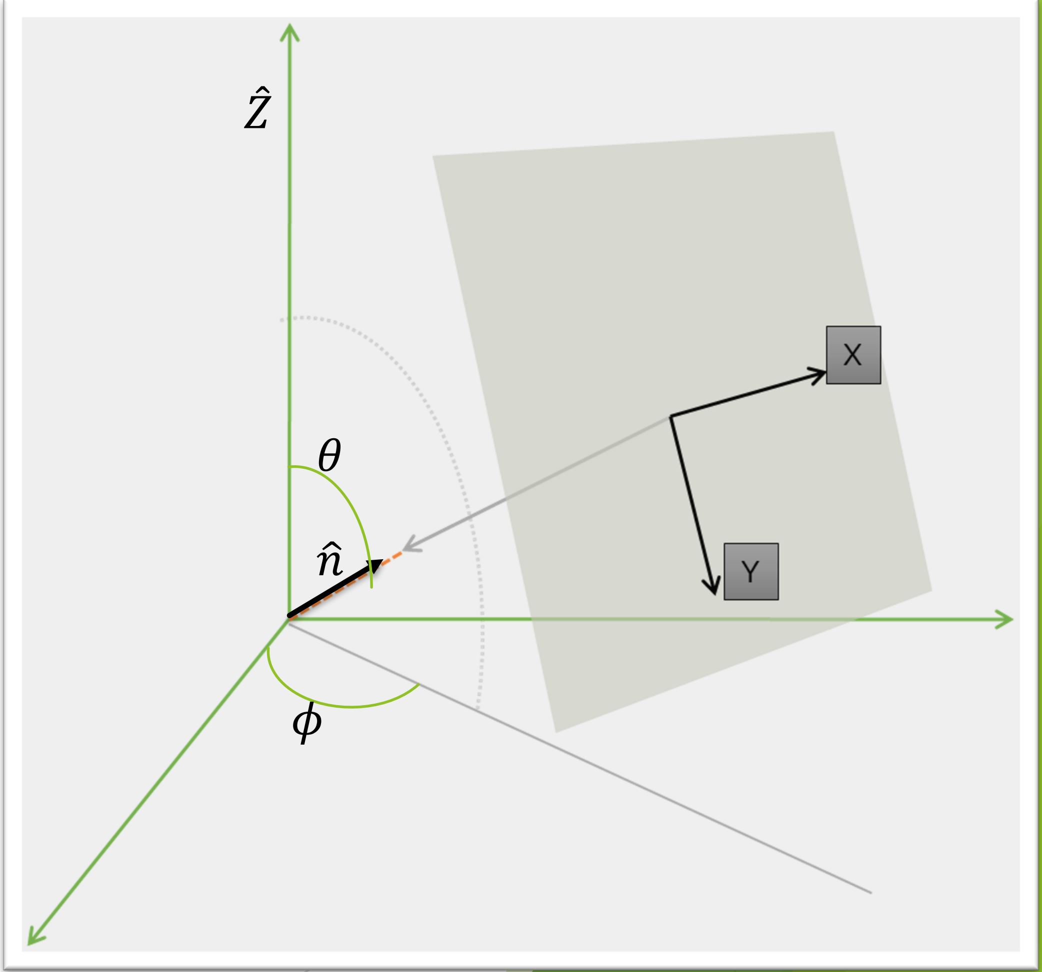


$$F_+(\theta, \phi) = \frac{1}{2}(1 + \cos^2\theta)\cos 2\phi$$



Antenna Patterns: Local frame←using tensors

- Use the expression for (a) polarization tensors, (b) Detector tensor, and (c) Contraction of polarization and detector tensors to obtain $F_{+,\times}$
- All tensor components must be expressed in a common frame before the tensors are contracted → express all unit vector components in a common frame
- We will use the detector frame as the common one:
  - Detector arm unit vectors and their components in the detector frame: $\hat{n}_X = (1,0,0)$, $\hat{n}_Y = (0,1,0)$
  - Detector frame Z vector: $\hat{Z} = (0,0,1)$,
  - Source direction vector in detector frame (for polar angles θ and ϕ): $\hat{n}$
- Wave frame unit vector components for polarization tensor calculation (burst GW convention):
  - Wave frame $\hat{x} \propto \hat{Z}\times\hat{n}$ (Note: must normalize)
  - Wave frame $\hat{y} = \hat{x}\times\hat{n}$
  - (Use GWSC/GW/vcrossprod.m to obtain vector cross product components numerically)



Strain signal

- Detector tensor:

$$\overleftrightarrow{D} = \frac{1}{2}(\hat{n}_X \otimes \hat{n}_X - \hat{n}_Y \otimes \hat{n}_Y)$$

- Wave tensor:

$$\overleftrightarrow{W} = h_+(t)\,\overleftrightarrow{e}_+ + h_\times(t)\overleftrightarrow{e}_\times$$

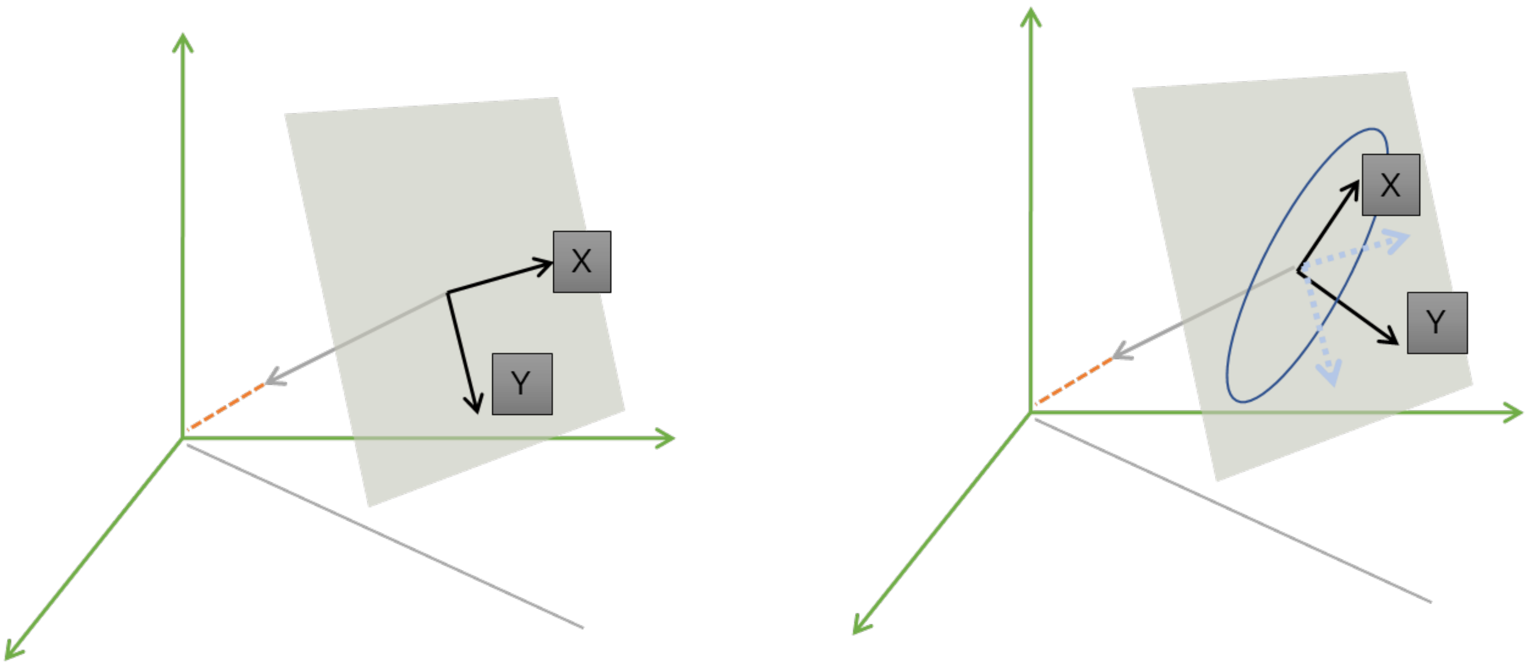
- Polarization tensors: $\overleftrightarrow{e}_+ = \hat{x}\otimes\hat{x} - \hat{y}\otimes\hat{y}$; $\overleftrightarrow{e}_\times = \hat{x}\otimes\hat{y} + \hat{y}\otimes\hat{x}$
- Matlab can calculate direct products of vectors:

$$\underbrace{\left.\begin{array}{r} a = [a_1, \quad a_2, \quad a_3] \\ b = [b_1, \quad b_2, \quad b_3] \end{array}\right\}}_{\text{2 row vectors in Matlab}} \rightarrow \underbrace{a' * b}_{Matlab} \rightarrow \hat{a}\otimes\hat{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

- Strain signal: “Contraction of wave and detector tensors”

$$s(t) = \sum_{i,j=1}^{3} W_{ij}D_{ij} = W^{ij}D_{ij} = \overleftrightarrow{W}:\overleftrightarrow{D} = h_+(t)\,\overbrace{\cancel{\overleftrightarrow{W}:\overleftrightarrow{e}_+}}^{F_+(\hat{n})} + h_\times(t)\,\overbrace{\cancel{\overleftrightarrow{W}:\overleftrightarrow{e}_\times}}^{F_\times(\hat{n})}$$

  - Contraction of matrices A and B in Matlab $\rightarrow \mathrm{sum}(A(:)\,.*\,B(:))$
- Compare the antenna patterns obtained using tensor contractions and analytical forms
- Demo codes: GWSC / GW/ detframefpfc.m and testdetframefpfc.m



Exercise: Wave frame conventions

Extend the previous exercise to include rotation due to polarization angle into the polarization tensors (Hint: the new wave frame X, Y vector components will be linear combinations of the old X, Y components)

Strain signal from a non-evolving binary

- Use the sinusoidal signal generation function to generate
$$h_+(t) = A\sin(2\pi f_0 t)$$
$$h_\times(t) = B\sin(2\pi f_0 t + \phi_0)$$
- Pick your own values of A, B, f_0, ϕ_0 (Respect Nyquist theorem!)
- Plot the strain signal for different values of θ, ϕ, and ψ

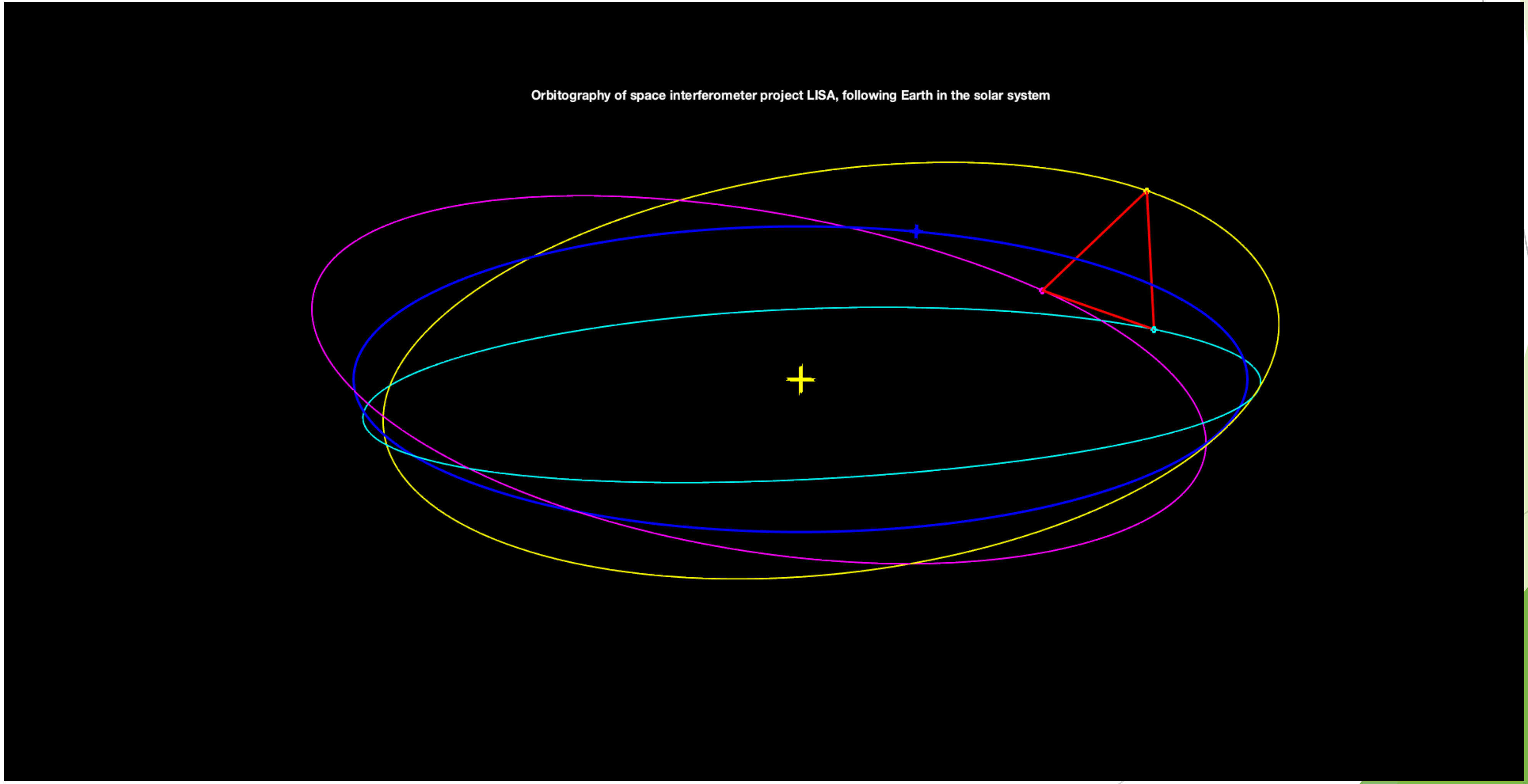
General strain signal for a static interferometer

- Write a function:
  - Inputs:
    - h_+ (vector) and $h_\times$ (vector): time series of the polarizations (don't have to be sinusoidal)
    - θ, ϕ
  - Output:
    - Strain signal (for perpendicular arm interferometer)

Antenna patterns for LISA

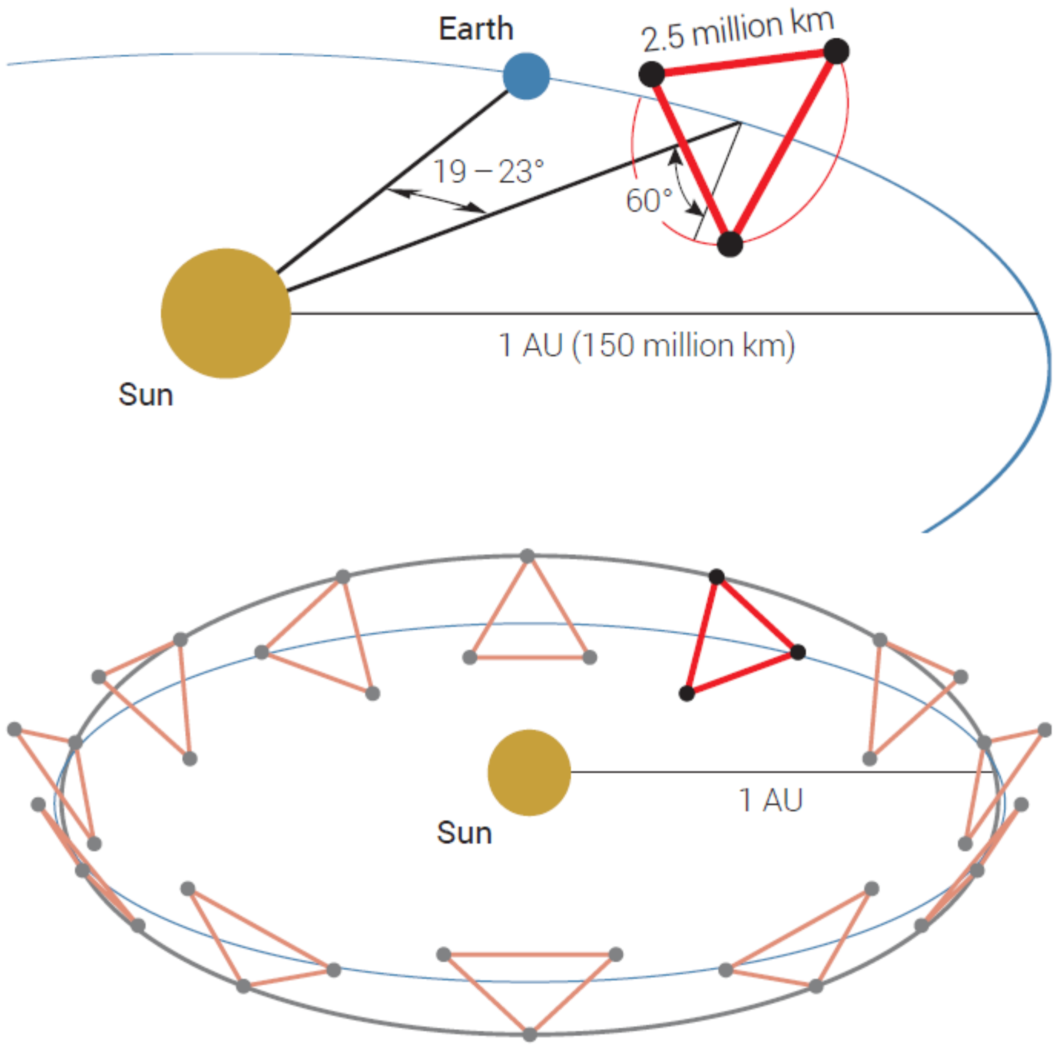
Toy LISA

- Toy LISA: Rigid equilateral triangle formation of three satellites
- Actual LISA cannot be rigid because the satellites must follow Keplerian orbits
- Toy LISA is good for practicing data analysis because it allows fast generation of signals and templates



Reference frames and rotations

- The common reference frame to use here is the Solar System Barycentric (SSB) frame
- The polarization tensors will be computed in the same way as for the detector frame (see previous exercises) but now the frame is the SSB
- We need to find the components of LISA arm unit vectors in the SSB frame and then obtain the detector tensor from the arm vectors
- Finally, contract the detector and polarization tensors to get antenna pattern functions



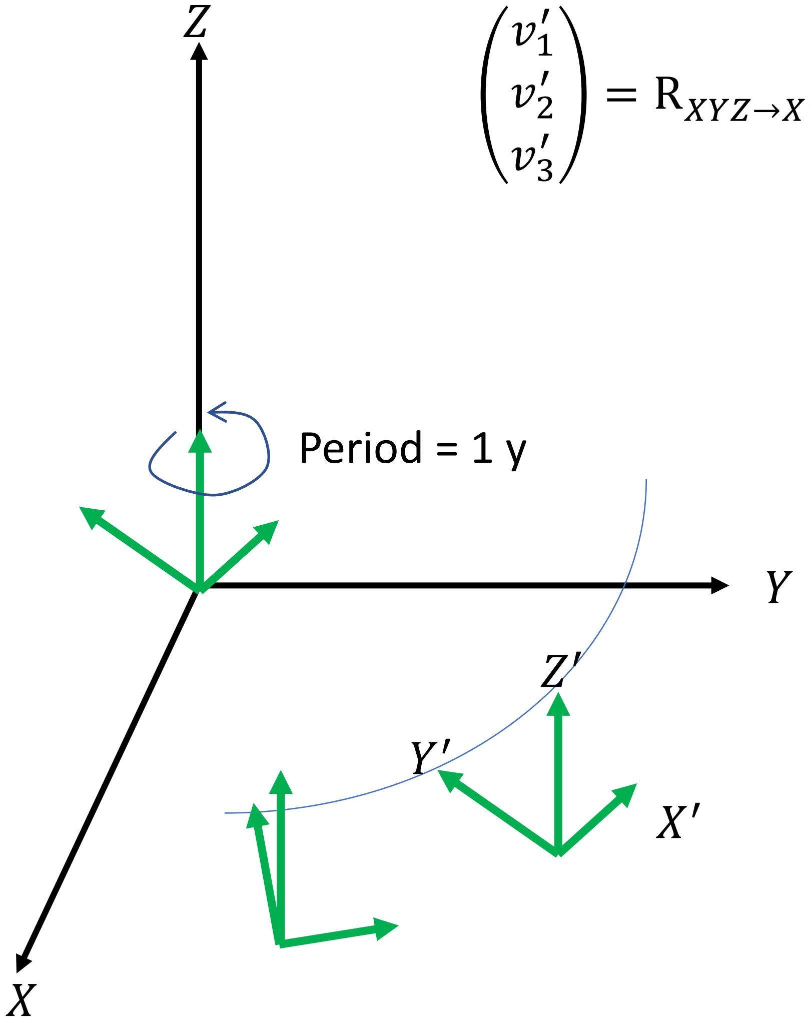
Obtain the arm components in the SSB frame

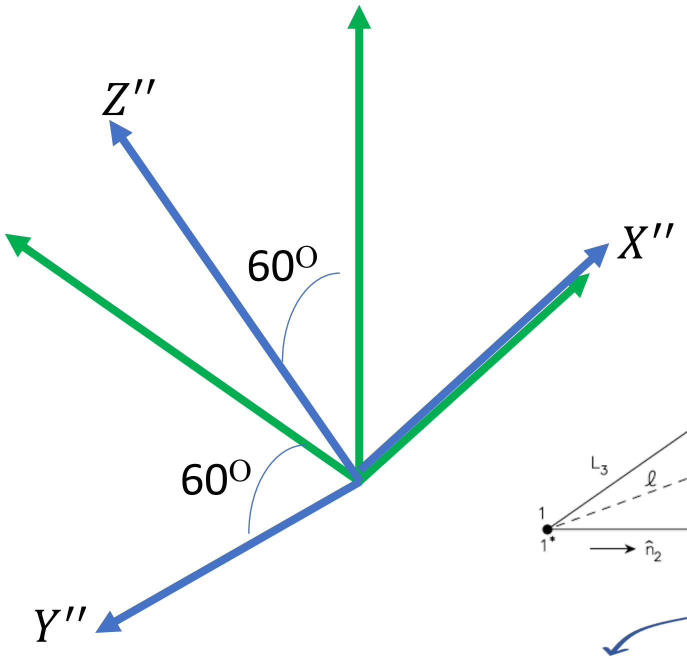
$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \mathrm{R}_{XYZ\to X'Y'Z'} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\mathrm{R}_{XYZ\to X'Y'Z'} = \begin{pmatrix} \hat{x}'.\hat{x} & \hat{x}'.\hat{y} & \hat{x}'.\hat{z} \\ \hat{y}'.\hat{x} & \hat{y}'.\hat{y} & \hat{y}'.\hat{z} \\ \hat{z}'.\hat{x} & \hat{z}'.\hat{y} & \hat{z}'.\hat{z} \end{pmatrix}$$

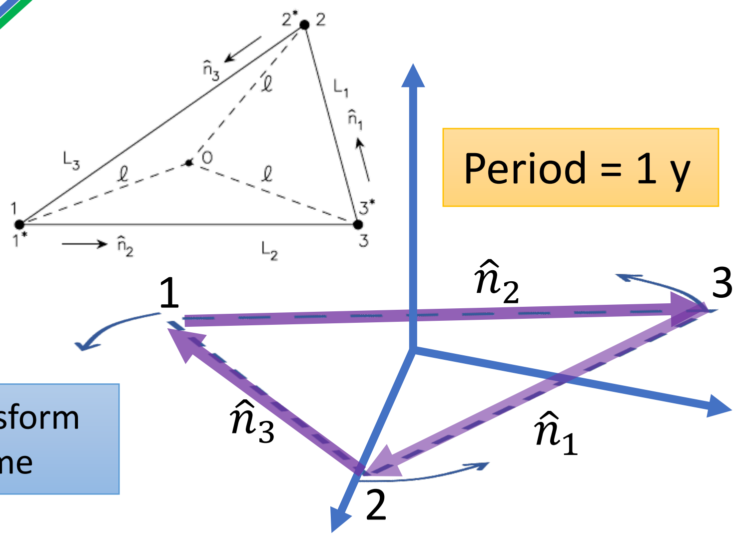
$$RR^T = I$$

2: get rotation matrix from blue to green frame





1: Get rotating arm unit vector components in Blue frame (See Fig. 4 of LDC1 manual)



3: get rotation matrix from green to black (SSB) frame

4: use product of rotation matrices to transform arm unit vector components to SSB frame

Antenna patterns

- Use the expressions in Sec IIIB of the paper arXiv:1207.4956v1 to obtain the detector tensors for the two **Michelson TDI combinations**

> the GW propagation direction. The two detector tensors are defined as $D_I \equiv \frac{1}{2}(\hat{n}_1 \otimes \hat{n}_1 - \hat{n}_2 \otimes \hat{n}_2), D_{II} \equiv \frac{1}{2\sqrt{3}}(\hat{n}_1 \otimes \hat{n}_1 + \hat{n}_2 \otimes \hat{n}_2 - 2\hat{n}_3 \otimes \hat{n}_3)$, where $\hat{n}_1, \hat{n}_2, \hat{n}_3$ denote the unit vectors along each arm of LISA. Here we assume

- Obtain the $F_{+,\times}(t;\hat{n})$ for each TDI combination by contracting the respective detector tensor above with each polarization tensor
- Write a code:
  - Inputs: Source direction, vector of sample times
  - Outputs: F_+, $F_\times$ time series for a Michelson TDI combination

Toy LISA response: Partial

- Write a Matlab script to do the following
- Generate h_+, $h_\times$ that are sinusoidal
  - The script should have user-specified sky location and polarization angle for the GW source
- Generate any one of the Michelson TDI response (no doppler shift included) using the code from the previous exercise to generate the $F_{+,\times}$ time series
$$s(t) = F_+(t;\theta,\phi)h_+(t) + F_\times(t;\theta,\phi)h_\times(t)$$
- Take FFT of the detector response and compare to the FFT of h_+

Toy LISA response: Full

- LISA detector response including doppler shift
  - $h_{+,\times}(t) \rightarrow h_{+,\times}(t - \frac{\hat{n}.\bar{x}_d}{c})$
  - $\hat{n}$: Wave propagation direction
  - $\bar{x}_d(t)$: LISA centroid
- Write a code to calculate the components of the position vector, $\bar{x}_d$, of the LISA centroid (simple circular orbit) in the SSB frame
- Plot any one of the LISA Michelson responses for a monochromatic source in the SSB frame
  - In SSB frame: $h_+(t) = A\sin(\omega_0 t)\,; h_\times(t) = \left(\frac{A}{2}\right)\cos\omega_0 t$
  - Generate doppler modulated sinusoids h_+, $h_\times$:

$$h_+(t) = A\sin\left(\omega_0\,(t - \frac{\hat{n}.\bar{x}_d}{c})\right)$$

$$h_\times(t) = B\cos\left(\omega_0\,(t - \frac{\hat{n}.\bar{x}_d}{c})\right)$$

  - One parameter is missing here: polarization angle (but we will ignore it)
- Compare FFT of the response to that of $h_{+,\times}(t)$ in the SSB frame

Effect of sky location

- Take the same SSB frame $h_+, h_\times$ but at a different sky location and verify that the LISA response is different
- This allows LISA to have directional sensitivity to long-lived sources.

Advanced

- Find out about the Tianqin detector configuration, which is a geocentric one
  - J. Luo et al., “TianQin: a space-borne gravitational wave detector," Class. Quant. Grav., vol. 33, no. 3, p. 035010, 2016.
- Write code using tensors to compute the response of Tianqin in the long wavelength approximation
- (Taiji configuration is essentially the same as LISA and, hence, does not lead to a very different result)