Lecture 4: Convexity

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Convex Sets

Definition

In real vector space V , a set $S \subset V$ is a convex set iff

$$\lambda x + (1-\lambda)y \in S$$

for any $\lambda \in [0,1]$ and $x, y \in S$.

For finitely many vectors $x_1, x_2, ... x_n$ in vector space V , a convex combination of $x_1, x_2, ..., x_n$ is a vector $\sum_{i=1}^{n} \lambda_i x_i$ for scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.

Separating Hyperplane Theorem

In $\mathbb{R}^n$, a hyperplane is defined as

$$H(p,c) = \{x \in \mathbb{R}^n : p \cdot x = c\}$$

where $p \in \mathbb{R}^n \backslash \{0\}$, $c \in \mathbb{R}$, and $\cdot$ is the dot product. A hyperplane $H(p,c)$ cuts the whole space $\mathbb{R}^n$ into halves. This is a generalization of a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$.

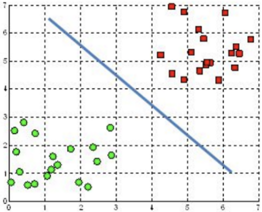
Theorem

(Minkowski's Separating Hyperplane). Let S_1 and S_2 be two disjoint nonempty and convex sets in $\mathbb{R}^n$. Then there exist $p \in \mathbb{R}^n \backslash \{0\}$ and $c \in \mathbb{R}$ s.t. $p \cdot x \geq c$ for any $x \in S_1$ and $p \cdot x \leq c$ for any $x \in S_2$.

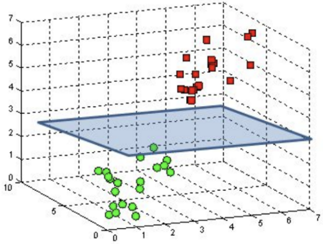
Minkowski's Separating Hyperplane is used in the proof of Second Welfare Theorem.

Example

A hyperplane in $\mathbb{R}^2$ is a line



A hyperplane in $\mathbb{R}^3$ is a plane



Brouwer's Fixed Point Theorem

Theorem

Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$, and consider a continuous function $f : X \to X$. Then there exists $x^* \in X$ s.t. $f(x^*) = x^*$.

Brouwer's fixed point theorem plays an important role in the existence of Walrasian equilibria in the general equilibrium theory and the existence of Nash equilibria in non-cooperative game theory.

Convex and Concave Functions

Definition

Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.

1. The function f is a convex function iff

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

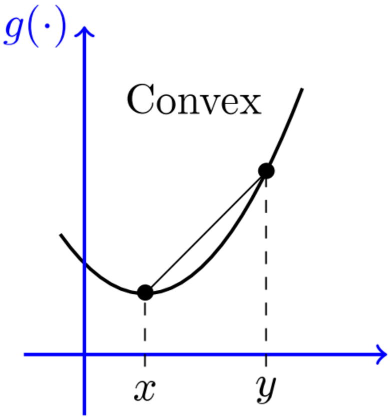
   for any $x, y \in S$ and $\lambda \in [0,1]$.

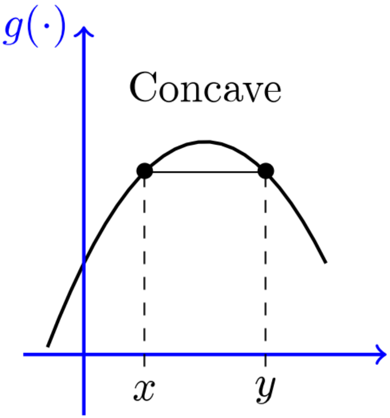
2. The function f is a concave function iff

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$

   for any $x, y \in S$ and $\lambda \in [0,1]$.

Strictly convex/concave will make $\geq / \leq$, $> / <$.


$g(\cdot)$
Convex
x
y


$g(\cdot)$
Concave
x
y

Jensen's Inequality

Theorem

(Jensen's Inequality). Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.

1. f is convex iff
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i-1}^{n} \lambda_i f(x_i)$$
for any $x_1, x_2, \ldots, x_n \in S$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.
2. f is concave iff
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i f(x_i)$$
for any $x_1, x_2, \ldots, x_n \in S$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.

Convex/Concave Property

Proposition

Consider two functions f and g from S to $\mathbb{R}$, where S is a convex set in vector space V. If f and g are both convex/concave functions, then

1. $f+g$ is a convex/concave function, and
2. cf is a convex/concave function, for any $c \in \mathbb{R}_+$.

Proposition

Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.
(1) If f is convex and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing and convex, then $\phi \circ f$ is convex.
(2) If f is concave and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing and concave, then $\phi \circ f$ is concave.

Proposition

Consider a finite family of functions $\{f_\alpha\}_{\alpha\in A}$ from S to $\mathbb{R}$, where S is a convex set in vector space V.

1. If all functions in the family are convex, and the set $\{f_\alpha\}_{\alpha\in A}$ is bounded from above for each $x \in S$, then $\sup\{f_\alpha\}_{\alpha\in A}$ is a convex function.
2. If all functions in the family are concave, and the set $\{f_\alpha(x) : \alpha \in A\}$ is bounded from below for each $x \in S$, then $\inf\{f_\alpha\}_{\alpha\in A}$ is a concave function.

Convexity/Concavity of Continuously Differentiable Functions.

Suppose the function $f : S \to \mathbb{R}$ is a C^1 function, where S is a convex and open set in $\mathbb{R}^n$.
(1) f is (strictly) convex iff

$$f(x')\ (>) \geq f(x) + \nabla f(x) \cdot (x' - x)$$

for any $x', x \in S$.
(2) f is (strictly) concave iff

$$f(x')\ (<) \leq f(x) + \nabla f(x) \cdot (x' - x)$$

for any $x', x \in S$.

Theorem

Suppose the function $f : S \to \mathbb{R}$ is a C^2 function, where S is a convex and open set in $\mathbb{R}^n$.

1. *f is convex iff its Hessian matrix $H(x)$ is positive semi-definite for any $x \in S$.*
2. *f is concave iff its Hessian matrix $H(x)$ is negative semi-definite for any $x \in S$.*
3. *f is strictly convex if its Hessian matrix $H(x)$ is positive definite for any $x \in S$.*
4. *f is strictly concave if its Hessian matrix $H(x)$ is negative definite for any $x \in S$.*

Quasi-convex and Quasi-concave Functions

Definition

Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.

1. The function f is a (strictly) quasi-convex function iff

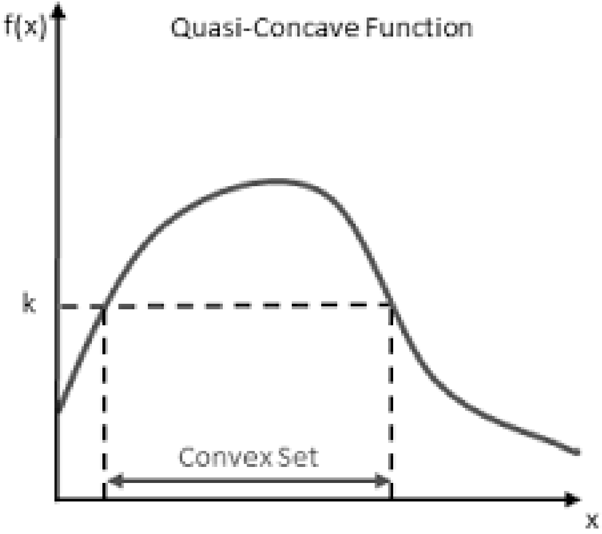
$$f(\lambda x + (1-\lambda)y)\ (<) \leq \max\{f(x), f(y)\}$$

   for any $x, y \in S$ and $\lambda \in [0,1]$.

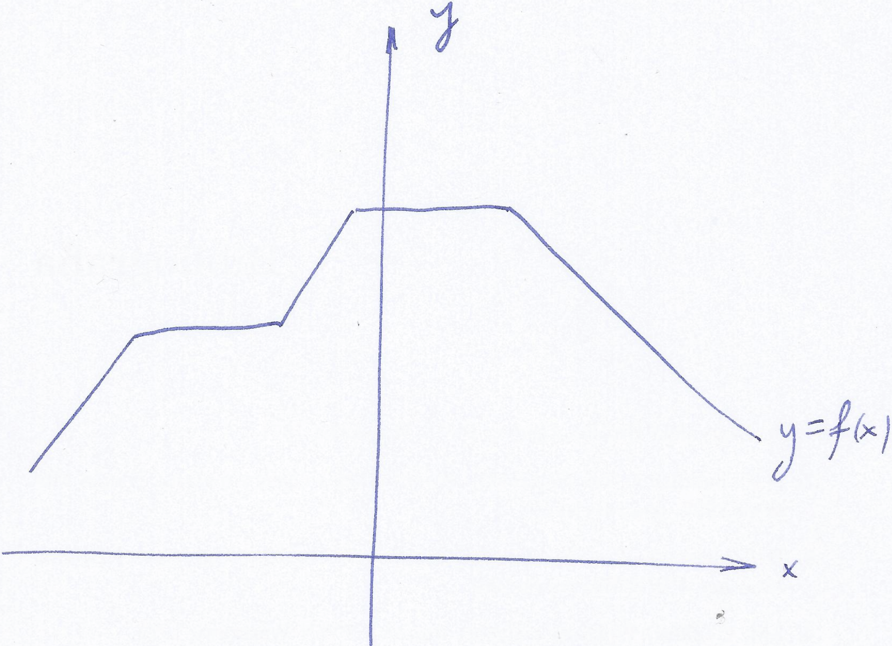
2. The function f is a (strictly) quasi-concave function iff

$$f(\lambda x + (1-\lambda)y)\ (>) \geq \min\{f(x), f(y)\}$$

   for any $x, y \in S$ and $\lambda \in [0,1]$.



f(x)
Quasi-Concave Function
k
Convex Set
x



y
$y=f(x)$
x

For a function $f : S \to \mathbb{R}$, define the **upper contour set** of f with cutoff a as

$$C^{+}(f, a) := \{x \in S : f(x) \geq a\}$$

and the **lower contour set** of f with cutoff a as

$$C^{-}(f, a) := \{x \in S : f(x) \leq a\}$$

Proposition

Consider a function $f : S \to \mathbb{R}$, *where* S *is a convex set in vector space* V.
(1) f *is quasi-concave iff its upper contour set* $C^{+}(f, a)$ *is a convex set in* V *for any* $a \in \mathbb{R}$;
(2) f *is quasi-convex iff its lower contour set* $C^{-}(f, a)$ *is a convex set in* V *for any* $a \in \mathbb{R}$.

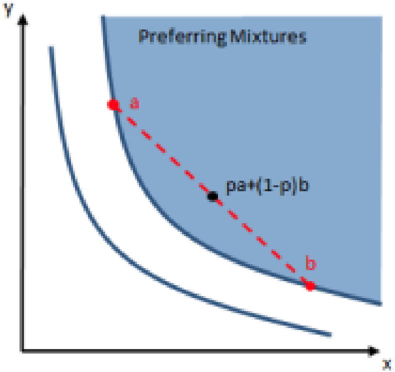
Proposition

Consider a function $f : S \to \mathbb{R}$, *where* S *is a convex set in vector space* V.
(1) If f *is quasi-convex and* $\phi : \mathbb{R} \to \mathbb{R}$ *is weakly increasing, then* $\phi \circ f$ *is quasi-convex.*
(2) If f *is quasi-concave and* $\phi : \mathbb{R} \to \mathbb{R}$ *is weakly increasing, then* $\phi \circ f$ *is quasi-concave.*

Economic Content of Quasiconcavity

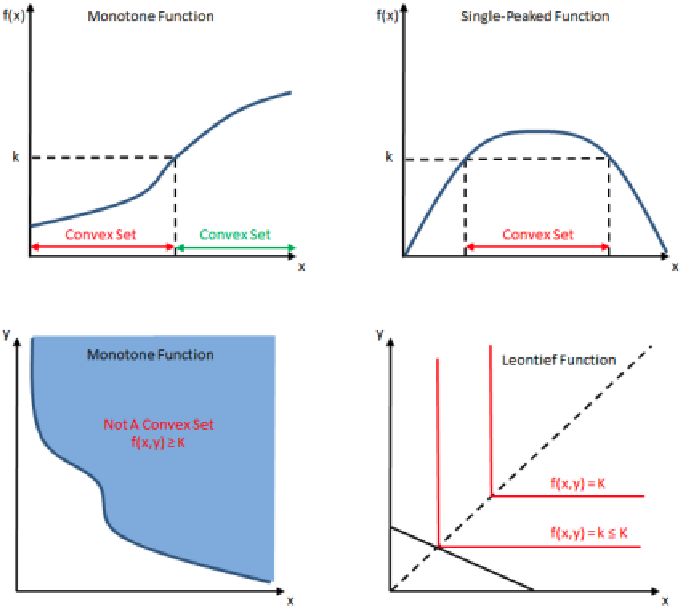
The convexity of uppercontour sets is a natural requirement for utility and production functions.

- consider an indifference curve of the concave utility function $f(x, y)$.
- The set of bundles which are preferred to them, is a convex set. In particular, the bundles that mix their contents are in this preferred set



Example

1. Any increasing transformation of a concave function is quasiconcave.
2. An monotone function $f(x)$ on $\mathbb{R}^1$ is both quasiconcave and quasiconvex.
3. A single peaked function $f(x)$ on $\mathbb{R}^1$ is quasiconcave.
4. The function $\min\{x, y\}$ is quasiconcave, as $C^+(f, a)$ is convex.


f(x)
Monotone Function
k
Convex Set
Convex Set
x
f(x)
Single-Peaked Function
k
Convex Set
x
Y
Monotone Function
Not A Convex Set
f(x,y) ≥ K
x
Y
Leontief Function
f(x,y) = K
f(x,y) = k ≤ K
x