Lecture2: Linear Algebra

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Some Notations

Let A be a matrix of M rows and N columns.

$$A_{M \times N} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}$$

The element in row i and column j is $[a_{ij}]$
It can also be written as a collection of N M-sized column vectors

$$\left[a^1, a^2, ..., a^N\right], \text{ where } a^i \in \mathbb{R}^M$$

Basic Definitions

- Real numbers can be seen as a 1×1 matrix.
- Matrices with as many rows as columns $m = n$ are called square matrices.
- The zero matrix of M_{MN} is the matrix with all entries equal to zero.
- A square matrix A is diagonal if all its non-diagonal elements are zero: $a_{ij} = 0$ for all i, j such that $i \neq j$. We note $A = diag(a_{11}, ..., a_{nn})$.
- The unit matrix of size n is the square matrix of size n having all its components equal to zero except the diagonal components, equal to 1. It is noted I_n.
- A square matrix A is upper-triangular if all its elements below its diagonal are nil: $a_{ij} = 0$ for all $i > j$.
- A square matrix A is lower-triangular if all its elements above its diagonal are nil: $a_{ij} = 0$ for all $i < j$.

Some Basic Matrix Algebra Operations

- Scalar multiplication

$$\alpha A = [\alpha A_{ij}]$$

- Addition

$$A + B = [a_{ij} + b_{ij}]$$

- Multiplication

$$A_{M \times N} \cdot B_{N \times Q} = D_{M \times Q} = [d_{ij}]$$

$$\text{where } d_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$$

Example

Multiplying Matrices

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

Laws of Matrix Algebra

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$

Transpose

- Definition

$$
\begin{aligned}
A_{M\times N} &= [a_{ij}] \\
A^T_{N\times M} &= \left[a^T_{ji}\right] = [a_{ij}] = A_{M\times N}
\end{aligned}
$$

Theorem

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(\alpha A)^T = \alpha A^T$
4. $(A^{-1})^T = (A^T)^{-1}$ (provided the inverse exists)

- Obviously, if we apply the transpose operator twice, we end up back on A: $(A^T)^T = A$. Note that a row vector is the transpose of a column vector.

Trace

Definition Given $A_{N\times N'}$

$$tr(A) = \sum_{i=1}^{N} a_{ii}$$

Theorem

Given: $A_{N\times N}, B_{N\times N}$

1. $tr(A+B) = tr(A) + tr(B)$
2. $tr(AB) = tr(BA)$
3. $tr(A^T) = tr(A)$

Determinants

- Given $A_{M \times N}$, $|A|$ denotes the determinant of A.

Definition (i,j)-th minor of A

$$A_{ij} : (M-1) \times (N-1)$$

Definition (i,j)- th cofactor of A

$$c_{ij}(A) = (-1)^{(i+j)} |A_{ij}|$$

and

$$|A| = \sum_{k=1}^{N} a_{ij} c_{ij}(A)$$

expansion along i-th row $(k=j)$ or j-th col $(k=i)$

Example

Calculate the determinants of matrix A

$$A = \left[\begin{array}{cc} 2 & 6 \\ 5 & 4 \end{array}\right]$$

we can get

$$|A| = 8 - 30 = -22$$

Property of Determinants

- $|A| = |A^T|$
- $|AB| = |A||B|$
- $|I_N| = 1$
- $|\lambda A| = \lambda^N |A|$
- $|A| = 0$ if A has a row or col of o's
- If we multiply a row(col) of A by λ to get $\hat{A}$, $|\hat{A}| = \lambda |A|$
- If we add a multiple of a row (col) to another row (col) to get $\hat{A}$, $|\hat{A}| = |A|$

Inverse

Definition Given $A_{N\times N}$, its inverse, denoted A^{-1}, is the $N\times N$ matrix such that

$$A^{-1}A = AA^{-1} = I_N$$

Theorem

1. A^{-1} exists $\Leftrightarrow |A| \neq 0$.
2. If A^{-1} exists, it is unique.
3. We say that A is non-singular if A^{-1} exists.

Properties of the Inverse

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $|A^{-1}| = \frac{1}{|A|}$
- $(A^T)^{-1} = (A^{-1})^T$

Computation of the Inverse

Definition the Adjoint of an $N \times N$ matrix is given by

$$Adj(A) = [C(A)]^T$$

where

$$C_{ij}(A) = [C_{ij}(A)]$$

$C(A)$ is a matrix whose element i, j is the (i, j)-th cofactor of A

$$c_{ij}(A) = (-1)^{(i+j)} |A_{ij}|$$

The inverse of A, A^{-1} is given by:

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

Inverse Example

Compute the Inverse of A

$$A = \left[\begin{array}{cc} 2 & 6 \\ 5 & 4 \end{array} \right]$$

We get

$$A^{-1} = \left[\begin{array}{cc} -2/11 & 3/11 \\ 5/22 & -1/11 \end{array} \right]$$

Linear Dependence and Independence

Definition Non-zero vector $\{x^1, ..., x^k\} \in \mathbb{R}^N$ are linearly dependent if $\exists \alpha_1, ..., \alpha_k$ not all zero such that

$$O_N = \sum_{i=1}^{k} \alpha_i x^i$$

where O_N is the N-dimensional zero vector.
Definition $\{x^1, ..., x^k\} \in \mathbb{R}^N$ are linearly independent if

$$O_N = \sum_{i=1}^{k} \alpha_i x^i \Rightarrow \alpha_i = 0 \ \forall i = 1, ..., K$$

Rank

Definition The column (row) rank of $A_{M \times N}$ is the max number of linearly

Theorem

independent cols (rows).

$$\textit{col rank } (A_{M \times N}) \leq N$$
$$\textit{row rank } (A_{M \times N}) \leq M$$

$$\textit{col rank}(A) = \textit{row rank}(A) = \textit{rank}(A)$$

Lemma

- $\textit{rank}(A) \leq \min\{N, M\}$
- $\textit{rank}(A^T) = \textit{rank}(A)$

Definition

$A_{M \times N},\ M \leq N$ is of full rank iff $\textit{rank}(A) = M$

A square matrix A of size n is invertible iff $\textit{rank}(A) = n$.

Systems of Linear Equations

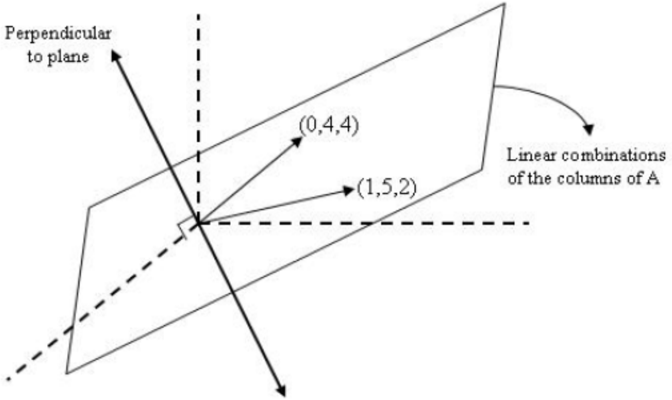
Consider the following system of linear equations in x:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where a_{ij} 's and b_i's are all elements of $\mathbb{R}$, and the unknowns $x_1, \ldots, x_n$ also take values in $\mathbb{R}$.
In matrix format, we have

$$A_{M\times N}x_{N\times 1} = b_{M\times 1}$$

Geometric Intuition



$Ax = b$ can be solved *iff* b lies in the plane that is spanned by the column vectors of A.

Example

Solving the system of linear equations

$$\begin{aligned} -12x_1 + 9x_2 &= 7 \\ 3x_1 - 4x_2 &= 2 \end{aligned}$$

Eigenvalues and Eigenvectors

Definition

Let A be an $m \times n$ matrix over $\mathbb{C}$. A scalar $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A iff $\exists x \in \mathbb{C}^n \backslash \{0\}$ s.t. $Ax = \lambda x$. A vector $x \in \mathbb{C}^n \backslash \{0\}$ is said to be an eigenvector of A iff $\lambda \in \mathbb{C}$ s.t. $Ax = \lambda x$.

Proposition

$\lambda \in \mathbb{C}$ is an eigenvalue of A iff $\det(\lambda I_n - A) = 0$.

By definition, $Ax = \lambda x$ has nonzero solution is equivalent to $(A - \lambda I_n)x = 0$ has a nonzero solution.

Theorem

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n, i.e. $P(\lambda) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0$, where $c_k \in \mathbb{C}$ for any $k = 0, 1, ..., n$ and $c_n \neq 0$. Then P has exactly n roots in $\mathbb{C}$, counted with multiplicity. That is, there exists $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ s.t.

$$P(\lambda) = c_n(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

Therefore, we can obtain all eigenvalues of A by setting the characteristic polynomial of A to 0 and solving for all its roots.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

First, we can find the eigenvalues of A by solving

$$\det(\lambda I - A) = 0$$

This gives $\lambda^2 + \lambda - 6 = 0$, which we find $\lambda_1 = 2$ and $\lambda_2 = -3$.
Plug in to the

$$(\lambda I - A)X = 0$$

for each eigenvalue and by normalization, we can find

$$X_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Diagonalization

Let A be an $n \times n$ matrix over $\mathbb{C}$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ are eigenvalues of A. Then $\det(A) = \lambda_1 \lambda_2 ... \lambda_n$

Definition

Let A be an $n \times n$ matrix over $\mathbb{C}$. The matrix A is diagonalizable in C iff there exists an $n \times n$ invertible matrix P over $\mathbb{C}$ and an $n \times n$ diagonal matrix over $\mathbb{C}$ s.t. $P^{-1}AP = \Lambda$. The matrix A is diagonalizable in $\mathbb{R}$ iff there exists an $n \times n$ invertible real matrix P and an $n \times n$ diagonal real matrix s.t. $P^{-1}AP = \Lambda$.

Let A be an $n \times n$ matrix over $\mathbb{C}$. Then A is diagonalizable in $\mathbb{C}$ iff A has n linearly independent eigenvectors.

Example

Following the previous question, given $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$, let do the Diagonalization.
From the previous question, we already learn that the eigenvalues are $\lambda_1 = 2$, and $\lambda_2 = -3$, with the corresponding eigenvectors

$$C = \begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}$$

So it is straightforward to have

$$A = C \times \Lambda \times C^{-1}$$

where $\Lambda = Diag(2, -3)$.