Exam

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1. (10 points) Set that are numerically equivalent to N are called countable. Can you show $N \times N$ is countable as well ? (hint: define a function from N to $N \times N$)

2. (10 points) Using the Chain Rule to calculate dz/dt for each of the following functions:

$$z = f(x, y) = \sqrt{x^2 - y^2},\ x = x(t) = e^{2t},\ y = y(t) = e^{-t}$$

3. (20 points) Please do the following calculation

   (a) Let $X \sim N(0, 1)$ $(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2})$, FInd $\int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

   (b) Find the eigenvalue and eigenvector of $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

4. (20 points) Remember CES demand system where

$$U = \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}}, \qquad 0 < \rho < 1 \tag{1}$$

and the Marshallian demand is

$$q(\omega_2) = \frac{I p(\omega_2)^{-\sigma}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}$$

If we define the aggregate price index as $P = \left(\int_0^n p(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}}$, can you simplify the Marshallian demand and write down the expression in terms of $p(\omega), P, I$. Moreover, can you further find the relationship between U and I and P?

5. (20 points) Suppose a consumer consumes two goods, x and y and has utility function $u(x, y) = x \cdot y$. He has a budget of \$400 . The price of x is $P_x = 10$ and the price of y is $P_y = 20$. Find his optimal consumption bundle using the Lagrange method. What if the price of y increase to $P_y = 25$?

6. (20 + 5 points) Assume a decision maker, labeled i, face J alternatives. The utility that the decision maker obtains from alternative j is decomposed into (1) a part labeled V_{ij} that is known by the researcher up to some parameters, and (2) an unknown part ϵ_{ij} that is treated by the researcher as random: $U_{ij} = V_{ij} + \epsilon_{ij},\ \forall j$. $\epsilon_{i,j}$ follows the type I extreme value distribution which has the following density function

$$f(\epsilon_{ij}) = e^{-\epsilon_{ij}} e^{-e^{-\epsilon_{ij}}}, \tag{2}$$

and please show that the cumulative distribution is

$$F(\epsilon_{ij}) = e^{-e^{-\epsilon_{ij}}} \tag{3}$$

Then, let $z = \epsilon_{iA} - \epsilon_{iB}$. Can you derive the distribution (CDF) of z? (10 points)
(hint: $\text{Prob}(z \leq \bar{z}) = \text{Prob}(\epsilon_{iA} - \epsilon_{iB} \leq \bar{z})$ which can be expressed as

$$\int_{-\infty}^{+\infty} f(\epsilon) F(\bar{z} + \epsilon) d\epsilon$$

If everything goes smoothly, you will find the connection to the logit distribution)
We now derive the logit choice probabilities, following McFadden (1974). The probability that decision maker n chooses alternative i is

$$\begin{aligned} P_{ik} &= \text{Prob}(V_{ik} + \epsilon_{Ik} > V_{ij} + \epsilon_{ij} \ \forall j \neq k) \\ &= \text{Prob}(\epsilon_{ij} < \epsilon_{ik} + V_{ik} - V_{ij} \ \forall j \neq k) \end{aligned}$$

If ϵ_{ik} is considered given, this expression is the cumulative distribution for each ϵ_{ij} evaluated at $\epsilon_{ik} + V_{ik} - V_{ij}$, which, according to 3, we have $\exp(-\exp(-(\epsilon_{ik} + V_{ik} - V_{ij})))$. Since ϵs are independent, P_{ik} becomes

$$P_{ik}|\epsilon_{ik} = \prod_{j \neq k} \exp(-\exp(-(\epsilon_{ik} + V_{ik} - V_{ij})))$$

The unconditional probability

$$P_{ik} = \int \left(\prod_{j \neq k} \exp(-\exp(-(\epsilon_{ik} + V_{ik} - V_{ij}))) \right) e^{-\epsilon_{ik}} e^{-e^{-\epsilon_{ik}}} d\epsilon_{ik}$$

Can you show that (10 + 5 points)

$$P_{ik} = \frac{e^{V_{ik}}}{\sum_j e^{V_{ij}}} \tag{4}$$

If this is too hard for you, can you derive the case when $J = 3$ (The consumer has 3 different alternatives) instead (10 points)?
If you can successfully derive the general form (4), you will get 5 points extra bonus!