Note Summary: Convexity

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1 Convex Sets

Definition 1. In real vector space V , a set $S \subset V$ is a convex set iff

$$\lambda x + (1 - \lambda) y \in S$$

for any $\lambda \in [0, 1]$ and $x, y \in S$.

Notice that it makes sense to talk about convex sets only in a vector space, since we need to be able to perform vector addition and scalar multiplication. In most applications, the vector space is $\mathbb{R}^n$. For finitely many vectors $x_1, x_2, ...x_n$ in vector space V , a convex combination of $x_1, x_2, ..., x_n$ is a vector $\sum_{i=1}^{n} \lambda_i x_i$ for scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$. Different from linear combination, convex combination requires that the coefficient i's are nonnegative and that they sum up to 1.
By definition, a set S is convex iff any combination of two vectors in S is still in S. However, the next result says that S is convex iff any combination of finitely many vectors in S is still in S.

Claim 1. In vector space V, the set $S \subset V$ is convex iff any convex combination of $x_1, x_2, ..., x_n \in S$ is also in S.

In vector space V, let $\{S_\alpha\}_{\alpha \in A}$ be a family of convex sets. Then $\cap_{\alpha \in A} S_\alpha$ is also convex.
In vector space V, the convex hull of set $S \subset V$ is

$$Co(S) \coloneqq \cap_{C \in \{X \subset V : X \text{ is convex and } X \supset S\}} C$$

Because intersection of convex sets is still convex, we know that Co (S) is convex, and therefore the convex hull can be also interpreted as the smallest convex set that covers S.
In vector space V , let $\{x_1, x_2, ..., x_n\}$ be a finite set of vectors. Then

$$C_0(\{x_1, x_2, ..., x_n\}) \\ = \left\{ \sum_{i-1}^{n} \lambda_i x_i : \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+, and \ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

[*]This note mainly borrows from math camp material in Columbia University, https://www.sites.google.com/site/mathcamp2018cu, and various online material. All errors are mine.

Separating Hyperplane Theorem In $\mathbb{R}^n$, a hyperplane is defined as

$$H(p,c) = \{x \in \mathbb{R}^n : p \cdot x = c\}$$

where $p \in \mathbb{R}^n \backslash \{0\}$, $c \in \mathbb{R}$, and $\cdot$ is the dot product. A hyperplane $H(p,c)$ cuts the whole space $\mathbb{R}^n$ into halves. This is a generalization of a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$.

Theorem 1. *(Minkowski's Separating Hyperplane). Let S_1 and S_2 be two disjoint nonempty and convex sets in $\mathbb{R}^n$. Then there exist $p \in \mathbb{R}^n \backslash \{0\}$ and $c \in \mathbb{R}$ s.t. $p \cdot x \geq c$ for any $x \in S_1$ and $p \cdot x \leq c$ for any $x \in S_2$.*

Minkowski's separating hyperplane theorem states that for any two disjoint nonempty and convex sets in $\mathbb{R}^n$, we can find a hyperplane $H(p,c)$ that weakly separates them, i.e. one of the two sets is contained in $H_+(p,c) \coloneqq \{x \in \mathbb{R}^n : p \cdot x \geq c\}$, and the other is contained in $H_-(p,c) \coloneqq \{x \in \mathbb{R}^n : p \cdot x \leq c\}$. Minkowski's Separating Hyperplane is used in the proof of Second Welfare Theorem.

Brouwer's Fixed Point Theorem

Theorem 2. *(Brouwer's Fixed Point). Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$, and consider a continuous function $f : X \to X$. Then there exists $x^* \in X$ s.t. $f(x^*\ ) = x^*$.*

The theorem states that a continuous self-map defined on a nonempty, compact, and convex set in $\mathbb{R}^n$ must have a fixed point. We will introduce its generalization, Kakutani's fixed point theorem, later when we discuss correspondences. Brouwer's fixed point theorem and Kakutani's fixed point theorem play an important role in the existence of Walrasian equilibria in the general equilibrium theory and the existence of Nash equilibria in non-cooperative game theory.

2 Convex and Concave Functions

Definition 2. Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.

1. The function f is a convex function iff

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

   for any $x, y \in S$ and $\lambda \in [0,1]$.

2. The function f is a concave function iff

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$

   for any $x, y \in S$ and $\lambda \in [0,1]$.

3. The function f is a strictly convex function iff

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$

   for any $x, y \in S$ with $x \neq y$ and $\lambda \in (0,1)$.

4. The function f is a strictly concave function iff

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$$

for any $x, y \in S$ with $x \neq y$ and $\lambda \in (0,1)$.

Theorem 3. *(Jensen's Inequality). Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.*

1. f is convex iff
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i-1}^{n} \lambda_i f(x_i)$$
for any $x_1, x_2, ..., x_n \in S$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.

2. f is concave iff
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i f(x_i)$$
for any $x_1, x_2, ..., x_n \in S$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.

For two vector spaces V and W, a function $f : V \to W$ is linear iff $f(x_1 + x_2) = f(x_1) + f(x_2)$ and $f(\lambda x) = \lambda f(x)$. Clearly, a linear function $f : V \to \mathbb{R}$, where V is a vector space, is both convex and concave, but not strictly convex or strictly concave.
Also, notice that f is (strictly) convex iff $-f$ is (strictly) concave. Consider a function $f : S \to \mathbb{R}$. Define the graph of f as

$$G(f) := \{(x, y) \in S \times \mathbb{R} : y = f(x)\}$$

Notice that this is in fact a redundant notation, because G(f) is exactly f. Define the epigraph of f as

$$G^+(f) := \{(x, y) \in S \times \mathbb{R} : y \geq f(x)\}$$

and the subgraph of f as

$$G^-(f) := \{(x, y) \in S \times \mathbb{R} : y \leq f(x)\}$$

The next result characterizes a convex/concave function using its epigraph/subgraph.

Proposition 1. *Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.*

1. *f is convex iff its epigraph $G^+(f)$ is a convex set in $V \times R$.*

2. *f is concave iff its subgraph $G^-(f)$ is a convex set in $V \times R$.*

Proposition 2. *Consider two functions f and g from S to $\mathbb{R}$, where S is a convex set in vector space V. If f and g are both convex/concave functions, then*

1. *$f + g$ is a convex/concave function, and*

2. *cf is a convex/concave function, for any $c \in \mathbb{R}_+$.*

The next result says: (1) a weakly increasing convex transformation of a convex function is still convex, and (2) a weakly increasing concave transformation of a concave function is still concave.

Proposition 3. *Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.*
(1) If f is convex and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing and convex, then $\phi \circ f$ is convex.
(2) If $f$is concave and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing and concave, then $\phi \circ f$ is concave.

A straightforward corollary of the proposition above is: (3) a weakly decreasing concave transformation of a convex function is concave, and (4) a weakly decreasing convex transformation of a concave function is convex.
Now let's consider a family $\{f_\alpha\}_{\alpha \in A}$ of real-valued functions defined on the same domain S. If the set $\{f_\alpha(x) : \alpha \in A\}$ is bounded from above for each $x \in S$, we can define the function $\sup\{f_\alpha\}_{\alpha \in A} : S \to \mathbb{R}$ as the pointwise supremum, i.e. for each $x \in S$,

$$\{\sup\{f_\alpha\}_{\alpha \in A}\}(x) \coloneqq \sup\{f_\alpha(x) : \alpha \in A\}$$

Similarly, if the set $\{f_\alpha(x) : \alpha \in A\}$ is bounded from below for each $x \in S$, we can define the function $inf\{f_\alpha\}_{\alpha \in A} : S \to \mathbb{R}$ as the pointwise infimum, i.e. for each $x \in S$,

$$(\inf\{f_\alpha\}_{\alpha \in A})(x) \coloneqq \inf\{f_\alpha(x) : \alpha \in A\}$$

Now let's state the following result.

Proposition 4. *Consider a finite family of functions $\{f_\alpha\}_{\alpha \in A}$ from S to $\mathbb{R}$, where S is a convex set in vector space V.*

1. *If all functions in the family are convex, and the set $\{f_\alpha\}_{\alpha \in A}$ is bounded from above for each $x \in S$, then $\sup\{f_\alpha\}_{\alpha \in A}$ is a convex function.*

2. *If all functions in the family are concave, and the set $\{f_\alpha(x) : \alpha \in A\}$ is bounded from below for each $x \in S$, then $\inf\{f_\alpha\}_{\alpha \in A}$ is a concave function.*

Shortly put, the proposition states that the sup of convex functions is still convex, and the inf of concave functions is still concave.
The proposition implies as a special case that $\max\{f, g\}$ is a convex function if f and g are both convex, and $\min\{f, g\}$ is a concave function if f and g are both concave. Because linear functions are both convex and concave, another important special case of the proposition above is that the sup of linear functions is convex, and the inf of linear functions is concave.
The next theorem provides a characterization of convexity/concavity of continuously differentiable functions.
Suppose the function $f : S \to \mathbb{R}$ is a C^1 function, where S is a convex and open set in $\mathbb{R}^n$.
(1) f is convex iff

$$f(x') \geq f(x) + \nabla f(x) \cdot (x' - x)$$

for any $x', x \in S$.

(2) f is concave iff

$$f(x') \leq f(x) + \nabla f(x) \cdot (x' - x)$$

for any $x', x \in S$.
(3) f is strictly convex iff

$$f(x') > f(x) + \nabla f(x) \cdot (x' - x)$$

for any $x', x \in S$ with $x' \neq x$.
(4) f is strictly concave iff

$$f(x') < f(x) + \nabla f(x) \cdot (x' - x)$$

for any $x', x \in S$ with $x' \neq x$.
The intuition of the theorem above is that the set

$$\{(x', y) \in S \times \mathbb{R} : y = f(x) + \nabla f(x) \cdot (x' - x)\}$$

is the hyperplane that is tangent to the graph of f at x. A convex/concave function should lie above/below this tangent plane.
The next theorem provides a characterization of convexity/concavity for twice continuously differentiable functions using the Hessian matrix.

Theorem 4. *Suppose the function $f : S \to \mathbb{R}$ is a C^2 function, where S is a convex and open set in $\mathbb{R}^n$.*

1. *f is convex iff its Hessian matrix $H(x)$ is positive semi-definite for any $x \in S$.*
2. *f is concave iff its Hessian matrix $H(x)$ is negative semi-definite for any $x \in S$.*
3. *f is strictly convex if its Hessian matrix $H(x)$ is positive definite for any $x \in S$.*
4. *f is strictly concave if its Hessian matrix $H(x)$ is negative definite for any $x \in S$.*

3 Quasi-convex and Quasi-concave Functions

Definition 3. Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.

1. The function f is a quasi-convex function iff

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$$

for any $x, y \in S$ and $\lambda \in [0, 1]$.

2. The function f is a quasi-concave function iff

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$$

for any $x, y \in S$ and $\lambda \in [0, 1]$.

3. The function f is a strictly quasi-convex function iff
$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$$
for any $x, y \in S$ with $x \neq y$ and $\lambda \in (0,1)$.

4. The function f is a strictly quasi-concave function iff
$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$$
for any $x, y \in S$ with $x \neq y$ and $\lambda \in (0,1)$.

Compare the definition of quasi-convex functions with that of convex functions in the previous section, clearly a convex function f is also quasi-convex, since
$$\lambda f(x) + (1-\lambda)f(y) \leq \max\{f(x), f(y)\}$$
Similarly, concavity implies quasi-concavity, strict convexity implies strict quasi-convexity, and strict concavity implies strict quasi-concavity. Clearly, f is (strictly) quasi-convex iff $-f$ is (strictly) quasi-concave.
For a function $f : S \to \mathbb{R}$, define the **upper contour set** of f with cutoff a as
$$C^+(f,a) := \{x \in S : f(x) \geq a\}$$
and the lower contour set of f with cutoff a as
$$C^-(f,a) := \{x \in S : f(x) \leq a\}$$
The next result characterizes a quasi-concave/quasi-convex function using its upper/ lower contour set.

Proposition 5. *Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.*
(1) f is quasi-concave iff its upper contour set $C^+(f,a)$ is a convex set in V for any $a \in \mathbb{R}$
(2) f is quasi-convex iff its lower contour set $C^-(f,a)$ is a convex set in V for any $a \in \mathbb{R}$.

Notice that the concept of upper/lower contour set is completely different from that of epigraph/subgraph. The upper/lower contour set is in the vector space V, but the epigraph/subgraph is in the vector space $V \times \mathbb{R}$. Using this characterization, it is not difficult to see that quasi-concavity is essentially a single peak condition, and that quasiconvexity is essentially a single trough condition.
The next result states that a weakly increasing transformation of a quasi-convex/quasiconcave function is still quasi-convex/quasi-concave.

Proposition 6. *Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V.*
(1) If f is quasi-convex and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing, then $\phi \circ f$ is quasi-convex.
(2) If f is quasi-concave and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing, then $\phi \circ f$ is quasi-concave.

A straightforward corollary of the proposition above is: (3) a weakly decreasing transformation of a quasi-convex function is quasi-concave, and (4) a weakly decreasing transformation of a quasi-concave function is quasi-convex. To see (3), suppose f is quasi-convex and ϕ is weakly decreasing, then $-\phi$ is weakly increasing. By (1), we have $(-\phi) \circ f$ is quasi-convex, and so $\phi \circ f = -(-\phi) \circ f$ is quasi-concave. (4) can be shown symmetrically. Although quasi-convexity/quasi-concavity is preserved under increasing transformations, the sum of two quasi-convex/quasi-concave functions may no longer be quasi-convex / quasi-concave.