Lecture: Principal of Mathematic(Analysis)

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Notation

- ▶ From a logical point of view, there is no difference between a lemma, proposition, theorem, or corollary - they are all claims waiting to be proved. However, we use these terms to suggest different levels of importance and difficulty.

- ▶ A **lemma** is an easily proved claim which is helpful for proving other propositions and theorems, but is usually not particularly interesting in its own right.

- ▶ A **proposition** is a statement which is interesting in its own right, while

- ▶ a **theorem** is a more important statement than a proposition which says something definitive on the subject, and often takes more effort to prove than a proposition or lemma.

- ▶ A **corollary** is a quick consequence of a proposition or theorem that was proven recently.

From Terence Tao (Analysis I, p. 25, n. 4)

Logic

▶ Logic is the business of evaluating arguments, sorting good ones from bad ones. A logical argument is structured to give someone a reason to believe some conclusion.

▶ In logic, we are only interested in sentences that can be as a premise or conclusion of an argument. **Generally, questions will not count as sentences, but answers will.**

    A It is raining

    B You have an umbrella

    C x is a prime number

Connectives and Truth Table

- and $\wedge$
- or $\vee$
- if then $\implies$
- not $\neg$
- if and only if $\Leftrightarrow$

Truth Table I

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$	$A \wedge \neg B$	$\neg(A \wedge \neg B)$
1	1	1	1	0	1	0	0	1
1	0	0	1	0	1	0	1	0
0	1	0	1	1	1	0	0	1
0	0	0	0	1	1	0	0	1

Truth Table II

A	B	$A \Rightarrow B$	$B \Rightarrow A$	A only if B	A if B	if A then B	$A \Leftrightarrow B$
1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0
0	1	1	0	1	0	1	0
0	0	1	1	1	1	1	1

Note that

- $A \Rightarrow B$ If A then B, $B \Rightarrow A$ A if B, $A \Rightarrow B$ A only if B

Truth Table III

A	B	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

$A \Rightarrow B$ and $\neg B \Rightarrow \neg A$ are contra-positive: one is true, the other is true; one is false, the other is false

Set Theory

- $\forall x, \ x \in A \cap B, \ \Leftrightarrow x \in A, \ \text{and } x \in B$
- $\forall x, \ x \in A \cup B, \ \Leftrightarrow x \in A, \ \text{or } x \in B$
- $\forall x, \ x \in A \triangle B, \ \Leftrightarrow x \in A, \ \text{exclusive or } x \in B$
- $\forall x, \ x \in A \backslash B, \ \Leftrightarrow x \in A, \ \text{and } x \notin B$
- $A \subseteq B \ \Leftrightarrow \forall x, \ x \in A \Rightarrow x \in B$
- $A = B \ \Leftrightarrow A \subseteq B \ \text{and } B \subseteq A$

Partition and Function Definitions

▶ Partition a partition of a set is a grouping of the set's elements into non-empty subsets, in such a way that every element is included in one and only one of the subsets.
$A = \{1, 2, 3, 4\}$, $P = \{\{1, 2\}, \{3, 4\}\}$ P is the partition of A

▶ Functions Given two sets X, Y, a FUNCTION $f : X \to Y$ assigns a unique element $y \in Y$ to each element of $x \in X$

  ▶ Injective function f is injective or one-to-one if
  $\forall a, b \in x, \ f(a) = f(b) \Rightarrow a = b$

  ▶ Surjective function f is surjective or onto if
  $\forall y \in Y, \ \exists x \in X, \ s.t. f(x) = y$

  ▶ Bijective function f is bijective if it is both injective and

Theorem

Let A and B be two sets and suppose that there exists injection of $f : A \to B$ and an injection $g : B \to A$. Then there exists a bijection $h : A \to B$

Russell's Paradox

- ▶ Let X be the set of all infinite sets.
- ▶ Let Y be the set of all sets that belong to themselves
- ▶ Let Z be the set of all sets that does not belong to themselves!

See reading material.

Numbers

▶ The set $\mathbb{N}$ of natural numbers consists of a starting element 1, the successor of 1 (denoted as 2), the successor of the successor of 1 (denoted as 3), and so on. Shortly put, $\mathbb{N} := \{1, 2, 3, ...\}$.
Some books define natural numbers $\mathbb{N} := \{0, 1, 2, 3, ...\}$,

▶ The set $\mathbb{Z}$ of integers consists of natural numbers and their negative counterparts, as well as a neutral element denoted as 0, i.e. $Z := \{..., -2, -1, 0, 1, 2, ...\}$.

▶ The set $\mathbb{Q}$ of rational numbers consists of ordered pairs (m, n) of integers, and treats (m, n) and (m', n') as the same element iff $m \cdot n' = m' \cdot n$. $\mathbb{Q} := \{m/n : m, n \in \mathbb{Z},\ m, n\ coprime\}$[1].

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

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[1] m, n coprime if the only positive integer that divides both m and n is 1.

Countability and Cardinality

Definition

A set X is countably infinite iff there exists a bijection between $\mathbb{N}$ and X.

Note that the countable infinite sets are the "smallest" infinite sets.

Claim

If X is countably infinite, then any infinite subset $Y \subset X$ is also countably infinite.

Definition

A set X is **countable** iff it is either finite or countably infinite. A set X is uncountable iff it is not countable.

Intuitively, uncountable sets are infinite sets which we can not find a bijection from $\mathbb{N}$ to. Therefore, they are considerably "larger" than countably infinite sets.

Q is countable, R is uncountable

Metric Spaces

Definition

Let X be a set. A function $d : X^2 \to \mathbb{R}_+$ is a **distance function**, or **metric,** on X iff it satisfies the following properties

1. (Positivity) for any $x, y \in X$, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;

2. (Symmetric) for any $x, y \in X$, $d(x, y) = d(y, x)$;

3. (Triangle inequality) for any $x, y, z \in X$, $d(x, y) \leq d(y, z) + d(z, y)$

A metric space (X, d) is a set X equipped with a metric d.

Example

1. Finite dimensional real space $(\mathbb{R}^l, d)$ is a metric space with the Euclidean distance $d(x, y) := \left\{ \sum_{i=1}^{l} (x_i - y_i)^2 \right\}^{1/2}$.

2. Let $a < b$ be real numbers. $C_b([a, b])$, the set of real-valued continuous (and bounded) functions on $[a, b]$ is a metric space, once we define the distance between $f, g \in C_b([a, b])$ as $d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$.

3. In $\mathbb{R}^k$, d_n metric can be defined as

$$d_n(x, y) := \left(\sum_{i=1}^{k} |x_i - y_i|^n \right)^{\frac{1}{n}}$$

Notice that (X, d_1) and (X, d_2) are two different metric spaces if d_1 and d_2 are two different distance functions.

Convergence

Definition

Let X be a set. The function $x : \mathbb{N} \to X$ is called a sequence in X.

Definition

Let (X, d) be a metric space. A sequence (X_n) in X is said to **converge** to a point $x \in X$, iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$ for all $n > N$. When the sequence (x_n) converges to x, the point x is called a limit of the sequence (x_n), and we use the notation $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

The next lemma establishes that the limit of a convergent sequence must be unique, and therefore it makes sense to talk about "the" limit of a convergent sequence

Lemma

Let (X, d) be a metric space. Suppose $x_n \to x$ and $x_n \to x'$, then $x = x'$.

A few more

Lemma

In (R, d_2), let there be two convergent sequences $x_n \to x$ and $y_n \to y$. If $x_n \leq y_n$ for any $n \in N$, then $x \leq y$.

Lemma

In (R, d_2), let there be two convergent sequences $x_n \to x$ and $y_n \to y$. Then

1. $x_n + y_n \to x + y$,
2. $x_n y_n \to xy$, and
3. if $x \neq 0$, then $1/x_n \to 1/x$

Corollary

In (R, d_2), let there be two convergent sequences $x_n \to x$ and $y_n \to y$. Then
(1) $x_n - y_n \to x - y$,
(2) if $y \neq 0$, then $x_n/y_n \to x/y$.

Examples & Application

1. Use the fact that $\lim_{n \to \infty} \frac{1}{n} = 0$ and the property of the convergence, show

$$\lim_{n \to \infty} a(n) = \lim_{n \to \infty} \frac{3n^2 + 2n - 1}{n^2 + n} = 3$$

1. Consider the sequence $(b_n)_{n \in \mathbb{N}}$ defined by $b_0 = 2$ and $b_{n+1} = 5 - \frac{4}{b_n}$.
   1.1 Find the first four terms of the sequence
   1.2 Can you find the limits of this sequence? (A sequence converges if and only if its elements approach each other)
   1.3 can you show the sequence is increasing by induction?

The next part will introduce the concept of continuity, correspondence, compactness. These knowledge helps to let us understand the Fixed Point Theory, which has a widely applications in the general equilibrium, game theory and dynamic programming.

Continuity of Functions

> ### Definition
> Let X, Y be metric spaces, $f : X \to Y$, and $x \in X$. f is continuous at x if for every open neighborhood $V \subset Y$ of $f(x)$, there exists an open neighborhood $U \subset X$ of x such that $U \subset f^{-1}(V)$. f is continuous if it is continuous at every $x \in X$.

> ### Definition
> Let X, Y be metric spaces, and $f : X \to Y$, $x \in X$. f is sequentially continuous at x if $f(x_n) \to f(x)$ in Y for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x$ in X. f is sequentially continuous if it is sequentially continuous at all $x \in X$.

Let X, Y be metric spaces, and $f : X \to Y$. f is continuous if and only if it is sequentially continuous.

(Epsilon–delta) of Continuous Functions

The continuity of $f : D \to \mathbb{R}$ at $x_0 \in D$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$:

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon.$$
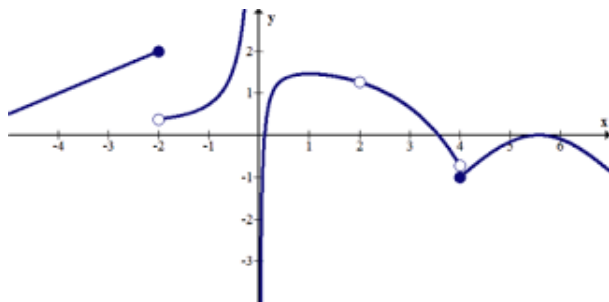
The two definitions are equivalent.

Example

Discontinuous Function
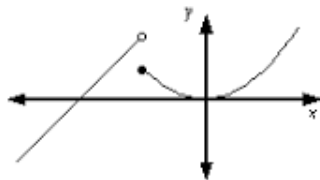
Continuous Function
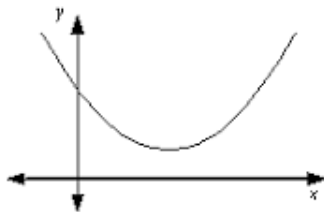
Correspondences

<div class="definition">

Definition

Let X, Y be nonempty sets. A correspondence Γ from X to Y, denoted by $\Gamma : X \twoheadrightarrow Y$, is a function from X to 2^Y, i.e., $\Gamma(x) \subset Y$ for every $x \in X$.

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Definition

Let $\Gamma : X \twoheadrightarrow Y$, and $E \subset Y$.

- The upper inverse of E is

$$\bar{\Gamma}^{-1}(E) = \{x \in X : \Gamma(x) \subset E\};$$

- The lower inverse of E is

$$\underline{\Gamma}^{-1}(E) = \{x \in X : \Gamma(x) \cap \neq \emptyset\}$$

</div>

Example

Example

$$f(x) = \begin{cases} \{\sqrt{x}, -\sqrt{x}\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$
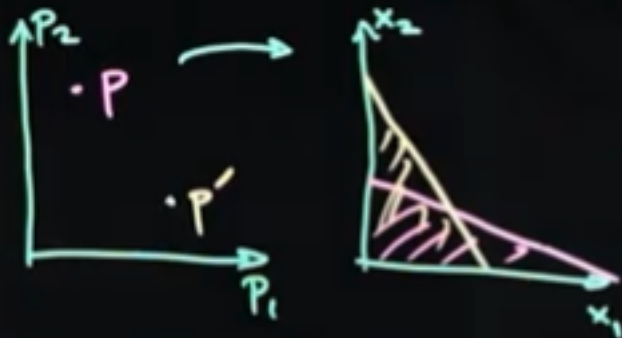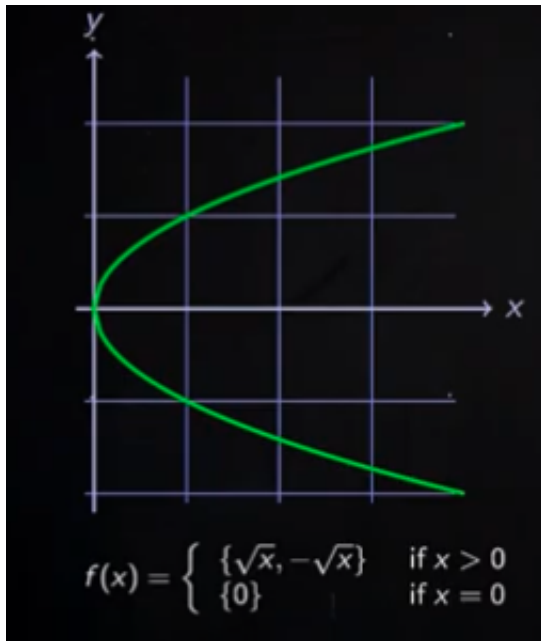
Upper and Lower Hemicontinuity

Proposition

Let X, Y be metric spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. Γ is lower hemicontinuous at x if and only if for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$ and all $y \in \Gamma(x)$, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Gamma(x_n)$ for all n, and $y_n \to y$.

Proposition

Let X, Y be metric spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. If for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ such that $x_n \to x$, $y_n \in \Gamma(x_n)$ for all n, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ that converges to some $y \in \Gamma(x)$, then Γ is upper hemicontinuous at x.
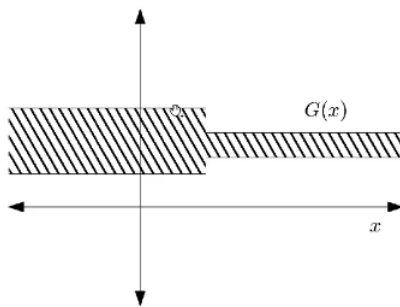
A closed-valued correspondence is UHC/LHC at x if no point suddenly appear/disappear in $f(x)$ as we move away from x (in any direction).

Graph Illustration

A UHC correspondence can get discontinuously **larger** but never discontinuously **smaller**.
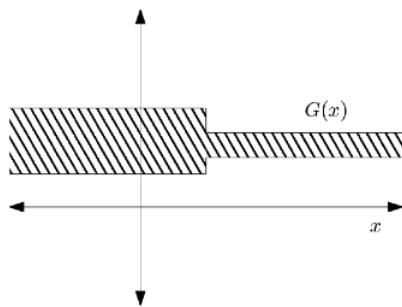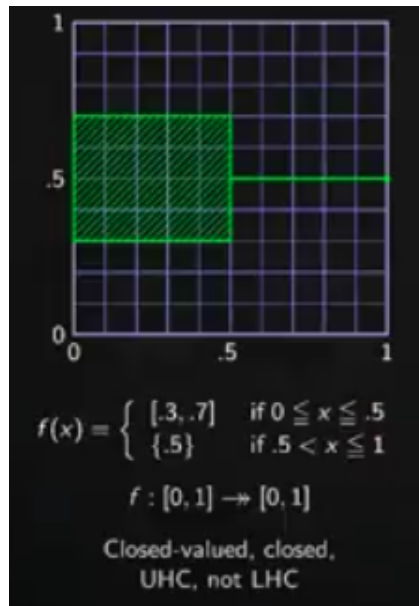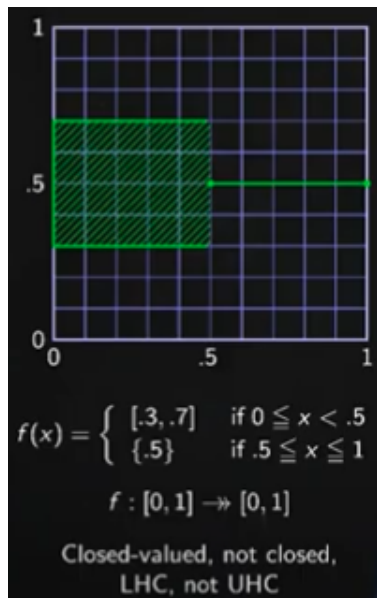

UHC   Not UHC

Upper Example

$$f(x) = \begin{cases} [.3, .7] & \text{if } 0 \leq x \leq .5 \\ \{.5\} & \text{if } .5 < x \leq 1 \end{cases}$$

$$f : [0, 1] \twoheadrightarrow [0, 1]$$

Closed-valued, closed,
UHC, not LHC
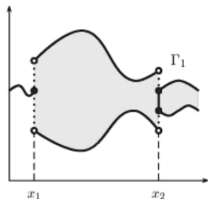
the image set around key point
0.5 does not suddenly enlarge
from left or right.
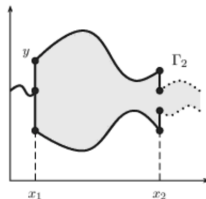From left to right, the image
set is closed.

LHC

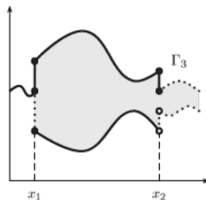From right to the left, the image set explode. only one point at 0.5

$$f(x) = \begin{cases} [.3, .7] & \text{if } 0 \leq x < .5 \\ \{.5\} & \text{if } .5 \leq x \leq 1 \end{cases}$$

$$f : [0, 1] \twoheadrightarrow [0, 1]$$

Closed-valued, not closed,
LHC, not UHC

Graph

Not upper hemicontinuous at x_1
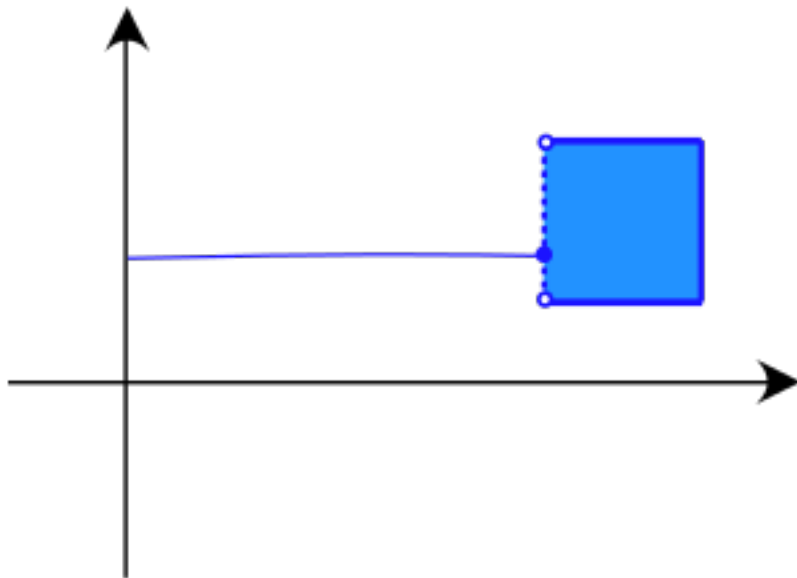Not upper hemicontinuous at x_2
Lower hemicontinuous

Not lower hemicontinuous at x_1
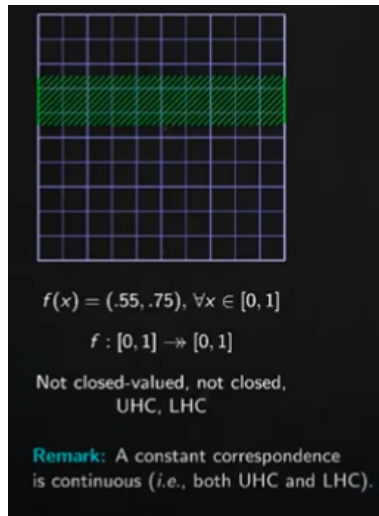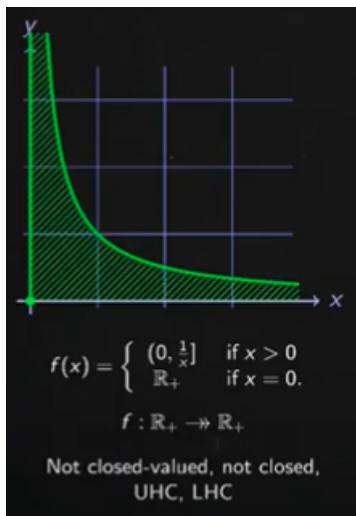Not lower hemicontinuous at x_2
Upper hemicontinuous

Not upper hemicontinuous at x_1 and x_2
Not lower hemicontinuous at x_1 and x_2

Graph

UHC & LHC

$f(x) = \begin{cases} (0, \frac{1}{x}] & \text{if } x > 0 \\ \mathbb{R}_+ & \text{if } x = 0. \end{cases}$

$f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+$

Not closed-valued, not closed,
UHC, LHC

$f(x) = (.55, .75), \forall x \in [0, 1]$

$f : [0, 1] \twoheadrightarrow [0, 1]$

Not closed-valued, not closed,
UHC, LHC

Remark: A constant correspondence
is continuous (*i.e.*, both UHC and LHC).

Compactness

Definition

Let (X, d) be a metric space, and S a subset of X. A family of open sets $\{E_\alpha\}_{\alpha \in A}$ is an open cover of S iff $\cup_{\alpha \in A} E_\alpha \supset S$.

Definition

Let (X, d) be a metric space, and S a subset of X. The set S is compact iff $\forall$ open cover $\{E_\alpha\}_{\alpha \in A}$ of S, $\exists$ a finite $B \subset A$ s.t. $\{E_\alpha\}_{\alpha \in B}$ is also an open cover of S.

In general metric spaces, we have shown that a compact set must be closed and bounded. In Euclidean spaces $(\mathbb{R}^k, d_2)$, the reverse is also true, i.e. a closed and bounded set in $(\mathbb{R}^k, d_2)$ must be compact.

Lemma

Any closed interval $[a, b]$ is compact in $(\mathbb{R}, d_2)$.

Example
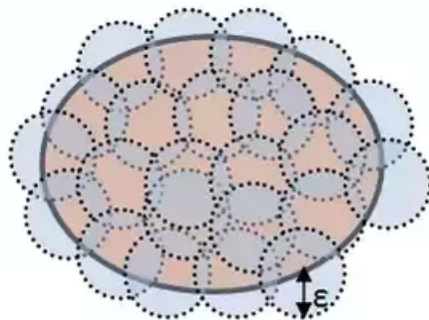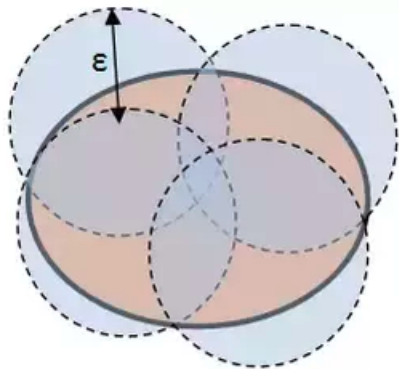
Totally bounded X : can be covered
with finitely many $N(x, \varepsilon)$, for any $\varepsilon > 0$.

Fixed Point Theory

Theorem

(Brouwer). Let $X \subset \mathbb{R}^I$ be nonempty, compact and convex ($I < \infty$), and $f : X \to X$ be continuous. Then f has a fixed point.

Theorem

(Kakutani). Let $X \subset \mathbb{R}^I$ be nonempty, compact and convex ($I < \infty$), and $\Gamma : X \twoheadrightarrow X$ be upper semicontinuous and such that $\Gamma(x)$ is nonempty and convex for all $x \in X$. Then Γ has a fixed point.

Example

▶ Suppose two identical strings are placed directly on top of one another.

▶ The bottom string remains unchanged, but the top string can be folded, stretched, and reshaped as long as it is replaced so that no part of the string hangs off the bottom string.

▶ There will be a point on the top string whose horizontal position has not changed!