Lecture: Principal of Mathematic(Analysis)

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UIBE Math Camp, 2022

Notation

- From a logical point of view, there is no difference between a lemma, proposition, theorem, or corollary - they are all claims waiting to be proved. However, we use these terms to suggest different levels of importance and difficulty.
- A **lemma** is an easily proved claim which is helpful for proving other propositions and theorems, but is usually not particularly interesting in its own right.
- A **proposition** is a statement which is interesting in its own right, while
- A **theorem** is a more important statement than a proposition which says something definitive on the subject, and often takes more effort to prove than a proposition or lemma.
- A **corollary** is a quick consequence of a proposition or theorem that was proven recently.

From Terence Tao (Analysis I, p. 25, n. 4)

Logic

- Logic is the business of evaluating arguments, sorting good ones from bad ones. A logical argument is structured to give someone a reason to believe some conclusion.
- In logic, we are only interested in sentences that can be as a premise or conclusion of an argument. **Generally, questions will not count as sentences, but answers will.**

    A It is raining

    B You have an umbrella

    C x is a prime number

Connectives and Truth Table

- and $\wedge$
- or $\vee$
- if then $\implies$
- not $\neg$
- if and only if $\Leftrightarrow$

Truth Table I

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$	$A \wedge \neg B$	$\neg(A \wedge \neg B)$
1	1	1	1	0	1	0	0	1
1	0	0	1	0	1	0	1	0
0	1	0	1	1	1	0	0	1
0	0	0	0	1	1	0	0	1

Truth Table II

A	B	$A \Rightarrow B$	$B \Rightarrow A$	A only if B	A if B	if A then B	$A \Leftrightarrow B$
1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0
0	1	1	0	1	0	1	0
0	0	1	1	1	1	1	1

Note that

- $A \Rightarrow B$ If A then B, $B \Rightarrow A$ A if B, $A \Rightarrow B$ A only if B

Truth Table III

A	B	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

$A \Rightarrow B$ and $\neg B \Rightarrow \neg A$ are contra-positive: one is true, the other is true; one is false, the other is false

Set Theory

- $\forall x,\ x \in A \cap B,\ \Leftrightarrow x \in A,\ and\ x \in B$
- $\forall x,\ x \in A \cup B,\ \Leftrightarrow x \in A,\ or\ x \in B$
- $\forall x,\ x \in A \backslash B,\ \Leftrightarrow x \in A,\ and\ x \notin B$
- $\forall x,\ x \in A \triangle B,\ \Leftrightarrow x \in A,\ exclusive\ or\ x \in B \Leftrightarrow A \backslash B \cup B \backslash A$
- $A \subseteq B\ \ \Leftrightarrow \forall x,\ x \in A \Rightarrow x \in B$
- $A = B\ \ \Leftrightarrow A \subseteq B\ and\ B \subseteq A$

Partition and Function Definitions

- A partition of a set is a grouping of the set's elements into non-empty subsets, in such a way that every element is included in one and only one of the subsets.
  $A = \{1,2,3,4\},\ P = \{\{1,2\},\{3,4\}\}$ P is the partition of A

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  $A = \{1,2,3,4\},\ P = \{\{1,2\},\{3,4\}\}$ P is the partition of A
- Given two sets X, Y , a FUNCTION $f\ : X \to Y$ assigns a unique element $y \in Y$ to each element of $x \in X$
  - Injective function f is injective or one-to-one if $\forall a, b \in x,\ f(a) = f(b) \Rightarrow a = b$
  - Surjective function f is surjective or onto if $\forall y \in Y,\ \exists x \in X,\ s.t. f(x) = y$
  - Bijective function f is bijective if it is both injective and surjective

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Theorem

Let A and B be two sets and suppose that there exists injection of $f : A \to B$ and an injection $g : B \to A$.
Then there exists a bijection $h : A \to B$.

Russell's Paradox

- Let X be the set of all infinite sets.
- Let Y be the set of all sets that belong to themselves
- Let Z be the set of all sets that does not belong to themselves!

See reading material.

Numbers

- The set $\mathbb{N}$ of natural numbers consists of a starting element 1, the successor of 1 (denoted as 2), the successor of the successor of 1 (denoted as 3), and so on. Shortly put, $\mathbb{N} := \{1, 2, 3, \dots\}$.
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- The set $\mathbb{Q}$ of rational numbers consists of ordered pairs (m, n) of integers, and treats (m, n) and (m', n') as the same element iff $m \cdot n' = m' \cdot n$. $\mathbb{Q} := \{m/n : m, n \in \mathbb{Z},\ m, n\ coprime\}$.
  - m, n *coprime* if the only positive integer that divides both m and n is 1

Different Size of Numbers

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Countability and Cardinality

Definition

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Definition

A set X is **countable** iff it is either finite or countably infinite. A set X is uncountable iff it is not countable.

Intuitively, uncountable sets are infinite sets which we can not find a bijection from $\mathbb{N}$ to.
Therefore, they are considerably "larger" than countably infinite sets.
e.g.Q is countable, R is uncountable

Metric Spaces

Definition

Let X be a set. A function $d : X^2 \to \mathbb{R}_+$ is a **distance function**, or **metric,** on X iff it satisfies the following properties

1. (Positivity) for any $x, y \in X$, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
2. (Symmetric) for any $x, y \in X$, $d(x, y) = d(y, x)$;
3. (Triangle inequality) for any $x, y, z \in X$, $d(x, y) \leq d(y, z) + d(z, y)$

A metric space (X, d) is a set X equipped with a metric d.

Why we need Triangle inequality

A simply and direct answer: for measurement and convergence

To show closeness, we usually see something like this if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) \leq d(x, y) + d(y, z) < \varepsilon + \varepsilon$; that is, if x is "close" to y and y is "close" to z, then x is "close" to z.

Example

1. Finite dimensional real space $(\mathbb{R}^l, d)$ is a metric space with the Euclidean distance $d(x,y) := \left\{\sum_{i=1}^{l}(x_i - y_i)^2\right\}^{1/2}$.
2. Let $a < b$ be real numbers. $C_b([a,b])$, the set of real-valued continuous (and bounded) functions on $[a,b]$ is a metric space, once we define the distance between $f, g \in C_b([a,b])$ as $d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$.
3. In $\mathbb{R}^k$, d_n metric can be defined as

$$d_n(x,y) := \left(\sum_{i=1}^{k} |x_i - y_i|^n\right)^{\frac{1}{n}}$$

Notice that (X, d_1) and (X, d_2) are two different metric spaces if d_1 and d_2 are two different distance functions.

Convergence

Definition

Let X be a set. The function $x : \mathbb{N} \to X$ is called a sequence in X.

Definition

Let (X, d) be a metric space. A sequence (X_n) in X is said to **converge** to a point $x \in X$, iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$ for all $n > N$. When the sequence (x_n) converges to x, the point x is called a limit of the sequence (x_n), and we use the notation $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Convergence

The next lemma establishes that the limit of a convergent sequence must be unique, and therefore it makes sense to talk about "the" limit of a convergent sequence

Lemma

Let (X, d) be a metric space. Suppose $x_n \to x$ and $x_n \to x'$, then $x = x'$.

A few more

Proposition

In (R, d_2)*, let there be two convergent sequences* $x_n \to x$ *and* $y_n \to y$*. If* $x_n \leq y_n$ *for any* $n \in N$*, then* $x \leq y$*.*

A few more

Proposition

In (R, d_2), *let there be two convergent sequences* $x_n \to x$ *and* $y_n \to y$. *If* $x_n \leq y_n$ *for any* $n \in N$, *then* $x \leq y$.

Proposition

In (R, d_2), *let there be two convergent sequences* $x_n \to x$ *and* $y_n \to y$. *Then*

1. $x_n + y_n \to x + y$,
2. $x_n y_n \to xy$, *and*
3. *if* $x \neq 0$, *then* $1/x_n \to 1/x$

Corollary

In (R, d_2), *let there be two convergent sequences* $x_n \to x$ *and* $y_n \to y$. *Then*
(1) $x_n - y_n \to x - y$,
(2) if $y \neq 0$, *then* $x_n/y_n \to x/y$.

Examples & Application

- Use the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ and the property of the convergence, show

$$\lim_{n\to\infty} a(n) = \lim_{n\to\infty} \frac{3n^2 + 2n - 1}{n^2 + n} = 3$$

Examples & Application

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$$\lim_{n\to\infty} a(n) = \lim_{n\to\infty} \frac{3n^2 + 2n - 1}{n^2 + n} = 3$$

- Consider the sequence $(b_n)_{n\in N}$ defined by $b_0 = 2$ and $b_{n+1} = 5 - \frac{4}{b_n}$.
  1. Find the first four terms of the sequence
  2. Can you find the limits of this sequence? (A sequence converges if and only if its elements approach each other)
  3. Can you show the sequence is increasing by induction?

The next part will introduce the concept of continuity, correspondence, compactness.
These knowledge helps to let us understand the **Fixed Point Theory**, which has a widely applications in the general equilibrium, game theory and dynamic programming.

Continuity of Functions

Definition

Let X, Y be metric spaces, $f : X \to Y$, and $x \in X$. f is continuous at x if for every open neighborhood $V \subset Y$ of $f(x)$, there exists an open neighborhood $U \subset X$ of x such that $U \subset f^{-1}(V)$. f is continuous if it is continuous at every $x \in X$.

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Definition

Let X, Y be metric spaces, and $f : X \to Y$, $x \in X$. f is sequentially continuous at x if $f(x_n) \to f(x)$ in Y for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x$ in X. f is sequentially continuous if it is sequentially continuous at all $x \in X$.

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Let X, Y be metric spaces, and $f : X \to Y$. f is continuous if and only if it is sequentially continuous.

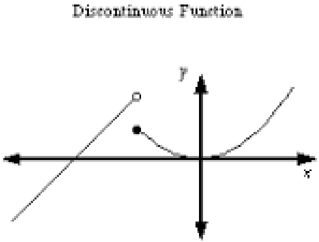
(Epsilon–delta) of Continuous Functions

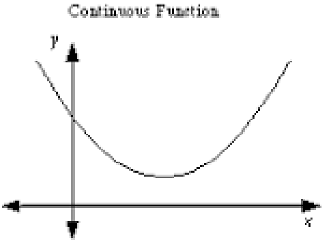
The continuity of $f : D \to \mathbb{R}$ at $x_0 \in D$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$:

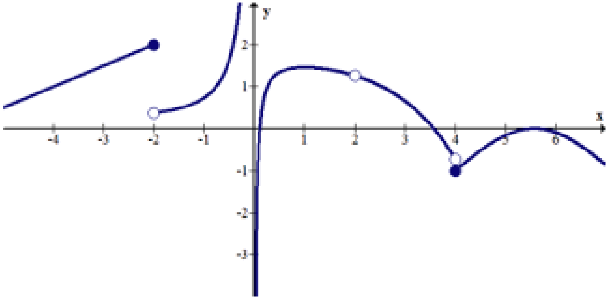
$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon.$$

The two definitions are equivalent.

Example






y
x
2
1
-1
-2
-3
-4
-3
-2
-1
1
2
3
4
5
6

Correspondences

Definition

Let X, Y be nonempty sets. A correspondence Γ from X to Y, denoted by $\Gamma : X \twoheadrightarrow Y$, is a function from X to 2^Y, i.e., $\Gamma(x) \subset Y$ for every $x \in X$.

Definition

Let $\Gamma : X \twoheadrightarrow Y$, and $E \subset Y$.

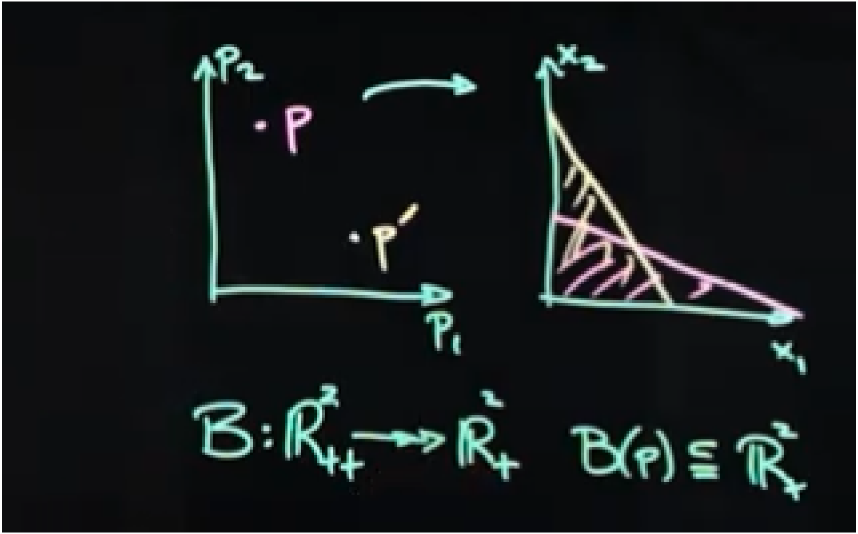
- The upper inverse of E is

$$\bar{\Gamma}^{-1}(E) = \{x \in X : \Gamma(x) \subset E\};$$

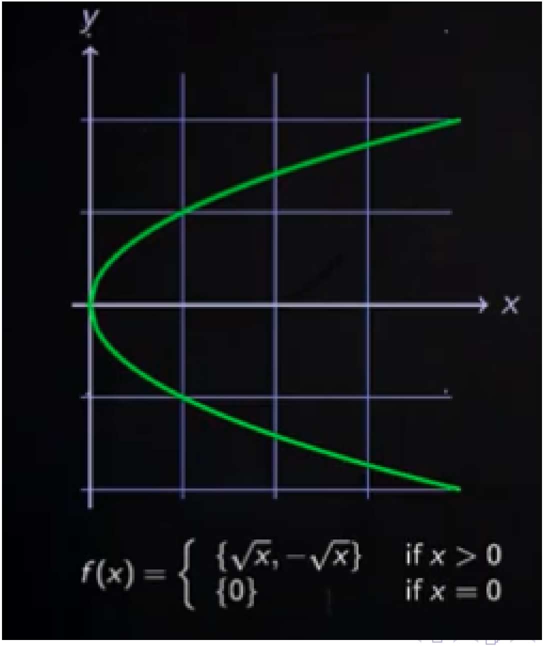
- The lower inverse of E is

$$\underline{\Gamma}^{-1}(E) = \{x \in X : \Gamma(x) \cap \neq \emptyset\}$$

Example



Example



Upper and Lower Hemicontinuity

Proposition

Let X, Y be metric spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. Γ is lower hemicontinuous at x if and only if for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$ and all $y \in \Gamma(x)$, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Gamma(x_n)$ for all n, and $y_n \to y$.

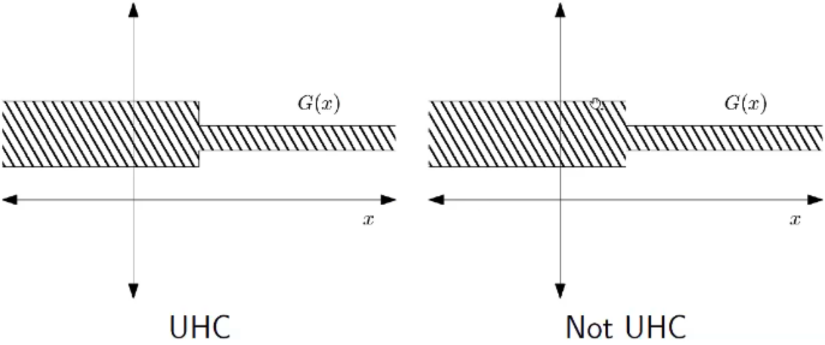
Proposition

Let X, Y be metric spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. If for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ such that $x_n \to x$, $y_n \in \Gamma(x_n)$ for all n, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ that converges to some $y \in \Gamma(x)$, then Γ is upper hemicontinuous at x.

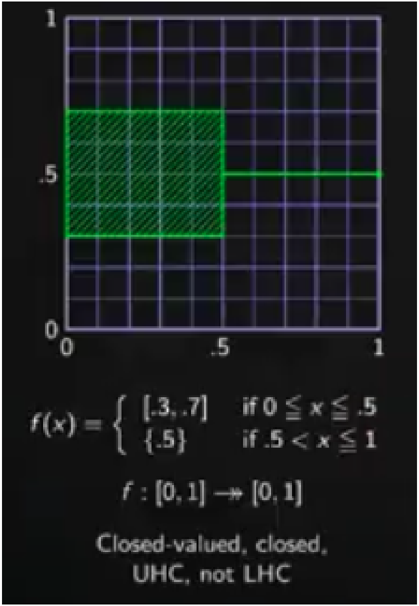
A closed-valued correspondence is UHC/LHC at x if no point suddenly appear/disappear in $f(x)$ as we move away from x (in any direction).

Graph Illustration

A UHC correspondence can get discontinuously **larger** but never discontinuously **smaller**.

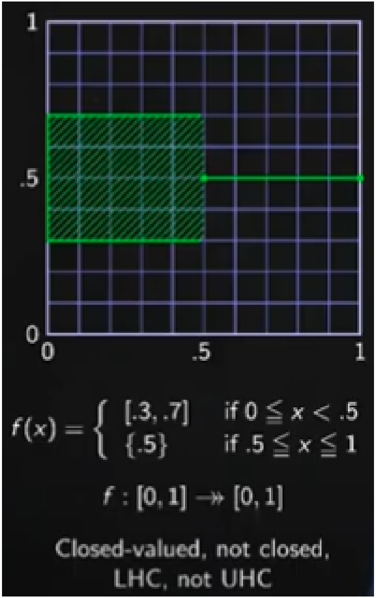


Upper Example



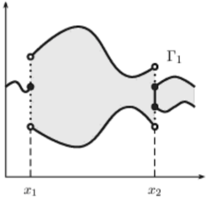
the image set around key point 0.5 does not suddenly enlarge from left or right.
From left to right, the image set is closed.

LHC

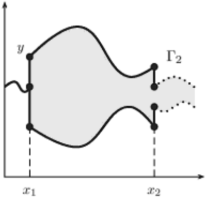


From right to the left, the image set explode.
only one point at 0.5

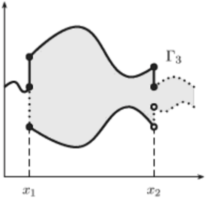
Graph



Not upper hemicontinuous at x_1
Not upper hemicontinuous at x_2
Lower hemicontinuous

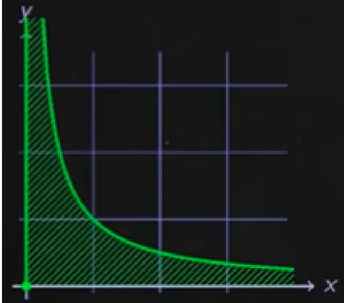


Not lower hemicontinuous at x_1
Not lower hemicontinuous at x_2
Upper hemicontinuous



Not upper hemicontinuous at x_1 and x_2
Not lower hemicontinuous at x_1 and x_2

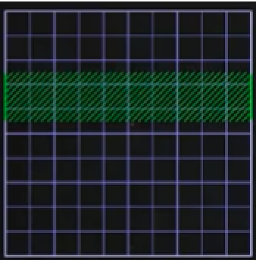
UHC & LHC



$$f(x) = \begin{cases} (0, \frac{1}{x}] & \text{if } x > 0 \\ \mathbb{R}_+ & \text{if } x = 0. \end{cases}$$

$$f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+$$

Not closed-valued, not closed,
UHC, LHC



$$f(x) = (.55, .75),\ \forall x \in [0, 1]$$

$$f : [0, 1] \twoheadrightarrow [0, 1]$$

Not closed-valued, not closed,
UHC, LHC

Remark: A constant correspondence is continuous (*i.e.*, both UHC and LHC).

Compactness

Definition

Let (X, d) be a metric space, and S a subset of X. A family of open sets $\{E_\alpha\}_{\alpha \in A}$ is an open cover of S iff $\cup_{\alpha \in A} E_\alpha \supset S$.

Definition

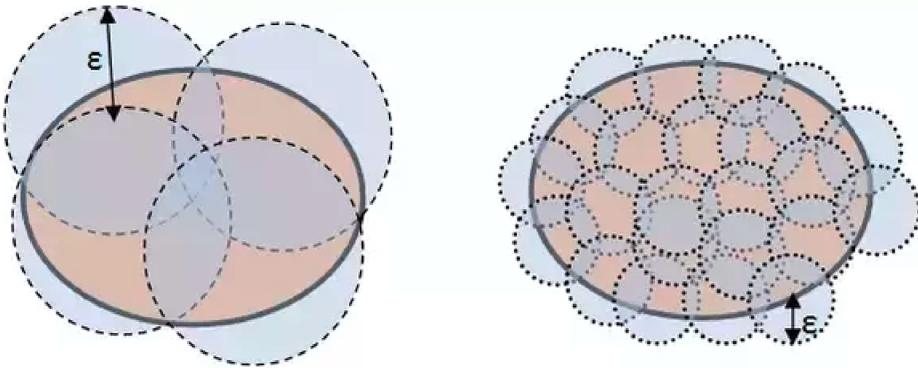
Let (X, d) be a metric space, and S a subset of X. The set S is compact iff $\forall$ open cover $\{E_\alpha\}_{\alpha \in A}$ of S, $\exists$ a finite $B \subset A$ s.t. $\{E_\alpha\}_{\alpha \in B}$ is also an open cover of S.

In general metric spaces, we have shown that a compact set must be closed and bounded. In Euclidean spaces $(\mathbb{R}^k, d_2)$, the reverse is also true, i.e. a closed and bounded set in $(\mathbb{R}^k, d_2)$ must be compact.

Lemma

Any closed interval $[a, b]$ is compact in $(\mathbb{R}, d_2)$.

Example



Totally bounded X : can be covered
with finitely many $N(x, \varepsilon)$, for any $\varepsilon>0$.

Fixed Point Theory

Theorem

(Brouwer). Let $X \subset \mathbb{R}^l$ be nonempty, compact and convex ($l < \infty$), and $f : X \to X$ be continuous. Then f has a fixed point.

Theorem

(Kakutani). Let $X \subset \mathbb{R}^l$ be nonempty, compact and convex ($l < \infty$), and $\Gamma : X \twoheadrightarrow X$ be upper semicontinuous and such that $\Gamma(x)$ is nonempty and convex for all $x \in X$. Then Γ has a fixed point.

Example

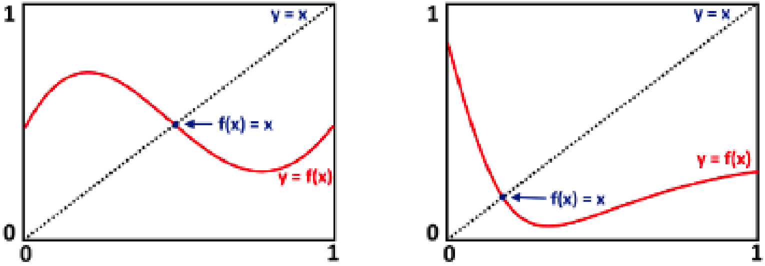
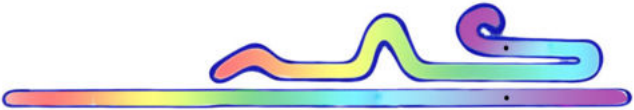


Figure 3.1. The idea behind the One-Dimensional Brouwer Fixed Point Theorem

Example

- Suppose two identical strings are placed directly on top of one another.
- The bottom string remains unchanged, but the top string can be folded, stretched, and reshaped as long as it is replaced so that no part of the string hangs off the bottom string.
- There will be a point on the top string whose horizontal position has not changed!



Economic Application

Here I present a example of application for the pure exchange economy and equilibrium prices.

- There is a system of m consumers, each of whom is initially endowed with fixed quantities of n different commodities. The consumers merely engaged in exchange.
- Trade takes place, because each consumer wishes to acquire a bundle of commodities that is preferred to the initial endowment.

Application: Environment Setup

- Assume a price vector $\mathbf{p} = (p_1, ..., p_n)$ is announced.
- For any commodity bundle $c = (c_1, ..., c_n)$

$$\mathbf{p} \cdot \mathbf{c} = \sum_{i=1}^{n} p_i c_i = p_1 c_1 + ... + p_n c_n$$

  is the market value of this bundle. We use the following notation. For each consumer j and commodity i:
  - w_i^j is j's initial endowment of i,
  - $x_i^j(\mathbf{p})$ is $j's$ final demand when the price vector is $\mathbf{p}$
  - $w_i = \sum_{j=1}^{m} w_i^j$ is the total endowment for i,
  - $x_i(\mathbf{p}) = \sum_{j=1}^{m} x_i^j(\mathbf{p})$ is the aggregate demand for i.
  - excess demand $g_i(\mathbf{p}) = x_i(\mathbf{p}) - w_i$ of commodity i.

The total value of consumer $j's$ initial endowment at price $\mathbf{p}$ is

$$\sum_{i=1}^{m} p_i w_i^j$$

and the budget constraint for j

$$\sum_{i=1}^{m} p_i x_i^j(\mathbf{p}) = \sum_{i=1}^{m} p_i w_i^j$$

Summing all budget constraints of the consumers

$$\sum_{i=1}^{m} p_i(x_i(\mathbf{p}) - w_i)$$

Question

Is it possible to find a price vector $\mathbf{p} = (p_1, \dots, p_n)$, a so called equilibrium price, which ensures that the aggregate demand does not exceed the corresponding aggregate endowment:

$$\forall i = 1, \dots, n: \; x_i(\mathbf{p}^*) \leq w_i \; \Leftrightarrow g_i(\mathbf{p}^*) \leq 0?$$

Obviously, only the relative prices matter in this economy. For the convenience, we can restrict our attention to the normalized price bundle $\mathbf{p}$ with $p_1 + ... + p_n = 1$ or shortly $\mathbf{p} \in \Delta^{n-1}$. Given excess demand functions are continuous on Δ^{n-1} and assume that

$$\sum_{i=1}^{n} p_i g_i(\mathbf{p}) = 0 \ \forall \mathbf{p} \in \Delta^{n-1}$$

The question is equivalent to find a $\mathbf{p}^* \in \Delta^{n-1}$ such that $g_i(\mathbf{p}^*) \leq 0, \ i = 1, ..., n$?

Proof

We shall use Brouwer's fixed point theorem to prove the existence.
To do so, we can construct a continuous mapping from Δ^{n-1} into itself
The trick here is how to construct the mapping.
if we construct something like following

$$p_i' = p_i + g_i(\mathbf{p}).$$

This mapping is not a self mapping of Δ^{n-1}, generally $p_1' + ... + p_n' = 1$.

A Modified Mapping

Now we formulate the mapping $\mathbf{p}' = \mathbf{F}(\mathbf{p})$like this

$$p_i' = \frac{1}{d(\mathbf{p})}(p_i + \max\{0, g_i(\mathbf{p})\}),$$

with

$$d(\mathbf{p}) = 1 + \sum_{k=1}^{n} \max\{0, g_i(\mathbf{p})\}$$

This map F is a continuous self mapping of $\Delta^{n-1}1$ into itself. By Brouwer's fixed point theorem there is a fixed point $\mathbf{p}^* \in \Delta^{n-1}$. This means that

$$p_i^* = \frac{1}{d(\mathbf{p}^*)}(p_i^* + max\{0, g_i(\mathbf{p}^*)\})$$

or

$$p_i^*(d(\mathbf{p}^*) - 1) = max\{0, g_i(\mathbf{p}^*)\}$$

$$p_i^*(d(\mathbf{p}^*)-1) = max\{0, g_i(\mathbf{p}^*)\}$$

If $d(\mathbf{p}^*) > 1$. Then for all i with p_i^* we have $\max\{0, g_i(\mathbf{p}^*)\} > 0$ which means that $\sum_{i=1}^n p_i^* g_i(p^*) > 0$, which is a contradiction to the budget constraint. So $d(\mathbf{p}^*) = 1$ and we see that $\max\{0, g_i(\mathbf{p}^*)\} = 0$ or $g_i(\mathbf{p}^*) \leq 0$ for all $i = 1, .., n$ which complete the proof.