Introduction to Dynamic Programming

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1 Preliminary Knowledge

Envelope Theorem

Proposition 1. *Let*

$$v(a) = \max_x f(x, a)$$

then:

$$\frac{dv(a)}{da} = \frac{\partial f(x, a)}{\partial a}|_{x=x^*(a)}$$

Let

$$m(a) = \max_x f(x, a) \ s.t. \ g(x, a) = 0 \text{ and } x \geq 0$$

Let $\mathcal{L}(x, a, \lambda)$ be the corresponding Lagrange function, and let $x^*(a)$and $\lambda^*(a)$ be the corresponding values, which solve the Kuhn-Tucker conditions. Then:

$$\frac{dm(a)}{da} = \frac{\partial \mathcal{L}}{\partial a}|_{x^*(a), \lambda^*(a)}$$

2 The "Cake Eating" Example: Direct Solution

2.1 Problem description

- Cake of the size W_1
- $t = 1, 2, ..., T$ periods
- In each period, a fraction of cake $(1 - \beta)$ will melt
- In each period, a bit of the cake can be eaten.

How does the cake have to be eaten optimally across time? What is the optimal consumption?

*This note is adapted mostly from book "Dynamic Economics: Quantitative Methods and Applications" by Jerome Adda and Russell W. Cooper. The MIT Press (October 12, 2003) and Prof. Dr. Aleksander Berentsen. All errors are mine.

c_t	Consumption in period t
$u(c_t)$	Utility in period t

2.2 Solve the problem

Let $u(c_t)$ be derivable, strictly monotonic and strictly concave. Then $\lim_{c\to 0} u'(c) \to \infty$. This so-called Inada condition makes sure that we eat at least a little bit of the cake in each period (i.e., no corner solution is possible).

The household's (discounted) utility is

$$\sum_{t=1}^{T} \beta^{t-1} u(c_t)$$

The law of motion for the cake across time is

$$W_{t+1} = W_t - c_t,\ t = 1, 2, ..., T$$

In direct solution, we list the Lagrange function as

$$\begin{aligned} &\max_{\{c_t\}_1^T, (W_t)_2^{T+1}} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \\ s.t.& \\ &W_{t+1} = W_t - c_t, \quad t = 1, 2, ..., T \qquad (1) \\ &c_t \geq 0 \\ &W_{T+1} \geq 0 \end{aligned}$$

W_T is for the consumption at the last period, so W_{T+1} is the nal stock.

The constraints (1) can be summarized as

$$\sum_{t=1}^{T} c_t + \sum_{t=1}^{T} W_{t+1} = \sum_{t=1}^{T} W_t$$

$$\sum_{t=1}^{T} c_t + W_{T+1} = W_1 \qquad (2)$$

Equation (2) means that the totality of consumption added to the final stock must be equal to the initial stock. Then the maximization problem can be formulated as

$$\max_{\{c_t\}_1^T, (W_t)_2^{T+1}} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \ s.t.\ (2) \text{ and } W_{T+1} \geq 0$$

Here we implicitly assume $c_t > 0$ because of the inada condition. The Lagrange

$$\mathcal{L} = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \lambda[W_1 - \sum_{t=1}^{T} c_t - W_{T+1}] + \phi W_{T+1}$$

FOC: c_t

$$\beta^{t-1}u'(c_t) = \lambda,\ t = 1, 2, ..., T$$

which we can get

$$u'(c_t) = \beta u'(c_{t+1})$$

the Euler equation

FOC: W_{T+1}

$$\lambda = \phi$$

This FOC tells us that in order to respect the Kuhn-Tucker conditions $\phi W_{T+1} = 0$ the following expression must

$$W_{T+1} = 0$$

Intuitively, we must empty the cake to obtain the maximal utility.

2.3 Interpretation of the Euler equation

At equilibrium, the marginal costs, when the consumption in one period is reduced, correspond exactly to the discounted marginal utility that is generated when the savings are consumed in the next period.

As for the indirect utility function, $V_T(W_1)$ describes the maximal utility that a household can reach, when the cake has a size of W_1 and the number of periods is T.

$$V_T(W_1) = \sum_{t=1}^{T} \beta^{t-1}u(c_t) + \lambda[W_1 - \sum_{t=1}^{T} c_t - W_{T+1}] + \phi W_{T+1}$$

What happened to the $V_T(W_1)$ when the cake changes (W_1)? By Envelope theorem, we have

$$\frac{dV(W_1)}{dW_1} = V_T'(W_1) = \lambda = \beta^{t-1}u'(c_t)$$

3 The "Cake Eating" Example: Dynamic Programming

3.1 Finite time horizon

Let us start with a very very simple case.

Assume $T = 1$. If there is only one period. The optimal solution for the consumer to maximize the utility is

$$V_1(W_1) = u(W_1) \tag{3}$$

We just eat all the cake.

Assume $T = 2$. Then, we will have

$$\begin{aligned}
V_2(W_1) = &\max_{c_1, c_2, W_2, W_3} u(c_1) + \beta u(c_2) \\
s.t.& \\
& W_2 = W_1 - c_1 \\
& W_3 = W_2 - c_2 \\
& W_3 \geq 0
\end{aligned}$$

With the KT condition or economic intuition, it is easy to see that $W_3 = 0$ and $W_2 = c_2$, which we have

$$V_2(W_1) = \max_{W_2} u(W_1 - W_2) + \beta u(W_2) \tag{4}$$

Comparing (3) and (4), It is natural to see that we can replace the second term in (4) with (3). So we have

$$V_2(W_1) = \max_{W_2} u(W_1 - W_2) + \beta V_1(W_2)$$

If $T = T$, we can do the same operation, and with the help of the Blackwell's sufficient condition, we know that there exists the solution and the solution is unique (u is concave).

If we assume $u(c) = \ln c$, we can solve that

$$V_1(W_1) = \ln(W_1)$$

and

$$V_2(W_1) = \max_{W_2} u(W_1 - W_2) + \beta V_1(W_2)$$

By FOC

$$\frac{1}{c_1} = \frac{\beta}{c_2}$$

Restriction

$$c_1 + c_2 = W_1$$

Therefore

$$c_1 = \frac{W_1}{1+\beta},\ c_2 = \frac{\beta W_1}{1+\beta}$$

Construction

$$V_2(W_1) = \ln(\frac{W_1}{1+\beta}) + \beta \ln(\frac{\beta W_1}{1+\beta})$$

3.2 Infinite time horizon

$$\max_{\{c_t\}_1^\infty, (W_t)_2^\infty} \sum_{t=1}^{T} \beta^{t-1} u(c_t)$$
$$s.t.$$
$$W_{t+1} = W_t - c_t \quad t = 1, 2, ...$$

Dynamic programming can be formulated as following where value function now is time independent:

$$V(W) = \max_{c \in [0,W]} u(c) + \beta V(W - c)$$

n. Where W is defined as a state variable and c as a control variable. Let

$$W' = W - c$$

We now can write:

$$V(W) = \max_{W' \in [0,W]} u(W - W') + \beta V(W')$$

which is also called the **Bellman equation**.

Usually, solving the Bellman equation is very hard. Most of time we can not get the analytical solution if the model is complex. But in this problem, somehow we can use guess and verify method to solve the value function.

Assume

$$V(W) = A + B\ln(W)$$

Then we have

$$A + B\ln W = \max_{W' \in [0,W]} \ln(W - W') + \beta V(A + B\ln W') \tag{5}$$

FOC

$$\frac{1}{W - W'} = \beta B \frac{1}{W'}$$
$$W' = \beta B W - \beta B W'$$
$$W' = \frac{\beta B W}{1 + \beta B}$$

Substitute in

$$A + B\ln(W) = \ln(\frac{W}{1 + \beta B}) + \beta \left[A + B\ln\left(\frac{\beta B W}{1 + \beta B}\right)\right]$$
$$= \underbrace{1 + \beta B}_{B} \ln(W) + \underbrace{- [1 + \beta B]\ln(1 + \beta B) + \beta(1 + B\ln(\beta B))}_{A}$$

And we get

$$B = \frac{1}{1-\beta}$$
$$A = \frac{1}{(1-\beta)^2}\left[\beta \ln(\frac{\beta}{1-\beta}) - \ln\frac{1}{1-\beta}\right]$$

And from the FOC, we can finally get

$$c = (1-\beta)W$$
$$W' = \beta W$$