Lecture 3: Multivariate Calculus

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UIBE Math Camp, 2022

Some Notations

A function $f : \mathbb{R}^N \to \mathbb{R}^M$ can be represented as:

$$f(x) = (f_1(x), ..., f_M(x))$$

since $f(x)$ is a point in $\mathbb{R}^M$; it can be represented as an $M \times 1$ matrix. Each of its coordinates is a function $f_m(x) : \mathbb{R}^N \to \mathbb{R}$ for $m = 1, ..., M$.

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_M(x) \end{bmatrix}$$

Derivatives

Definition

Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \quad (a < t < b, \; t \neq x)$$

and define

$$f'(x) = \lim_{t \to x} \phi(t),$$

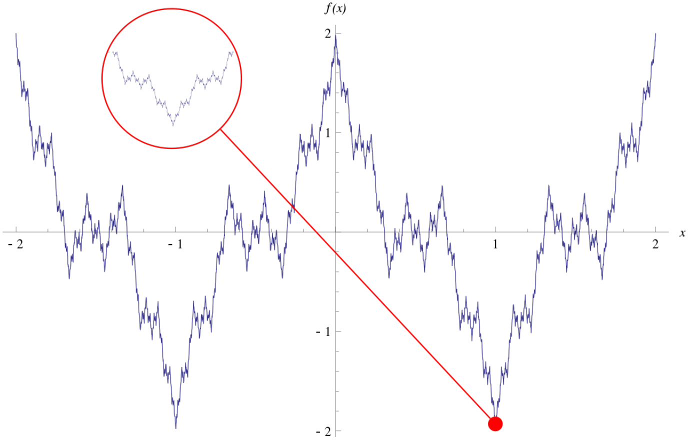
provided this limit exists. We thus associate with the function f' whose domain is the set of points x at which the limit exists; f' is called the derivative of f. If f' is defined at a point x, f is differentiable at x.

Continuity and Differentiability

Clearly, if a function f is differentiable at x, then it is continuous at x. This is because

$$\begin{aligned}
\lim_{x' \to x} \left| f(x)' - f(x) \right| &= \lim_{x' \to x} \left[\frac{f(x') - f(x)}{x' - x} \cdot (x' - x) \right] \\
&= \lim_{x' \to x} \left[\frac{f(x') - f(x)}{x' - x} \right] \cdot \lim_{x' \to x} \left[x' - x \right] \\
&= f'(x) \cdot 0 = 0
\end{aligned}$$

Example: Weierstrass_function



Some properties

if f and g are both differentiable at x, then $f+g$ is also differentiable at x, And
$(f+g)'(x) = f'(x) + g'(x)$. Also we have

1. $(\lambda f)' = \lambda f'$
2. $(fg)' = f'g + fg'$
3. $(f/g)' = \frac{f'g - fg'}{g^2}$

L'Hospital Rule

Theorem

Let $-\infty < a < b < +\infty$*, and* $f : (a, b) \to \mathbb{R}$ *and* $g : (a, b) \to \mathbb{R} \backslash \{0\}$ *are differentiable in* (a, b)*. If* $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *are both* 0 *or* $\pm\infty$*, and* $\lim_{x \to a} f'(x)/g'(x)$ *has a finite value or is* $\pm\infty$*, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The statement is also true for $x \to b$*.*

Example

The function is hard to see the limit directly $(\ln x)/\sqrt{x}$ when x diverges to $+\infty$

$$\frac{(\ln x)'}{(\sqrt{x})'} = \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{2}{\sqrt{x}} \to 0$$

Total Derivatives (multivariate functions)

Definition

Let $A \subset \mathbb{R}^n$ and $x \in int(A)$. A function $f : A \to \mathbb{R}^m$ is said to be differentiable at x iff $\exists$ an $m \times n$ real matrix C s.t.

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ch\|}{\|h\|} = 0$$

In this case, define the (total) derivative of f at x as the matrix C, denoted as $f'(x)$, or $Df(x)$.

A function $f : A \to \mathbb{R}$ is said to be differentiable iff A is open and f is differentiable at any $x \in A$.

C^k functions

We say that f from $A \subset \mathbb{R}^n$ to $\mathbb{R}^m$ is k-th continuously differentiable at x iff $x \in int(A_k)$ and $f^{(k)}(x)$ is continuous at x, where A_k is the set of points at which $f^{(k-1)}$ is differentiable. In this case, f is said to be C^k at x. We say that f is k-th continuously differentiable iff A is open and f is k-th continuously differentiable at all $x \in A$. In this case, f is said to be C^k.

- $u(x) = \frac{x^{1-\alpha}-1}{1-\alpha}$
- $u(x) = e^{-\alpha x}$

Partial and Directional Derivatives

Definition

Let $A \subset \mathbb{R}^n$ and $x \in int(A)$. For a function $f : A \to \mathbb{R}^m$, the partial derivative of $f_m$with respect to the n-th argument, x_n, evaluated at the point x, is

$$D_n f_m(x) = \frac{\partial f_m(x)}{\partial x_n} = \lim_{t \to 0} \frac{f_m(x_1, \ldots, x_n + t, \ldots x_N) - f_m(x)}{t}$$

assuming that the limit exists.

The vector $(x_1, \ldots, x_n + t, \ldots x_N)$ is a deviation from x only in the n-th argument. Therefore, intuitively, the partial derivative $\frac{\partial f_m}{\partial x_n}$ measures the sensitivity of the m-th coordinate f_m of the function f w.r.t. the n-th argument x_n.

Jacobian

The matrix of partial derivatives of all the coordinate functions $(1, ..., m)$ fm with respect to all the x_n evaluated at the point $x(1, ..., N)$ is called Jacobian of f at x.

$$Jf(x) = \begin{bmatrix} D_1 f_1(x) & \cdots & D_N f_1(x) \\ \vdots & \ddots & \vdots \\ D_1 f_M(x) & \cdots & D_N f_M(x) \end{bmatrix}_{M \times N}$$

Notice that all partial derivatives exist does not imply the existence of the total derivatives

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Directional Derivatives

Definition

Let $A \subset \mathbb{R}^n$ and $x \in int(A)$. For a function $f : A \to \mathbb{R}^m$, and a vector $z \in \mathbb{R}^n$ with $||z|| = 1$, the directional derivative of f along the vector $z \in \mathbb{R}^n$at $x \in A$ is

$$f'_z(x) := \frac{d}{dt} f(x+tz)|_{t=0} = \begin{bmatrix} \frac{d}{dt} f_1(x+tz)|_{t=0} \\ \frac{d}{dt} f_2(x+tz)|_{t=0} \\ \vdots \\ \frac{d}{dt} f_m(x+tz)|_{t=0} \end{bmatrix}$$

if the right-hand side derivative exists.

Notice that partial derivatives are a special case of directional derivative.$z = (0, ...1, ..., 0)$

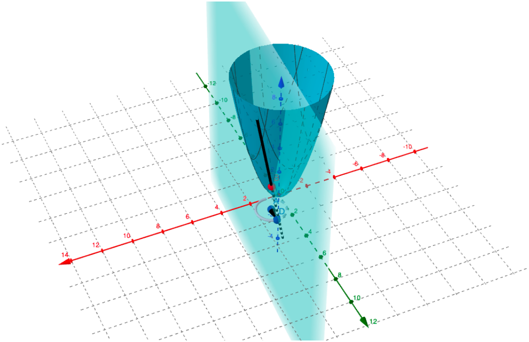
Example

$f(x) = 3x_1 + x_1 x_2, \hat{x} = (1,1),\ v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\begin{aligned} D_v f(\hat{x}) &= \lim_{t \to 0} \frac{f(\hat{x} + tv) - f(\hat{x})}{t} \\ &= \frac{5}{\sqrt{2}} \end{aligned}$$

Directional Derivatives and Gradient

Let f be a function from $A \subset \mathbb{R}^n$ to $\mathbb{R}$ that is differentiable at $x \in int(A)$, and $\nabla f(x) \neq 0$. Then the directional derivative $f'_z(x)$ is maximized when $z = \frac{\nabla f(x)}{||\nabla f(x)||}$, and the maximized directional derivative is $||\nabla f(x)||$.



Gradient Example

$f : \mathbb{R}^2 \to \mathbb{R}$
$f(x) = 3 \ln x_1 + \ln x_2, \; x_0 = (2,2)$
$\nabla f(x_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x_0) \\ \frac{\partial f}{\partial x_2}(x_0) \end{bmatrix} = \begin{bmatrix} \frac{3}{x_1} \\ \frac{1}{x_2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$ so, the norm of the $\nabla f(x_0)$ is $\frac{\sqrt{10}}{2}$
The directional (unit) vector is $v = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right)$.

Chain Rule

Let $S \in \mathbb{R}^n$, $x \in int(S)$, and $f : S \to \mathbb{R}^m$. Let T be s.t. $f(S) \subset T \subset \mathbb{R}^m$ and $f(x) \in int(T)$, and let $g : T \to \mathbb{R}^k$. If f is differentiable at x, and g is differentiable at $f(x)$, then g is differentiable at $f(x)$, then $g \circ f : S \to \mathbb{R}^k$ is differentiable at x. Furthermore, we have

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

In the equation $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$, the equality between the (i, j)-th entries of the matrices on two sides is

$$\frac{\partial (g \circ f)_i}{\partial x_j}(x) = \sum_{l=1}^{m} \left[\frac{\partial g_i}{\partial y_l}(f(x)) \cdot \frac{\partial f_l}{\partial x_j}(x) \right]$$

Example

$z = f(x,y) = 4x^2 + 3y^2$, $x = x(t) = \sin t$, $y = y(t) = \cos t$, calculate dz/dt
We need to calculate $\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt

- $\frac{\partial z}{\partial x} = 8x$
- $\frac{\partial x}{\partial t} = \cos t$
- $\frac{\partial z}{\partial y} = 6y$
- $\frac{dy}{dt} = -\sin t$

Now we can utilize the chain rule to calculate

$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\
&= 8x \cdot \cos t + 6y(-\sin t) \\
&= 8x \cos t - 6y \sin t \\
&= 2 \sin t \cos t
\end{aligned}
$$

Mean Value Theorem

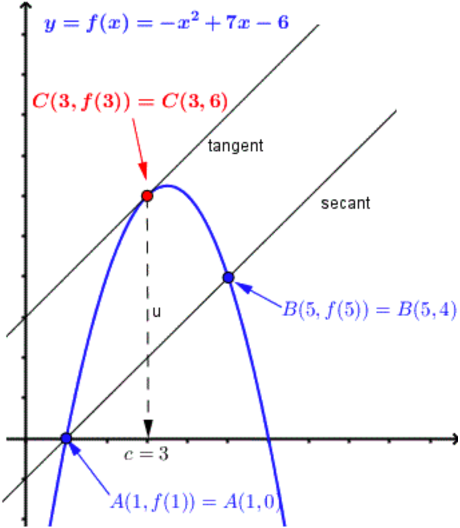
Theorem

Let $f : [a, b] \to \mathbb{R}$*, differentiable on* (a, b)*, and continuous on* $[a, b]$*. Then there exists* $x \in (a, b)$ *s.t.*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

One implication of mean value theorem is: if $f' > (\geq) 0$ on (a, b), then f is strictly (weakly) increasing on (a, b). If we have $f' < (\leq)$, then f is strictly (weakly) decreasing on (a, b).

Example



Higher Order Derivatives:Hessian

The second derivative of the real-valued function f at x is also known as the Hessian matrix of f at x, denoted as $H_f(x)$:

$$H_f(x) := f''(x) = (\nabla f)'(x) = \begin{bmatrix} \left(\nabla \frac{\partial f}{\partial x_1}\right)(x) \\ \left(\nabla \frac{\partial f}{\partial x_1}\right)(x) \\ \vdots \\ \left(\nabla \frac{\partial f}{\partial x_1}\right)(x) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial\left(\frac{\partial f}{\partial x_1}\right)}{\partial(x_1)}(x) & \frac{\partial\left(\frac{\partial f}{\partial x_1}\right)}{\partial(x_2)}(x) & \cdots & \frac{\partial\left(\frac{\partial f}{\partial x_1}\right)}{\partial(x_n)}(x) \\ \frac{\partial\left(\frac{\partial f}{\partial x_2}\right)}{\partial(x_1)}(x) & \frac{\partial\left(\frac{\partial f}{\partial x_2}\right)}{\partial(x_2)}(x) & \cdots & \frac{\partial\left(\frac{\partial f}{\partial x_2}\right)}{\partial(x_n)}(x) \\ \vdots & \vdots & & \vdots \\ \frac{\partial\left(\frac{\partial f}{\partial x_n}\right)}{\partial(x_1)}(x) & \frac{\partial\left(\frac{\partial f}{\partial x_n}\right)}{\partial(x_2)}(x) & \cdots & \frac{\partial\left(\frac{\partial f}{\partial x_n}\right)}{\partial(x_n)}(x) \end{bmatrix}$$

Example

Compute the Hessian of $f(x,y) = x^3 - 2xy - y^6$ at the point $(1,2)$:

$$f_x(x,y) = 3x^2 - 2y$$
$$f_y(x,y) = -2x - 6y^5$$

Then, we have

$$f_{xx} = 6x,\ f_{xy} = -2,\ f_{yx} = -2,\ f_{yy} = -30y^4$$

The Hessian matrix now is

$$Hf(x,y) = \begin{bmatrix} 6x & -2 \\ -2 & -30y^4 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & -480 \end{bmatrix}$$

Taylor Expansion

Theorem

Let $f : [a, b] \to \mathbb{R}$ *be* C^{n-1} *and* $f^{(n)}(t)$ *exists at every* $t \in (a, b)$. *Let a and b be distinct points in* $[a, b]$, *and define*

$$P_{n-1}(t) := f(\alpha) + f'(\alpha)(t-\alpha) + \frac{f''(\alpha)}{2}(t-\alpha)^2 + \ldots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}$$

Then there exists x strictly between α *and* β *s.t.*

$$f(\beta) = P_{n-1}(\beta) + \frac{f^{(n)}(x)}{n!}$$

First and Second Order Taylor Expansion

Let f be a function from $A \subset \mathbb{R}^n$ to $\mathbb{R}$, and f is C^2 at $x \in int(A)$. Then we have

$$f(x+h) = f(x) + \nabla f(x)h + o(||h||)$$

If f is C^3 at x, we have

$$f(x+h) = f(x) + \nabla f(x)h + \frac{1}{2}h^T H_f(x) + o(||h||^2)$$

remark: $f(n) = o(g(n))$ means $\lim f(n)/g(n) = 0, \ when \ n \to \infty$

Log-linearization

Consider multivariate function $f : A \subset \mathbb{R}^n \to \mathbb{R}$, we want to approximate it around point $x^* = (x_1^*, x_2^*, ..., x_n^*)$ s.t. $x_i^* \neq 0$, $\forall i$, for each variable x_i, we define

$$\hat{x}_i := \ln(x_i / x_i^*)$$

to be its log-deviation when x_i and x_i^* have the same sign (which is reasonable when x is "near" x^*).

Often we want to log-linearize an equation (part of a system at its steady state)

$$f(\mathbf{x}) = f(x_1, x_2, ..., x_n) = 0$$

So we have

$$f_1'(\mathbf{x}^*)x_1^*\hat{x}_1 + f_2'(\mathbf{x}^*)x_2^*\hat{x}_2 + ... + f_n'(\mathbf{x}^*)x_n^*\hat{x}_n = 0$$

Log-linearization II

If $f(\mathbf{x}^*) \neq 0$, define $\eta_i := \frac{f_i'(\mathbf{x}^*)x_i^*}{f(\mathbf{x}^*)}$ $(i = 1, 2, ..., n)$ the elasticity of f w.r.t x_i at x , we can also write:

$$f(\mathbf{x}) = f(\mathbf{x}^*)[1 + \eta_1 \hat{x}_1 + \eta_2 \hat{x}_2 + ... + \eta_n \hat{x}_n]$$

therefore

$$\frac{f(\mathbf{x}) - f(\mathbf{x}^*)}{f(\mathbf{x}^*)} = \eta_1 \hat{x}_1 + \eta_2 \hat{x}_2 + ... + \eta_n \hat{x}_n$$

Now we define the log-deviation of function f around some point $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)$ s.t. $f(\mathbf{x}^*) \neq 0$:

$$f\hat{(x)} := \ln(f(\mathbf{x})/f(\mathbf{x}^*))$$

(when $f(\mathbf{x})$ and $f(\mathbf{x})^*$ have the same sign). Notice that $\ln(f(\mathbf{x})/f(\mathbf{x}^*)) \approx \frac{f(\mathbf{x}) - f(\mathbf{x}^*)}{f(\mathbf{x}^*)}$, we then have

$$\widehat{f(x)} := \eta_1 \hat{x}_1 + \eta_2 \hat{x}_2 + ... + \eta_n \hat{x}_n$$

Shortcuts of the Log-linearization

- $\widehat{\alpha x} = \hat{x}$
- $\widehat{x_1 + x_2} = \frac{x_1^*}{x_1^* + x_2^*}\hat{x}_1 + \frac{x_2^*}{x_1^* + x_1^*}\hat{x}_2$
- $\widehat{x_1 x_2} = \hat{x}_1 + \hat{x}_2$
- $\widehat{x^\alpha} = \alpha \hat{x}$
- $\hat{c} = 0$ where c is a constant

Example

Consider the equation

$$y_t = s z_t k_t^{\alpha}$$

First we can get

$$y(1+\tilde{y}_t) = sz(1+\tilde{z}_t)k^{\alpha}(1+\alpha\tilde{k}_t)$$

Utilize the equation for the steady state $y = szk^{\alpha}$ for the simplification, we have

$$\begin{aligned}(1+\tilde{y}_t) &= (1+\tilde{z}_t)(1+\alpha\tilde{k}_t)\\ \tilde{y}_t &= 1+\tilde{z}_t+\alpha\tilde{k}_t+\alpha\tilde{k}_t\tilde{z}_t-1\\ &= \tilde{z}_t+\alpha\tilde{k}_t\end{aligned}$$

Since $\alpha\tilde{k}_t\tilde{z}_t \sim 0$.

Implicit Function Theorem

Theorem

(Implicit Function). Let f be a function from $A \subset \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^m$. Let $(x_0, y_0) \in int(A)$ s.t. $f(x_0, y_0) = 0$. If f is C^1 at (x_0, y_0) and the $m \times m$ Jacobian matrix $f'_y(x_0, y_0)$ is invertible, then there exist an open ball B_x around x_0 and an open ball B_y around y_0 s.t. $\forall x \in B_x$ there exists a unique $y \in B_y$ s.t. $f(x, y) = 0$. Therefore, the equation $f(x, y) = 0$ implicitly defines a function $g : B_x \to B_y$ with the property

$$f(x, g(x)) = 0$$

for any $x \in B_x$. Furthermore, we know that the function g is differentiable at any $x \in B_x$, and

$$g'(x) = -\left[f'_y(x, g(x))\right]^{-1} f'_x(x, g(x))$$

Two Dimension Case

In the two dimension case, $\hat{x}_2 = \psi(\hat{x}_1)$ and

$$\frac{d\psi}{dx_1}(\hat{x}_1) = -\frac{\frac{\partial F}{\partial x_1}(\hat{x})}{\frac{\partial F}{\partial x_2}(\hat{x})}$$

Example

Find the derivative of the polynomial equation $2x^2 - 4y^2 = 6$ using implicit function theorem.
Based on the the above formula $F(x,y) = 2x^2 - 4y^2 - 6 = 0$

$$\frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = -8y$$

So

$$f'(x) = \frac{x}{2y}$$

Fundamental Theorem of Calculus

Theorem

(Fundamental Theorem of Calculus). If f is (Riemann) integrable w.r.t. x on $[a, b]$, and if there is a differentiable function F on $[a, b]$ s.t. $F^0 = f$, then

$$\int_a^b f dx = F(b) - F(a)$$

F is called the antiderivative (or indefinite integral) of f on $[a, b]$, noted $\int f(x)dx$.

Useful Calculation Skills for Single integrals

1. Differentiation of α:

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

2. Change of variable:

$$\int_a^b f(\phi(x))d\alpha(\phi(x)) = \int_{\phi(a)}^{\phi(b)} f(y)d\alpha(y)$$

3. Integration by part:

$$\int_a^b f(x)dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)df(x)$$

Change of Variables in multiple integrals

Consider double integral $\int_A f(x,y)dxdy$. Suppose that

$$x = g(u,v),\ y = h(u,v)$$

defines a one-to-one C^1 transformation from an open and bounded set A' in the uv-plane onto an open and bounded set A in the xy-plane, and assume the Jacobian determinant

$$\frac{\partial(g,h)}{\partial(u,v)} := det\left(\begin{bmatrix} \partial g/\partial u & \partial g/\partial v \\ \partial h/\partial u & \partial h/\partial v \end{bmatrix}\right)$$

is bounded on A'.

Change of Variables in multiple integrals

Assume f is bounded and continuous on A. Then

$$\int_A f(x,y)dxdy = \int_{A'} f(g(u,v),h(u,v))d|\frac{\partial(g,h)}{\partial(u,v)}$$

where $|\frac{\partial(g,h)}{\partial(u,v)}|$ is the absolute value of the Jacobian determinant.

Example

Evaluate

$$\int_{D_c} e^{-x^2-y^2}\,dxdy$$

where D_c is region in the first quadrant of the xy-plane where $x^2+y^2 \leq c^2$
Solution:
Change to Polar coordinates. Region is sector $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq c$

$$\begin{aligned}
\int_{D_c} e^{-x^2-y^2}\,dxdy &= \int_0^{\pi/2}\int_0^c e^{-r^2}\,rdrd\theta \\
&= \int_0^{\pi/2} -\frac{1}{2}e^{-r^2}|_0^c d\theta \\
&= \frac{\pi}{4}(1-e^{-c^2})
\end{aligned}$$

Derivatives of Integrals

Theorem

(Leibniz's Formula). Let f be a function from a subset A of $\mathbb{R}^2$ to $\mathbb{R}$. Let rectangle $E := [a,b] \times [c,d] \subset A$ with $a < b$ and $c < d$. Let u and v be two C^1 functions from $[a,b]$ to $[c,d]$. If $\frac{\partial f}{\partial x}(x,t)$ exists for any $(x,t) \in E$ and $\frac{\partial f}{\partial x}$ is continuous on E, then $I(x) := \int_{u(x)}^{v(x)} f(x,t)dt$ is differentiable on $[a,b]$, and

$$I'(x) = f(x, v(x))v'(x) - f(x, u(x))u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)dt$$

Homogeneous Functions

Definition

A set C in real vector space V is said to be a **cone**, iff $\lambda v \in C$ for any $\lambda \in \mathbb{R}_{++}$ and $v \in C$.

Definition

Let C be a cone in real vector space V, and let W be another real vector space. For $k \in \mathbb{R}$, a function $f : C \to W$ is said to be homogeneous of degree k iff $f(\lambda v) = \lambda^k f(v)$ for any $\lambda \in \mathbb{R}_{++}$ and $v \in C$.

In most applications, C is a cone in $\mathbb{R}^n$ (usually $C = \mathbb{R}^n_{++}$ or $\mathbb{R}^n_{+}$), $W = \mathbb{R}$, and k is a non-negative integer.

Example

$f(k,l) = k^{\alpha} l^{1-\alpha}, \ \alpha \in (0,1)$

Homogeneous Functions

Definition

Let C be a cone in $\mathbb{R}^n$, and $f : C \to \mathbb{R}$ homogeneous of degree k. Let $x \in int(C)$ and $\lambda > 0$. If $\frac{\partial f}{\partial x_i}$ exists at x, then $\frac{\partial f}{\partial x_i}$ exists at λx, and we have

$$\frac{\partial f}{\partial x_i}(\lambda x) = \lambda^{k-1} \frac{\partial f}{\partial x_i}(x)$$

Shortly put, the theorem says that a partial of a function homogeneous of degree k is homogeneous of degree k − 1, if the partial exists.

Homogeneous Functions

Theorem

(Euler's Equation). Let C be a cone in $\mathbb{R}^n$, and $f : C \to \mathbb{R}$ homogeneous of degree k and differentiable at $x \in int(x)$, and then we have

$$\nabla f(x) \cdot x = kf(x)$$

Definition

Let C be a cone in real vector space V. A function $f : C \to \mathbb{R}$ is said to be homothetic iff there exists $h : C \to \mathbb{R}$ homogeneous of some degree k and $g := \mathbb{R} \to \mathbb{R}$ strictly increasing s.t. $f = g \circ h$.

Application in Economics

Intuitively, it says that the "marginal rate of substitution" of a homothetic function is preserved under scalar multiplication
Let C be a cone in $\mathbb{R}^n$, and $f : C \to \mathbb{R}$ homothetic. Let $x \in int(C)$ and $\lambda > 0$. If f is differentiable at x and λx, and $\frac{\partial f}{\partial x_j}(\lambda x)$ and $\frac{\partial f}{\partial x_j}(x)$ are not zero, then we have

$$\frac{\frac{\partial f}{\partial x_i}(\lambda x)}{\frac{\partial f}{\partial x_j}(\lambda x)} = \frac{\frac{\partial f}{\partial x_j}(x)}{\frac{\partial f}{\partial x_j}(x)}$$

Utility functions having constant elasticity of substitution (CES) are homothetic. They can be represented by a utility function such as:

$$u(x, y) = \left(\left(\frac{x}{w_x} \right)^r + \left(\frac{y}{w_y} \right)^r \right)^{1/r}$$