# Note Summary: Principal of Mathematics (Analysis)

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## 1 Logic

Logic is the business of evaluating arguments, sorting good ones from bad ones. A logical argument is structured to give someone a reason to believe some conclusion.
**Basic Sentence**
In logic, we are only interested in sentences that can be as a premise or conclusion of an argument. **Generally, questions will not count as sentences, but answers will.**

**A** It is raining

**B** You have an umbrella

**C** x is a prime number

**Connectives**

- and $\wedge$
- or $\vee$
- if then $\implies$
- not $\neg$
- if and only if $\Leftrightarrow$

**premise-indicators:** since, because, given that

**conclusion-indicators:** therefore, hence, thus, then, so

To be perfectly general, we can define an argument as a series of sentences. The sentences at the beginning of the series are premises. The final sentence in the series is the conclusion. If the premises are true and the argument is a good one, then you have a reason to accept the conclusion.

*This note mainly borrows from math camp material in Columbia University. https://www.sites.google.com/site/mathcamp2018cu

It is raining heavily.

If you do not take an umbrella, you will get soaked

Therefore you should take an umbrella.

The above case is an argument

**TRUTH TABLE**

| $A$ | $B$ | $A \wedge B$ | $A \vee B$ | $\neg A$ | $A \vee \neg A$ | $A \wedge \neg A$ | $A \wedge \neg B$ | $\neg(A \wedge \neg B)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |

| $A$ | $B$ | $A \Rightarrow B$ | $B \Rightarrow A$ | $A$ only if $B$ | $A$ if $B$ | if $A$ then $B$ | $A \Leftrightarrow B$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Note that

- $A \Rightarrow B$ If A then B
- $B \Rightarrow A$ A if B
- $A \Rightarrow B$ A only if B

**Definition 1.** Deductive Validity
An argument is deductively valid if and only if it is impossible for the premises to be true and the conclusion false. The crucial thing about a valid argument is that it is impossible for the premises to be true at the same time that the conclusion is false.

**Definition 2.** Inductive arguments
There can be good arguments which nevertheless fail to be deductively valid. Consider this one :
In January 1997, it rained in San Diego.
In January 1998, it rained in San Diego.
In January 1999, it rained in San Diego.
$\vdots$
$\therefore$It rains every January in San Diego.
This is an inductive argument, because it generalizes from many cases to a conclusion about all cases.

**Definition 3.** Logical Truth
1. It is raining. Logically speaking, it might be either true or false. Sentences like this are called **contingent sentences**.
2. Either it is raining, or it is not. This sentence is logically true. A **logically true** sentence is called a **tautology**.
3. It is both raining and not raining. The third sentence is **logically false** ; it is false regardless of what the world is like. A logically false sentence is called a **contradiction**.

**Definition 4.** Logical equivalence
When two sentences necessarily have the same truth value, we say that they are logically equivalent

**Definition 5.** SL And **QL**

- SL, which stands for sentential (symbolic) logic. In SL, the smallest units are sentences themselves. Simple sentences are represented as letters and connected with logical connectives like 'and' and 'not' to make more complex sentences.

- QL, which stands for quantified logic. In QL, the basic units are objects, properties of objects, and relations between objects.

**Definition 6.** Consistent and Inconsistent Sentences
A set of sentences is consistent if it is logically possible for all the members of the set to be true at the same time; it is inconsistent otherwise.

| $A$ | $B$ | $A \Rightarrow B$ | $\neg B \Rightarrow \neg A$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 |

$A \Rightarrow B$ and $\neg B \Rightarrow \neg A$ are contra-positive One is true, the other is true. One is false, the other is false

# 2 Set Theory

Some basic notations of Set

- $\forall x,\ x \in A \cap B,\ \Leftrightarrow x \in A,\ and\ x \in B$

- $\forall x,\ x \in A \cup B,\ \Leftrightarrow x \in A,\ or\ x \in B$

- $\forall x,\ x \in A \bigtriangleup B,\ \Leftrightarrow x \in A,\ exclusive\ or\ x \in B$

- $\forall x,\ x \in A \backslash B,\ \Leftrightarrow x \in A,\ and\ x \notin B$

- $A \subseteq B\ \Leftrightarrow \forall x,\ x \in A \Rightarrow x \in B$

- $A = B \ \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$

**Partition** a partition of a set is a grouping of the set's elements into non-empty subsets, in such a way that every element is included in one and only one of the subsets.

$A = \{1, 2, 3, 4\}, \ P = \{\{1, 2\}, \{3, 4\}\}$ P is the partition of $A$

**Functions** Given two sets $X, Y$ , a FUNCTION $f \ : X \to Y$ assigns a unique element $y \in Y$ to each element of $x \in X$

**Injective** function $f$ is injective or one-to-one if $\forall a, b \in x, \ f(a) = f(b) \Rightarrow a = b$

**Surjective** function $f$ is surjective or onto if $\forall y \in Y, \ \exists x \in X, \ s.t. f(x) = y$

**Bijective** function $f$ is bijective if it is both injective and surjective

**Russell's Paradox**

- Let X be the set of all infinite sets.
- Let Y be the set of all sets that belong to themselves
- Let Z be the set of all sets that does not belong to themselves!

**Definition 7.** $Im \ f \ \in B, \ y \in Im \ f \ \Leftrightarrow \exists x \in A, \ s.t. \ y = f(x)$

**Theorem 1.** *Cantor-Bernstern-Sehroder*
*Let $A$ and $B$ be two sets and suppose that there exists injection of $f: \ A \to B$ and an injection $g: \ B \to A$. Then there exists a bijection $h: \ A \to B$*

*Proof.* Take any $a \in A$ and construct the sequence : $a \to f(a) \to g(f(a)) \to \cdots$
If $f$ is injective, we can define, $f^{-1} : Im \ f \to A$ such that $f^{-1}(y) = x \ \Leftrightarrow f(x) = y$ and the sequence turns to

$$\cdots f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to \cdots$$

Define a partition of A by

- $a \in S_A$ if the sequence stops at some element in A.
- $a \in S_B$ if the sequence stops at some element in B.
- $a \in S_\infty$if the sequence never stops

Similarly, we can define a partition of B by $T_A, T_B, T_\infty$
*Claim* 1. $f$ is a bijection between $S_A$ and $T_A$
If $a \in S_A$, then $f(a) \in T_A$. Since $f(a)$ belong to the same sequence as a. If $b \in T_A, \exists a \in A.$ s.t. $f(a) = b$ (recall the sequence). Therefore, $f : S_A \to T_A$ is injective and surjective, implying $f \ : \ S_A \to T_A$ is bijective
*Claim* 2. $g$ is a bijection between $S_B$ and $T_B$

*Claim* 3. Claim $f$ *or* $g$ is a bijection between $S_\infty$ and $T_\infty$

Define $h\ :\ A \to B$ by

$$h(a) = \begin{cases} f(a) & a \in S_A \\ g(a) & a \in S_B \\ f(a)\ or\ g(a) & a \in S_\infty \end{cases}$$

□

## 2.1 Orders and relations

Ordered Pair $(a, b)$ $a \neq b \Rightarrow (a, b) \neq (b, a)$ and $(a, b) = (a', b')\ \Rightarrow a = a'\ b = b'$

**Definition 8.** Cartesian Product
given two sets A and B , their Cartesian Product or simply product of $X$ and $Y$, is the set $A \times B = \{(a, b); a \in A,\ and\ b \in B\}$

**Definition 9.** Relations
A relation on a set X is a set $\mathcal{R} \subseteq X \times X,\ x\mathcal{R}y \Leftrightarrow (x, y) \in \mathcal{R}$

**Properties**

- Reflexive $\forall x,\ x\mathcal{R}x$
- Transitive $\forall x, y, z\ x\mathcal{R}y,\ y\mathcal{R}z \Rightarrow\ x\mathcal{R}z$
- Antisymmetric $\forall x, y\ x\mathcal{R}y\ and\ y\mathcal{R}x\ \Rightarrow x = y$
- symmetric $\forall x, y\ x\mathcal{R}y \Rightarrow y\mathcal{R}x$
- Complete (connex) $\forall x, y\ x\mathcal{R}y\ or\ y\mathcal{R}x$

Examples:

- $\geq\ in\ \mathbb{R}\ \mathbb{Z}\ \mathbb{N}$ is reflexive, transitive, Antisymmetic complete But
- $\geq\ in\ \mathbb{R}^2$ is not complete : (1,2) and (0,4) are neither $(1, 2) \geq (0, 4)$ or $(0, 4) \geq (1, 2)$
- $A\mathcal{R}B \Leftrightarrow \forall a \in A,\ \exists b\ b \geq a$ it is reflexive but if $\forall b\ b \geq a$ it is not reflexive
- $\mathcal{R} \in \mathbb{N} \times \mathbb{N}\ (a, b)\mathcal{R}(c, d)\ \Leftrightarrow ad = bc$ it is equivalence

| | Reflexive | Transitive | Anti-symmetric | Symmetric | Complete |
|---|---|---|---|---|---|
| Equivalence | Y | Y | Y | | |
| Pre-Order | Y | Y | | | |
| Partial-Order | Y | Y | | Y | |
| Order | Y | Y | | Y | Y |

**Definition 10.** Quotient Sets
Given an equivalence relation $\sim$ the equivalence class of $x \in X$ is $[x] = \{y \in X,\ y \sim x\}$

examples
$(x, y)\mathcal{R}(x', y')\ \Leftrightarrow\ x + y = x' + y'\ [(5, 2)] = \{(x, y) :\ x + y = x' + y'\}$

**Definition 11.** Upper Bound and Least Upper Bound
An upper bound for a subset $A \subseteq X$ is an $x \in X\ s.t.\ x \geq a,\ \forall a \in A$
Given subset $A \subseteq X$, the supremum of A is the lease upper bond of A

## 2.2 Numbers

The set $\mathbb{N}$ of natural numbers consists of a starting element 1, the successor of 1 (denoted as 2), the successor of the successor of 1 (denoted as 3), and so on. Shortly put, $\mathbb{N} := \{1, 2, 3, ...\}$. However, be aware that the notation is not universal. Some books define natural numbers $\mathbb{N} := \{0, 1, 2, 3, ...\}$, but here we stick to the definition of $\mathbb{N} := \{1, 2, 3, ...\}$.
The set $\mathbb{Z}$ of integers consists of natural numbers and their negative counterparts, as well as a neutral element denoted as 0, i.e. $Z := \{..., -2, -1, 0, 1, 2, ...\}$.
The set $\mathbb{Q}$ of rational numbers consists of ordered pairs $(m, n)$ of integers, and treats $(m, n)$ and $(m', n')$ as the same element iff $m \cdot n' = m' \cdot n$. $\mathbb{Q} := \{m/n : m, n \in \mathbb{Z},\ m, n \textit{ coprime}\}$[1].

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

**Construction of Real Number** In real number, we have the following important axiom

**Axiom** $(R, \leq)$ has the least upper bound property[2].

**Definition 12.** A model for the real number system is a tuple $(\mathbb{R}, 0, 1, +, X, \leq)$ such that:

1. $\mathbb{R}$ together with $+, \times$ is a field, and $0, 1$ are the additive and multiplicative identies, respectively.
2. $(\mathbb{R}, \leq)$ is a totally ordered set;
3. $x \leq y, z \in \mathbb{R} \ \Rightarrow x + z \leq y + z$;
4. $0 \leq x, 0 \leq y \Rightarrow 0 \leq xy$;
5. (Order completeness) For every $A \subset \mathbb{R}$, if it has an upper bound, it has a least upper bound.

Remark. The model for the real number system is unqiue up to an "isomorphism" (i.e., a bijection that preserves the two identities, two operations, and the total order). We conveniently pick the one that "extends" $\mathbb{Q}$ (i.e., $\mathbb{Q} \subset \mathbb{R}$ and $+, \times, \leq$ perform the same way on $\mathbb{Q}$). This definition is equivalent to defining real numbers as equivalence classes of Cauchy sequences in $\mathbb{Q}$. Indeed, the last condition in the above definition ensures the completeness of the real numbers as a metric space.

**Complex numbers** The set of complex numbers $\mathbb{C}$ is at its core just the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the set of all ordered pairs of $\mathbb{R}$. Instead of noting $z = (a, b) \in \mathbb{R}^2$, we note $z = a + ib$, where $i$ is the imaginary unit. We call a the real part, noted $Re(z)$, and $b$ the imaginary part of $a + ib$, noted $Im(z)$. We see the real line as the subset of $\mathbb{C}$ whose numbers have imaginary part equal to zero. The set really becomes $\mathbb{C}$ once we define operations on it: an addition and

[1] $m, n$ *coprime* if the only positive integer that divides both $m$ and $n$ is 1.

[2] the $\leq$ indicates the order relation in the real number domain. The real numbers are complete in the sense that every set of reals which is bounded above has a least upper bound and every set bounded below has a greatest lower bound. The rationals do not have this property because there is a "gap" at every irrational number.

a multiplication, like on $\mathbb{R}$. The operations are defined so that they generalize the operations on $\mathbb{R}$.

- The addition: $(a+ib)+(c+id)=(a+b)+i(b+d)$
- The multiplication: we define $i^2=-1$ so that $(a+ib)(c+id)=(ac-bc)+i(bd+ad)$.

We define the conjugate of a complex number $z=a+ib$ to be $\bar{z}=a-ib$. We define the modulus of a complex number $z=a+ib$ to be $|z|=\sqrt{a^2+b^2}$. The modulus extends the notion of absolute value $|x|=\max(x,-x)$ on $\mathbb{R}$. We have that $z\bar{z}=|z|^2$.

## 2.3 Countability and Cardinality

It is natural to use the number of elements in a set to get an idea about the size of it. However, this approach does not work if the set has infinitely many elements, and we need a more sophisticated approach.

**Definition 13.** A set $X$ is countably infinite iff there exists a bijection between $\mathbb{N}$ and $X$.

The countable infinite sets are the "smallest" infinite sets.

*Claim* 4. If $X$ is countably infinte, then any infinte subset $Y\subset X$ is also countably infinite.

**Definition 14.** A set $X$ is **countable** iff it is either finite or countably infinite. A set $X$ is uncountable iff it is not countable.

Intuitively, uncountable sets are infinite sets which we can not find a bijection from $\mathbb{N}$ to. Therefore, they are considerably "larger" than countably infinite sets.

**Theorem 2.** *The countable union of countable sets is countable.*

**Corollary 1.** *$Q$ is countable.*

*Proof.* * Partition $\mathbb{Q}$ by the value of the denominators of fractions: $\mathbb{Q}=\cup_{q\in\mathbb{N},q\neq 0}Q_p$ with $Q_p=\{\frac{p}{q},p\in\mathbb{N}\}$ . Each $Q_p$ is countable, so $\mathbb{Q}$ is a countable union of countable sets. □

**Theorem 3.** *The cartesian product of finitely many countable sets is countable.*

*Claim* 5. $\mathbb{R}$ is uncountable.

**Definition 15.** Let $S$ and $T$ two sets.

- $S$ and $T$ have equal cardinality if there exists a bijection between them.
- $S$ has higher cardinality than $T$ if there exists an injection from $T$ onto $S$.
- $S$ has strictly higher cardinality than $T$ if it has a higher but not equal cardinality than $T$.

**Proposition 1.** $\mathbb{R}^n$ *(and so* $\mathbb{C}$*) has the same cardinality as* $\mathbb{R}$*.*

# 3 Spaces

## 3.1 Metric Spaces

**Definition 16.** Let $X$ be a set. A function $d : X^2 \to \mathbb{R}_+$ is a **distance function**, or **metric,** on $X$ iff it satisfies the following properties

1. (Positivity) for any $x, y \in X$, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;

2. (Symmetric) for any $x, y \in X$, $d(x, y) = d(y, x)$;

3. (Triangle inequality) for any $x, y, z \in X$, $d(x, y) \leq d(y, z) + d(z, y)$

A metric space $(X, d)$ is a set $X$ equipped with a metric $d$.

**Example.** 1

1. Finite dimensional real space $(\mathbb{R}^l, d)$ is a metric space with the Euclidean distance $d(x, y) \coloneqq \left\{ \sum_{i=1}^{l} (x_i - y_i)^2 \right\}^{1/2}$.

2. Let $a < b$ be real numbers. $C_b([a, b])$, the set of real-valued continous (and bounded) functions on $[a, b]$ is ametric space, once we define the distance between $f, g \in C_b([a, b])$ as $d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$.

Elements of a metric space are often called **points**. Note that the distance function d is an inseparable part of a metric space. $(X, d_1)$ and $(X, d_2)$ are two different metric spaces if $d_1$ and $d_2$ are two different distance functions. We can denote a metric space simply as $X$ only when there is no ambiguity regarding what distance function is used. As an example, the set $\mathbb{R}^k$ can be endowed with a natural metric, the **Euclidean distance** function $d_2$, defined as

$$d_2(x, y) \coloneqq \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

for any $x, y \in \mathbb{R}^k$. This distance function is natural in sense that it conssitent with how we understand “distance” in our real life where $k = 3$. Once can easily verify that $d_2$ satisfies properties $(1), (2), (3)$. The metric space $(\mathbb{R}^k, d_2)$ is often called a Euclidean space.
Examples of other metrics on $\mathbb{R}^k$:

1. $d_n$ metric:

   $$d_n(x, y) \coloneqq \left( \sum_{i=1}^{k} |x_i - y_i|^n \right)^{\frac{1}{n}}$$

   where $k$ can be any positive integer. This subsumes the Euclidean distance $d_2$ as a special case. Notice that when $k = 1$, all $d_n$'s reduce to the same absolute distance function $d(x, y) \coloneqq |x - y|$.

2. discrete metric:

$$d(x,y) := \begin{cases} 0, & \text{if } x \neq q \\ 1, & \text{if } x \neq q \end{cases}$$

Given a metric space $(X, d)$ and a subset $S \subset X$, we can restrict ourselves to the subset $S$ to get a smaller metric space, still using the distance function defined on the larger space $X$. Formally, define a new distance function $d|_S : S^2 \to \mathbb{R}_+$ as $d|_S(x, y) := d(x, y)$ for any $x, y \in S$. That is, $d|_S$ is the original distance function $d$ restricted in the subset $S$. Clearly, $d|_S$ is a valid metric on $S$, and so $(S, d|_S)$ is a valid metric space. Sometimes, we write the metric space $(S, d|_S)$ as $(S, d)$ for simplicity, but keep in mind that rigorously speaking the new metric $d$ in $(S, d)$ has a different domain from the original metric.

## 3.2 Topological Spaces

An alternative method is to define "neighborhood", i.e., a list of elements who are "close" to a given element. This structure is weaker than metric, yet strong enough to study convergence.

**Definition 17.** A topology on a set $X$ is a family $\mathcal{T}$ of subsets, called opens, satisfying:

1. $X \in \mathcal{T}$ , $\emptyset \in \mathcal{T}$;
2. (Closedness under finite intersections) If $E_1, E_2 \in \mathcal{T}$, then $E_1 \cap E_2 \in \mathcal{T}$.
3. (Closedness under unions) If $\{E_\alpha\}_{\alpha \in \mathcal{I}} \subset \mathcal{T}$, then $\cup_{\alpha \in \mathcal{I}} E_\alpha \in \mathcal{T}$, where $\mathcal{I}$ is some index set.

A topological space $(X, \mathcal{T})$ is a set $X$ equipped with a topology $\mathcal{T}$.

**Example.** The following topologies can be de ned on any nonempty set X:

1. (Trivial topology) $\mathcal{T}_1 = \{\emptyset, X\}$;
2. (discrete topology) $\mathcal{T}_2 = \mathcal{P}(X)$.

It should be apparent that metric spaces are topological spaces. But not all topological spaces are compatible with a metric. Below is a trivial (counter-)example.
Let $X = \{a, b\}$; and $\mathcal{T} = \{\emptyset, \{a\}, X\}$. $(X, \mathcal{T})$ is a topological space, but it is not compatible with any metric (just check the definition of metric space).

**Definition 18.** (open ball) Let $(X, d)$ be a metric space. The open ball centered at $x \in X$ with radius $r > 0$ is defined as the set

$$B_r(x) := \{z \in X : d(z, x) < r\}$$

Keep in mind that an open ball $B_r(x)$ depends both on the whole space $X$ and the distance function $d$. For example, in the metric space $(\mathbb{R}, d_2)$, the open ball $B_1(0) = (-1, 1)$; however, in $(\mathbb{R}_+, d_2)$, the open ball $B_1(0) = [0, 1)$. If we use the discrete metric $d$, then in $(\mathbb{R}, d)$ the open ball $B_1(0) = \{0\}$. Therefore, when we write down the notation for an open ball like $B_1(0)$, we have to be clear about which metric space we are working with.

**Definition 19.** Let $(X, d)$ be a metric space and $S$ be a subset of $X$. The set $S$ is said to be bounded iff there exists $x \in X$ and $r > 0$ s.t. $B_r(x) \supset S$.

That is, a set $S$ is bounded iff we can bound it using an open ball.

**Closed and Open Sets** Let $(X, \mathcal{T})$ be a given topological space. Closed sets are naturally de ned as complements of opens, and some corollaries are immediate from the De Morgan's law.

**Definition 20.** Let $(X, d)$ be a metric space, and S a subset of $X$
A point $x \in X$ is an interior point of $S$ iff $\exists r > 0,\ s.t.\ B_r(x) \subset S$. The set of interior points of S is denoted as $int(S)$.
The set $S$ is an open set iff $S \subset int(S)$, i.e., all points in $S$ are interior points.

Clearly, any interior point of $S$ is a point in $S$, and therefore, $S \subset int(S)$ is equivalent to $S = int(S)$. By definition, the empty set $\emptyset$ is open, because $int(\emptyset) = \emptyset$. Also, the whole space $X$ is also open, because $int(X) = X$. Keep in mind that whether a set is open depends on the metric space. For example, $[0, 1)$ is not an open set in $(\mathbb{R}, d_2)$, since 0 is not an interior point. However, $[0, 1)$ is an open set in $(R_+, d_2)$, since 0 is not an interior point. However, $[0, 1)$ is an open set in $(\mathbb{R}_+, d_2)$. The point 0 becomes an interior point of $[0, 1)$ because an open ball centered at 0 is now $B_r(0) = [0, r)$ instead of $(-r, r)$. As its name suggests, an open ball is open.

*Claim* 6. In metric space $(X, d)$, any open ball is an open set.

*Proof.* Take any open ball $B_r(x)$ in the metric space, and take any point $z \in B_r(x)$. let $\epsilon := r - d(z, x)$. First, because $z \in B_r(z)$, we have $d(z, x) < r$ and thus $\epsilon > 0$. Second, take any $y \in B_\epsilon(z)$, we have

$$d(y, x) \leq d(y, z) + d(z, x) < \epsilon + d(z, x) = r$$

and therefore $y \in B_r(x)$. □

Note that open intervals are open in $(R, d_2)$, because they are special cases of open balls. The next proposition is an important property of open sets. It states that an arbitrary union of open sets is open, and that a finite intersection of open sets is also open.

**Proposition 2.** *In metric space $(X, d)$:*

1. *Let $\{E_\alpha\}_{\alpha \in A}$ be an arbitrary family of open sets (potentially uncountably many of them). Then their union $\cup_{\alpha \in A} E_\alpha$ is also open.*

2. *Let $\{E_i\}_{i=1}^n$ be a finite family of open sets. Then their intersection $\cap_{i=1}^n E_i$ is also open.*

*Proof.* So

1. Take any $x \in \cup_{\alpha \in A} E_\alpha$, we need to find $r > 0$ s.t. $B_r(x) \subset \cup_{\alpha \in A} E_\alpha$. By definition of union, $\exists \hat{\alpha} \in A$ s.t. $x \in E_{\hat{\alpha}}$. Because $E_{\hat{\alpha}}$ is open, we can find $r > 0$ s.t. $B_r(x) \subset E_{\hat{\alpha}}$ . This is an $r$ we need to find because $B_r(x) \subset E_{\hat{\alpha}} \subset \cup_{\alpha \in A} E_\alpha$.

2. Take any $x \in \cap_{i=1}^n E_i$, we need to find $r > 0$ s.t. $B_r(x) \subset \cap_{i=1}^n E_i$. By definition of intersection, $x \in E_i$ for any $i = 1, 2, ..., n$. For each $i$, because $E_i$ is open, $\exists r_i > 0$ s.t. $B_{r_i}(x) \subset E_i$. Let $r := \min\{r_1, r_2, ..., r_n\}$ and this is an $r$ we need to find. First, clearly $r > 0$. Second, $B_r(x) \subset B_{r_i}(x) \subset E_i$ for any $i$, and therefore $B_r(x) \subset \cap_{i=1}^n E_i$.

$\square$

Note that an infinite intersection of open sets may not be open. For example, consider $E_n = (-1/n, 1/n)$ , and we have $\cap_{n=1}^{+\infty} E_n = \{0\}$.

**Definition 21.** Let $(X, d)$ be a metric space, and $S$ a subset of $X$.
A point $x \in X$ is a limit point of $S$ iff $(B_r(x) \backslash \{x\}) \cap S \neq \emptyset,\ \forall r > 0$. The set of limit points of $S$ is denoted as $S'$.
The set $S$ is a closed set iff $S \supset S'$, i.e. $S$ contains all of its limit points.

The condition $(B_r(x) \backslash \{x\}) \cap S \neq \emptyset,\ \forall r > 0$ states that the open ball $B_r(x)$ with the center removed always contains some points in the set $S$, no matter how small the radius $r$ is. That is, a point $x$ is a limit point of $S$ iff we can use points in $S$ to approximate $x$ arbitrarily well (the point $x$ itself may be a point in $S$, but we are not allowed to use $x$ to approximate itself). Notice that not every point in $S$ is necessarily a limit point, and so $S \supset S'$ is not equivalent to $S = S'$. For example in $(\mathbb{R}, d_2)$, the "isolated" point 2 in the set $S' = [0, 1] \cup \{2\}$ is not a limi point of $S$. The set $S$ is indeed closed by definition, since $S' = [0, 1]$ which is a proper subset of $S$.
By definition, the empty set $\emptyset$ is closed, because $\emptyset' = \emptyset$. Also, the whole space $X$ is also closed, because a limit point of $X$, by definition, must be a point in $X$ in the first place. Keep in mind that whether a set is closed depends on the metric space. For example, $(0, 1]$ is not a closed set in $(\mathbb{R}, d_2)$, since it does not contain its limit point 0. However, $(0, 1]$ is a closed set in $(\mathbb{R}_{++}, d_2)$. The point 0 is no longer a limit point of $(0, 1]$, since it is not even a point because it is not in the metric space.
The following proposition establishes two characterizations of closed sets.

**Proposition 3.** *Let $(X, d)$ be a metric space, and $S$ a subset of $X$. Then the following three statements are equivalent.*

1. *$S$ is a closed set.*
2. *(sequential definition) For any sequence $(x_n)$ in $S$ convergent to some point $x \in X$, we have $x \in S$.*
3. *(topological definition) $S^c$ is an open set.*

A corollary of (1) $\Leftrightarrow$ (3) is that $S$ is an open set iff $S^c$ is a closed set. Simply put, the complement of an open set is closed, and the complement of a closed set is open. For closed set, an arbitrary intersection of closed sets is closed, and that a finite union of closed sets is also closed.

**Proposition 4.** *In metric space $(X, d)$:*

1. *Let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary family closed sets (potentially uncountably many of them). Then their intersection $\cap_{\alpha \in A} F_\alpha$ is also closed.*
2. *Let $\{F_i\}_{i=1}^n$ be a finite family of closed sets. Then their union $\cup_{i=1}^n F_i$ is also closed.*

As a final note, again keep in mind that open sets and closed sets are not "absolute" concepts. They rely on the metric space we are working with. When there is ambiguity regarding which metric space we are using, we have to be explicit about it by saying "set S is open/closed in the metric space $(X, d)$" instead of simply saying "$S$ is open/closed". Also, notice that under discrete metric, all sets in the metric space are both open and closed (exercise).

| | $[0, +\infty)$ | $(0, +\infty)$ | $1/n : n \in N$ |
|---|---|---|---|
| in $(\mathbb{R}, d)$ | not open, but closed | open, not closed | not open, not closed |
| in $(\mathbb{R}_+, d)$ | open and closed | open, not closed | not open, not closed |
| in $(\mathbb{R}_{++}, d)$ | NA | open and closed | not open, but closed |

# 4 Convergence

**Definition 22.** Let $X$ be a set. The function $x : \mathbb{N} \to X$ is called a sequence in $X$.

Sequences are simply a special case of functions. The value of the function x evaluated at $1, x(1)$, is called the first **term** of the sequence, and the value of the function evaluated at $2, x(2)$, is called the second term, and so on. By convention, we often use subscripts and write $x1, x2, ...$ instead of $x(1), x(2), ...$, and the whole sequence is often denoted as $(x_n)$ instead of $x$.
Note that there is no distance function involved in the definition above, since we don't need a concept of distance to talk about sequences. However, we do need distance to talk about convergence.

**Definition 23.** Let $(X, d)$ be a metric space. A sequence $(X_n)$ in $X$ is said to **converge** to a point $x \in X$, iff $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n > N$. When the sequence $(x_n)$ converges to $x$, the point $x$ is called a limit of the sequence $(x_n)$, and we use the notation $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Notice that the requirement $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B_\epsilon(x)$. Another way to describe convergence is that the sequence $(x_n)$ will eventually go into the open ball $B_\epsilon(x)$, no matter how small the ball is. The next claim establishes that the limit of a convergent sequence must be unique, and therefore it makes sense to talk about "the" limit of a convergent sequence

*Claim* 7. Let $(X, d)$ be a metric space. Suppose $x_n \to x$ and $x_n \to x'$, then $x = x'$.

*Proof.* We prove this claim by contradiction. Suppose $x \neq x'$. By property (1) of $d$, we have $d(x, x') > 0$. Let $\epsilon := d(x, x')/2$. Because $x_n \to x$, there exists $N$ s.t. $d(x_n, x') < \epsilon$ for any $n > N$. Because $x_n \to x'$, there exists $N'$ s.t. $d(x_n, x') < \epsilon$ for any $n > N'$. Let $\hat{n} := \max\{N, N'\} + 1$, and so we have $\hat{n} > N$ and $\hat{n} > N'$. Therefore, we have $d(x_{\hat{n}}, x) < \epsilon$ and $d(x_{\hat{n}}, x') < \epsilon$, and thus

$$d(x_{\hat{n}}, x) + d(x_{\hat{n}}, x') < 2\epsilon = d(x, x')$$

where contradicts triangle inequality of $d$. Therefore we must have $x = x'$. □

A sequence is said to be **bounde**d iff its range $x_1, x_2, ...$ is a bounded set. The next claim establishes that a convergent sequence must be bounded.

*Claim* 8. Let the limit of $(x_n)$ be a metric space. If $(x_n)$ is a convergent sequence in $X$, then $(x_n)$ must be bounded.

*Claim* 9. In $(R, d_2)$, let there be two convergent sequences $x_n \to x$ and $y_n \to y$. If $x_n \leq y_n$ for any $n \in N$, then $x \leq y$.

**Proposition 5.** *Let $(x_n)$ be a sequence in $(\mathbb{R}^k, d_2)$. The sequence $(x_n)$ converges to $x \in \mathbb{R}^k$ iff the sequence $(x_n^i)$ converges to $x^i$ in $(R, d_2)$ for any $i \in 1, 2, ..., k$.*

Here we use superscript to index coordinates of vectors, since we have used subscript to index terms of the sequences.

**Proposition 6.** *In $(R, d_2)$, let there be two convergent sequences $x_n \to x$ and $y_n \to y$. Then*

1. $x_n + y_n \to x + y$ ,
2. $x_n y_n \to xy$, *and*
3. *if* $x \neq 0$, *then* $1/x_n \to 1/x$

**Corollary 2.** *In $(R, d_2)$, let there be two convergent sequences $x_n \to x$ and $y_n \to y$. Then*
*(1)* $x_n - y_n \to x - y$,
*(2) if* $y \neq 0$, *then* $x_n/y_n \to x/y$.

Because sequences are special cases of functions, the definition of monotonicity of functions applies to sequences. A sequence is increasing iff xm  xn for any m  n; is decreasing iff xm  xn for any m  n; is monotone iff it is increasing or decreasing. We can also define the strict versions of increasing/decreasing sequences in the natural way as we did for functions.

**Theorem 4.** *(Monotone Convergence Theorem). Every increasing (decreasing) and bounded from above (below) real sequence $(x_n)$ is convergent in $(R, d_2)$.*

Notice that an increasing/decreasing sequence is automatically bounded from below/ above by its first term. Therefore, the theorem can also be stated that every monotone and bounded real sequence is convergent in (R, d2).

**Lemma 1.** *Every sequence in $R$ has a monotone subsequence.*

*Proof.* Take any sequence $(x_n)$ in $\mathbb{R}$. Call the term $x_n$ a dominant term if $x_n \geq x_m$ for any $m \geq n$.
Case 1: $(x_n)$ has infinitely many dominant terms Then these dominant terms constitute a decreasing subsequence.
Case 2: $(x_n)$ has finitely many dominant terms Let $x_N$ be the last dominant term. Let $n_1 = N + 1$, and so $x_{n1}$ is not a dominant term. By definition, there exists $n_2 > n_1$ s.t. $x_{n2} > x_{n1}$ . The term $x_{n2}$ itself is not dominant either, and so there exists $n_3 > n_2$ s.t. $x_{n3} > x_{n2}$ ... Therefore, we obtain a strictly increasing subsequence.
Case 3: $(x_n)$ has no dominant term Let $n_1 = 1$, and construct a strictly increasing subsequence as in Case 2. Therefore, we can always find a monotone subsequence. □

**Theorem 5.** *(Bolzano-Weierstrass). Every bounded sequence in $(\mathbb{R}^k, d_2)$ has a convergent subsequence.*

# 5 Continuity of functions and correspondences

**Definition 24.** Let $X, Y$ be topological spaces, $f : X \to Y$, and $x \in X$. $f$ is continuous at $x$ if for every open neighborhood $V \subset Y$ of $f(x)$, there exists an open neighborhood $U \subset X$ of $x$ such that $U \subset f^{-1}(V)$. $f$ is continuous if it is continuous at every $x \in X$.

That is, a function $f : X \to Y$ is continuous at a point $x \in X$ if the pre-image of every neighborhood of $f(x)$ is a neighborhood of $x$. The following definition

**Proposition 7.** *Let $X, Y$ be topological spaces, and $f : X \to Y$. $f$ is continuous if every open set in $Y$ has a pre-image that is open in $X$.*

Sequential Continuity

**Definition 25.** Let $X, Y$ be topological spaces, and $f : X \to Y$, $x \in X$. $f$ is sequentially continuous at $x$ if $f(x_n) \to f(x)$ in $Y$ for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x$ in $X$. $f$ is sequentially continuous if it is sequentially continuous at all $x \in X$.

**Proposition 8.** *Continuous functions are sequentially continuous.*

*Proof.* Let $f : X \to Y$ a continuous function, $x \in X$, and $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$. Take any open $E$ in $Y$ such that $f(x) \in E$. By assumption, $f^{-1}(E)$ is open in X. Therefore, there exists $N > 0$ such that $x_n \in f^{-1}(E)$ for all $n \geq N$. Hence, $f(x_n) \in E$ for all $n \geq N$, i.e. $f(x_n) \to f(x)$. $\square$

For metric space, these two definitions coincide. Since open balls form a basis of topology in metric spaces, $x_n \to x$ can also be stated in the usual $\epsilon - \delta$ way, i.e. $(\forall \epsilon > 0)(\exists N > 0)(\forall n \geq N)(d(x, x_n) < \epsilon)$.

**Proposition 9.** *Let $X, Y$ be metric spaces, and $f : X \to Y$. $f$ is continuous if and only if it is sequentially continuous.*

Correspondences and Hemicontinuity

**Definition 26.** Let $X, Y$ be nonempty sets. A correspondence $\Gamma$ from $X$ to $Y$, denoted by $\Gamma : X \twoheadrightarrow Y$, is a function from $X$ to $2^Y$, i.e., $\Gamma(x) \subset Y$ for every $x \in X$.

**Definition 27.** Let $\Gamma : X \twoheadrightarrow Y$, and $E \subset Y$.

- The upper inverse of $E$ is

$$\bar{\Gamma}^{-1}(E) = \{x \in X : \Gamma(x) \subset E\};$$

- The lower inverse of $E$ is

$$\underline{\Gamma}^{-1}(E) = \{x \in X : \Gamma(x) \cap \neq \emptyset\}$$

Similarly to continuity of functions, we can define upper and lower hemicontinuity for correspondences using upper inverse and lower inverse, respectively.

**Definition 28.** Let $X, Y$ be topological spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. $\Gamma$ is upper hemicontinuous at $x$ if for every open set $V \subset Y$ such that $\Gamma(x) \subset V$, there exists an open neighborhood $U \subset X$ of $x$ such that $U \subset \bar{\Gamma}^{-1}(V)$. $\Gamma$ is lower hemicontinuous at $x$ if for every open set $V \subset Y$ such that $U \subset \underline{\Gamma}^{-1}(V) \cap V \neq \emptyset$, there exists an open neighborhood $U \subset X$ of $x$ such that $U \subset \underline{\Gamma}^{-1}(V)$. $f$ is upper (lower) hemicontinuous if it is upper (lower, respectively) hemicontinuous at every $x \in X$. A correspondence is continuous if it is both upper and lower hemicontinuous.

That is, a correspondence $\Gamma : X \twoheadrightarrow Y$ is upper hemicontiuous at a point $x \in X$ if the upper inverse of every open neighborhood of $\Gamma(x)$ is a neighborhood of $x$; it is lower hemicontinuous at a point $x \in X$ if whenever $x$ is in the lower inverse of an open set, so is every element in a neighborhood of $x$.

**Proposition 10.** *Let $X, Y$ be topological spaces, and $\Gamma : X \twoheadrightarrow Y$. $\Gamma$ is upper hemicontinuous if and only if the upper inverse of every open set in $Y$ is open in $X$. $\Gamma$ is lower hemicontinuous if and only if the lower inverse of every open set in $Y$ is open in $X$.*

Intuitively, lower hemicontinuous correspondences can "shrink" but not "explode", while upper hemicontinuous correspondences can "explode" but not "shrink". The following graph contains a correspondence $\mathbb{R} \to \mathbb{R}$ that is upper but not lower hemicontinuous at $x_1$, and lower but not upper hemicontinuous at $x_2$.

**Proposition 11.** *Let $X, Y$ be metric spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. $\Gamma$ is lower hemicontinuous at $x$ if and only if for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$ and all $y \in \Gamma(x)$, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Gamma(x_n)$ for all $n$, and $y_n \to y$.*

**Proposition 12.** *Let $X, Y$ be metric spaces, $\Gamma : X \twoheadrightarrow Y$, and $x \in X$. If for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ such that $x_n \to x$, $y_n \in \Gamma(x_n)$ for all $n$, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ that converges to some $y \in \Gamma(x)$, then $\Gamma$ is upper hemicontinuous at $x$. If $\Gamma(z)$ is a compact subset of $Y$ for all $z$ in a neighborhood of $x$, then the converse is also true.*

Semincontinuity

**Definition 29.** Let $X$ be a topological space, $\Gamma : X \to \mathbb{R}$ a real-valued function on $X$, and $x \in X$. $f$ is upper semicontinuous at $x$ if for every $\epsilon > 0$, there exits a neighborhood $U \subset X$ of $x$ such that $f(z) \leq f(x) + \epsilon$ for all $z \in U$. $f$ is lower semicontinuous at $x$ if for every $\epsilon > 0$, there exits a neighborhood $U \subset X$ of $x$ such that $f(z) \geq f(x) - \epsilon$ for all $z \in U$. $f$ is upper (lower) semicontinuous if it is upper (lower, respectively) semicontinuous at every $x \in X$.

Intuitively, upper semicontinuous functions can "jump" but not "dip", while lower semicontinuous functions can "dip" but not "jump". The following result is also obvious from definition.

**Proposition 13.** *Let $X$ be a topological space, and $\Gamma : X \to \mathbb{R}$. $f$ is continuous if and only if $f$ is both upper and lower semicontinuous.*

**Proposition 14.** *Let $X$ be a topological space, and $\Gamma : X \to \mathbb{R}$. $f$ is upper (lower) semicontinuous if and only if every upper (lower) contour set is closed, i.e. $\{x \in X : f(x) \geq (\leq) a\}$ is closed in $X$ for all $\alpha \in \mathbb{R}$.*

# 6 Compactness

Compactness is a stronger notion than closedness. It is also a crucial concept, because compact sets have many desirable properties that closed sets don't have.

**Definition 30.** Let $(X, d)$ be a metric space, and $S$ a subset of $X$. A family of open sets $\{E_\alpha\}_{\alpha\in A}$ is an open cover of $S$ iff $\cup_{\alpha\in A} E_\alpha \supset S$.

**Definition 31.** Let $(X, d)$ be a metric space, and $S$ a subset of $X$. The set $S$ is compact iff $\forall$ open cover $\{E_\alpha\}_{\alpha\in A}$ of $S$, $\exists$ a finite $B \subset A$ s.t. $\{E_\alpha\}_{\alpha\in B}$ is also an open cover of $S$.

To illustrate the definition of compact sets, let's verify that the open interval $(0, 1)$ is not a compact set in $(R, d_2)$. To do this, it is sufficient to provide an open cover that does not have a finite subcover. Consider the family of open sets $\{(1/n, 1 - \frac{1}{n})\}_{n=1}^{+\infty}$ . This covers $(0, 1)$ because any point strictly between 0 and 1 will be eventually covered by $(1/n, 1-1/n)$ when n is large enough. There is no finite subcover, since any finite family of $(1/n, 1-1/n)$ has a largest one, and it does not cover $(0, 1)$. By definition, the concept of compactness relies on the metric space we are working with, just like openness and closedness. A set can be compact in one metric space, but not in another metric space. However, compactness behaves much better than openness and closedness, in the sense that enlarging or shrinking the whole space does not affect compactness as long as we use the same metric. This result is formulated below.

**Proposition 15.** *Let $(X, d)$ be a metric space, and $S \subset Y \subset X$. Then $S$ is compact in $(X, d)$ iff $S$ is compact in $(Y, d)$.*

If we let $Y \coloneqq S$, we have that $S$ is compact in $(X, d)$ iff $S$ is compact in $(S, d)$. A metric space $(X, d)$ is said to be a **compact metric space** iff $X$ is a compact set in $(X, d)$. So we know that if S is compact in $(X, d)$, then $(S, d)$ itself is a compact metric space, and vice versa.

**Theorem 6.** *Let $(X, d)$ be a metric space, and $S$ a subset of $X$. If $S$ is compact in $(X, d)$, then $S$ is closed in $(X, d)$.*

The above theorem illustrates that compactness is stronger than closedness.

**Theorem 7.** *Let $(X, d)$ be a metric space, and $S$ a subset of $X$. If $S$ is compact in $(X, d)$, then $S$ is bounded in $(X, d)$.*

Combining the two aforementioned Theorems, we conclude that a compact set must be **closed and bounded**. The next theorem provides a way to prove compactness. It states that a closed set contained in a compact set is also compact. Therefore, in order to show that $S$ is compact, we can instead show that $S$ is closed and that some other set containing $S$ is compact.

**Theorem 8.** *Let $(X, d)$ be a metric space, and $S \subset Y \subset X$. If $S$ is closed in $(X, d)$ and $Y$ is compact in $(X, d)$, then $S$ is compact in $(X, d)$.*

The above discussions apply to general metric spaces. Now the next part s dedicated to Euclidean spaces $(\mathbb{R}^k, d_2)$, and establishes more results.

**Heine-Borel Theorem in** $(\mathbb{R}^k, d_2)$ In general metric spaces, we have shown that a compact set must be closed and bounded. In Euclidean spaces $(\mathbb{R}^k, d_2)$, the reverse is also true, i.e. a closed and bounded set in $(\mathbb{R}^k, d_2)$ must be compact. Therefore, in Euclidean spaces, compactness is equivalent to closedness plus boundedness, and this equivalence is known as Heine-Borel theorem. We establish this result in several steps.

**Lemma 2.** *Any closed interval* $[a, b]$ *is compact in* $(\mathbb{R}, d_2)$ *.*

Now we extend the lemma to $(\mathbb{R}^k, d_2)$

**Lemma 3.** *Every k-cell* $[a_1, b_1] \times [a_2, b_2] \times ... \times [a_k, b_k]$ *is compact in* $(\mathbb{R}^k, d_2)$*.*

Now let's consider a closed and bounded set S in $(\mathbb{R}^k, d_2)$. By definition of boundedness, S can be bounded by an open ball. Clearly, an open ball in $(\mathbb{R}^k, d_2)$ can be bounded by a k-cell. So $S$ is a subset of a k-cell, which is compact in $(\mathbb{R}^k, d_2)$ by the lemma above. Furthermore, because $S$ is assumed to be closed in $(\mathbb{R}^k, d_2)$, by Theorem 8, we know that $S$ is compact itself. Therefore we have proved the following theorem.

**Theorem 9.** *In* $(\mathbb{R}^k, d_2)$*, a closed and bounded set* $S$ *must be compact.*

Now we can havve the Heine-Borel Theorem.

**Theorem 10.** *In* $(\mathbb{R}^k, d_2)$*, a set* $S$ *is compact iff it is closed and bounded.*

Heine-Borel theorem states that in $(\mathbb{R}^k, d_2)$, to check whether a set is compact, we can instead check whether the set is closed and bounded. This greatly simplifies our job, since the definition of compactness involving open covers is not easy to check in most cases. Keep in mind that Heine-Borel theorem only works in Euclidean spaces $(\mathbb{R}^k, d_2)$. In general metric spaces $(X, d)$, although compactness always implies closedness plus boundedness, the reverse is not true in general. For example, in $(\mathbb{R}_{++}, d_2)$, the set $(0, 1]$ is closed and bounded, but not compact (consider the open cover $(1/n, +\infty)_{n=1}^{\infty}$). For another example, in $(\mathbb{R}, d)$, where $d$ is the discrete metric, the set $[0, 1]$ is closed and bounded, but not compact. In fact, under the discrete metric, a set is compact iff it is finite (exercise).

**Sequential Compactness** Let $(X, d)$ be a metric space, and $S$ a subset of $X$. The set $S$ is **sequentially compact** iff any sequence $(x_n)$ in $S$ has a subsequence convergent to some $x^* \in S$.

**Theorem 11.** *Let* $(X, d)$ *be a metric space, and* $S$ *a subset of* $X$*. The set* $S$ *is compact iff it is sequentially compact.*

## 6.1 Cauchy Sequences and Completeness

**Definition 32.** In metric space $(X, d)$, a sequence $(x_n)$ is a **Cauchy sequence** iff $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$d(x_m, x_n) < \epsilon$$

for any $m, n > N$.

**Proposition 16.** *In metric space $(X, d)$, a convergent sequence $(x_n)$ is a Cauchy sequence.*

However, a Cauchy sequence may fail to be convergent, for example the sequence $(1/n)$ in $(\mathbb{R}_{++}, d_2)$. If the metric space $(X, d)$ has the property that every Cauchy sequence converges, then we call it a complete metric space. The metric space $(\mathbb{R}_{++}, d_2)$ is not complete.

**Definition 33.** Let $(X, d)$ be a metric space, and $S$ a subset of $X$. The set $S$ is a complete set iff any Cauchy sequence in S converges to a limit point in S.
A metric space $(X, d)$ is a **complete metric space** iff $X$ is a complete set in $(X, d)$.

Completeness is stronger than closedness, but weaker than compactness. A complete set $S$ must be closed. Otherwise we can find a sequence $(x_n)$ in $S$ convergent to a point outside S. Because $(x_n)$ is Cauchy, this contradicts the completeness of $S$. The next result states that a compact set $S$ must be complete.

**Proposition 17.** *Let $(X, d)$ be a metric space, and $S$ a subset of $X$. If the set $S$ is compact, then it is complete.*

**Proposition 18.** *The Euclidean space $(\mathbb{R}^k, d_2)$ is a complete metric space.*

# 7 Fixed Point Theory

We introduce three fixed point theorems. Brouwer's fixed point theorem is used to show existence of Walrasian equilibrium; Kakutani's fixed point theorem is used to show existence of Nash equilibrium; Banach's fixed point theorem (i.e. contraction mapping theorem) is used in dynamic programming. We only prove the last result, and we will return to dynamic programming later in more details.

**Theorem 12.** *(Brouwer). Let $X \subset \mathbb{R}^l$ be nonempty, compact and convex $(l < \infty)$, and $f : X \to X$ be continuous. Then $f$ has a fixed point.*

**Theorem 13.** *(Kakutani). Let $X \subset \mathbb{R}^l$ be nonempty, compact and convex $(l < \infty)$, and $\Gamma : X \twoheadrightarrow X$ be upper semicontinuous and such that $\Gamma(x)$ is nonempty and convex for all $x \in X$. Then $\Gamma$ has a fixed point.*

**Banach's fixed point theorem**
A function is called a self-map iff it maps its domain to itself, i.e. $f : X \to X$. Note that a self-map need not be surjective or injective.
A point $x^* \in X$ is called a **fixed point** of the self-map $f : X \to X$, iff $f(x^*) = x^*$. Intuitively, the fixed point x does not "move away" if we apply f to it.

**Theorem 14.** *(Contraction Mapping Theorem). Let $(X, d)$ be a complete metric space, and $f : X \to X$ a contraction. Then $f$ has a unique fixed point $x$. Further, for any $x \in X$, we have $\lim_{n\to\infty} f^n(x) = x^*$.*

The notation $f^n(x)$ means to apply $f$ to $x$ for $n$ times.

*Proof.* Let $x_1 \in X$ and define a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ recursively, such that $x_{n+1} = f(x_n)$ for all $n$. Since $f$ is a contraction mapping, $d(x_{n+1}, x_n) \leq q^{n-1} d(x_2, x_1)$ for all $n$. Hence, by the triangle inequality, for all $n \geq m \geq 1$,

$$d(x_n, x_m) \leq \sum_{k=m}^{\infty} d(x_{k+1}, x_k) = \frac{\beta^{m-1}}{1-\beta} d(x_2, x_1) \to 0$$

as $m \to \infty$. That is, $\{x_n\}_{n=1}^{\infty}$ is Cauchy. By completeness, $x_n \to x^*$ for some $x^* \in X$. Since $f$ is a contraction mapping, it must be continuous, so at $x^*$, $f(x^*) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$, i.e., $x^*$ is a fixed point of $f$. Existence is shown.
Suppose that $x^* \neq y^*$ are both fixed points of $f$, then $d(f(x^*), f(y^*)) = d(x^*, y^*) > \beta \cdot d(x^*, y^*)$. This contradicts the assumption that $f$ is a contraction mapping. Hence, uniqueness is proved. □

**Blackwell's sufficient condition for contraction mapping.**
Let $B(X)$ denote the set of bounded real-valued functions on some set $X$. It is a metric space with metric $d$ such that $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.

**Theorem 15.** *(Blackwell). Let $X \subset \mathbb{R}^d$, and $T: B(X) \to B(X)$. $T$ is a contraction mapping if the following are satisfied:*

- *(Monotonocity) For all $f, g \in B(X)$ such that $f \leq g, Tf \leq Tg$;*
- *(Discontinuing) There exists $\beta \in [0, 1)$ such that for all $f \in B(X)$ and $a \geq 0, T(f+g) \leq Tf + \beta a$, where $(f + g)(\cdot) \coloneqq f(\cdot) + a$.*