Problem Set

Guoxuan Ma

1. Let P and Q be two statements such that $P \Rightarrow Q$ is false. Find the truth values for the following statement s:

   - $\neg P \Rightarrow Q$
   - $P \vee Q$
   - $Q \Rightarrow P$

   Answer:

   It is straightforward to see that if $P \Rightarrow Q$ is false, P is true and Q is false. Then, we can learn that

   - $\neg P \Rightarrow Q$ is true since $\neg P$ is false
   - $P \vee Q$ is true since P is true
   - $Q \Rightarrow P$ is true since Q is false

2. Read the method of proof and use the mathematical induction to prove:

   Let n be a positive integer, $1 + 3 + ... + (2n - 1) = n^2$.

   Answer:

   In the base step $n = 1$, on the left hand side (LHS), we have $(2 \cdot 1 - 1) = 1$. And on the right hand side (RHS), we have $1^2 = 1$. So the equation is true for $n = 1$

   Induction step: Suppose
   $$1 + 3 + ... + (2n - 1) = n^2$$
   holds, we must show that
   $$1 + 3 + ... + (2(n + 1) - 1) = (n + 1)^2$$
   To do so, we start from the LHS
   $$\begin{aligned} LHS &= 1 + 3 + ... + (2(n + 1) - 1) \\ &= 1 + 3 + ... + (2n - 1) + (2(n + 1) - 1) \\ &= n^2 + 2n + 1 \text{ (by induction hypothesis)} \\ &= (n + 1)^2 = RHS. \end{aligned}$$
   So by mathematical induction, $1 + 3 + ... + (2n - 1) = n^2$ for all $n \in N_0$

3. Set that are numerically equivalent to N are called countable. Can you show Z is countable? (hint: define a function from N to Z)

   Without loss of generality, let us define a function from N to Z where the function can be like the following property:
   $$\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \end{pmatrix}$$
   where $\left\lfloor \frac{n}{2} \right\rfloor$ is the greatest integer less than or equal to x.

4. Use the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ and the property of the convergence, show

   (a) $\lim_{n\to\infty} a(n) = \lim_{n\to\infty} \frac{3n^2+2n-1}{n^2+n} = 3$

   (b) and what is $\lim_{n\to\infty} \frac{6n^2+8n-1}{3n^2+2n}$

   Answer:

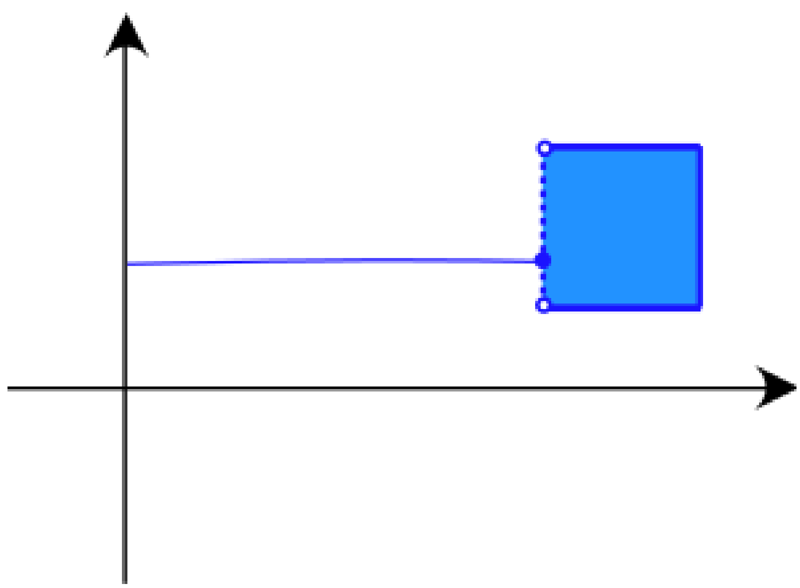
   (a) $\lim_{n\to\infty} \frac{3n^2+2n-1}{n^2+n} = 3$ it is easy to have

$$\lim_{n\to\infty} \frac{3n^2+2n-1}{n^2+n} = 3 - \frac{n+1}{n(n+1)} = 3$$

   (b) and

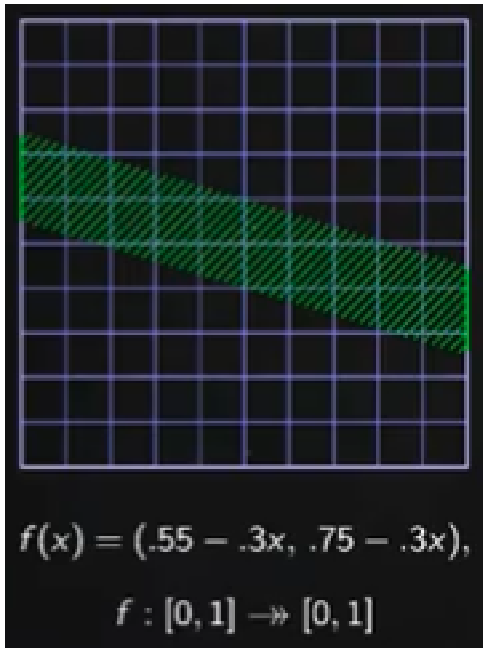
$$\begin{aligned}\lim_{n\to\infty} \frac{6n^2+8n-1}{3n^2+2n} &= 2 + \frac{4n-1}{3n^2+2n} \\ &= 2 + \frac{4-\frac{1}{n}}{3n+2} \\ &= 2\end{aligned}$$

5. In the following graph, can you tell whether it is UHC or LHC?



   It is LHC not UHC.

6. How about this one UHC or LHC



It is LHC not UHC

7. Read the related material to understand Brouwer fixed point theory
8. Calculate the following matrix operations

   (a) $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -7 & 1 & 4 & 3 \\ 2 & 3 & 1 & -1 \end{bmatrix}$

   (b) $\left| \begin{bmatrix} 4 & 1 & 0 \\ 5 & 2 & 3 \\ -1 & 2 & -3 \end{bmatrix} \right|$

   (c) The inverse of $\begin{bmatrix} -3 & 4 \\ -2 & -6 \end{bmatrix}$

   Answer:

   (a) $\begin{bmatrix} 11 & 5 & -2 & -5 \\ -13 & 15 & 16 & 5 \end{bmatrix}$

   (b) -36

   (c) $\begin{bmatrix} -\frac{6}{26} & -\frac{4}{26} \\ \frac{2}{26} & -\frac{3}{26} \end{bmatrix}$

9. Calculate the eigenvalues and eigenvectors of

$$\begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$$

answer

Using the function $|\lambda I - A| = 0$, we can get

$$\lambda(\lambda - 4)(\lambda - 2) = 0$$

Then we let

$$Av_1 = \lambda v_1$$

If $\lambda = 0$

$$\begin{aligned} 4v_1 + 6v_2 + 10V_3 &= 0 \\ 3v_1 + 10v_2 + 13v_3 &= 0 \\ -2v_1 - 6v_2 - 8v_3 &= 0 \end{aligned}$$

$$v_1 + v_3 = 0$$

So $v_1 = 1,\ v_3 = -1,\ v_2 = 1$ when $\lambda = 0$

If $\lambda = 4$

$$\begin{aligned} 4v_1 + 6v_2 + 10V_3 &= 4v_1 \\ 3v_1 + 10v_2 + 13v_3 &= 4v_2 \\ -2v_1 - 6v_2 - 8v_3 &= 4v_3 \end{aligned}$$

Let $v_3 = 1$, then $v_1 = -1$ and $v_2 = -\frac{5}{3}$.

If $\lambda = 2$

$$\begin{aligned} 4v_1 + 6v_2 + 10V_3 &= 2v_1 \\ 3v_1 + 10v_2 + 13v_3 &= 2v_2 \\ -2v_1 - 6v_2 - 8v_3 &= 2v_3 \end{aligned}$$

Let $v_3 = 1$, then $v_2 = -2, v_1 = 1$

So

$$\Lambda = \begin{bmatrix} 0 & & \\ & 4 & \\ & & 2 \end{bmatrix}, \ V = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -\frac{5}{3} & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

10. CES function. This question derives some of the most basic “Dixit-Stiglitz lite" equations
The representative consumer’s utility function is

$$U = \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}}, \qquad 0 < \rho < 1 \tag{1}$$

where $q(\omega)$ is consumption of variety ω, n is the mass of varieties available to consumers, and ρ is a measure of sustainability. The consumer has “taste for variety” in that he or she prefers to consume a diversified bundle of goods. More details on this below. Notice that the budget constraint is implicitly assumed to be $\int_0^n p(\omega)q(\omega)d\omega = I$, where I is the total endowment.

(a) Demand
The consumer’s constrained maximization problem may be solved by the Lagrangian $\mathcal{L} = U^\rho - \lambda(\int_0^n p(\omega)q(\omega)d\omega - I)$Take first derivatives on $\partial q(\omega)$, we have

$$\frac{\partial \mathcal{L}}{\partial q(\omega)} = \rho q(\omega)^{\rho-1} - \lambda p(\omega) = 0 \tag{2}$$

Rearranging terms yields the Frisch demand function:

$$q(\omega) = \left(\frac{\lambda p(\omega)}{\rho} \right)^{\frac{1}{\rho-1}} \tag{3}$$

Taking the ratio of Frisch demands for two varieties ω_1 and ω_2 yields relative demand:

$$\frac{q(\omega_1)}{q(\omega_2)} = \left(\frac{p(\omega_1)}{p(\omega_2)}\right)^{\frac{1}{\rho-1}} \tag{4}$$

At this stage, it will be useful to introduce $\sigma = \frac{1}{1-\rho}$ in order to keep notation concise. From (4), it is evident that the elasticity of substitution is the constant: $\sigma = \frac{-d\ln q(\omega_1)/q(\omega_2)}{d\ln p(\omega_1)/p(\omega_2)}$, hence this is a CES demand function. Using σ and multiplying both sides by $q(\omega_2)$ yields:

$$q(\omega_1) = q(\omega_2)\left(\frac{p(\omega_1)}{p(\omega_2)}\right)^{-\sigma} \tag{5}$$

Based on the above results, **can you show that**

$$q(\omega_2) = \frac{Ip(\omega_2)^{-\sigma}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}$$

hint(multiply both sides by $p(\omega_1)$ and take the integral with respect to ω_1. Then utilize the budget constraint relationship)

Answer:

Now multiply both sides by $p(\omega_1)$ and take the integral with respect to ω_1.

$$\int_0^n p(\omega_1)q(\omega_1)d\omega_1 = \int_0^n q(\omega_2)p(\omega_1)^{1-\sigma}p(\omega_2)^{\sigma}d\omega_1$$

The left-hand side is the consumer's total expenditure on all varieties - the consumer's income.

$$I = q(\omega_2)p(\omega_2)^{\sigma}\int_0^n p(\omega_1)^{1-\sigma}d\omega_1$$

To obtain Marshallian demad for ω_2 in terms of prices and income, divided by $p(\omega_2)^{\sigma}\int_0^n p(\omega_1)^{1-\sigma}d\omega_1$:

$$q(\omega_2) = \frac{Ip(\omega_2)^{-\sigma}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}$$