Metric and Topological Spaces ①

Introduction

$f: \mathbb{R} \to \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

f is continuous if it is continuous at a, $\forall a \in \mathbb{R}$.

$f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\underline{a} \in \mathbb{R}^n$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\|\underline{x} - \underline{a}\| < \delta \Rightarrow \|f(\underline{x}) - f(\underline{a})\| < \varepsilon$,
where $\|\underline{x} - \underline{y}\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$

One can go further, and discuss metric spaces, which are ones with a notion of distance between points.

Topology is the study of continuity without distance being mentioned. Formally, we model the idea of being "nearby" by saying that our sets come with preferred subsets (open sets), and that two points are "close" if they belong to the same open set.

Definition

A topological space is a pair $(X, \mathcal{T}_X)$ comprising a set X, and a collection $\mathcal{T}_X \in \text{Power}(X)$ of preferred open subsets, called open sets, such that:

i) $\emptyset, X$ are open (i.e. $\emptyset, X \in \mathcal{T}_X$)

ii) If $\{U_i\}_{i \in I}$ are open, then $\bigcup_{i \in I} U_i$ is open (I is any indexing set)

iii) If $\{V_j\}_{j=1}^{n}$ are open, then $\bigcap_{j=1}^{n} V_j$ is also open.

Definition

A function $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous, if for every open set V in Y, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X, i.e. "Preimages of open sets are open."

N.B. $f^{-1}(V)$ is the set-theoretic pre-image; we are not assuming that $f^{-1}(x)$ exists.

Definition

Two spaces X and Y are homeomorphic if $\exists f: X \to Y$ and $g: Y \to X$ with $f \circ g = i$, $g \circ f = i$, both of which are continuous. i.e. via f, g, X and Y have exactly the same open sets.

Example

The Euclidean Topology on $\mathbb{R}$ (or $\mathbb{R}^n$), $\mathcal{T}_{Eucl}$ is defined:
$U \subseteq \mathbb{R}$ is open if $\forall a \in U$, $\exists \varepsilon > 0$ (depending on a) such that $(a-\varepsilon, a+\varepsilon) \subseteq U$ (or for $\mathbb{R}^n$, $B_a(\varepsilon) \subseteq U$)
Then, axioms i) and ii) are trivial, and iii) is easy. If $U_1, \dots, U_n$ are open in $\mathcal{T}_{Eucl}$, and $a \in \bigcap_{i=1}^n U_i$, $a \in U_i$ $\Rightarrow \exists \varepsilon_i > 0$ such that $(a+\varepsilon_i, a-\varepsilon_i) \subseteq U_i$, as U_i is open.
Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then $(a-\varepsilon, a+\varepsilon) \subseteq \bigcap_{i=1}^n U_i$
Therefore, U_i is open.

Exercise

$f: (\mathbb{R}, \mathcal{T}_{Eucl}) \to (\mathbb{R}, \mathcal{T}_{Eucl})$ is continuous in the sense that pre-images of open sets are open $\iff$ f is "ε-δ" continuous at a $\forall a \in \mathbb{R}$.

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Lemma

Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces. If $f: X \to Y$, $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is also continuous.

Proof

Let $V \subseteq Z$ be open. We are interested in $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. But $g^{-1}(V)$ is open in Y by continuity of g, so $f^{-1}(g^{-1}(V))$ is open in X, by continuity of X. Therefore $g \circ f$ is continuous.

Example

The Zariski Topology on $\mathbb{R}$, $\tau_{Zariski}$, has open sets:

i) U is open if $\mathbb{R} \setminus U$ is finite ii) $\emptyset$ is open.

Check: i) $\emptyset$, $\mathbb{R}$ are obviously open.

ii) If $\{U_i\}_{i \in I}$ are open, $\mathbb{R} \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} \mathbb{R} \setminus U_i$ which is $\emptyset$, $\mathbb{R}$ or finite, and therefore ~~open~~. $\bigcup_{i \in I} U_i$ is open

iii) If $\{V_j\}_{j=1}^{n}$ are open, $\mathbb{R} \setminus \bigcap_{j=1}^{n} V_j = \bigcup_{j=1}^{n} (\mathbb{R} \setminus V_j)$ is finite on $\mathbb{R}$, and therefore ~~open~~. $\bigcap_{j=1}^{n} V_j$ is open.

Remark

Later on, we will prove that this topology is "incompatible" with any reasonable notion of distance.

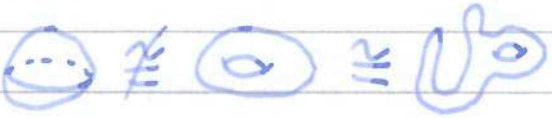
Remark

De Morgan's Laws: For $f: X \to Y$

$$f(\cup C_i) = \cup f(C_i) \qquad f^{-1}(\cup D_i) = \cup f^{-1}(D_i)$$
$$f(\cap C_i) \subseteq \cap f(C_i) \qquad f^{-1}(\cap D_i) = \cap f^{-1}(D_i)$$
$$f^{-1}(Y \setminus D_i) = X \setminus f^{-1}(D_i)$$

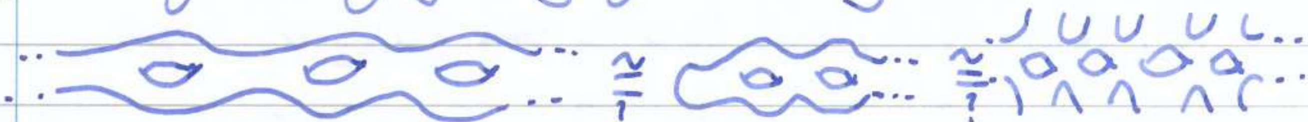
Topology is the study of all spaces up to homeomorphism, and we especially like spaces with some "local good structure", e.g. locally homeomorphic to $\mathbb{R}^n, \tau_{Eucl}$ or $\mathbb{R}^n, \tau_{Zariski}$ if $\forall x$, $\exists$ an open set $U \ni x$ such that U is homeomorphic to one of these.

Questions

i)  Why ? 

ii) What is the difference between and  (Trefoil knot) ?

iii) A surface is of genus g if it has g holes.



iv) Questions in dynamics related to the chaotic behaviour of maps.

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Recall: A topological space is a set X and a collection τ_X of open subsets such that:
i) $\emptyset, X \in \tau_X$ ii) Arbitrary unions, finite intersections are open.

$f : X \to Y$ (i.e. $f(X, \tau_X) \to (Y, \tau_Y)$) is ~~open~~ continuous if f^{-1}(open) is open.

Examples

1. The indiscrete topology on X has $\tau_X = \{\emptyset, X\}$.

2. The discrete topology on X has $\tau_X = \text{Power}(X)$, i.e. every subset of X is open. ($\Leftrightarrow \forall x \in X$, $\{x\}$ is open).

3. $X = \mathbb{R}^n$, $U \in \tau_{Eucl}$ if $\forall u \in U$, $\exists \delta_u > 0$ such that $B_u(\delta_u) = \{ y \in \mathbb{R}^n \mid \|y - u\| < \delta_u \} \subseteq U$

Lemma

$f(\mathbb{R}, \tau_{Eucl}) \to (\mathbb{R}, \tau_{Eucl})$ is continuous $\Leftrightarrow$ f is "ε-δ" continuous.

Proof

Suppose f is continuous in the sense that f^{-1}(open) is open. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, consider $B_{f(a)}(\varepsilon)$. This is open, so $f^{-1}(B_{f(a)}(\varepsilon))$ is open in $(\mathbb{R}, \tau_{Eucl})$, so it contains an open neighbourhood of each of its points. So $\exists \delta_a > 0$ such that $B_a(\delta_a) \subseteq f^{-1}(B_{f(a)}(\varepsilon))$, so f is ε-δ continuous for any $a \in \mathbb{R}$.

Conversely, if f satisfies the ε-δ definition, and $U \subseteq \mathbb{R}$ is τ_{Eucl} open we would like $f^{-1}(U)$ open. Let $b \in f^{-1}(U)$, then $f(b) \in U$.
U open $\Rightarrow \exists \varepsilon > 0$ such that $B_{f(b)}(\varepsilon) \subseteq U \Rightarrow f^{-1}(B_{f(b)}(\varepsilon)) \subseteq f^{-1}(U)$
Now, $\exists \delta > 0$ such that $f(B_b(\delta)) \subseteq B_{f(b)}(\varepsilon) \Rightarrow B_b(\delta) \subseteq f^{-1}(B_{f(b)}(\varepsilon))$
So $f^{-1}(U)$ contains an open ball around each point, and is τ_{Eucl} open.

Remark

"$U \subseteq \mathbb{R}$ is open" obviously only makes sense after specifying the topology. $\{0\} \subseteq (\mathbb{R}, \tau_{discrete})$ is open but $\{0\} \in (\mathbb{R}, \tau_{Eucl})$ is not

$id : (\mathbb{R}, \tau_{discrete}) \to (\mathbb{R}, \tau_{Eucl})$ is continuous.
$id : (\mathbb{R}, \tau_{Eucl}) \to (\mathbb{R}, \tau_{discrete})$ is not continuous.

Notation

A set $V \subseteq (X, \tau_X)$ is closed if $X \setminus V$ is open.

i) $\emptyset, X$ are closed

ii) Arbitrary intersections, finite unions of closed sets are closed

iii) $f : X \to Y$ is continuous $\Leftrightarrow$ $f^{-1}(\text{closed})$ is closed.

Note

"Most" sets are neither open nor closed. For example, in τ_{Eucl}, $[0, 2)$ is neither open nor closed, $\mathbb{Q}$ is neither open nor closed, etc.

Example

The Zariski Topology on $\mathbb{R}$ or $\mathbb{C}$ has closed sets which are the finite sets and the entire set. So, we can check that $\tau_{Zariski}$, τ_{Eucl} and $\tau_{discrete}$ are distinct, because the identity is not a homeomorphism, and in fact there is no homeomorphism between any pair of $(\mathbb{R}, \tau_{Zariski})$, $(\mathbb{R}, \tau_{Eucl})$, $(\mathbb{R}, \tau_{discrete})$

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Examples of Maps

1. If X, Y are any topological spaces, then a constant map, $f: X \to Y$, $f(x) = y_0 \in Y \ \forall x \in X$ is continuous.

$y_0 \in U \Rightarrow f^{-1}(U) = X$, $y_0 \notin U \Rightarrow f^{-1}(U) = \emptyset$ } Both open in X for $U \subseteq Y$ open, so f is continuous.

2. There is only one topology on a singleton set $\{p\}$, so any map $\{p\} \to (X, \tau_X)$ is continuous.

Remark

Counting the number of non-homeomorphic topologies on a finite set is hard.

Subspaces

Let (X, τ_X) be any topological space, and $A \subseteq X$ a subset. There is an induced topology on A; the subspace topology.

$\tau_A = \{ U \subseteq A \mid U = A \cap W, W \in \tau_X \text{ i.e. } W \text{ open in } X \}$
We claim that τ_A is a topology.

Proof

i) $\emptyset = \emptyset_X \cap A$, $A = A \cap X$ are both open by the definition of τ_A.

ii) If $\{U_i\}_{i \in I}$ are open in A, $\exists \{W_i\}_{i \in I}$ open in X such that $U_i = A \cap W_i$. Then $\bigcup_{i \in I} U_i = \bigcup_{i \in I} (A \cap W_i) = A \cap (\bigcup_{i \in I} W_i)$

$\bigcup_{i \in I} W_i$ is open in X by the axioms for τ_X. So $\bigcup_{i \in I} U_i \in \tau_A$.

iii) If $\{V_j\}_{j=1}^{n}$ are open in A, then $V_j = A \cap Z_j$, Z_j open in X.
$\bigcap_{j=1}^{n} V_j = A \cap \left(\bigcap_{j=1}^{n} Z_j\right)$, and $\bigcap_{j=1}^{n} Z_j$ is open in X.

<u>Lemma</u>

If X, Y are topological spaces, $f: X \to Y$ is continuous, and $A \subseteq X$ is given the subspace topology, then $f|_A : A \to Y$ is also continuous.

The proof is essentially tautological.

<u>Remark</u>

If $\mathbb{R}^k \subseteq \mathbb{R}^n$, $k < n$, is used as a linear subspace, the subspace topology induced on $\mathbb{R}^k$ from $(\mathbb{R}^n, \mathcal{T}_{Eucl})$ is $\mathcal{T}_{Eucl}$ on $\mathbb{R}^k$.

Proof Ingredient: The intersection of an open set (ball) in $\mathbb{R}^n$ with linear $\mathbb{R}^k \subseteq \mathbb{R}^n$ is an open set (ball) in $\mathbb{R}^k$.

<u>Example</u>

1. $(0,1) \subseteq (\mathbb{R}, \mathcal{T}_{Eucl})$ inherits a Euclidean Topology via the tan function and its inverse. $(0,1) \cong (0,\infty) \cong \mathbb{R}$ are homeomorphic with their Euclidean Topologies. So <u>boundedness</u> is not a topological property (but is a <u>metric</u> property).

2. $[0,2]$ in $\mathbb{R}$ inherits Euclidean Topology. $[0,1)$ is open in $([0,2], \mathcal{T}_{Eucl})$ but <u>not</u> in $\mathbb{R}$. $[0,1) = (-1,1) \cap [0,2]$

3.  $\subseteq \mathbb{R}^3$ inherits a Euclidean Topology.

4. The Cantor Set $= C = \bigcap_{n \geq 1} C_n$
where $C_n = \left[0, \frac{1}{3^{n-1}}\right] \cup \left[\frac{2}{3^{n-1}}, \frac{3}{3^{n-1}}\right] \cup \dots \cup \left[\frac{3^{n-1}-1}{3^{n-1}}, 1\right]$
This is a closed subset of $(\mathbb{R}, \mathcal{T}_{Eucl})$

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Products and Quotients

Definition

Let (X, τ_X), (Y, τ_Y) be topological spaces. The product topology on $X \times Y$ has open sets arbitrary unions of sets in the form $U \times V$, with $V \subseteq X$ open in X, $U \subseteq Y$ open in Y.

Remark

Clearly $\emptyset$, $X \times Y$ are of the form $U \times V$, $U \in \tau_X$, $V \in \tau_Y$. If we have open sets $U_i \times V_i$, U_i, V_i open, then $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

So finite intersections of open sets are open. The "unions" axiom is trivial so we do get a topology.

e.g. $(\mathbb{R}, \tau_{Eucl}) \times (\mathbb{R}, \tau_{Eucl}) \cong (\mathbb{R}^2, \tau_{Eucl})$. However, note that $\{z \in \mathbb{C} \mid \frac{1}{3} < |z-1| < \frac{2}{3}\} = U \subseteq \mathbb{R}^2$ is open. Obviously, the general open set in $X \times Y$ is not of the form $U \times V$ with $U \in \tau_X$, $V \in \tau_Y$.

Notation

A **basis** of a topology on X is a collection of open sets such that the general open set is an arbitrary union of "basic" open sets.

A **sub-basis** for a topology is a collection of open sets such that the general open set is an arbitrary union or finite intersection of "basic" open sets.

Example

In $(\mathbb{R}, \tau_{Eucl})$, the open intervals (a, b) form a basis $(a < b) \in \mathbb{R}$. The semi infinite intervals $(-\infty, a)$, (b, ∞) form a sub-basis.

Remark

Given any set X, a collection $\mathcal{B}$ of subsets of X, we can create a topology $\tau_{\mathcal{B}}$ such that $\mathcal{B}$ is a sub-basis, by taking $U \in \tau_{\mathcal{B}}$ if U is some union of finite intersections of elements of $\mathcal{B}$. This is the "smallest" topology (i.e. with fewest open sets) such that each $A \in \mathcal{B}$ is open.

$\{U \times V,\ U \in \tau_X,\ V \in \tau_Y\}$ is a basis and sub-basis for product topology.

Lemma

i) Projection maps $X \times Y \xrightarrow{\pi_1} X$, $X \times Y \xrightarrow{\pi_2} Y$ are both continuous.

ii) If Z is any space, then $f : Z \to X \times Y$ is continuous $\Leftrightarrow$ $\pi_1 \circ f$, $\pi_2 \circ f$ are continuous.

Proof

i) If $U \subseteq X$ is open, $\pi^{-1}(U) = U \times Y$ is a basic open set for $X \times Y$.

ii) If f is continuous, $\pi_1 \circ f$ is continuous, since we showed that compositions of continuous functions are continuous. Suppose then, $\pi_1 \circ f$, $\pi_2 \circ f$ are continuous as maps $Z \to X$, $Z \to Y$ respectively.

$U \in \tau_X, V \in \tau_Y \Rightarrow f^{-1}(U \times V) = (\pi_1 \circ f)^{-1}(U) \cap (\pi_2 \circ f)^{-1}(V)$. This is an intersection of open sets, which is therefore open. So
$f^{-1}(\text{open}) = f^{-1}(\bigcup_i U_i \times V_i) = \bigcup_i f(U_i \times V_i)$ is also open. $\square$

Examples

1. Let $f, g : X \to (\mathbb{R}, \tau_{Eucl})$ be continuous. $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, so $f + g : X \to \mathbb{R}$ is continuous.

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2. If $f: X \to Y$ is continuous, the graph $\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}$, $\Gamma_f \subseteq X \times Y$. With subspace topology, $\Gamma_f \cong X$.

Proof:

We have maps $X \to \Gamma_f \overset{\pi_1}{\to} X$. $X \overset{\phi}{\to} \Gamma_f \subseteq X \times Y$ is continuous, since $\pi_1 \circ \phi = id_X$, $\pi_2 \circ \phi = f$ are given as continuous. $\pi_1 : \Gamma_f \to X$ is the subspace of a projection map, so this is continuous. □

Example

The graph of $\sin\left(\frac{1}{x}\right)$ for $x > 0$ is homeomorphic to $(0, \infty) \cong \mathbb{R}$.

Warning

If $(X_i)_{i \in I}$ is an infinite collection of topological spaces, the product topology on $\prod_{i \in I} X_i$ is defined to have basic open sets $\prod_{i \in I} U_i$ with $U_i \in \mathcal{T}_{X_i}$ and all but finitely many $U_i = X_i$

Definition

If $(X, \mathcal{T}_X)$ is a topological space, Y is a set, and $f: X \to Y$ is a surjective map of sets, then the quotient topology on Y be definition has open sets as follows:

$U \in \mathcal{T}_{Quot(Y)} \iff f^{-1}(U) \in \mathcal{T}_X$, i.e. we put in as few open sets to Y as possible subject to forcing f to be continuous.

Check: $\emptyset$, Y are obviously open (since $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} = X$)
If $\{U_i\}_{i \in I}$ are open in Y, $f^{-1}(U_i) \in \mathcal{T}_X \ \forall i \in I$, so $\bigcup_{i \in I} f^{-1}(U_i)$ is open in X. But this is $f^{-1}(\bigcup_{i \in I} U_i)$ so $\bigcup_{i \in I} U_i \in \mathcal{T}_Y$. This is similar for finite intersections □

Remark

We assumed that $f: X \to Y$ is surjective. This isn't needed for the definition, but if f is not surjective, the topology on $Y \setminus \text{image}(f)$ is discrete.

Lemma

If $f: X \to Y$ is a quotient map, Y has the quotient topology with respect to f, and Z is any topological space, then:
$g: Y \to Z$ continuous $\Leftrightarrow$ $g \circ f: X \to Z$ continuous

Proof

If g is continuous, since f is continuous, $g \circ f$ is continuous.

Conversely, suppose $g \circ f: X \to Z$ is continuous. Let $U \in \mathcal{T}_Z$. Then $(g \circ f)^{-1}(U)$ is open in X, but this is $f^{-1}(g^{-1}(U))$. By definition of $\mathcal{T}_Y$ as a quotient topology, $g^{-1}(U) \in \mathcal{T}_Y$. U is arbitrary, so g is continuous $\square$

Example

A quotient map $f: X \to Y$ is just the same as an equivalence relation, $x \sim x' \Leftrightarrow f(x) = f(x')$. So Y is the set of equivalence classes, and f is the canonical projection $x \mapsto [x]$ $\leftarrow$ (the equivalence class of x)

For instance, quotient maps arise from group actions e.g. $\mathbb{Z}$ acts on $(\mathbb{R}, \mathcal{T}_{Eucl})$, and we will see that $\mathbb{R}/\mathbb{Z} \cong S^1$, a circle.

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~~Hausdorff~~ Spaces

A sub-basis for a topology is a collection of open sets such that the general open set is given by unions and finite intersections. Given a set X, and a collection $\mathcal{A}$ of subsets, there exists a minimal topology τ_A with subsets $\mathcal{A} \cup X$.

Remarks

We defined quotient spaces using $f: X \to Y$, a surjective map of sets, (X, τ_X) a topological space, and Y having quotient topology if $U \in \tau_Y \Leftrightarrow f^{-1}(U) \in \tau_X$. An equivalence relation on X defines a quotient map $X \to X/\sim$

Example

Take $\sim$ on $\mathbb{R}$: $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$, or alternatively, $\mathbb{Z}$ acts on $\mathbb{R}$ by translation, $x \sim y \Leftrightarrow x$ and y are in the same coset for this action. Then $\mathbb{R}/\sim \cong S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{|z| < 1\} \subseteq \mathbb{C}$
Every $x \in \mathbb{R}$ is equivalent to some element of $[0,1)$ or every x is equivalent to a point of $[0,1]$ subject to $0 \sim 1$.

Proofs

Define $\rho : \mathbb{R} \to S^1$, $x \mapsto e^{2\pi i x}$. This is certainly continuous. We just need to check that $\pi_i \circ \rho$ are continuous. Clearly $\rho(x) = \rho(x \pm 1)$. ρ descends to a map $\mathbb{R}/\sim \to S^1$ at the level of sets. The map out of a quotient is continuous $\Leftrightarrow$ the map from the space "above" the quotient is continuous: e.g. $\bar{\rho} : \mathbb{R}/\sim \to S^1$ i.e. $\rho : \mathbb{R} \to S^1$ is continuous.

We conclude that $\bar{\rho} : \mathbb{R}/\sim \to S^1$ is continuous and a bijection.

Note : To show that $\bar{p}$ is a homeomorphism, it is sufficient to show that it is an <u>open</u> map i.e. $\bar{p}(U) \in S^1$ is open whenever $U \in \mathbb{R}/_{\mathbb{Z}}$ is open. (Most continuous functions are not open).

To see that $\bar{p}$ is open, since $\bar{p}(\bigcup_i U_i) = \bigcup_i \bar{p}(U_i)$, it is enough to take $U \subseteq \mathbb{R}/_{\mathbb{Z}}$ to be a <u>basic</u> open set e.g. an open interval of length $< \frac{1}{4}$, say. $U = \bar{p}(x, x+\delta)$, $\delta < \frac{1}{4}$. Clearly, for such small intervals, $\bar{p}(U) = S^1 \cap$ (Some open set in $\mathbb{R}^2$) $\square$

<u>Corollary</u>

$\mathbb{R}^2/_{\mathbb{Z}^2} \cong S^1 \times S^1 =$ 

<u>Definition</u>

A topological space is <u>Hausdorff</u>, if given $x \neq y \in X$, $\exists$ disjoint open sets $U \ni x$, $V \ni y$ (i.e. the two points can be "housed off" from one another). Obviously $X \cong Y \Rightarrow$ both are Hausdorff, or neither is.

<u>Example</u>

1. $(\mathbb{R}, \mathcal{T}_{Eucl})$ is Hausdorff. If $x \neq y$, $\|x-y\| = \delta > 0$. Then let $U = B_x(\delta/3)$, $V = B_y(\delta/3)$.

2. $(\mathbb{R}, \mathcal{T}_{Zariski})$ is <u>not</u> Hausdorff. $U \subseteq \mathbb{R}$, $U \in \mathcal{T}_{Zariski}$ if $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite. So every two non-empty open sets intersect.

<u>Corollary</u>

$(\mathbb{R}, \mathcal{T}_{Eucl}) \not\cong (\mathbb{R}, \mathcal{T}_{Zariski})$

Lemma

i) Subspaces of Hausdorff spaces are Hausdorff.

ii) Products of Hausdorff spaces are Hausdorff.

iii) Quotients of Hausdorff spaces need not be Hausdorff.

Proof.

i) and ii) are exercises in the definition.

For iii), let X be two copies of the real line.
We define $(x,0) \sim (x,1)$ if $x \neq 0$. We claim that the quotient space $X/\sim$ is not Hausdorff. Why?

Let $x = (0,0)$, $y = (0,1)$. $x \neq y$ in $X/\sim$. Let $U \ni x$, $V \ni y$, be open sets in $X/\sim$. If $p : X \to X/\sim$, $p^{-1}(U) \ni (0,0)$ is open in X so contains some open interval $(-\delta, \delta) \in \mathbb{R}_{y=0} \subseteq X$, and $p^{-1}(V)$ contains some $(-\delta', \delta') \subseteq \mathbb{R}_{y=1} \subseteq X$. So in the quotient space, U and V intersect non-trivially, so $X/\sim$ is not Hausdorff. $\square$

Metric and Topological Spaces ⑤

Metric Spaces

Definition

A metric space comprises a set X and a "distance function" $d : X \times X \to \mathbb{R}$ satisfying :

i) $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$ (Non-degeneracy)

ii) $d(x,y) = d(y,x)$ (Symmetry)

iii) $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle Inequality)

Examples

1) On $\mathbb{R}^n$, the function $d(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$ defines a metric space structure.

2) Discrete metric spaces : Given any set X, define $d_{discrete}$ by $d_{discrete}(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ which satisfies axioms i) and ii)

If x, y, z are pairwise distinct, then $1 \leq 2$ so iii) is satisfied.

3. Let $B([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is bounded} \}$
Define $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$, and this is a metric.

Proof

If $|f(x)| \leq k_f$, $|g(x)| \leq k_g$, $x \in [a,b]$, then $|f(x) - g(x)| \leq k_f + k_g$
so the supremum makes sense and the metric is well defined.

$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$
$\Rightarrow |f(x) - h(x)| \leq d(f,g) + d(g,h)$

This is true for all x, so $d(f,h) \leq d(f,g) + d(g,h)$

Definition

If (X,d) is any metric space, the open ball
$B_x(\delta) = \{ y \in X \mid d(x,y) < \delta \}$

Lemma

i) If (X,d) is a metric space, d defines canonically a topology $\mathcal{T}_d$ on X, by defining $U \subseteq X$ open if $\forall x \in U$, $\exists \delta_x > 0$ such that $B_x(\delta_x) \subseteq U$.

ii) With respect to the topology $\mathcal{T}_d$, a function is continuous if $\forall x$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $B_x(\delta) \subseteq f^{-1}[B_{f(x)}(\varepsilon)]$. The proof is exactly as for $(\mathbb{R}, \mathcal{T}_{Eucl})$.

Definition

A topological space $(X, \mathcal{T}_X)$ is metrisable if $\exists$ a metric d on X for which $\mathcal{T}_X = \mathcal{T}_d$

Examples

1. The Euclidean topology $\mathcal{T}_{Eucl}$ is metrisable.

2. The discrete topology on a set X is metrisable, induced by the discrete metric.

Proof

Let X have discrete metric d. $B_x(\frac{1}{2})$ is open in $\mathcal{T}_d$, but $B_x(\frac{1}{2}) = \{x\}$
Now if $U \subseteq X$, $U = \bigcup_{x \in U} \{x\}$ is open, so $\mathcal{T}_d = \mathcal{T}_{discrete}$

Metric and Topological Spaces ⑤

Lemma

If (X,d) is any metric space, $\mathcal{T}_d$ is Hausdorff.

Proof

If $x \neq y \in X$, then $d(x,y) = \delta > 0$. Then, $x \in B_x(\frac{\delta}{3})$ and $y \in B_y(\frac{\delta}{3})$. These are open neighbourhoods of x and y. They must be disjoint, by the triangle inequality for the metric d.

Example

$\mathcal{T}_{Zariski}$ in $\mathbb{R}$ is not metrisable.

Definition

Metrics d_1 and d_2 on a set X are equivalent if $id : (X, \mathcal{T}_{d_1}) \to (X, \mathcal{T}_{d_2})$ and $id : (X, \mathcal{T}_{d_2}) \to (X, \mathcal{T}_{d_1})$ are continuous (so they are inverse homeomorphisms).

Note

This means that $U \subseteq X$ is open with $d_1 \Leftrightarrow$ it is open with d_2. This happens if and only if every d_1 open ball contains some d_2 open ball and vice versa. In particular, if d_1 and d_2 are equivalent, $(X, \mathcal{T}_{d_1}) \cong (X, \mathcal{T}_{d_2})$

Examples

On $\mathbb{R}^n$, the following are equivalent:

- $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$
- $d_2(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$
- $d_\infty(x,y) = \max_i |x_i - y_i|$

It is sufficient to know that $d_1 \geq d_2 \geq d_\infty \geq \frac{d_2}{\sqrt{n}} \geq \frac{d_1}{n}$
Then obviously open balls in one of the above metrics contain open balls in the others.

Example

Let $C[0,1] = \{ f : [0,1] \to \mathbb{R} \mid f$ is continuous with respect to $\mathcal{T}_{Eucl}$ on $[0,1]$ and $\mathbb{R} \}$

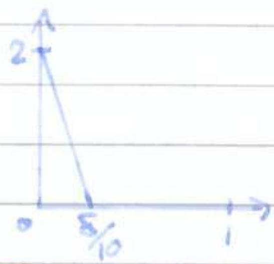
$d(f,g) = \int_0^1 |f(t) - g(t)| \cdot dt$ and

$d_\infty(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$ are both metrics. We claim that they are not equivalent.

Proof

Let $\underline{0}$ denote the constant function 0. We claim that $B_{\underline{0}}(1)^{d_\infty}$ contains no open ball about $\underline{0}$ in d_1-topology.

Suppose $B_{\underline{0}}(\delta)^{d_1} \subseteq B_{\underline{0}}(1)^{d_\infty}$



Graph of $f \in C[0,1]$

Then $d_1(f, \underline{0}) < \delta$, but $d_\infty(f, \underline{0}) = 2$

Metric and Topological Spaces (6)

Closure

In a metric space (X, d), we can talk about limits.

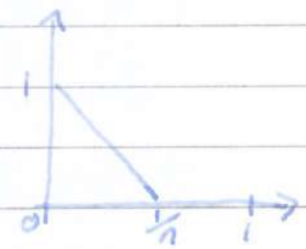
"$x_n \to x_\infty$" means (by definition) that $d(x_n, x_\infty) \to 0$

Caveats

For $(0,1) \subset (\mathbb{R}, \tau_{Eucl})$, $x_n = \frac{1}{n}$ does not converge in $(0,1)$ in $X = (0,1)$ although $d(\frac{1}{n}, 0) \to 0$.

2. $X_1 = (C[0,1], d_1) = \{\text{continuous functions } [0,1] \to \mathbb{R}\}$
$d_1(f,g) = \int_0^1 |f(t) - g(t)|\, dt$
$d_\infty(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$

$X_1 \neq X_2 = (C[0,1], d_\infty)$



Let f_n be continuous with the graph shown.

In X_1, (f_n) is a convergent subsequence, since pointwise, $f_n(x) \to \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$ so $d_1(f_n, 0) \to 0$ as $n \to \infty$.

However, $(f_n) \not\to 0$ in X_2 (and does not even converge).

In general spaces, we might guess that $x_n \to a$ (in (X,d)) means that $\forall \varepsilon > 0$, $\exists N : n > N \Rightarrow x_n \in B_a(\varepsilon)$.

Or for topological spaces: $x_n \to a$ in (X, τ_X) if $\forall$ open neighbourhood of a, U, $\exists N_U$ such that $x_n \in U$ $\forall n > N_U$.

This is not very useful as it may be that $x_n \to a, b$ with $a \neq b$. For example, in $(X, \tau_{indiscrete})$, every sequence converges to every point. If X is Hausdorff though, a sequence has at most one limit.

Definition

Let (X, τ_X) be a space and $A \subseteq X$ a subspace.

i) $x \in X$ is a limit point of A if $\forall$ open $U \subseteq X$ containing x, $(U \cap A) \setminus \{x\}$ is non-empty. This is shorthand for : $U \cap A$ meets A in a point $\neq x$. x need not be in A.

ii) The closure of A, written $cl(A)$ or $\bar{A}$, is the union of A and all its limit points.

Examples

1. $X = \mathbb{R}$, $A = \{1\} \cup (2,3)$. Then, the limit points of A are $\{2,3\}$ and $cl(A) = \{1\} \cup [2,3]$

2. $X = \mathbb{R}$, $A = \mathbb{Q}$. Then, $Cl(A) = \mathbb{R}$.

If $A \subseteq X$ has the property that $Cl(A) = X$, A is dense in X.

Properties of the Closure Operation

Let X be a space and H, K subspaces of X.

i) $H \subseteq K \Rightarrow Cl(H) \subseteq Cl(K)$

ii) $Cl(Cl(H)) = Cl(H)$

iii) H closed $\Leftrightarrow H = Cl(H)$

iv) $Cl(H)$ is closed.

Metric and Topological Spaces ⑥

Proofs

i) Observe that $x \in Cl(A) \Leftrightarrow$ Every open neighbourhood of $x \in X$ meets A.

ii) $A \subseteq Cl(A) \Rightarrow Cl(A) \subseteq Cl(Cl(A))$

Suppose that $x \in Cl(Cl(A))$. Let $U \ni x$ be an open neighbourhood of x in X. So since $x \in Cl(Cl(A))$, then U meets $Cl(A)$. Then U is an open set meeting $Cl(A)$, so U must meet A and $x \in Cl(A)$.
$\Rightarrow Cl(A) = Cl(Cl(A))$

iii) Let H be closed, and $x \in X \setminus H$. Note that by definition, $X \setminus H$ is open. So x belongs to an open set not meeting H i.e. $x \notin Cl(H)$.
$\therefore H$ closed $\Rightarrow Cl(H) \subseteq H \subseteq Cl(H)$.

Conversely, suppose $H = Cl(H)$. Take $x \in X \setminus H$. Since $x \notin Cl(H)$ $\exists$ an open set $U_x \ni x$ such that $U_x \cap H = \emptyset$, i.e. $U_x \subseteq X \setminus H$. Then $X \setminus H = \bigcup_{x \in X \setminus H} U_x$ is open, so H is closed.

iv) ii), iii) $\Rightarrow$ iv)

Corollary

The closure $Cl(A)$ of a set A is the smallest closed set containing A, i.e.

$$Cl(A) = \bigcap_{K \subseteq X,\ A \subseteq K,\ K \text{ closed}} K$$

Proof

$Cl(A)$ is a closed set containing A. But if V is closed, $A \subseteq V$, then $Cl(A) \subseteq Cl(V) = V$

Example

Let $(X, \mathcal{T}_X)$ be a space and let $(Y, \mathcal{T}_Y)$ be a Hausdorff Space. Let $f: X \to Y$ be continuous. If $A \subseteq X$, $f|_A$ is constant then $f|_{Cl(A)}$ is constant.

First, we claim that since Y is Hausdorff, the singleton sets $\{p\}$ are closed sets of Y. Pick $y \in Y$. If $y' \neq y$, $\exists$ an open set $V_{y'} \ni y'$ with $y \notin V_{y'}$. So $\bigcup_{y' \neq y} V_{y'}$ is open and equals $Y \setminus \{y\}$. Now f is continuous, and $\{y\} \subseteq Y$ is closed, so $f^{-1}(y) \subseteq X$ is closed. Pick $y = f(A)$. Then $A \subseteq f^{-1}(y)$ and $f^{-1}(y)$ is closed. So, $Cl(A) \subseteq f^{-1}(y)$, and indeed, f is constant on $Cl(A)$.

Digression

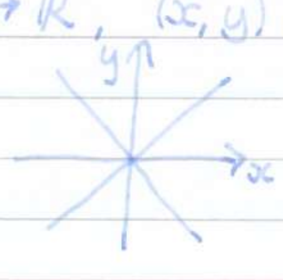
Recall that $A \subseteq X$ is dense if $Cl(A) = X$. Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is continuous, and infinitely differentiable in each variable. We say that $x \in \mathbb{R}^n$ is a regular point if at least one partial derivative $\frac{\partial f}{\partial x_i}\Big|_x$ is not zero, and that $y \in \mathbb{R}$ is a regular value if $\forall x \in f^{-1}(y)$, x is a regular point.

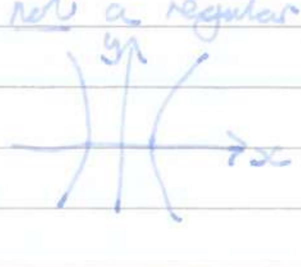
Theorem : Regular Values are dense in $\mathbb{R}$.

Inverse Function Theorem : If $y \in \mathbb{R}$ is regular, $f^{-1}(y)$ is a manifold (locally homeomorphic to $\mathbb{R}^{n-1}$)

Example

$f: \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 - y^2$. The origin is not a regular point

$f^{-1}(0)$ 

If $\omega \neq 0$, $f^{-1}(\omega)$ 

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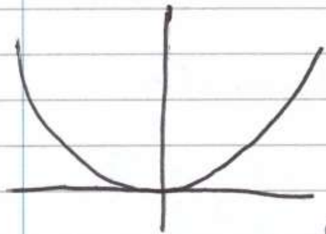
Connectedness

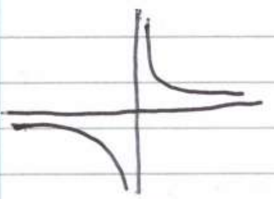
Definition A space $(X, \mathcal{T}_X)$ is connected, if wherever $X = A \cup B$, with A, B open, and $A \cap B = \emptyset$, then either A or B is empty.

Note Equivalently, $X = A \cup B$ with A, B closed and $A \cap B = \emptyset$ $\Rightarrow A = \emptyset$ or $B = \emptyset$ or: the only subsets of X which are both open and closed are X, $\emptyset$.

Examples



$\{y = x^2\} \subseteq \mathbb{R}^2$ is homeomorphic to $\mathbb{R}$, and $\mathbb{R}$ is connected.



$\{y = \frac{1}{x}\} \in \mathbb{R}^2$ is disconnected as it is the union of subspaces $\{x > 0\}$ and $\{x < 0\}$

Note $[0,1] = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ are not disconnections.

Definition The maximal connected subsets of a space X are called its connected components.

Lemma/Definition A space X is connected if wherever $f \to \{0, 1\}$ is a continuous map from X to a 2 point discrete space, then f is constant.

Proof Let X be a space, and $f : X \to \{0, 1\}$ continuous. Then if f is onto, $X = f^{-1}(0) \cup f^{-1}(1)$, so X is not connected. Conversely, if we can write $X = A \cup B$, A, B disjoint and non empty, then $f : X \to \{0, 1\}$, $f|_A = 0$, $f|_B = 1$ is continuous.

We observe that connectedness is a topological property, i.e.
$X \cong Y$ then X connected $\Leftrightarrow$ Y connected.

In fact, connectivity behaves well under maps.

Lemma If $f: X \to Y$ is ~~connected~~ continuous, and X is connected, then $f(X) \subseteq Y$ is connected. In particular, quotients of connected spaces are connected.

Proof

If $f(X) = A \cup B$ is a union of disjoint open sets, then (from the definition of the subspace topology), $A = U \cap f(X)$, $B = V \cap f(X)$ for U, V open in Y. Then $X = f^{-1}(U) \cup f^{-1}(V)$, open in X and disjoint. So $f(X)$ disconnected $\Rightarrow$ X disconnected.

Theorem

$I \subseteq (\mathbb{R}, \mathcal{T}_{Eucl}) \iff I$ is an interval.

meaning $x, y, z \in \mathbb{R}$, $x < y < z$, $x, z \in I \Rightarrow y \in I$.
$\{a\}$, $(1, \infty)$, $(-3, 1]$, all intervals

Proof

$I \ni x$

If I is not an interval, $\exists x, y, z \in \mathbb{R}$ such that $x, z \in I$, $y \notin I$, Then $I = I \cap \underset{\ni x}{(-\infty, y)} \cup I \cap \underset{\ni z}{(y, \infty)}$ exhibits I as not connected.

Conversely, suppose that I is an interval and also $I = A \cup B$, with A, B open/closed and disjoint. Suppose $A, B \neq \emptyset$. Let $a \in A$, $b \in B$. WLOG, $a < b \Rightarrow [a, b] \subseteq I$. Let $A' = A \cap [a, b]$ $B' = B \cap [a, b]$. So A', B' disjoint, both closed and cover $[a, b]$.

Consider $(x, y) \mapsto d_{Eucl}(x, y)$ as a function $A' \times B' \to \mathbb{R}$.

$A' \times B' \subseteq \mathbb{R}^2$ is obviously a closed and bounded set in $\mathbb{R}^2$ with respect to the usual Euclidean distance. $d_{Eucl}: \mathbb{R}^2 \to \mathbb{R}$ is continuous. From IA Analysis, A continuous function on a closed, bounded interval in $\mathbb{R}^n$ is bounded and attains it's bounds

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So $A' \times B' \to \mathbb{R}$, $(x,y) \mapsto d(x,y)$ has a minimum, achieved at some $(a',b') \in A' \times B'$. A', B' disjoint $\Rightarrow$ this min is strictly positive. Let $c = \frac{a'+b'}{2}$. Then $c \in [a,b] \in I$. But $d(c,b') = \left|\frac{a'-b'}{2}\right| < d(a',b')$ so $c \notin A'$. Similarly, $d(a',c) < d(a',b') \Rightarrow c \notin B'$.

Remark In 2-3 lectures time, we'll reprove the Heine-Borel Theorem in a more general context.

Corollary If X is a connected space, and $f: X \to \mathbb{R}$ is continuous, then $f(X)$ is an interval (cf the Intermediate Value Theorem).

Example

$\mathbb{R} \not\cong \mathbb{R}^k$ for $k > 1$. A sketch of the proof: Suppose $\varphi: \mathbb{R} \to \mathbb{R}^k$ is a homeomorphism. Then $\varphi: \mathbb{R}\setminus\{0\} \to \mathbb{R}^k \setminus \{\varphi(0)\}$ is a homeomorphism. $\mathbb{R}\setminus\{0\}$ is not connected, so it suffices to prove that if $k>1$, $\mathbb{R}^k\setminus\{\varphi(0)\}$ is connected. If $(x,y), (x',y') \in \mathbb{R}^k \setminus\{0\}$, obviously one can join them by a path i.e. $\exists \gamma: [0,1] \to \mathbb{R}^k\setminus\{0\}$, $\gamma(0) = (x,y)$, $\gamma(1) = (x',y')$. Suppose in fact $\mathbb{R}^k\setminus\{0\}$ had been disconnected, say $f: \mathbb{R}^k \setminus \{0,1\}$ was continuous and onto. $(x,y) \in f^{-1}(0)$, $(x',y') \in f^{-1}(1)$. Then $f \circ g : [0,1] \xrightarrow{\text{continuous}} \{0,1\}$, $0 \mapsto 0$, $1 \mapsto 1$, disconnecting $[0,1]$.
So $\mathbb{R}^k\setminus\{0\}$ is connected for $k>1$.

Similarly, $[0,1] \not\cong [0,1]\times[0,1]$, but there do exist space filling curves $\gamma: [0,1] \to [0,1]^2$, continuous and onto.

(In fact $\mathbb{R}^k \not\cong \mathbb{R}^l$ when $k \neq l$, but more topology is needed).

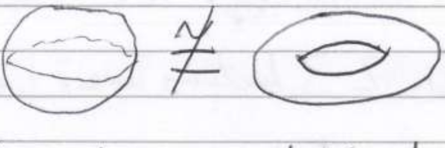
Digression

the loop space of X

If X is a space, $LX = \{\text{Maps } S^1 \to X \text{ which are continuous}\}$

If $X \subseteq (\mathbb{R}^n, \tau_{Eucl})$, then LX inherits a metric (X is a metric space)

$d(f, g) = \max_{\theta} d_X(f(\theta), g(\theta))$. This is well defined by Heine-Borel. Then



LX connected $\qquad$ LX not connected.

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We defined connectedness via $X = A \cup B$, A, B disjoint, open.

Then X connected $\Rightarrow$ A or B is empty. Equivalently, if

$f : X \to \{0,1\}$ continuous $\Rightarrow$ constant.

<u>Definition</u>

A space (X, τ_X) is <u>path-connected</u> if $\forall x, y \in X$, $\exists$ a path $j : [0,1] \to X$ such that $j(0) = x$, $j(1) = y$

The maximal path connected subsets are the path components of X.

<u>Lemma</u>

"Being joined by a path" defines an equivalence relation on points of X.

<u>Proof</u> i) $x \sim x$ via a constant path $j : [0,1] \to X$, $j(t) = t \ \forall t$

ii) If $x \sim y$ meaning $\exists j : [0,1] \to \cancel{X}$, continuous, $j(0) = x$, $j(1) = y$, then $t \mapsto j(1-t)$ defines a path $[0,1] \to X$ with start point x and end point y.

iii) If $x \sim y$ and $y \sim z$, we have $j : [0,1] \to X$, $j(0) = x$, $j(1) = y$ and $\tau : [0,1] \to X$, $\tau(0) = y$, $\tau(1) = z$.

Then define $j * \tau : [0,1] \to X$ by $j * \tau = \begin{cases} j(2t) & 0 \leq t \leq \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$

So $j * \tau : [0,1] \to X$ is continuous, and $x \sim z$ $\square$

Recall:

If H is connected, then $cl(H)$ is connected. If $f|_H$ is constant, then $f|_{cl(H)}$ is constant. Using this, one can show that in general, connectedness and path connectedness are not equivalent.

Example (Topologists' sine curve)

Let $X \in \mathbb{R}^2$ be $A \cup B$, $A = \{ (0, t) \mid -1 \leq t \leq 1 \}$

$B = \{ (x, \sin \frac{1}{x}) \mid x > 0 \}$. Then X is connected but not path connected.

However, path connectedness $\Rightarrow$ connectedness by the argument we gave for $\mathbb{R}^2 \setminus \{0\}$.

Proof

i) X is connected: B is a graph of a continuous function $(0, \infty) \to \mathbb{R}$

So, from our knowledge of product spaces, $B \cong (0, \infty)$, so B is connected. Therefore $Cl(B)$ is connected. So it suffices to prove that $A \subseteq Cl(B)$

By symmetry and closedness, we will show that $\{(0, y) \mid 0 < y < 1\} \subseteq Cl(B)$

Fix $\varepsilon > 0$, and choose n such that $\frac{1}{2\pi n} < \varepsilon$. Then $\sin \frac{\pi}{2}(4n+1) = 1$, $\sin \frac{\pi}{2}(4n+3) = -1$ $\Rightarrow$ $\sin \frac{1}{x}$ takes the value y at some $x_0 \in \left[\frac{2}{(4n+3)\pi}, \frac{2}{(4n+1)\pi} \right]$. But then by choice of n, $d((x_0, y), (0, y)) < \varepsilon$

so $\forall \varepsilon > 0$, $B_{(0,y)}(\varepsilon) \cap B \neq \emptyset$. So $(0, y) \in Cl(B)$ as required.

ii) X is not path connected: Suppose the contrary, and $\exists$ a path

$j : [0, 1] \to X$, $j(0) = (0, 0)$, $j(1) = (1, \sin(1))$.

Let $C = \sup \{ t \mid j(t) \in A \subseteq y\text{axis} \}$. A is closed, so in fact $j(C) \in A$.

$\Rightarrow C < 1$.

of path j at $c \in [0, 1]$

Note that by continuity that $\exists \delta > 0$ such that $j|_{[c, c+\delta]} \subseteq B_{j(c)}(\frac{1}{3})$

We consider $\bar{j} : [0, 1] \to \mathbb{R}$, $t \mapsto \pi_x \circ j(t)$, continuous.

$\bar{j}(c) = 0$, but $\bar{j}(t) > 0$ $\forall t > c$.

$\pi_x : (x, y) \mapsto x$

$\tilde{j}$ is continuous, on a connected set, so it has a connected image, so $\tilde{j}$ has image on an interval. Necessarily, the image of $\tilde{j}$ contains $[0, \alpha]$ for some $\alpha > 0$, but $\sin(\frac{1}{x})$ takes on all values between -1 and 1 in any interval $(0, \alpha]$. Therefore $j|_{[c,c+\delta]}$ must contain points of X with y-coordinate 1 and -1, violating $j|_{[c,c+\delta]} \subseteq B_{j(c)}(\frac{1}{3})$. So in fact no such j exists, and X is <u>not</u> path connected. $\square$

The above is the exception of the rule.

<u>Theorem</u>

Connected open subsets of Euclidean space <u>are</u> path connected.

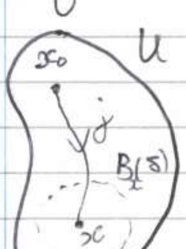
(Note: Manifolds, i.e. spaces locally $\cong \mathbb{R}^n$ have connectedness $\Leftrightarrow$ path connectedness)

<u>Proof</u> Let $U \subseteq \mathbb{R}^n$ be open and connected.

Fix $x_0 \in U$. Let $A = \{x \in U \mid \exists \text{ a path, } x_0 \text{ to } x\}$

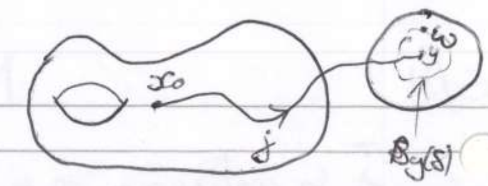
$B = \{y \in U \mid \nexists \text{ a path, } x_0 \text{ to } y\}$. $A \cap B = \emptyset$. Moreover, $x_0 \in A$. We will prove that A, B, are both open. Then $U = A \cup B$, a union of disjoint open sets, and connectness $\Rightarrow B = \emptyset$. Then, our equivalence relation property $\Rightarrow$ path connectedness.

If $x \in A$, $\exists \delta > 0$ such that $B_x(\delta) \subseteq U$, as $U \subseteq \mathbb{R}^n$ is <u>open</u>.



If $z \in B_x(\delta)$, $\exists$ a radial path from z to x, so by concatenation with j from x_0 to x, so $\exists$ a path x_0 to x, i.e. $x \in A \Rightarrow \exists \delta > 0$ such that $B_x(\delta) \subseteq A$, i.e. A is open.

Analogously, if $y \in B$, $B_y(\delta) \subseteq U$:



If $w \in B_y(\delta)$ lies in A, it's joined to x_0 by a path, we attach a radial path $\rightsquigarrow$ path $x_0 \rightsquigarrow y$ ※. So $B_y(\delta) \subseteq B$, B is also open. $\square$

Remark

In studying differential equations, we want to solve an equation $Df = 0$. D is built out of "differential operators" e.g. $D = \left(\frac{d}{dx}\right)^3 + 2\frac{d}{dx} + \sin(x)$.

We introduce a family of equations $D^{(s)} f = 0$, $0 \le s \le 1$, such that $D^{(0)} = D$, and $D^{(1)} f$ is easy to solve.

Then consider $\{ s \in [0,1] \mid \exists$ a solution $f^{(s)}$ of the equation$\}$. If one can show this is both open and closed, and we've solved $D^{(1)}$, then it must be all of $[0,1]$, and D admits a solution. ("Continuity method").

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Definition

Let $(X, \mathcal{T}_X)$ be a topological space. An open cover of X is a collection $\{U_\alpha\}_{\alpha\in A}$ of open subsets of X such that $X = \bigcup_\alpha U_\alpha$

X is compact if every open cover admits a finite sub-cover, i.e. $\exists \{\alpha_1, \alpha_2, \dots, \alpha_N\} \subseteq A$ such that $X = \bigcup_{i=1}^N U_{\alpha_i}$

Remarks

Compactness is a "local to global" principle. $f: X \to \mathbb{R}$ continuous. $\forall x \in X, f(x) \in \mathbb{R}$, so $\exists U_x \ni x$ open such that on U_x, $|f(y)| < |f(x)| + 1$. If X is compact,
$X = \bigcup_{i=1}^n U_{x_i} \Rightarrow f$ bounded. $|f(y)| < \max_i \{|f(x_i)|+1\}$

Key Theorems - Proofs are to come

i) $A \subseteq (\mathbb{R}^n, \mathcal{T}_{Eucl})$ is compact $\Leftrightarrow$ A is closed and bounded.

ii) (X, d) metric space is compact $\Leftrightarrow$ Every sequence in X has a convergent subsequence

Note Compactness is a topological property. If X is compact, $X \cong Y$, then Y is compact. If $A \subseteq X$, "A is compact" is a property of $(A, \mathcal{T}_A)$, the subspace topology.

Examples

1. $(X, \mathcal{T}_{discrete})$ has every subset open, so $X = \bigcup_{x\in X} \{x\}$.
So X compact $\Leftrightarrow$ X finite.

2 $(\mathbb{R}, \mathcal{T}_{Zariski})$ has open subsets complements of finite sets. This is compact.

Proof Let $\{U_j\}_{j\in J}$ be an open cover. $\bigcup_j U_j = \mathbb{R}$. Write $U_j = \mathbb{R} \setminus Z_j$, where Z_j is finite (discard any empty U_j)

$\bigcup U_j = \mathbb{R} \Rightarrow \bigcap_{j\in J} Z_j = \emptyset$. Since the Z_j are finite, $\exists j_1, j_2, \ldots, j_N$ such that $\bigcap_{i=1}^N Z_{j_i} = \emptyset$.

So $\mathbb{R} = \bigcup_{k=1}^N U_{j_k}$ ← $\nexists$ a finite subcover

3. $(\mathbb{R}, \tau_{Eucl})$ is <u>not</u> compact. $U_j = (-j, j) \Rightarrow \mathbb{R} = \bigcup_{j\in\mathbb{N}} U_j$

4. If (X, d) is compact, then X is <u>bounded</u> i.e. $\sup_{x,y\in X} d(x,y) < \infty$

Proof: Pick $x_0 \in X$. Consider $\bigcup_{n\geq 1} B_{x_0}(n)$ is an open cover of (X,d) $\square$

<u>Lemma</u>

If $f: X \to Y$ is a continuous map of topological spaces. Then if X is compact, $f(X)$ is compact.

<u>Proof</u>

Let $\{V_j\}_{j\in J}$ be an open cover of $f(X)$. So $V_j = f(x) \cap W_j$, for W_j open in Y. Then $f^{-1}(V_j) = f^{-1}(W_j)$ is therefore open in X. X compact and $X = \bigcup f^{-1}(W_j) \Rightarrow \exists j_1, \ldots, j_N$ such that $X = \bigcup_{k=1}^N f^{-1}(W_{j_k})$

So $f(X) = \bigcup_{k=1}^N V_{j_k}$ has a finite subcover. $\square$

<u>Corollary</u>

If $f: X \to \mathbb{R}$ is continuous, X is compact, then $f(X)$ is bounded. $\square$

Most compactness arguments make use of:

<u>Lemma</u>

i) A closed subspace of a compact space is compact.

ii) A compact subspace of a Hausdorff space is closed.

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Metric and Topological Spaces 9

Proofs

i) $A \subseteq X$, X compact, A closed. Take an open cover of A.
$A = \bigcup_{j\in J} V_j$. Write $A \cap W_j$, W_j open in X.
Then $\{W_j\}_{j\in J} \cup (X \setminus A)$ forms an open cover of A. Since X is compact, there is a finite subcover of X, $\{W_{j_1}, \dots, W_{j_n}\} \cup (X\setminus A)$.
Then $\bigcup_{i=1}^{n} V_{j_i}$ forms a finite subcover of $A \Rightarrow A$ is compact. $\square$

ii) Now, suppose X is Hausdorff, $Z \subseteq X$ is compact.
We claim that Z is closed, so $X\setminus Z$ is open. So it suffices to show that if $x \in X\setminus Z$, $\exists$ some open neighbourhood $U_x \ni x$ with $U_x \subseteq X \setminus Z$.
Then $X\setminus Z = \bigcup_{x\in X\setminus Z} U_x$ will be open.
Fix $x_0 \in X\setminus Z$. If $z \in Z$, the Hausdorff condition
$\Rightarrow \exists$ disjoint open sets, open in X, $U_z \ni x_0$, $V_z \ni z$, $U_z \cap V_z = \emptyset$
For each $z\in Z$, we have an open V_z, so Z is covered by the sets $V_z \cap Z$
Compactness of $Z \Rightarrow \exists$ a finite subcover, say $Z \subseteq V_{z_1} \cup \dots \cup V_{z_N}$, for some $\{z_1, z_2, \dots, z_N\} \subseteq Z$. So we have open neighbourhoods $U_{z_1}, U_{z_2}, \dots, U_{z_n}$ of x_0. Define $U_{x_0} := \bigcap_{i=1}^{N}$, a finite intersection of open sets, and therefore open.

Claim: $U_{x_0} \subseteq X \setminus Z$

Proof: If $w \in U_{x_0}$, then $w \in U_{z_i}$ $\forall i$, $1 \le i \le N$.
$U_{z_i} \cap V_{z_i} = \emptyset \Rightarrow w \in V_{z_i}$, $1 \le i \le N$. $\Rightarrow w \in Z$. $\square$

<u>Corollary</u> Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.

<u>Proof</u> Let $g = f^{-1}$ be the inverse function, so we need to know g is continuous. From lectures, it suffices to know that g^{-1}(closed set) is closed. i.e. it suffices to know that f is a closed mapping, taking closed sets to closed sets. But if $A \subseteq X$ closed, X compact $\Rightarrow$ A compact
f continuous $\Rightarrow$ $f(A)$ compact in Y. Y Hausdorff $\Rightarrow$ $f(A)$ closed. $\square$

Note: We proved $\mathbb{R}/\mathbb{Z} \cong S^1$,  $\cong$  $\frac{\mathbb{R}\times\mathbb{R}}{\mathbb{Z}\times\mathbb{Z}} \cong S^1 \times S^1$

<u>Digression</u> Let X be a space such that

i) $X \underset{\text{locally}}{\cong} \mathbb{R}^3$, X a 3-manifold   ii) X compact, connected, simply connected

<u>Theorem</u> (Perelman)

$$X \cong S^3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$$

$\uparrow$

Formerly the

Poincaré Conjecture !!!

20/05/11 Metric and Topological Spaces (10)

Heine-Borel

X is compact if every open cover admits a finite subcover.

Theorem 1 In $(\mathbb{R}, \mathcal{T}_{Eucl})$, $[a, b]$ is compact

Theorem 2 If X, Y are compact, $X \times Y$ is compact (with product topology)

Corollary (Heine-Borel Theorem)

$A \subseteq (\mathbb{R}^n, \mathcal{T}_{Eucl})$ is compact $\Longleftrightarrow$ it is closed and bounded

Proof

If A is compact, certainly it is bounded, and compact subspaces of Hausdorff spaces are closed.

Conversely, if A is closed and bounded, $\exists\, a, b \in \mathbb{R}$ such that $A \subseteq [a, b]^n$.

Then A ~~closed~~ is a compact space by Theorems 1 and 2, so A is compact $\square$

Corollary If X is compact and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is bounded and attains its bounds.

Proof

X is compact, so closed and bounded, so $\sup f$, $\inf f$ exist, so by closedness, these belong to $f(X)$ $\square$

Proof of Theorem 1

Let $\{U_i\}_{i \in I}$ be an open cover of $[a, b]$. Consider

$A = \{ t \mid t \in [a, b],\ [a, t] \text{ has a finite subcover wrt } \{U_i\}_{i \in I} \}$

We wish to show that $b \in A$.

Note Clearly A is bounded above by b, $a \in [a, b]$ lies in some set U_i, so $a \in A$, and $A \neq \emptyset$

Moreover, if $a \in U_i$, U_i is open so $[a, a + \delta) \subseteq U_i$ for some $\delta > 0$

so $[a, a + \frac{\delta}{2}]$ has a finite subcover $\Rightarrow$ $c = \sup(A)$ satisfies $c > a$.

$\varepsilon > 0$

Suppose for contradiction $c < b$, $c \in [a, b]$, so $\exists\, i_0$ such that $c \in U_{i_0}$. So openness $\Rightarrow$ $(c - \varepsilon, c + \varepsilon) \subseteq U_{i_0}$. ~~Consider $c - \frac{\varepsilon}{2}$ has the property that~~ $[a, \omega]$ ~~$[a, c - \frac{\varepsilon}{2}]$~~ has some finite subcover wrt $\{U_i\}$

$\exists \in (c - \varepsilon, c)$

Throw in U_{i_0} to see $[a, c + \frac{\varepsilon}{2}]$ has some finite subcover, violating $c = \sup A$

So we deduce that $c = b$. $b \in U_k$ for some k, with U_k open so $(b-\delta, b] \subseteq U_k$. Certainly, $\exists$ a point $\omega \in (b-\delta, b)$ such that $[a, \omega]$ has a finite subcover, Since $b = \sup A$. Add in U_k to see that $[a, b]$ has a finite subcover, so $b \in A$. $\square$

<u>Proof of Theorem 2</u> X, Y compact $\Rightarrow X \times Y$ compact

Let $\{U_i\}_{i \in I}$ be an open cover of $X \times Y$. Fix $x \in X$. We aim to show that $\exists$ an open set $A_x \ni x$, such that $A_x \times Y$ is covered by finitely many of the $\{U_i\}$.

If $y \in Y$, $(x, y) \in U_i$ for some i. By definition of product topology, $\exists$ open sets $\left.\begin{array}{l} A_y \ni x \text{ of } x \text{ in } X \\ B_y \ni y \text{ of } y \text{ in } Y \end{array}\right\}$ such that $(x,y) \in A_y \times B_y \subseteq U_i$.

$\forall y \in Y$, we get an open $B_y \ni y \Rightarrow \{B_y\}_{y \in Y}$ is an open cover of $\{x\} \times Y$.

$\exists$ a finite subcover $B_{y_1}, \ldots, B_{y_k}$ of $\{x\} \times Y$.

Let $A_x := \bigcap_{j=1}^{k} A_{y_j}$

The A_{y_i}'s were open neighbourhoods of $x \in X$, so the finite intersection A_x is an open neighbourhood of $x \in X$. Moreover, $\{A_x \times B_{y_j} \mid j = 1, \ldots, k\}$ actually covers $A_x \times Y$, and the sets $A_x \times B_{y_j}$ each belong to an element of the original cover $\{U_i\}$. So A realises our aim.

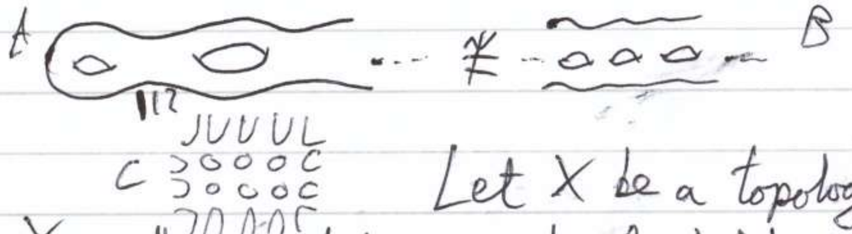
We now restart. For each $x \in X$, we can pick such an A_x, satisfying our aim. Now $\{A_x \mid x \in X\}$ forms an open cover of X. By compactness, $\exists$ a finite subcover $A_{x_1} \ldots A_{x_p}$ of X by such sets. Then $X \times Y = \bigcup_{i=1}^{p} A_{x_i} \times Y$, a finite union of sets, each of which admits a finite subcover, wrt $\{U_i\}$. So we are done. $\square$

<u>Remarks</u>

1. The product topology on an infinite product is defined such that basic open sets in $\prod_{i \in I} X_i$ are of the form $\prod_{i \in I} U_i$, with all but finitely many U_i the whole of U_x. Then, if $\{X_i\}_{i \in I}$, then $\prod_{i \in I} X_i$ is compact. Note that because of Cantor-type sets, the general compact set in $\mathbb{R}$ is very wild.

2. Heine-Borel completes our proof that intervals $\Leftrightarrow$ connected in $\mathbb{R}$.

__Definition__



Let X be a topological space. The __ends__ of X are the connected components of $X \setminus K$ where $K \subseteq X$ is compact. We say that ends of $X \setminus K_1$ and $X \setminus K_2$ are equivalent, if for $K_1 \subseteq K_2$, the natural map $X \setminus K_1 \to X \setminus K_2$ takes one to the other.
ends $\Leftrightarrow$ connected components of complements of arbitrarily large compact sets.

In surfaces A, C : Given any compact set K, $\exists$ a larger compact set K' such that $X \setminus K'$ is connected (A, C have one end, B has two).

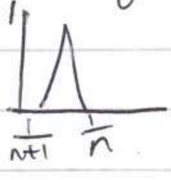
__Warning__

In $\mathbb{R}^n$, the closed unit ball is compact. This fails in general metric spaces

__Example__

In $(C[0,1], d_\infty) = \{$Continuous functions $[0,1] \to \mathbb{R}$, $d_\infty(f,g) = \sup_t |f-g|$

one can consider a sequence



$d_\infty(f_n, 0) = 1$

but $d_\infty(f_n, f_m) = 1$, $n \neq m$

So $\nexists$ a convergent subsequence ($\Rightarrow$ (why?) that $B_0(1)$ is not compact).

23/05/11 # Metric and Topological Spaces (11)

Sequential Compactness

Definition: A metric space (X,d) is sequentially compact if every sequence $(x_n) \subseteq X$ has a convergent subsequence.

Note: If $C \subseteq (X,d)$ is a subset, it inherits the structure of a metric space, say C is sequentially compact if $(x_n) \subseteq C$ has a subsequence converging to a point of C.

Observation If $x_n \to x_\infty$ wrt metric d, and d' is an equivalent metric, then $x_n \to x_\infty$ wrt d'. So sequential compactness of X is a property of the underlying topological space.

Theorem (X,d) sequentially compact $\Leftrightarrow$ (X,d) compact
$(0,1)$ is not sequentially compact, $(\frac{1}{n})$ has no convergent subsequence.
$[0,1]$ is sequentially compact, but $0,1,0,1\ldots$ is not itself convergent.

Lemma Let X be a metric space and $C \subseteq X$ a sequentially compact subset. Then C is closed in X.

Proof Closed sets are those which are equal to their closures. We show that if $x \in Cl(C)$, then $\exists (x_n) \subseteq C$ such that $x_n \to x$. If $x \in Cl(C)$, every open neighbourhood of x meets C. In particular, $\forall n\ \exists x_n$ such that $x_n \in B_x(\frac{1}{n}) \cap C$. Then $x_n \to x$ (since $d(x_n, x) < \frac{1}{n} \to 0$) so $x \in C$ by sequential compactness.

Proposition Compact $\Rightarrow$ Sequentially Compact

Proof Let (X,d) be a metric space and $C \subseteq X$ a compact subspace. Either $S = \{(\text{members of } x_n)\}$ is either finite, or infinite. In the 1st case, $\exists$ a constant subsequence, so assume $|S| = \infty$. We claim that S has a limit point in C.

Suppose S has no limit point in C. Then, $\forall x \in C$, $\exists\, \varepsilon(x) > 0$

$\exists\, \varepsilon(x) > 0$ such that $B_x(\varepsilon(x)) \cap S$ is empty or $\{x\}$.
(exactly the definition of limit points and metric space topology)
So $\{B_x(\varepsilon(x)) \mid x \in C\}$ gives an open cover of C, then $\exists$ a finite subcover. But each $B_{x_i}(\varepsilon(x_i))$ contains at most on point of S.
$\Rightarrow |S| < \infty$ $\times\!\!\times$
So indeed, S has a limit point in C, say x_∞.

We now build a subsequence of (x_n) tending to x_∞. WLOG, $x_\infty \notin S$. (Check that the other case works). Now iteratively, pick $x_{n(1)} \in B_{x_\infty}(1)$.
Suppose we have $n(1) < n(2) < n(3) < \ldots < n(k)$ such that $x_{n(i)} \in B_{x_\infty}(\frac{1}{i})$
Let $\varepsilon = \min\{\frac{1}{k+1}, d(x_j, x_\infty), 1 \leq j \leq n(k)\}$. Note that $\varepsilon \neq 0$, as $x_\infty \notin S$. So $\exists\, n(k+1)$ such that $x_{n(k+1)} \in B_{x_\infty}(\varepsilon)$
Definition of $\varepsilon \Rightarrow$ i) $x_{n(k+1)} \in B_{x_\infty}(\frac{1}{k+1})$ ii) $n(k) < n(k+1)$
Then $x_{n_j} \to x_\infty$

<u>Corollary</u> A bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
<u>**Proof**</u> (x_n) lies in a bounded set, then $Cl(\{x_n\})$ is closed and bounded, so compact, and we use the previous result.

For the converse, "Sequentially Compact $\Rightarrow$ Compact" we use the following.

<u>Definition</u> Let (X, d) be a metric space and $\{U_j\}_{j\in J}$ an open cover of X. A <u>Lebesgue Number</u> for the cover is $\delta > 0$ such that $\forall x \in X$, $B_x(\delta) \subseteq U_{j(x)}$ for some $j(x) \in J$.

<u>Example</u> For $(0,1)$, write $(0,1) = \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 1)$. This open cover has <u>no</u> Lebesgue Number. If $\delta > 0$, pick $k > \frac{1}{\delta}$, then $B_{\frac{1}{k}}(\delta) = (0, \frac{1}{k}+\delta) \not\subseteq (\frac{1}{n}, 1)$ for any $n \in \mathbb{N}$.

<u>Lemma</u>
Let X be a sequentially compact metric space.
i) Any open cover $\{U_j\}_{j\in J}$ of X has a Lebesgue number.
ii) $\forall \varepsilon > 0$, $\exists$ finite $S \subseteq X$ such that $X = \bigcup_{x\in S} B_x(\varepsilon)$

<u>Sketch of proof for ii)</u>
Take $\varepsilon > 0$ and suppose <u>no</u> such finite S exists. Take any x, and note that $B_x(\varepsilon)$ doesn't cover X. Call $x = x_1$. Pick $x_2 \notin B_{x_1}(\varepsilon)$.
$X \neq B_{x_1}(\varepsilon) \cup B_{x_2}(\varepsilon)$, so $\exists x_3$ outside these. Iteratively, we construct a sequence such that x_i has distance $\geq \varepsilon$ from all preceding members.
So $\nexists$ a convergent subsequence (or it would be "Cauchy", terms would get close).
Part i) is similar.

<u>Proposition</u> Sequentially Compact $\Rightarrow$ Compact
<u>Proof</u>
Let (X,d) be sequentially compact, and let $\{U_j\}_{j\in J}$ be an open cover of X. By i) we can pick a Lebesgue number $\delta > 0$.
By ii) $\exists \{x_1, \dots, x_n\}$ in X such that $X = \bigcup_{i=1}^{n} B_{x_i}(\delta)$
But for each $1 \leq i \leq n$, $\exists$ $j(i)$ such that $B_{x_i}(\delta) \subseteq U_{j(i)}$
So $X = \bigcup_{i=1}^{n} U_{j(i)}$, and we have a finite subcover.

<u>Example</u> Let (X,d) be a compact metric space. Suppose $f: X \to X$, such that $\forall x, y \in X$, $d(f(x), f(y)) = d(x,y)$, then f is a homeomorphism.

<u>Proof</u> f is obviously continuous and injective (check!). We claim that f is surjective. Then we have a continuous bijection from a compact space to a Hausdorff space.

Let $x \in X$. Let $x_n = f^{(n)}(x) = \overbrace{f \circ f \circ \dots \circ f}^{n \text{ times}}(x)$ which defines a sequence $(x_n) \subseteq X$. Sequential Compactness $\Rightarrow$ $\exists$ a convergent subsequence $x_{n(r)} \to x_\infty \in X$. But $d(f^{n(r)}(x), f^{n(s)}(x)) = d(f^{n(r)-n(s)}(x), x)$ when $n(r) > n(s)$, by the isometric nature of f. Let $r, s \to \infty$.
Deduce $x = x_\infty$. So x_∞ is a limit of points in $f(X)$, so $\in f(X$

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Metric and Topological Spaces (12)

A metric space (X, d) is compact $\Leftrightarrow$ it is sequentially compact.

<u>Definition</u> A <u>Cauchy Sequence</u> in (X, d) is a sequence (x_n) such that $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall n, m > N_\varepsilon$, $d(x_n, x_m) < \varepsilon$

(X, d) is complete if every Cauchy Sequence in (X, d) converges (to a point of X).

<u>Note</u> Completeness is <u>not</u> a topological property (in contrast to compactness, connectedness

E.g. $(0,1)$, d_{euc} is <u>not</u> complete, $(\frac{1}{n})$ is Cauchy.

$(\mathbb{R}, d_{euc})$ <u>is</u> complete (by axiom / definition)

But these spaces <u>are</u> homeomorphic.

<u>Example</u> $(X, d_{discrete})$ is always complete.

<u>Proof</u> What do Cauchy sequences look like? Take $\varepsilon = \frac{1}{2}$. If $(x_n) \subseteq X$ is Cauchy, $\exists N_{\frac{1}{2}}$ such that $\forall n, m > N_{\frac{1}{2}}$, $d(x_n, x_m) < \frac{1}{2}$.

$\Rightarrow (x_n)$ is eventually constant, so it converges.

<u>Exercises</u>

i) If (X, d) is compact, (and so sequentially compact) then it is complete.

ii) If (X, d) is complete, and $Y \subseteq X$ is a closed subspace, then $(Y, d_{X \times Y})$ is complete.

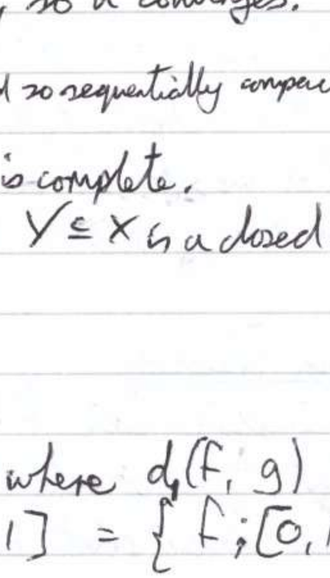
<u>Example</u>

a) $(C[0,1], l^1)$, a metric where $d_1(f, g) = \int_0^1 |f(t) - g(t)| \, dt$ is <u>not</u> complete. $C[0,1] = \{ f ; [0,1] \to \mathbb{R}, f \text{ continuous} \}$

b) $(C[0,1], l^\infty)$, $d(f,g) = \sup_t |f(t) - g(t)|$. This is complete.

<u>Proof</u>

a) Consider (f_n) with



$f_n \in C[0,1]$

Clearly, f_n is Cauch wrt d_1.

Suppose for contradiction, $f_n \to g \in C[0,1]$, and $\exists x \in (0, \frac{1}{2})$ such that $g(x) \neq 0$. Then $\exists \delta > 0$ such that on $(x - \delta, x + \delta)$, $|g(y)| > \varepsilon = \frac{|g(x)|}{2} > 0$.

Then, if n is sufficiently large, e.g. $\frac{1}{2} - \frac{1}{n} > x$, then
$d_1(f_n, g) = \int_0^1 |f_n(t) - g(t)| dt \geq \int_{x-\delta}^{x+\delta} |f_n(t) - g(t)| dt \geq 2\delta\varepsilon$
So $d_1(f_n, g) \not\to 0$ ※
Analogously, $g(x) = 1 \; \forall x \in (\frac{1}{2}, 1)$, but $g \in C[0,1]$ ※

b) Let (f_n) be Cauchy in $(C[0,1], d_\infty)$. If $t \in [0,1]$
$|f_n(t) - f_m(t)| < \sup_t |f_n(t) - f_m(t)| = d_\infty(f_n, f_m) \to 0$ as $n, m \to \infty$
So $(f_n(t))_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$. $\mathbb{R}$ is complete so $f_n(t)$ converges, say to $F(t)$. This defines $F : [0,1] \to \mathbb{R}$.
We need F continuous, and $f_n \xrightarrow{d_\infty} F$

$|F(t_1) - F(t_2)| \leq |F(t_1) - f_n(t_1)| + |f_n(t_1) - f_n(t_2)| + |f_n(t_2) - F(t_2)|$
$\forall \varepsilon > 0, \exists \delta > 0$ such that $|t_1 - t_2| < \delta$, $n \gg 0$, each term $< \frac{\varepsilon}{3}$
So F is continuous, and in fact uniformly continuous.

$|f_n(t) - F(t)| \leq |f_n(t) - f_m(t)| + |f_m(t) - F(t)| \leq d(f_n, f_m) + |f_m(t) - F(t)|$
So now given $\varepsilon > 0$, choose n, m large enough so that $d(f_n, f_m)$ is negligible.
Now if $\varepsilon > 0$, $\exists m$ such that $|f_m(t) - F(t)| < \frac{\varepsilon}{2}$
Now this holds $\forall n > m$, $n, m \gg 0$ (fill in the details).

<u>Contraction Mapping Theorem</u>
Let (X, d) be complete. Let $f : X \to X$ be such that
$\exists K < 1$, such that $\forall x, y$ $d(f(x), f(y)) < K d(x, y)$ for $x \neq y$.
Then f has a unique fixed point i.e. $\exists x_0 \in X$ such that $f(x_0) = x_0$.

<u>Remark</u> $x \mapsto x + \frac{1}{x}$ on $[1, \infty]$. $\forall x, y$ not equal
$|f(x) - f(y)| < |x - y|$, and obviously there is no fixed point.

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Metric and Topological Spaces (12)

Proof

Note that being a contraction $\Rightarrow$ f continuous. Let $x_0 \in X$. Let $x_1 = f(x_0)$, $x_n = f(x_{n-1})$. $d(x_r, x_{r-1}) \leq K d(x_{r-1}, x_{r-2})$

$d(x_r, x_{r-1}) \leq \ldots \leq K^{r-1} d(x_1, x_0)$

and $\underset{m>n}{d(x_m, x_{n-1})} \leq \sum_{j=0}^{m-n} d(x_{m-j}, x_{m-j-1}) \leq (K^{m-1} + K^{m-2} + \ldots + K^{n-1}) \times d(x_1, x_0)$

$$\leq K^{n-1}\left(\frac{1-K^{m-n}}{1-K}\right) d(x_1, x_0) \leq \frac{K^{n-1}}{1-K} d(x_1, x_0)$$

Since $K^{n-1} \to 0$ as $n \to \infty$, ~~and~~ we infer that x_n is a Cauchy Sequence. So $x_n \to x_\infty$. But $f(x_n) \to f(x_\infty)$ by continuity of f. So $f(x_\infty) = x_\infty$

If $x \neq y$, $f(x) = x$, $f(y) = y$, Then $\underset{d(x,y)}{\underbrace{d(f(x), f(y))}} < K d(x,y) < d(x,y)$ ※

So the fixed point is unique.

<u>Application : Picard's Theorem</u>

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Solve $\frac{du}{dt} = F(t, u(t))$ with $u(t_0) = u_0$ (initial value problem for ODEs).

<u>Claim</u> : $\exists \delta > 0$ such that a unique solution exists on $(t_0 - \delta, t_0 + \delta)$

<u>Idea of proof</u>

u solves the equation $\iff u(t) = u_0 + \int_{t_0}^{t} F(s, u(s))\,ds$ by FTC

Define $\phi : \underset{\text{with } d_\infty}{C[t_0 - 1, t_0 + 1]} \circlearrowleft$ by $\phi(u) = u_0 + \int_{t_0}^{t} F(s, u(s))\,ds$

We want u such that $\phi(u) = u$. By the contraction mapping theorem it is sufficient to show that ϕ is a contraction on some $C[t_0 - \delta, t_0 + \delta]$ (for some $\delta > 0$) wrt d_∞

Since F is nice, $|F(s, u(s)) - F(s, v(s))| < L|u(s) - v(s)|$ "F is Lipschitz in its 2nd variable", L is max $|\partial F|$ say.

$$|\phi(u) - \phi(v)| = \left|\int_{t_0}^{t} F(s, u(s)) - F(s, v(s))\right| ds$$
$$\leq \int_{t_0}^{t} L|u(s) - v(s)| \leq L|t_0 - t|\, d_\infty(u, v)$$

So if δ is such that $L\delta < \frac{1}{4}$ then $d_\infty(\phi(u), \phi(v)) < \frac{1}{2} d_\infty(u,v)$ and ϕ is a contraction.