Chapter 1

A Primer on Continuous-time Economic Dynamics

1.1 Linear Differential Equation Systems

1.1.1 Simplest case

We begin with the simple linear first-order differential equation,

$$\dot{x} = ax, \quad x(0) = x_0$$

The general solution is

$$x(t) = c_0 e^{at},$$

and the initial condition is satisfied by the particular solution,

$$x(t) = x_0 e^{at}$$

The growth rate ofxis given by a.

Next, we solve the two-dimensional system given by

$$\dot{x} = ax, \quad x(0) = x_0$$

and

$$\dot{y} = by, \quad y(0) = y_0.$$

It is useful to write these in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The solution for this two-dimensional system is simply

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{at} \\ y_0 e^{bt} \end{bmatrix}.$$

Each of these problems, the one-dimensional and the two-dimensional, are examples of initialvalue problems. This system possesses a single steady state, $x_0 = 0$ and $y_0 = 0$. A phase diagram plots out $y(t)$ as a function of $x(t)$ for different possible initial values of x and y. The resulting locus relating y to x at each time given any particular initial values, x_0 and y_0, depicts a trajectory along which x and y move over time. Arrows are usually drawn to depict the direction of motion in the $xy-$plane as t increases.

Consider three cases. In the first, the parameters a and b are both positive. For any initial point, (x_0, y_0), that is not the steady state, a trajectory moves away from the origin. This is the unstable case. In the second, a and b are both negative and any trajectory converges to the steady state. This is the stable case. In this case, trajectories asymptotically approach the steady state parallel to the $y-$axis if $b < a < 0$, and conversely. In the third case, one root, a or b is positive and the other is negative. For the example in which $a > 0 > b$, any trajectory such that $x_0 = 0$ converges to the steady state at the rate b. Any trajectory such that $x_0 \neq 0$ moves away from the steady state and converges towards the $x-$axis asymptotically. This is the saddle-path stable case.

Consider an example of saddle-path dynamics such that $a = 1$ and $b = 1$. The solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^t \\ y_0 e^{-t} \end{bmatrix}$$

As t grows without bound, $y(t)$ approaches zero as $x(t)$ grows without bound as $x_0 e^t$.

1.1.2 General case

Let z be an $n - \text{dimensional}$ column vector over the reals and $\dot{z} = Az$ where $z(0) = z_0$. The matrix, A, is $n \times n$ with constant real coefficients. Impose the restriction that A is nonsingular.

For example, consider the two-dimensional differential equation system,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & c \\ d & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

We handle this problem by changing coordinates. We want to find a change of basis from $\begin{bmatrix} x \\ y \end{bmatrix}$ to a new basis, $\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$ such that the initial-value problem becomes

$$\frac{d}{dt} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \hat{a} & 0 \\ 0 & \hat{b} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}, \quad \begin{bmatrix} \hat{x}(0) \\ \hat{y}(0) \end{bmatrix} = \begin{bmatrix} \hat{x}_0 \\ \hat{y}_0 \end{bmatrix}.$$

which has the solution given in section 1.1.1,

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \hat{x}_0 e^{\hat{a}t} \\ \hat{y}_0 e^{\hat{b}t} \end{bmatrix}$$

This is done by finding a nonsingular 2×2 matix, M, with constant coefficients such that

$$M^{-1} \begin{bmatrix} a & c \\ d & b \end{bmatrix} M = \begin{bmatrix} \hat{a} & 0 \\ 0 & \hat{b} \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}.$$

We can always do this given the assumptions madeso far. The differential equation systems are identical but expressed in different coordinates.