Chapter 1

Distribution

1.1 Γ distribution

A random variable X is said to have a Γ distribution with parameters α, β if its probability density function is given by

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}, \quad \alpha,\beta \geq 0, \quad x \geq 0. \tag{1.1}$$

The quantity $\Gamma(\alpha)$ is known as the Γ function and it is equal to:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \tag{1.2}$$

α is *shape*, β is called *scale*, and $\theta = \frac{1}{\beta}$ is called *rate*.
Some useful results:

$$\mathbb{E}[X] = \alpha\beta, \quad \mathbb{V}[X] = \alpha\beta^2, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(n) = (n-1)!.$$

If we set $\alpha = 1$ and $\beta = \frac{1}{\lambda}$, we get $f(x) = \lambda e^{-\lambda x}$. We see that the exponential distribution is a special case of the Γ distribution.

1.1.1 Moment Generating function

Moment generating function of $X \sim \Gamma(\alpha,\beta)$ is

$$M_X(t) = (1-\beta t)^{-\alpha} \tag{1.3}$$

Proof:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx}\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}\,dx = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-x(\frac{1-\beta t}{\beta})}\,dx \tag{1.4}$$

Let $y = x\big(\frac{1-\beta t}{\beta}\big)$, then $x = \big(\frac{\beta}{1-\beta t}\big)y$, and $dx = \big(\frac{\beta}{1-\beta t}\big)dy$. Substitute these in the expression above

$$
\begin{aligned}
M_X(t) &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \left(\frac{\beta}{1-\beta t}\right)^{\alpha-1} y^{\alpha-1} e^{-y} \frac{\beta}{1-\beta t}\, dy && (1.5)\\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^{\alpha} \int_0^\infty y^{\alpha-1} e^{-y}\, dy && (1.6)\\
&= (1-\beta t)^{-\alpha} && (1.7)
\end{aligned}
$$

1.2 χ^2 distribution

Let $Z_1, Z_2, \ldots, Z_k$ be independent random variables with $Z_i \sim \mathcal{N}(0,1)$ (iid), then

$$Z = Z_1^2 + Z_2^2 + \cdots + Z_k^2 = \sum_{i=1}^{k} Z_i^2 \sim \chi_k^2 \tag{1.8}$$

χ^2 is a class of distribution indeXed by its degree of freedom, like the t-distribution. In fact, χ^2 has a relation with t.

If $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim \mathcal{N}(\mu, \sigma)$, then

$$X = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2 \tag{1.9}$$

Let $X_1 \sim \chi_n^2$ and $X_2 \sim \chi_m^2$. If X_1 and X_2 are independent, then

$$X_1 + X_2 \sim \chi_{n+m}^2. \tag{1.10}$$

Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathcal{N}(\mu, \sigma)$. Define the sample variance as

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{1.11}$$

Then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2. \tag{1.12}$$

1.2.1 shape of χ^2 distribution

Figure 1.1: χ^2 with different df