Tutorial 1

Matrices

A rectangular array of numbers consisting of m rows and n columns is called a matrix

→ The numbers in the matrix are called entries

→ The order of a matrix with m rows and n columns is $m \times n$.

Columns

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

Rows

The order of A is 2×3.

In a matrix, an entry is often denoted by a_{ij} (or any letter with the subscript ij) where i and j are numbers, $1 \le i \le m$, $1 \le j \le n$.

So, in the above matrix A,

$$a_{11} = 1, \quad a_{12} = 4, \quad a_{13} = 7$$

$$a_{21} = 2, \quad a_{22} = 5, \quad a_{23} = 8$$

Types of matrices

1) Row matrix: Matrix with only one row ($m = 1$)

eg: $\begin{bmatrix} 2 & 4 & 8 \end{bmatrix}$ order: 1×3

2) Column matrix: Matrix with only one column ($n = 1$)

eg: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ order: 2×1

3) **Rectangular matrix**: Matrix with m columns and n rows, $m \neq n$.

eg: $\begin{bmatrix} 1 & 4 & 3 & 7 \\ 2 & 5 & 8 & 6 \end{bmatrix}$ Order: 2×4

4) **Square Matrix**: Rectangular matrix, but with $m = n$.

eg: $\begin{bmatrix} 1 & 9 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 5 \end{bmatrix}$ Order: 3×3

5) **Diagonal matrix**: Only the main diagonal can have non zero elements. Mathematically, $a_{ij} = 0$ if $i \neq j$.

eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ → only diagonal elements can be nonzero.

Note: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is also diagonal

Order: 3×3

6) **Upper triangular matrix**: Only the main diagonal and every element above the diagonal can be nonzero. That is, $a_{ij} = 0$ for $i > j$.

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 10 \\ 0 & 0 & 7 \end{bmatrix}$

7) Lower Triangular matrix: The opposite of upper triangular. Only the main diagonal and every element below the diagonal can be non zero.

That is, $a_{ij} = 0$ for $i < j$.

eg:
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 10 & 7 \end{bmatrix}$$

8) Zero matrix: All entries of the matrix are zero.

eg:
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9) Identity matrix: Diagonal matrix, but with all the diagonal entries $= 1$.

That is

$$a_{ij} = 1 \quad \text{if } i = j$$
$$a_{ij} = 0 \quad \text{if } j \neq i$$

eg:
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We often denote them as I_n where n is the number of columns (which is equal to number of rows).

Trace: Denoted by $tr(A)$ or $sp(A)$, it is the sum of the entries on the main diagonal

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad tr(A) = 1+5+9 = \underline{\underline{15}}.$$

Only defined for sq. matrices.

Matrix Operations

1. Transposition

Inter changing rows and columns.

Simply stated, $a_{ij} \to a_{ji}$.

$$A \begin{bmatrix} 2 & 4 & 8 \\ 6 & 12 & 18 \end{bmatrix}, \quad \text{then } A^T = \begin{bmatrix} 2 & 6 \\ 4 & 12 \\ 8 & 18 \end{bmatrix}$$

If $A^T = A$, A is called symmetric.

eg: Any diagonal square matrix,

$$B = \begin{bmatrix} 10 & 80 \\ 80 & 10 \end{bmatrix}, \quad B^T = \begin{bmatrix} 10 & 80 \\ 80 & 10 \end{bmatrix}$$

Properties

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(KA)^T = KA^T$, K is scalar constant
4. $(A \cdot B)^T = B^T A^T$

2. Addition

If A, B have same order, then,

$C = A \pm B \Rightarrow c_{ij} = a_{ij} \pm b_{ij}$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1+3 & 0+4 \\ 0+5 & 1+6 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 5 & 7 \end{bmatrix}.$$

→ If A & B have same order and $a_{ij} = b_{ij}$ $\forall$ i, j, then $A = B$.

3. Scalar Multiplication

Multiplying a matrix with a real number.

$A = [a_{ij}]_{m \times n}$. $c \in \mathbb{R}$.

Then,

$$cA = [c a_{ij}]_{m \times n}.$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ $c = 2$.

Then, $cA = \begin{bmatrix} 2\times 1 & 2\times 2 & 2\times 3 \\ 2\times 4 & 2\times 5 & 2\times 6 \\ 2\times 3 & 2\times 2 & 2\times 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 6 & 4 & 2 \end{bmatrix}$$

→ Linear Combination

$A_1, A_2, A_3, \ldots, A_p$ are matrices of the same size and $c_1, c_2, c_3, \ldots, c_p$ are real numbers.

$$c_1A_1 + c_2A_2 + c_3A_3 + \cdots + c_pA_p$$

is called a linear combination.

If

$$c_1A_1 + c_2A_2 + c_3A_3 + \cdots + c_pA_p = 0,$$

the linear combination is trivial.

Eg:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 6 & 0 & 10 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 10\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Matrix multiplication

$C_{m\times n}$ is the product of $A_{m\times p}$ and

$B_{l \times n}$ if C_{ij} is the dot product of the ith row of A and jth column of B.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

Each column multiplies with each row to give one entry.

$$C = \begin{bmatrix} (1\times2)+(2\times3) & (1\times4)+(2\times5) \\ (4\times2)+(5\times3) & (4\times4)+(5\times5) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 14 \\ 23 & 85 \end{bmatrix}.$$

Properties:

① In most cases, $A \cdot B \neq B \cdot A$.

② If $E \cdot B = E \cdot C$, then it does not necessarily mean $B = C$.

③ If $E \cdot D = 0$, then it does not necessarily mean that either $E = 0$ or $D = 0$

④ Any matrix multiplied with identity is that matrix itself.

$A_{n\times n} \cdot I_n = A_{n\times n} = I_n \cdot A_{n\times n}$

⑤ Any matrix multiplied with its inverse gives you the identity.

$$A_{n\times n} \cdot A^{-1}_{n\times n} = I_n = A^{-1}_{n\times n} \cdot A_{n\times n}.$$

More on this next time.

Elementary Row operations

We can perform row operations to transform $A \to B$.

(a) Inter change two rows.

$$\begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} - & R_3 & - \\ - & R_2 & - \\ - & R_1 & - \end{bmatrix}$$

(b) Replace a row with a sum or difference of that row and another row

$$\begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{bmatrix} \underset{R_2 \to R_2 + R_3}{\sim} \begin{bmatrix} - & R_1 & - \\ - & R_2 + R_3 & - \\ - & R_3 & - \end{bmatrix}$$

(c) Scale a row with a nonzero constant.

$$\begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{bmatrix} \underset{R_1 \to R_1 \cdot c}{\sim} \begin{bmatrix} - & cR_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{bmatrix}$$

A is said to be the row equivalent of B if B can be obtained by a sequence of row operations on A.

Row - Echelon Form (REF)

1. All non zero rows are above any row of all zeroes.
2. Each leading entry of a row is in the column to the right of the leading entry of the row above it.
3. All entries in columns below a leading entry are 0.

To get REF, satisfy 1, 2, 3.

Reduced Row - Echelon Form (RREF)

4. The leading entry in each nonzero row is 1
5. Each leading 1 is the only nonzero entry in its column.

Satisfy 1, 2, 3, 4, 5 to get RREF.

eg: REF:
$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pivots.

A pivot position in a matrix is the location of leading 1 in the RREF form of A.

The column containing pivot position is called pivot column.