

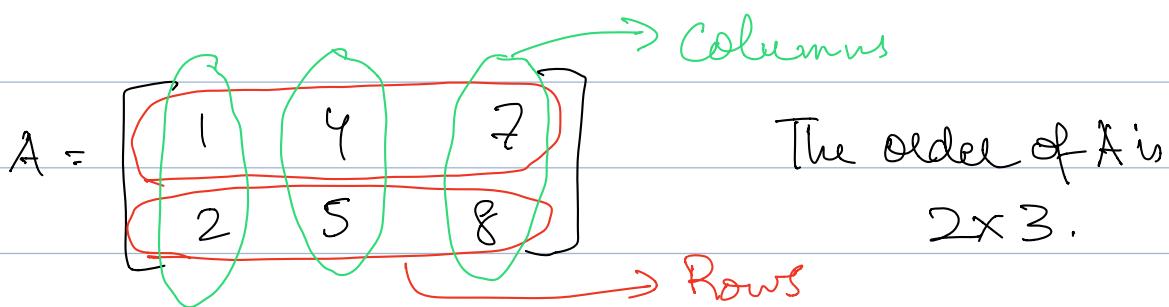
Tutorial 1

Matrices

A rectangular array of numbers consisting of m rows and n columns is called a matrix.

→ The numbers in the matrix are called entries.

→ The order of a matrix with m rows and n columns is $m \times n$.



In a matrix, an entry is often denoted by a_{ij} (or any letter with the subscript ij) where i and j are numbers, ($1 \leq i \leq m$, $1 \leq j \leq n$).

So, in the above matrix A ,

$$a_{11} = 1, a_{12} = 4, a_{13} = 7$$

$$a_{21} = 2, a_{22} = 5, a_{23} = 8$$

Types of matrices

1) Row matrix: Matrix with only one row ($m=1$)

eg: $[2 \ 4 \ 8]$ order: 1×3

2) Column matrix: Matrix with only one column ($n=1$)

eg: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ order: 2×1

3) Rectangular matrix: Matrix with m columns and n rows, $m \neq n$.

Eg:
$$\begin{bmatrix} 1 & 4 & 3 & 7 \\ 2 & 5 & 8 & 6 \end{bmatrix}$$
 Order: $\underline{2 \times 4}$

4) Square Matrix: Rectangular matrix, but with $m = n$.

Eg:
$$\begin{bmatrix} 1 & 9 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 5 \end{bmatrix}$$
 Order: $\underline{3 \times 3}$

main

5) Diagonal matrix: Only the diagonal can have non-zero elements. Mathematically, $a_{ij} = 0$

if $i \neq j$.

Eg:
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$
 \rightarrow only diagonal elements can be nonzero.

Note:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
 is also diagonal

Order: 3×3

6) Upper triangular matrix: Only the diagonal and every element above the diagonal can be nonzero. That is, $a_{ij} = 0$ for $i > j$.

Eg:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 10 \\ 0 & 0 & 7 \end{bmatrix}$$

7) Lower Triangular matrix: The opposite of upper triangular. Only the main diagonal and every element below the diagonal can be non zero.

That is, $a_{ij} = 0$ for $i < j$.

Eg:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 10 & 7 \end{bmatrix}$$

8) Zero matrix: All entries of the matrix are zero.

Eg: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

9) Identity matrix: Diagonal matrix, but with all the diagonal entries = 1.

That is

$$a_{ij} = 1 \text{ if } i=j$$

$$a_{ij} = 0 \text{ if } i \neq j$$

Eg: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We often denote them as I_n where n is the number of columns (which is equal to number of rows).

Trace: Denoted by $\text{tr}(A)$ or $\text{sp}(A)$, it is the sum of the entries on the main diagonal

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{tr}(A) = 1+5+9 = 15$$

only defined for square matrices.

Matrix Operations

1. Transposition

Inter changing rows and columns.

Simply stated, $a_{ij} \rightarrow a_{ji}$.

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 12 & 18 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 2 & 6 \\ 4 & 12 \\ 8 & 18 \end{bmatrix}$$

If $A^T = A$, A is called symmetric.

Eg: Any diagonal square matrix,

$$B = \begin{bmatrix} 10 & 80 \\ 80 & 10 \end{bmatrix}, \quad B^T = \begin{bmatrix} 10 & 80 \\ 80 & 10 \end{bmatrix}$$

Properties

$$\textcircled{1} \quad (A^T)^T = A$$

$$\textcircled{2} \quad (A \pm B)^T = A^T \pm B^T$$

$$\textcircled{3} \quad (kA)^T = kA^T, \quad k \text{ is scalar constant}$$

$$\textcircled{4} \quad (A \cdot B)^T = B^T A^T$$

2. Addition

If A, B have same order, then,

$$C = A \pm B \Rightarrow C_{ij} = a_{ij} \pm b_{ij}$$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1+3 & 0+4 \\ 0+5 & 1+6 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 5 & 7 \end{bmatrix}.$$

\rightarrow If A & B have same order

and $a_{ij} = b_{ij} + i_{ij}$, then $A=B$.

3. Scalar Multiplication

Multiplying a matrix with a real number.

$$A = [a_{ij}]_{m \times n}, C \in \mathbb{R}.$$

Then,

$$cA = [ca_{ij}]_{m \times n}.$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ $c = 2$.

Then, $cA = \begin{bmatrix} 2 \times 1 & 2 \times 2 & 2 \times 3 \\ 2 \times 4 & 2 \times 5 & 2 \times 6 \\ 2 \times 3 & 2 \times 2 & 2 \times 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 6 & 4 & 2 \end{bmatrix}$$

\rightarrow Linear Combination

$A_1, A_2, A_3, \dots, A_p$ are matrices of the same size and $c_1, c_2, c_3, \dots, c_p$ are real numbers.

$$c_1 A_1 + c_2 A_2 + c_3 A_3 + \dots + c_p A_p$$

is called a linear combination.

If

$$c_1 A_1 + c_2 A_2 + c_3 A_3 + \dots + c_p A_p = 0,$$

the linear combination is trivial.

Eg:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 6 & 0 & 10 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 10 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

H. Matrix multiplication

$C_{m \times n}$ is the product of $A_{m \times l}$ and

$B_{l \times n}$ if C_{ij} is the dot product of the i th row of A and j th column of B .

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

Each column multiplies with each row to give one entry.

$$B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} (1 \times 2) + (2 \times 3) & (1 \times 4) + (2 \times 5) \\ (4 \times 2) + (5 \times 3) & (4 \times 4) + (5 \times 5) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 14 \\ 23 & 85 \end{bmatrix}.$$

Properties : ① In most cases, $A \cdot B \neq B \cdot A$.

② If $E \cdot B = E \cdot C$, then it does not necessarily mean

$$B = C.$$

③ If $E \cdot D = 0$, then it does not necessarily mean that either

$$E = 0 \text{ or } D = 0$$

④ Any matrix multiplied with identity is that matrix itself.

$$A_{nxn} \cdot I_n = A_{nxn} = I_n \cdot A_{nxn}$$

- ⑤ Any matrix multiplied with its inverse gives you the identity.

$$A_{nxn} \cdot A_{nxn}^{-1} = I_n = A_{nxn}^{-1} \cdot A_{nxn}.$$

More on this next time.

Elementary Row Operations

We can perform row operations to transform $A \rightarrow B$.

- (a) Inter change two rows.

$$\left[\begin{array}{c|c} -R_1- \\ -R_2- \\ -R_3- \end{array} \right] \underset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{c|c} -R_3- \\ -R_2- \\ -R_1- \end{array} \right]$$

- (b) Replace a row with a sum or difference of that row and another row

$$\left[\begin{array}{c|c} -R_1- \\ -R_2- \\ -R_3- \end{array} \right] \underset{R_2 \rightarrow R_2 + R_3}{\sim} \left[\begin{array}{c|c} -R_1- \\ -R_2 + R_3- \\ -R_3- \end{array} \right]$$

- (c) Scale a row with a nonzero constant.

$$\left[\begin{array}{c|c} -R_1- \\ -R_2- \\ -R_3- \end{array} \right] \underset{R_1 \rightarrow R_1 \cdot C}{\sim} \left[\begin{array}{c|c} -CR_1- \\ -R_2- \\ -R_3- \end{array} \right]$$

A is said to be the row equivalent of B
 if B can be obtained by a sequence of
 row operations on A.

Row-Echelon Form (REF)

- ① All non-zero rows are above any row of all zeroes.
- ② Each leading entry of a row is in the column to the right of the leading entry of the row above it.
- ③ All entries in columns below a leading entry are 0.

To get REF, satisfy ①, ②, ③.

Reduced Row-Echelon Form (RREF)

- ④ The leading entry in each non-zero row is 1
- ⑤ Each leading 1 is the only nonzero entry in its column.

Satisfy ①, ②, ③, ④, ⑤ to get RREF.

Eg: REF:

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF :

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Pivots.

A pivot position in a matrix is the location of leading 1 in the RREF form of A.

The column containing pivot position is called pivot column.

Tutorial 2

For Elementary row operations, REF and RREF forms, refer to tutorial 1 notes.

System of linear equations

An equation of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ is called a linear equation in n variables, x_1, x_2, \dots, x_n . a_1, a_2, \dots, a_n are coefficients and are numbers.

Eg: $2x + 7y - 10z = 30$ is a linear equation.

A system of linear equations is a set of linear equations involving the same variables.

Eg: $2x - 8y + 30z = 10$

$$x + 20y - \left(\frac{8\sqrt{2}}{\pi}\right)z = 2e^2$$

This is a system of linear equations because
 ① They involve the same variables, i.e. x, y, z
 ② The powers on x, y, z is 1, so they are linear.

Solution set

The set of numbers that solve each equation in the system (ie upon substitution in place of unknown variables, they make the equation a true statement) is called the solution set of the linear system.

In systems of two variables, each equation represents a straight line and the intersection of these lines is the solution set.

$$\left. \begin{array}{l} \text{Eq: } 2x - 3y = 10 \\ 3x + 4y = 5 \end{array} \right\} \begin{array}{l} \text{The system has the solution } x=5, y=0 \text{ because} \\ \text{it satisfies each equation, and } (5, 0) \text{ is the intersection} \\ \text{of both lines on the } x-y \text{ plane.} \end{array}$$

In 3 dimensions, this corresponds to the intersection of planes.

A linear system can have either

- $\left. \begin{array}{l} \text{consistent:} \\ \text{① 1 unique solution (the intersection of two lines, eg: } (0,0) \text{ for } \begin{array}{l} x-y=0 \\ x+y=0 \end{array} \text{)} \\ \text{② Infinitely many solutions (overlap of two lines, eg: } x+y=2, 3x+3y=6 \text{)} \end{array} \right\}$
- $\left. \begin{array}{l} \text{inconsistent:} \\ \text{③ No solution (The two lines are parallel, eg: } x+y=2, x+y=10 \text{)} \end{array} \right\}$

Matrix notation

Linear systems can be represented in the form of a matrix equation

$Ax = B$. If your linear system is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

\vdots

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

Then in matrix form, this can be represented as $Ax = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

For a system $A\mathbf{x} = \mathbf{B}$, we represent an augmented matrix

$A|B$ as

$$A|B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

Two linear systems are said to be equivalent if they have the same solution set. Systems have the same solution set if one of the matrices can be obtained by using row operations on the other, i.e. they are row equivalent.

Methods of solving linear systems in matrix notation

I. Gaussian Elimination ($m=n$)

This method consists of two phases: ① Forward & ② Backward.

① Forward phase

In forward phase, you take the augmented matrix and apply row operations to convert to REF form.

Say your augmented matrix is of the form

$$A|B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right]$$

Apply row transforms to take this to

$$\text{REF}(A|B) = \left[\begin{array}{ccccc|c} c_{11} & c_{12} & \dots & c_{1n} & d_1 \\ 0 & c_{22} & \dots & c_{2n} & d_2 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & c_{nn} & d_n \end{array} \right]$$

② Backward phase

Take the REF(A|B), convert it back to system of linear equations. Solve the equations to get the final answer.

Say your REF(A|B) is

$$\text{REF}(A|B) = \left[\begin{array}{cccc|c} c_{11} & c_{12} & \dots & c_{1n} & d_1 \\ 0 & c_{22} & \dots & c_{2n} & d_2 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & c_{nn} & d_n \end{array} \right]$$

then this converts to:

$$c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = d_1$$

$$c_{22}x_2 + \dots + c_{2n}x_n = d_2$$

⋮

$$c_{nn}x_n = d_n$$

Solve for x_n first, substitute in the above equation to find x_{n-1} and repeat the process till you get x_1 .

Cases you might encounter:

No solution $c_{nn} = 0, d_n \neq 0$

$$\Rightarrow 0 \cdot x_n = d_n$$

x_n has no solution. Thus the entire system won't have a solution.

Infinite solutions: $c_{nn} = 0, d_n = 0$.

$$\Rightarrow 0 \cdot x_n = 0$$

This will probably lead you to infinite solutions for one of your variables, thus giving you infinite solutions for the system.

eg: $\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$ This will give you
 $x_3 = 3, x_2 = -3, x_1 = ?$ (x_1 can be anything)

This gives you infinite solutions.

Unique solution: $C_{nn} \neq 0$

$$\Rightarrow x_n = d_n / C_{nn}.$$

You will end up with a unique solution.

II. Gauss-Jordan Elimination ($m=n$)

This method only consists of one forward phase, which is to take your augmented matrix $A|B$ to its RREF form.

Since in your augmented matrix is now in RREF form, it will look something like this:

$$\text{RREF}(A|B) = \left[\begin{array}{cccc|c} C_{11} & 0 & \dots & 0 & d_1 \\ 0 & C_{22} & \dots & 0 & d_2 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C_{nn} & d_n \end{array} \right]$$

where the C_{ij} is either 1 or 0.

that is $C_{11}, C_{22}, \dots, C_{nn}$ is either 1 or 0.

The solution is simply then:

$$\left. \begin{array}{lll} C_{11}x_1 & = d_1 \\ C_{22}x_2 & = d_2 \\ \vdots & \vdots \\ C_{nn}x_n & = d_n \end{array} \right\} \begin{array}{l} C_{ii} \text{ is either} \\ 1 \text{ or } 0. \end{array}$$

Again you have the same cases as above.

No solution: $C_{nn} = 0, d_n \neq 0$

$\therefore x_n$ has no solution.

Unique soln: $C_{nn} = 1$

$x_n = d_n$.

Infinite solutions $C_{nn} = 0, d_n = 0$

\therefore one or more of x_i has ∞ solutions.

III. Gaussian elimination ($m < n$)

Used when you have more variables than equations.

You take the A|B to RREF. you will end up with

a matrix like

$$\text{RREF}(A|B) = \left[\begin{array}{cccc|c} c_{11} & 0 & 0 & \dots & c_{1n} & d_1 \\ 0 & c_{21} & \dots & 0 & \dots & c_{2n} & d_2 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_{m1} & \dots & c_{mn} & d_m \end{array} \right]$$

That is, you will have one free variable and

$n-1$ pivots. Then you will have ∞ solutions

with the solution in terms of your free variable.

Example:
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 4 & 5 & 6 & 10 \end{array} \right]$$

RREF $\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -5/3 \\ 0 & 1 & 2 & 10/3 \end{array} \right]$

x_1 and x_2 have pivots. x_3 is free.

let $x_3 = t$. Then,

$$x_1 - t = -5/3 \Rightarrow x_1 = -5/3 + t$$

$$x_2 + 2t = 10/3 \Rightarrow x_2 = 10/3 - 2t$$

\therefore Solution is:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5/3 + t \\ 10/3 - 2t \\ t \end{bmatrix}, t \in \mathbb{R}$$

IV. Inverse matrix method

The inverse matrix of a square matrix A is denoted by A^{-1} .

Then $AA^{-1} = I$ and $A^{-1}A = I$. (like how $2 \times \frac{1}{2} = 1$ or $2 \times 2 = 1$).

Definition: Matrix A is said to be invertible if there is an $n \times n$ matrix C such that $AC = I = CA$. C is called the unique inverse of A and denoted by A^{-1} .

If A is 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{then inverse is } A^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For all other matrices, for now, to find A^{-1} , use the following method.

We know $AA^{-1} = I$. Augment A and I.

$$A | I = \left[\begin{array}{cccc|cccccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & - & - & - \end{array} \right]$$

Row reduce till you end up with the P.R.E.F.

If inverse exists, then the left side becomes

I and the matrix you get on the right is your inverse.

Inverse of a matrix may or may not exist.

If it doesn't exist, A is called singular or non-invertible

If it exists, A is non-singular or invertible.

Note: There is a much easier way to compute A^{-1} ,

we will see that next tutorial. The above method is called the elementary matrix method.

If inverse exists, then, for a system

$$AX = B,$$

the solution is given by

$$X = A^{-1}B.$$

Note: This always leads to unique solutions.

If inverse doesn't exist, there are most likely no
solutions or ∞ many solutions.

Use Gaussian elimination or Gauss-Jordan elimination
to find the solution then!

