Problem Set 2, Math 191 Fall '15

This problem set is due Tuesday, September 8, 2014 at **the beginning of class**. All class guide rules apply. **Please remember to set aside "self-work time"** before consulting Piazza or working with others.

1. Let G be a directed graph (possibly with loops). For any vertex v, its *neighbors* are vertices (possibly itself, if there are loops) that v can reach by moving exactly once. Its *extended neighbors* are vertices that v can reach by moving exactly twice. Prove that for some vertex v, v has at least as many extended neighbors as neighbors.

2. 200 students take a math test with 6 problems on it. Every problem was solved correctly by at least 120 people. Prove that there exist two people such that between them they solved all problems (i.e. every problem was solved correct by at least one of them).

3. Let $a_0, a_1, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$ (i, j, k are not necessarily distinct!). Find a_{2014}.

4. Evaluate the sum
$$S(n) = \sum_{k=0}^{n-1} (-1)^k \cos^n(k\pi/n),$$
where n is a positive integer.

5. Let M be a 4×4 matrix where each entry is 2 or -1. Prove $27 | \det(M)$.

6. A lattice point $(x, y) \in \mathbf{Z}^2$ is "visible" from the origin if x and y are coprime. Prove that for any positive number C there exists a lattice point (a, b) whose distance from every visible point is greater than C.

7. Show that every positive integer is a sum of distinct Fibonacci numbers.

8. We want to cut a $4 \times 4 \times 4$ cube into 64 identical unit cubes with an infinitely long plane knife (we can cut several pieces at the same time, if necessary). What is the minimal number of cuts that can be used to achieve this?

9. **(required)** How much time (including self-work time) did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, etc.)