Problem Set 9, Math 191 Fall '14

This problem set is due Tuesday, Oct 27, 2014 at **the beginning of class**. All class guide rules apply. **Please remember to set aside "self-work time"** before consulting Piazza or working with others.

(Before doing the problems, look up and understand "Intermediate Value Theorem" in Wikipedia/Google for 20 minutes.)

1. Show that if a, b, c are positive real numbers such that $abc \geq 2^9$, then
$$\frac{1}{\sqrt{1+(abc)^{1/3}}} \leq (1/3)(\frac{1}{\sqrt{1+a}} + \frac{1}{\sqrt{1+b}} + \frac{1}{\sqrt{1+c}}).$$

2. Given $nm+1$ closed finite intervals on the real line, show that either some $(m+1)$ of them have a point in common or some $(n+1)$ of them are pairwise disjoint.

3. Can we have nine non-overlapping unit squares touch a given unit square S? If you are picky: touch means have any point in common; the squares are closed sets. For example, in a chessboard, each square not on the boundary touches 8 other squares.

4. For a positive integer n, denote by $f(n)$ the number of choices of signs $\pm$ such that $\pm 1 \pm 2 \pm \cdots \pm n = 0$. Show that
$$f(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos(t)\cos(2t)\cdots\cos(nt)dt.$$

5. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous, and suppose that there exists some real number a such that $f(f(f(a))) = a$. Show that there is some b where $f(b) = b$. must the sequences converge? If so, find their limit. If not, show why they don't have to converge.

6. If every point of the plane is colored with one of three colors, are there necessarily at least two points of the same color exactly one meter apart?

7. We repeatedly take real numbers chosen uniformly in $[0, 1]$ and add until we get to at least 1. What is the expected number of numbers we need?

8. For integers $n \geq 2$ and $0 \leq k \leq n-2$, compute the determinant of the $n \times n$ matrix M, where $M_{ij} = (i+j-1)^k$.

9. **(required)** How much time (including self-work time) did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, etc.) Please confirm you have the basic idea of the boxed principle above (and would be able to show me on the board a basic example and some properties if I called on you =D). You can, of course, ask people to explain specific questions on Piazza.