Problem Set 12, Math 191 Fall '15

This problem set is due Tuesday, November 17, 2015 at **the beginning of class**. All class guide rules apply. **Please remember to set aside "self-work time"** before consulting Piazza or working with others.

(Before doing the problems, look up and understand "Perpetual Motion" in Wikipedia/Google for 20 minutes. (Just do it.))

1. For nonnegative $a_1, \ldots, a_n, b_1, \ldots, b_n$, show that

$$((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n} \geq (a_1 a_2 \cdots a_n)^{1/n}$$

2. Alice and Bob play a game on a 3×3 empty matrix. Alice enters a 1 somewhere empty. Bob plays a 0 in an empty spot, etc. until the matrix is filled. If the determinant is 0, Bob wins. If it is 1, Alice wins. Supposing optimal play, who wins and how?

3. Find all integral solutions of $x^2 + y^2 + z^2 + 2xyz = 0$.

4. Suppose you have 127 potato chips, labeled by the non-empty subsets of $\{1, 2, \ldots, 7\}$ (don't worry; no Death Chips). Each day, you pick a uniformly random still-noneaten potato chip and eat it. The caveat is if you eat a chip labeled A, you must also eat all chips labeled A' with $A' \subset A$ that you still have remaining. Find the expected number of days it takes to eat all your chips.

5. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

6. There are b boys and g girls present at a party, where $g \geq 2b - 1$, and both are positive integers. Prove there is a way for all the boys to each ask a different girl to dance such that each boy is either dancing with a girl that he knows, or that all the girls he knows are not dancing. (Alice: "Boys and girls... is this Hall's Theorem?" Bob: "If so, why is he making it so obvious by having us have this dialogue? It must be a red herring." Alice: "Naw man you are thinking too much. I think he is using reverse reverse psychology.")

7. One day, a Pastafarian monk climbed from Evans Hall to the top of the mountain of MSRI, starting her trek at 6AM and ending at 9AM. The next day, the monk came back down along the same path, again starting at 6AM and ending at 9AM. Prove that there is a point on the mountain where she was in the exact same place at the same time on both days. Pontificate.

8. Let $P(x)$ be a polynomial with positive real coefficients. Prove that $\sqrt{P(a)P(b)} \geq P(\sqrt{ab})$.

9. **(required)** How much time (including self-work time) did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, etc.)