1) Take any non-negative $a_1, \cdots, a_n, b_1, \cdots, b_n$.

$$\begin{aligned}(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n) &= \\ &= a_1 a_2 \cdots a_n + C \text{for some non-negative C} \\ &\geq a_1 a_2 \cdots a_n\end{aligned}$$

Taking the nth root on both sides, we get

$$\begin{aligned}((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n))^{\frac{1}{n}} &= \\ &\geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \ \square\end{aligned}$$

3) Given the set up $x^2 + y^2 + z^2 = -2xyz$, we know that out of x, y, and z: either two of the numbers are odd or they are all even.

Case 1: 2 numbers are odd] WLOG let x and y be odd and z even.

For some integers k, m, n, we get $x = 2k + 1, y = 2m + 1$, and $z = 2n$.

Furthermore:

$$\begin{aligned} x^2 + y^2 + z^2 &= \\ &= (2k+1)^2 + (2m+1)^2 + (2n)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 + 4m + 1 + 4n^2 \\ &\equiv 2 \mod 4 \end{aligned}$$

But:

$$\begin{aligned} -2xyz &= \\ &= -2(2k+1)(2m+1)(2n) \\ &\equiv 0 \mod 4 \end{aligned}$$

Thus we have a contridiction and two numbers can't be odd and 1 even.

Case 2: All three numbers are even]

For some integers k, m, n, we get $x = 2k, y = 2m$, and $z = 2n$.

From the original equation we get:

$$\begin{aligned} x^2 + y^2 + z^2 &= -2xyz \\ (2k)^2 + (2m)^2 + (2n)^2 &= -2(2k)(2m)(2n) \\ 4(k^2 m^2 + n^2) &= -4(4)(kmn) \\ k^2 + m^2 + n^2 &= -4kmn \end{aligned}$$

Using a similar argument about odd and evenness, we know that k, m, and n all must be even so we can again rewrite them. So we end up infinitely descending to smaller even numbers. However there aren't an infinite number of smaller even numbers and we'd only stop if we ever hit $(0, 0, 0)$ but unless we start with with those number, we'll never end up there since each time we are only dividing through by 2. Thus the only solution is $(0, 0, 0)$. $\square$

5) Let $m(x)$ be a monic polynomial of degree n with real coefficients.

Pick n real points x_i such that $x_1 < \cdots < x_n$.

Since m has real cofficients, we can pick n real points y_i such that: $\begin{cases} \min(2m(x_i), 0) > y_i & \text{for } i \text{ odd} \\ \max(2m(x_i), 0) < y_i & \text{otherwise} \end{cases}$

Use Langrange Interpolation on the the points $(x_i, y_i - x_i^n)$, we can get an n degree monic polynomial r.

Now let $s(x) = x^n + r(x)$. Note that s is a n degree monic polynomial with such that $s(x_i) = y_i$.

Note that by the way we defined y_i, y_i is always negative for i odd and y_i is always positive for i even. Thus we know that s crosses the x-axis $n-1$ times and thus has $n-1$ real roots. But s has n roots and can't have just 1 complex root so it has n real roots.

Now let $t(x) = 2m(x) - s(x)$. Note t is a monic polynomial of degree n,

By the way we defined y_i, $t(x_i)$ is always positive for i odd and $t(x_i)$ is always negative for i even. Thus we know that t crosses the x-axis $n-1$ times and thus has $n-1$ real roots. But t has n roots and can't have just 1 complex root so it has n real roots.

Rewriting t, we get $m(x) = \frac{t(x)+s(x)}{2}$. So we've found for any monic polynomial of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots. $\square$

7) Let f and g be functions on R such that input corresponds to times and output the monk's distance from Evans at that time. Let f be the function correpsonding to day 1 and g the function corresponding to the monk's movement on day 2.

So f by construction is an increasing continuous function on $[a, b]$.

And g by construction is a decreasing continuous function on $[a, b]$ whose image is in $[g(b), g(a)]$ which equals $[f(a), f(b)]$.

Suppose g didn't intersect f.

That means g is always strictly greater than f on the interval $[a, b]$ or always strickly less than f on the interval $[a, b]$ since both functions are continuous.

Since $g(a) = f(b)$ ie the image of g's first point is greater than the image of f's first point, g must always be greater than f.

But we know that $g(b) = f(a)$ ie the image of g's last point is less than the image of f's first point so that would indicate g must always be greater than f which is a contradiction so g indeed intersects f.

Since g indeed intersects f, we know that there exists some time t where $g(t) = f(t)$. $\square$

8) Proof for real a and b:

Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ sucht that $c_i \in \mathbf{R}_{>0}$

Take any $a, b \in \mathbf{R}$.

Using the Cauchy Schwartz Inequality on $(\sqrt{c_n a^n}, \sqrt{c_{n-1}a^{n-1}}, \cdots, \sqrt{c_0})$ and $(\sqrt{c_n b^n}, \sqrt{c_{n-1}b^{n-1}}, \cdots, \sqrt{c_0})$ where for the first vector the ith component is the product of ith power of a and its coefficient in $p(a)$ and for the second vector the ith component is the product of ith power of b and its coefficient in $p(b)$, we get:

$$\begin{aligned} & (c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0)(c_n b^n + c_{n-1}b^{n-1} + \cdots + c_0) \geq (c_n(\sqrt{ab})^n + c_{n-1}(\sqrt{ab})^{n-1} + \cdots + c_0)^2 \\ \iff & \sqrt{(c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0)(c_n b^n + c_{n-1}b^{n-1} + \cdots + c_0)} \geq (c_n(\sqrt{ab})^n + c_{n-1}(\sqrt{ab})^{n-1} + \cdots + c_0) \\ \iff & \sqrt{p(a)p(b)} \geq p(\sqrt{ab}) \ \square \end{aligned}$$

9) I spent probably 12 or so hours on it. I think the problems were easier this week since I solved more of them.

Perpetual motion is an action that infinitely continues without an external source of energy. This in practice seems impossible since there is always some energy loss.