Problem Set 3, Math 191 Fall '15

This problem set is due Tuesday, September 15, 2014 at **the beginning of class**. All class guide rules apply. **Please remember to set aside "self-work time"** before consulting Piazza or working with others.

1. Show if $\sum_n a_n$ converges, where $\{a_n\}$ form a decreasing sequence of nonnegative real numbers, then
$$\lim_{n\to\infty} na_n = 0.$$

2. Prove or disprove: every positive integer n can be written as some
$$e_1 1^2 + e_2 2^2 + e_3 3^2 + \cdots + e_k k^2,$$
where each e_i equals 1 or -1.

3. Show that, for nonnegative r_i,
$$(1 + (r_1 r_2 \cdots r_n)^{1/n})^n \leq \prod_k (1 + r_k) \leq (1 + (r_1 + r_2 + \cdots + r_n)/n)^n.$$

4. Prove or disprove: for any $n \times n$ matrix A with real entries, we have
$$\det(I + A^2) \geq 0,$$
where I is the identity matrix of dimension n.

5. There are 33 rooks on a 8×8 chessboard (bet you are sick of chessboards right now, but they'll keep coming!). Prove that you can find 5 of them such that no rook is attacking any other one.

6. For any positive integer n, prove there is a Fibonacci number that is a multiple of n.

7. Suppose $p(x)$ is a polynomial of degree n and for all x, $p(x) \geq 0$. Show that for all x, we have
$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \geq 0.$$

8. Find (or prove one doesn't exist) a square $ABCD$ in the plane and a point X such that the sides of the square are integral and the distance from X to each vertex A, B, C, and D are also integral.

9. **(required)** How much time (including self-work time) did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, etc.)