Contributions: Eric Severson helped me get to the answer on the first question. 2 hours

1) Suppose for a contradiction that every vertex has strictly less extended neighbors than neighbors.

Let v_i be the ith neighbor of v if you can get to v from v_i by traversing exactly i edges.

[**Lemma 0**] Every vertex has $\deg(v) > 0$.
Proof: Suppose that a vertex has no neighbors, then that vertex would have 0 neighbors and 0 extended neighbors which is a contradiction.

[**Lemma 1**] If v_i is the ith neighbor of some vertex v where $\deg(v) = n$, then $\deg v_i < n-i+1$
Proof: Since every vertex has strictly less extended neighbors than neighbors, for all v_1, $\deg(v_1) < n$ (otherwise just that v_1 extended neighbor would result in there being at least as many extended neighbors as neighbors of v).

But then the same argument can be made for the neighbors of all v_1, the 2cd neighbors of v, and we get $\deg(v_2) < \deg(v_1) < n$.

So then extending the process out the ith neighbor of v, we know that $\deg(v_i) < \deg(v_{i-1}) < \cdots < \deg(v_1) < n$.

Since the degree is strictly decreasing and starts at n, we know that $\deg(v_i) < n - i + 1$.

Okay now getting back to the main proof.

We know that the $n + 1$ neighbor of v must exist since each vertex has positive degree and thus you can always traverse $n + 1$ edges and get to some vertex.

By **Lemma 1**, we know that v_{n+1}, the $n+1$th neighbor of v has the property $\deg(v_{n+1}) < 0$.

But by **Lemma 0**, we know that $\deg(v_{n+1}) > 0$ so we have a contradiction and thus there exists some vertex with at least as many extended neighbors as neighbors. $\square$

2) Partial Progress:

Suppose there didn't exist 2 people such that between them they solved all the problems.

My approach assigns each person to a set of problems so that we have k sets of problems and then k_i people in each set such that each person can only be associated with 1 set of problems (but each problem can appear in multiple sets).

Clearly there can't exist any person who solved all the problems (since then that person with any other person would have solved all the problems).

There also can't exist a person who has solved 5 of the problems. Suppose not. Then there exists some person, X, who solved all the problems but say problem a. Now no person could have solved problem a or then we could pick the person who solved problem a and person X and we'll cover all the problems. But this is a contradiction since at least 120 people solved each problem so it's not possible 0 people solved problem a.

Now, if any person solved 4 problems let's call them p_0, p_1, p_2, and p_3. If anyone else solved a different set of 4 problems, then we know that 3 must overlap with p_0, p_1, p_2, and p_3 (if 2 or less overlapped you'd cover all the problems). Thus if 4 problems were solved, we can say generally that there are at most 3 types of people who solved 4 problems, those that solved p_0, p_1, p_2, and p_3, those that solved p_0, p_1, p_2, and p_4, and those that solved p_0, p_1, p_2, and p_5, where the p_i is randomly assigned to each problem number.

Using similar logic, if any person solved 3 problems let's call them p_0, p_1, p_2. If anyone else solved a different set of 3 problems, then we know that only 2 could have overlapped with p_3, p_4, p_5. Thus if 3 problems were solved, we can say generally that there are at most 18 types of people who solved 3 problems.

If any person solved 2 problems, we can take any combo of 2 which is at most 30 sets.

And if any person solved 1 problem, then we have at most 6 sets.

If every person did 3 problems, to get each problem count to at least 120, we would need 120 people more people to reach our count of at least 720. Thus at least 40 people do 4 problems (since no one could have done 5 or 6 problems). But I'm not sure how much that helps.

It seems to me given these configurations (and especially because the sets go down if you choose multiple number) that you can't get 120 people at least in each group but I couldn't get the logic to work out. :/

3) $\boxed{8144865729}$

Progress:

Let (a_n) be the sequence such that $a_i = \begin{cases} (i-1)^3 & \text{if } i \text{ is odd} \\ (i-2)^3+1 & \text{o.w.} \end{cases}$

Note that (a_n) is an increasing sequence of nonnegative integers.

Now we'll do a proof by induction.

I propose that for all $n \in \mathbb{Z}_{\geq 0}$, n can be written uniquely in the form $a_i + 2a_j + 4a_k$ where i, j, k are not neccesarily distinct.

Our base case for $n = 0$ we get trivially from **Statement 1**.

Now we'll show for all $k > n$ for some $n > 1$, our proposition holds and that implies that the $k+1$ case holds.

We know by our inductive hypothesis we can write k $= a_i + 2a_j + 4a_k$.

If a_i is even, $k + 1 = (a_i + 1) + 2a_j + 4a_k$.

Otherwise, $a_i = (i-1)^3$. I'm not actually really sure where to go from here. I tried writing it out to look for patterns:

000
100
010
110
001
101
011
111
800
900
810
910
801
901
811
911
080
180
090

But I wasn't able to see anything too obvious.

4) Partial Progress:
Here is what I figured out.

It was helpful to me rewording the problem as being a cicle with radius c not touching any visible points centered at a lattie point.

You need a 1 off ascending sequence of c integers where each number in the the sequence shares a factor with another 1 off ascending sequence of c integers. Both sequences clearly can't have any prime numbers.

Both sequences can't have any prime numbers which is where I got stuck since I found it difficult to generally find a sequence of c integers (I'm pretty sure this is actually a very very hard problem to solve).

6) Partial Progress:

Doing out examples I noticed that the sum came to be something odd and descending on top and the bottom was a power of 2.

7) We'll do a prove by induction.
Proposition: For all $n \in \mathbb{Z}_{>0}$, n can be written as a the sum of distinct Fibonacci numbers.

We trivially get our bases cases for $n = 1$ and $n = 2$.

Now we'll show for all $k > n$ for some $n > 2$, our proposition holds and that implies that the $k + 1$ case holds.

Suppose that $k + 1$ is a fibonacci number. Then we are done since that is trivially is the sum of distinct Fibonacci numbers.

Now suppose that $k + 1$ is not a F number.

Let's call j and i the biggest and second biggest F number strictly smaller than $k + 1$, respectively.

We know by our inductive hypothesis that we can write all integers less than i as a sum of distinct Fibonacci numbers (and clearly F numbers that are less than i).

We also know that $0 < k + 1 - j < i$ since $j + i$ is the F number that occurs after $k + 1$.

Thus for each number in $(j, j + i]$, we can write them as $j + l$ where $l \in [1, i]$.

We know by our inductive hypothesis that we can write all $l \in [1, i]$ as a sum of distinct Fibonacci numbers (and clearly using F numbers that are less than i).

Since $k + 1$ is in $(j, j + i]$, we have shown that $k + 1$ can be written as the sum of a distinct F numbers.

Thus our inductive hypothesis is proven and our proof is done. $\square$

9) I spent probably around 15 hours on the problem set (though honestly I should have kept better track, I tend to work on it a little bit each day and also think about the problems when I go walking or something). 7 was definitely the easiest problem. I thought the coprime question was pretty difficult and for number 3 I had a hard time articulating in logic my answer. And I really ran out of time/steam for 4 and 6.