2) The statement for all $k \in \mathbb{Z}_{>0}$, for all $n \in \mathbb{Z}_{>0}$, $n = \sum_{i=1}^{k} e_i i^2$ where $e_i \in \{-1, 1\}$ is false. A counter example is for $k = 1$, clearly $n = 10$ fails.

Thus the statement I'll work with is for all $n \in \mathbb{Z}_{>0}$, there exists a $k \in \mathbb{Z}_{>0}$ such that $n = \sum_{i=1}^{k} e_i i^2$ where $e_i \in \{-1, 1\}$.

[**Lemma 0**] For any 4 consecutive postive integers, x, x+1, x+2, x+3, we can get 4 using the construction $(-1) * (x + 1)^2 + (1) * (x + 2)^2 + (-1) * (x + 3)^2 + (1) * (x + 4)^2$
Proof:
Take some $x \in \mathbb{Z}_{>0}$.

$$\begin{aligned}
&(1) * (x)^2 + (-1) * (x + 1)^2 + (-1) * (x + 2)^2 + (1) * (x + 3)^2 \\
&= (1) * (x)^2 + (-1) * (x^2 + 2x + 1) + (-1) * (x^2 + 4x + 4) + (1) * (x^2 + 6x + 9) \\
&= x^2 - x^2 - 2x - 1 - x^2 - 4x - 4 + x^2 + 6x + 9 \\
&= 4 \ \square
\end{aligned}$$

Okay now getting back to proving the main statement. Note that the following is true:
[$\mathbf{n = 1}$] $1 = 1 * 1$
[$\mathbf{n = 2}$] $2 = (-1) * 1 + (-1) * 4 + (-1) * 9 + (1) * 16$
[$\mathbf{n = 3}$] $3 = (-1) * 1 + (1) * 4$
[$\mathbf{n = 4}$] $4 = (-1) * 1 + (-1) * 4 + (1) * 9$

Now take any $n \in \mathbb{Z}_{>4}$, I'll show you can construct that n using some $k \in \mathbb{Z}_{>0}$ such that $n = \sum_{i=1}^{k} e_i i^2$ where $e_i \in \{-1, 1\}$.

First let r = n mod 4.

Clearly r is $0, 1, 2$, or 3. Let that remainder correspond to the cases n equals $4, 1, 2$, or 3, respectively.

So now $n - r$ is some multiple of 4. Thus for some $m \in \mathbb{Z}_{\geq 0}$, $4m = c + 1 - r$.

Now after the j powers in the base cases in which we get r, we can take m sets of 4 consecutive integers to get $4m$ because [**Lemma 0**] holds.

Thus we have shown for any $n \in \mathbb{Z}_{>4}$, we can construct that n using some $k \in \mathbb{Z}_{>0}$ such that $n = \sum_{i=1}^{k} e_i i^2$ where $e_i \in \{-1, 1\}$. And we already showed explicitly the cases for $n = 1, 2, 3, 4$ so our proof is complete. $\square$

7) [**Lemma 0**] p is an even degree polynomial with a positive leading coefficient.
Proof:
Suppose not.

p is neccesarily even with a negative leading coefficient, odd with a negative leading coefficient, or odd with a positive leading coefficient.

If p is even with a negative leading coefficient, then as x goes to ∞, $p(x)$ goes to $-\infty$, which is a contradiction because p is always non-negative.

If p is odd with a negative leading coefficient, then as x goes to ∞, $p(x)$ goes to $-\infty$, which is a contradiction because p is always non-negative.

If p is odd with a positive leading coefficient, then as x goes to $-\infty$, $p(x)$ goes to $-\infty$, which is a contradiction because p is always non-negative. $\square$

Let $f(x) = \sum_{i=0}^{n} p^{(i)}(x)$.

[**Lemma 1**] $f(x) = p(x) + f'(x)$
Proof: Since $f'(x) = \sum_{i=1}^{n} p^{(i)}(x)$, $f(x) = p(x) + f'(x)$. $\square$

[**Lemma 2**] f is is an even degree polynomial with a positive leading coefficient
Proof:
Since derivatives of a polynomial are of a lesser degree, we know for each integer $i \in [1, n]$, $\deg(p^{(i)}(x)) < \deg(p(x))$.

Since f is the sum of a bunch of polynomials and p is strictly the highest degree polynomial in the sum, we know the degree of f is the same as the degree of p and f has the same leading coefficient as p.

By [**Lemma 0**], we know that p is an even degree polynomial with a positive leading coefficient and thus f is an even degree polynomial with a positive leading coefficient. $\square$

Suppose now that, f is always greater than or equal to 0. Then, we're done.

So now suppose that, f is less than 0 at some point.

Since f is an even degree polynomial, it must cross $y = 0$ at least twice.

Let's label the times f crosses $y = 0$ as $x_0, \cdots, x_n$.

Let $x_{\min}$ be min of $\{x_0, \cdots, x_n\}$ and $x_{\max}$ be max of $\{x_0, \cdots, x_n\}$.

Since f has a postive leading coefficient, for all x outside the interval $[x_{\min}, x_{\max}]$, f is will always be greater.

Since f is continuous on a closed bounded interval $[x_{\min}, x_{\max}]$ and f is less than 0 at some point, we know the global minimum on the interval is at some point z where $f'(z) = 0$.

We actually know $f(z)$ is the global minimum that this will actually be the global minimum across the entire function since outside that interval f is always greater.

Thus for all x,

$$\begin{aligned} f(x) &\leq f(z)[\textbf{ By the construction of the point z}]] \\ &= p(z) + f'(z)[\textbf{Lemma 1}]] \\ &= p(z) + 0[\textbf{ By the construction of the point z}]] \\ &\leq 0[\textbf{ We are given p is non-negative}]] \end{aligned}$$

$\square$

9) I think I probably spent around 20 hours on the problem set, around 7 hours on problem 7. I think I need to force myself to do ones involving linear algebra since I always do them last and end up not giving them my full effort cause I run out of time. I use to be scared of polynomial ones but after having spent so much time on the one I did, I no longer feel as scared. I'm also getting better intuition as how to solve the inductive ones where you prove all positives numbers (or something) are of a particular form.