Problem Set 7, Math 191 Fall '15

This problem set is due Tuesday, Oct 13, 2014 at **the beginning of class**. All class guide rules apply. **Please remember to set aside "self-work time"** before consulting Piazza or working with others.

(Before doing the problems, look up and understand "Determinant" in Wikipedia for 20 minutes. Note the change in Problem 9.)

1. $n_1, n_2, n_3, \ldots$ is a sequence of positive integers with the property $n_{k+1} > n_{n_k}$. Find all such sequences.

2. In this problem, $\times$ denotes the cross product and the v_i are vectors. Prove or disprove: given any two parenthetizations of $v_1 \times v_2 \times \ldots \times v_n$ (for example, $(a \times b) \times (c \times d)$ and $a \times ((b \times c) \times d)$ are two different parenthetizations of $a \times b \times c \times d$), there is always an assignment of i, j, k (recall these are the three mutually orthogonal unit vectors in $\mathbf{R}^3$) to each v_x such that the two expressions are equal and nonzero.

   Example: $i \times (j \times i) = (i \times j) \times i = j$.

3. Does there exist a sequence of $e_n \in \{1, -1\}$ such that
$$\sum_{n=0}^{\infty} e_n/n = e?$$

4. Evaluate D_n, the determinant of
$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & n \end{bmatrix}.$$

5. On a table are 100 coins. A and B try to remove coins from the table by turns. Each turn they can remove 2, 5, or 6 coins. The first one that cannot make a move loses. Find who has a winning strategy if A plays first.

6. Let $f_1, f_2, \ldots, f_n$ be linearly independent, differentiable functions. Prove that some $(n-1)$ of their derivatives $f_1', f_2', \ldots, f_n'$ are linearly independent.

7. For a closed interval $[a, b]$, call a function "baby convex" if
$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}.$$
   Prove that baby convex functions satisfy
$$f(\frac{\sum a_i}{n}) \leq \frac{\sum f(a_i)}{n}$$
   for any $a_1, \ldots, a_n$ in $[a, b]$.

8. Cut out a square of a $2^n \times 2^n$ chessboard. Show that the remaining $2^{2n} - 1$ squares can be tiled with L-tiles, where an L-tile is a 3-square big piece that looks like a 2×2 box with a square removed.

9. **(required)** How much time (including self-work time) did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, etc.) **Please confirm you have the basic idea of "Determinant" (and would be able to show me on the board a basic example and some properties if I called on you =D). You can, of course, ask people to explain on Piazza. I will be understanding if you have not taken linear algebra.**