Practical Course: GPU Programming in Computer Vision

Mathematics 2: Structure Tensor

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The Structure Tensor of an Image

Given an input image $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^k$, compute the smoothed image $s : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^k$ defined as

$$s := G_\sigma \star u, \tag{1}$$

where $\sigma > 0$ is called the inner scale. The structure tensor T of u is defined as

$$T := G_\rho \star M, \tag{2}$$

where $\rho > 0$ is called the outer scale and where we use (1) to define

$$M := \nabla s \cdot \nabla s^T = \begin{pmatrix} (\partial_x s)(\partial_x s) & (\partial_x s)(\partial_y s) \\ (\partial_y s)(\partial_x s) & (\partial_y s)(\partial_y s) \end{pmatrix} \tag{3}$$

The Structure Tensor of an Image

Plugging (3) into (2) we end up with

$$T = G_\rho \star \begin{pmatrix} (\partial_x s)(\partial_x s) & (\partial_x s)(\partial_y s) \\ (\partial_y s)(\partial_x s) & (\partial_y s)(\partial_y s) \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} G_\rho \star (\partial_x s)(\partial_x s) & G_\rho \star (\partial_x s)(\partial_y s) \\ G_\rho \star (\partial_y s)(\partial_x s) & G_\rho \star (\partial_y s)(\partial_y s) \end{pmatrix} . \tag{5}$$

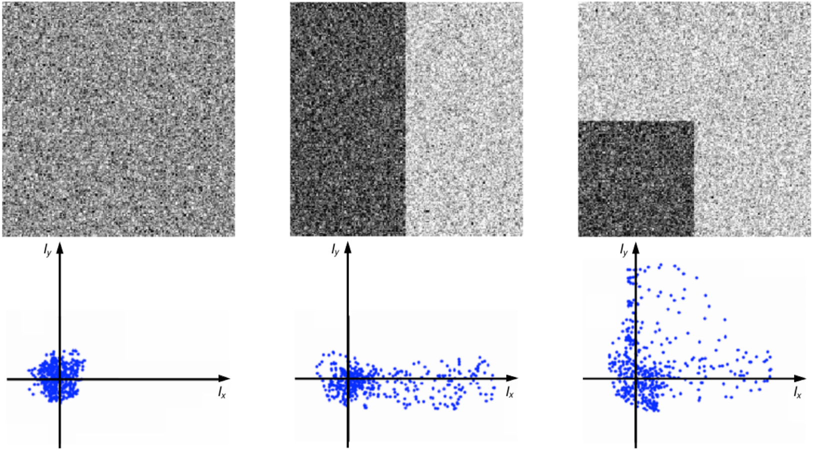
Thus we know that per pixel $T \in \mathbb{R}^{2\times 2}$, symmetric and positive definite.
Symmetric:

$$T^t = T \tag{6}$$

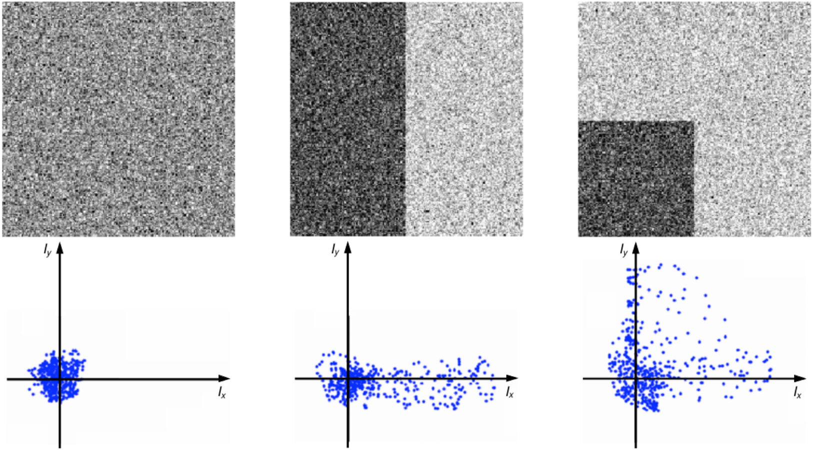
Positive definite:

$$x^t T x > 0 \ \ \forall x \in \mathbb{R}^2 \tag{7}$$

Interpretation of the structure tensor



Interpretation of the structure tensor



Relation between edges/corners and the tensor T?

Eigenvalues Revisited

Theorem

$\lambda \in \mathbb{R}$ is said to be an eigenvalue and $0 \neq v \in \mathbb{R}^2$ is said to be an eigenvector of the matrix $T \in \mathbb{R}^{2\times 2}$, iff

$$Tv = \lambda v \tag{8}$$

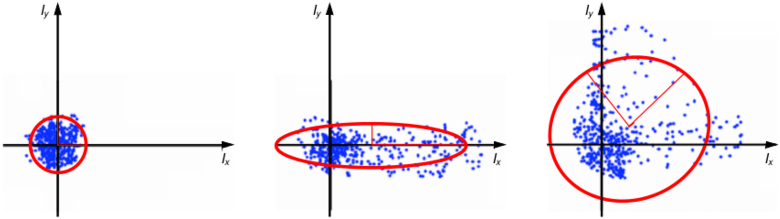
holds.

Theorem

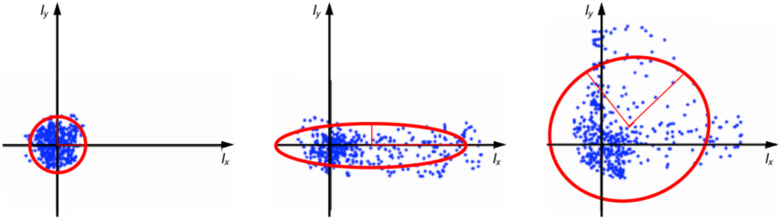
All eigenvalues of a positive definite matrix are positive.

$\Longrightarrow$ T has two positive eigenvalues λ_1 and λ_2.

Interpretation of the structure tensor

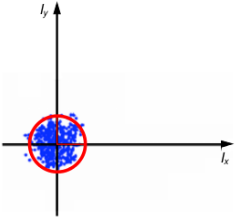


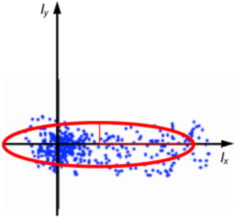
Interpretation of the structure tensor

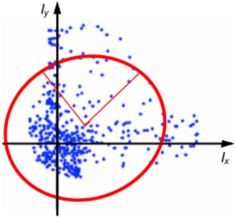


$$\lambda_1 \approx \lambda_2$$

Interpretation of the structure tensor



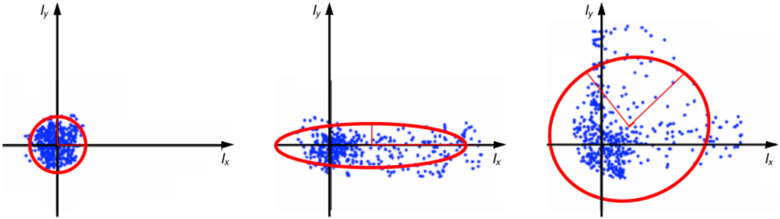




$\lambda_1 \approx \lambda_2$

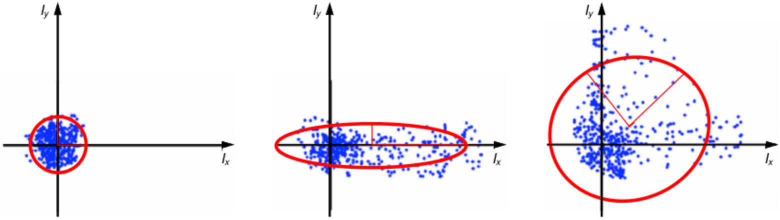
$\lambda_1 \gg \lambda_2$

Interpretation of the structure tensor



$\lambda_1 \approx \lambda_2$ $\lambda_1 \gg \lambda_2$ $\lambda_1, \lambda_2 \gg 0$

Interpretation of the structure tensor



$\lambda_1 \approx \lambda_2$ $\qquad$ $\lambda_1 \gg \lambda_2$ $\qquad$ $\lambda_1, \lambda_2 \gg 0$

- The eigenvectors are the directions of principal axes and the eigenvalues the length of the principal axes.
- Yields simple edge/corner detector.

Applying the structure tensor

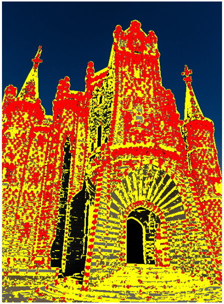
(a) Input image (b) $(\partial_x s)(\partial_x s)$ (c) $(\partial_x s)(\partial_y s)$ (d) $(\partial_y s)(\partial_y s)$

Figure: Visualization of the matrix $M \in \mathbb{R}^{2\times 2}$

Applying the structure tensor

Resulting in "Harris Corner Detector"

(a) Input image



(b) Marking corners and edges per pixel according to λ_1 and λ_2

Figure: C. Harris, M. Stephens, A combined corner and edge detector, Proc. of Fourth Alvey Vision Conference, 1988