Modeling and Control of a Planar Three-link Bipedal Robot

Divyansh Gupta
MS Robotics
Northeastern University
Boston, USA
gupta.divy@northeastern.edu

Prem Sukhadwala
MS Robotics
Northeastern University
Boston, USA
sukhadwala.p@northeastern.edu

***Abstract*—This project is a culmination of four mini-projects over the semester. The project's goal was to model and control a three-link planar bipedal robot. It involves modeling the robot using Lagrange formalism, obtaining the zero-dynamics, optimization-based gait design, and designing a non-linear controller for the biped in question.**

I. Introduction

Legged locomotion in Robotics is heavily inspired by biology–human and animal walking/running. Bipedal robots are a subclass of this topic in robotics. Humans are bipeds and have a very efficient gait cycle. Bipedal robots are–in a sense–a way to mimic human gait cycles and traverse over difficult terrains where wheeled robots would fail. Popular examples of bipedal robots are Boston Dynamics' ATLAS, Honda's ASIMO, and MABEL from the University of Michigan. In this project, to make things less complicated, we have addressed a planar three-link biped with point feet contact. MABEL, although 3D has restricted motion in the transversal plane. Thus it becomes a very good real-life example of a planar robot. This simplification helps us to only focus on the sagittal plane motion and have no knee joints. Fig. 1. shows the schematic of the biped with the choice of generalized coordinates.

Work of Prof. Jessy Grizzle [1] has been used as a reference for this project. We derived equations of motions for single-support phase, using forward kinematics. Obtained the Inertia (D), Coriolis (C) and Gravity (G) matrices using Lagrangian Mechanics and obtained the state-space equations. We obtained the zero dynamics for the model using Zero Dynamics Manifold and developed a gait using Bezier polynomials and optimization.

II. Parameters of the robot

Parameter	Value
Mass of swing leg, m_1	5 kg
Mass of stance leg, m_2	5 kg
Mass of hip, m_h	15 kg
Mass of Torso, m_t	10 kg
Length of Torso, l	0.5 m
Length of each leg, r	1 m

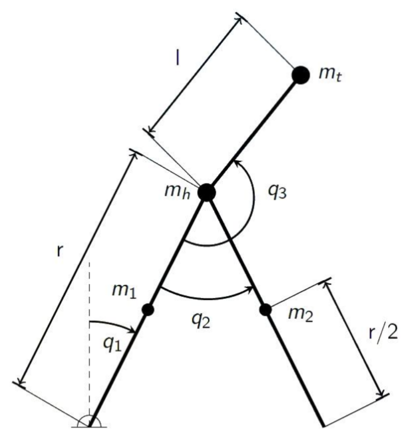

Fig. 1. Bipedal schematic.

The robot's mass is concentrated into single points, each leg has a length of "r" and a mass located at "r/2". The distance from the hips to the center of mass of the robot's torso is represented by "l". The robot is shown in body (or shape) coordinates, with q_1 and q_2 measured relative to the body, and only q_3 measured to the world frame.

III. Assumptions

We would like to declare some important underlying assumptions we have made:

- Robot has point feet contact
- No slipping or rebounds when in contact with the ground
- Impact is instantaneous
- Only one leg in contact with the ground at any instant of time
- Two symmetric legs connected to a common point called the hip
- Independently actuated at each of the revolute joints
- Both the joints are at the same location, the hip
- Unactuated at the point of contact between the stance leg and the ground

- In each step, the swing leg starts strictly behind the stance leg and is placed strictly in front of the stance leg at impact
- Walking is from left to right on a level surface
- Positive angles are counter-clockwise

IV. Kinematic Model

The kinematic model of the biped was derived using forward kinematics. It is a process to obtain the position and velocity of the end-effector using joint angles and angular velocities–in the case of all revolute joints. Vector q is the body angle vector where q_1 is the cyclic variable and is an absolute angle. Vector dq is the vector consisting of the time derivatives of the body angles.

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \; \dot{q} = dq = \begin{bmatrix} dq_1/dt \\ dq_2/dt \\ dq_3/dt \end{bmatrix} \tag{1}$$

For simplicity we represent the vector dq as following -

$$dq = \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \end{bmatrix} \tag{2}$$

We have used homogeneous transformations to calculate the positions of all the lumped masses and the leg-end of the swing leg, with respect to the origin which is the point where the stance leg touches the ground. The following equations below denote the position of m_h, m_t, m_1, m_2 and CoM, respectively.

$$pm_{\mathrm{h}} = \begin{bmatrix} -rsin(q_1) \\ rcos(q_1) \end{bmatrix} \tag{3}$$

$$pm_{\mathrm{t}} = \begin{bmatrix} lsin(q_1+q_3) - rsin(q_1) \\ rcos(q_1) - lcos(q_1+q_3) \end{bmatrix} \tag{4}$$

$$pm_1 = \begin{bmatrix} \frac{-rsin(q_1)}{2} \\ \frac{rcos(q_1)}{2} \end{bmatrix} \tag{5}$$

$$pm_2 = \begin{bmatrix} \frac{rsin(q_1+q_2)}{2} - rsin(q_1) \\ rcos(q_1) - \frac{rcos(q_1+q_2)}{2} \end{bmatrix} \tag{6}$$

$$pcm = \begin{bmatrix} \frac{m_1 \cdot pm_1 + m_2 \cdot pm_2 + m_{\mathrm{h}} \cdot pm_{\mathrm{h}} + m_{\mathrm{t}} \cdot pm_{\mathrm{t}}}{m_1 + m_2 + m_{\mathrm{h}} + m_{\mathrm{t}}} \end{bmatrix} \tag{7}$$

Using these positions, we found the velocities of each lumped mass using (8)

$$v = \frac{\partial(position)}{\partial q} \cdot \frac{dq}{dt} \tag{8}$$

The following equations denote the velocities of m_h, m_t, m_1, m_2 and CoM, respectively.

$$vm_{\mathrm{h}} = \begin{bmatrix} -dq_1 \cdot rcos(q_1) \\ -dq_1 \cdot rsin(q_1) \end{bmatrix} \tag{9}$$

$$vm_{\mathrm{t}} = \begin{bmatrix} dq_1 \cdot (lcos(q_1+q_3) - rcos(q_1)) + dq_3 \cdot lcos(q_1+q_3) \\ dq_1 \cdot (lsin(q_1+q_3) - rsin(q_1)) + dq_3 \cdot lsin(q_1+q_3) \end{bmatrix} \tag{10}$$

$$vm_1 = \begin{bmatrix} \frac{-dq_1 \cdot rsin(q_1)}{2} \\ \frac{-dq_1 \cdot rcos(q_1)}{2} \end{bmatrix} \tag{11}$$

$$vm_2 = \begin{bmatrix} dq_1 \cdot (\frac{rcos(q_1+q_2)}{2} - rcos(q_1)) + \frac{dq_2 \cdot rcos(q_1+q_2)}{2} \\ dq_1 \cdot (\frac{rsin(q_1+q_2)}{2} - rsin(q_1)) + \frac{dq_2 \cdot rsin(q_1+q_2)}{2} \end{bmatrix} \tag{12}$$

$$vcm = \frac{\partial(pcm)}{\partial q} \cdot \frac{dq}{dt} \tag{13}$$

V. Dynamic Model

To drive the dynamic model of the robot we used the Lagrangian formalism, which describes the motion of the robot using an energy based approach. The Lagrangian is given by (14) where K is the total kinetic energy and V is the total potential energy of all links and the general Lagrange's equation is given by (15) where Γ is the vector of generalized forces and torques.

$$\mathcal{L}(q,\dot{q}) = K(q,\dot{q}) - V(q) \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \Gamma \tag{15}$$

After obtaining the position and velocities of our lumped masses, we get the kinetic and potential energies using the following equations -

$$K = \sum_{i}^{n} \frac{1}{2} m {v_{\mathrm{i}}}^2 \qquad V = \sum_{i}^{n} mgh_{\mathrm{i}} \tag{16}$$

where h_i is the vertical component of the position vectors.

The Lagrange's equation (15) is equivalent to the following second-order differential equation -

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \Gamma \tag{17}$$

where,

$$D(q) = \frac{\partial(\frac{\partial K}{\partial \dot{q}})}{\partial \dot{q}} \tag{18}$$

$$C_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{\partial D_{kj}}{\partial q_i} + \frac{\partial D_{ki}}{\partial q_j} - \frac{\partial D_{ij}}{\partial q_k}\right)\dot{q}_i \tag{19}$$

$$G(q) = \frac{\partial V(q)}{\partial q} \tag{20}$$

In equation (17) Γ can be replaced by $B(q) \cdot u$, where B is a matrix that maps the joint torques to generalized forces and u is the input torques.

$$B(q) = \left(\frac{\partial}{\partial q}\begin{bmatrix} \theta_1^{rel} \\ \vdots \\ \theta_{N-1}^{rel} \end{bmatrix}\right)^T \tag{21}$$

The state-space representation of the swing-phase model can be described as (22), where the subscript "s" denotes swing-phase.

$$\dot{x_{\mathrm{s}}} = f_{\mathrm{s}}(x) + g_{\mathrm{s}}(x) \cdot u \tag{22}$$

$$f_{\mathrm{s}}(x) = \begin{bmatrix} \dot{q}_s \\ D^{-1}(q_s) \cdot (-C(q_s, \dot{q}_s) \cdot \dot{q}_s - G(q_s)) \end{bmatrix} \tag{23}$$

$$g_{\mathrm{s}}(x) = \begin{bmatrix} 0 \\ 0 \\ D^{-1}(q_s) \cdot B(q_s) \end{bmatrix} \tag{24}$$

VI. Impact Map

We can describe the swing-phase motion model using the equations derived till now. But walking involves swing-phase and impact phase when the swing leg gets in contact with the ground. Hence, we need a way to model the impact condition which cannot be described by our swing-phase model. The development of the impact model involves reaction forces at the leg-ends and hence require an unpinned model of the robot [1].

For this model we require an additional set of coordinates in addition to the body coordinates (q) used for the previous model. We denote this additional coordinates as $p_e = \begin{bmatrix} p_e^h & ; p_e^v \end{bmatrix}^T$. Any point on the robot's body can be chosen for this purpose and to keep things simple we choose *pcm* (coordinates of CoM) as our additional set of coordinates. We extend the body coordinates by appending the additional set of coordinates in the q vector and define generalized coordinates as $q_e = \begin{bmatrix} q & ; p_e \end{bmatrix}^T$. Using Lagrange's method we get

$$D_e(q_e)\ddot{q}_e + C_e(q_e, \dot{q}_e)\dot{q}_e + G_e(q_e) = B_e(q_e)u + \delta F_{ext} \tag{25}$$

where, δF_{ext} represents the external force acting on the robot when swing-leg comes in contact of the ground. Following our assumption that impact forces are instantaneous, integrating (25) over the entire duration of the impact, we obtain

$$D_e(q_e^+)q_e^+ - D_e(q_e^-) = F_{ext} \tag{26}$$

where, $F_{ext} := \int_{t^-}^{t^+} \delta F_{ext}(\tau)d\tau$, is the result of integrating the impulsive force. The velocity before the impact is give by $\dot{q}_e^-$, while the velocity after the impact is given by $\dot{q}_e^+$.

During impact the p_{e} can be determined from q_{s} and is denoted by $p_e = \gamma_e(q_s)$. Thus, pre-impact states can be given as

$$q_e^- = \begin{bmatrix} q_s^- \Upsilon_e(q_s^{-1}) \end{bmatrix} \tag{27}$$

$$\dot{q}_e^- = \begin{bmatrix} I_{NxN} \\ \frac{\partial}{\partial q_s}\Upsilon_e(q_s^{-1}) \end{bmatrix} \cdot \dot{q}_s^{-1} \tag{28}$$

Using principle of virtual work we get

$$F_{ext} = E_2(q_e^-)^T F_2 \tag{29}$$

where, $E_2(q_e) = \frac{\partial}{\partial q_e} p_2(q_e)$. Here, $p_2(q_e)$ is the position of the swing-leg end. F_2 is the force vector acting at the end of the swing leg. Now, we assumed no slipping and rebound conditions on contact so,

$$E_2(q_e^-)\dot{q_e}^+ = 0 \tag{30}$$

Using (26) and (30) we obtain

$$\begin{bmatrix} D_{\mathrm{e}}(q_{\mathrm{e}}^-) & -E_2(q_{\mathrm{e}}^-)' \\ E_2(q_{\mathrm{e}}^-) & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{q}_{\mathrm{e}}^+ \\ F_2 \end{bmatrix} = \begin{bmatrix} D_{\mathrm{e}}(q_{\mathrm{e}}^-) \begin{bmatrix} I_{N\times N} \\ \frac{\partial}{\partial q_{\mathrm{s}}}\Upsilon_{\mathrm{e}}(q_{\mathrm{s}}^-) \end{bmatrix} \\ 0_{2\times N} \end{bmatrix} \dot{q}_{\mathrm{s}}^- \tag{31}$$

solving (31) gives us

$$\begin{bmatrix} \dot{q_e}^+ \\ F_2^+ \end{bmatrix} = \begin{bmatrix} \bar{\Delta}_{\dot{q}_e}(q^-) \\ \Delta_{F_2}(q^-) \end{bmatrix} \dot{q_{\mathrm{s}}}^- \tag{32}$$

where,

$$\Delta_{F_2} = -\left(E_2 D_e^{-1} E_2^T\right)^{-1} E_2 \begin{bmatrix} I_{\mathrm{N\ x\ N}} \\ \frac{\partial}{\partial q}\Upsilon_e \end{bmatrix} \tag{33}$$

$$\bar{\Delta}_{\dot{q}_e} = D_e^{-1} E_2^T \Delta_{F_2} + \begin{bmatrix} I_{\mathrm{N\ x\ N}} \\ \frac{\partial}{\partial q}\Upsilon_e \end{bmatrix} \tag{34}$$

As we are assuming symmetric gait cycle, the roles of the swing and stance leg swaps post-impact. So we need to relabel the coordinates after impact using the following expression

$$x^+ = \Delta(x^-) \tag{35}$$

where, x^+ denotes the states pre-impact and x^- denotes the states post-impact.

$$\Delta(x^-) := \begin{bmatrix} \Delta_{q_s} q_s^- \\ \Delta_{\dot{q}_s}(q_s^-)\dot{q_s}^- \end{bmatrix} \tag{36}$$

where,

$$\Delta_{q_s} := R, \textit{ R is a circular matrix} \tag{37}$$

$$\Delta_{\dot{q}_s}(q_s^-) := \begin{bmatrix} R & 0_{Nx2} \end{bmatrix} \bar{\Delta}_{\dot{q}_e}(q_s^-) \tag{38}$$

This marks the complete model during impact phase. Now we can have an expression for the complete model of walking that involves swing-phase, impact phase and switching condition. It is given as follows -

$$\sum : \begin{cases} \dot{x} = f_s(x) + g_s(x)u & x^- \notin \mathcal{S} \\ x^+ = \Delta(x^-) & x^- \in \mathcal{S} \end{cases} \tag{39}$$

$$\mathcal{S} := \left\{(q_s, \dot{q}_s) \in \mathcal{TQ}_s | p_2^v(q) = 0, p_2^h(q) > 0\right\} \tag{40}$$

VII. Zero Dynamics

The concept of zero dynamics follows from the notion that a geometric task or configuration of a robot can be encoded into a set of outputs so that zeroing the outputs is asymptotically equivalent to achieving the task. The system's internal dynamics of the system that is compatible with the output being zero is called zero dynamics. [1]. In bipeds, this approach helps exploit the natural dynamics of the system as observed in humans and generates an efficient and stable natural-looking gait.

A. Swing-phase Zero Dynamics

During the swing phase of stable walking, if we consider any point on the biped, its position and velocity during the gait cycle will follow a cyclic pattern. The variables defining the state of this point are called cyclic variables. During the swing phase of stable walking, these cyclic variables lie on a zero-dynamics manifold. The equations defining the robot dynamics in terms of these cyclic variables (reduced order model) are given by (22), (23) and (24).

$$\eta_1 = h(q), \quad \eta_2 = L_f h(q, \dot{q}), \tag{41}$$
$$\xi_1 = \theta(q), \quad \xi_2 = L_f \theta(q, \dot{q}), \tag{42}$$

$$\dot{\eta}_1 = \eta_2, \quad \dot{\eta}_2 = L_f^2 h + L_g L_f h u \tag{43}$$
$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = L_f^2 \theta + L_g L_f \theta u \tag{44}$$
$$y = \eta_1 \tag{45}$$

B. Hybrid Zero Dynamics

During impact in bipeds, we need to extend our swing-phase dynamics model to incorporate the effect of this impact on robot dynamics. The equation incorporating the effect of impact on robots' zero dynamics is defined here. Suppose $\mathcal{Z}$ and $\dot{z} = f_{\text{zero}}(z)$ be the associated zero dynamics manifold and zero dynamics of the swing phase model respectively.

$$\Sigma_{\text{zero}} : \begin{cases} \dot{z} = f_{\text{zero}}(z), & z^- \notin \mathcal{S} \cap \mathcal{Z} \\ z^+ = \Delta(z^-), & z^- \in \mathcal{S} \cap \mathcal{Z}, \end{cases}$$

VIII. Optimization-based Gait Design

In gait design, we employ the reduced order model defined by Zero Dynamics and run it through an optimizer to obtain an optimal gait design that minimizes the cost function.

A. Parametrization of configuration variables

To implement this, we started by defining the cyclic variable q_1 for our system and further parametrizing the robot configuration variables q_2 and q_3 as a function of the selected cyclic variables using virtual-holonomic constraints. In our case, we have used Bezier polynomials for parametrizing configuration variables because of their interesting properties. Some key properties of Bezier Polynomial are defined below:

Bezier curve properties:

- Confined in Convex Hull: Bezier curves are confined by the control points which helps in stabilizing optimization algorithms ensuring the curve or in this case our virtual constraints do not go beyond the control points.
- Symmetry: Bezier curves can be made symmetric by placing the control points in a symmetric pattern, which can be useful in optimizing the curve using fewer control points.

Equations for parametrization of configuration variables $q2andq3$ using Bezier Polynomial are defined as follows:

$$b_1(s) = \sum_{k=0}^{4} \alpha_k \frac{4!}{k!(4-k)!} s^k (1-s)^{(4-k)}$$
$$b_2(s) = \sum_{k=0}^{4} \gamma_k \frac{4!}{k!(4-k)!} s^k (1-s)^{(4-k)}$$

$$\frac{\partial b_1(s)}{\partial s} = \sum_{k=0}^{3} (\alpha_{k+1} - \alpha_k) \frac{4!}{k!(4-k-1)!} s^k (1-s)^{(4-k-1)}$$
$$\frac{\partial b_2(s)}{\partial s} = \sum_{k=0}^{3} (\gamma_{k+1} - \gamma_k) \frac{4!}{k!(4-k-1)!} s^k (1-s)^{(4-k-1)}$$

Subsequently, we define the output function as $y = h(q, q) - hd(\theta)$ and generate zero dynamics for our biped using the equations described in earlier sections.

B. Optimizer

Finally, we run an optimizer to obtain optimal parametrizing parameters and initial conditions that minimize energy consumption during walking. In our study, we use *fmincon*, a built-in function in MATLAB, to minimize an objective function subject to constraints.

1) Objective Function: In [1], multiple cost functions are defined for bipeds, and each cost function employs different constraints to the optimizer. Our study focused on minimizing the energy spent per distance traveled and used the cost function described in the equation below.

$$J(\alpha) = \frac{1}{\text{steplength}} \int_0^{tf} \|u\|^2 dt$$

2) Equality Constraints: In a practical system, pre-impact states obtained through trajectory optimization will not be the same during each gait cycle. Applying the same impact map for different pre-impact conditions will add instability to the system, and the system will eventually fail. Thus, to minimize this effect, we incorporated the norm between initial post-impact conditions and final post-impact conditions at the end of the gait cycle as equality constraints. This ensures the obtained trajectory remains in the same zero-dynamics manifold and is stable.

3) Inequality Constraints: An inequality constraint is added to the system to ensure that the obtained gait design is symmetric about the origin and follows the following hypothesis for bipeds as defined in [1].

4) Initial values: The defined cost function is non-linear and is subjected to getting stuck at local minima based on the initialized conditions. So it becomes critical that physically feasible conditions are used to initialize the optimizer. In our study, we used the following initial conditions.

$$q^- = -20 \deg \; \dot{q}^- = -210 \deg / sec$$
$$alpha_{q_2} = [0.436, 0.70, 0.785]$$
$$alpha_{q_3} = [3.14, 3.40, 3.49]$$

IX. Linearized Feedback Controller Design

The control action required for the objective function of the optimizer in the gait design section is computed using a hybrid zero dynamics equation, assuming the cyclic variable remains on the zero-dynamics manifold. However, we need a feedback controller that continuously drives the model to Zero Dynamics Manifold. The equations below define the linearized feedback controller needed to keep the cyclic variables on the zero-dynamics manifold.

$$u = u^* + v \tag{46}$$

here,

$$u^* = -\left(L_g L_f h(q)\right)^{-1} L_f^2 h \tag{47}$$

$$v = -\left(L_g L_f h(q)\right)^{-1} \left(K_D L_f h + K_P h\right) \tag{48}$$

X. Results

After successfully obtaining the zero dynamics of the robot, we plot the phase portrait of q_1 and dq_1 given in Fig. 2.

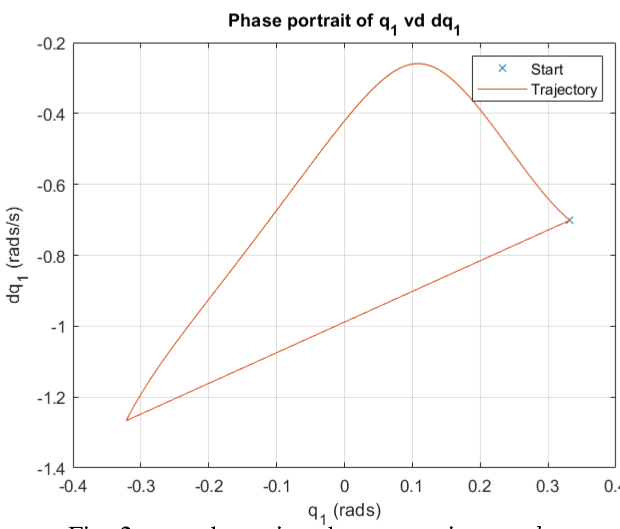

Fig. 2. zero dynamics phase portrait q vs dq_1

Fig. 3. shows the phase portrait of the full model of the robot and fig. 4. shows the plot of joint angles vs time and joint velocities vs time.

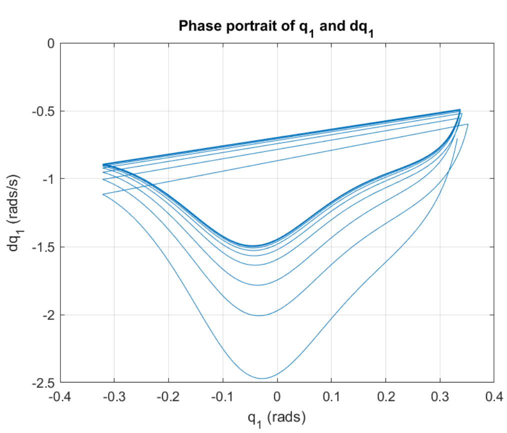

Fig. 3. Full phase portrait q vs dq_1

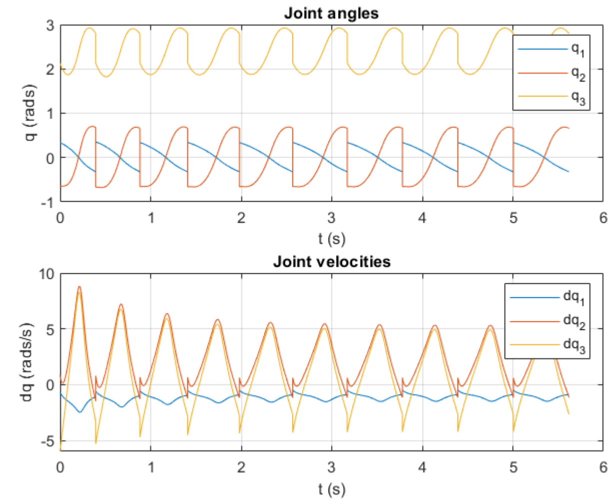

Fig. 4. Joint angles and Join velocities

Simulation results: Fig. 5 and Fig. 6 shows the start and end of the simulation, respectively. It can be observed that the torso of the robot is bent more than normal. This can be fixed using smartly chose initial values for gait design and is a part of our future work.

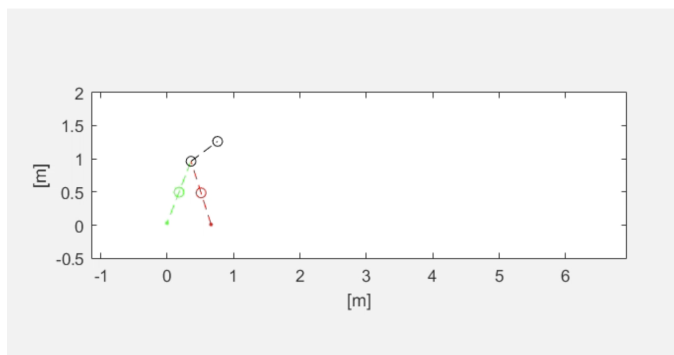

Fig. 5. Start of the simulation

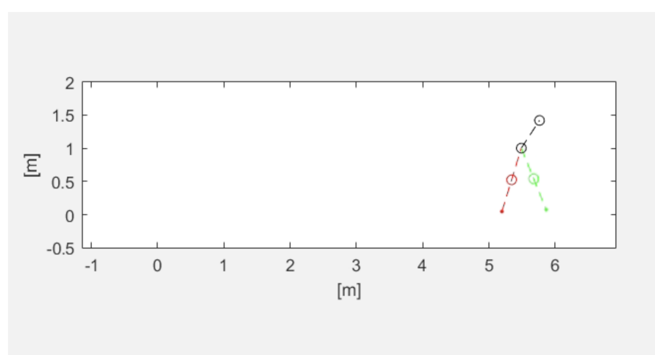

Fig. 6. End of the simulation

References

[1] E. R. Westervelt, J. W. Grizzle, C. Chevallereau, J. H. Choi, and B. Morris "Feedback Control of Dynamic Bipedal Robot Locomotion".