Solution of GATE-ST 2023 Q58

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Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \qquad t = 1, 2, \ldots, n \tag{1}$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

1) There exists an unbiased estimator of α_1
2) There exists an unbiased estimator of α_2
3) There exists an unbiased estimator of α_0
4) There exists an unbiased estimator of μ

Solution: Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t \tag{2}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & A_{11} & A_{12} \\ 1 & A_{21} & A_{22} \\ \vdots & & \\ 1 & A_{n1} & A_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \tag{3}$$

Finding mean of the $\mathbf{y}$,

$$\mathbf{y} = \mathbf{A}\mathbf{a} + \boldsymbol{\epsilon} \tag{4}$$

$$\begin{aligned} E(\mathbf{y}) &= E(\mathbf{A}\mathbf{a} + \boldsymbol{\epsilon}) && (5) \\ &= \mathbf{A}\mathbf{a} + E(\boldsymbol{\epsilon}) && (6) \end{aligned}$$

$$E(\boldsymbol{\epsilon}) = \begin{bmatrix} E(\epsilon_1) \\ E(\epsilon_2) \\ \vdots \\ E(\epsilon_n) \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{7}$$

$$\mathbf{C_y} = \mathbf{C}_{\boldsymbol{\epsilon}} \tag{8}$$

Since the $\boldsymbol{\epsilon}_i$'s are independent and identical vectors,

$$\text{Cov}(\epsilon_i, \epsilon_j) = 0, \qquad \forall i \neq j \tag{9}$$

$$\text{Var}(\epsilon_i) = 1, \qquad \forall 1 \leq i \leq n \tag{10}$$

$$\mathbf{C_y} = I_{n \times n} \tag{11}$$

$$\mathbf{y} \sim \mathcal{N}(\mathbf{A}\mathbf{a} + E(\boldsymbol{\epsilon}), I) \tag{12}$$

$$p_{\mathbf{y}} = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C_y})}} \exp -\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_Y)^T \mathbf{C_y}^{-1}(\mathbf{y} - \boldsymbol{\mu}_Y) \tag{13}$$

where $\mathbf{C_y}$ is the covariance matrix for $\mathbf{y}$, and $\boldsymbol{\mu}_Y$ is the expectance vector for $\mathbf{y}$ The maximum likelihood function can be written as:

$$L(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{(\mathbf{y}-\boldsymbol{\mu}_Y)^T(\mathbf{y}-\boldsymbol{\mu}_Y)}{2}} \tag{14}$$

$$\ln L(\mathbf{y}) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_Y)^T(\mathbf{y} - \boldsymbol{\mu}_Y) \tag{15}$$

$$\frac{\partial \ln L(\mathbf{y})}{\partial \mathbf{a}} = \mathbf{A}^T\mathbf{A}\mathbf{a} - \mathbf{A}^T\mathbf{y} \tag{16}$$

The normal equation is

$$\frac{\partial \ln L(\mathbf{y})}{\partial \mathbf{a}} = 0 \tag{17}$$

$$\mathbf{A}^T\mathbf{A}\mathbf{a} = \mathbf{A}^T\mathbf{y} \tag{18}$$

$$\mathbf{a} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y} \tag{19}$$