Solution of GATE-ST 2023 Q18

SUJAL GUPTA - EE22BTECH11052

Suppose that X has the probability density function

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \lambda > 0 \\ 0 & otherwise \end{cases} \tag{1}$$

where $\alpha > 0$ and $\lambda > 0$. Which one of the following statements is NOT true?

1) $E(X)$ exists for all $\alpha > 0$ and $\lambda > 0$
2) Variance of X exists for all $\alpha > 0$ and $\lambda > 0$
3) $E(\frac{1}{X})$ exists for all $\alpha > 0$ and $\lambda > 0$
4) $E(ln(1+X))$ exists for all $\alpha > 0$ and $\lambda > 0$

Solution:

1)

$$E(X) = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2}$$
$$= \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \tag{3}$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} \tag{4}$$
$$\tag{5}$$

since we know that

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha} \qquad \text{for } \lambda > 0, \alpha > 0 \tag{6}$$

$$E(X) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \tag{7}$$

Using the relation

$$\Gamma(x+1) = \Gamma(x)x \tag{8}$$

$$E(X) = \frac{\alpha}{\lambda} \tag{9}$$

Thus $E(X)$ exists for all $\alpha > 0$ and $\lambda > 0$.

2)

$$Var(X) = E(X^2) - E(X)^2 \tag{10}$$

$$E(X^2) = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \tag{11}$$
$$= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{(\alpha+2)-1} e^{-\lambda x} dx \tag{12}$$
$$= \int_0^\infty \frac{1}{\lambda^2} \frac{\lambda^{\alpha+2}}{\Gamma(\alpha)} x^{(\alpha+2)-1} e^{-\lambda x} dx \tag{13}$$

$$E(X^2) = \int_0^\infty \frac{\alpha(\alpha+1)}{\lambda^2} \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\lambda x} dx \tag{14}$$

using the density of the gamma distribution, we get

$$E(X^2) = \frac{\alpha(\alpha+1)}{\lambda^2} \tag{15}$$

$$Var(X) = \frac{\alpha^2 + \alpha}{\lambda^2} - \frac{\alpha^2}{\lambda} \tag{16}$$
$$= \frac{\alpha}{\lambda^2} \tag{17}$$

Thus, Variance of X exists for all $\alpha > 0$ and $\lambda > 0$

3)

$$E\left(\frac{1}{X}\right) = \int_0^\infty \frac{1}{x} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \tag{18}$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-2} e^{-\lambda x} \tag{19}$$

For this, $\alpha > 1$ is a must condition. Hence C is not a correct option. Hence C is the answer.

4) For $E(ln(1+X))$,

$$E(ln(1+X)) = E(X) - \frac{E(X^2)}{2} + \frac{E(X^4)}{4} - .. \tag{20}$$

We write the general expression for $E(X^n)$

$$E(X^n) = \frac{(\alpha)(\alpha+1)\ldots(\alpha+n-1)}{\lambda^n} \tag{21}$$

So by applying the ratio test to check the convergence of the sequence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \tag{22}$$

$$\left|\frac{E(X^{n+2})}{E(X^n)}\right| = \frac{\frac{(\alpha)(\alpha+1)...(\alpha+n+1)}{\lambda^{n+2}}}{\frac{(\alpha)(\alpha+1)...(\alpha+n-1)}{\lambda^n}} \tag{23}$$

$$= \frac{(\alpha+n)(\alpha+n+1)}{\lambda^2} \tag{24}$$

$$\lim_{n\to\infty}\left|\frac{E(X^{n+2})}{E(X^n)}\right| = \infty \tag{25}$$

Thus $E(ln(1+X))$ generates a divergent function and hence $E(ln(1+X))$ does not exist for all $\alpha > 0$ and $\lambda > 0$.