Solution of GATE-ST 2023 Q18

SUJAL GUPTA - EE22BTECH11052

Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \qquad t = 1, 2, \dots, n \tag{1}$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

1) There exists an unbiased estimator of α_1
2) There exists an unbiased estimator of α_2
3) There exists an unbiased estimator of α_0
4) There exists an unbiased estimator of μ

Solution: Let $X1 = XandX_2 = X^2$
Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \epsilon_t \tag{2}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & & \\ y1 & x_{n1} & x_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \tag{3}$$

$$y = X\alpha + \epsilon \tag{4}$$

Let B be the set of all possible vectors α. The object is to find a vector b from B that minimizes the sum of squared deviations of ϵ's i.e.,

$$S(\alpha) = \sum_{t=1}^{n} \epsilon^2 \tag{5}$$

$$= \epsilon^T \epsilon \tag{6}$$

$$= (y - X\alpha)^T (y - X\alpha) \tag{7}$$

Differentiate $S(\alpha)$ wrt α

$$\frac{\partial S(\alpha)}{\partial \alpha} = 2X^T X\alpha - 2X^T y \tag{8}$$

The normal equation is

$$\frac{\partial S(\alpha)}{\partial \alpha} = 0 \tag{9}$$

$$X^T X\alpha = X^T y \tag{10}$$

$$b = \left(X^T X\right)^{-1} X^T y \tag{11}$$