Solution of GATE-ST 2023 Q58

SUJAL GUPTA - EE22BTECH11052

Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \qquad t = 1, 2, \ldots, n \tag{1}$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

1) There exists an unbiased estimator of α_1
2) There exists an unbiased estimator of α_2
3) There exists an unbiased estimator of α_0
4) There exists an unbiased estimator of μ

Solution: Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t \tag{2}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & A_{11} & A_{12} \\ 1 & A_{21} & A_{22} \\ \vdots & & \\ 1 & A_{n1} & A_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \tag{3}$$

Finding mean of the $\boldsymbol{\epsilon}$,

$$\mathbf{y} = A\boldsymbol{x} + \boldsymbol{\epsilon} \tag{4}$$

$$E(\boldsymbol{\epsilon}) = \begin{bmatrix} E(\epsilon_1) \\ E(\epsilon_2) \\ \vdots \\ E(\epsilon_n) \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}$$

The covariance matrix for $\boldsymbol{\epsilon}$ is:

$$\mathbf{C}_{\boldsymbol{\epsilon}} = \mathbf{E}[(\boldsymbol{\epsilon} - \mathbf{E}\boldsymbol{\epsilon})(\boldsymbol{\epsilon} - \mathbf{E}\boldsymbol{\epsilon})^{\mathbf{T}}] \tag{5}$$

$$= E \begin{bmatrix} (\epsilon_1 - E(\epsilon_1))^2 & (\epsilon_1 - E(\epsilon_1))(\epsilon_2 - E(\epsilon_2)) & \ldots & (\epsilon_1 - E(\epsilon_1))(\epsilon_n - E(\epsilon_n)) \\ (\epsilon_2 - E(\epsilon_2))(\epsilon_1 - E(\epsilon_1)) & (\epsilon_2 - E(\epsilon_2))^2 & \ldots & (\epsilon_2 - E(\epsilon_2))(\epsilon_n - E(\epsilon_n)) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ (\epsilon_n - E(\epsilon_n))(\epsilon_1 - E(\epsilon_1)) & (\epsilon_n - E(\epsilon_n))(\epsilon_2 - E(\epsilon_2)) & \ldots & (\epsilon_n - E(\epsilon_n))^2 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \mathrm{Var}(\epsilon_1) & \mathrm{Cov}(\epsilon_1, \epsilon_2) & \ldots & \mathrm{Cov}(\epsilon_1, \epsilon_n) \\ \mathrm{Cov}(\epsilon_2, \epsilon_1) & \mathrm{Var}(\epsilon_2) & \ldots & \mathrm{Cov}(\epsilon_2, \epsilon_n) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \mathrm{Cov}(\epsilon_n, \epsilon_1) & \mathrm{Cov}(\epsilon_n \epsilon_2) & \ldots & \mathrm{Var}(\epsilon_n) \end{bmatrix} \tag{7}$$

Since the $\boldsymbol{\epsilon}$'s are independent vectors,

$$\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0, \qquad \forall i \neq j \tag{8}$$

$$\mathrm{Var}(\epsilon_i) = 1, \qquad \forall 1 \leq i \leq n \tag{9}$$

Hence,

$$\mathbf{C}_{\boldsymbol{\epsilon}} = I_{n\times n} \tag{10}$$

$$E(\mathbf{y}) = E(A\boldsymbol{x} + \boldsymbol{\epsilon}) \tag{11}$$
$$= A\mathbf{x} + E(\boldsymbol{\epsilon}) \tag{12}$$

$$\mathbf{C}_{\mathbf{y}} = I_{n\times n} \tag{13}$$

$$\mathbf{y} \sim \mathcal{N}\left(A\mathbf{x} + E(\boldsymbol{\epsilon}), I\right) \tag{14}$$

$$f(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp -\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{y} - \boldsymbol{\mu}) \tag{15}$$

where Σ is the covariance matrix for $\mathbf{y}$, and $\boldsymbol{\mu}$ is the expectance vector for $\mathbf{y}$

$$f(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^n}} \exp -\frac{1}{2}(\boldsymbol{\epsilon} - E(\boldsymbol{\epsilon}))^T (\boldsymbol{\epsilon} - E(\boldsymbol{\epsilon})) \tag{16}$$

The maximum likelihood function can be written as:

$$L(\boldsymbol{\epsilon}) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{(\boldsymbol{\epsilon} - E(\boldsymbol{\epsilon}))^T (\boldsymbol{\epsilon} - E(\boldsymbol{\epsilon}))}{2}} \tag{17}$$

$$lnL(\boldsymbol{\epsilon}) = -\frac{n}{2} ln(2\pi) - \frac{(\boldsymbol{\epsilon} - E(\boldsymbol{\epsilon}))^T (\boldsymbol{\epsilon} - E(\boldsymbol{\epsilon}))}{2} \tag{18}$$