Solution of GATE-ST 2023 Q18

SUJAL GUPTA - EE22BTECH11052

Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \qquad t = 1, 2, \dots, n \quad (1)$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

1) There exists an unbiased estimator of α_1
2) There exists an unbiased estimator of α_2
3) There exists an unbiased estimator of α_0
4) There exists an unbiased estimator of μ

Solution: Let $X_1 = X$ and $X_2 = X^2$
Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \epsilon_t \quad (2)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & & \\ y1 & x_{n1} & x_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \quad (3)$$

$$\mathbf{y} = X\boldsymbol{\alpha} + \epsilon \quad (4)$$

Here the variable y_i is distributed as

$$y_i \sim \mathcal{N}\left(x_i \boldsymbol{\alpha}^T, \sigma^2\right) \quad (5)$$

The maximum likelihood function can be written as:

$$L(\boldsymbol{\alpha}, \mu) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{-(\mathbf{y}-X\alpha)^T(\mathbf{y}-X\alpha)}{2}} \quad (6)$$

$$lnL(\boldsymbol{\alpha}, \mu) = -\frac{n}{2} ln(2\pi) - \frac{1}{2}(\mathbf{y} - X\boldsymbol{\alpha})^T(\mathbf{y} - X\boldsymbol{\alpha}) \quad (7)$$

To find the maximum log likelihood

$$\frac{\partial lnL(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = X^T X\boldsymbol{\alpha} - X^T\mathbf{y} \quad (8)$$

$$= 0 \quad (9)$$

$$X^T X\boldsymbol{\alpha} = X^T\mathbf{y} \quad (10)$$

$$\hat{\boldsymbol{\alpha}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} \quad (11)$$