Solution of GATE-ST 2023 Q58

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Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \qquad t = 1, 2, \ldots, n \tag{1}$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

1) There exists an unbiased estimator of α_1
2) There exists an unbiased estimator of α_2
3) There exists an unbiased estimator of α_0
4) There exists an unbiased estimator of μ

Solution: Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t \tag{2}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & A_{11} & A_{12} \\ 1 & A_{21} & A_{22} \\ \vdots & & \\ 1 & A_{n1} & A_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \tag{3}$$

Finding mean of the $\mathbf{y}$,

$$\mathbf{y} = \mathbf{A}\mathbf{a} + \boldsymbol{\epsilon} \tag{4}$$

$$E(\mathbf{y}) = E(\mathbf{A}\mathbf{a} + \boldsymbol{\epsilon}) \tag{5}$$

$$= \mathbf{A}\mathbf{a} + E(\boldsymbol{\epsilon}) \tag{6}$$

$$E(\boldsymbol{\epsilon}) = \begin{bmatrix} E(\epsilon_1) \\ E(\epsilon_2) \\ \vdots \\ E(\epsilon_n) \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} \tag{7}$$

$$\boldsymbol{\mu}_{\boldsymbol{\epsilon}} = \mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{8}$$

Let $\mathbf{A}_i$ represent the i^{th} row of $\mathbf{A}$

$$\mathrm{Cov}(y_i, y_j) = E\left((y_i - E(y_i))\left(y_j - E(y_j)\right)\right) \tag{9}$$

$$= E\left((\mathbf{A}_i\mathbf{a} + \epsilon_i - E(\mathbf{A}_i\mathbf{a} + \epsilon_i))(\mathbf{A}_j\mathbf{a} + \epsilon_j - E(\mathbf{A}_j\mathbf{a} + \epsilon_j))\right) \tag{10}$$

$$= E\left((\epsilon_i - E(\epsilon_i))(\epsilon_j - E(\epsilon_j))\right) \tag{11}$$

$$= \mathrm{Cov}(\epsilon_i, \epsilon_j) \tag{12}$$

$$\mathbf{C_y} = \mathbf{C}_{\boldsymbol{\epsilon}} \tag{13}$$

Since the ϵ_i's are independent and identical vectors,

$$\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0, \qquad \forall i \neq j \tag{14}$$

$$\mathrm{Var}(\epsilon_i) = 1, \qquad \forall 1 \leq i \leq n \tag{15}$$

$$\mathbf{C_y} = I_{n\times n} \tag{16}$$

$$\mathbf{y} \sim \mathcal{N}\left(\mathbf{Aa} + \boldsymbol{\mu}_{\boldsymbol{\epsilon}}, I\right) \tag{17}$$

$$p_{\mathbf{y}} = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C_y})}} \exp\frac{-(\mathbf{y} - \boldsymbol{\mu}_{\boldsymbol{\epsilon}} - \mathbf{Aa})^T \mathbf{C_y}^{-1} (\mathbf{y} - \boldsymbol{\mu}_{\boldsymbol{\epsilon}} - \mathbf{Aa})}{2} \tag{18}$$

where $\mathbf{C_y}$ is the covariance matrix for $\mathbf{y}$
The maximum likelihood function can be written as:

$$L(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{-(\mathbf{y}-\boldsymbol{\mu}_{\boldsymbol{\epsilon}}-\mathbf{Aa})^T(\mathbf{y}-\mathbf{Aa}-\boldsymbol{\mu}_{\boldsymbol{\epsilon}})}{2}} \tag{19}$$

$$\ln L(\mathbf{y}) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_{\boldsymbol{\epsilon}} - \mathbf{Aa})^T(\mathbf{y} - \boldsymbol{\mu}_{\boldsymbol{\epsilon}} - \mathbf{Aa}) \tag{20}$$

$$\frac{\partial \ln L(\mathbf{y})}{\partial \mathbf{a}} = \frac{\partial(-\mathbf{y}^T\mathbf{Aa} - \mathbf{a}^T\mathbf{A}^T\mathbf{y} + \mathbf{a}^T\mathbf{A}^T\mathbf{Aa} + \mathbf{a}^T\mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}} + \boldsymbol{\mu}_{\boldsymbol{\epsilon}}{}^T\mathbf{Aa})}{\partial \mathbf{a}} \tag{21}$$

$$= -2\mathbf{A}^T\mathbf{y} + 2\mathbf{A}^T\mathbf{Aa} + 2\mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}} \tag{22}$$

The normal equation is

$$\frac{\partial \ln L(\mathbf{y})}{\partial \mathbf{a}} = 0 \tag{23}$$

$$\mathbf{a} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\left(\mathbf{A}^T\mathbf{y} - \mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}}\right) \tag{24}$$

For unbiased estimator,

$$E(\mathbf{a}) = \mathbf{a} \tag{25}$$

$$E(\mathbf{a}) = E\left(\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\left(\mathbf{A}^T\mathbf{y} - \mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}}\right)\right) \tag{26}$$

$$= \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\left(\mathbf{A}^T E(\mathbf{y}) - \mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}}\right) \tag{27}$$

$$= \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\left(\mathbf{A}^T\left(\mathbf{Aa} + \boldsymbol{\mu}_{\boldsymbol{\epsilon}}\right) - \mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}}\right) \tag{28}$$

$$= \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{a} \tag{29}$$

$$= \mathbf{a} \tag{30}$$

Hence there exist unbiased estimator for $\alpha_0, \alpha_1, \alpha_2$
For Maximum Likelihood Estimator of μ

$$\frac{\partial \ln L(\mathbf{y})}{\partial \boldsymbol{\mu}_{\boldsymbol{\epsilon}}} = \frac{\partial(-\mathbf{y}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}} - \mathbf{a}^T\mathbf{A}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}} + \boldsymbol{\mu}_{\boldsymbol{\epsilon}}\mathbf{y} + \boldsymbol{\mu}_{\boldsymbol{\epsilon}}\mathbf{Ax} + \boldsymbol{\mu}_{\boldsymbol{\epsilon}}{}^T\boldsymbol{\mu}_{\boldsymbol{\epsilon}})}{\partial \boldsymbol{\mu}_{\boldsymbol{\epsilon}}} \tag{31}$$

$$= -2\mathbf{y} + 2\mathbf{Aa} + 2\boldsymbol{\mu}_{\boldsymbol{\epsilon}} \tag{32}$$

The normal equation is

$$\frac{\partial \ln L(\mathbf{y})}{\partial \boldsymbol{\mu}_{\epsilon}} = 0 \tag{33}$$

$$\boldsymbol{\mu}_{\epsilon} = \mathbf{y} - \mathbf{Aa} \tag{34}$$

$$E(\boldsymbol{\mu}_{\epsilon}) = E(\mathbf{y} - \mathbf{Aa}) \tag{35}$$

$$= E(\mathbf{y}) - \mathbf{Aa} \tag{36}$$

$$= E(\boldsymbol{\epsilon}) \tag{37}$$

$$= \boldsymbol{\mu}_{\epsilon} \tag{38}$$

Since,

$$\boldsymbol{\mu}_{\epsilon} = \mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{39}$$

Hence there exists an unbiased estimator for μ as well.