

# GATE-ST 2023 Q18

SUJAL GUPTA - EE22BTECH11052

EE2102 - IITK (IIT KANDI)

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# Question

Suppose that  $X$  has the probability density function

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \lambda > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where  $\alpha > 0$  and  $\lambda > 0$ . Which one of the following statements is NOT true?

- ①  $E(X)$  exists for all  $\alpha > 0$  and  $\lambda > 0$
- ② Variance of  $X$  exists for all  $\alpha > 0$  and  $\lambda > 0$
- ③  $E(\frac{1}{X})$  exists for all  $\alpha > 0$  and  $\lambda > 0$
- ④  $E(\ln(1 + X))$  exists for all  $\alpha > 0$  and  $\lambda > 0$

1

$$E(X) = \int_{-\infty}^{\infty} xp_X(x)dx \quad (2)$$

$$= \int_0^{\infty} x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad (3)$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^\alpha e^{-\lambda x} \quad (4)$$

$$(5)$$

since we know that

$$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha} \quad \text{for } \lambda > 0, \alpha > 0 \quad (6)$$

## Solution II

$$E(X) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \quad (7)$$

Using the relation

$$\Gamma(x+1) = \Gamma(x)x \quad (8)$$

$$E(X) = \frac{\alpha}{\lambda} \quad (9)$$

Thus  $E(X)$  exists for all  $\alpha > 0$  and  $\lambda > 0$ .

2

$$\text{Var}(X) = E(X^2) - E(X)^2 \quad (10)$$

$$E(X^2) = \int_0^{\infty} x^2 \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \quad (11)$$

$$= \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{(\alpha+2)-1} e^{-\lambda x} dx \quad (12)$$

$$= \int_0^{\infty} \frac{1}{\lambda^2} \frac{\lambda^{\alpha+2}}{\Gamma(\alpha)} x^{(\alpha+2)-1} e^{-\lambda x} dx \quad (13)$$

$$E(X^2) = \int_0^{\infty} \frac{\alpha(\alpha+1)}{\lambda^2} \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\lambda x} dx \quad (14)$$

using the density of the gamma distribution, we get

$$E(X^2) = \frac{\alpha(\alpha + 1)}{\lambda^2} \quad (15)$$

$$\text{Var}(X) = \frac{\alpha^2 + \alpha}{\lambda^2} - \frac{\alpha^2}{\lambda} \quad (16)$$

$$= \frac{\alpha}{\lambda^2} \quad (17)$$

Thus, Variance of  $X$  exists for all  $\alpha > 0$  and  $\lambda > 0$

## Solution V

3

$$E\left(\frac{1}{X}\right) = \int_0^{\infty} \frac{1}{x} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad (18)$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-2} e^{-\lambda x} \quad (19)$$

For this,  $\alpha > 1$  is a must condition. Hence C is not a correct option.  
Hence C is the answer.

4 For  $E(\ln(1 + X))$ ,

$$E(\ln(1 + X)) = E(X) - \frac{E(X^2)}{2} + \frac{E(X^4)}{4} - \dots \quad (20)$$

We write the general expression for  $E(X^n)$

$$E(X^n) = \frac{(\alpha)(\alpha + 1) \dots (\alpha + n - 1)}{\lambda^n} \quad (21)$$

## Solution VI

So by applying the ratio test to check the convergence of the sequence

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad (22)$$

$$\left| \frac{E(X^{n+2})}{E(X^n)} \right| = \frac{\frac{(\alpha)(\alpha+1)\dots(\alpha+n+1)}{\lambda^{n+2}}}{\frac{(\alpha)(\alpha+1)\dots(\alpha+n-1)}{\lambda^n}} \quad (23)$$

$$= \frac{(\alpha + n)(\alpha + n + 1)}{\lambda^2} \quad (24)$$

$$\lim_{n \rightarrow \infty} \left| \frac{E(X^{n+2})}{E(X^n)} \right| = \infty \quad (25)$$

Thus  $E(\ln(1 + X))$  generates a divergent function and hence  $E(\ln(1 + X))$  does not exist for all  $\alpha > 0$  and  $\lambda > 0$ .