

Solution of GATE-ST 2023 Q18

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Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \quad t = 1, 2, \dots, n \quad (1)$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t 's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

- 1) There exists an unbiased estimator of α_1
- 2) There exists an unbiased estimator of α_2
- 3) There exists an unbiased estimator of α_0
- 4) There exists an unbiased estimator of μ

Solution: Let $X_1 = X$ and $X_2 = X^2$

Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \epsilon_t \quad (2)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \quad (3)$$

$$y = X\alpha + \epsilon \quad (4)$$

Let B be the set of all possible vectors α . The object is to find a vector α from B that minimizes the sum of squared deviations of ϵ 's i.e.,

$$S(\alpha) = \sum_{t=1}^n \epsilon^2 \quad (5)$$

$$= \epsilon^T \epsilon \quad (6)$$

$$= (y - X\alpha)^T (y - X\alpha) \quad (7)$$

Differentiate $S(\alpha)$ wrt α

$$\frac{\partial S(\alpha)}{\partial \alpha} = 2X^T X\alpha - 2X^T y \quad (8)$$

The normal equation is

$$\frac{\partial S(\alpha)}{\partial \alpha} = 0 \quad (9)$$

$$X^T X\alpha = X^T y \quad (10)$$

$$\alpha = (X^T X)^{-1} X^T y \quad (11)$$