Solution of GATE-ST 2023 Q58

SUJAL GUPTA - EE22BTECH11052

Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t, \qquad t = 1, 2, \dots, n \tag{1}$$

where α_0 , α_1 and α_2 are unknown parameters and ϵ_t's are independent and identically distributed random variables each having $\mathcal{N}(\mu, 1)$ distribution with μ unknown. Then which of the following statements is/are true?

1) There exists an unbiased estimator of α_1
2) There exists an unbiased estimator of α_2
3) There exists an unbiased estimator of α_0
4) There exists an unbiased estimator of μ

Solution: Assuming that the model is

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t \tag{2}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & A_{11} & A_{12} \\ 1 & A_{21} & A_{22} \\ \vdots & & \\ 1 & A_{n1} & A_{n2} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \tag{3}$$

Finding mean of the ϵ,

$$\mathbf{y} = A\boldsymbol{x} + \epsilon \tag{4}$$

$$E\epsilon = \begin{bmatrix} E\epsilon_1 \\ E\epsilon_2 \\ \vdots \\ E\epsilon_n \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}$$

The covariance matrix for ϵ is:

$$\mathbf{C}_\epsilon = \mathbf{E}[(\epsilon - \mathbf{E}\epsilon)(\epsilon - \mathbf{E}\epsilon)^{\mathbf{T}}] \tag{5}$$

$$= E \begin{bmatrix} (\epsilon_1 - E\epsilon_1)^2 & (\epsilon_1 - E\epsilon_1)(\epsilon_2 - E\epsilon_2) & \dots & (\epsilon_1 - E\epsilon_1)(\epsilon_n - E\epsilon_n) \\ (\epsilon_2 - E\epsilon_2)(\epsilon_1 - E\epsilon_1) & (\epsilon_2 - E\epsilon_2)^2 & \dots & (\epsilon_2 - E\epsilon_2)(\epsilon_n - E\epsilon_n) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ (\epsilon_n - E\epsilon_n)(\epsilon_1 - E\epsilon_1) & (\epsilon_n - E\epsilon_n)(\epsilon_2 - E\epsilon_2) & \dots & (\epsilon_n - E\epsilon_n)^2 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \mathrm{Var}(\epsilon_1) & \mathrm{Cov}(\epsilon_1, \epsilon_2) & \dots & \mathrm{Cov}(\epsilon_1, \epsilon_n) \\ \mathrm{Cov}(\epsilon_2, \epsilon_1) & \mathrm{Var}(\epsilon_2) & \dots & \mathrm{Cov}(\epsilon_2, \epsilon_n) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \mathrm{Cov}(\epsilon_n, \epsilon_1) & \mathrm{Cov}(\epsilon_n \epsilon_2) & \dots & \mathrm{Var}(\epsilon_n) \end{bmatrix} \tag{7}$$

Since the ϵ's are independent vectors,

$$\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0, \qquad \forall i \neq j \tag{8}$$

$$\mathrm{Var}(\epsilon_i) = 1, \qquad \forall 1 \leq i \leq n \tag{9}$$

Hence,

$$\mathbf{C}_\epsilon = I_{n \times n} \tag{10}$$

$$\begin{aligned} E(\mathbf{y}) &= E(A\boldsymbol{x} + \epsilon) && (11) \\ &= AE(\boldsymbol{x}) + E(\epsilon) && (12) \end{aligned}$$

$$\begin{aligned} Var(\mathbf{y}) &= Var(A\boldsymbol{x} + \epsilon) && (13) \\ &= AVar(\boldsymbol{x}) + Var(\epsilon) && (14) \\ &= AVar(\boldsymbol{x}) + I && (15) \end{aligned}$$