appears as small high-frequency oscillations in Figure 7.8) now appears in the form of "freckles" that are visible in all three Tikhonov solutions. We can safely identify six (perhaps seven?) of the nine spots, while the smallest ones are lost in the inverted noise; we conclude that the resolution limit is about 4×4 pixels in this problem.

7.7 Tomography in 2D*

Tomography can be characterized as the science of computing reconstructions in 2D and 3D from projections, i.e., data obtained by integrations along rays (typically straight lines) that penetrate a domain Ω—typically a rectangle in 2D, and a box in 3D.

Here we consider a 2D model problem on the square domain $\Omega = [0,1] \times [0,1]$ (in arbitrary units), in which we are given an unknown function $f(\mathbf{t}) = f(t_1, t_2)$ that we wish to reconstruct. We assume that this function represents some material parameter, in such a way that the damping of a signal penetrating an infinitesimally small part $d\tau$ of a ray at position $\mathbf{t}$ is proportional to the product to $f(\mathbf{t})\, d\tau$. The data in the tomography problem consist of measurements of the damping of signals following well-defined rays through the domain Ω. See, e.g., [7, Section 7.4] for details of the mathematical formulation.

In this model problem, the ith observation b_i, for $i = 1, \dots, m$, represents the damping of a signal that penetrates Ω along a straight line, which we refer to as ray^i; see Figure 7.10 for an example. All the points $\mathbf{t}^i$ on ray^i are given by

$$\mathbf{t}^i(\tau) = \mathbf{t}^{i,0} + \tau\, \mathbf{d}^i, \qquad \tau \in \mathbb{R},$$

where $\mathbf{t}^{i,0}$ is an arbitrary point on the ray, and $\mathbf{d}^i$ is a (unit) vector that points in the direction of the ray. Due to the above assumption, the damping is proportional to the integral of the function $f(\mathbf{t})$ along the ray. Specifically, for the ith observation, and ignoring a problem-specific constant, the damping associated with the ith ray is given by

$$b_i = \int_{-\infty}^{\infty} f\big(\mathbf{t}^i(\tau)\big)\, d\tau, \qquad i = 1, \dots, m,$$

where $d\tau$ denotes the integration along the ray.

We can discretize this problem by dividing Ω into an $N \times N$ array of pixels, and in each pixel (k, ℓ) we assume that the function $f(\mathbf{t})$ is a constant $f_{k\ell}$:

$$f(\mathbf{t}) = f_{k\ell} \qquad \text{for} \qquad t_1 \in I_k \text{ and } t_2 \in I_\ell,$$

where we have defined the interval $I_k = [(k-1)/N,\ k/N]$, $k = 1, \dots, N$ (and similarly for I_ℓ). With this assumption about $f(\mathbf{t})$ being piecewise constant, the expression for the kth measurement takes the simpler form

$$b_i = \sum_{(k,\ell) \in \text{ray}^i} f_{k\ell}\, \Delta L_{k\ell}^{(i)}, \qquad \Delta L_{k\ell}^{(i)} = \text{length of ray}_i \text{ in pixel } (k, \ell)$$

for $i = 1, \dots, m$. Again, see Figure 7.10 for clarification, for the case $N = 6$.

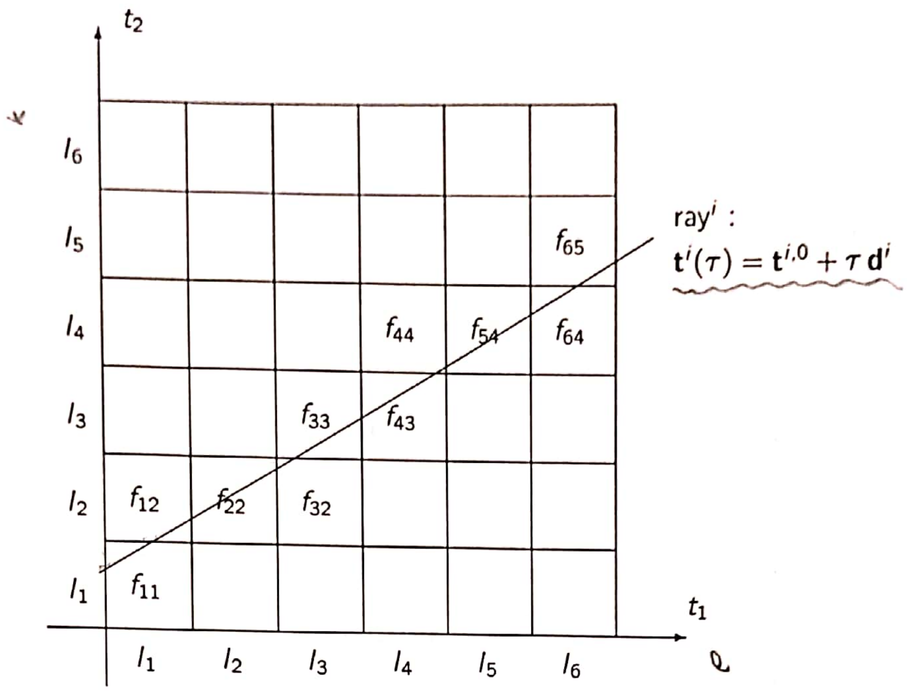

Figure 7.10. *Example of a discretized tomography problem with* $N = 6$*. The* i*th ray intersects a total of* 10 *pixels, and thus the* i*th row of the matrix* A *has* 10 *nonzero elements (in columns* 1, 2, 8, 14, 15, 21, 22, 28, 34, *and* 35*).*

The above equation is, in fact, a linear system of equations in the N^2 unknowns $f_{k\ell}$. To arrive at a more convenient notation for this system, we introduce the vector x of length $n = N^2$ whose elements are the (unknown) function values $f_{k\ell}$, ordered as follows:

$$x_j = f_{k\ell}, \qquad j = (\ell - 1)\,N + k.$$

This corresponds to stacking the columns of the $N \times N$ matrix F whose elements are the values $f_{k\ell}$. Moreover, we organize the measurements b_i into a vector b.

There is clearly a linear relationship between the data b_k and the unknowns in the vector x, meaning that we can always write

$$b_i = \sum_{j=1}^{n} a_{ij}\,x_j, \qquad i = 1, \ldots, m.$$

With the above definitions it then follows that we arrive at a linear system of equations $Ax = b$ with an $m \times n$ matrix whose elements are given by

$$a_{ij} = \begin{cases} \Delta L_{k\ell}^{(i)}, & (k,\ell) \in \text{ray}_i \\ 0 & \text{else.} \end{cases}$$

We recall that index i denotes the ith observation (corresponding to ray$_i$) and j denotes the pixel number in an ordering with $j = (\ell - 1)\,N + k$. The matrix A is very sparse,

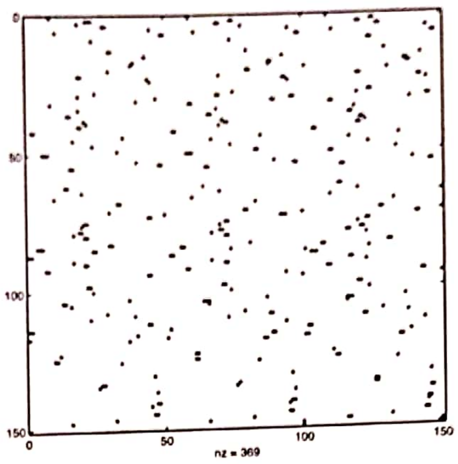

Figure 7.11. *An arbitrary* 150×150 *submatrix of a sparse tomography matrix* A *for the case* $N = 50$ *leading to a matrix of dimensions* 2500×2500.

and the number of nonzero elements in any row is bounded above by $2N$. Figure 7.11 shows a typical example of a 150×150 submatrix of A for the case $N = 50$.

In the image deblurring problem described in the previous section, one would never form the coefficient matrix explicitly. On the other hand, for tomography problems it is often feasible and convenient to explicitly form the very sparse matrix A. For both problems one will use an iterative method, and the classical methods from Section 6.1 have proved to be quite efficient for tomography problems (while they may be very slow when applied to general discrete inverse problems with a dense coefficient matrix). Exercise 7.8 illustrates this.

7.8 Depth Profiling and Depth Resolution*

We have already seen a variety of examples of how inverse problems can be used to reveal hidden information, that would somehow involve opening or destroying the object. Therefore, inverse problems play an important role in nondestructive testing, a common analysis technique used in science and industry to evaluate the properties of a material, component, or system without causing damage to it.

The example in this section, which acts as a prelude to the geophysical prospecting problem considered in the next section, deals with reconstruction of hidden layers of paint in an image via solving an inverse problem that involves X-ray measurements. The technique is called Particle-Induced X-ray Emission Spectroscopy (PIXE) depth profiling; see, e.g., [55] for details.

At a specific position, an X-ray source sends a beam at an angle s into the material, and the energy penetrates to a depth of $d\cos(s)$, where d depends on the energy in the beam. The beam that penetrates the material is partially reflected according to the material parameter $f(t)$, where t denotes the depth into the material, with $0 \le t \le d$. A detector located perpendicular to the surface records the reflected signal $g(s)$, which depends on the incidence angle s and the material parameter $f(t)$