Signal Analysis, Design of Experiments and system Identification
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4^{th} Computer Exercise
Regularisation methods

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We are concerned with discretisations of large linear inverse problems of the form:

$$Ax \simeq b, \quad A \in \mathbb{R}^{M\times N},$$

Where: b is the vector representing the measured data (typically with noise), and A is a matrix presenting the forward mapping.
Given A and b, the aim is to compute an approximation of the unknown vector x. Such problems usually arise in computed tomography, geoscience and image deblurring. Due to the large problem dimensions, the previous equation can be ill-posed problem in the sense that the singular values of A gradually decay and cluster at zero and hence the computed x is very sensitive to the errors in b. Regularization methods are therefore used by solving a penalised least-squares problem of the form:

$$\min_x \|Ax - b\|_2^2 + \lambda^2 \Omega(x)$$

In the case where $\Omega(x) = \|x\|_2^2$ and $\Omega(x) = \|Lx\|_2^2$, we obtain the classical Tikhonov regularization problem. Iterative method consits, however, to terminate the iterations when semi-convergence is achieved, which means to terminate when the desired approximation is obtained, but before noise starts to show up in the solution. The truncated singular value decomposition (TSVD) consists on decomposing the matrix A to the form $A = \sum_{i=1}^{n} u_i \sigma_i v_i^T$ so that the singular values decay gradually to zero.

1. No regularization is implemented, i.e. solving the problem $x = A^{-1}b$. Run the code inverseTomo_classicalMatrixInversion.m. Compare the plot with the reference plot generated by the tomography and make interpretation.
   Remark N counts the number of pixels. In case the computations may take too long, reduce the number of pixels, e.g. by setting N=30.

2. Methods: Truncated singular value decomposition (TSVD), iterative and Tikhonov. Run the codes inverseTomo_TSVD.m, inverseTomo_Iterative.m and inverseTomo_Tikhonov.m

3. Vary the regularization parameter in each approach, and try to find the values where the to tomography image is clear and better. Make a conclusion about the values: should they be too high or too low for the individual approaches?

4. Implement the L-curve criterion to find an optimal trade off between accuracy and regularization (LCurve_TSVD.m, LCurve_Iterative and LCurve_Tikhonov.m). For this, draw for different choices of the regularization parameter the norm of the solution x over the norm of the residual. Does an L-curve occur? If no, try to plot the squared values of the norms or try a logarithmic scale.

5. Optionally: Vary the amount of noise added to the ,,measurements" b. What needs to be done for the regularization parameters if the noise level increases / decreases?