We will prove that every amount of postage greater than or equal to 12¢ can be formed using 4¢ and 5¢ stamps. Another way to express this is to say for all $n \geq 12¢, n¢ = a \cdot 4¢ + b \cdot 5¢$ for $a \geq 0$ and $b \geq 0$. Thus we will let P(n) be the proposition that $n¢ = a \cdot 4¢ + b \cdot 5¢ \wedge a \geq 0 \wedge b \geq 0$.

Base Case: $(P(12))$:

$$12¢ = 3 \cdot 4¢ + 0 \cdot 5¢$$

Thus the base case is true.

Inductive Hypothesis $(P(k) \rightarrow P(k+1))$, where P(k) is the assumption that

$\mathrm{k}¢ = \mathrm{a} \cdot 4¢ + \mathrm{b} \cdot 5¢ \wedge a \geq 0 \wedge b \geq 0$.

We assume that P(k) is true, but we will break P(k) down into 2 possibilities: either we use at least one 4¢ stamp to make k¢ of postage, or we do not use any 4¢ stamps to make k¢ of postage. We prove each case separately. First we formally define the cases:

a. $P(k)$		Assumption
b. $P(k) \wedge \mathrm{T}$	a	Identity
c. $P(k) \wedge$ (uses 4¢ stamps $\vee$ uses no 4¢ stamps)	b	Negation
d. $(P(k) \wedge$ uses 4¢ stamps) $\vee$ $(P(k) \wedge$ uses no 4¢ stamps)	c	Distribution
e. $(P(k) \wedge \mathrm{a} \geq 1) \vee (P(k) \wedge \mathrm{a} = 0)$	d	(Alternate expression using definition of P(k)

Thus we have two cases to consider. For case 1, we will explain in English, then show the proof.

Case 1: The assumption is that we can make k¢ postage using some combination of 4¢ and 5¢ stamps, and we have used at least one 4¢ stamp. We take away one 4¢ stamp, which gives us (k-3)¢ in postage, then substitute a 5¢ stamp which brings us to (k+1)¢. Thus make correct (k+1)¢ postage using only 4¢ and 5¢ stamps.

Formal proof:

a. $P(k) \land$ uses 4¢ stamps		Assumption
b. $P(k) \land (a \geq 1)$		(Equivalent expression)
c. $P(k)$	b	Simplification
d. $k¢ = a \cdot 4¢ + b \cdot 5¢ \land a \geq 0 \land b \geq 0$	c	Definition of P(k)
e. $k¢ = a \cdot 4¢ + b \cdot 5¢$	d	Simplification
f. $(k+1)¢ = a \cdot 4¢ + b \cdot 5¢ + 1¢$	e	Math (add 1¢ to both sides)
g. $(k+1)¢ = (a\text{-}1) \cdot 4¢ + b \cdot 5¢ + 4¢ + 1¢$	f	Math
h. $(k+1)¢ = (a\text{-}1) \cdot 4¢ + b \cdot 5¢ + 5¢$	g	Math
i. $(k+1)¢ = (a\text{-}1) \cdot 4¢ + (b+1) \cdot 5¢$	h	Math
Let $a' = a - 1$ and $b' = (b+1)$		
j. $a \geq 1$	b	Simplification
k. $a' \geq 0$	j	Math
l. $b \geq 0$	d	Simplification
m. $b' \geq 0$	l	Math
n. $(k+1)¢ = a' \cdot 4¢ + b' \cdot 5¢$	i	Substitution / Math
o. $(k+1)¢ = a' \cdot 4¢ + b' \cdot 5¢ \land a' \geq 0 \land b' \geq 0$	k,m,n	Conjunction
p. P(k+1)	o	Definition of P(k+1)

Case 2, we will explain in English, then show the proof. The assumption is that we can make k¢ postage using some combination of 4¢ and 5¢ stamps, but we do not use any 4¢ stamps. Since our claim is for postages of ≥12¢, there must be at least 3 x 5¢ stamps to reach a value ≥12¢, which implies that $k \geq 3 \text{ x } 5 = 15$. We remove 3 x 5¢ stamps, which gives us (k-15)¢ in postage, then substitute a 4 x 4¢ stamps which adds 16¢ in postage and brings us to (k+1)¢. Thus we can make correct (k+1)¢ postage using only 4¢ and 5¢ stamps.

To prove this case we use the following lemma:
Lemma 1: if there are no 4¢ stamps, then there are at least three 5¢ stamps. Alternately, using the definition of P(k), this can be expressed as: $(P(k) \wedge \mathrm{a}{=}0) \rightarrow (b \geq 3)$.

Lemma 1:

a. $P(k) \wedge \mathrm{a}{=}0$		Assumption
b. $\mathrm{a} = 0$	a	Simplification
c. P(k)	a	Simplification
d. $\mathrm{k}¢ = \mathrm{a} \cdot 4¢ + \mathrm{b} \cdot 5¢ \wedge a \geq 0 \wedge b \geq 0$	c	Definition of P(k)
e. $\mathrm{k}¢ = \mathrm{a} \cdot 4¢ + \mathrm{b} \cdot 5¢$	d	Simplification
f. $k¢ = a \cdot 4¢ + b \cdot 5¢ \wedge a = 0$	b,e	Conjunction
g. $k \geq 12$		Definition of P(k)
h. $k¢ = a \cdot 4¢ + b \cdot 5¢ \wedge a = 0 \wedge k \geq 12$	f,g	Conjunction
i. $k¢ = 0 \cdot 4¢ + b \cdot 5¢ \wedge k \geq 12$	h	Math
j. $k¢ = 0 \cdot 4¢ + b \cdot 5¢ \geq 12¢$	i	Math
k. $b \cdot 5¢ \geq 12¢$	j	Math
l. $b \geq \frac{12¢}{5¢}$	k	Math
m. $b \geq 3$	l	Math (since b is an integer)

We can now prove Case 2:

n. $P(k) \wedge$ a=0		Assumption
o. $P(k)$	n	Simplification
p. $k¢ = a \cdot 4¢ + b \cdot 5¢ \wedge a \geq 0 \wedge b \geq 0$	o	Definition of P(k)
q. $k¢ = a \cdot 4¢ + b \cdot 5¢$	p	Simplification
r. $(k+1)¢ = a \cdot 4¢ + b \cdot 5¢ + 1¢$	q	Math (add 1¢ to both sides)
s. $(k+1)¢ = a \cdot 4¢ + (b-3) \cdot 5¢ + 15¢ + 1¢$	r	Math
t. $(k+1)¢ = a \cdot 4¢ + (b-3) \cdot 5¢ + 16¢$	s	Math
u. $(k+1)¢ = (a+4) \cdot 4¢ + (b-3) \cdot 5¢$	t	Math
Let $a' = a + 4$ *and* $b' = (b$ - 3)		
v. $a = 0$	n	Simplification
w. $b \geq 3$	v	Lemma 1
x. $b' \geq 0$	w	Math
y. $a' \geq 0$	v	Math
z. $(k+1)¢ = a' \cdot 4¢ + b' \cdot 5¢$	u	Substitution / Math
aa. $(k+1)¢ = a' \cdot 4¢ + b' \cdot 5¢ \wedge a' \geq 0 \wedge b' \geq 0$	x,y,z	Conjunction
bb. P(k+1)	aa	Definition of P(k+1)

Thus we have shown that $P(k) \rightarrow P(k+1)$ in both cases.