Mihir Gupta
101172281

ASSINGMENT #2
COMP-1805 B
Robert Collier
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(1)

(a)

Let “M” be “You get regular maintenance on your car
Let “B” be “It will break down.”
Let “C” be “You care for your car.”

Given:

- If you get regular maintenance on your car, it won’t break down.
  $M \rightarrow \neg B$
- If you care about your car, you make sure it gets regular maintenance.
  $C \rightarrow M$
- Riley’s car broke down.
  B

To Prove:

- He did not care about his car.
  $\neg C$

1)	$M \rightarrow \neg B$	(given)
2)	$C \rightarrow M$	(given)
3)	B	(given)
4)	$C \rightarrow \neg B$	(1,2 Hypothetical syllogism)
5)	$\neg(\neg B)$	(3 Double Negation)
6)	$\neg C$	(4,5 Modus Tollens)

Hence, he did not break his car.
Hence, the argument is valid.

(b)

Let “J” be “Jordan rides his bike in winter.”
Let “R” be “Jordan ruins his bike.”
Let “S” be “It is spring.”
Let “F” be “Jordan gets his bike fixed.”

Given:

- If Jordan rides his bike in the winter, he ruins his bike.
  $J \rightarrow R$
- If Jordan ruins his bike and it is spring, he gets his bike fixed.
  $(R \wedge S) \rightarrow F$
- It is spring but Jordan does not get his bike fixed.
  $S \wedge \neg F$

To Prove:

- Jordan did not ride his bike.
  $\neg J$

(1)	$J \rightarrow R$	Given
(2)	$(R \wedge S) \rightarrow F$	Given
(3)	$S \wedge \neg F$	Given
(4)	$\neg J \vee R$	(1), Implication equivalence
(5)	S	(3), Simplification
(6)	$\neg F$	(3), Simplification
(7)	$\neg (R \wedge S)$	(6,2) Modus Tollens
(8)	$\neg R \vee \neg S$	(7) De Morgan's Law
(9)	$\neg S \vee \neg R$	(8) Commutativity
(10)	$\neg(\neg S)$	(5) Double Negation
(11)	$\neg R$	(9,10) Disjunctive syllogism
(12)	$R \vee \neg J$	(4) Commutativity
(13)	$\neg J$	(11,12) Disjunctive Syllogism

Hence, Jordan did not ride his bike.
Hence, the argument Is valid.

(c)

Let "W" be "Office floor is wet."
Let "D" be "Office floor is dirty."
Let "S" be "I wore my boots to school."
Let "R" be "It is raining outside."
Let "C" be "It is cold outside."

Given:

- If my office floor is wet, or my office floor is dirty, it is because I wore my boots to school. $(W \vee D) \rightarrow S$ (It is invalid because "because" is not a truth functional.)
- I wore my boots to school if it is raining outside or if it is cold outside.

$(R \vee C) \to S$

- It is not raining outside, but it is cold.

$\neg R \wedge C$

To Prove:

- My office floor is wet.

W

It is because 'because' is not truth functional.

That is, knowing the truth-values of P and Q does not tell you the truth-value of 'P because of Q'

For example, the two statements 'Grass is green,' and 'Snow is white' are both true, but 'Grass is green because snow is white' is an invalid argument, and hence, as a statement as to the validity of that argument, a false statement.

On the other hand, 'Grass is green because grass is green' is a true statement as to the validity of this as an argument, but yet again it involves two true statements.

This shows that with P and Q both being true, the statement 'P because of Q' can either be true or false, and hence it is not truth functional.

(2) Prove $\exists x\, E(x)$

1) $\forall x \neg B(x) \to \neg F(x)$	
2) $\forall x D(x) \vee F(x)$	
3) $\exists x D(x) \vee C(x)$	
4) $\forall x A(x) \to C(x)$	
5) $\exists x (B(x) \vee C(x)) \to E(x)$	
6) $\forall x \neg D(x) \wedge \neg A(x)$	
7) $D(\mathrm{k}) \vee C(\mathrm{k})$	(3), Existential Instantiation
8) $\neg D(\mathrm{k}) \wedge \neg A(\mathrm{k})$	(6), Universal Instantiation
9) $\neg D(\mathrm{k})$	(8), Conjunction
10) $C(\mathrm{k})$	(7,9), Disjunctive Syllogism
11) $B(\mathrm{k}) \vee C(\mathrm{k})$	(10), Addition
12) $(B(\mathrm{k}) \vee C(\mathrm{k})) \to E(\mathrm{k})$	(5), Existential Instantiation
13) $\mathrm{E}(\mathrm{k})$	(11,12), Modus Ponens
14) $\exists x\, \mathrm{E}(x)$	(13), Existential Generalization

Hence proved.

(3)

To prove: [(n is an integer) $\wedge$ (3n+3 is odd)] $\to$ n is even

For the indirect proof, we need to take contrapositive and assume the antecedent of the contrapositive. And prove the consequent of the contrapositive.

So the contrapositive of "[(n is an integer) $\wedge$ (3n+3 is odd)] $\rightarrow$ n is even" is:

$= \neg$ (n is even) $\rightarrow \neg$ [(n is an integer) $\wedge$ (3n+3 is odd)]	(by contrapositive)
$= \neg$ (n is even) $\rightarrow$ [$\neg$ (n is an integer) $\vee \neg$ (3n+3 is odd)]	(by De Morgan's law)

= n is not even (i.e. odd) $\rightarrow$ [(n is not an integer) $\vee$ (3n+3 is not odd (i.e. even))]
= n is odd $\rightarrow$ [(n is not an integer) $\vee$ (3n+3 is even)]

So, now we have to prove the consequent i.e. either n is not integer or 3n+3 is even or both while assuming n is odd.

1) n is odd	(by assumption)
2) $\exists k\ n = 2k + 1$	(by definition of odd)
3) $n = 2i + 1$	(by Existential Instantiation)
4) 3n+3	
5) 3(2i+1) +3	(3,4 by substitution)
6) 6i+3+3	(5, by math)
7) 6i + 6	(6, by math)
8) 6(i+1)	(7, by math)
9) 2(3(i+1))	(8, by math)
10) $\exists k\ 3n + 3 = 2m$	(9, by existential generalization)
11) 3n+3 is even	(10, by definition of even)
12) (3n+3 is even) $\vee$ (n is not an integer)	(11, by addition)

Hence the truth value of the contrapositive of the statement is equal to the truth value of the actual statement. Hence the statement [(n is an integer) $\wedge$ (3n+3 is odd)] $\rightarrow$ n is even is proved using indirect proof method.

(4) To prove: [$(n \in Z) \wedge (n \geq 2) \wedge (n^2$-3 is odd)] $\rightarrow$ n is even

To prove the implication statement $P \rightarrow Q$ true by contradiction, as $P \rightarrow Q$ is equal to ($\neg P \vee Q$), we would assume the negation (i.e. $\neg(\neg P \vee Q)$) and show that this is false and this would follow that if negation is false, then the original proposition must be true.

Finding the negation of the implication statement.

1) [$(n \in Z) \wedge (n \geq 2) \wedge (n^2$-3 is odd)] $\rightarrow$ n is even	(To prove)
2) $\neg$ [$(n \in Z) \wedge (n \geq 2) \wedge (n^2$-3 is odd)] $\vee$ (n is even)	(by implication equ.)
3) $\neg$ {$\neg$ [$(n \in Z) \wedge (n \geq 2) \wedge (n^2$-3 is odd)] $\vee$ (n is even)}	(Negating 1)

4) $\neg \{\neg [(n \in Z) \wedge (n \geq 2) \wedge (n^2\text{-3 is odd})]\} \wedge \neg$ (n is even) (2, De Morgan's law)
5) $[(n \in Z) \wedge (n \geq 2) \wedge (n^2\text{-3 is odd})] \wedge \neg$ (n is even) (3, Double negation)
6) $[(n \in Z) \wedge (n \geq 2) \wedge (n^2\text{-3 is odd})] \wedge$ (n is odd) (4, definition of odd)

Proof:

1) $[(n \in Z) \wedge (n \geq 2) \wedge (n^2\text{-3 is odd})] \wedge$ (n is odd)	(by assumption)
2) $[(n \in Z) \wedge (n \geq 2) \wedge (n^2\text{-3 is odd})]$	(1, by simplification)
3) (n^2-3 is odd)	(2, by simplification)
4) n is odd	(1, by simplification)
5) $\exists k\ n = 2k + 1$	(4, definition of odd)
6) $n = 2i + 1$	(5, Existential Inst.)
7) $n^2 = (2i+1)^2$	(by Math)
8) $n^2 = 4i^2 + 4i + 1$	(by math)
9) $n^2 = 2(2i^2 + 2i) + 1$	(by math)
10) $n^2 - 3 = 2(2i^2 + 2i) + 1 - 3$	(by math)
11) $n^2 - 3 = 2(2i^2 + 2i) - 2$	(by math)
12) $n^2 - 3 = 2(2i^2 + 2i - 1)$	(by math)
13) $\exists k\ n^2 - 3 = 2k$	(by existential general.)
14) $n^2 - 3$ is even	(by definition of even)
15) $n^2 - 3$ is even $\wedge$ (n^2-3 is odd)	(3,14 conjunction)
16) False	(15, by negation)

Hence Contradiction.

Therefore, the Original proposition is correct.

(5) Prove that $\sqrt{27}$ is an irrational number using a proof by contradiction.

1) $\sqrt{27}$ is a rational number.	By Assumption
2) $\sqrt{27} = c/d \wedge$ (c/d is in lowest form)	By definition of rational number
3) $\sqrt{27} = 3. \sqrt{3}$	2, by Math
4) (3 is rational) $\wedge$ ($\sqrt{3}$ should be rational)	by definition of rational number
5) $\sqrt{3} = a/b \wedge$ (a/b is in lowest form)	By definition of rational number
6) $\sqrt{3} = a/b$	(2), Simplification
7) $3 = a^2/b^2$	(3), By Squaring both sides
8) $3b^2 = a^2$	(4), Multiplying both sides by b^2
9) $3(b^2) = a^2$	(5) By Math
10) a^2 is divisible by 3.	(6) By Math
11) a is divisible by 3.	By Lemma #1
12) $\exists k\ a = 3k$	(8), Existential Generalization
13) $a = 3c$	(9), Existential Instantiation

14) $a^2 = 9c^2$ (10) Squaring both sides
15) $3(b^2) = 9c^2$ (6,11) By substitution
16) $b^2 = 3c^2$ by math
17) b^2 is divisible by 3 by math
18) b is divisible by 3 by Lemma #1
19) a and b both divisible by 3 (11,18) conjunction
20) a/b is not in its lowest form by definition
21) a/b is in lowest form (5), by simplification
22) (a/b is in lowest form) ∧ (a/b is not in lowest form) by conjunction

(If we let p = a/b is in lowest form, this becomes $p \wedge \neg p$)

23) FALSE (by negation)

The assumption that $\sqrt{3}$ is a rational number has led to a contradiction the conclusion? the assumption that $\sqrt{3}$ is a rational number must be incorrect ∴ $\sqrt{3}$ is an irrational number.

24) $\sqrt{3}$ is an irrational (proved above)
25) $\sqrt{3}$ is rational number (4) Simplification
26) ($\sqrt{3}$ is a rational) ∧ ($\sqrt{3}$ is an irrational number) (by conjunction)
27) ($\sqrt{3}$ is a rational) ∧ ¬ ($\sqrt{3}$ is a rational number)
28) False (by negation)

Hence contradiction

Hence, $\sqrt{27}$ is irrational number, as multiplication of rational and irrational number is an irrational number.

Hence proved.

Proof of Lemma #1:

Prove that if a^2 is divisible by 3 → a is also divisible by 3.

We will use an indirect proof here with contrapositive (a is not divisible 3) → (a^2 is divisible by 3).

#Finding Contrapositive:

1) ¬ (a is divisible by 3) → ¬ (a^2 is divisible by 3)
(by definition of contrapositive)
2) (a is not divisible by 3) → (a^2 is not divisible by 3) (by negation)

Proving Lemma #1 here:

There will be two cases to prove that a is not divisible by 3. The two cases are: a = 3k +1 and a = 3k+2

 # if a = 3k+1

1) a is not divisible by 3 (by assumption)
2) a = 3k +1 (by definition) (Given above)
3) $a^2 = (3k + 1)^2$ (by math)
4) $a^2 = (9k^2 +6x+1)$ (by math)
5) $a^2 = 3(3k^2 + 2k) +1$ (by math)

If we let $y = 3k^2 + 2k$, this becomes

6) $a^2 = 3y +1$ (by math)
7) a^2 is not divisible by 3 (by definition of factor of 3)

#if a = 3k+2

1) a is not divisible by 3
2) a = 3k +2 (Given above) (by definition)
3) $a^2 = (3k + 2)^2$ (by math)
4) $a^2 = (9k^2 +12k+4)$ (by math)
5) $a^2 = (9k^2 +12k+3 +1)$ (by math)
6) $a^2 = 3(3k^2 +4k+1) +1$ (by math)

If we let $z = 3k^2 + 4k +1$, this becomes

7) $a^2 = 3(z) +1$ (by math)
8) a^2 is not divisible by 3 (by definition of factor of 3)

Hence, in both the cases a^2 is not divisible by 3

Hence, Lemma has been proven.

(6)

To prove: $1 + 3 + 9 + 27 + \text{----} + 3^{n-1} = (3^n - 1)/2$ (by induction)

To prove the above expression, we need to prove the following to expression.

1) P (1) (basis step)

2) $\forall k \in N\ (P(k) \rightarrow P(k+1))$ (inductive step)

Proof of base step: P (1):

1) $3^{n-1} = (3^n - 1)/2$
2) $n = 1$
3) $3^{1-1} = (3^1 - 1)/2$ (1,2 by math)
4) $3^0 = (3-1)/2$ (3, by math)
5) $1 = (2)/2$ (4, by math)
6) $1 = 1$ (5, by math)

Hence left-hand side is equal to right hand side.
Hence base step is proved.

Proof of inductive step: $\forall k \in N\ (P(k) \rightarrow P(k+1))$

1) P (k) (by assumption)
2) $1 + 3 + 9 + 27 + \text{----} + 3^{k-1} = (3^k - 1)/2$ (by definition)
3) $1 + 3 + 9 + 27 + \text{----} + 3^{k-1} + 3^{(k+1)-1} = (3^{(k+1)} - 1)/2$ (by math
4) $(3^k - 1)/2 + 3^{(k+1)-1} = (3^{(k+1)} - 1)/2$ (2,3 substitution)
5) $(3^k - 1)/2 + 3^k = (3^{(k+1)} - 1)/2$ (4 by math)
6) $(3^k - 1)/2 + 2(3^k)/2 = (3^{(k+1)} - 1)/2$ (multiplying by 2/2)
7) $½[(3^k - 1) + 2(3^k)] = (3^{(k+1)} - 1)/2$ (6, by math)
8) $½[3(3^k) - 1] = (3^{(k+1)} - 1)/2$ (7, by math)
9) $½[(3^{k+1}) - 1] = (3^{(k+1)} - 1)/2$ (8 by math)
10) $(3^{(k+1)} - 1)/2 = (3^{(k+1)} - 1)/2$ (9 by math)

Hence left-hand side is equal to right hand side.
Hence inductive step is proved.

Hence by induction $1 + 3 + 9 + 27 + \text{----} + 3^{n-1} = (3^n - 1)/2$ has been proved equal.

(7)

We will prove that every amount of product greater than or equal to \$2 can be formed using \$2 and \$3. Another way to express this is to say for all $n \geq \$2$, $\$n = a \cdot \$2 + b \cdot \$3$ for $a \geq 0$ and $b \geq 0$. Thus, we will let P(n) be the proposition that $\$n = a \cdot \$2 + b \cdot \$3 \wedge a \geq 0 \wedge b \geq 0$.

Base Case: (P (2)):

$$\$2 = 1.\ \$2 + 0.\ \$3$$

Thus, the base case is true.

Inductive Hypothesis ($P(k) \rightarrow P\ (k+1)$), where P(k) is the assumption that \$c = $a \cdot$ \$2 + $b \cdot$ \$3 $\wedge\ a \geq 0 \wedge b \geq 0$. We assume that P(k) is true, but we will break P(k) down into 2 possibilities: either we use at least one \$3 coin to make \$k of product, or we do not use any \$3 coins to make \$k of product. We prove each case separately. First, we formally define the cases:

1) P(k) (by assumption)
2) P(k) $\wedge$ T (1, Identity)
3) P(k) $\wedge$ (uses \$3 coin ∨ no \$3 coin) (2, Negation)
4) (P(k) $\wedge$ uses \$3 coin) ∨(P(k) $\wedge$ no \$3 coin) (3, Distribution)
5) ($P(k) \wedge$ b ≥ 1) ∨ ($P(k) \wedge$ b = 0) (4, alternate expression using definition)

Thus, we have two cases to consider. For case 1, we will explain in English then show the proof:

Case1: The assumption is that if we used \$2 coins to make \$k then, we take away one \$2 coin, which gives us \$(k-1) and then substitute by \$3 that brings us \$(k+1).

1) P(k) ∧ uses \$2 coins (Assumption)
2) P(k) $\wedge$ (a ≥ 1) (Equivalent expression)
3) P(k) 2, simplification
4) \$k = $a \cdot$ \$2 + $b \cdot$ \$3 ∧ $a \geq 0 \wedge b \geq 0$ Definition of P(k)
5) \$k = $a \cdot$ \$2 + $b \cdot$ \$3 4, simplification
6) \$(k+1) = $a \cdot$ \$2 + $b \cdot$ \$3 +\$1 by math
7) \$(k+1) = (a-1) · \$2 + $b \cdot$ \$3 +\$1 + \$2 by math
8) \$(k+1) = (a-1) · \$2 + $b \cdot$ \$3 +\$3 by math
9) \$(k+1) = (a-1) · \$2 + (b+1) · \$3 by math
   Let $a' = a - 1$ *and* $b' = (b + 1)$
10) $a \geq 1$ 2, simplification
11) a'≥ 0 by math
12) b≥0 4, simplification
13) b'≥0 by math
14) \$(k+1) = a' · \$2 + b' · \$3 by math
15) \$(k+1) = a' · \$2 + b' · \$3 ∧ $a' \geq 0 \wedge b' \geq 0$ 14,11,12 conjunction
16) P(k+1) definition of P(k+1)

Case2: We will explain in English, then show the proof. The assumption is that we make \$k by using some combination of \$2 and one \$3 coin. If we substitute \$3 by two \$2 coins that brings us \$(k+1).

1) P(k) ⋀ uses \$2 coins ⋀ \$3 coin	(Assumption)
2) P(k) ⋀ (b ≥ 1) ⋀ (a ≥ 0)	(equivalent expression)
3) P(k)	simplification
4) \$k = $a \cdot \$2 + b \cdot \$3 \wedge a \geq 0 \wedge b \geq 0$	definition of P(k)
5) \$k = $a \cdot \$2 + b \cdot \$3$	4, simplification
6) \$(k+1) = $a \cdot \$2 + b \cdot \$3 + \$1$	by math
7) \$(k+1) = $(a) \cdot \$2 + (b\text{-}1) \cdot \$3 + \$1 + \$3$	by math
8) \$(k+1) = $(a) \cdot \$2 + (b\text{-}1) \cdot \$3 + \$4$	by math
9) \$(k+1) = $(a+2) \cdot \$2 + (b\text{-}1) \cdot \$3$	by math
Let $a' = a + 2$ *and* $b' = (b - 1)$	
10) b≥1	2, by simplification
11) b'≥0	by math
12) a ≥ 0	2, simplification
13) a'≥ 0	by math
14) \$(k+1) = $a' \cdot \$2 + b' \cdot \$3$	by math
15) \$(k+1) = $a' \cdot \$2 + b' \cdot \$3 \wedge a' \geq 0 \wedge b' \geq$	14,11,13 conjunction
16) P(k+1)	definition of P(k+1)

Thus, we have proved P(k) → P(k+1) in both the cases.

Hence, proved by induction