Please ensure that you include your full name and student number *on your submission.*

Your submission ***must be saved as a "pdf"*** *document and* ***have the filename "lastname_studentid_a2.pdf"*** *(using your last name and student number)*

Your submission ***must have been created using Microsoft Word, Google Docs, or LaTeX****.*

Do NOT compress your submissions *into a "zip" archive.*

Late assignments will not be accepted *and will* ***receive a mark of 0****.*

Submissions ***written by hand****,* ***compressed into an archive*** *or* ***submitted in the wrong format*** *(i.e., not "pdf") will also be penalized and* ***may receive a mark of 0****.*

The due date for this assignment is Saturday, February 15, 2020, by 11:00pm.

1. Determine whether the following arguments are valid or invalid. If it is valid, then you must state the rules of inference used to prove validity, but if it is not then you must provide an example to outline precisely why it is invalid. (n.b., You do not need quantified predicate logic expressions to complete this problem.) (12 marks

   a. If you get regular maintenance on your car, it won't break down. If you care about your car, you make sure it gets regular maintenance. Riley's car broke down. Therefore, he did not care about his car.

Let M be "get regular maintenance on your car"
Let B be "the car breaks down"
Let C be "caring about the car"

1. $M \rightarrow \neg B$		
2. $C \rightarrow M$		
3. B		
4. $\neg M$	1,3	Modus Tollens
$\therefore \neg C$	2,4	Modus Tollens

1. $M \rightarrow \neg B$		
2. $C \rightarrow M$		
3. B		
4. $C \rightarrow \neg B$	1,2	Hypothetical Syllogism
5. $\neg\neg B$	3	Double Negation
$\therefore \neg C$	4,5	Modus Tollens

a. If Jordan rides his bike in the winter, he ruins his bike. If Jordan ruins his bike and it is spring, he gets his bike fixed. It is spring but Jordan does not get his bike fixed. Therefore, Jordan did not ride his bike in the winter.

Let W be "ride bike in winter"
Let R be "ruins bike"
Let S be "it is spring"
Let F be "get the bike fixed"

1. $W \rightarrow R$		
2. $(R \wedge S) \rightarrow F$		
3. $S \wedge \neg F$		
4. $\neg F$	3	Simplification
5. $\neg(R \wedge S)$	2,5	Modus Tollens
6. $\neg R \vee \neg S$	5	DeMorgan's
7. S	3	Simplification
8. $\neg R$	6, 7	Disjunctive Syllogism
$\therefore \neg W$	1,8	Modus Tollens

b. If my office floor is wet, or my office floor is dirty, it is because I wore my boots to school. I wore my boots to school if it is raining outside or if it is cold outside. It is not raining outside, but it is cold. Therefore, my office floor is wet.

Let W be "my office floor is wet"

Let D be "my office floor is dirty"

Let B be "I wore my boots to school"

Let R be "it is raining outside"

Let C be "it is cold outside"

1. $B \rightarrow (W \vee D)$
2. $(R \vee C) \rightarrow B$
3. $\neg R \wedge C$
4. $\therefore W$

Consider the case that I wore my boots to school and my floor is dirty, and it is cold outside but it is not raining or wet. Then we can assign the following values for each proposition:

a) B is True
b) W is False
c) D is True
d) C is True
e) R is False

Then we have for premise 1:	$B \rightarrow (W \vee D)$ $T \rightarrow F \vee T$ $T \rightarrow T$ T

For premise 2:	$(R \vee C) \rightarrow B$ $F \vee T \rightarrow T$ $T \rightarrow T$ T

For premise 3:	$\neg R \wedge C$ $\neg F \wedge T$ $T \wedge T$ T

All the premises are true, yet the conclusion, 4, is False. Therefore this is not a valid argument.

1. Starting from the six numbered premises below (which are assumed to be true) and using only the rules of inference (including the instantiation and generalization rules) and the logical equivalences (as both were presented in class), show that ∃x E(x). Make sure that you include both the rule and the line number(s) to which that rule is applied. (6 marks)

   a. $\forall x \neg B(x) \rightarrow \neg F(x)$
   b. $\forall x D(x) \vee F(x)$
   c. $\exists x D(x) \vee C(x)$
   d. $\forall x A(x) \rightarrow C(x)$
   e. $\exists x \big(B(x) \vee C(x)\big) \rightarrow E(x)$
   f. $\forall x \neg D(x) \wedge \neg A(x)$

g. $\big(B(y) \vee C(y)\big) \rightarrow E(y)$	e	Existential Instantiation
h. $\neg D(y) \wedge \neg A(y)$	f	Universal Instantiation
i. $\neg D(y)$	h	Simplification
j. $D(y) \vee F(y)$	b	Universal Instantiation
k. $F(y)$	i, j	Disjunctive Syllogism
l. $\neg B(y) \rightarrow \neg F(y)$	a	Universal Instantiation
m. $B(y)$	k, l	Modus Tollens
n. $B(y) \vee C(y)$	m	Addition
o. $E(y)$	g, n	Modus Tollens
p. ∃x E(x)	o	Existential Generalization

2. Prove, by indirect proof, that if n is an integer and 3n + 3 is odd, then n is even. Show all your work. (4 marks)

Prove: If n is an integer and 3n+3 is odd then n is even.

$(n \text{ is an integer} \wedge (3n+3) \text{ is odd}) \rightarrow n \text{ is even}$

$\neg n \text{ is even} \rightarrow \neg(n \text{ is an integer} \wedge (3n+3) \text{ is odd})$	by Contraposition
$\neg n \text{ is even} \rightarrow (\neg n \text{ is an integer} \vee \neg(3n+3) \text{ is odd})$	by DeMorgan's
$n \text{ is odd} \rightarrow (\neg n \text{ is an integer}) \vee (3n+3) \text{ is even}$	by Definition of Odd/Even

a. n is odd		Assumption
b. $n = 2k + 1$	a	Definition of Odd
c. $3n + 3 = 3(2k+1) + 3$	b	Math
d. $3n + 3 = 6k + 3 + 3$		Math
e. $3n + 3 = 6k + 6$		Math
Let $k' = 3k + 3$		
f. $3n + 3 = 2k'$	c	Math
g. $(3n+3)$ is even	d	Definition of even
h. $(3n+3)$ is even $\vee$ $(\neg n$ is an integer$)$	e	Addition

3. Prove, using a proof by contradiction, that if n is an integer and $n^2 - 3$ is an odd number for $n \geq 2$, then n is an even number. Show all your work. (4 marks)

Prove: $\forall n \geq 2, n \;\; is\ an\ integer \;\wedge (n\text{^}2 - 3) \;\; is\ odd \rightarrow (n \;\; is\ even)$

$$\neg(n \;\; is\ an\ integer \;\wedge (n^2 - 3)\ is\ odd \rightarrow (n \;\; is\ even))$$

$\neg(\neg(n \;\; is\ an\ integer \;\wedge (n^2-3)\ is\ odd) \vee (n \;\; is\ even))$	by Implication Equivalence
$\neg((\neg n \;\; is\ an\ integer \vee \neg(n^2-3)\ is\ odd) \vee (n \;\; is\ even))$	by DeMorgan's
$(\neg(\neg n \;\; is\ an\ integer \vee \neg(n^2-3)\ is\ odd) \wedge \neg(n \;\; is\ even))$	by DeMorgan's
$((\neg\neg n \;\; is\ an\ integer \wedge \neg\neg(n^2-3)\ is\ odd) \wedge \neg(n \;\; is\ even))$	by DeMorgan's
$((n \;\; is\ an\ integer \wedge (n^2-3)\ is\ odd) \wedge n\ is\ odd)$	by Double Negation

Contradiction, thus assume the negation:

$(n \text{ is an integer}) \wedge (n^2 - 3) \text{ is odd} \wedge (n \text{ is odd})$

a. $(n$ is an integer$) \wedge (n^2 - 3)$ is odd $\wedge$ $(n$ is odd$)$		Assumption
b. n is odd	a	Simplification
c. $n = 2k + 1$	b	Definition of odd
d. $(n^2 - 3) = (2k+1)^2 - 3$	c	Math
e. $(n^2 - 3) = 4k^2 + 4k + 1 - 3$		Math

f. $(n^2 - 3) = 4k^2 + 4k - 2$		Math
Let $k' = 2k^2 + 2k - 1$		
g. $(n^2 - 3) = 2k'$	d	Math
h. $(n^2 - 3)$ *is even*	e	Definition of even
i. $(n^2 - 3)$ is odd	a	Simplification
j. $(n^2 - 3)$ *is even* $\wedge (n^2 - 3)$ is odd	f, g	Conjunction
k. *False*		

Therefore we have reached a contradiction, which implies that $\forall n \geq 2, n \text{ is an integer} \wedge (n^2 - 3) \text{is odd} \rightarrow (n \text{ is even})$. (Formally stating the conclusion is optional.) Note that $\forall n \geq 2$ simply keeps everything positive, but is not necessary for the proof, since the proof works for any value n.

4. The definition of a rational number is a number that can be written with the form a/b with the fraction a/b being in lowest form. Prove that $\sqrt{27}$ is an irrational number using a proof by contradiction. You MUST use the approach described in class (and on the supplemental material on cuLearn) and your solution MUST include a lemma demonstrating that if a^2 is divisible by 3 then a is divisible by 3. Hint: reduce $\sqrt{27}$ to the product of two numbers and recall that the product of two rational numbers is a rational number. (8 marks)

If we know that $\sqrt{3}$ is irrational, we can use 1 and 2 to deduce that $\sqrt{27}$ is also irrational. Thus to show $\sqrt{27}$ is irrational, it is sufficient to show that $\sqrt{3}$ is irrational. We use a proof by contradiction, and assume $\sqrt{3}$ is rational. We will call this Lemma 1.

1.	$\sqrt{3}$ is rational	by Assumption
2.	$\sqrt{3} = \frac{a}{b} \wedge \frac{a}{b}$ *in lowest form*	by Definition of Rational, 1
3.	$\sqrt{3} = \frac{a}{b}$	by Simplification, 2
4.	$3 = \frac{a^2}{b^2}$	Math
5.	$3b^2 = a^2$	
6.	$\exists x\ 3(x) = a^2$	by Existential Generalization 5
7.	a^2 *is divisible by* 3	by Definition of Divisible

Lemma 2

(Indirectly) Prove that $a^2 \text{ is divisible by } 3 \rightarrow a \text{ is divisible by } 3$

$\neg a \text{ is divisible by } 3 \rightarrow \neg a^2 \text{ is divisible by } 3$ Contraposition

$a \text{ is NOT divisible by } 3 \rightarrow a^2 \text{ is NOT divisible by } 3$ by Definition of Not Divisible

1. $a \text{ is NOT divisible by } 3$ — by Assumption
2. $(\exists k\ a = 3k + 1) \lor (\exists k\ a = 3k + 2)$ — by Definition of Not Divisible by 3

Case 1:

3. $\exists k\ a = 3k + 1$
4. $a = 3x + 1$ — by Existential Instantiation 3
5. $a^2 = (3x + 1)^2 = (9x^2 + 6x + 1) = 3(3x^2 + 2x) + 1$
6. let $y = (3x^2 + 2x)$
7. $a^2 = 3y + 1$
8. $(\exists k\ a^2 = 3k + 1)$ — by Existential Generalization 7
9. $(\exists k\ a^2 = 3k + 1) \lor (\exists k\ a^2 = 3k + 2)$ — by Addition 8

Case 2:

10. $\exists k\ a = 3k + 2$
11. $a = 3x + 2$ — by Existential Instantiation 10
12. $a^2 = (3x + 2)^2$
13. $a^2 = (9x^2 + 12x + 4) = (9x^2 + 12x + 3 + 1) = 3(3x^2 + 4x + 1) + 1$
14. let $y = (3x^2 + 4x + 1)$
15. $a^2 = 3y + 1$
16. $(\exists k\ a^2 = 3k + 1)$ — by Existential Generalization 15
17. $(\exists k\ a^2 = 3k + 1) \lor (\exists k\ a^2 = 3k + 2)$ — by Addition 16

18. $(\exists k\ a^2 = 3k + 1) \lor (\exists k\ a^2 = 3k + 2)$ — Proof by Cases
19. $a^2 \text{ is NOT divisible by } 3$ — by Definition of Not Divisible

8. $a^2 \text{ is divisible by } 3$ — (repeating 7 from before Lemma)
9. $a \text{ is divisible by } 3$ — by Lemma 2
10. $\exists k\ a = 3k$ — by Definition of Divisible by 3
11. $a = 3z$ — by Existential Instantiation
12. $3b^2 = (3z)^2$ — (by substitution into 5)
13. $3b^2 = 9z^2$
14. $b^2 = 3z^2$
15. $\exists x\ 3(x) = b^2$ — by Existential Generalization
16. $b^2 \text{ is divisible by } 3$ — by Definition of Divisible by 3
17. $b \text{ is divisible by } 3$ — by Lemma
18. $\frac{a}{b} \text{ NOT in lowest form}$ — by Definition of Lowest Form
19. $\frac{a}{b} \text{ in lowest form}$ — by Simplification 2
20. $\frac{a}{b} \text{ in lowest form} \land \frac{a}{b} \text{ NOT in lowest form}$ — by Conjunction 18, 19
21. $False$ — by Negation 20

We can now use Lemma 1 to prove the main result.

1. $\sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$ 2. If an irrational number is multiplied by an integer the product is an irrational number 3. $\sqrt{3}$ is irrational 4. $\sqrt{27}$ is irrational	Math Known fact (given) Lemma 1 1,2,3 Modus Ponens

Use a proof by induction to show that the sum $1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$. (6 marks)

Let $P(n)$ be the proposition that $1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \frac{3^n-1}{2}$

Basis Step / Base Case (n = 1):

$$1 = \frac{3^1 - 1}{2}$$

$$1 = \frac{3 - 1}{2}$$

$$1 = \frac{2}{2}$$

$$1 = 1$$

Thus the base case holds.

Inductive Hypothesis: $P(k) \rightarrow P(k+1)$

I use sigma notation below to save space. You had not learned it at the time of your assignment, so you may not have used it. The meaning is the same, but it saves space.

b. $P(k)$		Assumption
c. $1 + 3 + 9 + 27 + \cdots + 3^{k-1} = \frac{3^k-1}{2}$	a	Definition of P(k)
d. $\sum_{i=0}^{k-1} 3^i = \frac{3^k-1}{2}$		Alternate expression

e. $\sum_{i=0}^{k-1} 3^i + 3^k = \frac{3^k-1}{2} + 3^k$	d	Math (add 3^k to both sides)
f. $\sum_{i=0}^{k} 3^i = \frac{3^k-1}{2} + 3^k$		Math (move 3^k into the sigma)
g. $\sum_{i=0}^{k} 3^i = \frac{3^k-1}{2} + \frac{2\cdot 3^k}{2}$		Math (multiply by 2/2)
h. $\sum_{i=0}^{k} 3^i = \frac{3^k+2\cdot 3^k-1}{2}$		Math (common denominator)
i. $\sum_{i=0}^{k} 3^i = \frac{3\cdot 3^k-1}{2}$		Math ($1 \cdot 3^k + 2 \cdot 3^k = 3 \cdot 3^k$)
j. $\sum_{i=0}^{k} 3^i = \frac{3^{k+1}-1}{2}$		Math ($3 \cdot 3^k = 3^{k+1}$)
k. $P(k+1)$		Definition of P(k+1)
Q.E.D.		

5. Assume you are visiting another country where everything you buy requires exact change, but the country only has \$2 and \$3 coins. Use a proof by induction to show that for any product that costs \$2 or more you can always make exact change. (6 marks)

Let $P(n)$ be the proposition that you can pay for any product costing >=\$2 using \$2 and \$3 coins. Thus $P(n)$ is the proposition that $\$k = a \cdot \$2 + b \cdot \$3 \wedge a \geq 0 \wedge b \geq 0$.

Base Case: ($P(2)$): pay \$2 using a \$2 coin.

$$\$2 = 1 \cdot \$2 + 0 \cdot \$3$$

$$\$2 = \$2$$

Thus the base case is true.

Inductive Hypothesis ($P(k) \rightarrow P(k+1)$):

a. $P(k)$		Assumption
b. $P(k) \wedge \mathrm{T}$	a	Identity
c. $P(k) \wedge$ (uses \$2 coins $\vee$ uses no \$2 coins)	b	Negation

d. $(P(k) \wedge$ uses \$2 coins$) \vee$ (P(k) $\wedge$ uses no \$2 coins)	c	Distribution
e. $(P(k) \wedge a \geq 1) \vee (P(k) \wedge a = 0)$	d	(Alternate expression)

In English: We assume we can make correct change for \$k, and break it down into 2 cases: either we use at least one \$2 coin to make \$k, or we do not use any \$2 coins to make \$k.

We have two cases to consider. Case 1:

a. $P(k) \wedge$ uses \$2 coins		Assumption
b. $P(k) \wedge (a \geq 1)$		(Alternate)
c. $(\$k = a \cdot \$2 + b \cdot \$3) \wedge (a \geq 1) \wedge$ $(b \geq 0)$	b	Definition of P(k)
d. $\$k = a \cdot \$2 + b \cdot \$3$	c	Simplification
e. $\$k+\$1 = a \cdot \$2 + b \cdot \$3 + \$1$	d	Math (add \$1 to both sides)
f. $\$(k+1) = (a-1) \cdot \$2 + b \cdot \$3 + \$2 + \$1$		Math
g. $\$(k+1) = (a-1) \cdot \$2 + b \cdot \$3 + \$3$		Math
h. $\$(k+1) = (a-1) \cdot \$2 + (b+1) \cdot \$3$		Math
Let $a' = a - 1$ *and* $b' = (b+1)$		
i. $a \geq 1$	c	Simplification
j. $a' \geq 0$	i	Math
k. $b \geq 0$	c	Simplification
l. $b' \geq 0$	k	Math
m. $\$(k+1) = a' \cdot \$2 + b' \cdot \$3 \wedge a' \geq 0 \wedge b' \geq 0$	h,j,l	Conjunction
n. P(k+1)		Definition of P(k+1)

In English: The assumption is that we can make the correct change for \$k, and we have used at least one \$2 coin. We take away one \$2 coin which gives us \$k-2, then substitute a \$3 coin which brings us to \$k+1. Thus we have correct change to pay \$k+1 using only \$2 and \$3 coins.

Case 2 (note that no \$2 coins implies that there is at least 1 \$3 coin, and students were permitted to use this without proof, i.e., $a = 0 \rightarrow b \geq 1$):

a. $P(k) \wedge \mathrm{a} = 0$		Assumption
b. $P(k)$	a	Simplification
c. $(\$k = a \cdot \$2 + b \cdot \$3) \wedge (\mathrm{a} \geq 0) \wedge (\mathrm{b} \geq$		
$0)$	b	Definition of P(k)
d. $\$k = a \cdot \$2 + b \cdot \$3$		
e. $\$k+\$1 = a \cdot \$2 + b \cdot \$3 + \$1$	c	Simplification
f. $\$(k+1) = a \cdot \$2 + (b-1) \cdot \$3 + \$3 + \$1$		Math (add $1 to both
g. $\$(k+1) = a \cdot \$2 + (b-1) \cdot \$3 + \$4$		sides)
h. $\$(k+1) = a \cdot \$2 + (b-1) \cdot \$3 + 2 \cdot \$2$		Math
i. $\$(k+1) = (a+2) \cdot \$2 + (b-1) \cdot \$3$		Math
j. $\mathrm{a} = 0$		Math
k. $b \geq 1$	a	Simplification
Let $a' = a + 2$ *and* $b' = (b - 1)$	j	Lemma $(a = 0 \rightarrow b \geq 1)$
l. $\mathrm{b}' \geq 0$		
m. $a' \geq 0$	k	Math
n. $\$(k+1) = a' \cdot 2 + b' \cdot 3 \wedge a' \geq 0 \wedge b' \geq 0$	j	Math
o. P(k+1)	k	Conjunction
	h,j,l	Definition of P(k+1)

In English: The assumption is that we can make correct change for $k using only $3 coins. Since $k \geq \$2$, we must use at least one $3 coin. We take away a $3 coin and have $k-3. We substitute 2 x $2 = $4, which brings us to $k+1. Thus we have correct change to pay $k+1 using only $2 and $3 coins.

Thus we have show that $P(k) \rightarrow P(k+1)$ in both cases.

There is an alternate solution that uses strong induction. There are two base cases:

P(2): $1 \cdot \$2 + 0 \cdot \$3$

P(3): $0 \cdot \$2 + 1 \cdot \$3$

Inductive Hypothesis: $P(k-2) \rightarrow P(k), \forall k \geq 4$

The student does not need to state this, but this proof works because when $k = 4, k - 2 = 2$, and when $k = 5, k - 2 = 3$, both of which are covered by our base cases. Then the induction follows $\forall k \geq 4$.

a. $P(k-2) \wedge k \geq 4$		Assumption ($k \geq 4$ is optional, see below)
b. $P(k-2)$	a	Simplification
c. $\$(k-2) = a \cdot \$2 + b \cdot \$3 \wedge a \geq 0 \wedge b \geq 0$		Definition of P(k-2)
d. $\$(k-2) = a \cdot \$2 + b \cdot \$3$		Simplification
e. $\$(k-2) + \$2 = a \cdot \$2 + b \cdot \$3 + \$2$	c	Math (add \$2 to both sides)
f. $\$k = (a+1) \cdot \$2 + b \cdot \$3$	d	Math
Let $a' = a - 1$		
g. $a \geq 0$	c	Simplification
h. $a' \geq 0$	g	Math
i. $b \geq 0$	c	Simplification
j. $\$k = a' \cdot \$2 + b \cdot \$3$	f	Math / Substitution
k. $\$k = a' \cdot \$2 + b \cdot \$3 \wedge a' \geq 0 \wedge b \geq 0$	j,h,i	Conjunction
l. $P(k)$	k	Definition of P(k)

$k \geq 4$ is optional, but some reference should be made to how the argument is constructed, i.e., how this argument covers all values of k. This could be implicit (a well worded argument will communicate this) or explicit as above. (No marks will be deducted if this is missing in this instance).