Please ensure that you ***include your name and student number*** *on your submission.*

Your submission ***must be created using Microsoft Word, Google Docs, or LaTeX****.*

Your submission ***must be saved as a "pdf"*** *document and* ***have the filename "lastname_studentid_a4.pdf"*** *(using your last name and student number)*

Do not compress your submission *into a "zip" file.*

Late assignments will not be accepted *and will receive a mark of 0.*

Submissions ***written by hand***, ***compressed into an archive****, or submitted in the* ***wrong format*** *(i.e., not "pdf") will be penalized and* ***may receive a mark of 0.***

The due date for this assignment is March 28, 2020, by 11:00pm.

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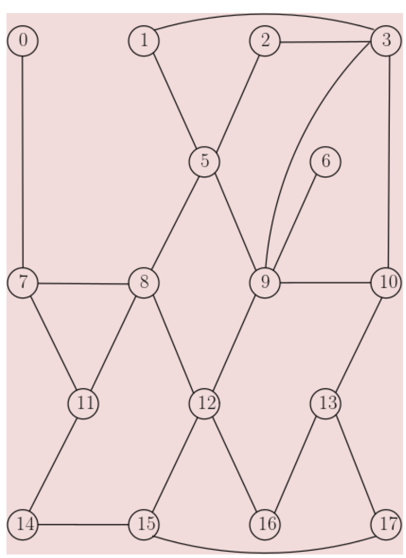
1. Consider the undirected graph on the left side of the following page and complete the exercises below. (4 marks total)

   a. If you were to remove vertex 4 from the graph (along with all the edges that connect to vertex 4) then which vertices in the resulting graph would be considered cut vertices?

This is the graph with vertex 4 removed.
The cut vertices are {7,9}



b. Provide the formal representation of a subgraph G' (of the graph G above) that is an element of the set K_n for some value of n greater than 2. Specify G' by listing the elements of both V' and E' such that G' = (V', E') and then provide an adjacency matrix representation of this graph as well.

There are many possible solutions, but all are K_3. Here is one example:

G' = (V', E')
V' = {7, 8, 11}
E' = { {7,8},{8,11},{7,11} }

	7	8	11
7	0	1	1
8	1	0	1
11	1	1	0

2. Consider the directed graph on the right side of the following page and complete the exercises below. When conducting a search, be very careful (since a small error early on can result in a large deduction of marks), and whenever you have a "choice" of which adjacent vertex to consider, you must consider the vertices in numerical order from least to greatest. (10 marks total)

a. Provide an adjacency list representation of this graph.

0: [2, 3]
1: [0,3]
2: [0, 4]
3: [5, 6, 7]

4: [2, 7]
5: [6, 7]
6: [5, 7]
7: []

b. Compute the depth-first search tree starting from vertex 0. You must provide your search tree as an adjacency list; do not "draw" your search trees.

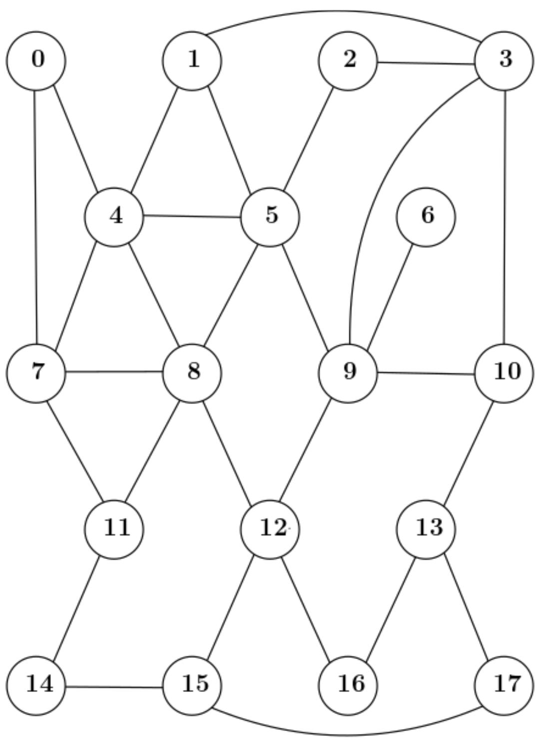
0: [2, 3]
2: [4, 6]
3: []
4: [7]
5: []
6: [5]
7: []

0: [2, 3]
2: [0, 4, 6]
3: [0]
4: [2, 7]
5: [6]
6: [2, 5]
7: [4]

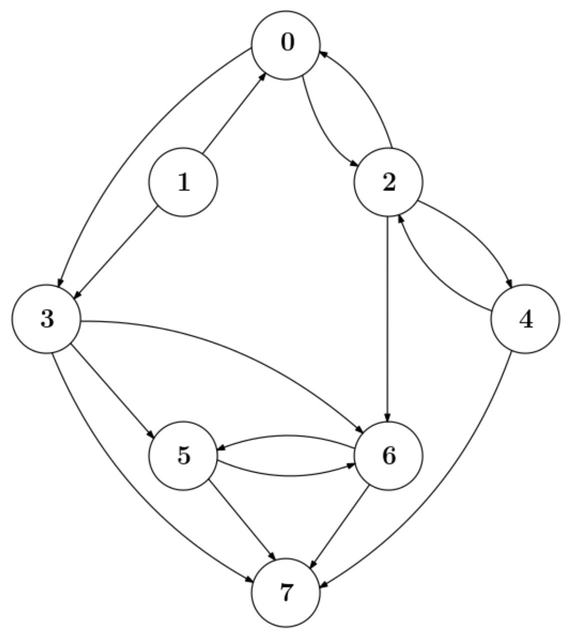
c. Compute the breadth-first search tree starting from vertex 0. You must provide your search tree as an adjacency list; do not "draw" your search trees.

0: [2, 3]
2: [4, 6]
3: [5, 7]
4: []
5: []
6: []
7: []

0: [2, 3]
2: [0, 4, 6]
3: [0, 5, 7]
4: [2]
5: [3]
6: [2]
7: [3]



Graph for Question 1



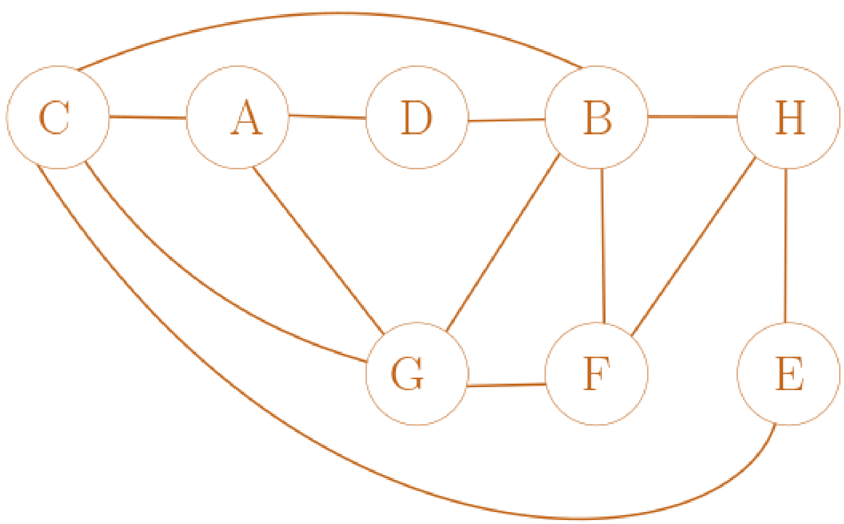
Graph for Question 2

3. Draw a planar representation of the graph corresponding to the following adjacency matrix:

(4 marks)

This is the corrected version of the adjacency matrix.

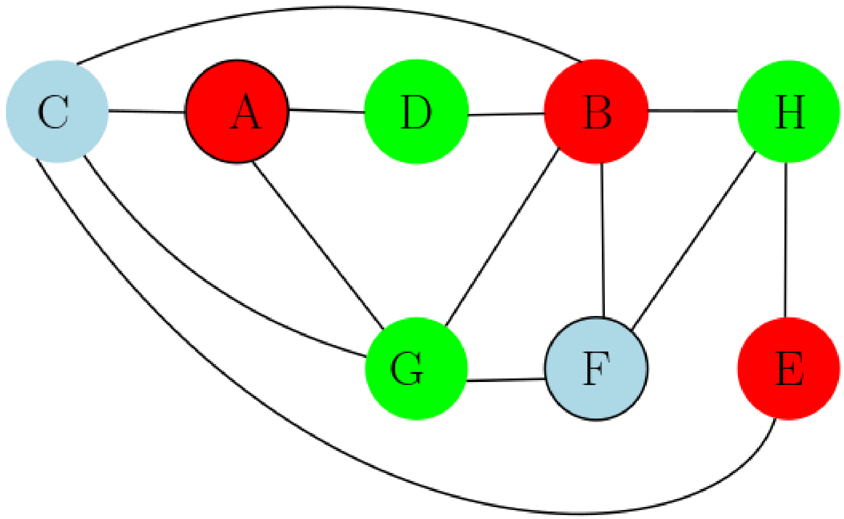
	A	B	C	D	E	F	G	H
A	0	0	1	1	0	0	1	0
B	0	0	1	1	0	1	1	1
C	1	1	0	0	1	0	1	0
D	1	1	0	0	0	0	0	0
E	0	0	1	0	0	0	0	1
F	0	1	0	0	0	0	1	1
G	1	1	1	0	0	1	0	0
H	0	1	0	0	1	1	0	0



This is one of many possible solutions

4. Find the chromatic number for the graph above by finding a valid graph colouring that uses the minimum number of colours, and then, once you have found this colouring, prove that you have used the minimum number of colours. You are not permitted to reference any theorems for this; you must find a valid colouring of n colours and then show that it is impossible to colour the graph in fewer than n colours.

(4 marks)



The subgraph of B, F, and H (and the edges which connect them) is an instance of K_3, and since any graph K_n (i.e., of the complete family, with n vertices) has chromatic number n, the minimum number of colours cannot be less than 3. (There are many other examples of K_3)

Alternatively, the subgraph of C, G, F, H, E (and the edges which connect them) is an instance of C_5, and since any graph C_n where n is an odd number (i.e., of the cycle family, with an odd number of vertices) has chromatic number 3, the minimum number of colours cannot be less than 3. (There are other examples of C_5.)

Since the graph contains a subgraph that cannot be coloured in fewer than 3 colours, and I have provided a colouring that uses exactly 3 colours, the chromatic number must be 3.

5. Consider the following function written in Python 3 (recalling that range(x, y), in Python, refers to the sequence of values starting at x, counting up by 1s, and stopping at y-1.

```
for a in range(1, n+1):
     foo()
     for b in range(1, n+1):
          bar()
     for c in range(1, a+1):
          bar()
          bar()
          for d in range(0, c):
               foo()
```

If n has a value of 2000, how many times will the function `foo()` be called and how many times will the function `bar()` be called? You must solve this problem using Sigma notation (i.e., you must derive an expression that uses Sigma notation to specify how many times each of these functions will be called, and then you must find a closed form for this expression and evaluate for n = 2000). You must show all your work.

(6 marks)

The foo function is called once inside the 'a' loop and once inside the nested 'd' loop, so we could represent the number of calls to foo using Sigma notation as:

$$\sum_{a=1}^{n}\left(1+\sum_{c=1}^{a}\left(\sum_{d=0}^{c-1}1\right)\right)$$

$$=\sum_{a=1}^{n}1+\sum_{a=1}^{n}\left(\sum_{c=1}^{a}\left(\sum_{d=0}^{c-1}1\right)\right)$$

$$=n+\sum_{a=1}^{n}\left(\sum_{c=1}^{a}(c)\right)$$

$$=n+\sum_{a=1}^{n}\left(\frac{a(a+1)}{2}\right)$$

$$=n+\sum_{a=1}^{n}\left(\frac{a^2}{2}\right)+\sum_{a=1}^{n}\left(\frac{a}{2}\right)$$

$$=n+\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}$$

$$n + \frac{2n^3 + 3n^2 + n}{12} + \frac{n^2 + n}{4}$$

$$= \frac{12n}{12} + \frac{2n^3 + 3n^2 + n}{12} + \frac{3n^2 + 3n}{12}$$

$$= \frac{2n^3 + 6n^2 + 16n}{12}$$

Optional

=1335336000

The bar function is called once inside the nested 'b' loop and twice inside the nested 'c' loop, so we could represent the number of calls to foo using Sigma notation as:

$$\sum_{a=1}^{n}\left(\sum_{b=1}^{n} 1 + \sum_{c=1}^{a} 2\right)$$

$$=\sum_{a=1}^{n}(n + 2a)$$

$$= \sum_{a=1}^{n} n + \sum_{a=1}^{n} 2a$$

$$=n^2 + n^2 + n$$

$$= 8002000$$

6. Determine whether or not the following are true and provide a full derivation explaining your answer for each. The domain of the functions of n below is the positive real numbers. For convenience, you may assume that the logs are in the base of your choice, but you should specify what base you are using in your derivation.

(2 marks each)

a. $3n^2 - 3n + 9$ is O(n)

Proof that $3n^2 - 3n + 9$ is NOT O(n) by Contradiction (thus assume it is true).

1. $\exists c\, \exists k\, \forall n \geq k\, (3n^2 - 3n + 9 \leq cn)$
2. $\exists k\, \forall n \geq k\, (3n^2 - 3n + 9 \leq an)$ by Existential Instantiation, 1
3. $\forall n \geq b\, (3n^2 - 3n + 9 \leq an)$ by Existential Instantiation, 2
4. $\forall n \geq b\, (3n^2 - 3n + 9 < 3n^2 - 3n)$ (since $9 > 0$)
5. $\forall n \geq b\, (3n^2 - 3n < an)$ (by transitivity, 3, 4)
6. $\forall n \geq b\, (3n - 3 < a)$ (divide all terms by n, 5)
7. $\forall n \geq b\, (3n < a + 3)$ (add 3 to both sides, 6)
8. $3(a + 3) < a + 3$ by Universal Instantiation, 7
9. False (since $3k \nless k$)

b. $n(3n + 4\, log\, n - 2)$ is $\Theta(n^2)$

$$n(3n + 4\,log\,n - 2) = 3n^2 + 4n\,log\,n - 2n$$

$when\ n \geq 1,\quad 3n^2 + 4n\,log\,n - 2n \leq 3n^2 + 4n\,log\,n \leq 3n^2 + 4n^2 \leq 7n^2$

$$is\ O(n^2)\ w/\ \ k = 1, c = 7$$

$when\ n \geq 1,\quad 3n^2 + 4n\,log\,n - 2n \geq 3n^2 + 4n\,log\,n - 2n^2 = n^2 + 4n\,log\,n \geq n^2$

$$is\ \Omega(n^2)\ w/\ \ k = 1, c = 1$$

c. $n^2(3\log n + 2)$ is $O(n^3)$

$$n^2(3\log n + 2) = 3n^2\,log\,n + 2n^2$$

$when\ n \geq 1,\quad 3n^2\,log\,n + 2n^2 \leq 3n^3 + 2n^3 \leq 5n^3$

$$is\ O(n^3)\ w/\ \ k = 1, c = 5$$

d. $n^3 + 5n^2 + \frac{log\, n}{n}$ is $O(n^3)$

$$when\ n \geq 1, \quad n^3 + 5n^2 + \frac{log\, n}{n} \leq n^3 + 5n^3 + n^3 \leq 7n^3$$

$$is\ O(n^3)\ \ w/\ \ k = 1, c = 7$$

e. $3n^2 - 4n - 5n\, log\, n$ is $\Omega(n^2)$

$$3n^2 - 4n - 5nlogn = 2n^2 + (n^2 - 4n - 5n\, log\, n)$$
$$\geq 2n^2$$

Is true when $(n^2 - 4n - 5n\, log\, n) \geq 0$ which is when $n \geq 20$, thus

$$is\ \Omega(n^2)\ \ w/\ \ k = 20, c = 2$$

f. $n^5 - 7n^3 + 5n^2 \log n$ is $\Omega(n^3)$

$$n^5 - 7n^3 + 5n^2 \log n \geq n^5 - 7n^3 \geq \frac{1}{2}n^5 + \left(\frac{1}{2}n^5 - 7n^3\right) \geq \frac{1}{2}n^5$$

all of which is true when n >= 1 and $\frac{1}{2}n^5 - 7n^3 \geq 0$, which is when $n \geq 4$

$$is\ \Omega(n^3)\ \ w/\ \ k = 4, c = \frac{1}{2}$$