State Variable Analysis - Introduction

The procedure of determining the state of a system is called state variable analysis.

<u>State</u> :- The state of a system at any time 't_0' is the minimum set of numbers $x_1, x_2, x_3 \ldots x_n$ which along with the input to the system for time $t \geq t_0$ is sufficient to determine the behaviour of the system for all $t \geq t_0$.

State Variable & State Vector

State Variables :- The state of a system at any time 't_0' is the minimum set of numbers $x_1, x_2, x_3 \ldots x_n$ which along with the input to the system for time $t \geq t_0$ is sufficient to determine the behaviour of the system for all $t \geq t_0$. This set of variables is called state variables.

State Vector :- If we need n variables to completely describe the behaviour of a given system, then there 'n' state variables may be considered as n component of a vector x. Such a vector is called state vector.

State Equations

State equations :-

In linear time-invariant systems, the general form of state equations are ;

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t) + D\,u(t)$$

where x = n-dimensional state vector

y = n- dimensional output vector

u = input vector (r- dimensional)

A = $n \times n$ system matrix

B = $n \times r$ control matrix

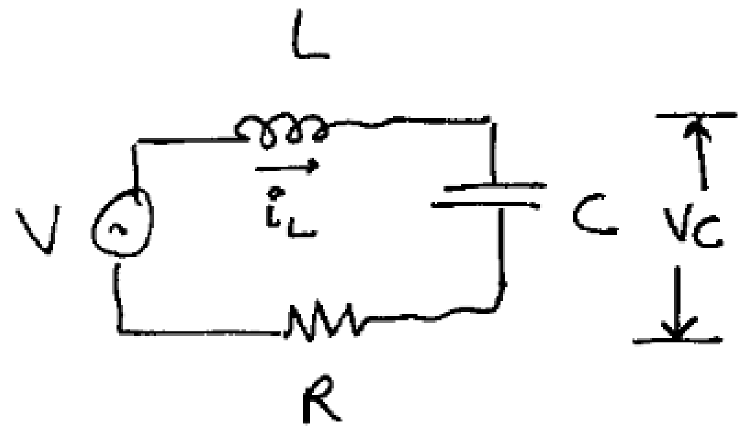
C = $n \times n$ output matrix

State Space Representation

Three Techniques

1. *For Electrical Network*
2. *Higher Order Differential Equation*
3. *From Transfer Function*

State Space Representation for Electrical Network



$$\frac{di_L}{dt} = -\frac{R}{L} i_L - \frac{V_C}{L} + \frac{V}{L} \quad -(1)$$

$$\frac{dV_C}{dt} = \frac{1}{C} i_L$$

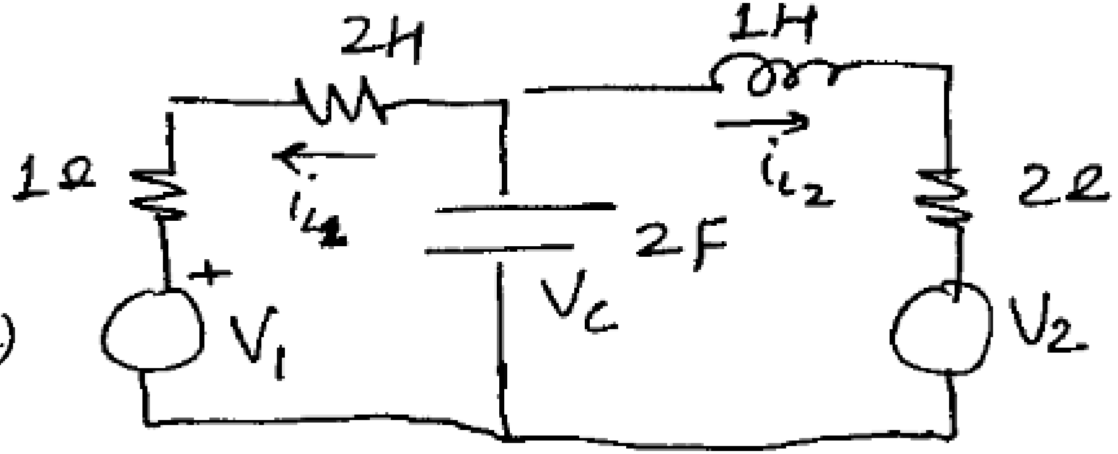
$$\begin{bmatrix} \dfrac{di_L}{dt} \\[2ex] \dfrac{dV_C}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\[2ex] \dfrac{1}{C} & 0 \end{bmatrix}}_{\text{system matrix}} \begin{bmatrix} i_L \\ V_C \end{bmatrix} + \underbrace{\begin{bmatrix} \dfrac{1}{L} \\[2ex] 0 \end{bmatrix}}_{\text{input matrix}} [V]$$

State Space Representation for Electrical Network

$$\frac{di_{L_1}}{dt} = -\frac{1}{2} i_{L_1} + \frac{1}{2} V_c - \frac{1}{2} V_1 \quad -(1)$$

$$\frac{di_{L_2}}{dt} = -2 i_{L_2} - V_2 + V_c \quad - (2)$$

$$\frac{dV_c}{dt} = \frac{1}{2} i_{L_1} + \frac{1}{2} i_{L_2} \quad - (3)$$



$$\begin{bmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -2 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_c \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

State Space Representation for Differential Equation

Ques:- A system is described by the following differential equation. Represent the system in state space.

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = u_1(t) + 3u_2 + 4u_3$$

and outputs are

$$y_1 = 4\frac{dx}{dt} + 3u_1$$

$$y_2 = \frac{d^2x}{dt^2} + 4u_2 + u_3$$

State Space Representation for Differential Equation

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = u_1(t) + 3u_2 + 4u_3$$

State Space Representation for Differential Equation

Soln:- Select state variables as

Let $x_1 = x$

$$\dot{x}_1 = \dot{x} = x_2 \quad -(1)$$

$$\dot{x}_2 = \ddot{x} = x_3 \quad -(2)$$

and $\dot{x}_3 = \dddot{x} = u_1 + 3u_2 + 4u_3 - 4x - 4\frac{dx}{dt} - 3\frac{d^2x}{dt^2}$

or $\dot{x}_3 = u_1 + 3u_2 + 4u_3 - 4x_1 - 4x_2 - 3x_3 \quad -(3)$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Ans

State Space Representation for Differential Equation

Now outputs are;

$$y_1 = 4\frac{dx}{dt} + 3u_1 = 4x_2 + 3u_1$$

$$y_2 = \frac{d^2x}{dt^2} + 4u_2 + u_3 = x_3 + 4u_2 + u_3$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

State Space Representation from Transfer Function

Ques:- For the given transfer function, obtain the state model.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K(c_2 s + c_1)}{s^3 + a_3 s^2 + a_2 s + a_1}$$

$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \cdot \frac{Y(s)}{X(s)} = \left[\frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}\right] \cdot \left[c_2 s + c_1\right]$$

State Space Representation from Transfer Function

$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \cdot \frac{Y(s)}{X(s)} = \left[\frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}\right] \cdot \left[c_2 s + c_1\right]$$

Consider, $\dfrac{X(s)}{U(s)} = \dfrac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$

or $s^3 X(s) + a_3 s^2 X(s) + a_2 s X(s) + a_1 X(s) = K U(s)$

taking inverse Laplace,

$$\frac{d^3 x(t)}{dt^3} + a_3 \frac{d^2 x(t)}{dt^2} + a_2 \frac{dx(t)}{dt} + a_1 x(t) = K u(t)$$

State Space Representation from Transfer Function

Let $x_1 = x$

then $\dot{x}_1 = \dfrac{dx}{dt} = x_2$

$$\dot{x}_2 = \frac{d^2x}{dt^2} = x_3$$

$$\dot{x}_3 = \frac{d^3x}{dt^2} = K\,u(t) - a_1 x - a_2 \frac{dx}{dt} - a_3 \frac{d^2x}{dt^2}$$

or $\dot{x}_3 = K\,u(t) - a_1 x_1 - a_2 x_2 - a_3 x_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

As

State Space Representation from Transfer Function

Now consider $\dfrac{Y(s)}{U(s)} = C_2 s + C_1$

or $Y(s) = sC_2 U(s) + C_1 U(s)$

Taking inverse Laplace;

$$y(t) = C_1 x(t) + C_2 \frac{dx(t)}{dt}$$

or $y = C_1 x_1 + C_2 x_2$

or $[y] = \begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Ans

State Model

$$\dot{x}_1 = \frac{dx}{dt} = x_2$$

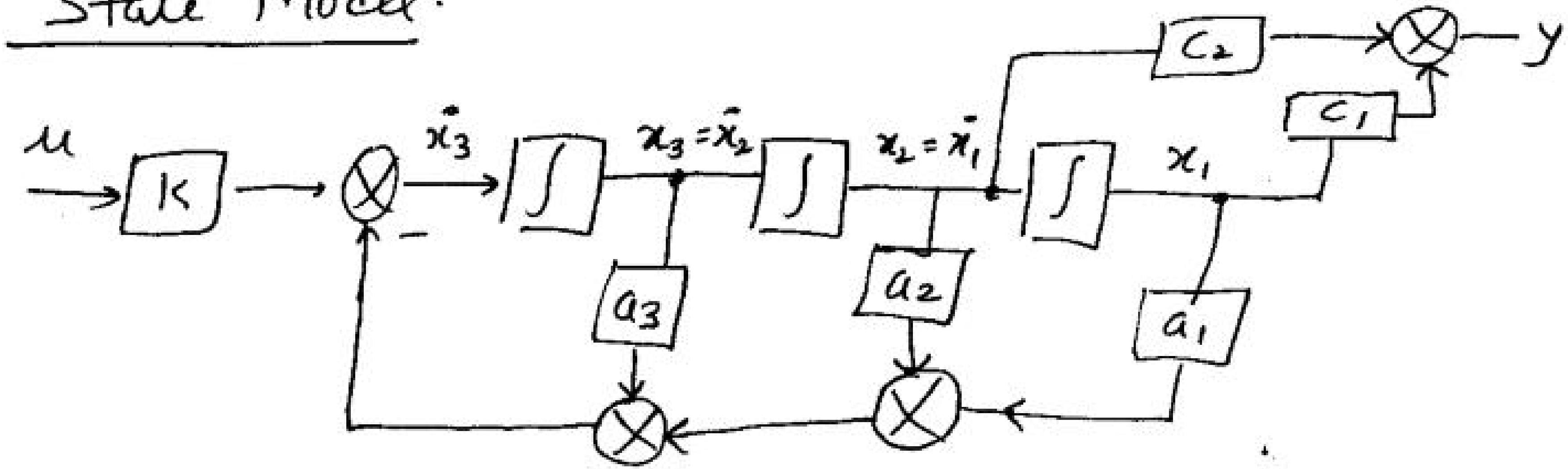
$$\dot{x}_2 = \frac{d^2x}{dt^2} = x_3$$

$$\dot{x}_3 = \frac{d^3x}{dt^2} = K\,u(t) - a_1 x - a_2 \frac{dx}{dt} - a_3 \frac{d^2x}{dt^2}$$

$$\dot{x}_3 = K\,u(t) - a_1 x_1 - a_2 x_2 - a_3 x_3$$

$$[y] = \begin{bmatrix} c_1 & c_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Model.



Solution of State Equation (Homogeneous Equation)

We know that

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$u(t) = 0$ for unforced response

then

$$\dot{x}(t) = Ax(t)$$

State Transition Matrix

Computed by Three Techniques

1. *Laplace Inverse Method*
2. *Series Summation*
3. *Modal Matrix*

State Transition Matrix (Laplace Inverse Method)

Ques:- Compute the STM when

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{Transpose of cofactor } [sI - A]}{\text{Mod of } [sI - A]}$$

$$[sI - A]^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}$$

State Transition Matrix (Laplace Inverse Method)

$$[sI - A]^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}$$

$$\phi(t) = L^{-1}[sI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{s+2}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{s+1}{(s+1)(s+2)} \end{bmatrix}$$

or $$\phi(t) = \begin{bmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$ Ans

State Transition Matrix (Series Summation Method)

$$\phi(t) = e^{At} = 1 + At + \frac{A^2t^2}{\lfloor 2} + \frac{A^3t^3}{\lfloor 3} + \cdots$$

Ques: Evaluate the STM by series summation method

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & 3 \\ 4 & -2 & 7 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 14 & -4 & 17 \\ 13 & 3 & 11 \\ 13 & 11 & 3 \end{bmatrix}$$

State Transition Matrix (Series Summation Method)

$$\phi(t) = e^{At} = I + At + \frac{A^2t^2}{\lfloor 2} + \frac{A^3t^3}{\lfloor 3} + \cdots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} t + \begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & 3 \\ 4 & -2 & 7 \end{bmatrix} \frac{t^2}{2}$$

$$+ \begin{bmatrix} 14 & -4 & 17 \\ 13 & 3 & 11 \\ 13 & 11 & 3 \end{bmatrix} \frac{t^3}{6}$$

$$= \begin{bmatrix} 1 + 2t + \frac{5t^2}{2} + \frac{14}{6}t^3 \cdots & -2t + \frac{3t^2}{2} - \frac{4t^3}{6} + \cdots & 3t + \frac{t^2}{2} + \frac{17t^3}{6} \cdots \\ t + 2t^2 + \frac{13}{6}t^3 + \cdots & 1 + t^2 + t + \frac{3t^3}{6} + \cdots & t + \frac{3t^2}{2} + \frac{11t^3}{6} + \cdots \\ t + 2t^2 + \frac{13t^3}{6} + \cdots & 3t - \frac{2t^2}{2} + 11\frac{t^3}{6} + \cdots & 1 - t + \frac{7t^2}{2} + \frac{t^3}{2} + \cdots \end{bmatrix}$$

Ans

State Transition Matrix (Modal Matrix Method)

$$\boxed{e^{At} = M e^{\Lambda t} M^{-1}}$$

where M = Modal matrix

Λ = diagonal matrix with eigen values on its main diagonal

M^{-1} = inverse of modal matrix.

Eigen Value

Ques:- Compute STM when $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$

For eigen values; $|\lambda I - A| = 0$

$$\lambda^2 + 4\lambda + 3 = 0$$

or $\lambda_1 = -1, \quad \lambda_2 = -3$

Eigen Vector

For eigen values; $|\lambda I - A| = 0$

$$\lambda^2 + 4\lambda + 3 = 0$$

or $\lambda_1 = -1, \quad \lambda_2 = -3$

Hence eigen vector;

$$P_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Modal Matrix

Modal Matrix $[M] = \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

and $[M^{-1}] = -\frac{1}{2}\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$

or $M^{-1} = \begin{bmatrix} 1.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$

Diagonal Matrix

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

STM using Modal Matrix

$$e^{At} = M e^{\Lambda t} M^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0.5e^{-t} - 0.5e^{-3t} \\ 1.5e^{-3t} - 1.5e^{-t} & 1.5e^{-3t} - 0.5e^{-t} \end{bmatrix}$$

Solution of State Equation (Non-Homogeneous Equation)

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

Find the time response of the system described by the equation

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix},\ u(t) = 1\,,\ t > 0$$

Solution of State Equation (Non-Homogeneous Equation)

Calculation of $\phi(t)$:

$$\phi(t) = \pounds^{-1}\phi(s) = \pounds^{-1}(sI - A)^{-1}$$

$$\phi(s) = [sI - A]^{-1}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \dfrac{s+2}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ 0 & \dfrac{s+1}{s^2+3s+2} \end{bmatrix}$$

$$\phi(t) = \pounds^{-1}\phi(s) = \begin{bmatrix} \pounds^{-1}\dfrac{s+2}{s^2+3s+2} & \pounds^{-1}\dfrac{1}{s^2+3s+2} \\ 0 & \pounds^{-1}\dfrac{s+1}{s^2+3s+2} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

Time Response of State Space System

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

$$x(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1\,d\tau$$

$$= \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$x_1(t) = -e^{-t} + \int_0^t e^{-(t-\tau)} - e^{-2(t-\tau)}\,d\tau$$

$$x_1(t) = \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t}$$

$$x_2(t) = \int_0^t e^{-2(t-\tau)}\,d\tau$$

$$x_2(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

Transfer Function of State Space System

The system equations are given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Find the transfer function of the system.

Transfer Function of State Space System

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \qquad \therefore \; [sI - A]^{-1} = \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ -\dfrac{2}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix}$$

$$C[sI - A]^{-1} b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ \dfrac{-2}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+3s+2}$$

Kalman's Test

A linear time invariant system described by the state equation

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is completely controllable if and only if the rank of the controllability matrix is defined as :

$$Q_c = [B \,;\, AB \,;\, A^2B \,;\, A^3B \,--\, A^{n-1}.B]$$

is equal to ~~rank~~ size of the matrix $[A]$.

Problem

Ques:. Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Test for controllability and observability.

Problem

Soln:- Here $A = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

size of matrix $A = 2 \times 2$

Thus, the size of Q_c and Q_o is 2×2.

Thus, $Q_c = [B \;;\; AB]$

$$Q_c = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

Rank of Q_c = no. of non zero rows = 1

size of matrix A = 2

$\therefore$ rank of $Q_c \neq$ size of matrix

Hence system is uncontrollable.

Also since $|Q_c| = 0$, hence system is uncontrollable.

Problem

A system characterised by the transfer function

$$\frac{Y(s)}{u(s)} = \frac{2}{s^3 + 6s^2 + 11s + 6}$$

Find the state and output equation in matrix form and also test the controllability and observability of the system.

Problem

A system characterised by the transfer function

$$\frac{Y(s)}{u(s)} = \frac{2}{s^3 + 6s^2 + 11s + 6}$$

Find the state and output equation in matrix form and also test the controllability and observability of the system.

Solution : $(s^3 + 6s^2 + 11s + 6)\ Y(s) = 2u(s).$

Taking inverse laplace

$$\dddot{Y}(t) + 6\ddot{Y}(t) + 11\dot{Y}(t) + 6Y(t) = 2u(t).$$

let

$$Y(t) = x_1$$

$$\dot{Y}(t) = \dot{x}_1 = x_2$$

$$\ddot{Y}(t) = \dot{x}_2 = x_3$$

$$\dddot{Y}(t) = \dot{x}_3 = x_4$$

$\therefore$

$$\dddot{Y}(t) = x_4 = \dot{x}_3 = 2u(t) - 6x_3 - 11x_2 - 6x_1$$

Problem

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)$$

$$Q = [b : Ab : A^2 b]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$Ab = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix}$$

$$A^2 b = \begin{bmatrix} 2 \\ -12 \\ 50 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & -12 \\ 2 & -12 & 50 \end{bmatrix}$$

$$|Q| \neq 0$$

system is controllable

Kalman's Test

A linear time invariant system described by the state equation

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ Also, the given system is observable if and only if the rank of the observability matrix is defined as:

$$Q_0 = [C^T ; A^T C^T ; (A^T)^2 \cdot C^T ; \ldots]$$

is equal to size of the matrix [A].

Problem

Ques:. Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Test for controllability and observability.

Problem

Here $A = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore Q_0 = \begin{bmatrix} C^T ; A^T \cdot C^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$

$\because$ rank of $Q_0 \neq$ size of matrix A

Hence system is unobservable.

Also since $|Q_0| = 0$, hence system is unobservable.

Problem

A system characterised by the transfer function

$$\frac{Y(s)}{u(s)} = \frac{2}{s^3 + 6s^2 + 11s + 6}$$

Find the state and output equation in matrix form and also test the controllability and observability of the system.

Problem

A system characterised by the transfer function

$$\frac{Y(s)}{u(s)} = \frac{2}{s^3 + 6s^2 + 11s + 6}$$

Find the state and output equation in matrix form and also test the controllability and observability of the system.

Solution : $(s^3 + 6s^2 + 11s + 6)\ Y(s) = 2u(s).$

Taking inverse laplace

$$\dddot{Y}(t) + 6\ddot{Y}(t) + 11\dot{Y}(t) + 6Y(t) = 2u(t).$$

let

$$Y(t) = x_1$$

$$\dot{Y}(t) = \dot{x}_1 = x_2$$

$$\ddot{Y}(t) = \dot{x}_2 = x_3$$

$$\dddot{Y}(t) = \dot{x}_3 = x_4$$

$\therefore$

$$\dddot{Y}(t) = x_4 = \dot{x}_3 = 2u(t) - 6x_3 - 11x_2 - 6x_1$$

Problem

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)$$

$$Q' = \left[C^T : A^T C^T : A^{T^2} C^T \right]$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A^{T^2} C^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of the matrix = 3, Hence the system observable.