Stability

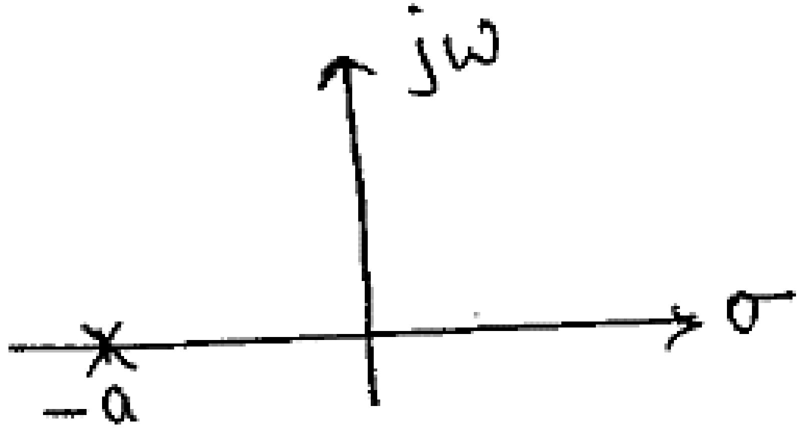
A linear time invariant (LTI) system is stable if

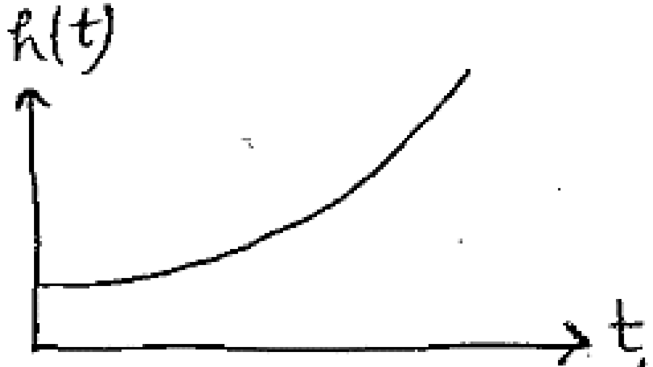
i) the system is excited by a bounded input, the output is also bounded. This is called Bounded input bounded output (BIBO) stability criteria.

ii) In the absence of the input, the output tends towards zero. This is known as asymptotic stable.

Effect of Poles Location on Stability

a) Poles on negative real axis :-



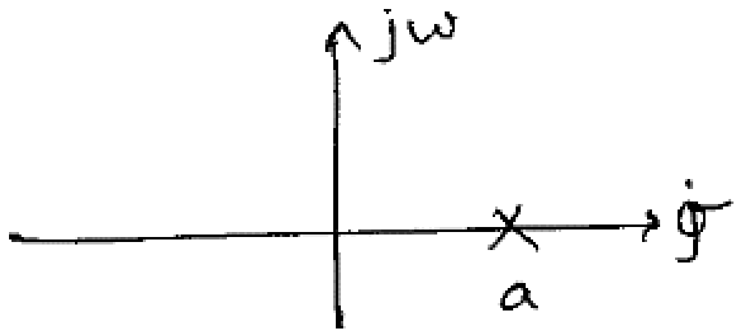


for $h(t) = e^{at} u(t)$

unstable.

Effect of Poles Location on Stability

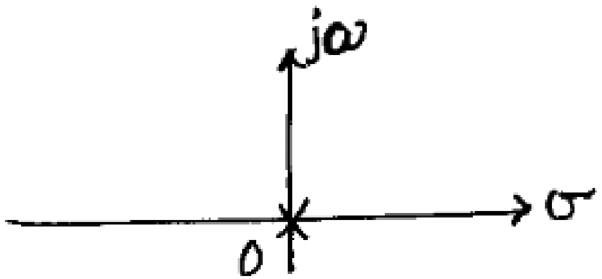
b) Poles on positive real axis :

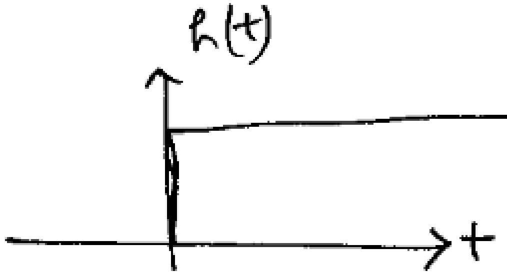


Effect of Poles Location on Stability

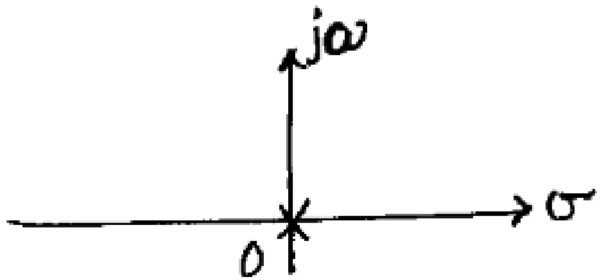
iii) Pole at origin :-

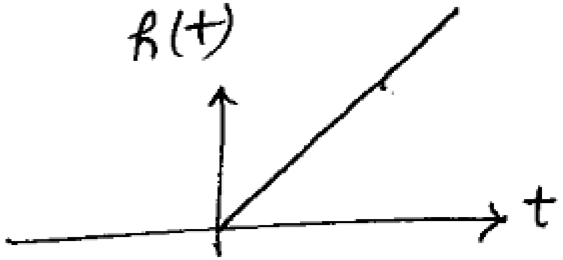
a) Single pole at origin





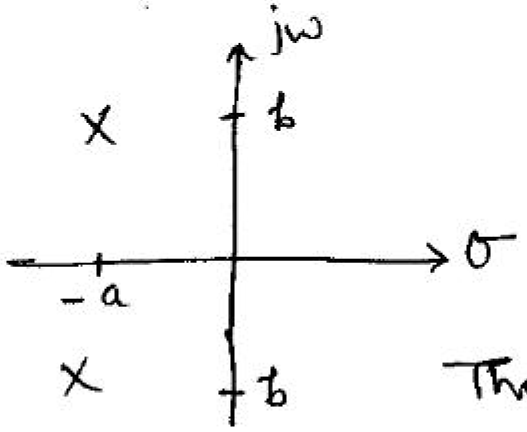
b) Double pole at origin

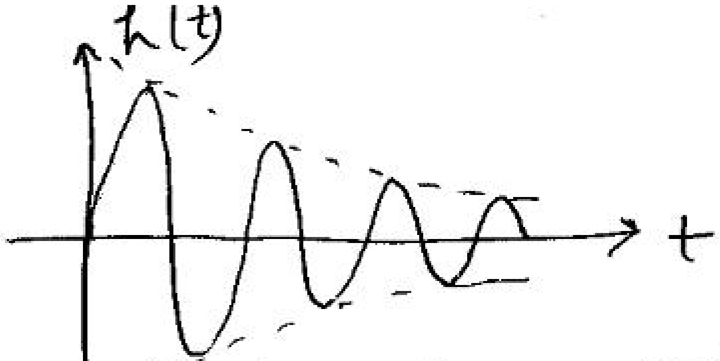




Effect of Poles Location on Stability

iv) complex pole in the left half of s-plane :-





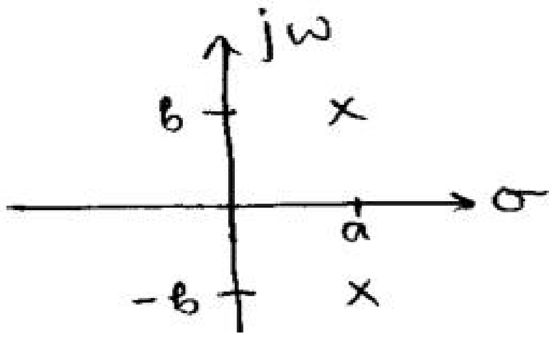
The response approaches towards zero and hence the system is stable

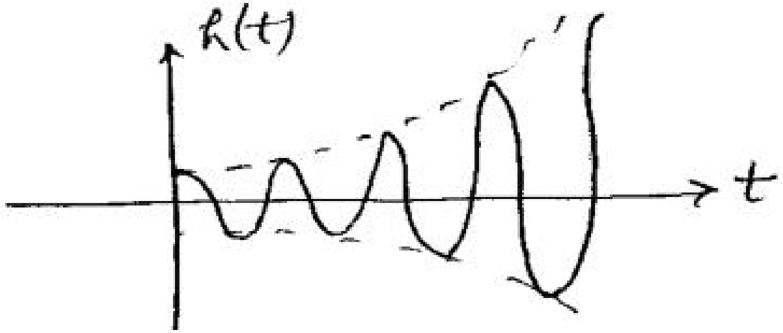
Thus, $H(s) = \frac{1}{(s+a+jb)(s+a-jb)} = \frac{1}{(s+a)^2+b^2}$

$h(t) = e^{-at} \cos bt$

Effect of Poles Location on Stability

v) complex poles in the right half of s-plane



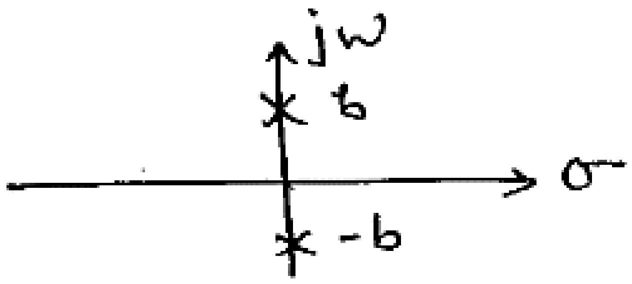


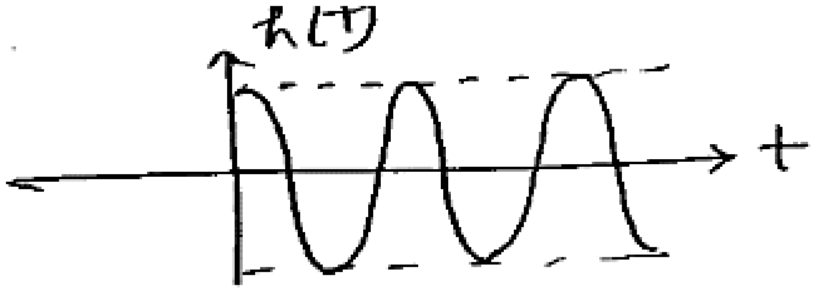
Thus, $$H(s) = \frac{1}{(s-a+jb)(s-a-jb)} = \frac{1}{(s-a)^2+b^2}$$

or $$h(t) = e^{at}\cos bt$$

Effect of Poles Location on Stability

vi) Poles on jw axis :-





Thus ; $H(s) = \frac{1}{(s^2+b^2)}$

$h(t) = \cos bt$

Necessary but not sufficient conditions for Stability

Consider a system with characteristic equation

$$a_0 s^m + a_1 s^{m-1} + a_2 s^{m-2} + \dots a_m = 0$$

i) All the coefficients of the equation should have same sign,

ii) There should be no missing term.

If any of the above two conditions are not satisfied the system will be unstable. But if all the coefficients are having same sign and there is no missing term, we have no guarantee that the system will be stable. For this, we use ROUTH-HURWITZ CRITERION.

Routh's Hurwitz Criterion

Ques:- Check the stability of the system whose characteristic equation is given by.

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0$$

s^4	1	6	1
s^3	2	4	
s^2	4	1	
s^1	3.5		
s^0	1		

Routh's Hurwitz Criterion - Statement

It states that the system is stable if and only if all the elements in the first column must have the same algebric sign. If all elements are not of the same sign then the number of sign changes of the elements in first column equals the number of roots of the characteristic equation in the right half of the s-plane.

Problem

Ques:- Investigate the stability

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	-6		
s^0	5		

No. of sign changes = 2

Hence, no. of roots on right half of s-plane = 2

Therefore, system is unstable.

Routh's Hurwitz Criterion – Special Case - I

Ques: Investigate the stability

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

Soln:-

s^5	1	2	3
s^4	1	2	5
s^3	0	−2	
s^2			
s^1			
s^0			

Special case – I

First element of the row becomes zero, then multiply the given equation by a factor $(s+1)$ and repeat the process of making routh's array.

Routh's Hurwitz Criterion – Special Case - I

Ques: Investigate the stability

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

Therefore, $(s+1)(s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5) = 0$

or $s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 8s + 5 = 0$

Routh's Hurwitz Criterion – Special Case - I

Therefore, $(s+1)(s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5) = 0$

or $s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 8s + 5 = 0$

s^6	1	3	5	5
s^5	2	4	8	
s^4	1	1	5	
s^3	2	-2		
s^2	2	5		
s^1	-7			
s^0	5			

Here, No. of sign changes in first column = 2

Hence, no. of roots on right half of s-plane = 2

Therefore, system is unstable.

Routh's Hurwitz Criterion – Special Case - II

Ques:. Investigate the stability

$$s^5 + 2s^4 + 24s^3 + 48s^2 + 25s - 50 = 0$$

Soln:

s^5	1	24	25
s^4	2	48	−50
s^3	0	0	
s^2			
s^1			
s^0			

Special case - II

All the elements of row becomes zero.

At this time, we consider the auxiliary polynomial i.e.

the polynomial whose coefficients are the element of the row just above the row of zeros in Routh array.

Routh's Hurwitz Criterion – Special Case - II

Here auxiliary polynomial $A(s) = 2s^4 + 48s^2 + 50$

$$\therefore \frac{dA(s)}{ds} = 8s^3 + 96s.$$

s^5	1	24	25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	79.3		
s^0	-50.		

Here all the sign are not positive. So the system is

Routh's Hurwitz Criterion – Special Case - II

<u>Question</u> :- Investigate the stability

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

s^6	6	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	$\boxed{0}$ 8	$\boxed{0}$ 24	$\boxed{0}$	
s^2	6	16		
s^1	2.67			
s^0	16			

$$A(s) = 2s^4 + 12s^2 + 16$$

$$\frac{dA(s)}{ds} = 8s^3 + 24s.$$

Routh's Hurwitz Criterion – Special Case - II

$A(s) = 0$

$\therefore \quad 2s^4 + 12s^2 + 16 = 0$

put $s^2 = x$;

$2x^2 + 12x + 16 = 0$

or $\quad x^2 + 6x + 8 = 0$

or $\quad (x+2)(x+4) = 0$

if $x = -2$ then $s = \pm j\sqrt{2}$

if $x = -4$ then $s = \pm 2j$

Application of Routh's Hurwitz in Control System

Ques: The open loop transfer function of unity feedback system is $\frac{K}{s(1+0.4s)(1+0.25s)}$. Find the restriction of K so that the closed loop system is absolutely stable.

i.e. $1 + \frac{K}{s(1+0.4s)(1+0.25s)} \cdot 1 = 0$

or $s^3 + 6.5s^2 + 10s + 10K = 0$

s^3	1	10
s^2	6.5	10K
s^1	$\frac{65-10K}{6.5}$	
s^0	10K	

Application of Routh's Hurwitz in Control System

s^3	1	10
s^2	6.5	$10K$
s^1	$\frac{65 - 10K}{6.5}$	
s^0	$10K$	

$\boxed{0 < K < 6.5}$ for stability

Problem-02

Ques: The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying Routh criterion, discuss the stability of the closed loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed loop system. What are the corresponding oscillation frequencies?

$$1 + \frac{K}{(s+2)(s+4)(s^2+6s+25)} = 0$$

or $s^4 + 12s^3 + 69s^2 + 198s + (200+K) = 0$

Problem-02

or $s^4 + 12s^3 + 69s^2 + 198s + (200+K) = 0$

s^4	1	69	$(200+K)$
s^3	12	198	
s^2	52.5	$200+K$	
s^1	$198 - \frac{12(200+K)}{52.5}$		
s^0	$200+K$		

$\therefore$ For stable ; $\boxed{-200 < K < 666.25}$

for unstable ; $\boxed{K > 666.25}$

for marginally stable ; $\boxed{K = 666.25}$

Oscillations occurs when $\boxed{K = 666.25}$

Problem-02

Ques: The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying Routh criterion, discuss the stability of the closed loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed loop system. What are the corresponding oscillation frequencies?

Problem-02

For oscillation frequency :

$A(s) \Rightarrow 52.5s^2 + (200+K) = 0$

or $52.5s^2 + (200+666.25) = 0$

or $s^2 = -16.5$

or $s = \pm j4.06 \mid j\omega = \pm j4.06$

or $\boxed{\omega = 4.06 \text{ rad/sec}}$

Frequency of sustained oscillation $\omega = \underline{4.06} \text{ rad/sec}$ Ans

Problem-03

Ques:. The characteristic equation of feedback control system is

$$s^4 + 20s^3 + 15s^2 + 2s + K = 0 .$$

a) Determine the range of K for the system to be stable.

b) Can the system be marginally stable? if so, find the required value of K and the frequency of sustained oscillation.

Problem-03

Soln:

s^4	1	15	K
s^3	20	2	0
s^2	14.9	K	
s^1	$\frac{29.8-20K}{14.9}$		
s^0	K		

a) For stability ; $K > 0$

and $\frac{29.8-20K}{14.9} > 0$ or $K < 1.49$

Hence for stability ; $\boxed{0 < K < 1.49}$

b) For marginally stable ; $K = 1.49$

Now $A(s) \Rightarrow 14.9s^2 + K = 0$ or $14.9s^2 = -1.49$

or $s^2 = -0.1$

or $s = \pm j\,0.316$ or $\omega = 0.316$ rad/sec.

$\therefore$ Freq. of sustained oscillation $\omega = \underline{\underline{0.316 \text{ rad/sec}}}$ Ans.

Root Locus (Root Path)

It is a graphical method in which roots of the characteristic equation are placed (plotted) in s-plane for the different values of parameter. The locus of the roots of the characteristic equation when gain is varied from zero to infinity is called root locus.

Root Locus (Root Path)

Consider a system with $G(s) = \frac{K}{s(s+2)}$, $H(s) = 1$

Then CE ; $1 + G(s)H(s) = 0$

or $1 + \frac{K}{s(s+2)} \cdot 1 = 0$

or $s^2 + 2s + K = 0$

The roots of the above eqn are ;

$p_1 = -1 + \sqrt{1-K}$ and $p_2 = -1 - \sqrt{1-K}$

As 'K' varies, the two roots give the loci in s-plane.

Problem (Design Steps)

Ques: The forward path transfer function of a unity feedback system is given by $\frac{K}{s(s+4)(s+5)}$. Sketch the root locus as k varies from zero to infinity.

Soln: Step-1. Plot the poles and zeros (i.e. pole zero plot).

poles are 0, -4, -5 and no zeros.

Step-2. Mark the path of root locus on real axis.

Problem (Design Steps)

Step-3. Calculate number of root locii

i.e. compute how many roots (poles) are terminate at ∞ or at zeros.

$$N = P - Z = 3 - 0 = 3$$

Step-4. Calculate centroid of asymtotes;

$$\sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{(0-4-5)-0}{3-0} = -3$$

where P = no. of poles, Z = no. of zeros.

Problem (Design Steps)

<u>step-5</u> Angle of asymtotes;

$$\phi = \left(\frac{2K+1}{P-Z}\right) \times 180^\circ \quad \text{where } K = 0, 1, 2, 3 \ldots$$

$K = 0$; $\phi_1 = \frac{1}{3} \times 180^\circ = 60^\circ$

$K = 1$; $\phi_2 = \frac{3}{3} \times 180^\circ = 180^\circ$

$K = 2$; $\phi_3 = \frac{5}{3} \times 180^\circ = 300^\circ$.

Problem (Design Steps)

Step-6. Calculation of breakaway point.

CE ; $1 + G(s)H(s) = 0$

or $1 + \frac{K}{s(s+4)(s+5)} \cdot 1 = 0$

or $s^3 + 9s^2 + 20s + K = 0$

or $K = -s^3 - 9s^2 - 20s$

$\frac{dK}{ds} = -3s^2 - 18s - 20 = 0$

or $3s^2 + 18s + 20 = 0$

or $s_1 = -1.47$, $s_2 = -4.52$

Problem (Design Steps)

Step-7 Calculation of intersection point

This is calculated by Routh Hurtwiz

Here, CE is $s^3 + 9s^2 + 20s + K = 0$

$$
\begin{array}{lll}
s^3 & 1 & 20 \\
s^2 & 9 & K \\
s & \frac{180-K}{9} & \\
s^0 & K &
\end{array}
$$

Problem (Design Steps)

For marginally stable; $\frac{180 - K}{9} = 0$ or $\underline{K = 180}$

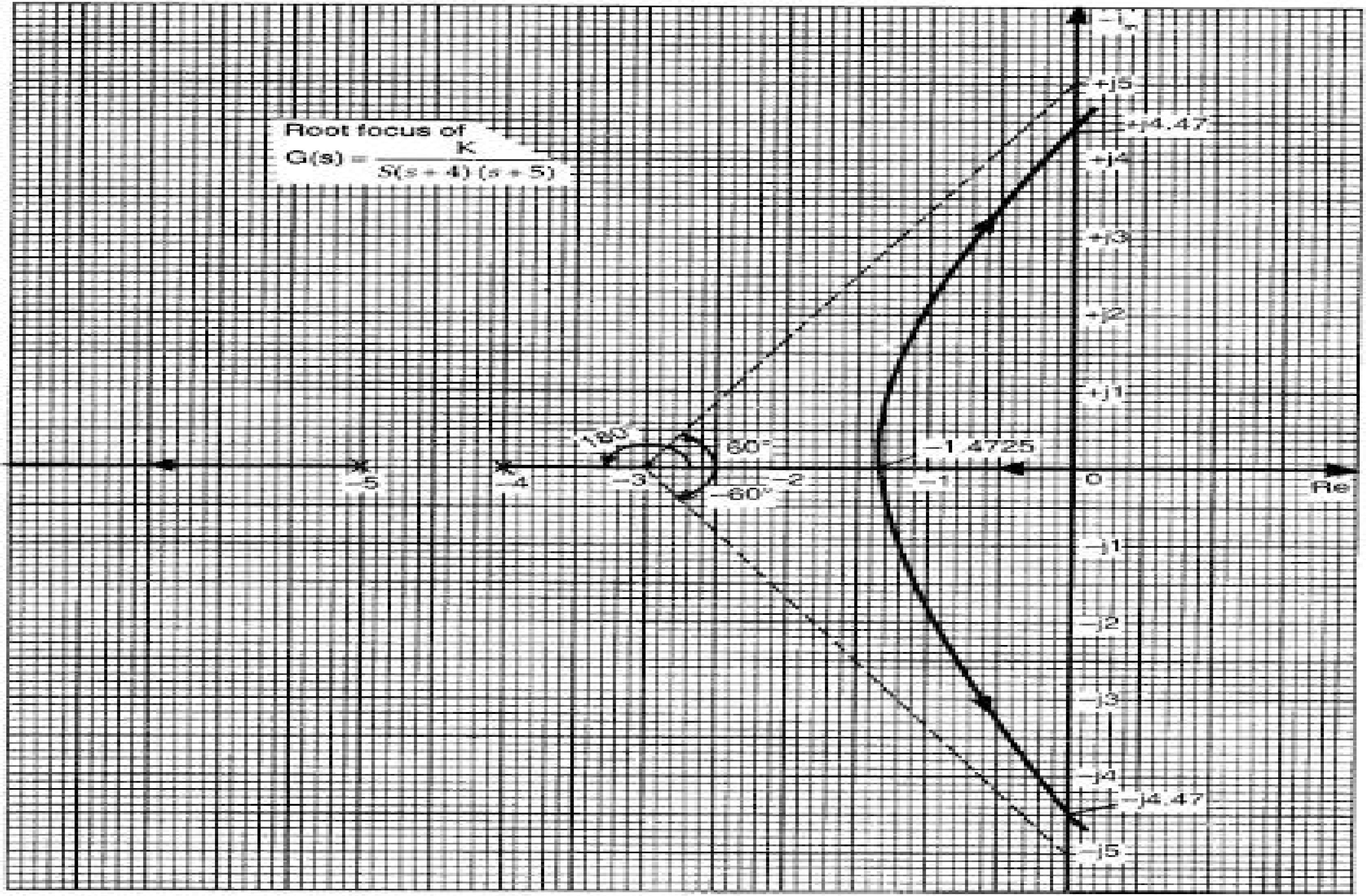
Now, auxiliary equation $A(s) = 9s^2 + K = 0$

or $9s^2 + 180 = 0$

or $s^2 = -20$

or $s = \pm j4.47$

Problem


Root locus of
$G(s) = \frac{K}{S(s+4)(s+5)}$
+j5
+j4.47
+j4
+j3
+j2
+j1
180°
60°
−60°
−1.4725
−5
−4
−3
−2
−1
0
Re
−j1
−j2
−j3
−j4
−j4.47
−j5

Root Locus (Root Path)

It is a graphical method in which roots of the characteristic equation are placed (plotted) in s-plane for the different values of parameter. The locus of the roots of the characteristic equation when gain is varied from zero to infinity is called root locus.

Problem with Complex Poles (Design Steps)

Ques:. For a unity feedback system the O.L.T.F is given by

$$G(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

a) Sketch the root locus for $0 < K \leq \infty$

b) At what value of 'K' the system becomes unstable.

c) determine the freq. of oscillation of the system.

Soln: 1) Plot the pole-zero plot

poles are $0, -2, -3+4j, -3-4j$

2) Mark the path of roots on real axis.

Problem with Complex Poles (Design Steps)

3) Calculate the number of root locii.

$$N = P - Z = 4$$

4) Calculate centroid of asymtotes.

$$\sigma_A = \frac{(0 - 2 - 3 + 4j - 3 - 4j) - 0}{4 - 0} = -2$$

5) Angle of asymitotes

$$\phi = \frac{2K+1}{P-Z} \times 180^\circ \quad \text{for } k = 0, 1, 2, 3 \ldots$$

for $k = 0$; $\phi_1 = \frac{1}{4} \times 180^\circ = 45^\circ$

for $k = 1$; $\phi_2 = \frac{3}{4} \times 180^\circ = 135^\circ$

for $k = 2$; $\phi_3 = \frac{5}{4} \times 180^\circ = 225^\circ$

for $k = 3$; $\phi_4 = \frac{7}{4} \times 180^\circ = 315^\circ$

Problem with Complex Poles (Design Steps)

(6) Breakaway Point

characteristic equation $1 + G(s)H(s) = 0$

$$1 + \frac{K}{s(s+2)(s^2+6s+25)} = 0$$

or $s^4 + 8s^3 + 37s^2 + 50s + K = 0$

or $K = -(s^4 + 8s^3 + 37s^2 + 50s)$

or $\frac{dK}{ds} = -4s^3 + 24s^2 + 74s + 50 = 0$

By hit & trial method $s = \underline{\underline{-0.89}}$ is valid breakaway point.

Problem with Complex Poles (Design Steps)

(7) Determination of intersection point.

We know CE ; $s^4 + 8s^3 + 37s^2 + 50s + K = 0$

s^4	1	37	K
s^3	8	50	
s^2	30.75	K	
s^1	$\frac{1537.5 - 8K}{30.75}$		
s^0	K		

For marginally stable ; $\frac{1537.5 - 8K}{30.75} = 0$

or $K = 192.18$

Now auxiliary equation $A(s) \Rightarrow 30.75s^2 + K = 0$

or $\boxed{s = \pm j2.5}$

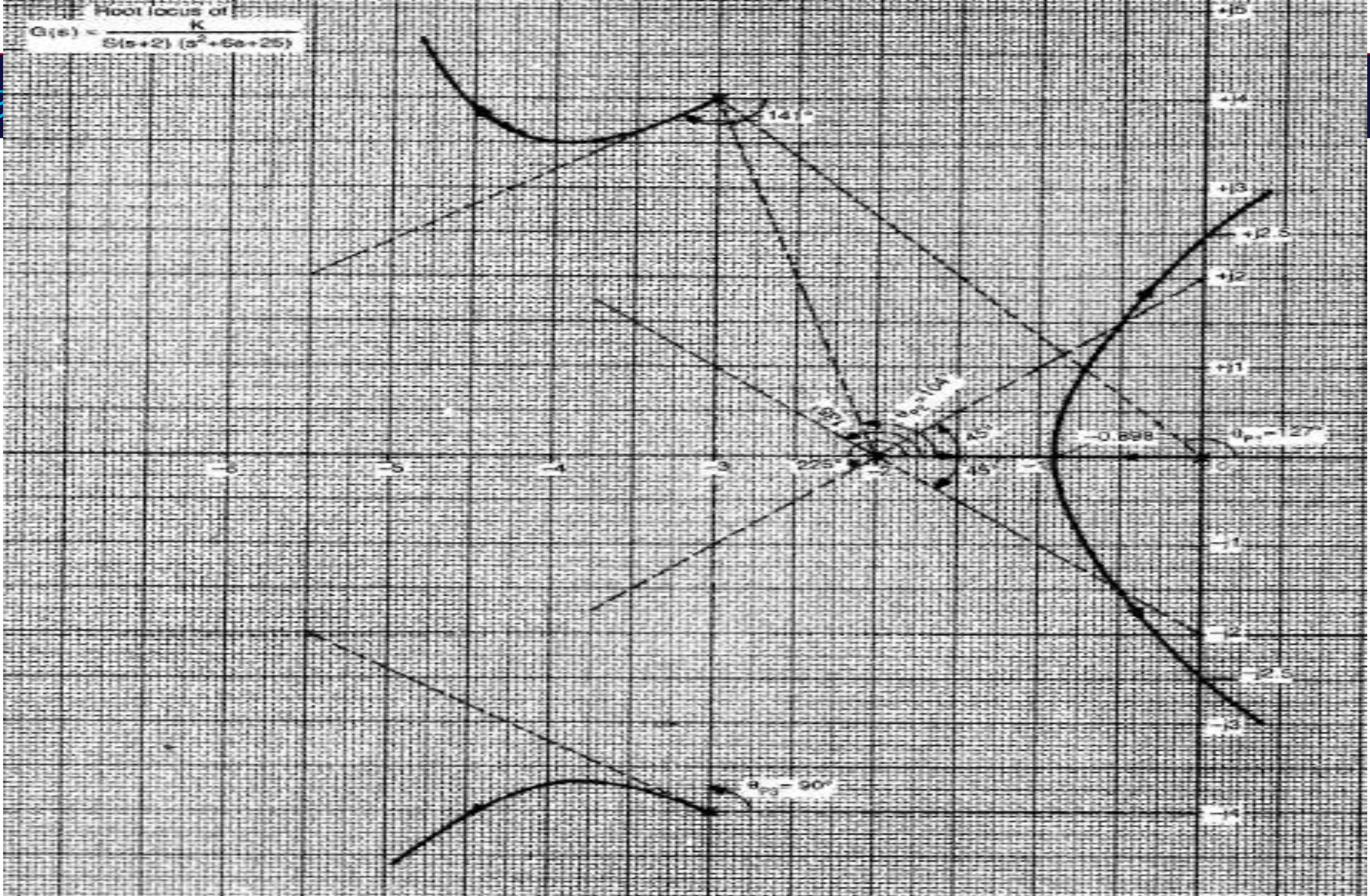
Problem with Complex Poles (Design Steps)

8) Calculation of angle of departure for complex poles.

$\phi_d = 180^\circ -$ (sum of angles of vectors drawn to this pole from other poles) + (sum of angles of vectors drawn to this pole from other zeros).

or $\phi_d = 180^\circ - (104^\circ + 90^\circ + 127^\circ) = -141^\circ$

Proble



Root locus of
G(s) = K / S(s+2) (s²+6s+25)
141°
+j4
+j3
+j2.5
+j2
+j1
−0.898
225°
45°
−6
−5
−4
−3
−2
−1
−j1
−j2
−j2.5
−j3
−j4
θp3 = 90°

Problem with Complex Poles (Design Steps)

Ques:. For a unity feedback system the O.L.T.F is given by

$$G(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

a) sketch the root locus for $0 \le K \le \infty$

b) At what value of 'K' the system becomes unstable.

c) determine the freq. of oscillation of the system.

Soln: 1) Plot the pole-zero plot

poles are $0, -2, -3+4j, -3-4j$

2) Mark the path of roots on real axis.

Problem with Complex Poles (Design Steps)

(7) Determination of intersection point.

We know CE ; $s^4 + 8s^3 + 37s^2 + 50s + K = 0$

s^4	1	37	K
s^3	8	50	
s^2	30.75	K	
s^1	$\frac{1537.5 - 8K}{30.75}$		
s^0	K		

For marginally stable ; $\frac{1537.5 - 8K}{30.75} = 0$

or $K = 192.18$

Now auxiliary equation $A(s) \Rightarrow 30.75s^2 + K = 0$

or $\boxed{s = \pm j2.5}$

Problem with Complex Poles (Design Steps)

(b) range of stability $0 < K < 192.18$

for unstable ; $K < 0$ & $K > 192.18$ Ans

(c) Freq of oscillation ;

For marginally stable ; $K = 192.18$

Thus, $A(s) \Rightarrow 30.75s^2 + K = 0$

or $s = \pm j2.5$

or $\omega = \underline{\underline{2.5}}$ rad/sec Ans