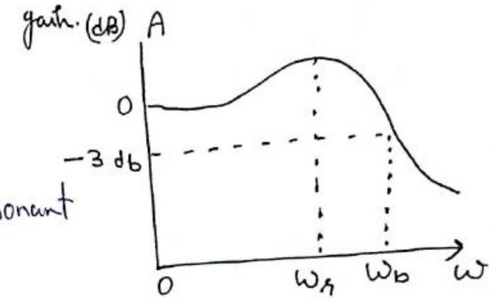
Frequency Domain Analysis: Mr (resonant peak) and ωr (resonant frequency)

FREQUENCY DOMAIN SPECIFICATIONS



a) Resonant Peak (M_r) :-
The maximum value of magnitude is known as resonant peak.

b) Resonant Frequency (ω_r) :-
The frequency at which magnitude has maximum value is known as resonant frequency (ω_r).

c) Bandwidth (ω_b) :- It is defined as the range of frequencies in which the magnitude of system (response) is higher than 70% of its maximum value. (zero frequency value).

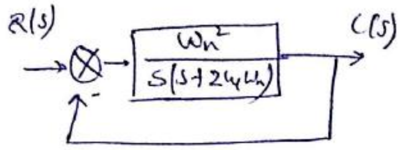
OR

It is defined as the range of frequencies over which the magnitude does not drop below −3 dB of the maximum value of gain.

d) Cut-off frequency :- The frequency at which the magnitude is 3 db below its zero frequency value is called cut-off frequency.

CORRELATION BETWEEN TIME & FREQUENCY RESPONSE

Consider the second order system



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Put $s = j\omega$.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j\, 2\zeta\omega_n \omega}$$

$$= \frac{\omega_n^2}{\omega_n^2\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\, 2\zeta \frac{\omega}{\omega_n}\right]}$$

or $$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1-u^2) + j\, 2\zeta u.} \quad \left[\begin{array}{l}\text{Let} \\ u = \frac{\omega}{\omega_n} \\ \text{normalized freq.}\end{array}\right]$$

Magnitude $$M = \left|\frac{C(j\omega)}{R(j\omega)}\right| = \frac{1}{\sqrt{(1-u^2)^2 + 4\zeta^2 u^2}} \quad -(1)$$

and $$\theta = \angle \frac{C(j\omega)}{R(j\omega)} = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$$

Expression for resonant frequency :⇒

For resonant frequency

$$\left.\frac{dM}{du}\right|_{u=u_r} = 0$$

or $$-\frac{1}{2}\left[(1-u^2)^2 + 4\zeta^2 u^2\right]^{-3/2}\left[2(1-u^2)(-2u) + 8\zeta^2 u\right] = 0$$

or $\quad -4u + 4u^3 + 8u\zeta^2 = 0$

or $\quad 4u\left(u^2 - 1 + 2\zeta^2\right) = 0$

or $\quad u = \sqrt{1 - 2\zeta^2}$

or $\quad \boxed{u_r = \sqrt{1 - 2\zeta^2}}$ normalised resonant frequency.

or $\quad \dfrac{\omega_r}{\omega_n} = \sqrt{1 - 2\zeta^2}$

or $\quad \boxed{\omega_r = \omega_n\sqrt{1 - 2\zeta^2}}$

This is the expression for resonant frequency.

Expression for Resonant Peak

For maximum value of M put $u = u_r = \sqrt{1 - 2\zeta^2}$ in eqⁿ (1),

$$\therefore \quad M_r = \frac{1}{\sqrt{\left\{1 - \left(1 - 2\zeta^2\right)\right\}^2 + 4\zeta^2\left(1 - 2\zeta^2\right)}}$$

$$\text{or} \quad M_r = \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}}$$

$$\text{or} \quad \boxed{M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}}$$

Bandwidth of the prototype second order system

Expression for Bandwidth or cut off frequency :-

For bandwidth put $u = u_b$ (normalised Bandwidth)

and $M = \frac{1}{\sqrt{2}}$ in eqn. (1).

$$\therefore \quad \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + 4\zeta^2 u_b^2}}$$

$$\text{or} \quad (1-u_b^2)^2 + 4\zeta^2 u_b^2 = 2$$

$$\text{or} \quad 1 + u_b^4 + 4\zeta^2 u_b^2 - 2u_b^2 - 2 = 0$$

$$\text{or} \quad u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 - 1 = 0$$

$$\text{or} \quad u_b^4 - 2u_b^2(1 - 2\zeta^2) - 1 = 0$$

Let $u_b^2 = x$.

then $x^2 - 2x(1-2\xi^2) - 1 = 0$

or $x = \dfrac{2(1-2\xi^2) \pm \sqrt{4(1-2\xi^2)^2 + 4}}{2}$

or $x = (1-2\xi^2) \pm \sqrt{1-4\xi^2+4\xi^4+1}$

or $x = (1-2\xi^2) \pm \sqrt{2-4\xi^2+4\xi^4}$

or $\boxed{u_b = \sqrt{(1-2\xi^2) + \sqrt{2-4\xi^2+4\xi^4}}}$

or $\omega_b = \omega_n \sqrt{(1-2\xi^2) + \sqrt{2-4\xi^2+4\xi^4}}$

Q- The forward path transfer function of a unity feedback control system is $G(s) = \dfrac{100}{s(s+6.54)}$

Find the M_r, ω_r and bandwidth of the closed loop system.

Soln:- Given $G(s) = \dfrac{100}{s(s+6.54)}$

$H(s) = 1$

$\therefore \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

or $\dfrac{C(s)}{R(s)} = \dfrac{100 / s(s+6.54)}{1 + \dfrac{100}{s(s+6.54)} \cdot 1}$

or $\dfrac{C(s)}{R(s)} = \dfrac{100}{s^2+6.54s+100}$

Compare this with standard T.F. i.e. $\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

Then, $\omega_n = \sqrt{100} = 10$ rad/sec.

and $2\xi\omega_n = 6.54$

or $\boxed{\xi = 0.327}$

$\therefore$ i) $\omega_r = \omega_n\sqrt{1-2\xi^2} = 8.86$ rad/sec.

ii) $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}} = 1.618.$

iii) $\omega_b = \omega_n\sqrt{1-2\xi^2 + (2-4\xi^2+4\xi^4)^{1/2}}$

$= 14.34$ rad/sec

Effects of adding a zero to the forward path, Effects of adding a pole to the forward path

Transfer Function: Prototype second-order forward path transfer function

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

Let us add a zero at $s = -1/T$ to the transfer function.

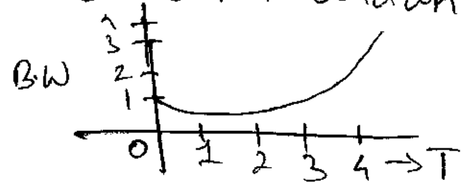
$$G(s) = \frac{\omega_n^2(1+Ts)}{s(s+2\zeta\omega_n)}$$

$$M(s) = \frac{\omega_n^2(1+Ts)}{s^2(2\zeta\omega_n + T\omega_n^2)s + \omega_n^2}$$

$$BW = \sqrt{\left(-b + \frac{1}{2}\sqrt{b^2 + 4\omega_n^4}\right.}$$

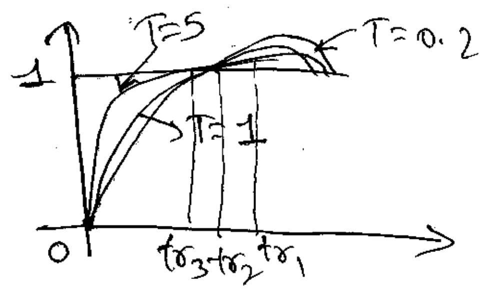
where, $b = 4\zeta^2\omega_n^2 + 4\zeta\omega_n^3 + T - 2\omega_n^2 - \omega_n^4$

$\zeta = 0.707$ and $\omega_n = 1$



The general effect of adding a zero to the forward path is to increase the bandwidth of closed loop system.

So bandwidth increases as T increases



$t_{r_3} < t_{r_2} < t_{r_1}$

So as T increases rise Time decreases.

Polar Plot

POLAR PLOT

The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as 'ω' is varied from zero to infinity.

Sketch Polar plot of Type - 0 system

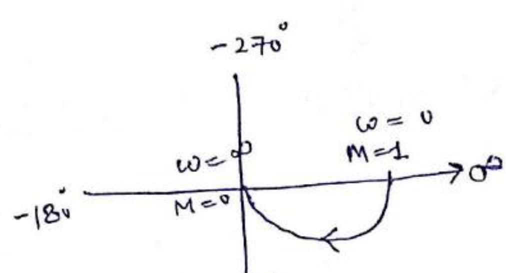
e.g. $G(s) = \frac{1}{(s+1)}$

put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega+1}$$

Now $M = \frac{1}{\sqrt{\omega^2+1}}$

$$\phi = -\tan^{-1}(\omega).$$



ω	M	ϕ
0	1	0
∞	0	-90°

With the help of this table draw polar plot.

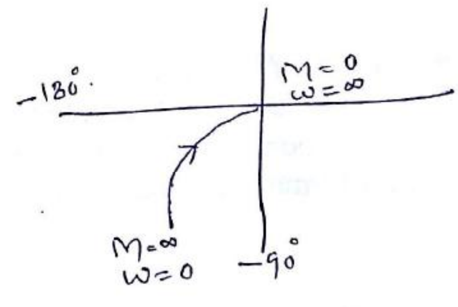
Polar Plot of Type-1 system :-

i) $G(s) = \frac{10}{s(s+1)}$

put $s = j\omega$ $\quad G(j\omega) = \frac{10}{j\omega(j\omega+1)}$

$$M = \frac{1}{\omega\sqrt{\omega^2+1}}, \quad \phi = -90^\circ - \tan^{-1}(\omega).$$

ω	M	ϕ
0	∞	-90°
∞	0	-180°

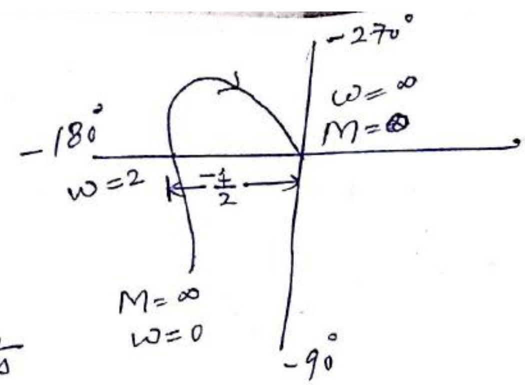


ii) $G(s) = \frac{10}{s(s+1)(s+4)}$

put $s = j\omega$ $\quad G(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+4)}$

$$M = \frac{10}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4^2}} \quad \phi = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

ω	M	ϕ
0	∞	-90°
∞	0	-270°



Since, here polar plot cuts the real axis. Hence, we have to determine the magnitude at this cut-point.

Since, polar plot crosses -180° axis. Hence

$$\phi = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

put $\phi = -180^\circ$.

$$-180^\circ = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

or $$\tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{4}\right) = 90^\circ$$

or $$\tan^{-1}\left[\frac{\omega + \frac{\omega}{4}}{1 - \omega \cdot \frac{\omega}{4}}\right] = 90^\circ$$

or $$\frac{\omega + \omega/4}{1 - \frac{\omega^2}{4}} = \tan 90^\circ = \infty = \frac{1}{0}$$

or $$1 - \frac{\omega^2}{4} = 0 \quad \text{or} \quad \omega = \pm 2$$

or $$\boxed{\omega = 2 \text{ rad/sec}}$$

put $\omega = 2$ in Magnitude equation;

$$M = \frac{10}{2\sqrt{4+1}\sqrt{4+4^2}} = \frac{10}{2\sqrt{5} \cdot \sqrt{20}} = \frac{1}{2}$$

Hence, at cross point $\omega = 2$ rad/sec.
$M = \frac{1}{2}$ A

Inverse Polar Plot, Gain Margin and Phase margin calculation from Polar Plot

Inverse Polar Plot

For inverse polar plot, just reverse the given function and then plot the polar plot for the reverse function.

For e.g. $G(s) = \frac{1+sT}{sT}$

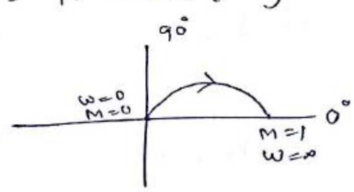
Plot inverse polar plot.

$\therefore \quad G(s)^{-1} = \frac{sT}{1+sT}$

put $s = j\omega \qquad [G(j\omega)]^{-1} = \frac{j\omega T}{1+j\omega T}$

$M = \frac{\omega T}{\sqrt{1+(\omega T)^2}} \quad , \quad \phi = 90^\circ - \tan^{-1}(\omega T)$

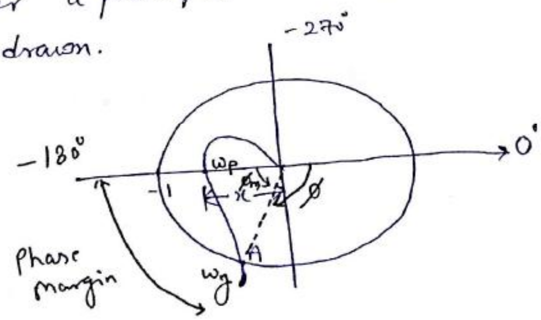
ω	M	ϕ
0	0	90°
∞	1	0



PHASE MARGIN, GAIN MARGIN AND STABILITY

The stability of control system is measured from the measurement of quantities like 'phase margin' and 'gain margin'.

For this, consider a polar plot on which a unit radius circle has been drawn.



Phase cross-over frequency :-

The frequency at which polar plot crosses the negative real axis is called "phase cross-over frequency".

Gain Margin :-

It is the reciprocal of the magnitude $|G(j\omega)|$ at phase cross over frequency.

$$\therefore \text{ Gain Margin} = 20 \log \frac{1}{|G(j\omega_p)|} \text{ decibels.}$$

Gain Cross over frequency :-

The frequency at which polar plot intersects with unit radius circle is called gain crossover frequency. It is denoted by ω_c.

Phase Margin :-

It is given by

$$\phi_m = 180^\circ + \phi$$

For stable stable, both G.M. and P.M. must be positive.
For M.stable system, the G.M and PM both are zero.
For unstable system, GM and PM are negative.

Nyquist stability criterion, problems based on Nyquist Plot

Nyquist Plot

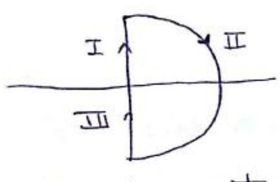
Nyquist Stability Criterion :⇒

The characteristic equation is given by

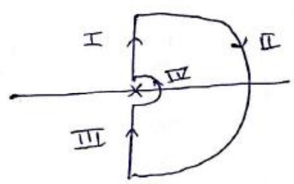
$$D(s) = 1 + G(s)\,H(s)$$

For stability, the necessary and sufficient condition is that all zeros (roots) of C.E. must lie in the left half of s-plane.

In order to determine the presence of zeros in right half of s-plane, we choose a contour called Nyquist contour as follows:-



For Type-0 system



Type-1, 2, 3, system

This contour is mapped in D(s) plane, which is known as Nyquist Plot.

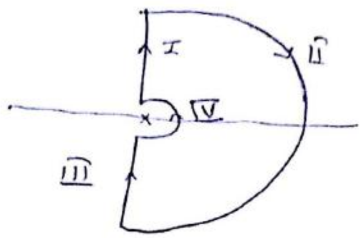
As per Nyquist stability criterion, the feedback system is stable if the number of anticlockwise encirclements of plot about the point $(-1+0j\omega)$ equals the number of poles of $G(s)H(s)$ in right half of s-plane.

In common case, a system is stable if the nyquist plot does not pass through the point $(-1+j\omega)$ or net encirclement is zero.

i.e. $\boxed{N = -P}$

Ques:- Use Nyquist criterion, determine the closed loop system stability $G(s)\,H(s) = \dfrac{1}{s(1+2s)(1+s)}$

Soln:- Here, it is type-1 system, Hence, the nyquist contour is



Part I mapping

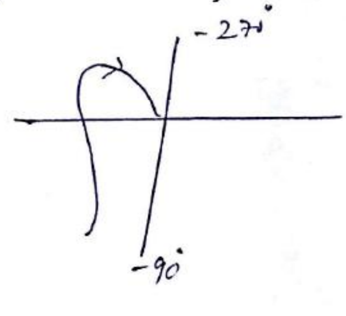
Polar Plot of function $G(s)\,H(s) = \dfrac{1}{s(1+2s)(1+s)}$

put $s=j\omega$ $\quad G(j\omega)\,H(j\omega) = \dfrac{1}{j\omega(1+2j\omega)(1+j\omega)}$

$$M = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}}$$

$$\phi = -90^\circ - \tan^{-1}(2\omega) - \tan^{-1}(\omega)$$

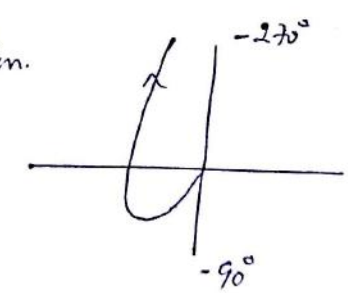
ω	M	ϕ
0	∞	-90°
∞	0	-270°



Part III mapping.

inverse polar plot of function.

simply mirror image of polar plot.



since, here polar plot cut the negative real axis;

$$\therefore \quad -180^\circ = -90^\circ - \tan^{-1}(2\omega) - \tan^{-1}(\omega)$$

$$\text{or} \quad \tan^{-1}\left(\frac{2\omega+\omega}{1-2\omega^2}\right) = 90^\circ$$

$$\text{or} \quad \frac{3\omega}{1-2\omega^2} = \tan 90 = \infty = \frac{1}{0}$$

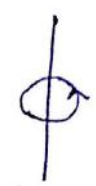
$$\text{or} \quad 1-2\omega^2 = 0 \quad \text{or} \quad \omega = \frac{1}{\sqrt{2}} = 0.707$$

$$\text{Now} \quad M\Big|_{at\ \omega=0.707} = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\sqrt{1+\frac{4}{2}}\sqrt{1+\frac{1}{2}}}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\cdot\sqrt{3}\cdot\sqrt{\frac{3}{2}}} = \frac{2}{3}.$$

Part - II mapping



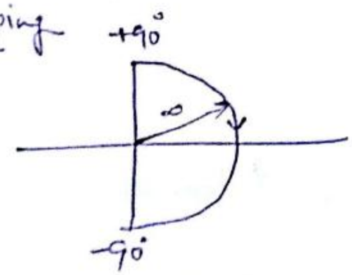
Since $(-1+j\omega)$ is not encircle.

$N = 0$.

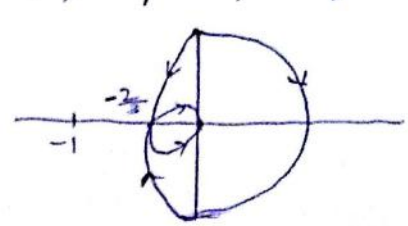
$P = 0$

$Z = 0$

Part IV mapping



Combining all the four parts, we get nyquist plot



System is stable.

Stability determination from Nyquist stability criterion

PHASE MARGIN, GAIN MARGIN AND STABILITY

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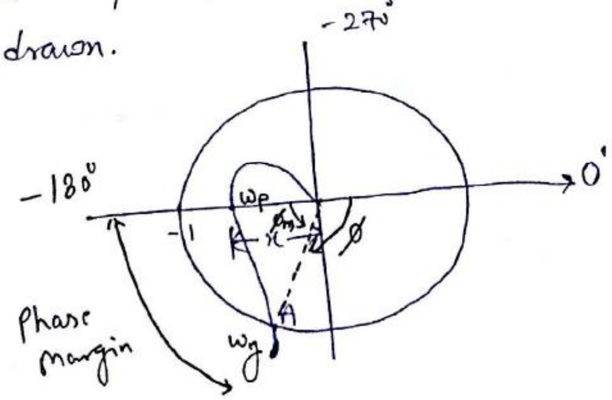
For this, consider a polar plot on which a unit radius circle has been drawn.



Phase cross-over frequency :-

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Gain Margin :- It is the reciprocal of the magnitude $|G(j\omega)|$ at phase cross over frequency.

$$\therefore \text{Gain Margin} = 20 \log \frac{1}{|G(j\omega_p)|} \text{ decibels.}$$

Gain Cross over frequency :-

The frequency at which polar plot intersects with unit radius circle is called gain crossover frequency. It is denoted by ω_c.

Phase Margin :-

It is given by

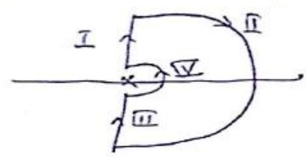
$$\phi_m = 180^\circ + \phi$$

For stable stable, both G.M. and P.M. must be positive.
For M.stable system, the G.M and PM both are zero.
For unstable system, GM and PM are negative.

Ques:- Using Nyquist criterion, determine the stability of the feedback system having open loop transfer function

$$G(s)\,H(s) = \frac{K}{s^2(Ts+1)}$$

Soln:- This is Type-2 system. Hence, Nyquist contour have four parts :-

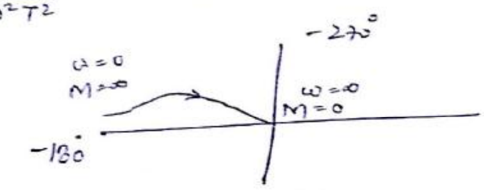


Part I mapping

$$G(j\omega) = \frac{K}{(j\omega)(j\omega)(1+j\omega T)}$$

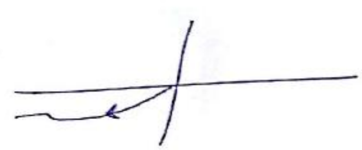
$$M = \frac{K}{\omega^2\sqrt{1+\omega^2T^2}}, \quad \phi = -180^\circ - \tan^{-1}(\omega T)$$

ω	ϕ	M
0	-180°	∞
∞	-270°	0

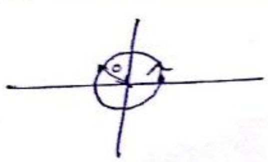


Part III mapping

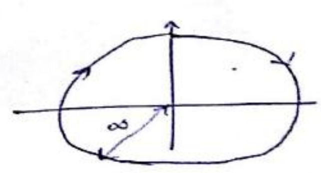
Inverse polar plot



Part II mapping.



Part IV mapping.



Bode plot : Magnitude Plot and Phase Plot

BODE PLOT :⇒

It is graphical representation of the transfer function for determining the stability. It consists of two plots:

i) Magnitude Plot

ii) Phase Plot.

Ques: Sketch the Bode Plot for the transfer function

$$G(s) = \frac{1000}{(1+0.1s)(1+0.001s)}$$

or

$$G(s) = \frac{10^7}{(s+100)(s+1000)}$$

Determine the
a) P.M.
b) G.M.
c) Stability of the system.

Soln:- put $s = j\omega$
and calculate magnitude and phase of the function.

$$G(j\omega) = \frac{1000}{(1+0.1j\omega)(1+0.001j\omega)}$$

Now, $M = \dfrac{1000}{\sqrt{1+(0.1\omega)^2}\cdot\sqrt{1+(0.001\omega)^2}}$

$$\phi = -\tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega)$$

1) Magnitude Plot

S.N.	Factor	Standard Slope	Total Slope	Corner Frequency
1.	$\frac{1}{s^0}$	0 dB/decade	0 dB/decade	10
2.	$\frac{1}{1+0.1j\omega}$	−20 dB/decade	−20 dB/decade	1000
3.	$\frac{1}{1+0.001j\omega}$	−20 dB/decade	−40 dB/decade	—

Starting Point $\Rightarrow 20\log K = 20\log(1000) = \underline{60 dB}$.

2) Phase Plot

$$\phi = -\tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega)$$

ω	$-\tan^{-1}(0.1\omega)$	$-\tan^{-1}(0.001\omega)$	ϕ
50	$-78.1°$	$-2.86°$	$-81.46°$
100	$-84.2°$	$-5.7°$	$-90°$
200	$-87.13°$	$-11.3°$	$-98°$
800	$-89.28°$	$-38.65°$	$-127.93°$
2000	$-89.72°$	$-63.43°$	$-153.15°$
8000	$-89.92°$	$-82.87°$	$-172.79°$

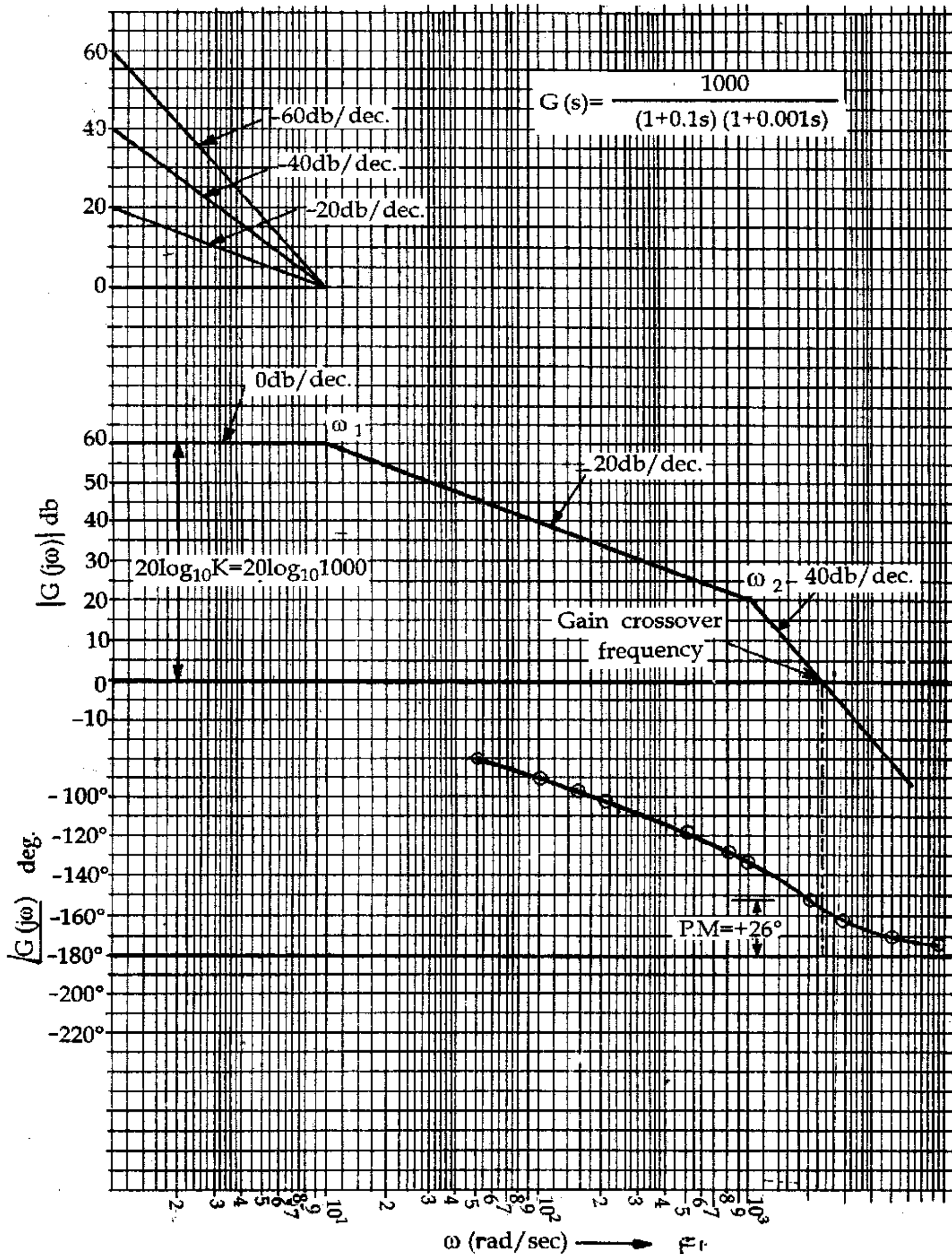
With the help of magnitude, phase and starting point the Bode Plot drawn on semi log graph paper.

From Bode Plot;

$GM = \infty$

$PM = +26°$

Hence system is unstable.



60
40
20
0
-60db/dec.
-40db/dec.
-20db/dec.
G (s)= 1000 / (1+0.1s) (1+0.001s)
0db/dec.
ω 1
-20db/dec.
ω 2
-40db/dec.
Gain crossover
frequency
20log10K=20log101000
|G (jω)| db
60
50
40
30
20
10
0
-10
-100°
-120°
-140°
-160°
-180°
-200°
-220°
/G (jω) deg.
P.M=+26°
ω (rad/sec)

Stability analysis with the Bode plot

4.11. PHASE MARGIN & GAIN MARGIN

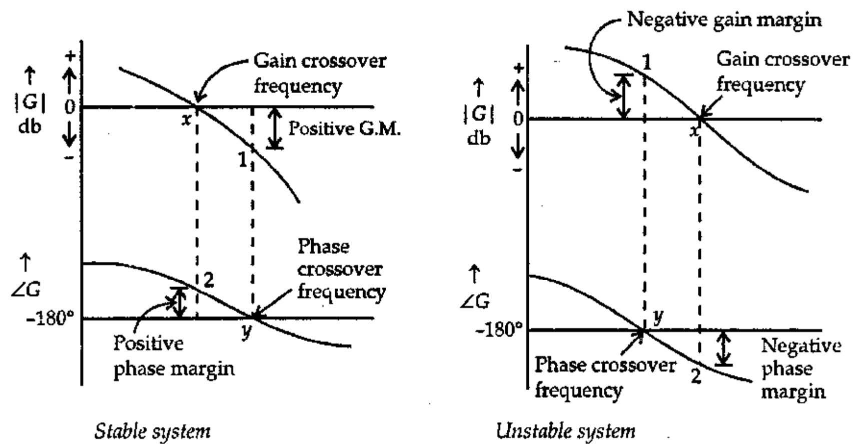


Fig. 4.29.

Positive gain margin means the system is stable and negative gain margin means the system is unstable. For minimum phase system both phase margin and gain margin must be positive for the system to be stable.

The point at which the magnitude curve crosses the 0db line is the gain crossover frequency. The phase crossover frequency is the point where the phase curve crosses the 180° line.

Gain Margin : Gain margin is defined as the margin in gain allowable by which gain can be increased till system reaches on the verge of instability. Mathematically gain margin is defined as the reciprocal of the magnitude of the $G(j\omega)\ H(j\omega)$ at phase cross-over frequency.

$$\therefore \qquad \text{G.M} = \frac{1}{G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_2}}}$$

where ω_{c_2} = phase cross-over frequency.

Generally, G.M is expressed in decibels

$$\therefore \text{ In decibels} \quad \text{G.M} = 20 \log \frac{1}{|G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_2}}}$$

$$\text{or,} \qquad \text{G.M.} = -20 \log_{10} |G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_2}}$$

Phase Margin : For gain the additional phase lag can be introduced without affecting the magnitude plot. Therefore, phase margin can be defined as the amount of additional phase lag which can be introduced in the system till system reaches on the verge of instability is called as phase margin (P.M.). Mathematically phase margin can be defined as

$$\text{P.M.} = \left[\angle G(j\omega)\ H(j\omega)\big|_{\omega=\omega_{c_1}}\right] - (-180°)$$

$$\text{P.M.} = 180° + \angle G(j\omega)\ H(j\omega)\big|_{\omega=\omega_{c_1}}$$

where ω_{c_1} = Gain cross-over frequency.

Ques: Sketch the Bode Plot for the transfer function

$$G(s) = \frac{1000}{s(1+0.1s)(1+0.001s)}$$

Determine i) Gain cross over frequency.

ii) Phase cross over frequency.

iii) G.M & P.M.

iv) Stability of the system.

Soln:- put $s = j\omega$

$$G(j\omega) = \frac{1000}{j\omega(1+0.1j\omega)(1+0.001j\omega)}$$

Magnitude Plot :-

S.N.	Factor	Standard Slope dB/decade	Total Slope dB/decade	Corner Frequency
1.	$\frac{1}{s}$	-20	-20	10
2.	$\left(\frac{1}{1+0.1s}\right)$	-20	-40	1000
3.	$\left(\frac{1}{1+0.001s}\right)$	-20	-60	—

Starting Point $\Rightarrow$ $20\log(1000) = \underline{\underline{60\ dB}}$.

Phase Plot

$$\phi = -90^\circ - \tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega).$$

ω	ϕ
1	-95.7°
5	-116.5°
10	-135.6°
50	-171.4°
100	-179.6°
200	-188°
500	-205.41°

From Bode Plot

GM = 0

PM = 0

Hence system is marginally stable.

i) Gain Cross over freq. ω_g = 100 rad/sec

ii) Phase crossover freq. ω_p = 100 rad/sec.