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INPUT EFFICIENCY MEASURES: A GENERALIZED, ENCOMPASSING FORMULATION

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Input Efficiency Measures: A Generalized, Encompassing Formulation

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Abstract

This contribution establishes a link between three well known input efficiency measures: the Debreu-Farrell measure, the Färe-Lovell measure and the asymmetric Färe measure. A new power min generalized input efficiency measure is defined. The three previously cited efficiency measures are shown to be limiting cases of this new, encompassing measure. The same applies to another new measure, which we term the multiplicative Färe-Lovell efficiency measure.

Keywords: Debreu-Farrell efficiency measure, Färe-Lovell efficiency measure, asymmetric Färe efficiency measure, input efficiency.

JEL: D21, D24

1 Introduction

Debreu (1951) and Farrell (1957) described the first concept of a radial efficiency measure. Their seminal work has since then been extended to several other efficiency measures. In particular, Färe and Lovell (1978) define a new Russell efficiency measure, that is also referred to as a Färe-Lovell efficiency measure in the literature. This measure was defined to reconcile efficiency measures with the notion of Koopmans' (1951) efficiency.[1] Färe (1975) earlier defined another efficiency measure, which we label the Asymmetric Färe

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[1]See also Russell (1985, 1988).

efficiency measure.[2] Most of this literature has been surveyed in Russell (1998).

In this paper, we define a new, power min generalization of these input efficiency measures that clearly demonstrates the links between the three previously cited measures.[3] Moreover, we show that a new, fourth efficiency measure which we term multiplicative Färe-Lovell efficiency measure belongs to this same family of efficiency measures. Our new measure therefore proves to offer an encompassing framework to the main existing efficiency measures in the economic literature. Furthermore, in line with the economic literature, this paper mainly concentrates on input efficiency. Obviously, these results can be easily transposed to other measurement orientations.

2 Assumptions on Technology and Definitions of Efficiency Measures

Technology describes all production possibilities to transform input vectors $x = (x_1, \cdots, x_N) \in \mathbb{R}_+^N$ into output vectors $y = (y_1, \cdots, y_M) \in \mathbb{R}_+^M$. Given our focus on input-oriented efficiency measurement, technology is represented by its input sets:

$$L(y) = \{x \in \mathbb{R}_+^N : x \text{ can produce } y\}. \tag{2.1}$$

The following standard conditions are imposed on $L(y)$ (see, e.g., Hackman (2008) for details):
L.1: $L(0) = \mathbb{R}_+^N$ and $y \neq 0 \Rightarrow 0 \notin L(y)$;
L.2: Let $y_n \in \mathbb{R}_+^N, n \geq 0$ such that $\lim_{n\to\infty} y_n = +\infty$, then $\cap_{n\to\infty} L(y_n) = \emptyset$;
L.3: $L(y)$ is a closed set;
L.4: $x \in L(y)$ and $x' \geq x \Rightarrow x' \in L(y)$.
Apart from the traditional regularity assumptions (possibility of inaction, boundedness, and closedness), assumption L.4 represents the strong or free disposability of inputs. Remark that we do not impose any convexity assumption on the input sets.

Now we can recall the definition of the Debreu (1951) and Farrell (1957) radial efficiency measure $DF : \mathbb{R}_+^N \times \mathbb{R}_+^M \longrightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$ as follows:

$$DF(x, y) = \begin{cases} \inf_{\delta \in \mathbb{R}_+} \{\delta : \delta x \in L(y)\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

This radial efficiency measure indicates the maximal equiproportionate reduction in all inputs which still allows production of the given output vector.

[2] Kopp (1981a,b) has been critical about both the Asymmetric Färe and the Färe-Lovell efficiency measures.

[3] See also Briec et al. (2011) for such links in the context of input directional efficiency measures.

From a debate on axiomatic properties of radial efficiency measures, the Färe-Lovell (1978) efficiency measure emerged. This function $FL : \mathbb{R}_+^N \times \mathbb{R}_+^M \longrightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$ can be defined as follows:

$$FL(x,y) = \begin{cases} \inf_{\beta \in \mathbb{R}_+^N} \left\{ \frac{1}{N} \sum_{i=1}^N \beta_i : \beta x \in L(y) \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

We also recall the asymmetric Färe (1975) input efficiency measure defined by $AF : \mathbb{R}_+^N \times \mathbb{R}_+^M \longrightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$,

$$AF(x,y) = \begin{cases} \min_{i \in I(x)} \inf \{\beta : \beta e_i \odot x \in L(y)\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$

where $\odot$ denotes the Schur product of two vectors (element by element product) and $I(x) = \{i : x_i > 0\}$.

Paralleling Färe and Lovell (1978), we establish a new Färe-Lovell type of input efficiency measure that is multiplicative rather than additive in nature. Let us define this function as follows: $MFL : \mathbb{R}_+^N \times \mathbb{R}_+^M \longrightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$ as

$$MFL(x,y) = \begin{cases} \inf \left\{ \prod_{i \in I(x)} (\beta_i)^{\alpha_i} : \beta \odot x \in L(y) \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

where β is the vector of $\mathbb{R}^{I(x)}$ whose elements are β_i for $i \in I(x)$, and $\alpha \in \mathbb{R}_{++}^{I(x)}$ is the vector whose elements are α_i for $i \in I(x)$ and such that $\sum_{i \in I(x)} \alpha_i = 1$. Since this function is the multiplicative equivalent of the Färe-Lovell (1978) measure of input efficiency, we term it a multiplicative Färe-Lovell input efficiency measure. Note that the minimization is computed only when $x \in L(y)$. Otherwise, some of the β parameters could be smaller and some greater than one at the same time.

As shown in the next section, these four input efficiency measures turn out to belong to a single family.

3 A New Power Min Generalization of Input Efficiency Measures and Its Limiting Cases

The Debreu (1951) and Farrell (1957), Färe-Lovell (1978), asymmetric Färe (1975), and multiplicative Färe-Lovell input efficiency measures can all be seen as special cases of an extended Färe-Lovell input efficiency measure.

Indeed, we can introduce the generalised Färe-Lovell measure $GFL_p : \mathbb{R}_+^N \times \mathbb{R}_+^M \longrightarrow [0,1]$ defined by

$$GFL_p(x,y) = \begin{cases} \inf \left\{ \left(\frac{1}{|I(x)|} \sum_{i \in I(x)} (\beta_i)^p \right)^{1/p} : \beta \odot x \in L(y) \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.1}$$

where $|I(x)|$ stands for the cardinality of $I(x)$ and for all $p \neq 0$.

Proposition 3.1 *Assume that B is a compact subset of $\mathbb{R}^N_{++}$ and that $p_0 \in [-\infty, +\infty]$. We have*

$$\lim_{p \longrightarrow p_0} \inf_{\beta \in B} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} = \inf_{\beta \in B} \lim_{p \longrightarrow p_0} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p}.$$

In particular, we have:

$$(a) \quad \lim_{p \longrightarrow 0} \inf_{\beta \in B} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} = \inf_{\beta \in B} \prod_{i=1}^{N} (\beta_i)^{1/N},$$

$$(b) \quad \lim_{p \longrightarrow -\infty} \inf_{\beta \in B} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} = \inf_{\beta \in B} \min_{i=1,\dots,N} \beta_i,$$

$$(c) \quad \lim_{p \longrightarrow +\infty} \inf_{\beta \in B} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} = \inf_{\beta \in B} \max_{i=1,\dots,N} \beta_i.$$

Proof: Let us denote $M_p(\beta) = \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p}$ for all $\beta \in B$ and all $p \neq 0$. If $p_0 \notin \{-\infty, 0, +\infty\}$, since $B \subset \mathbb{R}^N_{++}$, we have:

$$\lim_{p \longrightarrow p_0} M_p(\beta) = M_{p_0}(\beta).$$

Since M_{p_0} is continuous over B, we deduce that the sequence $\{M_p\}_{p \in \mathbb{N}}$ converges uniformly to M_{p_0}. Since B is compact, there is some $\beta_p \in B$ that achieves the minimum for each p, i.e $M_p(\beta_p) = \inf_{x \in B} M_p(\beta)$. Moreover, there is some $\beta_{p_0} \in B$ which achieves the minimum of M_{p_0} over B. From the continuity of M_{p_0} and M_p for each p, the result follows immediately.
(a) Suppose that $p_0 = 0$. In such a case, we have:

$$\lim_{p \longrightarrow 0} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} = \prod_{i=1}^{N} (\beta_i)^{1/N}.$$

Since the function $M_0 : B \longrightarrow \mathbb{R}_{++}$ defined by $M_0(\beta) = \prod_{i=1}^{N} (\beta_i)^{1/N}$ is continuous over B, we deduce the statement for $p_0 = 0$.
(b) Suppose that $p_0 = -\infty$. First, notice that for all $p < 0$, the function M_p is continuous over B. In such a case, we have:

$$\lim_{p \longrightarrow -\infty} \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} = \min_{i=1,\dots,N} \beta_i.$$

Since the function $M_{-\infty} : B \longrightarrow \mathbb{R}_{++}$ defined by $M_{-\infty}(\beta) = \min_{i=1,\dots,N} \beta_i$ is continuous over B, we deduce the statement for $p_0 = -\infty$.
(c) Finally, the proof of the case ($p_0 = +\infty$) is very similar and is thus omitted . $\square$

Proposition 3.2 *Under L.1 to L.4, if the inputs are essential and $y \neq 0$, then for all $x \in L(y)$ we have:*

$$\begin{array}{ll} (a) & GFL_1(x,y) = FL(x,y) \\ (b) & \lim_{r \longrightarrow -\infty} GFL_r(x,y) = AF(x,y) \\ (c) & \lim_{r \longrightarrow 0} GFL_r(x,y) = MFL(x,y) \\ (d) & \lim_{r \longrightarrow +\infty} GFL_r(x,y) = DF(x,y) \end{array}$$

Proof: Let $\beta^* = \arg\min \left\{ \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} : \beta \odot x \in L(y) \right\}$. Let us prove that for all $p \in \mathbb{R}\backslash\{0\}$ there is a compact B independent of p such that $\beta^* \in B$. By definition, we have $\beta^* \leq 1_n$. Let us denote

$$\bar{\beta}_i = \inf\{\beta : x + (\beta - 1)x \odot e_i \in L(y)\}.$$

Since the inputs are essential, we have $\bar{\beta}_i > 0$ for $i = 1, ..., N$. Moreover, we have from the strong disposal assumption $\beta \geq \bar{\beta}$. Thus, the optimal value of β lies in the subset

$$B = \{\beta \in \mathbb{R}^N_+ : \bar{\beta} \leq \beta \leq 1_n, \beta \odot x \in L(y)\} \subset \mathbb{R}^N_{++}.$$

Since B is a closed and bounded subset of $\mathbb{R}^N$, it is compact. Consequently, for all $p \neq 0$, we have

$$GFL_p(x,y) = \inf_{\beta \in B} \left\{ \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_i)^p \right)^{1/p} \right\}.$$

We then immediately deduce $(a), (b)$ and (c) from Proposition 3.1. Let us prove (d). From Proposition 3.1, we have

$$\lim_{r \longrightarrow +\infty} GFL_r(x,y) = \inf_{\beta \in B} \max_{i=1,...,N} \beta_i.$$

Let $V(\delta) = \{\beta \in B : \max_{i=1,...,N} \beta_i = \delta\}$. We have by definition

$$\lim_{r \longrightarrow +\infty} GFL_r(x,y) = \inf \left\{ \delta : V(\delta) \neq \oslash \right\}.$$

However, since the strong disposal assumption holds, $\beta \in V(\delta)$ implies that $\delta 1_n \geq \beta$ and $\delta 1_n \in V(\delta)$. Therefore:

$$\lim_{r \longrightarrow +\infty} GFL_r(x,y) = \inf \{\delta : \delta 1_n \odot x \in L(y)\},$$

which can be rewritten

$$\lim_{r \longrightarrow +\infty} GFL_r(x,y) = \inf_{\delta} \{\delta : \delta x \in L(y)\} = DF(x,y). \quad \square$$

4 Concluding Comments

We have introduced a new generalized power min input efficiency measure. Thereafter, we have established that one can obtain the Debreu (1951) and Farrell (1957) input efficiency measure when p tends to infinity, the Färe-Lovell measure for $p = 1$, the asymmetric Färe measure when p tends to minus infinity and the multiplicative Färe-Lovell measure when $p = 0$.

Table 1 sums up how to obtain the main economic efficiency measures from different values of the parameter p in the power min input efficiency measure:

Value of p	Efficiency measure
$-\infty$	Asymmetric Färe
0	Multiplicative Färe-Lovell
1	Färe-Lovell
$+\infty$	Debreu-Farell

Table 1: Summary of Main Result

This generalized power min input efficiency measure offers a broader framework for empirical applications and its encompassing nature provides a natural framework for testing.

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