Topology of Sequences up to the Indices of Repeated Elements

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Introduction

It is difficult to determine the dynamic behavior of recursively defined sequences. Even determining whether a k-regular sequence has a repeated term is undecideable [1]. We intend to study sequences as a whole and develop a topology measuring how elements in a sequence can repeat.

Given a recursively defined sequence with varying initial condition, it is often possible to rewrite these sequences as a polynomial sequence evaluated at some initial condition.

Repeated Indices

Let $q \in \mathbb{Q}[x]^{\mathbb{N}}$. We say that (i, j) is a induced repeated index if $q_i = q_j$. We say that (i, j) is a variable repeated index at $x \in \mathbb{R}$ if $q_i(x) = q_j(x)$ and $q_i \neq q_j$.

Let $q \in \mathbb{Q}[x]$ be defined by $q_1 = x$ and $q_{2n} = 2q_n + 1$ and $q_{2n+1} = 4q_n + 3$ for $n \geq 1$. Observe that

	1	2	3	4	5	
$q =$	x,	$2x+1$,	$4x+3$,	$4x+3$,	$8x+7$,	$\ldots$

Observe that $q_2 = q_3$. Then $(5, 6)$ is an induced repeated index. See that $q_2(-1) = q_3(-1)$ and $q_2 \neq q_3$. Then $(2, 3)$ is a variable repeated index at -1.

Setting up a Topology

Equivalence Classes

Let A be some set such that $|A| \geq |\mathbb{N}|$. Let $a, b \in A^{\mathbb{N}}$. Define an equivalence relation $\sim$ on $A^{\mathbb{N}} \times A^{\mathbb{N}}$ by

$$a \sim b \text{ if } a_i = a_j \iff b_i = b_j \text{ for all } i, j \in \mathbb{N}.$$

Let $S = \{[a] : a \in A^{\mathbb{N}}\}$.

Let $A = \{!, *, \#, \%, ?, x_1, x_2, x_3, \ldots\}$. Define $a, b, c \in A^{\mathbb{N}}$ by

	1	2	3	4	5	6	
$a =$	$*$,	$*$,	$?$,	$*$,	x_1,	x_1,	$\cdots$
$b =$	x_2,	x_2,	$\#$,	x_2,	$?$,	$?$,	$\cdots$
$c =$	$*$,	$*$,	$?$,	$*$,	x_2,	$?$,	$\cdots$

Then $a \sim b$ but $a \not\sim c$.

Ordering Elements

Let $[a], [b] \in S$ then we define $[a] < [b]$ if there exists $i, j \in \mathbb{N}$ so that $a_i = a_j$ and $b_i = b_k$ implies that $j < k$ for all $k \in \mathbb{N}$. Additionally $a_m = a_n \iff b_m = b_n$ for $m, n < i$.

Let $A = \{!, *, \#, \%, ?, x_1, x_2, x_3, \ldots\}$. Define $a, c \in A^{\mathbb{N}}$ by

	1	2	3	4	5	6	
$a =$	$*$,	$*$,	$?$,	$*$,	x_2,	$?$,	$\cdots$
$b =$	$*$,	$*$,	$?$,	$*$,	x_1,	x_1,	$\cdots$

Let $i = 6$ and $j = 3$ and that $a_i = a_j$. Observe that $a_m = a_n \iff b_m = b_n$ for $m, n < 6$. Notice that $b_6 = b_5$ and that $j = 3 < 5$. Then $[c] < [a]$.

Initial Index Representation

Given $a \in A^{\mathbb{N}}$ we say that its initial index representation is $\phi \in \mathbb{N}^{\mathbb{N}}$, defined by

$$\phi_i = \min\{j \in \mathbb{N} : a_i = a_j\}.$$

for all $i \in \mathbb{N}$.

Let $A = \{!, *, \#, \%, ?, x_1, x_2, x_3, \ldots\}$. Define $a, c \in A^{\mathbb{N}}$ by

	1	2	3	4	5	6	
$a =$	$*$,	$*$,	$?$,	$*$,	x_2,	$?$,	$\cdots$
$\phi =$	1,	1,	3,	1,	5,	3,	$\ldots$
$b =$	$*$,	$*$,	$?$,	$*$,	x_1,	x_1,	$\cdots$
$\psi =$	1,	1,	3,	1,	5,	5,	$\ldots$

Observe that $[a] < [b]$ and that $\phi < \psi$ in the lexicographical ordering on $\mathbb{N}^{\mathbb{N}}$.

Proposition

Let $f : S \to \mathbb{N}^{\mathbb{N}}$ be defined $f([a]) = \phi$ where ϕ is the initial index representation for a. Then f is well-defined and

$$f(S) = \{b \in \mathbb{N}^{\mathbb{N}} : \forall k \in \mathbb{N}, b_k = \min\{j \in \mathbb{N} : b_j = b_k\}\} = S' \subseteq \mathbb{N}^{\mathbb{N}}$$

Given the lexicographical ordering, S' is order isomorphic to S via F. Additionally S and S' are homeomorphic when each given their respective order topologies.

Topology of S'

Topological Properties

$S' = \{b \in \mathbb{N}^{\mathbb{N}} : \forall k \in \mathbb{N}, b_k = \min\{j \in \mathbb{N} : b_j = b_k\}\}$ under the lexicographical order in the order topology has the following topological properties:

- metrizable
- complete (metric)
- has the least upper bound property
- compact
- totally disconnected
- perfect.

Characterization of Cantor Spaces

The Cantor set is the only totally disconnected, perfect compact metric space (up to homeomorphism) [2, 30.4 Corollary].

Corollary

S' is homeomorphic to the Cantor Set.

Issues with the Topology on S'

Continuous Functions From Connected Spaces

Let S' be the space of Initial Index Representations under the lexicographical order topology. Let $f : [0, 1] \to S'$ be continuous where $[0, 1] \subseteq \mathbb{R}$ is given the standard subspace topology. Then $f(x) = a$ for all $x \in [0, 1]$ and some $a \in S'$.

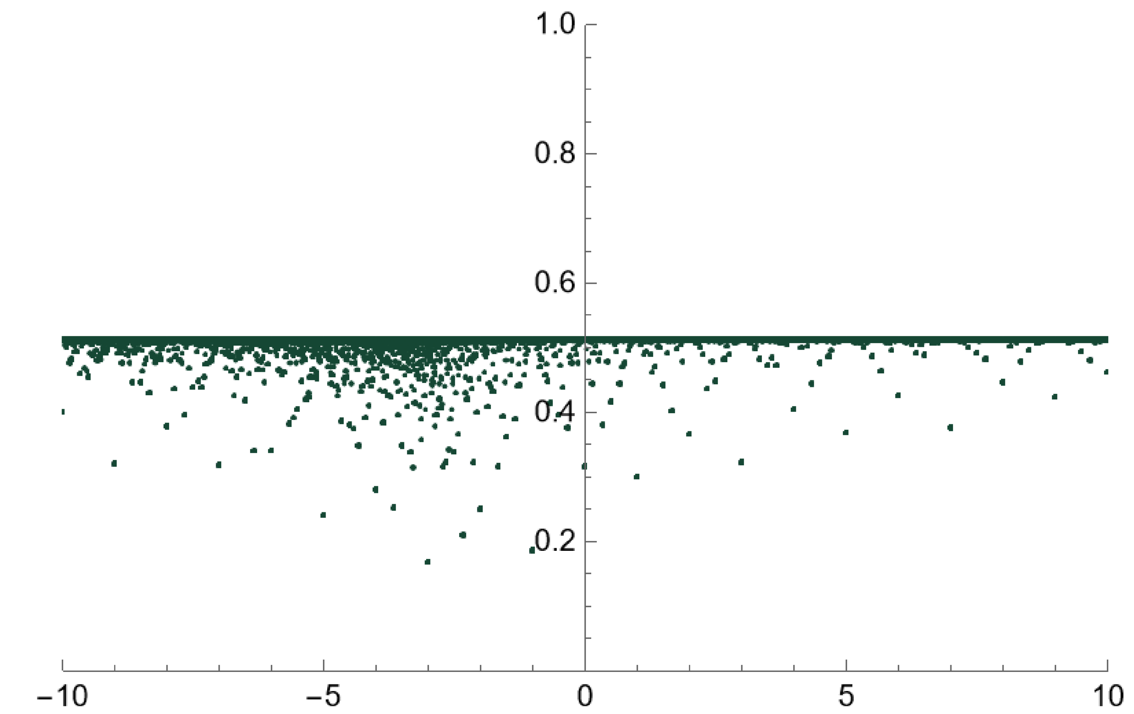

Figure 1: Let $f : \mathbb{R}^{\mathbb{N}} \to S'$ send $a \in \mathbb{R}^{\mathbb{N}}$ to its initial index representation. Let $g : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ be defined $g(x)_1 = x$, $g(x)_{2n} = \frac{g(x)_n}{2} - 3$, and $a_{2n+1} = 2g(x)_n + 3$ for $n \geq 1$. We plot $\frac{\text{new terms}}{\text{total terms}}$ for 500 terms.

Points of Continuity

Let $f : \mathbb{R}^{\mathbb{N}} \to S'$ send $a \in \mathbb{R}^{\mathbb{N}}$ to its initial index representation. Let $q \in \mathbb{Q}[x]^{\mathbb{N}}$. Let $\psi \in S'$ be the initial index representation for q. Now let $g : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ be evaluation so that $g(x) = q_1(x), q_2(x), q_3(x), \ldots$. Then $f \circ g : \mathbb{R} \to S'$ is continuous at $x \in \mathbb{R}$ if and only if $(f \circ g)(x) = \psi$.

The proof of this fact resembles the proof that Thomae's function is continuous at $x \in [0, 1]$ if and only if x is irrational.

Conclusion

In the end we find that mapping continuously into S' is difficult. This is not surprising as dynamical behavior is often chaotic. Further investigation has revealed that the discretization of elements makes continuity difficult with respect to functions from $\mathbb{R}$. In the future we hope to examine sequence representations with minimum distances to give more flexibility.

References

[1] Daniel Krenn and Jeffrey Shallit. "Decidability and k-regular sequences". In: *Theoretical Computer Science* 907 (Mar. 2022), 34–44. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2022.01.018.

[2] Stephen Willard. *General topology*. Dover Publications, 1998, p. 217.

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