General Counting Method For Aid In Computing The Distance Zeta Function On Modified Sierpiński Carpets

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Abstract. Complex dimensions, as introduced by Lapidus, Radunović, and Ẑubrinić in their book *Fractal Zeta Functions and Fractal Drums*, provide a nuanced understanding of the component pieces of any geometric object, by giving a set of complex numbers corresponding to all geometric dimensions present. The set of values is particularly useful for fractals, where the complex periodicity correlates to geometric oscillations. We illustrate this by computing the complex dimensions for modifications of the standard 3×3 Sierpiński carpet. By breaking sections of removal into "path connected components", called substructures, we use the recursive definition of Sierpiński carpets to develop a system of recurrence relations that count all of the substructures appearing in the carpet at each possible scale. This information is turned into a system of rational generating functions, where partial fraction decomposition is used to extract explicit n-th term formulas. These formulas can be used to compute the Distance Zeta Function for each carpet, from which its component complex dimensions can be extracted.

1. Introduction

1.1. The Distance Zeta Function

In their book titled *Fractal Zeta Functions and Fractal Drums* [1], authors Michel Lapidus, Goran Radunović, and Darko Ẑubrinić introduced two crucial tools for understanding the complex dimensions of various fractals; namely, the distance zeta function:

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N}\, dx$$

and the tubular zeta function:

$$\tilde{\zeta}_A(s) = \int_0^\delta t^{s-N-1} |A_t|\, dt.$$

We will largely focus on the former in this paper. The poles of the distance zeta function tells us the Minkowski dimensions of the components that make up a Lebesgue-measurable set $A \subseteq \mathbb{R}^N$.

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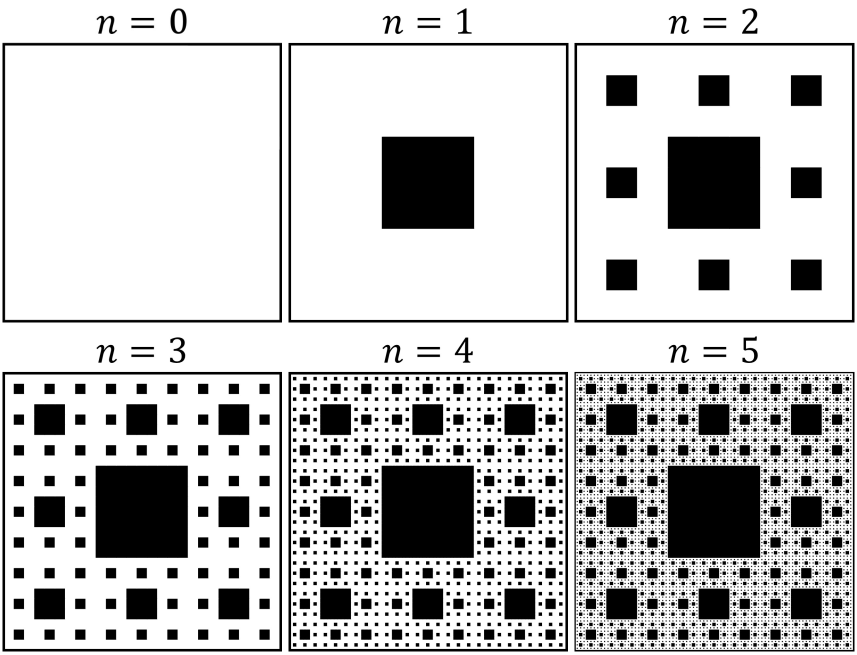


Figure 1.1. Depicts iterations 0 – 5 of the Sierpiński Carpet.

1.2. Motivating Example

We will construct the classic Sierpiński Carpet fractal thusly: start with the unit square $[0, 1]^2 \subseteq \mathbb{R}^2$, divide it into a uniform 3×3 grid, and then remove the center square. Subdivide each of the leftover squares into uniform 3×3 grids of their own and remove their respective center squares. The process is continued indefinitely. The first 5 stages are shown in Figure 1.1: We can analyze a self-similar fractal like this one by analyzing each finite stage of the process. Using n as the "construction level" of the carpet (as described in 1.1), we can see that each time we increment n, we introduce 8^{n-1} new squares, each with side length $\frac{1}{3^n}$. We will embed our Sierpiński Carpet in $\mathbb{R}^2$, so we will be using $N = 2$ in the zeta function formulas. For the sake of convenience, we will ensure $\delta > \frac{1}{6}$ so that A_δ is simply-connected. Each square of side length λ will have the following contribution to the distance zeta function:

$$C_\lambda = 4 \int_0^{\frac{\lambda}{2}} \int_x^{-x+\lambda} x^{s-2} \, dy \, dx = \lambda^s \cdot \frac{8}{2^s (s)(s-1)}$$

The outside of the carpet is unaffected by our recursive process, so we can treat it like an ordinary unit square:

$$O = 4\int_0^{\delta}\int_0^{1} y^{s-2}\,dx\,dy + 4\int_0^{\frac{\pi}{2}}\int_0^{\delta} r^{s-2} r\,dr\,d\theta = \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s}$$

Thus, combining all of our information gets the following for construction level n of the carpet:

$$\begin{aligned}\zeta_{A_n}(s) =& O + \sum_{k=1}^{n}\left[8^k C_{\frac{1}{3^k}}\right]\\ =& \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} + \sum_{k=1}^{n}\left[8^k\left(\frac{1}{3^k}\right)^s \cdot \frac{8}{2^s(s)(s-1)}\right]\\ =& \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} + \frac{8}{2^s(s)(s-1)} \cdot \sum_{k=1}^{n}\left(\frac{8}{3^s}\right)^k\end{aligned}$$

As $n \to \infty$, we can evaluate the geometric series:

$$\zeta_A(s) = \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} + \frac{8}{2^s(s)(s-1)} \cdot \frac{8}{3^s - 8}$$

The geometric series converges absolutely when $3^{\Re(s)} > 8$ where $\Re(s)$ is the real part of s, and each of the integrals involved converge for $\Re(s) > 1$. However, we can use the principle of analytic continuation to get a meromorphic continuation to all of $\mathbb{C}$. In doing so, we see that we have simple poles at $s = 0, 1, \log_3(8) + \frac{2\pi i z}{\ln(3)}$ for every $z \in \mathbb{Z}$. These poles tell us the dimension of each of our components: we have vertices (dimension 0), edges (dimension 1), and a periodic component of dimension $\log_3(8)$.

When we divide the squares into their uniform 3×3 grids, we are not limited to only removing the middle square. By Polya-Redfield Theory, we know there are exactly 100 different fractals that can be generated through removing some arrangement of squares in the 3×3 grid (unique up to rotation and reflection). These were all given names, explored, and analyzed in [2]. Figure 1.2 contains some particularly thought provoking examples.

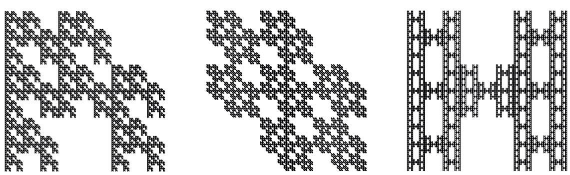


Figure 1.2. Left to Right: The Sus Carpet, The Hexagonal Snowflake Carpet, The Doom Carpet

In our Classic Sierpiński Carpet computation, we knew from the start that additional squares appear at an 8^n rate as n increments. For fractals such as the ones in Figure 1.2, determining that rate of increasing can get complicated, thus motivating the need for a combinatorial tool that will generate a sequence to count those interior portions. We will describe the counting method, and show it in action on two different carpets: the Cornercopia and the Creeper Carpet (named after its visual resemblance to a creature in the popular game *Minecraft*).

2. General Counting Method

2.1. Motivations and Assumptions

In order to count the number of different substructures that appear at different scales throughout the carpet, we will build a system of recurrence relations and solve the system with generating functions.

The basic idea is as follows: first, we count the number of new boxes that appear at each level of the carpet. We can keep track of how many boxes get pulled into substructures that already existed on the carpet in previous iterations and subtract this count off our total boxes. Then we can divide this quantity by the number of boxes it takes to start a substructure ($\frac{\text{\# of total boxes - \# of absorbed boxes}}{\text{\# of starting boxes}}$). This leaves us with the total number of substructures at that level.

In order to make things easier to count, we will identify the top and bottom edges of the carpet as well as the left and right edges of the carpet. This leaves us with a torus and defines away troublesome edge behavior that can be refactored in later. We will also assume we are working on carpets where a unique collection of starting boxes corresponds to a unique substructure growth.

In Figure 2.1 we see a carpet with different patterns of growth arising from the same starting boxes.

In Figure 2.2 we see a carpet with substructure types that have repeated boxes. More precisely the collection of largest boxes in a connected component have repeated boxes. Cutting out this case makes things easier and it is often easy to count them by inspection.

2.2. Definitions for Counting

Definition 2.1. A substructure type is a pattern of removal in a generalized Sierpiński carpet. For this paper, we assume there is a bijection between initial boxes and substructure type. Additionally each substructure types contains at most one box of each type at each scale. Additionally each substructure must contain at most one type of

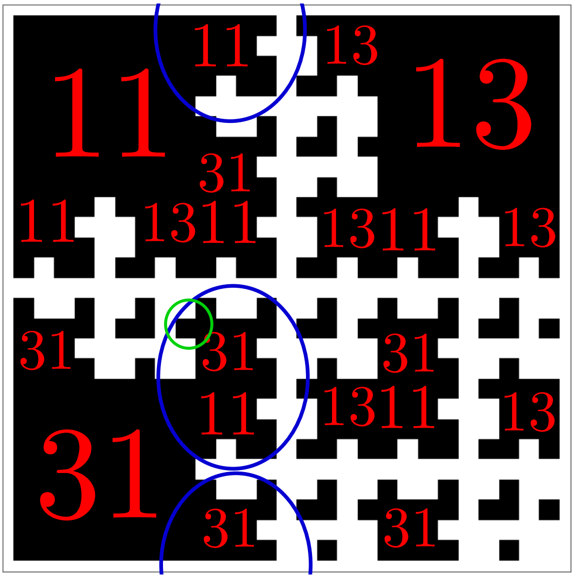


Figure 2.1. The antenna carpet, formed by removing the 11, 13, and 31 boxes, with blue circles indicating substructures of the same type, $\{11,31\}$, with different growth patterns. A green circle indicates growth one substructure has and the other substructure does not.

each box at each scale. We denote substructures according to a set containing the types of initial boxes a substructure grows off of as the carpet iterates. These initial boxes are labeled by where they appear in the 3×3 grid at the level of detail they appear, according to the standard labeling of a 3×3 matrix. Any sections of the carpet connected via cardinal steps at a finite iteration are considered part of the same substructure.

Example 2.1. See Figure 2.3 to see how removed boxes are named.

Definition 2.2. An isolated substructure is a substructure that is not connected to any larger substructure at any finite iteration of the carpet. Two sections of removal are connected if there is a path cardinal direction path from one section to the other in some finite iteration of the carpet.

Example 2.2. Look at Figure 2.4 to see isolated substructures of type $\{12\}$ in red and of type $\{21,23\}$ in blue. Remember opposite edges are identified.

Definition 2.3. An edge substructure is a substructure that is connected to a larger substructure at some finite iteration of the carpet.

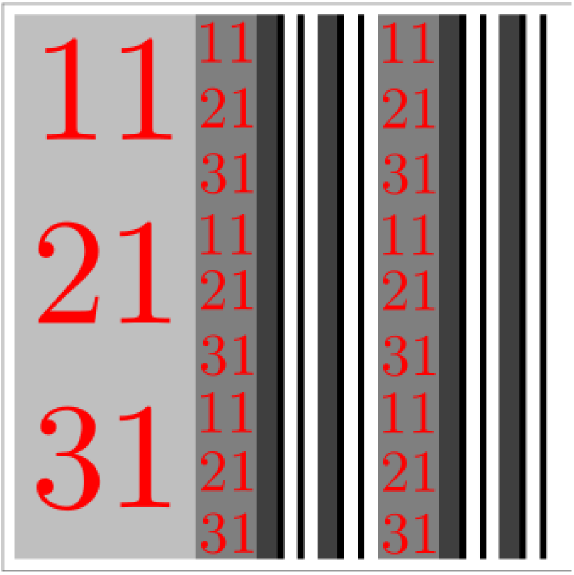


Figure 2.2. The carpet formed by removing the 11, 21, and 31 boxes. Box types are labeled in red showing how this carpet has connected components where the collection of largest boxes has repeated elements.

Definition 2.4. A scale d substructure is a substructure that has largest boxes of length d. Equivalently, a substructure that grew off of initial boxes of length d. A substructure of type p and scale d may be referred to as p_d.

Example 2.3. See Figure 3.2 to see $\{13\}_{\frac{1}{3}}$ substructures in light gray, $\{13\}_{\frac{1}{9}}$ in a slightly darker gray and so on.

The goal of this counting method is to count the number of isolated substructures for each type and scale.

Definition 2.5. $S_p(n)$ is defined as the number of isolated scale $\frac{1}{3^n}$ and type p substructures in a carpet. Then $S_p(n)$ is the number of isolated $p_{\frac{1}{3^n}}$ substructures in a carpet. This assumes the standard carpet construction on $[0, 1]^2 \subseteq \mathbb{R}^2$ with $\frac{1}{3}$ scaling.

From these $S_p(n)$ values, the Distance Zeta calculations can be completed, see [2]. In order to calculate $S_p(n)$ we will also need to keep track of the edge substructures. This way we can subtract all the boxes absorbed by edge substructures from the total count of boxes. Then we can solve for the number of isolated substructures from the leftover boxes.

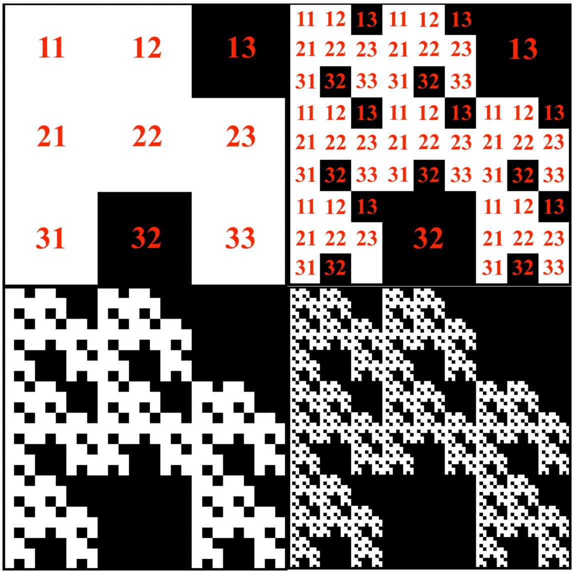


Figure 2.3. Sus Carpet with matrix grid at $n = 1, 2, 3, 4$, removed box types only labeled at $n = 1$ and $n = 2$ for readability

Definition 2.6. We define a representative set of all the scale t and type q substructures on a scale s and type p substructure to be $q_t \sigma p_s$ for some specific p_s structure. This is read q_t's on p_s's. This will allow us to refer to the behavior of scale t and type q substructures lying on a scale s and type p substructure. Note $|q_t \sigma p_s|$ is the number of scale t and type q substructures on a scale s and type p substructure.

Definition 2.7. $E_{p,q}(n)$ is the number of type q and scale $\frac{1}{3^n}$ edge substructures that appear on a scale 1 substructure of type p for $n \geq 1$. Then $E_{p,q}(n) = q_{\frac{1}{3^n}} \sigma p_1$.

Lemma 2.1. *Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let p, q be substructure types. Suppose that $|q_{\frac{1}{3^t}} \sigma p_{\frac{1}{3^s}}| = n$ for some $n \in N$. Then*

$$|q_{\frac{1}{3^t}} \sigma p_{\frac{1}{3^s}}| = |q_{\frac{3^s}{3^t}} \sigma p_1| = E_{p,q}(t - s)$$

Proof. Let p, q be substructure types. Suppose that $|q_{\frac{1}{3^t}} \sigma p_{\frac{1}{3^s}}| = n$. The construction is by $\frac{1}{3}$ scaling. Imagine scaling the whole carpet by 3^s. Then our $p_{\frac{1}{3^s}}$ structure would be scale 1 and our $q_{\frac{1}{3^s}}$ structure scale $\frac{3^s}{3^t}$. We conclude that

$$|q_{\frac{1}{3^t}} \sigma p_{\frac{1}{3^s}}| = |q_{\frac{3^s}{3^t}} \sigma p_1| = E_{p,q}(t - s).$$

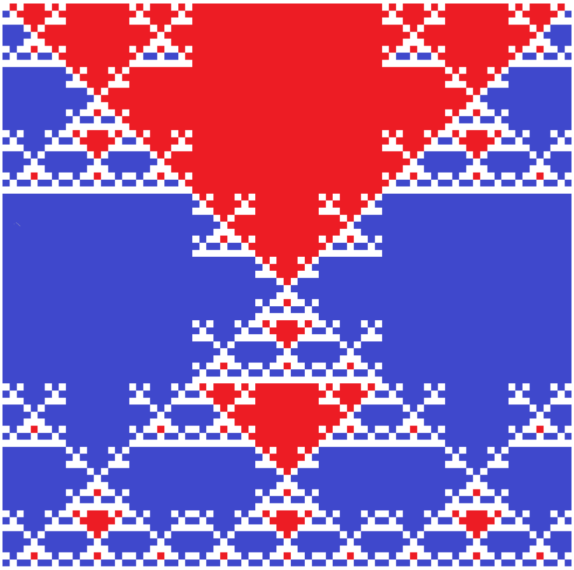


Figure 2.4. Creeper carpet iteration 4 where independent $\{12\}$ structures are colored red and independent $\{21, 23\}$ structures are labeled blue

■

In the following sections we will go through some examples to help depict what our $S_p(n)$ and $E_{p,q}(n)$ sequences look like.

3. Counting on Carpets with One Type of Substructure

3.1. Counting Single Type Isolated Substructures

We will start out "easy" by counting the carpet formed by removing the 13 box, which we called the Cornercopia:

This carpet only has one substructure type, but requires recurrence relations to come up with a formula for counting our edge substructures with $E_{p,q}(n)$. We leave this task for Section 3.2. Let $\mathcal{T} = \{\{13\}\}$ be the set of all possible substructures that form in this carpet.

Figure 3.2 shows removed boxes of scale $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, and $\frac{1}{81}$ being shaded from light grey to dark grey. The non-removed sections are shown in white. Additionally, each of the boxes from the first three iterations are labeled according to their relative position at removal.

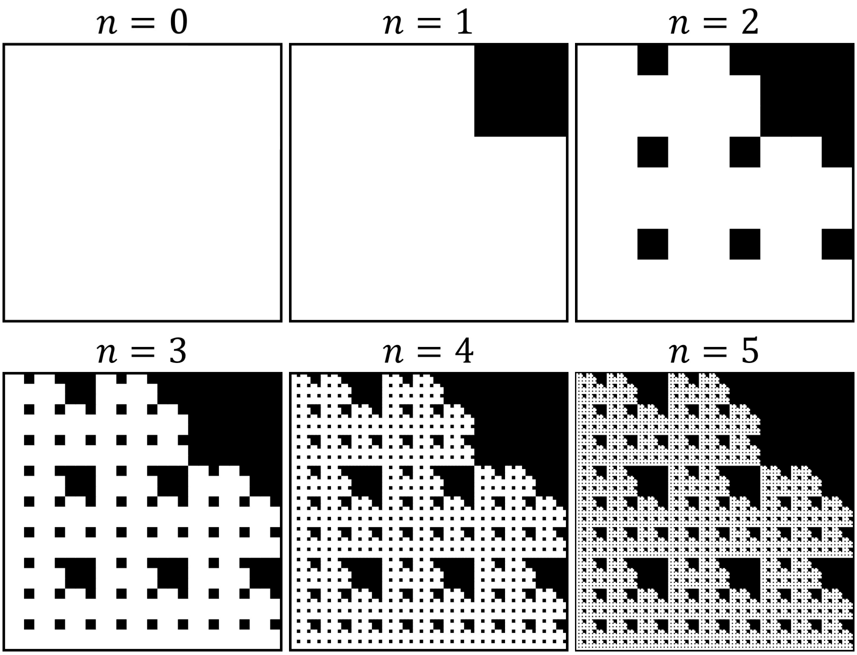


Figure 3.1. Depicts the iterations 0 – 5 of the Cornercopia.

For now, we count by inspection. Refer to Figure 3.2 to see that the corner carpet has 8 boxes of type 13 at scale $\frac{1}{9}$. This is our total number of boxes.

Also referring to Figure 3.2 we see that $|\{13\}_{\frac{1}{9}}\sigma\{13\}_{\frac{1}{3}}| = 2$. Then by Lemma 2.1 $E_{\{13\},\{13\}}(1) = 2$. Furthermore, see that the only boxes being absorbed by a larger structure are those that we just counted. Finally notice it only takes one box of type 13 to spawn a $\{13\}$ substructure. We take the total number of boxes, subtract those being absorbed and divide by the number of boxes it takes to make a $\{13\}$ substructure to get

$$S_{\{13\}}(2) = \frac{8 - E_{\{13\},\{13\}}(1)}{1} = 6.$$

We compute $S_{\{13\}}(3)$ with the same method. There are 64 13 boxes that could contribute. This time, we have to subtract off all boxes of the right type attached to a scale $\frac{1}{3}$ or $\frac{1}{9}$ substructure. For each $\{13\}_{\frac{1}{3}}$ structure we must subtract off $|\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{3}}|$ boxes. There are $S_{\{13\}}(1)$ such structures. For each $\{13\}_{\frac{1}{9}}$ structure we must subtract off $|\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{9}}|$ boxes. There are $S_{\{13\}}(2)$ such structures. Inspect Figure 3.2 to see the following and apply Lemma 2.1 to get

$$E_{\{13,13\}}(2) = |\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{3}}| = 6$$

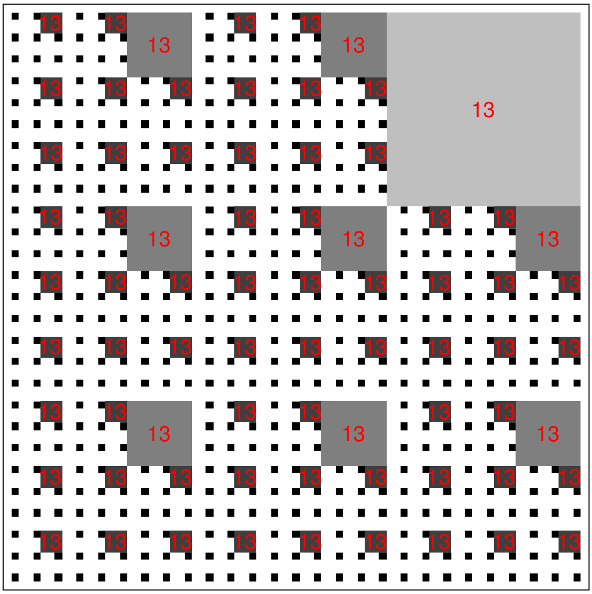


Figure 3.2. An iteration 5 corner carpet. Boxes are labeled according to type and shades of gray corresponding to scale, the darker the smaller.

$$E_{\{13,13\}}(2) = |\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{9}}| = 2.$$

It follows that

$$\begin{aligned} S_{\{13\}}(3) &= \frac{64 - S_{\{13\}}(1)E_{\{13\},\{13\}}(2) - S_{\{13\}}(2)E_{\{13\},\{13\}}(1)}{1} \\ &= \frac{64 - 1 \cdot 6 - 6 \cdot 2}{1} = 46. \end{aligned}$$

3.2. Counting Single Type Edge Substructures

Now that we have had a look at counting independent substructures, we can see that we need to be able to count all the smaller edge structures attached to each structure. We will start with counting single edge substructures on the corner carpet.

Look again at Figure 3.2. We will be examine the only $\{13\}_{\frac{1}{3}}$ independent substructure and call it our parent structure. We will split edge substructures into two groups: those directly attached to their parent structure and those that are not.

Definition 3.1. A substructure a is directly attached to a substructure b if b has a larger scale than a and at least of of b's initial boxes is adjacent to to some box part of a.

Definition 3.2. We define a representative set of all the scale t and type q substructures directly attached to a scale s and type p substructure to be $q_t \phi p_s$ for some specific p_s structure. This is read q_t's directly on p_s's. Note $|q_t \phi p_s|$ is the number of scale t and type q substructures directly attached to a scale s and type p substructure.

Example 3.1. Referring to Figure 3.2 we can see $|\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{3}}| = 6$ and $|\{13\}_{\frac{1}{27}} \phi \{13\}_{\frac{1}{3}}| = 2$.

Lemma 3.1. *Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let p, q be substructure types. Then*

$$|q_{\frac{1}{3^{n+1}}} \phi p_{\frac{1}{3^n}}| = |q_{\frac{1}{3^{n+1}}} \sigma p_{\frac{1}{3^n}}|$$

for all $n \in \mathbb{N}$.

Proof. Suppose s, a $p_{\frac{1}{3^{n+1}}}$ structure, is connected a $p_{\frac{1}{3^n}}$. It is either directly connected or not. If it is not directly connected it must be connected to some r_l structure where $l > n+1$. l Structures of the same size cannot be connected. If they are they must be the same structure. Similarly, the r_l structure is connected to a $p_{\frac{1}{3^n}}$ structure so $l < n$. There is no such natural number. Then s must be directly connected to $p_{\frac{1}{3^n}}$. This implies that $|q_{\frac{1}{3^{n+1}}} \phi p_{\frac{1}{3^n}}| = |q_{\frac{1}{3^{n+1}}} \sigma p_{\frac{1}{3^n}}|$ as desired. ■

We resume counting. Further examining Figure 3.2 we find

$$|\{13\}_{\frac{1}{9}} \phi \{13\}_{\frac{1}{3}}| = 2$$
$$|\{13\}_{\frac{1}{27}} \phi \{13\}_{\frac{1}{3}}| = 2$$
$$|\{13\}_{\frac{1}{81}} \phi \{13\}_{\frac{1}{3}}| = 4$$
$$|\{13\}_{\frac{1}{243}} \phi \{13\}_{\frac{1}{3}}| = 8.$$

For $n \geq 2$, it appears this pattern continues and we conjecture

$$|\{13\}_{\frac{1}{3 \cdot 3^n}} \phi \{13\}_{\frac{1}{3}}| = 2 \cdot 2^{n-2}.$$

We would now like to apply this knowledge and find a recurrence for $|\{13\}_{\frac{1}{3 \cdot 3^n}} \sigma \{13\}_{\frac{1}{3}}|$, equivalently $E_{\{13\},\{13\}}(n+1)$.

When considering $|\{13\}_{\frac{1}{3 \cdot 3^n}} \sigma \{13\}_{\frac{1}{3}}|$ notice that the count can be split into parts: those boxes attached directly to a $\frac{1}{3}$ structure and those indirectly attached. Observe that

$$E_{\{13\},\{13\}}(2) = |\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{3}}| = |\{13\}_{\frac{1}{9}} \sigma \{13\}_{\frac{1}{3}}||\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{9}}| + |\{13\}_{\frac{1}{9}} \phi \{13\}_{\frac{1}{3}}|$$

$$
\begin{aligned}
&= 4 + 2 = 6 \\
E_{\{13\},\{13\}}(3)|\{13\}_{\frac{1}{81}}\sigma\{13\}_{\frac{1}{3}}| &= |\{13\}_{\frac{1}{9}}\sigma\{13\}_{\frac{1}{3}}||\{13\}_{\frac{1}{81}}\sigma\{13\}_{\frac{1}{9}}| \\
&\quad + |\{13\}_{\frac{1}{27}}\phi\{13\}_{\frac{1}{3}}||\{13\}_{\frac{1}{81}}\sigma\{13\}_{\frac{1}{27}}| \\
&\quad + |\{13\}_{\frac{1}{81}}\phi\{13\}_{\frac{1}{3}}| \\
&= 12 + 4 + 4 = 20.
\end{aligned}
$$

Continuing this pattern, the general formula is as follows:

$$
\begin{aligned}
E_{\{13\},\{13\}}(n) &= |\{13\}_{\frac{1}{3\cdot 3^n}}\sigma\{13\}_{\frac{1}{3}}| = |\{13\}_{\frac{1}{3\cdot 3^n}}\phi\{13\}_{\frac{1}{3}}| \\
&\quad + \sum_{k=1}^{n-1} |\{13\}_{\frac{1}{3\cdot 3^k}}\phi\{13\}_{\frac{1}{3}}||\{13\}_{\frac{1}{3\cdot 3^n}}\sigma\{13\}_{\frac{1}{3\cdot 3^k}}|.
\end{aligned}
$$

Applying Lemma 2.1 and Lemma 3.1 we find for $n \geq 2$ that

$$
\begin{aligned}
E_{\{13\},\{13\}}(n) &= E_{\{13\},\{13\}}(1)E_{\{13\},\{13\}}(n-1) + 2\cdot 2^{n-2} \\
&\quad + \sum_{k=2}^{n-1} 2\cdot 2^{k-2}E_{\{13\},\{13\}}(n-k).
\end{aligned}
$$

We keep the 2 separate because each of the $\{13\}_{\frac{1}{27}}\phi\{13\}_{\frac{1}{3}}$ will correspond to a geometric series of substructures. This is left for Section 5. Unfortunately, this is the main pattern that fails in $m \times n$ carpets, and more initial conditions must be considered.

Definition 3.3. Let $EN_{p,q}$ denote $|q_{\frac{1}{9}}\phi p_1|$.

Example 3.2. Look at Figure 3.3 to see an example of $EN_{\{13\},\{13\}}$

Now we rewrite our equation from above:

$$
\begin{aligned}
E_{\{13\},\{13\}}(n) &= E_{\{13\},\{13\}}(1)E_{\{13\},\{13\}}(n-1) \\
&\quad + EN_{\{13\},\{13\}}2^{n-2} + \sum_{k=2}^{n-1} EN_{\{13\},\{13\}}2^{k-2}E_{\{13\},\{13\}}(n-k).
\end{aligned}
$$

4. Counting on Carpets with Multiple Types of Substructures

4.1. Counting Multiple Type Isolated Substructures

Now we would like to count carpet with multiple substructure types. Let us take a look at the creeper carpet. It is formed by removing the 12, 21, and 23 boxes. There are 4 substructure types. So our set of substructures is $\mathcal{T} = \{\{12\}, \{21\}, \{23\}, \{21, 23\}\}$.

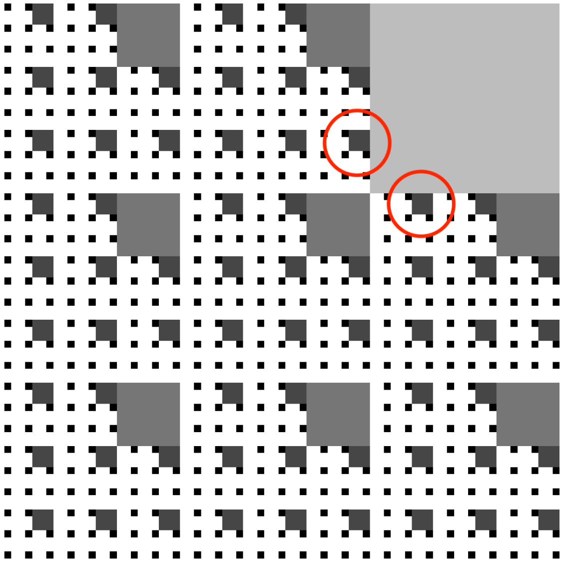


Figure 3.3. A iteration five corner carpet with $EN_{\{13\},\{13\}}$ represented by two red circles marking an example.

Remember that the left and right edges as well top and bottom edges are identified; by inspection:

$$S_{\{21,23\}}(1) = 1$$
$$S_{\{21,23\}}(2) = 4$$
$$S_{\{21\}}(n) = S_{\{23\}}(n) = 0 \text{ for } n \geq 1.$$

Whenever a substructure's set of initial boxes is a subset of another substructure's initial boxes we should expect there to be no isolated substructures of that type.

We can compute $S_{\{21,23\}}(2)$ with recurrences. We can see that there are $3 \cdot 6^1$ boxes that could contribute to a $\{21, 23\}_{\frac{1}{9}}$ structure. However there is a stark difference from the corner-carpet as potential boxes can be attached to the edge of 4 different types of substructures.

When counting boxes contributing to some type of independent substructure we must subtract boxes from our total count corresponding to (the number of initial boxes the counted independent structure shares with each type of edge substructure) times (the number of times each edge substructure occurs on its parent structure at the proper scale).

Since boxes of $\frac{1}{9}$ scale can only be absorbed by substructures of scale $\frac{1}{3}$ we can restrict to looking at substructures of the form $p_{\frac{1}{3}}$ where $p \in \mathcal{T}$. Note that by inspection

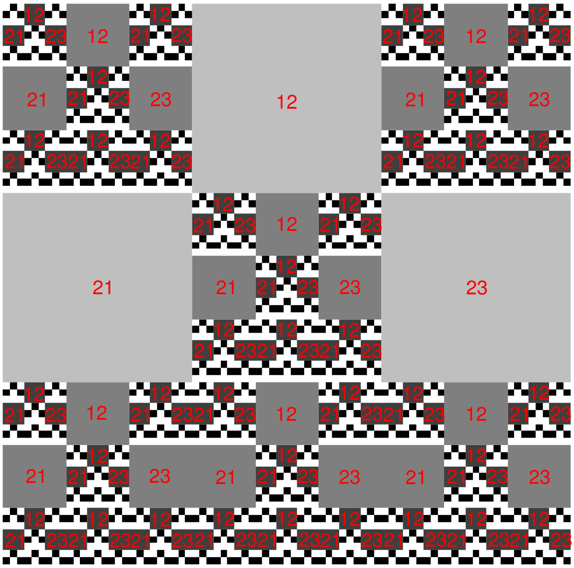


Figure 4.1. An iteration 5 creeper carpet. Boxes are labeled according to type and shades of gray corresponding to scale, with darker shades used for smaller scale.

of Figure 4.1 and Lemma 2.1 that

$$
\begin{aligned}
|\{12\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| &= E_{\{12\},\{12\}}(1) = 1\\
|\{21\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| &= E_{\{12\},\{21\}}(1) = 1\\
|\{23\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| &= E_{\{12\},\{23\}}(1) = 1\\
|\{21,23\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| &= E_{\{12\},\{21,23\}}(1) = 0
\end{aligned}
$$

$$
\begin{aligned}
|\{12\}_{\frac{1}{9}}\sigma\{21\}_{\frac{1}{3}}| &= E_{\{21\},\{12\}}(1) = 1\\
|\{21\}_{\frac{1}{9}}\sigma\{21\}_{\frac{1}{3}}| &= E_{\{21\},\{21\}}(1) = 1\\
|\{23\}_{\frac{1}{9}}\sigma\{21\}_{\frac{1}{3}}| &= E_{\{21\},\{23\}}(1) = 0\\
|\{21,23\}_{\frac{1}{9}}\sigma\{21\}_{\frac{1}{3}}| &= E_{\{21\},\{21,23\}}(1) = 0
\end{aligned}
$$

$$
\begin{aligned}
|\{12\}_{\frac{1}{9}}\sigma\{23\}_{\frac{1}{3}}| &= E_{\{23\},\{12\}}(1) = 1\\
|\{21\}_{\frac{1}{9}}\sigma\{23\}_{\frac{1}{3}}| &= E_{\{23\},\{21\}}(1) = 0\\
|\{23\}_{\frac{1}{9}}\sigma\{23\}_{\frac{1}{3}}| &= E_{\{23\},\{23\}}(1) = 1
\end{aligned}
$$

$$|\{21,23\}_{\frac{1}{9}}\sigma\{23\}_{\frac{1}{3}}| = E_{\{23\},\{21,23\}}(1) = 0$$

$$|\{12\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = E_{\{21,23\},\{12\}}(1) = 2$$
$$|\{21\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = E_{\{21,23\},\{21\}}(1) = 1$$
$$|\{23\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = E_{\{21,23\},\{23\}}(1) = 1$$
$$|\{21,23\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = E_{\{21,23\},\{21,23\}}(1) = 0.$$

Suppose we have boxes of scale t and we want to find the boxes absorbed by a single q_t structure that is connected to a p_s structure. Then for each type q_t structure we must remove the number of boxes q and p share. This amounts to computing $|p \cap q| \cdot |q_t \sigma p_s|$. Then to get $S_{\{21,23\}}(2)$ we consider the possible boxes that could contribute. We loop through all the types of structures of each scale whose boxes could appear and subtract off boxes at each applicable scale. We do this by further looping through all the types of substructures that could appear on each type of structure, then divide the resulting total by the number of boxes it takes to create the substructure we are looking at. See that

$$\begin{aligned} S_{\{21,23\}}(2) =& \frac{2\cdot 6^1 - \sum_{p,q\in\mathcal{T}} S_p(1)|q\cap\{21,23\}||q_{\frac{1}{9}}\sigma p_{\frac{1}{3}}|}{|\{21,23\}|} \\ =& \frac{2\cdot 6^1 - \sum_{p,q\in\mathcal{T}} S_p(1)|q\cap\{21,23\}|E_{p,q}(1)}{|\{21,23\}|} \\ =& \frac{12 - 1[0(1)+1(1)+1(1)+2(0)] - 0[0(1)+1(1)+1(0)+2(0)]}{2} \\ & \frac{-0[0(1)+1(0)+1(1)+2(0)] - 1[0(2)+1(1)+1(1)+2(0)]}{2} = 4. \end{aligned}$$

Here we take a closer look at the first summand:

$$\begin{aligned} &\sum_{q\in\mathcal{T}} S_{\{13\}}(1)|q\cap\{21,23\}||q_{\frac{1}{9}}\sigma\{13\}_{\frac{1}{3}}| = \\ &S_{\{12\}}(1)\Big[|\{12\}\cap\{21,23\}||\{12\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| \\ &+|\{21\}\cap\{21,23\}||\{21\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| \\ &+|\{23\}\cap\{21,23\}||\{23\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}| \\ &+|\{21,23\}\cap\{21,23\}||\{21,23\}_{\frac{1}{9}}\sigma\{12\}_{\frac{1}{3}}|\Big]. \end{aligned}$$

We now compute $S_{\{21,23\}}(3)$. We need to consider all the boxes of size $\frac{1}{27}$. Then for each $p, q \in \mathcal{T}$ we must subtract off $|q_{\frac{1}{27}}\sigma p_{\frac{1}{3}}|$ and $|q_{\frac{1}{27}}\sigma p_{\frac{1}{9}}|$. These values correspond to $E_{p,q}(1)$ and $E_{p,q}(2)$. We find that

$$S_{\{21,23\}}(3) = \frac{2\cdot 6^1 - \sum_{p,q\in\mathcal{T}}\sum_{i=1}^{2} S_p(i)|q\cap\{21,23\}||q_{\frac{1}{27}}\sigma p_{\frac{1}{3^i}}|}{|\{21,23\}|}$$
$$= \frac{2\cdot 6^1 - \sum_{p,q\in\mathcal{T}}\sum_{i=1}^{2} S_p(i)|q\cap\{21,23\}|E_{p,q}(3-i)}{|\{21,23\}|}.$$

4.2. Counting Multiple Type Edge Substructures

We will now switch back to looking at the creeper carpet.

Lets try to count $\{12\}_{\frac{1}{3\cdot 3^n}}\sigma\{21,23\}_{\frac{1}{3}}$, aka $E_{\{21,23\},\{12\}}(n)$. Notice that $\{12\}$, $\{21\}$, $\{21\}$, and $\{21,23\}$ substructures could appear on the edge of our initial structure. Looking at 4.1 we can see that

$$|\{12\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = 2$$
$$|\{21\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = 1$$
$$|\{23\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = 1$$
$$|\{21,23\}_{\frac{1}{9}}\sigma\{21,23\}_{\frac{1}{3}}| = 0.$$

It follows from Lemma 2.1 that

$$E_{\{21,23\},\{12\}}(1) = 2$$
$$E_{\{21,23\},\{21\}}(1) = 1$$
$$E_{\{21,23\},\{23\}}(1) = 1$$
$$E_{\{21,23\},\{21,23\}}(1) = 0.$$

We will also need $EN_{\{21,23\},\{12\}}$, $EN_{\{21\},\{21,23\}}$, $EN_{\{23\},\{21,23\}}$, and $EN_{\{21,23\},\{21,23\}}$. See that

$$EN_{\{21,23\},\{12\}} = 4$$
$$EN_{\{21,23\},\{21\}} = 2$$
$$EN_{\{21,23\},\{23\}} = 2$$
$$EN_{\{21,23\},\{0\}} = 0.$$

Building off the last section we can make a similar recurrence for $E_{\{21,23\},\{12\}}$ but now we must consider each type of structure possible in the carpet. It should be as follows:

$$E_{\{21,23\},\{12\}}(n) = EN_{\{21,23\},\{12\}}2^{n-2}$$
$$+ E_{\{12\},\{12\}}(1)E_{\{12\},\{12\}}(n-1) + \sum_{k=2}^{n-1} EN_{\{12\},\{12\}}2^{k-2}E_{\{12\},\{21,23\}}(n-k)$$

$$+ E_{\{12\},\{21\}}(1)E_{\{21\},\{21,23\}}(n-1) + \sum_{k=2}^{n-1} EN_{\{12\},\{21\}}2^{k-2}E_{\{21\},\{21,23\}}(n-k)$$
$$+ E_{\{12\},\{23\}}(1)E_{\{23\},\{21,23\}}(n-1) + \sum_{k=2}^{n-1} EN_{\{12\},\{21\}}2^{k-2}E_{\{23\},\{21,23\}}(n-k).$$

The recurrence will mostly stay the same for other substructures except for the last subscript of $\{12\}$ being switched out depending on which edge substructure is being counted.

5. Counting Edge Substructures

Given a carpet C and a set of possible substructure types $\mathcal{T}$ with $p, q \in \mathcal{T}$ we wish to find $E_{p,q}(n)$ for $n \in \mathbb{N}$. As explored in Section 4.2, We will break our count into two pieces.

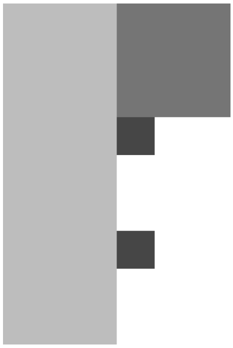
Since we want to find $E_{p,q}(n)$ we consider p_1. We can break the substructure attached to p_1 into those directly attached to p_1 and those that are indirectly attached. We will first count structures that are directly attached; their sum will resemble a geometric series.

Lemma 5.1. *Let $\mathcal{T}$ be the set of substructure types for some carpet. Let $p, q \in \mathcal{T}$ then*

$$|q_{\frac{1}{3^n}} \phi p_1| = 2^{n-2}EN_{p,q} \text{ for } n \geq 2.$$

Proof. Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure types for C and $p, q \in \mathcal{T}$. There a finite number of ways a $q_{\frac{1}{3^n}}$ structure can attach to a structure p_1. There are six cases to analyze as shown in Figure 5.1.

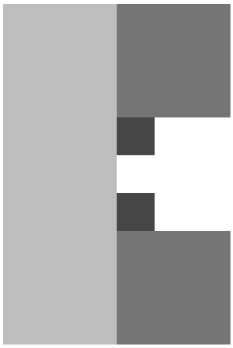
Case (a): Suppose the attach point is on a corner. Consider a general $\frac{1}{3^k}$ length segment of the remaining edge of a substructure at the k-th iteration from Figure 5.1a. In the same figure, the boxes added in the $(k+1)$-th iteration are shown in a darker gray. The $(k+1)$-th iteration splits the k-th iteration into two sections of $\frac{1}{3^{k+1}}$ scale. There is one box in each section. In one of the sections the box gets absorbed and, in the other, the box starts a new substructure. Let the number of absorbed sections be represented by a_i and the number of sections that do not get absorbed be b_i on that remaining edge, where i represents the i-th iteration past the k-th. Then $a_1 = 1$, $b_1 = 1$, $a_{i+1} = a_i + b_i$ and $b_{i+1} = a_i + b_i$, since the total number of sections at the i-th iteration past the k-th is $2(a_i + b_i)$.



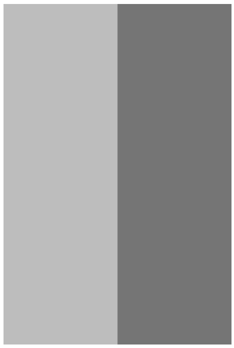
(a) One Corner Removed

(b) One Middle Removed

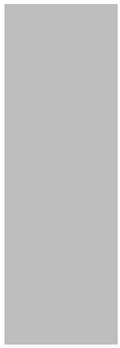
(c) One Corner One Middle Removed



(d) Two Corners Removed



(e) Three Boxes removed



(f) Zero Boxes Removed

Figure 5.1. A collection of figures showing how sections of removal may grow on the edge of a substructure.

Solving this recurrence and verifying with induction gives $a_i = 2^{i-1}$ and $b_i = 2^{i-1}$ for $i \geq 1$. So within this section there are 2^{i-1} substructures of scale $\frac{1}{3^{k+i}}$ where $i \geq 1$.

Case (b): Suppose the attach point is a middle box. Consider a general $\frac{1}{3^k}$ length segment of the remaining edge of a q substructure at the k-th iteration from Figure 5.1b. In the same figure the boxes added in the $(k+1)$-th iteration are shown in a darker gray. We can use a similar process to the last case, with the difference that no boxes in any section get absorbed. There are 2^i substructures of scale $\frac{1}{3^{k+i}}$ where $i \geq 1$. Or in each section of the two sections there are 2^{i-1} substructures of scale $\frac{1}{3^{k+i}}$.

Case (c): Suppose the attach point is one middle on corner box. Notice that each new iteration of boxes gets completely absorbed by the last. So there are zero additional substructures at the $(k+i)$-th iteration.

Case (d): The attach point is two corner boxes. Notice that each new iteration of boxes gets completely absorbed. So there are zero additional substructures at the $(k+i)$-th iteration.

Case (e): Suppose the attach point is all three boxes. Notice that each new iteration of boxes gets completely covers the edge. So there are zero additional substructures at the $(k+i)$-th iteration.

Case (f): Suppose there is no attachment. Notice that each new iteration places zero boxes on this kind of edge. So there are zero additional substructures at the $(k+i)$-th iteration.

Now notice that each $|q_{\frac{1}{9}}\phi p_1| = EN_{p,q}$ represents an edge segment of length $\frac{1}{3}$ as described in the above cases. Since each segment is length $\frac{1}{3}$ then each segment contains exactly 2^{i-1} boxes of scale $\frac{1}{3^{i+1}}$. Then $|q_{\frac{1}{3\cdot 3^n}}\phi p_1| = 2^{n-1}EN_{p,q}$. Reindexing, we get $|q_{\frac{1}{3^n}}\phi p_1| = 2^{n-2}EN_{p,q}$ for $n \geq 2$ as desired. ∎

Theorem 5.2. *Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure types for C and $p, q \in \mathcal{T}$ then for $n \geq 1$*

$$E_{p,q}(n) = 2^{n-2}EN_{p,q} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)E_{a,q}(n-1) + \sum_{k=2}^{n-1} 2^{k-2}EN_{p,a}E_{a,q}(n-k)\right).$$

Proof. Let $\mathcal{T}$ be the types of substructures for some carpet. Then let $p, q \in \mathcal{T}$. Consider $q_{\frac{1}{3^n}}\sigma p_1$ for $n \in \mathbb{N}$. Each $s \in q_{\frac{1}{3^n}}\sigma p_1$ must either lie directly on p_1 or lie on a substructure that directly lies on p_1. In Equation 5.1 the first terms counts the number of $q_{\frac{1}{3^n}}$ substructures directly attached to p_1. The double summation counts the $q_{\frac{1}{3^n}}$ indirectly attached to p_1 by looping through all the substructures directly connected to p_1 and all the substructures on them with boxes of the right scale. Within the double summation in Equation 5.1, $|a_{\frac{1}{3^k}}\phi p_1||q_{\frac{1}{3^n}}\sigma a_{\frac{1}{3^k}}|$ counts the number of substructures of type $a_{\frac{1}{3^k}}$ directly lying on a p_1 structure times the number of desired $q_{\frac{1}{3n}}$ structures lying on an $a_{\frac{1}{3^k}}$ structure. The sum ranges from $k = 1$ to $n - 1$ as a $p_{\frac{1}{3^n}}$ structure could lie directly on a $a_{\frac{1}{3^k}}$ structure directly lying on p_1 for $1 \leq k \leq n - 1 \in \mathbb{N}$.

$$\begin{aligned} E_{p,q}(n) =& |q_{\frac{1}{3^n}} \sigma p_1| = |q_{\frac{1}{3^n}} \phi p_1| + \sum_{a \in \mathcal{T}} \sum_{k=1}^{n-1} |a_{\frac{1}{3^k}} \phi p_1||q_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}| \\ =& |q_{\frac{1}{3^n}} \phi p_1| + \sum_{a \in \mathcal{T}} \left(|a_{\frac{1}{3}} \phi p_1||q_{\frac{1}{3^n}} \sigma a_{\frac{1}{3}}| + \sum_{k=2}^{n-1} |a_{\frac{1}{3^k}} \phi p_1||q_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}| \right) \\ =& |q_{\frac{1}{3^n}} \phi p_1| + \sum_{a \in \mathcal{T}} \left(|a_{\frac{1}{3}} \sigma p_1||q_{\frac{1}{3^n}} \sigma a_{\frac{1}{3}}| + \sum_{k=2}^{n-1} |a_{\frac{1}{3^k}} \phi p_1||q_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}| \right) \\ =& 2^{n-2} EN_{p,q} + \sum_{a \in \mathcal{T}} \left(E_{p,a}(1) E_{a,q}(n-1) + \sum_{k=2}^{n-1} 2^{k-2} EN_{p,a} E_{a,q}(n-k) \right). \end{aligned} \tag{5.1}$$

by Lemma 2.1, Lemma 5.1, and Lemma 3.1. We work under the convention that $\sum_{k=m}^{n} w_k$ is zero if $n < m$. Then $n = 2$ will imply the second term inside the sum is zero. ■

6. Counting Isolated Substructures

We want to get a recurrence for $S_p(n)$ for $n \geq 2$, which is the number of $p_{\frac{1}{3^n}}$ independent substructures. To do so, we find the total number of boxes that could contribute to the desired substructure of the proper scale, then we subtract off all the boxes that are part of an edge substructure, and finally we divide the quantity by the number of boxes it takes to create the desired substructure. The resulting number should be the number of said substructures.

Theorem 6.1. *Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure types for C, $p, q \in \mathcal{T}$, and r the number of boxes initially removed, then for $n \geq 2$*

$$S_p(n) = \frac{|p|(9-r)^{n-1} - \sum_{a,b \in \mathcal{T}} \sum_{k=1}^{n-1} |b \cap p| E_{a,b}(n-k) S_a(k)}{|p|}.$$

Proof. Let $\mathcal{T}$ be the set of substructure types for some carpet. Let $p, q \in \mathcal{T}$. Let r be the number of squares removed in the first iteration of C.

We can partition the boxes of type in p and scale $\frac{1}{3^n}$ in C into those part of an independent substructure or part of an edge substructure. If we get the total number of boxes and subtract off those part of an edge substructure we get the number of boxes part of an independent substructure. Since unique collections of boxes lead to unique structures we can divide by the number of boxes it takes to make a type p substructure to get the number of type $p_{\frac{1}{3^n}}$ independent substructures.

In each iteration of C each non-removed square is divided into 9 squares and r are removed. So if we let a_n be the number of non-removed squares at the n-th iteration then $a_{n+1} = (9-r)a_n$. Each non-removed square is split into 9 parts and r boxes are removed from each. Notice $a_0 = 1$ and so $a_n = (9-r)^n$. There are a_{n-1} non-removed boxes of scale $\frac{1}{3^{n-1}}$ in the $(n-1)$-th iteration, and in each of these non-removed sections there are r removed boxes, but only $|p|$ of them are the right type to contribute to $p_{\frac{1}{3^n}}$. The number of boxes that could contribute to a $p_{\frac{1}{3^n}}$ is then $|p|a_{n-1}$. So the total number of boxes that could contribute to a $p_{\frac{1}{3^n}}$ independent substructure is $|p|(9-r)^{n-1}$.

Suppose $a \in \mathcal{T}$ is some independent substructure type. We want to count all of the boxes absorbed by type a substructures of all scales and remove them from the total count of boxes. Remember we are looking for scale $\frac{1}{3^n}$ boxes. To do so we consider all the substructures on type a substructures and subtract off the correct scale boxes. Let $b \in \mathcal{T}$. For any $a_{\frac{1}{3^k}}$ there are $S_a(k)$ such structures. For each we must subtract off boxes for each $b_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}$. Since $|b_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}|$ counts substructures and not boxes we must actually subtract off $|p \cap b||b_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}|$ boxes for each $b_{\frac{1}{3^n}}$ lying on a $a_{\frac{1}{3^k}}$. By Lemma 2.1 we must then subtract off $|p \cap b|E_{a,b}(n-k)$ for each type of b and one specific $a_{\frac{1}{3^k}}$. When considering all $a_{\frac{1}{3^k}}$ structures we must subtract off $S_p(k)|p \cap b|E_{a,b}(n-k)$. Then to make sure we subtract off boxes from all structures in the carpet we must iterate a, b over $\mathcal{T}$ and $1 \le k \le n-1$. (This is all the structures larger than scale $\frac{1}{3^n}$ that could have scale $\frac{1}{3^n}$ boxes lying on them as children). Then for $n \ge 2$ we get

$$\sum_{a,b \in \mathcal{T}} \sum_{k=1}^{n-1} |b \cap p| E_{a,b}(n-k) S_a(k)$$

boxes are removed. For the boxes it takes to make a type p substructure it is clearly $|p|$.

Putting it all together we get

$$S_p(n) = \frac{|p|(9-r)^{n-1} - \sum_{a,b\in\mathcal{T}} \sum_{k=1}^{n-1} |b \cap p| E_{a,b}(n-k) S_a(k)}{|p|}.$$

as desired. ∎

7. Computing S_p on the Creeper Carpet

Note that the types of substructures for the creeper carpet is

$$\mathcal{T} = \{\{12\}, \{21\}, \{23\}, \{21, 23\}\}$$

$E_{p,q}(1)$	$q = \{12\}$	$q = \{21\}$	$q = \{23\}$	$q = \{21, 23\}$
$p = \{12\}$	1	1	1	0
$p = \{21\}$	1	1	0	0
$p = \{23\}$	1	0	1	0
$p = \{21, 23\}$	2	1	1	0

Table 7.1. A table of values for $E_{p,q}(1)$ in the creeper carpet

$EN_{p,q}$	$q = \{12\}$	$q = \{21\}$	$q = \{23\}$	$q = \{21, 23\}$
$p = \{12\}$	2	2	2	0
$p = \{21\}$	2	2	0	0
$p = \{23\}$	2	0	2	0
$p = \{21, 23\}$	4	2	2	0

Table 7.2. A table of values for $EN_{p,q}$ in the creeper carpet

In order to define our recurrences relations for $S_p(n)$ we draw values from Figure 7.1, Figure 7.2, and Theorem 5.2. Then we must solve our recurrence relations. In order to do this we will look at the ordinary generating functions for each.

Definition 7.1. Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure types for C. Let $p, q \in \mathcal{T}$. Then

$$e_{p,q}(x) = \sum_{n=1}^{\infty} E_{p,q}(n)x^n.$$

It will follow from Theorem 6.1 that

Corollary 7.0.1. *Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure types for C. Let $p, q \in \mathcal{T}$. Then*

$$\begin{aligned} e_{p,q}(x) =& E_{p,q}(1)x + EN_{p,q}\frac{x^2}{1-2x} \\ &+ \sum_{a \in \mathcal{T}} \left(E_{p,a}(1)e_{a,q}(x)x + EN_{p,a}\frac{x^2}{1-2x}e_{a,q}(x) \right) \end{aligned}$$

Proof. Using Theorem 5.2, Definition 7.1, and the Cauchy product we have

$$
\begin{aligned}
&e_{p,q}(x) - E_{p,q}(1)x = \\
=&\sum_{n\geq 2}\left(2^{n-2}EN_{p,q} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)E_{a,q}(n-1) + \sum_{k=2}^{n-1}2^{k-2}EN_{p,a}E_{a,q}(n-k)\right)\right)x^n \\
=&EN_{p,q}\sum_{n\geq 2}2^{n-2}x^n \\
&+\sum_{a\in\mathcal{T}}\left(E_{p,a}(1)\sum_{n\geq 2}E_{a,q}(n-1)x^n + \sum_{n\geq 2}\sum_{k=2}^{n-1}2^{k-2}EN_{p,a}E_{a,q}(n-k)x^n\right) \\
=&EN_{p,q}x^2\sum_{n\geq 2}2^{n-2}x^{n-2} \\
&+\sum_{a\in\mathcal{T}}\left(E_{p,a}(1)x\sum_{n\geq 2}E_{a,q}(n-1)x^{n-1} + \sum_{n\geq 3}\sum_{k=2}^{n-1}2^{k-2}EN_{p,a}E_{a,q}(n-k)x^n\right) \\
=&EN_{p,q}x^2\sum_{n\geq 0}2^{n}x^{n} \\
&+\sum_{a\in\mathcal{T}}\left(E_{p,a}(1)x\sum_{n\geq 1}E_{a,q}(n)x^{n} + x^3\sum_{n\geq 3}\sum_{k=2}^{n-1}2^{k-2}EN_{p,a}E_{a,q}(n-k)x^{n-3}\right) \\
=&EN_{p,q}\frac{x^2}{1-2x} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)e_{a,q}(x)x + x^3\sum_{n\geq 3}\sum_{k=0}^{n-3}2^{k}EN_{p,a}E_{a,q}(n-2-k)x^{n-3}\right) \\
=&EN_{p,q}\frac{x^2}{1-2x} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)e_{a,q}(x)x + x^3\sum_{n\geq 0}\sum_{k=0}^{n}2^{k}EN_{p,a}E_{a,q}(n+1-k)x^{n}\right) \\
=&EN_{p,q}\frac{x^2}{1-2x} \\
&+\sum_{a\in\mathcal{T}}\left(E_{p,a}(1)e_{a,q}(x)x + x^2EN_{p,a}\left(\sum_{n\geq 0}2^n x^n\right)\left(\sum_{n\geq 0}E_{a,q}(n+1)x^{n+1}\right)\right) \\
=&EN_{p,q}\frac{x^2}{1-2x} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)e_{a,q}(x)x + EN_{p,a}\frac{x^2}{1-2x}e_{a,q}(x)\right).
\end{aligned}
$$

■

Using the values from Table 7.1 and Corollary 7.0.1 ,we compute:

Definition 7.2. Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure

$e_{p,q}(x)$	$q = \{12\}$	$q = \{21\}$	$q = \{23\}$
$p = \{12\}$	$\frac{x^2-x}{-7x^2+6-x-1}$	$\frac{2x^2-x}{-7x^2+6-x-1}$	$\frac{2x^2-x}{-7x^2+6-x-1}$
$p = \{21\}$	$\frac{x(2x-1)}{-7x^2+6-x-1}$	$\frac{x\left(5x^2-5x+1\right)}{(3x-1)\left(-7x^2+6-x-1\right)}$	$\frac{x^2(2x-1)}{(3x-1)\left(7x^2-6+x+1\right)}$
$p = \{23\}$	$\frac{2x^2-x}{-7x^2+6-x-1}$	$\frac{x^2(2x-1)}{(3x-1)\left(7x^2-6x+1\right)}$	$\frac{5x^3-5x^2+x}{(3x-1)\left(-7x^2+6-x-1\right)}$
$p = \{21, 23\}$	$\frac{2x(2x-1)}{-7x^2+6-x-1}$	$\frac{(x-1)x}{-7x^2+6-x-1}$	$\frac{(x-1)x}{-7x^2+6-x-1}$

Table 7.3. A table holding values for $e_{p,q}(x)$. Note that $e_{p,\{21,23\}}(x) = 0$ for each $p \in \mathcal{T}$.

types for C. Let $p \in \mathcal{T}$. Then

$$s_p(x) = \sum_{n=1}^{\infty} S_p(n)x^n$$

Using Theorem 6.1 it will follow that

Corollary 7.0.2. *Let C be a generalized 3×3 Sierpiński carpet such that there is a bijection between initial boxes and substructure types. Additionally each substructure types contains at most one box of each type at each scale. Let $\mathcal{T}$ be the set of substructure types for C. Let $p \in \mathcal{T}$. Then*

$$s_p(x) = S_p(1)x + \frac{(9-r)x^2}{1-(9-r)x} - \sum_{a,b\in\mathcal{T}} \left(\frac{|b \cap p|}{|p|} e_{a,b}(x) s_a(x) \right).$$

Proof. Using Theorem 6.1, Definition 7.2, and the Cauchy product we have

$$s_p(x) - S_p(1) = \sum_{n\geq 2} \left(\frac{|p|(9-r)^{n-1} - \sum_{a,b\in\mathcal{T}} \sum_{k=1}^{n-1} |b \cap p| E_{a,b}(n-k) S_a(k)}{|p|} \right) x^n$$

$$= \sum_{n\geq 2} (9-r)^{n-1} x^n - \sum_{a,b\in\mathcal{T}} \left(\frac{|b \cap p|}{|p|} \sum_{n\geq 2} \sum_{k=1}^{n-1} E_{a,b}(n-k) S_a(k) x^n \right)$$

$$= (9-r)x^2 \sum_{n\geq 2} (9-r)^{n-2} x^{n-2} - \sum_{a,b\in\mathcal{T}} \left(\frac{|b \cap p|}{|p|} \sum_{n\geq 2} \sum_{k=0}^{n-2} E_{a,b}(n-1-k) S_a(1+k) x^n \right)$$

$$= \frac{(9-r)x^2}{1-(9-r)x} - \sum_{a,b\in\mathcal{T}} \left(\frac{|b \cap p|}{|p|} \sum_{n\geq 0} \sum_{k=0}^{n} E_{a,b}(n+1-k) S_a(1+k) x^{n+2} \right)$$

$$= \frac{(9-r)x^2}{1-(9-r)x} - \sum_{a,b\in\mathcal{T}} \left(\frac{|b \cap p|}{|p|} \left(\sum_{n\geq 0} E_{a,b}(n+1) x^{n+1} \right) \left(\sum_{n\geq 0} S_a(n+1) x^{n+1} \right) \right)$$

$$=\frac{(9-r)x^2}{1-(9-r)x} - \sum_{a,b\in\mathcal{T}} \left(\frac{|b\cap p|}{|p|} e_{a,b}(x) s_a(x) \right).$$

■

Using the values from Table 7.3 and Corollary 7.0.2, we compute:

$s_{\{12\}}(x)$	$s_{\{21\}}(x)$	$s_{\{23\}}(x)$	$s_{\{21,23\}}(x)$
$\frac{-5x^2+x}{12x^2-8x+1}$	0	0	$\frac{-4x^2+x}{12x^2-8x+1}$

Figure 7.1. A table of values for $s_p(n)$ in the creeper carpet.

Finally, we can use partial fraction decomposition to reduce these generating functions to rational functions, representing geometric series. After computing the coefficients with Mathematica we find for $n \geq 1$ that

$S_{\{12\}}(n)$	$S_{\{21\}}(n)$	$S_{\{23\}}(n)$	$S_{\{21,23\}}(n)$
$3 \cdot 2^{n-3} + 2^{n-3}3^{n-1}$	0	0	$2^{n-2} + 2^{n-2}3^{n-1}$

Figure 7.2. A table of values for $S_p(n)$ in the creeper carpet.

We have finally acquired $S_p(n)$ for each $p \in \mathcal{T}$ as desired.

8. Further Notes

The current counting method relies on uniqueness of initial boxes. We believe expanding the number of types allowed and using similar methods could solve the counting problem for carpets where initial boxes do not uniquely determine long term behavior and $m \times n$ carpets. The idea is this: keep track of the number and types of boxes growing on a substructure at the 0th, 1st, 2nd, and 3rd smaller relative scales (initial boxes correspond to the 0th level). The 1st level must be tracked manually since it is special because edge substructures can wrap around corners at this scale. The second level is needed to count the EN values. Then the 3rd level is need to calculate the common ratio for the geometric series. The geometry of 3×3 carpets limits this to 2, but $m \times n$ carpets can have multiple different common ratios on the same substructure corresponding to different types of edge substructures. We conjecture that knowing these behaviors will determine unique long term behavior allowing us to define unique types from finite observations.

Further there is a Mathematica file associated with this paper that can compute substructure sequences given the initial removal of the carpet (given there is a bijection between initial boxes and substructure types and each substructure types contains at most one box of each type at each scale). Access the notebook at https://www.notebookarchive.org/2024-09-3a9ffb0.

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