General Counting Method For Aid in Computing the Distance Zeta Function on Modified Sierpiński Carpets

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A Brief and Informal Introduction to Fractal Dimension

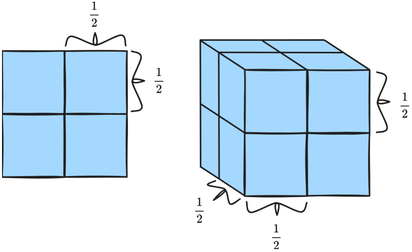


Figure: A figure showing that a square is made up of 4, $\frac{1}{2}$ scale copies of itself and a cube is made up of 8, $\frac{1}{2}$ scale copies of itself

$$N = \epsilon^{-D}$$

$$4 = (1/2)^{-D} \implies 4 = 2^D \implies D = 2$$

$$8 = (1/2)^{-D} \implies 8 = 2^D \implies D = 3.$$

Hausdorff Dimension of the Sierpiński Carpet.

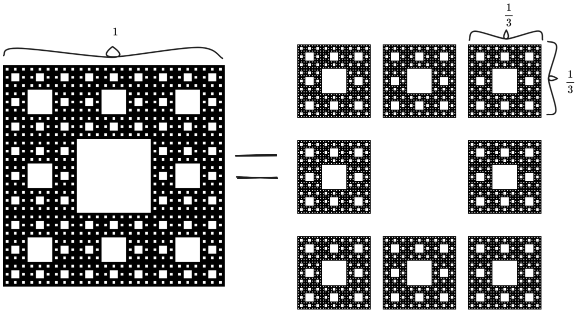


Figure: A depiction of the Sierpiński Carpet, demonstrating that it is made up of 8, $\frac{1}{3}$ scale copies of itself.

$$8 = \left(\frac{1}{3}\right)^{-D} \implies 8 = 3^D \implies 3\ln(2) = D\ln(3) \implies D = \frac{\ln(8)}{\ln(3)}.$$

Construction of the Sierpiński Carpet

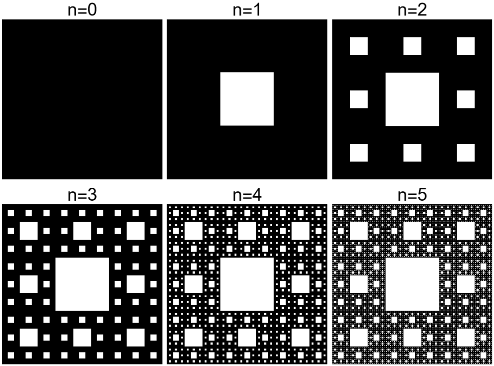


Figure: A figure depicting the first 5 steps of the Sierpiński Carpet Construction

Construction of the Corner Carpet

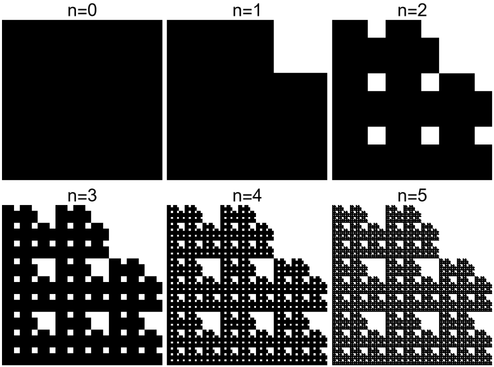


Figure: A figure depicting the first 5 steps of the modified Sierpiński Carpet construction where we remove the top right square

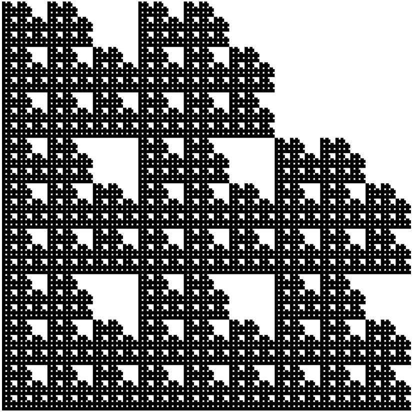


Figure: A figure showing the 5th iteration of the corner carpet.

$$8 = \left(\frac{1}{3}\right)^{-D} \implies 8 = 3^D \implies 3\ln(2) = D\ln(3) \implies D = \frac{\ln(8)}{\ln(3)}.$$

Distance Zeta Function

Definition

Given a set bound subset A of the N-dimensional Euclidean space $\mathbb{R}^N$, where $n \geq 1$, the distance zeta function is

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} dx$$

where $d(x, A)$ is the distance of x to A and A_δ is a δ-neighborhood of A.

The poles of the distance zeta function are the Minkowski dimensions of the components making up a Lebesgue measurable set $A \subseteq \mathbb{R}^N$.

Using the Distance Zeta Function on the Sierpiński Carpet

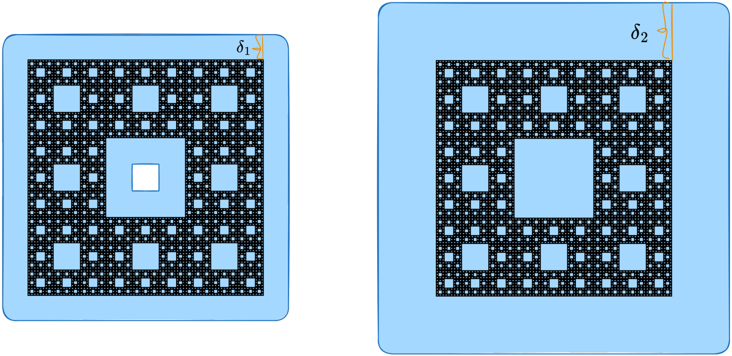


Figure: A figure showing δ inflated neighborhoods of the Sierpiński Carpet.

Choosing $\delta > \frac{1}{6}$ ensures that A_δ is simply connected.

Breaking the Integral Into Pieces

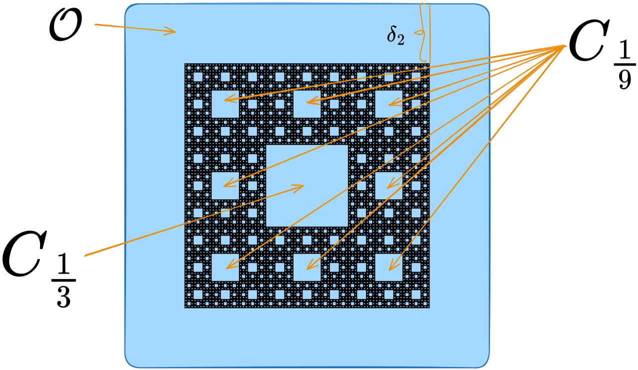


Figure: A figure showing how to break up the values obtained from different nonzero parts of the distance zeta function.

$$
\begin{aligned}
\zeta_{A_\delta}(s) &= \mathcal{O} + \sum_{k=1}^{n}\left[8^k C_{\frac{1}{3^k}}\right] \\
&= \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} + \sum_{k=1}^{n}\left[8^k\left(\frac{1}{3^k}\right)^s \cdot \frac{8}{2^s(s)(s-1)}\right] \\
&= \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} + \frac{8}{2^s(s)(s-1)} \cdot \sum_{k=1}^{n}\left(\frac{8}{3^s}\right)^k \\
&= \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} + \frac{8}{2^s(s)(s-1)} \cdot \frac{8}{3^s-8}
\end{aligned}
$$

Then s is a pole if

$$
s \in \left\{0, 1, \frac{\ln(8)}{\ln(3)} + \frac{2\pi i z}{\ln(3)}\right\} \quad \text{for } z \in \mathbb{Z}.
$$

The Distance Zeta Function Records More Information

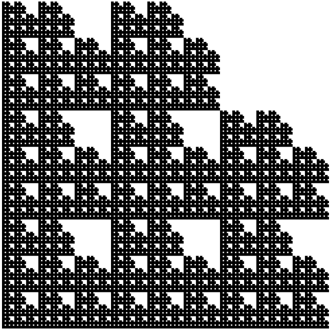


Figure: A figure showing the 5th iteration of the corner carpet.

$$\left\{0, 1, \frac{\ln(2)}{\ln(3)} + \frac{2\pi i z}{\ln(3)}, \frac{\ln(8)}{\ln(3)} + \frac{2\pi i z}{\ln(3)}, \frac{\ln(2 \pm \sqrt{2})}{\ln(3)} + \frac{2\pi i z}{\ln(3)}\right\} \quad \text{for } z \in \mathbb{Z}.$$

Interesting Carpets

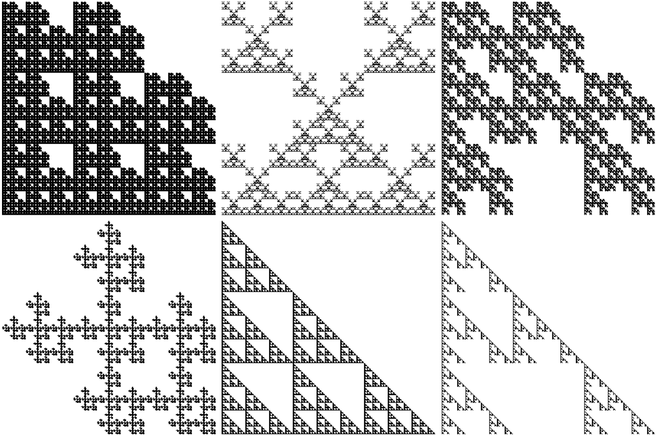


Figure: Some examples of possible modified Sierpiński Carpets.

Isolated Substructures

Definition

Within a modified Sierpiński Carpet, an isolated substructure is a finite cardinal-direction connected component within the set of removed points. Two points are finite cardinal-direction connected if there exists some finite iteration of the carpet where there is a path between the respective boxes containing the points that only moves in cardinal directions and stays within the set of removed points.
Substructures are named by a set denoting the relative position of their biggest boxes, following the naming scheme of a 3×3 matrix.

Definition

$S_p(n)$ is the number of isolated scale $\frac{1}{3^n}$ and type p substructures in a modified carpet. (Assuming the standard carpet on $[0,1]^2 \subseteq \mathbb{R}^2$, that is we assume the side lengths of the initial box to be 1).

Counting Independent Substructures on the Sierpiński Carpet

n=4

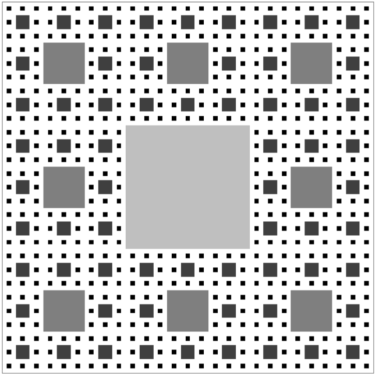


Figure: A picture of the Sierpiński carpet where smaller substructures smaller substructures are shaded darker gray.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

n	$S_{\{22\}}(n)$
1	1
2	8
3	64
4	512

Counting Independent Substructures on the Corner Carpet

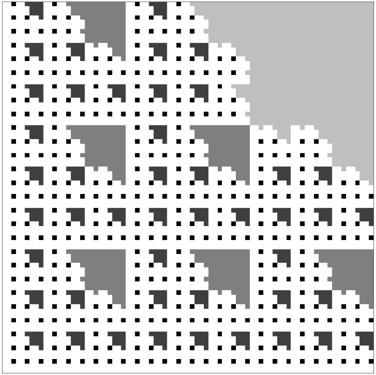


Figure: A picture of the Sierpiński carpet where smaller substructures smaller substructures are shaded darker gray.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

n	$S_{\{13\}}(n)$
1	1
2	6
3	46
4	364

Obtaining the Number of Independent Substructures Recursively

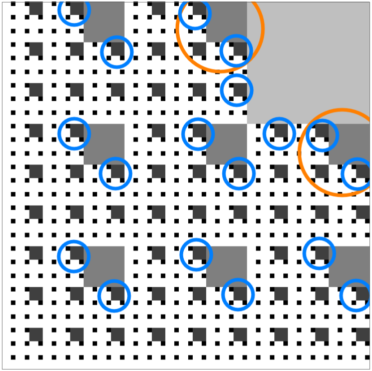


Figure: The 4th iteration of the Sierpiński Carpet where smaller boxes are shaded darker gray. Boxes of scale $\frac{1}{3}$ attached to larger structures are circled orange while such $\frac{1}{9}$ boxes are circled blue.

$$S_{\{13\}}(2) = \begin{array}{c}\text{total}\\\text{boxes}\end{array} - \begin{array}{c}\text{boxes attached to a}\\\text{larger substructure}\end{array}$$

$$= 8 - 2 = 6$$

$$S_{\{13\}}(3) = 8^2 - 18 = 46.$$

Edge Substructures

Definition

An edge substructure of scale s is a pattern of removal in a modified Sierpiński carpet that is itself an independent substructure when discounting boxes larger than scale s.

Definition

We define a shorthand for the number of scale t and type q substructures lying on a scale s and type p substructure. This is denoted

$$|q_t \sigma p_s|.$$

This is read the number of q_t's on p_s's.

Obtaining the Number of Independent Substructures Recursively

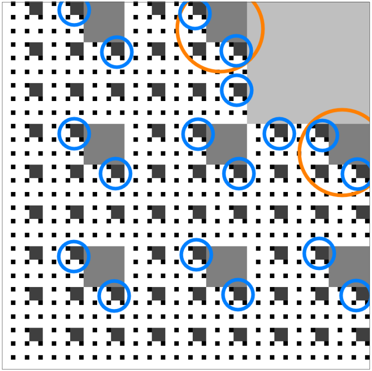


Figure: The 4th iteration of the Corner Carpet with removed boxes colored darker shades of gray the smaller the boxes.

$$\begin{aligned}
S_{\{13\}}(2) &= 8 - 2 = 6 \\
&= 8 - |\{13\}_{\frac{1}{9}} \sigma \{13\}_{\frac{1}{3}}| \\
S_{\{13\}}(3) &= 8^2 - 18 = 46 \\
&= 8^2 - 6 - 6(2) \\
&= 8^2 - |\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{3}}| \\
&\quad - 6|\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{9}}| \\
&= 8^2 - S_{\{13\}}(1)|\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{3}}| \\
&\quad - S_{\{13\}}(2)|\{13\}_{\frac{1}{27}} \sigma \{13\}_{\frac{1}{9}}|.
\end{aligned}$$

Edge Substructure Sequences

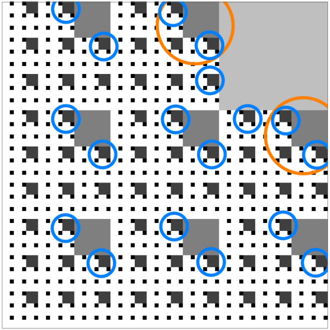
Definition

$E_{p,q}(n)$ is the number of type q and scale $\frac{1}{3^n}$ edge substructures lying on a scale 1 substructure of type p for $n \geq 1$. That is $E_{p,q}(n) = |q_{\frac{1}{3^n}} \sigma p_1|$.

Lemma

Let p and q be substructure types and suppose that $|q_{s^t} \sigma p_{3^s}| = n$. Then

$$|q_{\frac{1}{3^t}} \sigma p_{\frac{1}{3^s}}| = |q_{\frac{3^s}{3^t}} \sigma p_1| = E_{p,q}(t-s).$$



$$S_{\{13\}}(3) = 8^2 - S_{\{13\}}(1)|\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{3}}| - S_{\{13\}}(2)|\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{9}}|$$
$$= 8^2 - S_{13}(1)E_{\{13\},\{13\}}(2) - S_{\{13\}}(2)E_{\{13\},\{13\}}(1).$$

$$S_{\{13\}}(n) = 8^{n-1} - \sum_{k=1}^{n-1} S_{\{13\}}(k)\left|\{13\}_{\frac{1}{3^n}} \sigma \{13\}_{\frac{1}{3^k}}\right|$$

$$S_{\{13\}}(n) = 8^{n-1} - \sum_{k=1}^{n-1} S_{\{13\}}(k) E_{\{13\},\{13\}}(n-k).$$

- We can determine new elements of $S_p(n)$ from previous terms in the sequence. But, how do we compute $E_{p,q}(n)$?

Directly Attached Substructures

Definition

We define a shorthand for the number of scale t and type q substructures lying "directly on" a scale s and type p substructure. This is denoted

$$|q_t \phi p_s|.$$

This is read the number of q_t's on p_s's.
An edge substructure lies directly on an independent substructure if every path to a largest box of the parent substructure passes through no other edge substructure.

Lemma

Let p and q be substructure types and suppose that $|q_{s^t} \sigma p_{3^s}| = n$. Then

$$|q_{\frac{1}{3^t}} \sigma p_{\frac{1}{3^s}}| = |q_{\frac{3^s}{3^t}} \sigma p_1|.$$

Lemma

Given a modified Sierpiński Carpet with types p and q. We have that

$$|p_{\frac{1}{9}}\phi q_{\frac{1}{3}}| = |p_{\frac{1}{3}}\phi q_1| = |p_{\frac{1}{3}}\sigma q_1| = E_{p,q}(1).$$

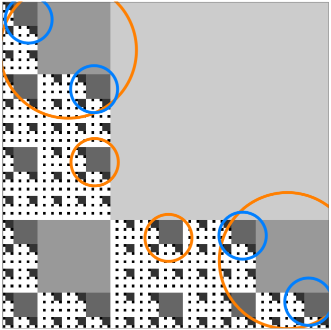
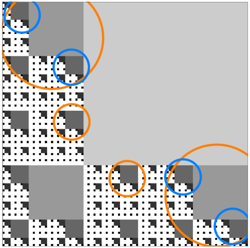


Figure: A zoomed in picture of the type $\{13\}$ substructure. Edge substructures of scale $\frac{1}{3}$ and $\frac{1}{9}$ are circled. Regular edge substructures are circled blue while edge substructures lying directly on another structure are circled orange.

$$|\{13\}_{\frac{1}{9}}\sigma\{13\}_{\frac{1}{3}}| = 2 \qquad |\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{3}}| = 6$$

$$|\{13\}_{\frac{1}{9}}\phi\{13\}_{\frac{1}{3}}| = 2 \qquad |\{13\}_{\frac{1}{27}}\phi\{13\}_{\frac{1}{3}}| = 2$$

Counting Edge Substructures Recursively

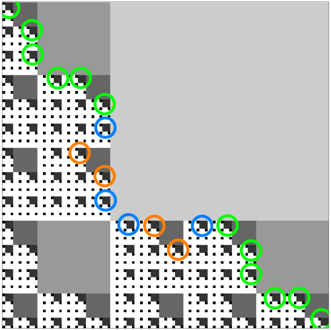


$$\left|\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{3}}\right| = \begin{matrix}\text{directly attached}\\ \text{substructures}\end{matrix} + \begin{matrix}\text{not directly attached}\\ \text{substructures}\end{matrix}$$

$$= 2 + 4 = 2 + 2(2) = 6$$

$$= \left|\{13\}_{\frac{1}{27}}\phi\{13\}_{\frac{1}{9}}\right| + \left|\{13\}_{\frac{1}{27}}\sigma\{13\}_{\frac{1}{9}}\right| \cdot \left|\{13\}_{\frac{1}{9}}\phi\{13\}_{\frac{1}{3}}\right|$$

Counting Edge Substructures Recursively



$$\left|\{13\}_{\frac{1}{3^4}}\sigma\{13\}_{\frac{1}{3}}\right| = 4 + 2(6) + 2(2) = 20$$

$$= \left|\{13\}_{\frac{1}{81}}\phi\{13\}_{\frac{1}{3}}\right| + \left|\{13\}_{\frac{1}{9}}\phi\{13\}_{\frac{1}{3}}\right| \cdot \left|\{13\}_{\frac{1}{81}}\sigma\{13\}_{\frac{1}{9}}\right|$$

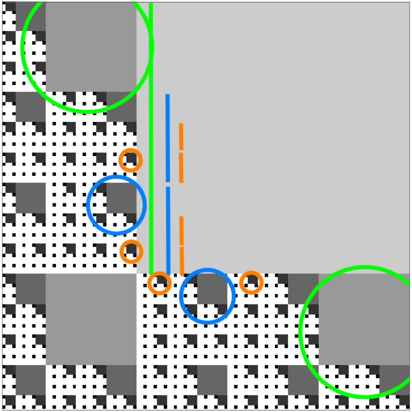
$$+ \left|\{13\}_{\frac{1}{27}}\phi\{13\}_{\frac{1}{3}}\right| \cdot \left|\{13\}_{\frac{1}{81}}\sigma\{13\}_{\frac{1}{27}}\right|$$

$$\left|\{13\}_{\frac{1}{3^{n+1}}}\sigma\{13\}_{\frac{1}{3}}\right| = \left|\{13\}_{3^{n+1}}\phi\{13\}_{\frac{1}{3}}\right| + \sum_{k=1}^{n-1}\left|\{13\}_{\frac{1}{3^{k+1}}}\phi\{13\}_{\frac{1}{3}}\right| \cdot \left|\{13\}_{\frac{1}{3^{n+1}}}\sigma\{13\}_{\frac{1}{3^{k+1}}}\right|$$

$$E_{\{13\},\{13\}}(n) = \left|\{13\}_{3^{n+1}}\phi\{13\}_{\frac{1}{3}}\right| + \sum_{k=1}^{n-1}\left|\{13\}_{\frac{1}{3^{k+1}}}\phi\{13\}_{\frac{1}{3}}\right| E_{\{13\},\{13\}}(n-k)$$

for $n \geq 2$.

Counting Directly Attached Substructures



n	$\left\lvert \{13\}_{\frac{1}{3^{n+1}}} \phi \{13\}_{\frac{1}{3}} \right\rvert$
1	2
2	2
3	4
4	8
5	64

Figure: A zoomed in picture of the $\{13\}_{\frac{1}{3}}$ structure of the corner carpet. Edge substructures of scales $\frac{1}{9}$, $\frac{1}{27}$, and $\frac{1}{81}$ are circled green, blue, and orange respectively.

We claim that

$$\begin{aligned}
\left|\{13\}_{\frac{1}{3^{n+1}}}\sigma\{13\}_{\frac{1}{3}}\right| &= 2^{n-2}\cdot 2\\
&= 2^{n-2}\left|\{13\}_{\frac{1}{27}}\phi\{13\}_{\frac{1}{3}}\right|\\
&= 2^{n-2}\left|\{13\}_{\frac{1}{9}}\phi\{13\}_{1}\right| \quad \text{for } n\geq 2.
\end{aligned}$$

Definition

Let $EN_{p,q}$ denote $|p_{\frac{1}{9}}\phi q_1|$.

It follows that

$$\left|\{13\}_{\frac{1}{3^{n+1}}}\sigma\{13\}_{\frac{1}{3}}\right| = 2^{n-2}EN_{\{13\},\{13\}}.$$

Note that $EN_{\{13\},\{13\}} = 2$.

$$
\begin{aligned}
E_{\{13\},\{13\}}(n) &= |\{13\}_{3^{n+1}}\phi\{13\}_{\frac{1}{3}}| + \sum_{k=1}^{n-1} |\{13\}_{\frac{1}{3^{k+1}}}\phi\{13\}_{\frac{1}{3}}| E_{\{13\},\{13\}}(n-k) \\
&= |\{13\}_{3^{n+1}}\phi\{13\}_{\frac{1}{3}}| + |\{13\}_{\frac{1}{9}}\phi\{13\}_{\frac{1}{3}}| E_{\{13\},\{13\}}(n-1) \\
&\quad + \sum_{k=2}^{n-1} |\{13\}_{\frac{1}{3^{k+1}}}\phi\{13\}_{\frac{1}{3}}| E_{\{13\},\{13\}}(n-k) \\
&= 2^{n-2} EN_{\{13\},\{13\}} + E_{\{13\},\{13\}}(1) E_{\{13\},\{13\}}(n-1) \\
&\quad + \sum_{k=2}^{n-2} 2^{k-2} EN_{\{13\},\{13\}} E_{\{13\},\{13\}}(n-k).
\end{aligned}
$$

What have we done so far?

- We have expressed the nth term of $S_{\{13\}}(n)$ as a linear combination of preceding terms from $S_{\{13\}}(n)$ and $E_{\{13\},\{13\}}(n)$:

$$S_{\{13\}}(n) = 8^{n-1} - \sum_{k=1}^{n-1} S_{\{13\}}(k)E_{\{13\},\{13\}}(n-k) \quad \text{for } n \geq 1.$$

- We have expressed the nth term of $E_{\{13\},\{13\}}(n)$ as a linear combination of preceding terms from $E_{\{13\},\{13\}}(n)$

$$\begin{aligned} E_{\{13\},\{13\}}(n) = 2^{n-2}EN_{\{13\},\{13\}} &+ E_{\{13\},\{13\}}(1)E_{\{13\},\{13\}}(n-1) \\ &+ \sum_{k=2}^{n-2} 2^{k-2}EN_{\{13\},\{13\}}E_{\{13\},\{13\}}(n-k) \quad \text{for } n \geq 2. \end{aligned}$$

- We can then compute $S_{\{13\}}(n)$ for any n! However, to use the distance zeta function we need an explicit formula.

Using Generating Functions

Definition

Given a sequence $(a_n)_{n=1}^{\infty}$, its ordinary generating function is

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Generating functions can be used to solve linear recurrence relations. Recall the identify for a geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Let $e(x) = \sum_{n=1}^{\infty} E_{\{13\},\{13\}}(n)x^n$. Let $a_n = E_{\{13\},\{13\}}(n)$ for brevity. Then

$$a_n = 2 \cdot 2^{n-2} + a_1 a_{n-1} - \sum_{k=2}^{n-1} 2 \cdot 2^{k-2} a_{n-k} \quad \text{for } n \geq 2.$$

It follows that

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} \left(2 \cdot 2^{n-2} + a_1 \cdot a_{n-1} - \sum_{k=2}^{n-1} 2 \cdot 2^{k-2} a_{n-k} \right) x^n \\
&= 2 \sum_{n=2}^{\infty} 2^{n-2} x^n + 2 \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} x^n \sum_{k=2}^{n-1} 2^{k-2} a_{n-k} \\
&= 2x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x \sum_{n=1}^{\infty} a_n x^n + 2 \sum_{n=2}^{\infty} x^n \sum_{k=0}^{n-3} 2^k a_{n-2-k} \\
&= 2x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x \sum_{n=1}^{\infty} a_n x^n + 2 \sum_{n=3}^{\infty} x^n \sum_{k=0}^{n-3} 2^k a_{n-2-k}.
\end{aligned}$$

$$\sum_{n=2}^{\infty} a_n x^n = 2x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x \sum_{n=1}^{\infty} a_n x^n + 2 \sum_{n=3}^{\infty} x^n \sum_{k=0}^{n-3} 2^k a_{n-2-k}$$

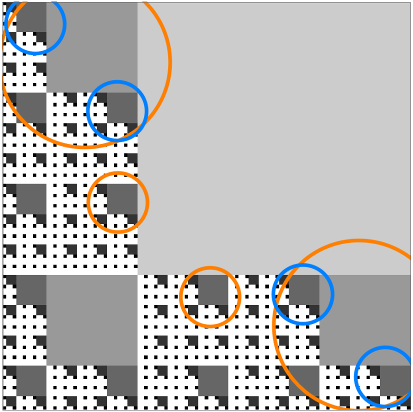
$$= 2x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x \sum_{n=1}^{\infty} a_n x^n + 2x^3 \sum_{n=0}^{n} x^n \sum_{k=0}^{n} 2^k a_{n+1-k}$$

$$= 2x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x \sum_{n=1}^{\infty} a_n x^n + 2x^3 \left(\sum_{n=0}^{\infty} 2^n \right) \left(\sum_{n=0}^{\infty} a_{n+1} x^n \right)$$

$$= 2x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \left(\sum_{n=0}^{\infty} 2^n \right) \left(\sum_{n=0}^{\infty} a_{n+1} x^{n+1} \right)$$

$$e(x) - 2x = \frac{2x^2}{1-2x} + 2xe(x) + \frac{2x^2}{1-2x} e(x) \quad .$$

$$
\begin{aligned}
e(x) &= \frac{\left(2x + \frac{2x^2}{1-2x}\right)}{1 - 2x - \frac{2x^2}{1-2x}} = \frac{2x - 2x^2}{2x^2 - 4x + 1} = \frac{2x - 2x^2}{2\left(x - \frac{2+\sqrt{2}}{2}\right)\left(x - \frac{2-\sqrt{2}}{2}\right)} \\
&= \frac{2x - 2x^2}{\left(1 - (2+\sqrt{2})x\right)\left(1 - (2-\sqrt{2})x\right)} \\
&= -1 + \frac{1}{2\left(1 - (2+\sqrt{2})x\right)} + \frac{1}{2\left(1 - (2-\sqrt{2})x\right)} \\
&= -1 + \sum_{n=0}^{\infty} \frac{1}{2}(2+\sqrt{2})^n x^n + \sum_{n=0}^{\infty} \frac{1}{2}(2-\sqrt{2})^n x^n \\
&= \sum_{n=1}^{\infty} \frac{\left((2+\sqrt{2})^n + (2-\sqrt{2})^n\right)}{2} x^n = \sum_{n=1}^{\infty} a_n x^n = e(x).
\end{aligned}
$$



$$E_{\{13\},\{13\}}(n) = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{2}.$$

We found that

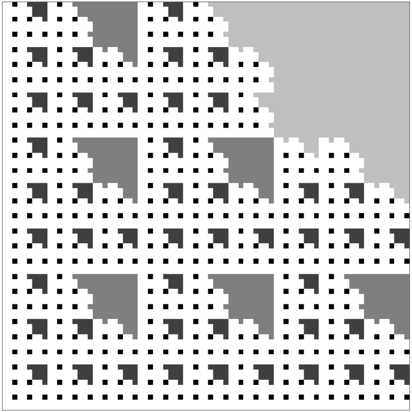
$$S_{\{13\}}(n) = 8^{n-1} - \sum_{k=1}^{n-1} S_{\{13\}}(k) E_{\{13\},\{13\}}(n-k) \quad \text{for } n \geq 1$$

$$\begin{aligned} E_{\{13\},\{13\}}(n) = 2^{n-2} EN_{\{13\},\{13\}} + E_{\{13\},\{13\}}(1) E_{\{13\},\{13\}}(n-1) \\ + \sum_{k=2}^{n-2} 2^{k-2} EN_{\{13\},\{13\}} E_{\{13\},\{13\}}(n-k) \quad \text{for } n \geq 2. \end{aligned}$$

If we let $s(x) = \sum_{n=1}^{\infty} S_{\{13\}}(n) x^n$ and $e(x) = \sum_{n=1}^{\infty} E_{\{13\},\{13\}}(n)$ we find that

$$e(x) = \frac{2x - 2x^2}{2x^2 - 4x + 1}$$

$$s(x) = \frac{x\left(2x^2 - 4x + 1\right)}{(2x-1)(8x-1)}.$$



$$S_{\{13\}}(n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{3} \cdot 2^{n-2} + \frac{17}{3} \cdot 2^{3n-6} & \text{if } n > 1 \end{cases}.$$

Carpets With Multiple Substructures

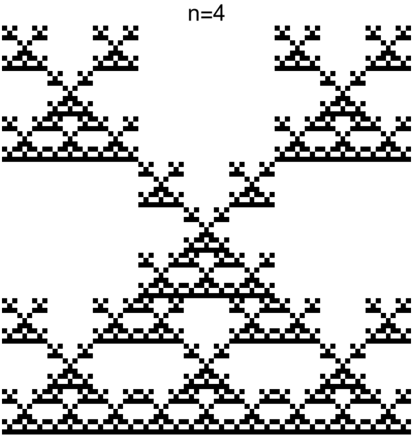
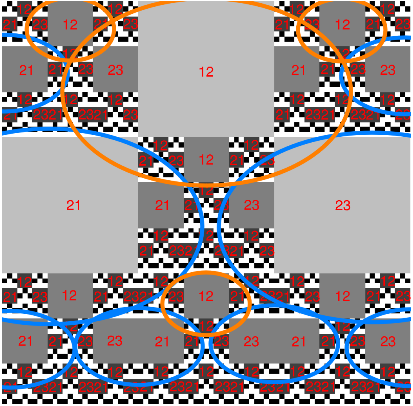


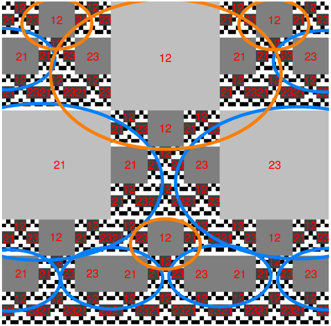
Figure: The 4th iteration of the modified Sierpiński Carpet removing the 12, 21, and 22 boxes



n	$S_{\{12\}}(n)$	$S_{\{21,23\}}$
1	1	1
2	3	4
3	12	20
4	60	112

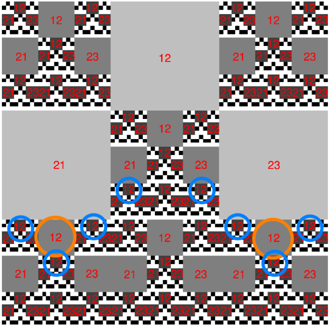
$$\mathcal{T} = \{\{12\}, \{21, 23\}, \{12\}, \{21\}, \{23\}\}$$

Figure: A picture of the modified Sierpiński carpet obtained by removing 12, 21, and 23 boxes with smaller boxes colored darker gray, boxes labeled, and substructures circled orange or blue due to type.



$$S_{\{21,23\}}(2) = \begin{array}{c}\text{total}\\\text{boxes}\end{array} - \begin{array}{c}\text{boxes attached to a}\\\text{larger substructure}\end{array} = \frac{2 \cdot 6^{2-1} - (1+1) - (1+1)}{2} = \frac{12-4}{2} = 4$$

$$S_{\{21,23\}}(n) = \frac{|\{21,23\}|(6)^{n-1} - \sum_{a,b \in \mathcal{T}} \sum_{k=1}^{n-1} |b \cap \{21,23\}| S_a(k) E_{a,b}(n-k)}{|\{21,23\}|}.$$



$$E_{\{21,23\},\{12\}}(2) = \begin{array}{c}\text{directly attached}\\ \text{substructures}\end{array} + \begin{array}{c}\text{not directly attached}\\ \text{substructures}\end{array} = 4 + 2(1) + 1(1) + 1(1) = 8$$

$$E_{\{21,23\},\{12\}}(n) = 2^{n-2}EN_{\{21,23\},\{12\}} + \sum_{a\in\mathcal{T}} \Bigg(E_{\{21,23\},a}E_{a,\{12\}}(n-1) + \sum_{k=2}^{n-1} 2^{k-2}EN_{\{21,23\},a}E_{a,\{12\}}(n-k) \Bigg).$$

Lemma

Let $\mathcal{T}$ be the set of substructures for a modified Sierpiński Carpet. Let $p, q \in \mathcal{T}$ then

$$|q_{\frac{1}{3^n}} \phi p_1| = 2^{n-2} EN_{p,q} \quad \textit{for } n \geq 2.$$

Theorem

Let $\mathcal{C}$ be a 3×3 Sierpiński Carpet with substructure types $\mathcal{T}$ so that there is a bijection between initial boxes and substructure types. Additionally each substructure types must contain at most one box of each type at each scale. Let r be the number of boxes initial removed and $p, q \in \mathcal{T}$, then for $n \geq 1$

$$E_{p,q}(n) = 2^{n-2} EN_{p,q} + \sum_{a \in \mathcal{T}} \left(E_{p,a}(1) E_{a,q}(n-1) + \sum_{k=2}^{n-1} 2^{k-2} EN_{p,a} E_{a,q}(n-k) \right).$$

General Formula

Proof.

$$
\begin{aligned}
E_{p,q}(n) &= |q_{\frac{1}{3^n}}\sigma p_1| = |q_{\frac{1}{3^n}}\phi p_1| + \sum_{a\in\mathcal{T}}\sum_{k=1}^{n-1}|a_{\frac{1}{3^k}}\phi p_1||q_{\frac{1}{3^n}}\sigma a_{\frac{1}{3^k}}| \\
&= |q_{\frac{1}{3^n}}\phi p_1| + \sum_{a\in\mathcal{T}}\left(|a_{\frac{1}{3}}\phi p_1||q_{\frac{1}{3^n}}\sigma a_{\frac{1}{3}}| + \sum_{k=2}^{n-1}|a_{\frac{1}{3^k}}\phi p_1||q_{\frac{1}{3^n}}\sigma a_{\frac{1}{3^k}}|\right) \\
&= |q_{\frac{1}{3^n}}\phi p_1| + \sum_{a\in\mathcal{T}}\left(|a_{\frac{1}{3}}\sigma p_1||q_{\frac{1}{3^n}}\sigma a_{\frac{1}{3}}| + \sum_{k=2}^{n-1}|a_{\frac{1}{3^k}}\phi p_1||q_{\frac{1}{3^n}}\sigma a_{\frac{1}{3^k}}|\right) \\
&= 2^{n-2}EN_{p,q} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)E_{a,q}(n-1) + \sum_{k=2}^{n-1}2^{k-2}EN_{p,a}E_{a,q}(n-k)\right)..
\end{aligned}
$$

$\square$

Theorem

Let $\mathcal{C}$ be a 3×3 Sierpiński Carpet with substructure types $\mathcal{T}$ so that there is a bijection between initial boxes and substructure types. Additionally each substructure types must contain at most one box of each type at each scale. Let $p, q \in \mathcal{T}$, then

$$S_p(n) = \frac{|p|(9-r)^{n-1} - \sum_{a,b \in \mathcal{T}} \sum_{k=1}^{n-1} |b \cap p| E_{a,b}(n-k) S_a(k)}{|p|}.$$

Proof.

$$S_p(n) = \frac{\overbrace{|p|(9-r)^{n-1}}^{\text{total boxes}} - \sum_{a,b \in \mathcal{T}} \sum_{k=1}^{n-1} \overbrace{|b \cap p|}^{\substack{\text{contributed} \\ \text{boxes}}} \overbrace{E_{a,b}(n-k)}^{|b_{\frac{1}{3^n}} \sigma a_{\frac{1}{3^k}}|} \overbrace{S_a(k)}^{\#\text{ of } a_{3^k}}}{\underbrace{|p|}_{\#\text{ of boxes needed to start } p}}.$$

$\square$

Computing the 12, 21, 23 Carpet

$E_{p,q}(1)$	$q = \{12\}$	$q = \{21\}$	$q = \{23\}$	$q = \{21, 23\}$
$p = \{12\}$	1	1	1	0
$p = \{21\}$	1	1	0	0
$p = \{23\}$	1	0	1	0
$p = \{21, 23\}$	2	1	1	0

Table: A table of values for $E_{p,q}(1)$ in the 12, 21, 23 carpet

$EN_{p,q}$	$q = \{12\}$	$q = \{21\}$	$q = \{23\}$	$q = \{21, 23\}$
$p = \{12\}$	2	2	2	0
$p = \{21\}$	2	2	0	0
$p = \{23\}$	2	0	2	0
$p = \{21, 23\}$	4	2	2	0

Table: A table of values for $EN_{p,q}$ in the 12, 21, 23 carpet

Let $e_{p,q}(x) = \sum_{n=1}^{\infty} E_{p,q}(n)x^n$ and $s_p(x) = \sum_{n=1}^{\infty} S_p x^n$. We get a system of rational equations for $p, q \in \mathcal{T} = \{\{12\}, \{21, 23\}, \{12\}, \{21\}, \{23\}\}$.

$$e_{p,q}(x) = EN_{p,q}\frac{x^2}{1-2x} + \sum_{a\in\mathcal{T}}\left(E_{p,a}(1)e_{a,q}(x)x + EN_{p,a}\frac{x^2}{1-2x}e_{a,q}(x)\right)$$

$$s_p(x) = \frac{(9-r)x^2}{1-(9-r)x} - \sum_{a,b\in\mathcal{T}}\left(\frac{|b\cap p|}{|p|}e_{a,b}(x)s_a(x)\right).$$

$e_{p,q}(x)$	$q = \{12\}$	$q = \{21\}$	$q = \{23\}$	$q = \{21, 23\}$
$p = \{12\}$	$\frac{x^2-x}{-7x^2+6-x-1}$	$\frac{2x^2-x}{-7x^2+6-x-1}$	$\frac{2x^2-x}{-7x^2+6-x-1}$	0
$p = \{21\}$	$\frac{x(2x-1)}{-7x^2+6-x-1}$	$\frac{x\left(5x^2-5x+1\right)}{(3x-1)\left(-7x^2+6-x-1\right)}$	$\frac{x^2(2x-1)}{(3x-1)\left(7x^2-6+x+1\right)}$	0
$p = \{23\}$	$\frac{2x^2-x}{-7x^2+6-x-1}$	$\frac{x^2(2x-1)}{(3x-1)\left(7x^2-6x+1\right)}$	$\frac{5x^3-5x^2+x}{(3x-1)\left(-7x^2+6-x-1\right)}$	0
$p = \{21, 23\}$	$\frac{2x(2x-1)}{-7x^2+6-x-1}$	$\frac{(x-1)x}{-7x^2+6-x-1}$	$\frac{(x-1)x}{-7x^2+6-x-1}$	0

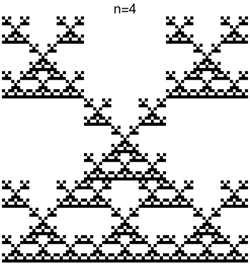
Table: A table holding values for $e_{p,q}(x)$

$s_{\{12\}}(x)$	$s_{\{21\}}(x)$	$s_{\{23\}}(x)$	$s_{\{21,23\}}(x)$
$\frac{-5x^2+x}{12x^2-8x+1}$	0	0	$\frac{-4x^2+x}{12x^2-8x+1}$

Figure: A table of values for $s_p(x)$ in the 12, 21, 23 carpet.

$S_{\{12\}}(n)$	$S_{\{21\}}(n)$	$S_{\{23\}}(n)$	$S_{\{21,23\}}(n)$
$3 \cdot 2^{n-3} + 2^{n-3}3^{n-1}$	0	0	$2^{n-2} + 2^{n-2}3^{n-1}$

Figure: A table of values for $S_p(n)$ in the 12, 21, 23 carpet.



Caveats

- For the method to work we need a bijection between substructure types and largest boxes. This is not always the case.
- Max of one box of each type per structure.
- It can be hard to tell when components are connected (especially for bigger carpets)
- The fact that $|q_{\frac{1}{3^n}}\phi p_1| = 2^{n-2}EN_{p,q}$ for $n \geq 2$ does not immediately generalize to bigger carpets.

A Carpet Lacking the Substructure Type and Largest Box Bijection

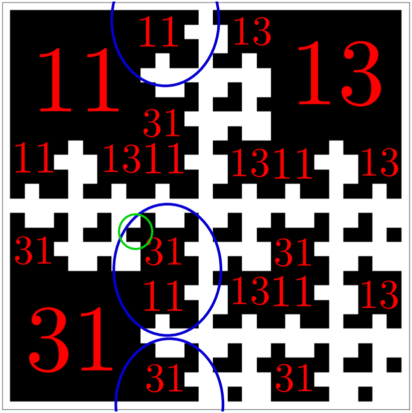


Figure: The 4th iteration of the 11, 13, 31 carpet. Two different edge substructures with largest boxes 31 and 11 are circled blue. A green circle shows that one of these substructure has one more box than the other.

A Late Merging Carpet

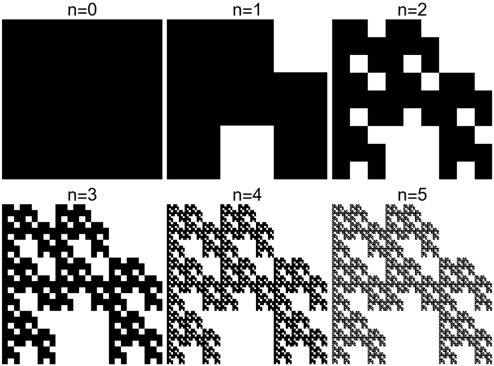


Figure: A table of pictures showing the first 5 iterations of the 13, 32 carpet.

Further Work

Things to do

- Get rid of the bijection condition
- Get rid of the max of one type of box per structure.
- Bigger Carpets. Start with a $m \times n$ removal instead of a 3×3 removal.
  - Computational intensive
  - To make sure we use enough types we could define substructures up to scaling, and translation of the plane, however, we want a nice way to refer to types so that a program can recognize them.

Possible Solutions

- Use the 4 iterations of a substructures boxes in order to label its type (+ enough iterations so that substructures of the right size are connected)
- Track a new value $G_{p,q} = \frac{E_{p,q}(3)}{E_{p,q}(2)}$ in hopes that $|q_{\frac{1}{3^n}} \phi p_1| = G_{p,q}^{n-2} EN_{p,q}$.

Thank You!