A Hundred And One Arabian Carpets

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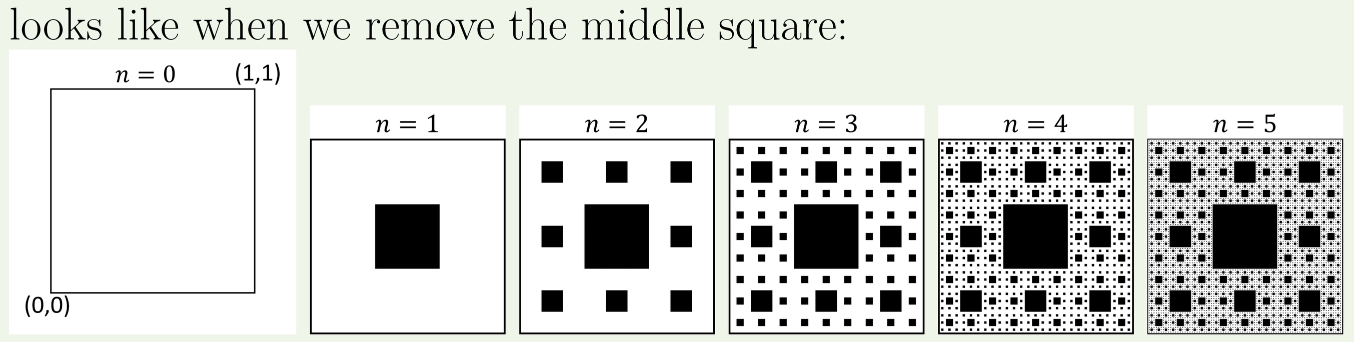
Goal

We wanted to study fractals, and specifically Sierpiński Carpet-type fractals. These can be generated by dividing a unit square into a 3×3 grid and removing a particular set of squares. On all the remaining squares, divide them into the same 3×3 grid and remove the same set of squares in each grid. Here is what it looks like when we remove the middle square:



Using Polya Theory, we found that there are exactly 100 non-trivial ways to remove squares from a 3×3 grid, unique up to rotation and reflection. We want to compute the dimensions of all of these modifications of the Sierpiński Carpet.

The Tools

We used two Fractal Zeta Functions, first published by Dr. M. L. Lapidus in 2017. First, we embed our set in some dimension (we will use $N = 2$). Then, we "inflate" our set, meaning we make a new set $A_t = \{x \in \mathbb{R}^N | d(x, A) < t\}$ where d is the Euclidean Metric in $\mathbb{R}^N$. Then, we compute one of the following Lebesgue integrals for a constant δ:

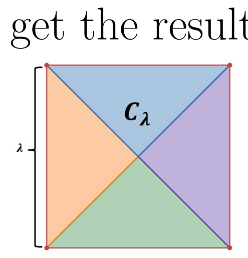
$$\text{Distance Zeta Function: } \zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N}\, dx$$

$$\text{Tubular Zeta Function: } \widetilde{\zeta_A}(s) = \int_0^\delta t^{s-N-1} m(A_t)\, dt$$

The variable s is complex, and m refers to the N-dimensional Lebesgue Measure. Once the integrals are computed, the poles of the function will tell us the dimensions of all the components that make up the set. So for instance, when A is a unit square, the Tubular Zeta Function gets poles at $s = 0, 1, 2$, indicating that a square is made up of points and lines, and it is a solid 2-dimensional surface.

Example (Normal Sierpiński)

The best way to compute the Distance Zeta Function for the Normal Sierpiński Carpet is to compute for the finite "level" of the carpet, and then let that level go to ∞. We can dissect a single closed square with side length λ into the following regions and get the result below:



$$C_\lambda(s) = 2\int_0^{\frac{\lambda}{2}} \int_y^{-y+\lambda} y^{s-2}\, dx\, dy + 2\int_0^{\frac{\lambda}{2}} \int_x^{-x+\lambda} x^{s-2}\, dy\, dx$$
$$= \lambda^s \cdot \frac{8}{2^s(s)(s-1)}$$

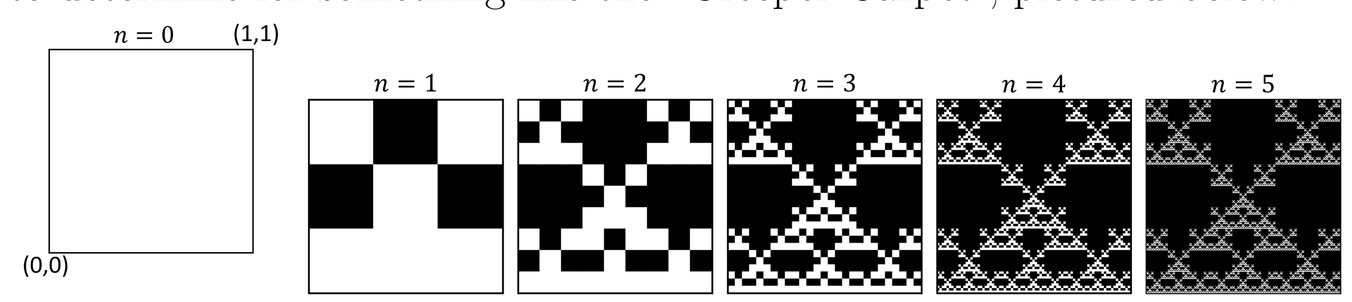
At each level of the carpet, new squares appear in clusters of 8 surrounding the squares generated in the previous level, and they appear with scale $\lambda = \frac{1}{3^n}$. Since we are uniformly inflating our set, we must consider the region outside of the carpet, which will simply be a unit square. This gets:

$$\zeta_A(s) = \lim_{n\to\infty} (\text{Unit Square}) + \sum_{k=1}^{n} \left[8^k C_{\frac{1}{3^k}}\right] = \frac{4\delta^{s-1}}{s-1} + \frac{8\delta^s}{s} + \frac{8}{2^s(s)(s-1)} \cdot \frac{8}{3^s - 8}$$

This function thus has poles at $s = 0, 1$, and $\log_3(8) + \frac{2\pi ik}{\ln(3)}$ for all $k \in \mathbb{Z}$. This matches known facts about the fractal dimension of the Sierpiński Carpet, and it tells us that the fractal consists of points and lines.

Breakthrough: General Counting Method

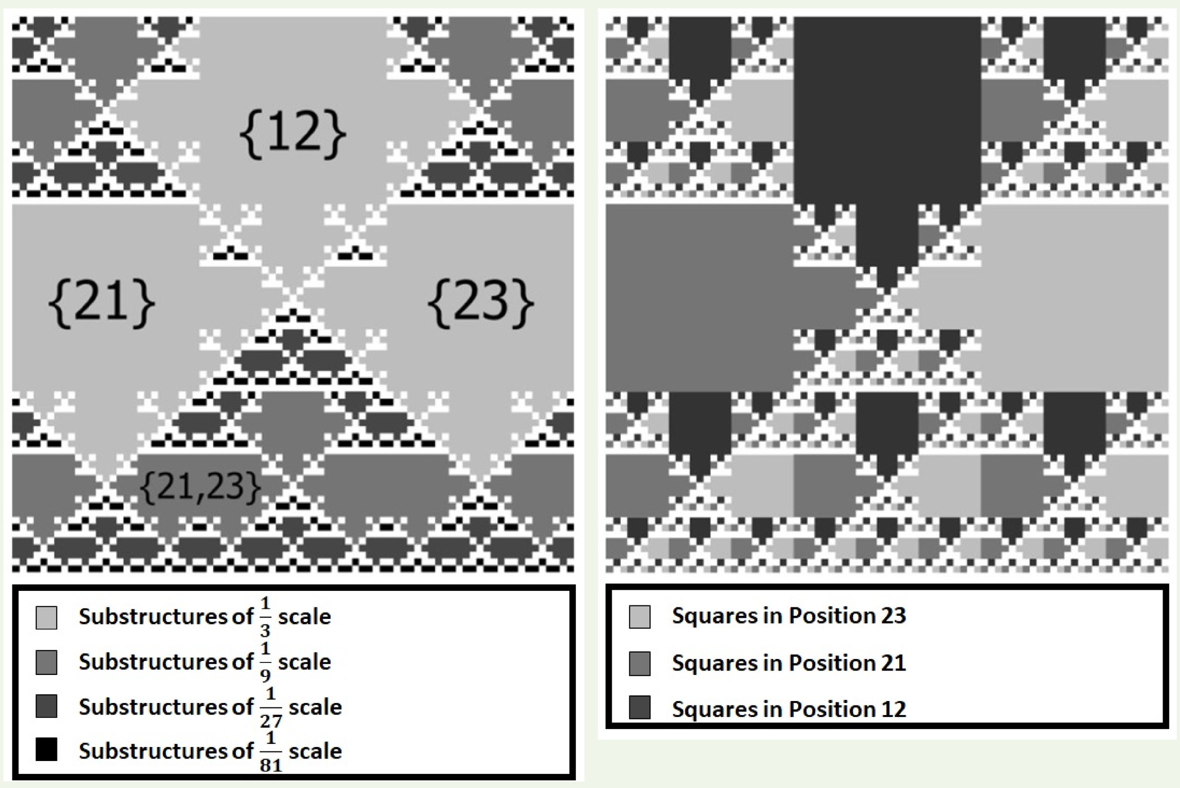
When we computed the Distance Zeta Function for the Normal Sierpiński Carpet, we already knew that new squares appeared at an 8^n rate. However, it is more difficult to determine for something like the "Creeper Carpet", pictured below:



To get accurate calculations, we needed to count both the structures protruding downwards and the double-square structures that appeared in the middle. We accomplished this by defining a new sequence:

Absorption Sequence

We first break up our carpet into Substructures, labeled based on which boxes in which positions are needed to start producing the Substructure:



From there, we define a sequence:
$E_{p,q}(n)$ = # of Substructures of type q of scale $\frac{1}{3^n}$ that are absorbed into a Substructure of type p that is at scale 1.
The sequence can be determined from two initial values. Below is an example from the Creeper Carpet:

$$E_{12,12}(n) = \frac{\left(\sqrt{2} - 4\right)\left(3 - \sqrt{2}\right)^n + \left(\sqrt{2} + 4\right)\left(3 + \sqrt{2}\right)^n}{14\sqrt{2}}$$

From there, the way we found the number of structures that appeared at each level was:

$$S_{(\text{Structure})}(n) = \frac{(\text{\# of new boxes}) - (\text{\# of boxes "absorbed"})}{(\text{\# of initial boxes in structure})}$$

For the Creeper Carpet, this comes out to:

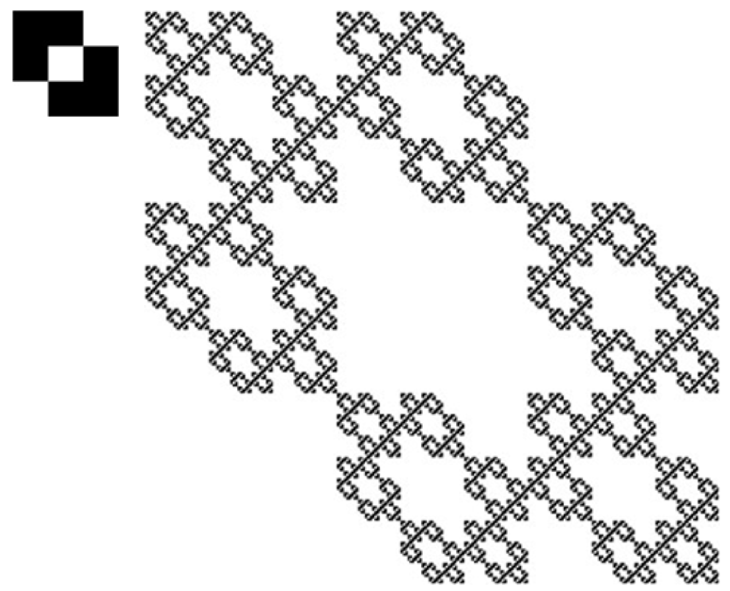
$$S_{12}(n) = \frac{1}{4}(3 \cdot 2^{n-1} + 6^{n-1}), n \geq 1$$
$$S_{21}(n) = S_{23}(n) = 2^{n-1}, n \geq 1$$
$$S_{\{21,23\}}(n) = \frac{1}{2}(-2^{n-1} + 6^{n-1}), n \geq 1$$

The dimensions of the Creeper Carpet came out to the following for all $k \in \mathbb{Z}$:

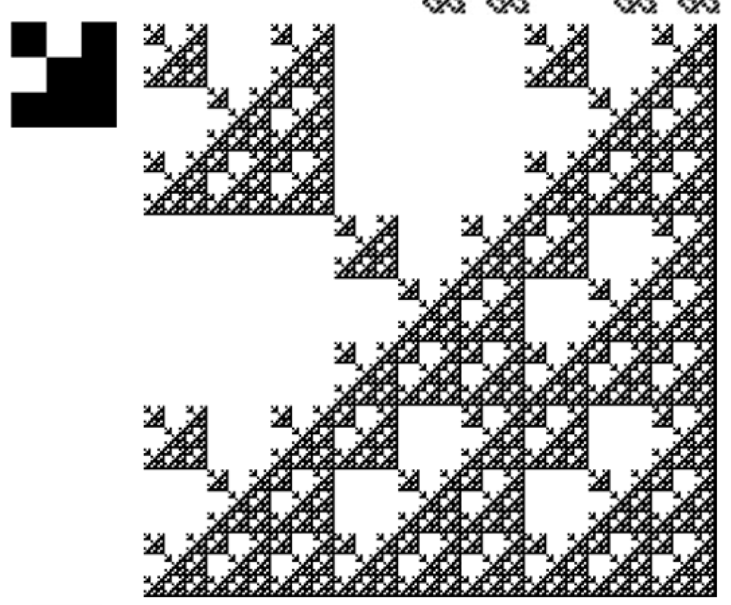
$$\boxed{\text{Dimensions} = \left\{0, \log_3(3 - \sqrt{2}) + \frac{2\pi ik}{\ln(3)}, \log_3(2) + \frac{2\pi ik}{\ln(3)}, \log_3(3 + \sqrt{2}) + \frac{2\pi ik}{\ln(3)}, \log_3(6) + \frac{2\pi ik}{\ln(3)}\right\}}$$

Interesting Carpets

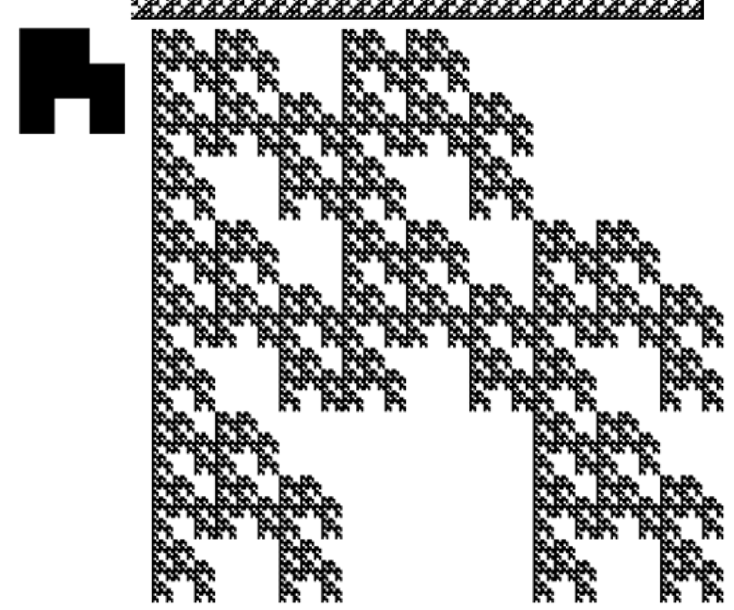
These dimensions were computed using the Max Metric in the Distance Zeta Function.



Koch Snowflake:
$0 + \frac{2\pi ik}{\ln(3)}$ *(order 2)*,
1,
$\log_3(4) + \frac{2\pi ik}{\ln(3)}$,
$\log_3(6) + \frac{2\pi ik}{\ln(3)}$
For all $k \in \mathbb{Z}$



Starship Troopers:
$\log_3(2) + \frac{2\pi ik}{\ln(3)}$,
1,
$\log_3(4) + \frac{2\pi ik}{\ln(3)}$,
$\log_3(7) + \frac{2\pi ik}{\ln(3)}$
For all $k \in \mathbb{Z}$



Sus Carpet:
$0 + \frac{2\pi ik}{\ln(3)}$,
$\log_3(2) + \frac{2\pi ik}{\ln(3)}$,
1,
$\log_3(7) + \frac{2\pi ik}{\ln(3)}$
For all $k \in \mathbb{Z}$

Future Research

We have extensively studied 85 of the 100 carpets generated from 3×3 grids. Once we complete the categorization of all 100 carpets, we hope to extend the research to larger grids, and possibly generalize to $n \times n$ grids. We also hope to extend the research to Bedford-McMullen Carpets, which are generated from dividing the squares into rectangular grids. We can also extend the findings into more than 2 dimensions, finding dimensions of fractals like the Menger Sponge. Since the Fractal Zeta Functions work for any set, a (admittedly lofty) goal would be to study the components of Julia Sets and Mandelbrot Sets.

Acknowledgements

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