Computation of Legendrian Contact Homology

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Introduction

In general it can be hard to distinguish knots up to isotopy classes. A common method of doing so is defining a knot invariant and showing that two knots have different values for the invariant. Doing so would imply that the two knots lie in two different isotopy classes, and should be considered different knots. One powerful invariant is Legendrian contact homology (LCH), which is an invariant for Legendrian knot isotopy classes (see [1]). It is difficult to compute the LCH of a given knot, even when linearization is possible. One step in computing the differential involves enumerating all of the disks with base points in a differential graded algebra. This alone is a difficult process, and it gets especially difficult with nested right cusps. In the past, programs computing LCH have converted knots with nested right cusps to isotopic knots without nested right cusps via Cromwell moves ([2],[4]). Our goal for this project was to write a program that could compute the LCH for knot diagrams that have nested right cusps.

What Are We Computing?

- Two knots are said to be *smoothly isotopic* if one can be transformed into the other by a sequence of Cromwell moves.
- Two knots are *Legendrian isotopic* if they can be related by a sequence of Cromwell moves that excludes certain types of moves.

The Legendrian Contact Homology

The Legendrian contact homology, (LCH) of a knot diagram associates a set of polynomials to that knot diagram. This set of polynomials is a *Legendrian* invariant.

Example. The knots $m(5_2)A$ and $m(5_2)B$ are smoothly isotopic knots with the same rotation number and Thurston Bennequin number: 0 and 1, respectively [5]. Due to this fact, it was not known whether the knots were Legendrian isotopic. However, their LCHs, a more recent Legendrian knot invariant, are distinct. This implies $m(5_2)A$ and $m(5_2)B$ are not Legendrian isotopic.

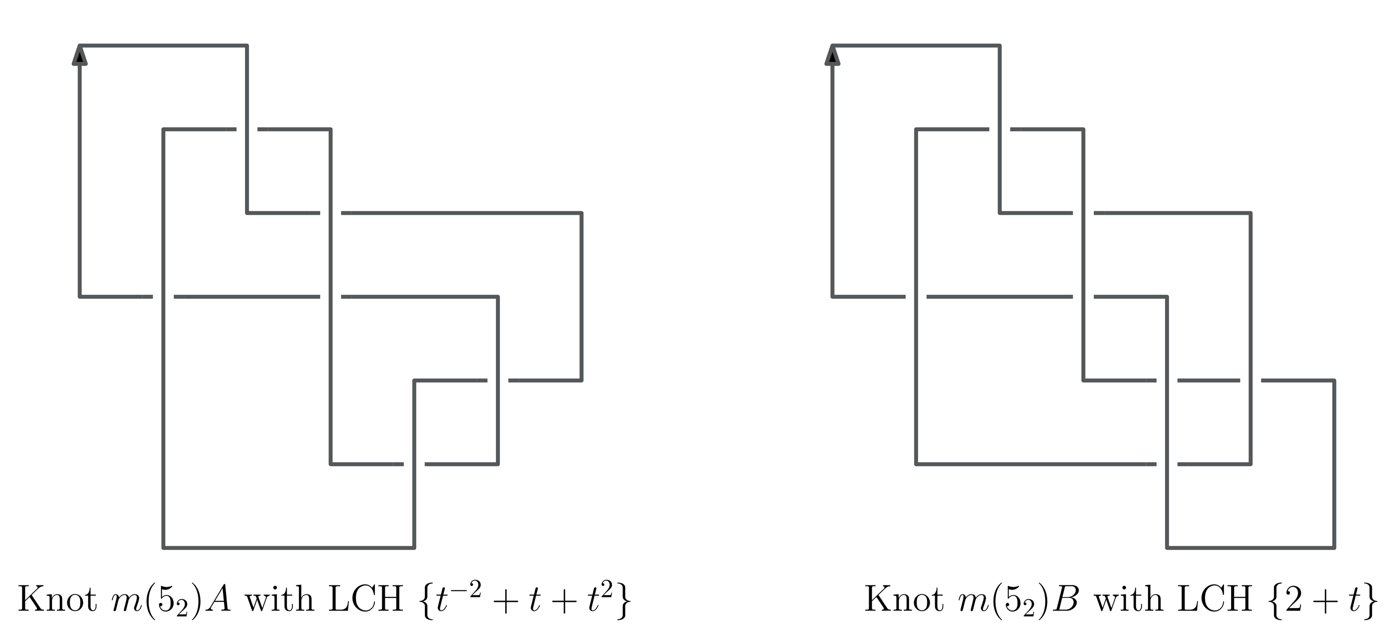


Figure: The two $m(5_2)$ knots and their LCH's

Computing the Differential

Part of computing the LCH for a knot diagram is computing the differential. This process involves traversing the knot diagram to find "disks."

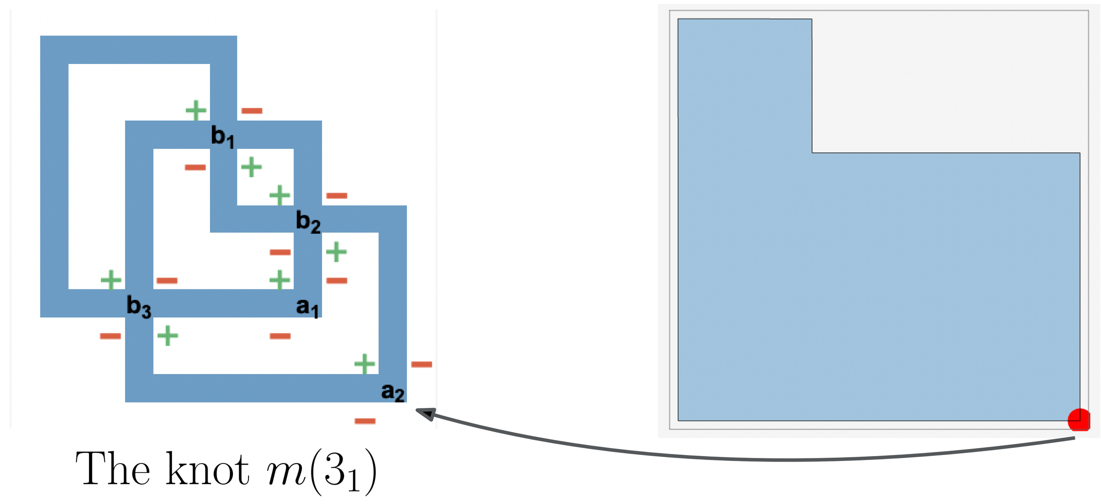


Figure: Example of disk for the knot $m(3_1)$ starting at a right cusp.

To find disks, start at a right cusp or a crossing and traverse the knot.

- Traversing a portion of the knot outlines the boundary of the disk, where the disk itself is on the left of the boundary.
- Certain turns and "moves" can be made at crossings or right cusps, each leading to a different boundary and by extension, a different disk.

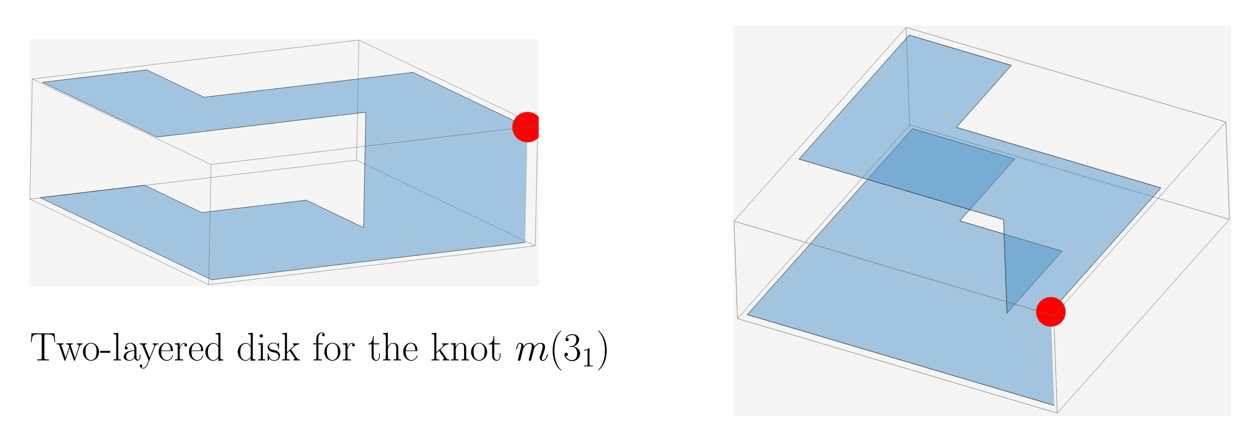


Figure: Disks can also loop under themselves at right cusps

- Each disk has a word associated to it based on the "moves" made while traversing the knot. From there, the differential at a generator is the sum of all of those words. If a knot diagram has many generators, the computation that finds all possible disks gets very large.

Example: Knot 13n1907

A more complex knot that demonstrates how big these disks can get is 13n1907. The disk below was obtained by traversing this knot.

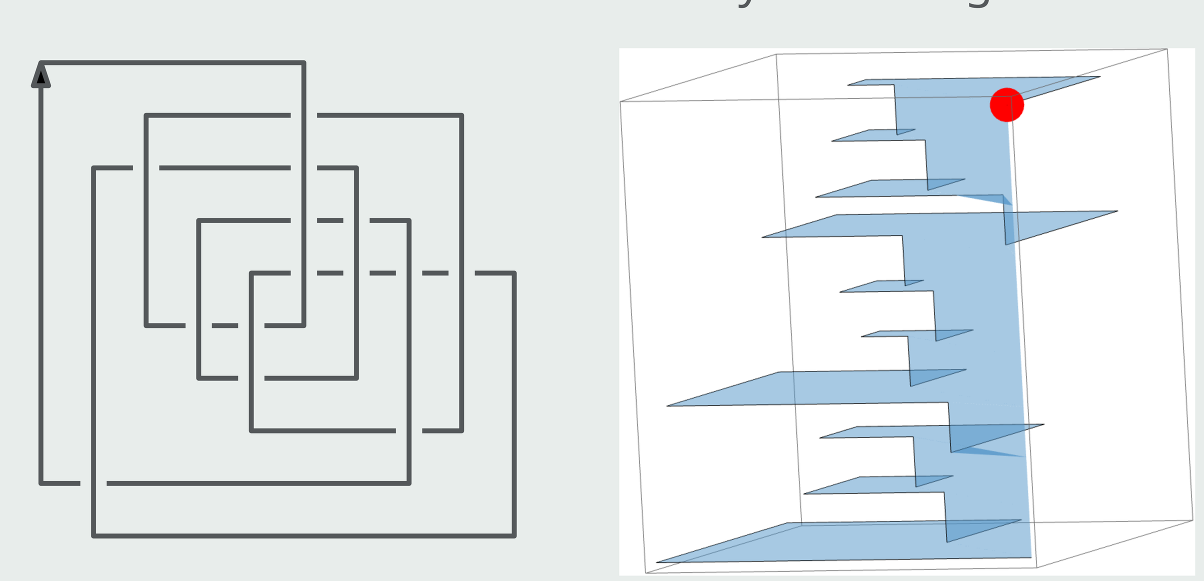
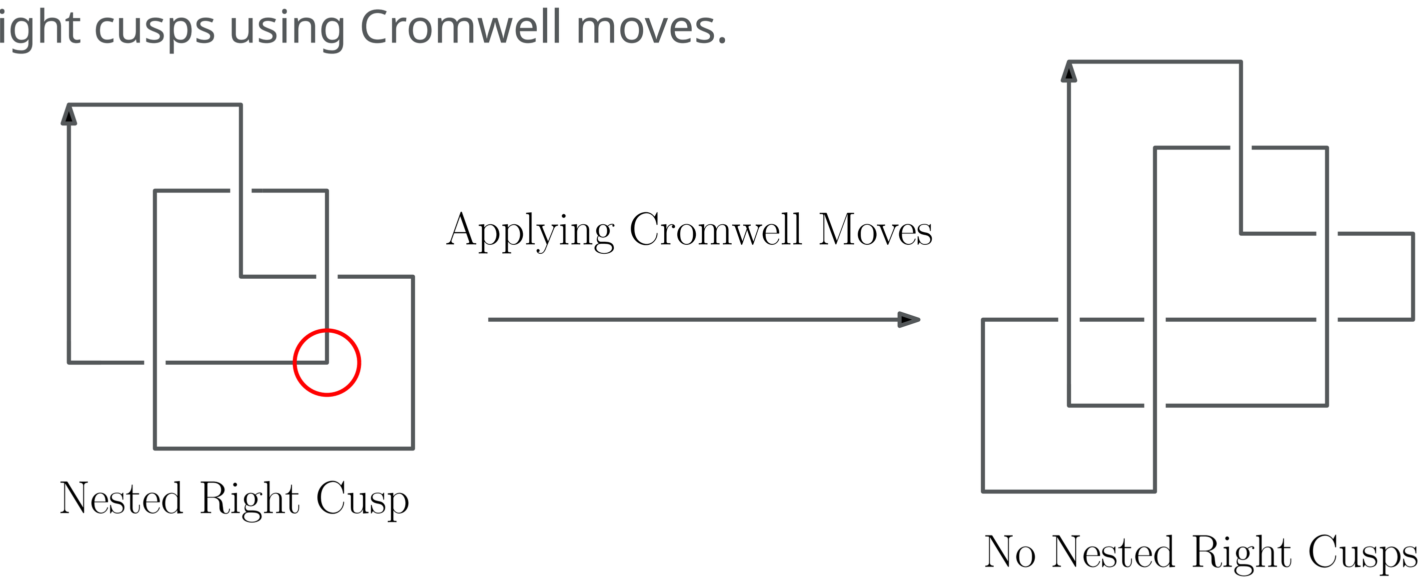


Figure: Knot 13n1907 with a 10 layer disk

The LSU Atlas

LSU has an atlas with many grid diagrams and their corresponding LCH's [3]. While the vast majority of their atlas' grid diagrams had their corresponding LCH, there was a small number of grids for which their LCH had not been successfully computed. To summarize this issue:

- Existing programs cannot handle "nested" right cusps.
- Instead, they convert to a different diagram that doesn't have nested right cusps using Cromwell moves.



- This process generally increases the amount of crossings and right cusps $\Longrightarrow$ harder to compute LCH

Results - 17 New LCHs!

Our program can handle those nested right cusps, meaning that when we computed the LCHs for all of the LSU grids, we were able to produce 17 new LCH computations!

As an example, the previous knot 13n1907 was one such knot diagram that we had computed the LCH for. While its LCH was originally unknown, we computed that its LCH was $\{6+t, 8+t^{-1}+2t\}$.

Future Work

Recently, the persistent LCH of a Legendrian embedding has been defined, but sample computations remain few. During the summer, we've started writing code which computes the persistent LCH associated to a Legendrian grid diagram. Once we fully finish programming this, we will then use the completed program to search for theoretical connections between persistent LCH and complexity of a knot diagram.

Acknowledgments and References

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