# Topology of Sequences up to the Indices of Elements

Elijah Guptill

California Polytechnic State University, San Luis Obispo

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# Some Recursive Sequences

Let $(b_n)_{n=1}^{\infty}$ be a sequence defined by $b_1 = 1$, $b_{2n} = 2b_n + 2$, and $b_{2n+1} = 3b_n + 4$. Additionally, let $(q_n(x))_{n=1}^{\infty}$ be a sequence of polynomials in the variable $x$ defined by $q_1(x) = 1$, $q_{2n}(x) = 2q_n(x) + x$, and $q_{2n+1}(x) = 3q_n(x) + 4$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $n$ in base 2 | 1 | 10 | 11 | 100 | 101 | 110 | 111 |
| $b_n = q_n(2)$ | 1 | 4 | 7 | 10 | 16 | 16 | 34 |
| $q_n(x)$ | 1 | $x+2$ | 7 | $3x+4$ | $3x+10$ | $x+14$ | 25 |

## Definition

Given a sequence a polynomials $q_1(x), q_2(x), q_3(x), \ldots$ a *shifting repetition* is a pair of numbers in $i$ and $j$ in $\mathbb{N} = 1, 2, 3, \ldots$ so that $q_i(x_0) = q_j(x_0)$ for some number $x_0$ but $q_i(x) \neq q_j(x)$ and $i \neq j$. Additionally, we say $x_0$ induces a shifting repetition.

- Given a polynomial sequence $(q_n(x))_{n=1}^{\infty} = q_1(x), q_2(x), q_3(x), \ldots$ If we know where terms repeat in the sequence $(q_n(c))_{n=1}^{\infty}$ can we figure out where terms repeat in $(q_n(x_0))_{n=1}^{\infty}$ for $x_0$ is really close to $c$?
- We will use the mathematical field of topology which generalizes continuous functions.

# Continuous functions

Let $f$ be a function from the real numbers, $\mathbb{R}$, to the real numbers. Then $f$ is continuous at a point $c$ if for all real numbers $\epsilon > 0$, there exists a real number $\delta > 0$ so that if $x$ is a real number and $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.
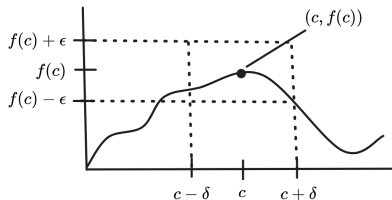


Figure: A graphical representation of a real function continuous at $c$.

**Definition**

Let $X$ and $Y$ be metric spaces with metrics $d_1$ and $d_2$. Then a function from $X$ to $Y$ is continuous at a point $c$ in $X$ if for all real numbers $\epsilon > 0$ there exists a real number $\delta > 0$ so that if $x$ is in $X$ and $d_1(x, c) < \delta$ then $d_2(f(x), f(c)) < \epsilon$.

## Definition

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then its Initial Index Representation is a sequence $(\phi_n)_{n=1}^{\infty}$ in $\mathbb{N} = \{1, 2, 3, \ldots\}$ where

$$\phi_n = \min\{j = 1, 2, 3, \ldots \text{ so that } a_j = a_n\} \quad \text{for all } n = 1, 2, 3, \ldots.$$

Let $(a_n)_{n=1}^{\infty}$ be defined by $a_1 = 1$, $a_{2n} = a_n$, and $a_{2n+1} = a_{n+1}$ for $n = 1, 2, 3, \ldots$ and let $(\phi_n)_{n=1}^{\infty}$ be its initial index representation.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 1 | 2 | 1 | 2 | 2 | 3 |
| $\phi_n$ | 1 | 1 | 3 | 1 | 3 | 3 | 7 |

## Definition

Given two initial index representations $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ we define a metric (distance function) $d$ by

$$d(\phi, \psi) = \begin{cases} 0 & \text{if } \phi = \psi \\ 2^{-\mu} & \text{if } \phi \neq \psi \end{cases}$$

where $\mu = \min\{j = 1, 2, 3, \ldots \text{ so that } \phi_j \neq \psi_j\}$.

Let $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ be initial index representations.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|------|---|---|---|---|---|---|---|
| $\phi_n$ | 1 | 1 | 3 | 1 | 3 | 3 | 7 |
| $\psi_n$ | 1 | 1 | 3 | 1 | 5 | 6 | 7 |

$$d(\phi, \psi) = 2^{-5}.$$

> **Definition**
>
> Let $(q_n(x))_{n=1}^{\infty}$ be a fixed polynomial sequence with real coefficients. We define a function $\Lambda$ from the real numbers to the space of initial index representations by $\Lambda(x_0) = (\phi_n(x_0))_{n=1}^{\infty}$ where $(\phi_n(x_0))_{n=1}^{\infty}$ is the initial index representation for $(q_n(x_0))_{n=1}^{\infty}$ and $x_0$ is a real number.

Let $(q_n(x))_{n=1}^{\infty}$ be a sequence of polynomials in the variable $x$ defined by $q_1(x) = 1$, $q_{2n} = 2q_n(x) + x$, and $q_{2n+1} = 3q_n(x) + 4$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $q_n(2)$ | 1 | 4 | 7 | 10 | 16 | 16 | 34 |
| $(\Lambda(2))_n = \phi_n(2)$ | 1 | 2 | 3 | 4 | 5 | 5 | 7 |
| $q_n(-1)$ | 1 | 1 | 7 | 1 | 7 | 13 | 25 |
| $(\Lambda(-1))_n = \phi_n(-1)$ | 1 | 1 | 3 | 1 | 3 | 6 | 7 |

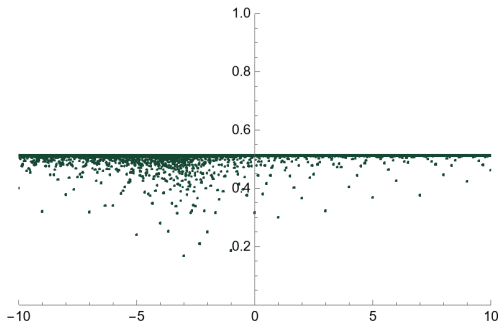# Characterization of Continuity



Figure: Let $(q_n(x))_{n=1}^{\infty}$ be a sequence of polynomials with real coefficients in defined by $q_1(x) = x$, $q_{2n}(x) = \frac{1}{2}q_n(x) - 3$ and $q_{2n+1}(x) = 2q_n(x) + 3$ for $n = 1, 2, 3, \ldots$. It would be difficult to graph initial index representations directly; instead we graph the ratio $\frac{\text{new terms}}{\text{total terms}}$ for the first 500 terms.

# Results

## Definition

Let $(q_n(x))_{n=1}^{\infty}$ be a fixed polynomial sequence with real coefficients. We define a function $\Lambda$ from the real numbers to the space of initial index representations by $\Lambda(x_0) = (\phi_n(x_0))_{n=1}^{\infty}$ where $(\phi_n(x_0))_{n=1}^{\infty}$ is the initial index representation for $(q_n(x_0))_{n=1}^{\infty}$ and $x_0$ is a real number.

## Theorem

$\Lambda$ is continuous at a real number $c$ if and only if $c$ does not induce a shifting repetition.

- We can use continuity to find sequences with little repetition given a sequence with minimal repetition.
- We cannot use a sequence with extra repetitions to find more sequences with extra repetitions.

# Acknowledgements

# Thank You!