Continuity of Mapping Evaluations of Function Sequences to Lexicographic Orders Measuring Repetition

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Abstract

In general it is difficult to determine whether a recursive sequence has a repeated term. We endeavor to study how the indices of repeated elements in the evaluations of polynomial sequences are influenced by the evaluated term. We define the notion of induced and shifting repetition indices. To study these properties, we introduce a topology measuring repeated symbols in a word. We put an equivalence class on words: two words are similar if and only if they have repeated symbols at the same indices. We put a total order on these equivalence classes and find an order isomorphism to a subset of $\mathbb{N}^{\mathbb{N}}$ under the lexicographical ordering. As a topological space with the order topology, we call this subset the space of initial index representations, S'. We show that S' is homeomorphic to the Cantor Set. Finally, we examine the function Λ that sends its input to the initial index representation of its evaluation for a fixed polynomial sequence. We find that Λ is continuous at x if and only if $\Lambda(x)$ is sent to the initial index repetition of the corresponding polynomial sequence. Equivalently, Λ is continuous at x if and only if x induces no shifting repetitions. The points of continuity in Λ resemble the points of continuity in the Thomae's function. We conclude by introducing the κ-initial index representation. We show that sending an element of a metric space to the κ-initial index representation of is evaluation relative to a fixed sequence of continuous functions from a metric space into an ultra-metric space is continuous.

1 Introduction

In this paper we will investigate repetition in sequences generated by the evaluation of a fixed sequence of continuous functions. We hope to impose a kind of continuity. If we know how terms in$\{f(c)\}_{n=1}^{\infty}$ repeat, can we determine how terms repeat in $\{f_n(x)\}_{n=1}^{\infty}$ where x is close to c?

We note some of the language and terminology to be used throughout this paper.

If X and Y are sets, then X^Y denotes all of the functions from from Y to X. We will consider elements of X^Y to be words with index set Y and symbols in X. In the case the $Y = \mathbb{N}$, we will call elements of X^Y sequences in X. For a larger look at words, consider [7].

We begin by examining repetition in some recursive sequences generated by evaluating recursive polynomial sequences. Then we will state the goal explored in this paper.

Example 1.1. Let $a_1 = 1$, $a_{2n} = a_n$, and $a_{2n+1} = a_n + 1$ for $n \in \mathbb{N}$.

n	1	2	3	4	5	6	7	8	9	10
n in base 2	1	10	11	100	101	110	111	1000	1001	1010
a_n	1	1	2	1	2	2	3	1	2	2

Table 1: A table containing the first 10 values of the $\{a_n\}_{n=1}^{\infty}$, which is A000120 from the OEIS.

In Table 1 we can see that $\{a_n\}_{n=1}^{\infty}$ has repetition: $a_3 = a_5 = a_9 = a_{10}$. Upon closer inspection, it can be shown that a_n counts the number of 1's in the binary expansion of n [5, A000120]. Due to this property, it is very easy to find indices of repetition in $\{a_n\}_{n=1}^{\infty}$. In fact $a_n = a_m$ for $n, m \in \mathbb{N}$ if and only if n and m have the same number of 1's in their respective binary representations.

Now let $b_1 = 1$, $b_{2n} = 2b_n + 2$, and $b_{2n+1} = 3b_n + 4$ for $n \in \mathbb{N}$. The first few values of $\{b_n\}_{n=1}^{\infty}$ are shown in Table 2.

n	1	2	3	4	5	6	7	8	9	10
n in base 2	1	10	11	100	101	110	111	1000	1001	1010
b_n	1	4	7	10	16	16	25	22	34	34

Table 2: A table showing the first ten binary expansions of n and the first ten values of $\{b_n\}_{n=1}^{\infty}$.

Similar to $\{a_n\}_{n=1}^{\infty}$, we see that $b_9 = b_{10}$. It is the case that $b_n = b_m$ if n and m have the same number of ones and zeros after the leading 1 in their binary expansions. To see why is is true, let $f(x) = 2x + 2$ and $g(x) = 3x + 4$. We can see that f and g commute.

$$f(g(x)) = 2(3x + 4) + 2 = 6x + 10 = 3(2x + 2) + 4.$$

If we applied f and g the same number of times for different indicies of the sequence we could iteratively use the fact that f and g commute to move all the f's to the left and g's to the right. At this point it becomes clear that these indices of the sequence have the same value. For example, 9 and 10 have the same number of 1's and 0's in their binary expansion. Observe that

$$b_9 = f(f(g(x)) = f(g(f(x)) = b_{10}.$$

That is, if n and m have the same number of 0's and 1's after the leading 1 in their binary expansions, then $b_n = b_m$. It should be easy to see that this property is lost when f and g do not commute.

Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of polynomials with real coefficients so that $q_0(x) = 1$, $q_{2n}(x) = 2q_n(x) + x$, and $q_{2n+1}(x) = 3q_n(x) + 4$ for all $n \in \mathbb{N}$. The first few values of are shown in Table 3

n	1	2	3	4	5	6	7	8
n in base 2	1	10	11	100	101	110	111	1000
b_n	1	4	7	10	16	16	25	34
$q_n(x)$	$x+2$	7	$3x+4$	$3x+10$	$x+14$	25	$7x+8$	$9x+16$

Table 3: A table showing the first ten values of the binary expansion of n, $\{b_n\}_{n=1}^{\infty}$, and $\{q_n(x)\}_{n=1}^{\infty}$.

First take note that $\{q_n(2)\}_{n=1}^{\infty} = \{b_n\}_{n=1}^{\infty}$. Here it is plain to see that indices $n, m \in \mathbb{N}$ having the same quantities of digits in their binary expansions will not ensure that $q_n(x) = q_m(x)$ for $x \neq 2$. Observe that $q_5(x) \neq q_6(x)$. For each x, $\{q_n(x)\}_{n=1}^{\infty}$ corresponds to a sequence constructed similarly to $\{b_n\}_{n=1}^{\infty}$, and repetition can manifest very different in $\{q_n(x)\}_{n=1}^{\infty}$. However, we might hope that for values of x very close to 2, that $\{q_n(x)\}_{n=1}^{\infty}$ exhibits repetitions similar to $\{b_n\}_{n=1}^{\infty}$. We will formalize this hope with topology.

We will now set up some definitions to describe the idea of continuously tracking repetitions in a one-parameter family of sequences. In Example 1.1, we had a sequence parameterized by x, and we can think of it as an initial condition.

Definition 1.2. Let X and Y be topological spaces. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from X to Y. Then evaluation corresponding to $\{f_n\}_{n=1}^{\infty}$ is the map $\mathrm{ev} : X \to Y^{\mathbb{N}}$ given by $x \mapsto \{f_n(x)\}_{n=1}^{\infty}$.

Now suppose we have topological spaces X and Y with a sequence of continuous functions from X to Y, $\{f_n\}_{n=1}^{\infty}$. Let ev be the corresponding evaluation map. We want to come up with a topological space Z and a map $\tau : Y^{\mathbb{N}} \to Z$ so that if $y \in Y^{\mathbb{N}}$ then $\tau(y)$ measures how and where distinct appear in y. Additionally, we want $\tau \circ ev$ to be continuous. If we can ensure this, given the way terms repeat in $\{f_n(c)\}_{n=1}^{\infty}$ we would know that for x near c that terms repeat similarly in $\{f_n(x)\}_{n=1}^{\infty}$.

Before moving on we should note that the sequences in Examples 1.1 are examples of k-regular sequences. k-regular sequences were first introduced by Allouche and Shallit in [2] as a generalization of automatic sequences, which are generated by finite automaton. In general it is an undecidable decision question to determine whether or not a k-regular sequences has a repeated element [6]. This indicates that determining whether there is repetition in such sequences is a difficult. This obstacle informs our goal to continuously measure repetition in a family of sequences derived from evaluation of a fixed sequence of continuous functions. If we understand the repetition induced at a particular evaluation, we hope to continuously extend this understanding to 'nearby' evaluations. It should be noted that this is a bold goal, and we will find many obstacles to this approach.

Furthermore if Z were an order topology and $\tau \circ \mathrm{ev}$ is continuous, we could apply the Topological Intermediate Value Theorem to show the existence of sequences with certain patterns of repetition.

Definition 1.3 ([4, p. 84])**.** Let X be a set with a partial order and assume that X has more than one element. Let $\mathfrak{B}$ the collection of all sets of the following types:

1. All open intervals of the form (a, b) in X.
2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathfrak{B}$ is a basis for a topology on X, this is the order topology.

Definition 1.4. Let X be a topological space. A separation of X is a pair of disjoint non-empty open sets U and V so that $U \cup V = X$. X is connected if there is no separation of X.

Theorem 1.5 ([4])**.** *Let X be a connected topological space and Y an order topology. If $f : X \to Y$ is continuous $y_1 < y_2 < y_3 \in Y$, $f(a) = y_1$, and $f(b) = y_3$, then there there exists $c \in X$ so that $f(c) = y_2$.*

Example 1.6. Let $f(x) = 2x^2 - 1$, $g(x) = 4x^3 - 3x$, and $h(x) = 8x^4 - 8x^2 + 1$. f, g, and h are the first three Chebyshev polynomials. It follows the f, g, and h commute pairwise. Now define $q_n(x, t) : \mathbb{R} \times [0, 1] \to \mathbb{R}$ by $q_{2n}(x, t) = f(q_n(x, t))$, $q_{2n+1}(x, t) = (1 - t)g(q_n(x, t)) + th(q_n(x, t))$, and $q_1(x) = x$. Just like Example 1.1 $q_n(x, t) = q_m(x, t)$ if n and m have the same number of 1's and 0's following the leading 1 in their binary expansions for $t \in \{0, 1\}$. If we know how terms repeat when $t = 0$ and when $t = 1$, that is we know how terms repeat in $\{q_n(x, 0)\}_{n=1}^{\infty} \in Y^{\mathbb{N}}$ and $\{q_n(x, 1)\}_{n=1}^{\infty} \in Y^{\mathbb{N}}$, and by extension we know the value of $\tau\left(\{q_n(x, 0)\}_{n=1}^{\infty}\right) \in Z$ and $\tau\left(\{q_n(x, 1)\}_{n=1}^{\infty}\right) \in Z$. In the case that $\tau\left(\{q_n(x, 0)\}_{n=1}^{\infty}\right) < \tau\left(\{q_n(x, 1)\}_{n=1}^{\infty}\right)$ we could use the Intermediate Value Theorem to prove there exists $t_0 \in (0, 1)$ so that $\tau\left(\{q_n(x, 0)\}_{n=1}^{\infty}\right) < \tau\left(\{q_n(x, 0)\}_{n=1}^{\infty}\right) < \tau\left(\{q_n(x, c)\}_{n=1}^{\infty}\right)$.

Example 1.6 demonstrates the potential of using the Intermediate Value Theorem, although it is a bold hope.

2 Only Recording where Terms Repeat

We will first attempt to measure repetition by recording where terms are repeat and how they are related. Later, this will be built into a topology. If A and B are sets, $|A|$ denotes the cardinality of A. Note that $|A| = |B|$ if there is a bijection between them and $|A| \geq |B|$ if there exists an injection from A to B. In order to measure repetition on words, we define a relation to throw away all information about a word, except for where symbols occur.

Definition 2.1. Let A be some set and I a totally ordered index set such that $|A| \geq |I|$. Let $\sim$ be a relation on $A^I \times A^I$ defined by $a \sim b$ if $a_i = a_j \iff b_i = b_j$ for all $i, j \in I$ and $a, b \in A^I$.

Colloquially, two sequences are similar if new terms appear at the same indices and terms repeat at the same indices.

Example 2.2. Let $A = \{x, y, z, w\}$ and $I = \{1, 2, 3\}$. Let $a = x, y, x$ be a sequence in A. Note that a can be treated as an element A^I. Now let $b, c \in A^I$ be defined by $b = z, w, z$ and $c = z, w, w$. Observe that $a \sim b$. The symbols in a and b appear and repeat in the same ways at the same indices, disregarding the actual symbols. Note that $a \not\sim c$ because $a_1 = a_3$ but $c_1 \neq c_3$. c is lacking a repetition that a has.

Proposition 2.3. $\sim$ *is an equivalence relation.*

Proof. We show that $\sim$ is reflexive, symmetric, and transitive.

First we show that $\sim$ is reflexive. Let $a \in I^I$. Then it is clear that $a_i = a_j \iff a_i = a_j$ for all $i, j \in I$. Then $a \sim a$. We have that $\sim$ is reflexive.

Second we show that $\sim$ is symmetric. Let $a, b \in I^I$ so that $a \sim b$. Then $a_i = a_j \iff b_i = b_j$ for all $i, j \in I$. It is clear that $b_i = b_j \iff a_i = b_j$ for all $i, j \in I$ as the converse of an if and only if is logically equivalent to the original proposition. We have that $b \sim a$ and that $\sim$ is symmetric.

Third we show that $\sim$ is transitive. Let $a, b, c \in I^I$ so that $a \sim b$ and $b \sim c$. Then $a_i = a_j \iff b_i = b_j$ and $b_i = b_j \iff c_i = c_j$ for all $i, j \in I$. We have that

$$a_i = a_j \iff b_i = b_j \iff c_i = c_j.$$

It follows that $a \sim c$ and $\sim$ is transitive. $\square$

Definition 2.4. Let the $\mathcal{S} = \{[a] : a \in A^I\}$ where $[a] = \{b \in A^I : a \sim b\}$, the equivalence class with representative a.

2.1 Introducing an Order

Given our equivalence relation $\sim$, we would like to find an order for $\mathcal{S}$ based upon where repetitions occur in a word. One intuitive idea would be to order elements of $\mathcal{S}$ by where repetitions occur. The more repetitions a sequence has at its beginning the smaller it is. We will model this definition off of the lexicographical order for a space of sequences.

Definition 2.5. Let A and I be sets. If $a, b \in A^I$, $a < b$ if there exists $\mu \in I$ so that $a_\mu < b_\mu$ and $a_i = b_i$ for all $i < \mu$. We say that $a \le b$ if $a < b$ or $a = b$. This is called the lexicographical ordering on A^I; $\le$ is a partial order.

Example 2.6. Let $A = I = \{1, 2, 2\}$ and $a, b \in A^I$ be defined $a = 1, 2, 1$ and $b = 1, 1, 3$. It follows that $b \le a$ because $b_2 = 1 < 2 = a_2$ and $a_i = b_i$ for all $i \in A$ so that $i < 2$.

Example 2.7. We attempt to emulate a lexicographical ordering on $\mathcal{S}$. Let $A = \{x, y, z\}$ and $I = \{1, 2, 3\}$ with $a, b \in A^I$ defined by $a = x, y, x$ and $b = x, x, y$. Imagine relabeling a and b by the indices of where each element first appears. We get $\hat{a} = 1, 1, 3$ and $\hat{b} = 1, 2, 1$ we can think of these elements as representatives of $[a]$ and $[b]$, which are elements of $\mathcal{S}$. Then $a < b$ in the lexicographical ordering. Let $c = x, x, x$ and $d = x, y, z$. We can see that the element in $\mathcal{S}$ with the most repetition, $[c]$, is represented by $\hat{c} = 1, 1, 1$ and the element with the least repetition, $[d]$, is represented by $\hat{d} = 1, 2, 3$. Then $\hat{c}$ is the minimum in the lexicographical order on A^I while $\hat{d}$ is the maximum.

We will attempt to replicate the behavior of the ordering alluded to in Example 2.7 without using a canonical element or a bijection with our set of equivalence classes. The key feature being that words with more repetition earlier are smaller.

Definition 2.8. Let $<$ be a relation on $\mathcal{S} \times \mathcal{S}$ defined by $[a] < [b]$ if there exists $\mu, j_0 \in I$ so that $a_\mu = a_{j_0}$, $b_k = b_\mu$ implies that $j_0 < k$ for all $k \in I$, and $a_m = a_n \iff b_m = b_n$ for all $m, n < \mu$.

Definition 2.8 says that $a < b$ if there is some index, μ, where a and b have the same pattern of repetition before μ but a_μ is equal to some element that has appeared in a before μ at some some index j_0 and b_μ appeared for the first time in b later than index j_0.

Example 2.9. Let $A = \{x, y, z\}$ and $I = \{1, 2, 3, 4, 5\}$. Let $a, b \in A^I$ be defined $a = x, y, x, x, y$ and $b = x, y, x, z, z$. Observe that $a_4 = a_3$ and $b_k = b_4$ implies that $k > 3$. Additionally $a_m = a_n \iff b_m = b_n$ for $m, n < 4$. Then $[a] < [b]$.

Observe from Example 2.9 that we did not need to use the first time x appears in a, $a_1 = x$, but just the occurrence of x before index 4 at index 3. Example 2.10 will show the generality of Definition 2.8 versus relabeling.

Example 2.10. The order in Definition 2.8 is applicable to more families of sequences/functions than the lexicographical relabeling technique. Consider $A = \mathbb{Z}$ and $I = \mathbb{Z}$. Let $a, b \in A^I$ be defined as in Table 4.

n	$\dots$	-6	-5	-4	-3	-2	-1	0	1	$\dots$
a_n	$\dots$	0	-5	0	-3	0	-1	0	0	$\dots$
b_n	$\dots$	0	-5	0	-3	0	-1	0	-3	$\dots$

Table 4: A table showing some values of $a, b \in \mathbb{Z}^{\mathbb{Z}}$, where $a_{2n} = b_{2n} = 0$ and $a_{2n+1} = b_{2n+1} = -2n + 1$ for $n \le -1$.

Because the set of indices where a takes on the value 0 is not bounded below we cannot relabel a_0 with the first time 0 occurs. Then the idea of relabeling a as $\hat{a}$, sequence of where indices first appear in a, and considering the lexicographical order is not applicable (see Example 2.7). On the other hand see that $a_1 = a_{-4}$ and $b_k = b_1$ implies that $k \geq -3 > -4$. Additionally $a_m = a_n \iff b_m = b_n$ for all $m, n < 0$. So $[a] < [b]$.

Now that we have expressed the generality of $<$ from Definition 2.8 we will show that $<$ does in fact create an order on $\mathcal{S}$.

Definition 2.11. Let $\leq$ be a relation on $\mathcal{S} \times \mathcal{S}$ defined by $[a] \leq [b]$ if $[a] < [b]$ or $[a] = [b]$.

Lemma 2.12. *$\leq$ is a partial order on $\mathcal{S}$.*

The proof of Lemma 2.12 can be found in Appendix A.

Despite the generality of Definition 2.11 we saw in Example 2.10, $\leq$ still has some restrictions. Concisely, when two words do not have a common prefix they are incomparable.

Example 2.13. If we do not put any further restrictions on I we will often find that $\mathcal{S} = A^I/\sim$ is a partial order and not a total order. Consider the case were $A = \mathbb{Z}$ and $I = \mathbb{Z}$. Then let $a, b \in \mathbb{Z}$ as defined in Table 5.

n	$\dots$	-6	-5	-4	-3	-2	-1	0	1	$\dots$
a_n	$\dots$	-6	-5	-4	-3	-2	-1	0	0	$\dots$
b_n	$\dots$	-3	-3	-2	-2	-1	-1	0	0	$\dots$

Table 5: A table showing some values of $a, b \in \mathbb{Z}^{\mathbb{Z}}$, where $a_i = i$ and $b_i = \left\lfloor \frac{i}{2} \right\rfloor$ for all $i \in \mathbb{Z}$.

There is no $\mu \in \mathbb{Z}$ so that $a_m = a_n \iff b_m = b_n$ for all $m, n < \mu$. It follows that $[a] \not< [b]$, $[a] \not> [b]$, and $[a] \neq [b]$. We conclude that $\leq$ is not a total order on $\mathcal{S}$.

The additional assumption that I is well-ordered will make $\mathcal{S}$ a total order under $\leq$. Assuming I is a well order ensures there is a first index where the patterns of repetition can disagree between two elements in A^I.

Theorem 2.14. *Suppose that I is a well-ordered set, then $\leq$ is a total order on $\mathcal{S}$.*

The proof of Theorem 2.14 can be found in Appendix A.

With the additional assumption that I is well-ordered we are able to show that $\mathcal{S}$ is a total order. A well-order is one in which every subset has a minimum element. Although we are not particularly interested in studying a total order over a partial order, it would be nice to come up with a canonical representation for each equivalence class $[a] \in \mathcal{S}$. Working directly with the elements of $\mathcal{S}$ can be difficult due to technical definition of $\leq$. It will turn out that when I is well-ordered ordered, we can easily come up with an element of I^I to represent

$[a]$.

Before moving on it should be noted that the lexicographical-type order on $\mathcal{S}$ can been seen as an extension of a monoid-induced order. Let A and I be sets so that $|A| \geq |I|$. The order in Definition 2.11 is actually an extension of the partial order induced by a monoid of equivalence classes similar to $\mathcal{S}$. Let $\mathcal{A} = \bigcup_{j=1}^{\infty} A^j$ where $A^j = \prod_{i=1}^{j} A$ is the Cartesian product. Define $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $a \times b \mapsto (a, b)$ where $a \in A^i$, $b \in A^j$, and $(a, b) \in A^{i+j}$. $(a, b) \in A^{i+i}$ is defined by $(a, b)_i = (a_i, b_i)$ for all $i \in I$. $\oplus$ is well defined on equivalence classes under $\sim$. It follows that $\mathcal{A}/\sim$ is a monoid under $\oplus$ and the order naturally produced by $\oplus$ is a restriction of the order on $\mathcal{S}$ from Definition 2.11.

2.2 An Order Isomorphism

Given two elements $[a]$ and $[b]$ in $\mathcal{S}$, using Definition 2.8 to show $[a] < [b]$ is verbose. In this section we introduce representative elements of $[a]$ and $[b]$ that can be compared to determine whether $[a] < [b]$. This will be done via an order isomorphism between $\mathcal{S}$ and the set of representatives.

Definition 2.15. Let A be a set such that $|A| \geq |I|$ where I is some index set. Let $a \in A^I$, $\phi \in I^I$ is an index representation of a if $a_i = a_j \iff \phi_i = \phi_j$ for all $i \in I$.

Example 2.16. Let $A = \{x, y, z\}$ and $I = \{1, 2, 3\}$. Observe that $a = x, x, y \in A$ and that $b_1 = 1, 1, 2$, $b_2 = 2, 2, 1$, and $b_3 = 1, 1, 3$ are all index representations of a.

Definition 2.17. Let A be a set such that $|A| \geq |I|$ where I is some index set and I is well-ordered. Let $a \in A^I$, then its initial index representation, $\phi \in I^I$, is defined

$$\phi_i = \min\{j \in I : a_i = a_j\} = \min\{j \in I : a_i = a_j, j \leq i\} \, \forall i \in I.$$

Note that this definition is only well-defined because we assumed that I is well-ordered.

Example 2.18. Let $A = \mathbb{N}$ and $I = \mathbb{N}$. If $a = 5, 5, 6, 5, 7, 5, 6, 8, \ldots \in A^I$ its initial index representation is $\phi = 1, 1, 3, 1, 5, 1, 3, 8, \ldots$. Observe that ϕ is an index representation for a.

Initial index representations will act as a way to represent equivalence classes $[a] \in S$.

Proposition 2.19. *Let A be a set such that $|A| \geq |I|$ where I is some well-ordered index set. Let $a \in A^I$ and $\phi \in I^I$ be its initial index representation. Then ϕ is an index representation of a. Additionally, if $b \in A^I$ and $\psi \in I^I$ is its Initial Index Representation, $a \sim b$ implies that $\phi = \psi$.*

Proof. First we show that an initial index representation is an index representation. Let $a \in A^I$ and $n, m \in I$.

$\implies$ Suppose that $a_n = a_m$. Observe that $\phi_n = \min\{j \in I : a_n = a_j\}$ and $\phi_m = \min\{j \in I : a_m = a_j\}$. Then $a_n = a_m$ and $\phi_n = a_n$ so $\phi_n = a_m$. Then $\phi_n \in \min\{j \in I : a_m = a_j\}$ so $\phi_m \leq \phi_n$. A similar argument shows that $\phi_n \leq \phi_m$. Then $\phi_n = \phi_m$.

$\Longleftarrow$ Suppose that $\phi_n = \phi_m$. Recall that $\phi_n = a_n$ and $\phi_m = a_m$. This and the fact that $\phi_n = \phi_m$ implies that $a_n = a_m$.

We have shown that $a_n = a_m \iff \phi_n = \phi_m$. Then ϕ is an index representation for a from the definition of initial index representation.

Now let $a, b \in A^I$ such that $a \sim b$ with corresponding initial index representations $\phi, \psi \in I^I$. Observe that

$$j \in \{j \in I : a_n = a_j\} \iff a_n = a_j \iff b_n = b_j \iff j \in \{j \in I : b_n = b_j\}.$$

Then $\{j \in I : a_n = a_j\} = \{j \in I : b_n = b_j\}$. The definition of initial index representation quickly implies that $\phi_n = \psi_n$. Because we started with arbitrary $n \in \mathbb{N}$, we have that $\phi = \psi$. $\square$

The second part of Proposition 2.19 implies that initial index representations are well-defined on equivalence classes under $\sim$. We will now show that initial index representations can represent elements of $\mathcal{S}$.

Definition 2.20. Let the ordered set of initial index representations be $\mathcal{S}'$ where

$$\mathcal{S}' = \{a \in I^I : a_k = \min\{j \in I : b_k = b_j\} \, \forall k \in I\}$$

and $\leq$ is the lexicographical order on I^I restricted to $S' \subseteq I^I$. Note that this set is just the image of A^I under the map sending an element of A^I to its initial index representation.

Theorem 2.21. *$\mathcal{S}$ is order isomorphic $\mathcal{S}'$ via $\tau : \mathcal{S} \to \mathcal{S}'$ where $[a] \mapsto \phi$ and $\phi \in I^I$ is the initial index representation for a.*

Proof. It is immediate from Proposition 2.19 that τ is well-defined.

We first show that τ is a bijection.

Lets star with injectivity. Suppose that $[a], [b] \in \mathcal{S}'$ so that $\tau([a]) = \tau([b]) = \phi \in I^I$. From Proposition 2.19 we have that ϕ is an index representation for a and b. By Definition 2.15 we have that

$$a_i = a_j \iff \phi_i = \phi_j \iff b_i = b_j \, \forall i, j \in I.$$

Then $a \sim b$ which implies that $[a] = [b]$.

Second, we show that τ is surjective. Suppose that $\phi \in \mathcal{S}'$. By assumption we have that $|A| \geq |I|$. Then there exists an injection $g : I \to A$. Now define $a \in A^I$ by

$$a_i = g(\phi_i) \, \forall i \in I.$$

It is clear that $a \in A^I$. It remains to show that a has initial index representation ϕ. Let $\psi \in I$ be the initial index representation for a. We will now that that $a_i = a_j \iff \phi_i = \phi_j$ for all $i, j \in I$.

$\implies$ Suppose that $a_i = a_j$. By the definition of a, $g(\phi_i) = g(\phi_j)$. Because g is an injection it follows that $\phi_i = \phi_j$.

$\impliedby$ Suppose that $\phi_i = \phi_j$. Then $g(\phi_i) = g(\phi_j)$ and the definition of a implies that $a_i = a_j$.

It has been shown that $a_i = a_j \iff \phi_i = \phi_j$ for all $i, j \in I$. It follows that $\{j \in I : a_n = a_j\} = \{j \in I : \phi_n = \phi_j\}$ for all $n \in I$. Note that the initial index representation of ϕ is ϕ.Then $\phi_n = \psi_n$ for all $n \in I$, $\phi = \psi$, and $\tau(a) = \psi = \phi$. We have shown that τ is surjective.

Then τ is a bijection. It remains to shown that τ preserves order in both directions.

Let $[a], [b] \in \mathcal{S}'$ with $\phi, \psi \in I^I$ being their corresponding initial index representations.

$\implies$ Suppose that $[a] \leq [b]$. If $[a] = [b]$ it is clear that $\tau([a]) = \phi = \psi = \tau([b])$ and so $\tau([a]) \leq \tau([b])$. We may now assume that $[a] < [b]$. Then there exists $\mu, j_0 \in I$ so that $a_\mu = a_{j_0}$ and $b_\mu = b_k$ implies that $j_0 < k$ for all $k \in I$. Additionally, $n, m < \mu \in I$ implies that $a_n = a_m \iff b_n = b_m$. We want to show that $\phi_n = \psi_n$ for $n < \mu$ and $\phi_\mu < \psi_\mu$. We begin with the former. Observe that $j_0 \in \{j \in I : a_\mu = a_j\}$. Then $\phi_\mu \leq j_0$. Next observe that $b_\mu = b_{\psi_\mu}$ which implies that $j_0 < \psi_\mu$. It follows that $\phi_\mu \leq j_0 < \psi_\mu$. Now let $n < \mu \in I$. Observe that that $\{j \in I : a_n = a_j, j \leq n\} = \{j \in I : b_n = b_j, j \leq n\}$ because $n < \mu$. It follows that $\phi_n = \psi_n$ for $n < \mu$. Then $\phi < \psi$ in the lexicographical order by definition. In either case $\phi \leq \psi$

$\impliedby$ Suppose that $\phi = \tau([a]) \leq \tau([b]) = \psi$. Suppose that $\phi = \psi$. Then $[a] = [b]$ because τ is an injection. We may now assume that $\phi < \psi$. Because $\phi < \psi$ there must be some index where ϕ is less than ψ. Because I is well-ordered there must be a first index. Then there exists $\mu \in I$ so that $\phi_n = \psi_n$ for all $n < \mu \in I$ and $\phi_\mu < \psi_\mu$. Recall that ϕ and ψ are index representations for a and b, it follows that

$$a_n = a_m \iff \phi_n = \phi_m \iff \psi_n = \psi_m \iff b_n = b_m.$$

for all $n, m < \mu \in I$. Observe that $a_\mu = a_{\phi_\mu}$. Now suppose that $k \in I$ and $b_\mu = b_k$. Notice that $k \in \{j \in I : b_\mu = b_j\}$. Then $\psi_\mu \leq k$. It follows that $\phi_\mu < \psi_\mu \leq k$. Then $[a] < [b]$. In either case $[a] \leq [b]$.

We have now shown that f is a bijective order isomorphism. $\square$

Recall Section 1. We want to take an indexed collection of functions $(Y^X)^I$ where I is ordered and track how terms repeat in the evaluations by composing $\mathrm{ev} : X \to Y^I$ with some map $Y^I \to Z$. We came up with $\mathcal{S}$ as a ordered set, ordering functions by how quickly new terms appear. Under the order topology, $\mathcal{S}$ is our candidate for Z. We have just shown with Theorem 2.21 that $\mathcal{S}$ and $\mathcal{S}'$ are order isomorphic. If $\mathcal{S}$ and $\mathcal{S}'$ are homeomorphic we can let $Z = S'$ instead of S. This will be nice as it is easier to do proofs with elements of S'. Then we can examine the continuity of $\tau \circ \mathrm{ev} : X \to \mathcal{S}'$ instead of mapping into $\mathcal{S}$ directly. It is a simple exercise to show the following result.

Lemma 2.22. *Suppose that $(X, \leq_2)$ and $(Y, \leq_2)$ are paritally ordered sets and $f : X \to Y$ is an order isomorphism. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are their respective order topologies then $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are homeomorphic.*

The desired result immediately follows.

Corollary 2.23. *$\mathcal{S}$ and $\mathcal{S}'$ are homeomorphic in their respective order topologies.*

We can now study $\mathcal{S}'$ instead of $\mathcal{S}$ to determine topological properties of these spaces as well as the behavior of $\tau \circ \mathrm{ev} X \to S'$. The order definition on $\mathcal{S}'$ is easier to work with than the order on $\mathcal{S}$.

3 Space of Initial Index Representations

In Corollary 2.23 we have showed that S and $\mathcal{S}'$ are homeomorphic (when I is well ordered). We may now investigate the topological properties of $\mathcal{S}'$ to study the topological properties of $\mathcal{S}$. In the next section we will assume that $I = \mathbb{N}$, but we will try to be more general for some of the results in this section.

Definition 3.1. Let the space of initial index representations be $\mathcal{S}'$ (see Definition 2.20 in the order topology induced by the lexicographical order.

From now on any reference to $\mathcal{S}'$ should be assumed to refer to S' as a topological space with the specified topology.

3.1 Topological Properties

When $I = \mathbb{N}$, we want to apply the following theorem S' characterize S'.

Theorem 3.2 ([9, 30.4 Corollary])**.** *The Cantor Set is the only totally disconnected perfect compact metric space.*

We will now show that $\mathcal{S}'$ is totally disconnected, perfect, metrizable, and compact. We will by proving that $\mathcal{S}'$ is compact.

Theorem 3.3 ([4, Theorem 27.1])**.** *Let X be a total order having the least upper bound property. In the order topology, each closed interval in X is compact.*

We will use Theorem 3.3 to show that S' is compact. First, we show that S' is a closed interval. Second, we show that S' has the least upper bound property.

Definition 3.4. Suppose that I is well-ordered and $o = \min(I)$. Then $\hat{o} \in I^I$ is defined $\hat{o}_i = o$ for all $i \in I$ and $\hat{n} \in I^I$ is defined $\hat{n}_i = i$ for all $i \in I$.

Proposition 3.5. *If I is well-ordered, $\mathcal{S}' = [\hat{o}, \hat{n}] = \{a \in \mathcal{S}' : \hat{o} \leq a \leq \hat{n}\}$, the closed interval in $\mathcal{S}'$ containing elements between $\hat{o}$ and $\hat{n}$.*

Proof. First, we show that $\hat{o} \in \mathcal{S}'$. Let $n \in I$. Then $\hat{o}_o = o$, $\hat{o}_n = o$. Note $\hat{o}_n = \hat{o}_o$ implies that $o \in \{j \in I : \hat{o}_n = \hat{o}_j\}$. From the fact $o = \min(I)$ it follows that $o = \min\{j \in I : \hat{o}_n = \hat{o}_j\}$. Then $\hat{o}_n = o = \min\{j \in I : \hat{o}_n = \hat{o}_j\}$ for all $n \in I$. From the definition of initial index representation, we have that $\hat{o} \in S'$.

Second, we show that $\hat{n} \in \mathcal{S}'$. Let $n \in I$. Recall that $\hat{n}_i = i < n$ for all $i < n \in I$. Then $\hat{n}_i \neq \hat{n}_n = n$ for all $i < n$. But notice that $\hat{n}_n = \hat{n}_n$. It follows that $n \in \{j \in I : \hat{n}_n = \hat{n}_j\}$ and that $n = \min\{j \in I : \hat{n}_n = \hat{n}_j\}$. Then $\hat{n}_n = \min\{j \in I : \hat{n}_n = \hat{n}_j\}$. From the definition of initial index representation, we have that $\hat{n} \in S'$.

Third, we show that $[\hat{o}, \hat{n}] = \mathcal{S}'$. One direction of set containment is easy. Observe that if $a \in [\hat{o}, \hat{n}]$ then $a \in \mathcal{S}'$ by definition. Now for the other direction. Suppose that $a \in \mathcal{S}'$. Observe that $\hat{o}_n \leq a_n$ for all $n \in I$. It follows that $\hat{o} \leq a$ in the lexicographical order. Additionally, because $a \in \mathcal{S}'$ we have that $a_n = \min\{j \in I : a_n = a_j\} \leq n = \hat{n}_n$ as $a_n = a_n$ for all $n \in I$. Then $a_n \leq \hat{n}_n$ for all $n \in I$ and it follows that $a \leq \hat{n}$. We have shown that $a \in [\hat{o}, \hat{n}]$. $\mathcal{S}' = [\hat{o}, \hat{n}]$ as desired. $\square$

Now that we have shown $\mathcal{S}'$ is a closed interval we must show that $\mathcal{S}'$ has the least upper bound property. In order to do so we will assume that $I = \mathbb{N}$. We will use that every bounded subset of $\mathbb{N}$ has a maximum.

Theorem 3.6. *Let $I = \mathbb{N}$, then $\mathcal{S}' \subseteq I^I$ has the least upper bound property. That is if $A \subseteq \mathcal{S}'$ is non-empty there exists $b \in S'$ such that $a \leq b$ for all $a \in A$. Additionally, if $a \leq c$ for all $a \in A$, then $b \leq c$.*

Proof. This proof was modified from [8], which proves long words with finite alphabet have the least upper bound property in the lexicographical order. When $I = \mathbb{N}$ elements of S' are long words with extra conditions and an infinite alphabet. The key is that even though infinitly many symbols can appear in an element of $\mathcal{S}'$ there is a maximum on the elements that are allowed to appear at each index.

Recall that $\mathcal{S}' = \{a \in \mathbb{N}^{\mathbb{N}} : a_k = \min\{j \in \mathbb{N} : a_k = a_j\} \, \forall k \in \mathbb{N}\}$. Let $n \in \mathbb{N}$ and observe that $b_n = b_n$. It follows that $n \in \{j \in \mathbb{N} : a_n = a_j\}$. Then $\phi_n \leq n$ for all $n \in \mathbb{N}$.

Now let $A \subseteq \mathcal{S}'$ be non-empty. We will use the above observation to explicitly construct the supremum.

Let $A_n = \{a_n \in A : a_k = b_k \, \forall k < n \in \mathbb{N}\}$ where

$$b_n = \begin{cases} \max(A_n) & \text{if } A_n \neq \emptyset \\ 1 & \text{if } A_n = \emptyset \end{cases}.$$

for all $n \in \mathbb{N}$. Observe that from the beginning observation that $A_n \subseteq \mathbb{N}$ that is bounded above by n so it must have a maximum. Colloquially, this is the set of all sequences in A that start with $b_1, \ldots, b_{n-1}$.

We claim that $b \in \mathbb{N}^{\mathbb{N}}$ defined by $b_1 = 1$ and $b_n = b_n$ for all $n > 1 \in \mathbb{N}$ is a supremum for A. We first show that $b \in \mathcal{S}'$. If $n = 1$ it is clear that $b_1 = 1 = \min\{j \in \mathbb{N} : b_1 = a_j\}$. We may now assume that $n > 1$. Observe that either $A_n = \emptyset$ or $A_n \neq \emptyset$.

Case 1: Suppose that $A_n = \emptyset$. Then $b_n = 1$ and $b_1 = 1$. If follows that $1 \in \{j \in \mathbb{N} : b_n = b_j\}$. Because the set is a subset of $\mathbb{N}$ it follows that $b_n = 1 = \min\{j \in \mathbb{N} : b_n = b_j\}$.

Case 2: Suppose that $A_n \neq \emptyset$. Then $b_n = a_n$ for some $a \in A_n \subseteq A \subseteq \mathcal{S}'$. Because $a \in \mathcal{S}'$ it follows that $a_n = \min\{j \in \mathbb{N} : a_n = a_j\}$. $a \in A_n$ implies that $a_i = b_i$ for $i < n$. Clearly, $a_i = b_i$ for $i \leq n$. It follows that

$$j \in \{j \in \mathbb{N} : b_n = b_j, j \leq n\} \iff j \in \{j \in \mathbb{N} : a_n = a_j, j \leq n\}.$$

Combining these facts we get that

$$\begin{aligned} b_n = a_n &= \min\{j \in \mathbb{N} : a_n = a_j\} \\ &= \min\{j \in \mathbb{N} : a_n = a_j, j \leq n\} \\ &= \min\{j \in \mathbb{N} : b_n = b_j, j \leq n\} \\ &= \min\{j \in \mathbb{N} : b_n = b_j\}. \end{aligned}$$

In either case $b_n = \min\{j \in \mathbb{N} : b_n = b_j\}$. Then $b_n = \min\{j \in \mathbb{N} : b_n = b_j\}$ for all $n \in \mathbb{N}$ and so $b \in \mathcal{S}'$ by definition.

We will now show that b is an upper bound for A via the contrapositive. That is, if $a \in \mathcal{S}'$ and $b < a$ then $a \notin A$. Let $a \in \mathcal{S}'$ so that $b < a$. There there exists $\mu \in \mathbb{N}$ so that $a_i = b_i$ for $i < \mu$ and $b_\mu < a_\mu$. Either $A_\mu = \emptyset$ or $A_\mu \neq \emptyset$.

Case 1: Suppose that $A_\mu = \emptyset$. Because $A_\mu \neq \emptyset$ $a' \in A$ implies that there exists $i < \mu$ so that $a'_i \neq b_i$. But $a_i = b_i$ for all $i < \mu$. It must be that $a \notin A$.

Case 2: Suppose that $A_\mu \neq \emptyset$. Then $\max(A_\mu) = b_\mu < a_\mu$. It follows that $a_\mu \notin A_\mu$. Because $a_i = b_i$ for all $i < n \in \mathbb{N}$ but $a_\mu \notin A_\mu$ it must be that $a \notin A$.

We have shown that $b < a$ implies $a \notin A$. It follows that $a \in A$ implies $a \leq b$.

All that remains is to show b is the least upper bound. Once again, we use the contrapositive. That is if $c \in \mathcal{S}'$ and $c < b$, there exists $a \in A$ so that $c < a$. Because $c < b$ there exists $\mu \in \mathbb{N}$ so that $c_i = b_i$ for all $i < \mu \in \mathbb{N}$ and $c_\mu < b_\mu$. Let $n \in \mathbb{N}$. Either $A_n = \emptyset$ or $A_n \neq \emptyset$.

Case 1: Suppose that $A_\mu = \emptyset$. Then $b_\mu = 1 \leq c_\mu$ as $c \in \mathbb{N}^{\mathbb{N}}$. This is a contradiction, so it must be the other case.

Case 2: Suppose that $A_\mu \neq \emptyset$. Then there exists $a \in A$ so that $a_i = b_i$ for all $i < \mu$ and $b_\mu = \max(A_\mu) = a_\mu$. By transitivity we have that $a_i = c_i$ for all $i < \mu \in \mathbb{N}$. Additionally $c_\mu < b_\mu = a_\mu$. Then $c < a$.

In the only possible case $c < a$. We have shown the contrapositive, so c is an upper bound for a implies that $b \leq c$. It follows that b is the supremum of A as desired. Every non-empty subset of $\mathcal{S}'$ has a supremum. $\square$

Corollary 3.7. *Let $I = \mathbb{N}$, the $\mathcal{S}'$ is compact.*

Proof. From Theorem 3.6 we have that $\mathcal{S}'$ has the least upper bound property. From Proposition 3.5 we have that $\mathcal{S}' = [\hat{o}, \hat{n}]$, a closed interval in $\mathcal{S}'$. From Theorem 3.3 it follows that S' is compact. $\square$

We will now show that S' is totally disconnected and perfect using the order topology.

Definition 3.8. Let X be a topological space. X is totally disconnected if a separation can be made around any two points. That is for each pair of distinct points $a, b \in X$, there exists open sets $A, B \subseteq X$ so that $a \in A$, $b \in B$, $A \cup B = X$, and $A \cap B = \emptyset$.

Theorem 3.9. *If I is well-ordered and unbounded, S' is totally disconnected.*

Proof. Let $a, b \in \mathcal{S}'$ so that $a \neq b$. Recall from Theorem 2.14 that $\mathcal{S}'$ is a total order. Without loss of generality we may assume that $a < b$. Proposition 3.5 states that $S = [\hat{o}, \hat{n}]$. If (a, b) is empty, $[\hat{o}, a)$ and $(b, \hat{n}]$ form a separation of $\mathcal{S}'$.

We may now assume that there exists $c \in (a, b) \subseteq \mathcal{S}'$. Then there exists $\mu_1, \mu_2 \in I$ so that $a_i = c_i$ for all $i < \mu_1$, $a_{\mu_1} < c_{\mu_1}$, $c_i = b_i$ for all $i < \mu_2$, and $c_{\mu_2} < b_{\mu_2}$. Let $\operatorname{suc}(A) = \min\{x \in I : y < x \,\forall y \in A\}$. Unboundedness ensure this is well-defined. Let $\omega = \operatorname{suc}(\mu_1, \mu_2)$. Now, define $\alpha, \beta \in \mathcal{S}'$ by

$$\alpha_n = \begin{cases} c_n & \text{if } n < \omega \\ o & \text{if}\, n = \omega \\ n & \text{if}\, n > \omega \end{cases}$$

$$\beta_n = \begin{cases} c_n & \text{if } n \leq \omega \\ \min\left(\{c_i : i < \omega\} \setminus \{o\}\right) & \text{if } n = \omega \\ o & \text{if}\, n > \omega \end{cases}$$

where $o = \min(I)$. $\alpha, \beta \in \mathcal{S}'$ as their terms come from elements of $\mathcal{S}'$ until they are maximal or minimal; see of proof of Proposition 3.5. Note that β_ω is well-defined because $a_{\mu_1} < c_{\mu_1}$ implies that $c_{\mu_1} > o$.

We show that $\alpha < \beta$. Now let $i \in I$ so that $i < \omega$. It is clear that $\alpha_i = c_i = \beta_i$. Additionally, $\alpha_\omega < \beta_\omega$. By definition, $\alpha < \beta$.

See that $i < \mu_1 < \omega$ implies that $a_i = c_i = \beta_i$. Additionally, $a_{\mu_1} < c_{\mu_1} = \beta_{\mu_1}$. Then $a < \beta$ and $a \in [\hat{o}, \beta)$. Similarly, $b \in (\alpha, \hat{n}]$. We want to show that $[\hat{o}, \beta)$ and $(\alpha, \hat{n}]$ form a separation around α and β. Recall that I being well-ordered, makes it a total order so $[\hat{o}, \beta) \cup (\alpha, \hat{n}] = \mathcal{S}'$. It remains to show that $[\hat{o}, \beta) \cap (\alpha, \hat{n}] = (\alpha, \beta) = \emptyset$.

Assume for purposes of contradiction that $x \in (\alpha, \beta)$. Then there exists $\nu_1, \nu_2 \in I$ so that $i < \nu_1$ implies $\alpha_i = x_i$, $\alpha_{\nu_1} < x_{\nu_1}$, $i < \nu_2$ implies $x_i = \beta_i$, and $x_{\nu_2} < \beta_{\nu_2}$. The definition of α implies that $\mu_1 \leq \omega$. The definition of β implies $\nu_2 \leq \omega$. If $\nu_1 < \omega$, it would imply that $x > \beta$, so $\nu_1 \geq \omega$. If $\nu_2 < \omega$, it would imply that $x < \alpha$, so $\nu_2 \geq \omega$. We conclude that $\nu_1 = \nu_2 = \omega$. This implies that $o = \alpha_\omega < x_\omega < \beta_\omega = \min(\{c_i : i < \omega\} \setminus \{o\})$.

Note that $x \in \mathcal{S}$, so $0 < x_\omega \leq \omega$. Since $0 < x_\omega < \min(\{c_i : i < \omega\} \setminus \{o\})$, it must be that $x_\omega \notin \{c_i : i < \omega\} \setminus \{o\}$. Because $c, x \in \mathcal{S}'$, $x_\omega = x_{x_\omega}$ and $c_{x_\omega} \leq x_\omega$. Note that $x_\omega \notin \{c_i : i < \omega\} \setminus \{0\}$ implies that $c_{x_\omega} < x_\omega = x_{x_\omega}$. Further, $c \in \mathcal{S}'$, so $c_i \leq i$ and $x_\omega < \min(\{c_i : i < \omega\} \setminus \{o\}) < \omega$. Finally, $x_\omega < \omega$ implies that $x_{x_\omega} = c_{x_\omega}$. This is a contradiction. It must be that $(\alpha, \beta) = \emptyset$.

Then $[\hat{o}, \beta)$ and $(\alpha, \hat{n}]$ are disjoint open sets containing a and b. In any case we were able to separate a and b with open sets so $\mathcal{S}'$ is total disconnected. □

Theorem 3.10. *If I is unbounded, $\mathcal{S}'$ is a perfect topological space. That is for each $c \in \mathcal{S}'$ any open set containing c contains some other element of $\mathcal{S}'$.*

Proof. Recall that S' has the standard order topology and $S' = [\hat{o}, \hat{n}]$ from Proposition 3.5. It will suffice to show that the basis open sets containing a point contain another point. The basis open sets of S' are of the form $[\hat{o}, b)$, $(a, \hat{n}]$, and (a, b) for $a, b \in S'$. We will show if $c \in S'$ and some basis open set of the form (a, b) contains c then (a, b) contains some other element of S'. Similar arguments work for basis open sets of the form $[\hat{o}, b)$ and $(a, \hat{n}]$. Let $\mu_1 = \min\{j \in I : a_j \neq c_j\}$ and $\mu_2 = \min\{j \in I : b_j \neq c_j\}$. We can do this because I is well-ordered and $a < c < b$ implies that $a \neq c$ and $b \neq c$. Using unboundedness, let

$$\mu = \min\{j \in I : j > \max\{\mu_1, \mu_2\}\}.$$

Define $\gamma \in I^I$ to be

$$\gamma_n = \begin{cases} c_n & \text{if } n < \mu \\ \begin{cases} o & \text{if } c_\mu \neq o \\ n & \text{if } c_\mu = o \end{cases} & \text{if } n = \mu \\ o & n > \mu \end{cases}.$$

It is clear that $\gamma \in S'$. Observe that $\mu_1 < \mu$ and so $\gamma_i = c_i$ for all $i \leq \mu_1$. Additionally $a_i = c_i$ for $i < \mu_1$ and $a_{\mu_1} < c_{\mu_1}$. Then $a_i = c_i = \gamma_i$ for all $i < \mu_1$ and $a_{\mu_1} < c_{\mu_1} = \gamma_{\mu_1}$. By definition $a < \gamma$. A similar argument shows that $\gamma < b$. Furthermore, the definition of γ_μ implies that $\gamma_\mu \neq c_\mu$. This is clear in the case that $c_\mu \neq o$. Otherwise, we must show that $n \neq o$. Observe that the definition of S' implies that $w_o = o$ for all $w \in S'$. We have that $c_o = a_o = o$. It must be that $\gamma_n = \mu > \mu_1 > o$. In either case $\gamma \neq c$. Then $\gamma \in (a, b)$ and $\gamma \neq c$. We have shown that S' is perfect.

□

Finally we observe that $\mathcal{S}'$ is metrizable.

Lemma 3.11. $\mathbb{N}^{\mathbb{N}}$ *in the lexicographical order with the order topology is metrizable by* $d : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ *given by*

$$d(a, b) = \begin{cases} 0 & \textit{if } a = b \\ 2^{-\mu} \ \textit{where } \mu = \min\{n \in \mathbb{N} : a_n \neq b_n\} & \textit{if } a \neq b \end{cases}.$$

Additionally, d is an ultra-metric.

Proof. [1, pp. 6–7] contains a proof of this result for words in a finite alphabet. A similar proof can show the result. □

Corollary 3.12. *Let $I = \mathbb{N}$, then $\mathcal{S}' \subseteq I^I$ is metrizable using $d\big|_{\mathcal{S}' \times \mathcal{S}'}$ where $d : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is the metric on $\mathbb{N} \times \mathbb{N}$ from Lemma 3.11.*

Proof. This proof is technical, simple, and left to the reader. □

Now we have shown that S' is totally disconnected, perfect, metrizable, and compact, we can apply Theorem 3.3.

Corollary 3.13. *Let $I = \mathbb{N}$, then $S' \subseteq I^I$ is homeomorphic to the Cantor Set.*

Proof. The result follows from Theorem 3.9, Theorem 3.10, Corollary 3.7, Corollary 3.12, and Corollary 3.2. □

3.2 Consequences on Evaluation Maps

In Theorem 3.9 we were able to show that $\mathcal{S}'$ is totally disconnected. Recall from Section 1 that we want to find topological spaces X, Y with so that evaluation $\mathrm{ev} : X \to Y^{\mathbb{N}}$ composed with some function $\tau : Y^{\mathbb{N}} \to Z$ is continuous. We let $Z = \mathcal{S}'$ and let $\tau : Y^{\mathbb{N}} \to \mathcal{S}'$ be the function sending an element of $Y^{\mathbb{N}}$ to its initial index representation. Finally, Recall the following classic result from topology: the image of a connected set under a continuous map is connected [4, Theorem 23.5].

Corollary 3.14. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, let $\mathrm{ev} : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ be evaluation corresponding to $\{f_n\}_{n=1}^{\infty}$, and $\tau : \mathbb{R}^{\mathbb{N}} \to \mathcal{S}'$ be the function sending a sequence in $\mathbb{R}$ to its initial index representation. Then $\Lambda = g \circ \mathrm{ev}$ is continuous if and only if Λ is constant.*

Proof.

- $\Longrightarrow$ Suppose that Λ is continuous. Then $\Lambda(\mathbb{R})$ is a connected subspace of $\mathcal{S}'$. Recall from Theorem 3.9 that S' is totally disconnected. The only connected subsets of a totally disconnected space are singletons. It must be that $\Lambda(\mathbb{R}) = \{a\}$ for some $a \in \mathcal{S}'$. Then Λ is a constant map.
- $\Longleftarrow$ Suppose that Λ is a constant map. Λ is trivially continuous.

□

Then we cannot use this $(X = \mathbb{R}, Y = \mathbb{R}, I = \mathbb{N}, Z = S', \tau)$ to study the repetition induced in evaluations of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Because Λ is continuous if and only if it is constant, we have no interesting continuous functions to study. It is clear that the connectedness of $\mathbb{R}$ is a problem from Theorem 3.14, but is connectedness the only thing we must avoid? It will turn out that the problem is not connectedness but that the number of possible ways elements in a sequence can be related. We will be able to characterize points of continuity in Λ if we restrict to polynomials. This is a reasonable restriction as dynamics over polynomials is already very difficult, so it must be very hard for Λ to be continuous.

Figure 1 will help us investigate whether there are any points of $\mathbb{R}$ on which Λ is continuous.

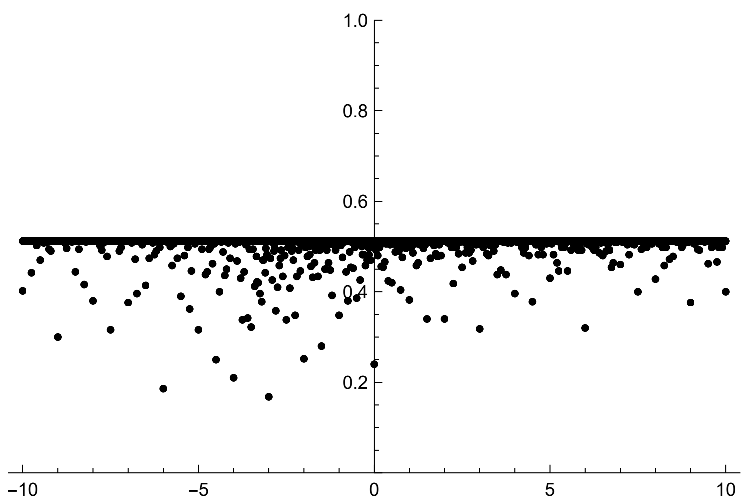


Figure 1: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of polynomials with coefficients in $\mathbb{R}$ defined by $f_1(x) = x$, $f_{2n}(x) = \frac{1}{2}f_n(x) - 3$ and $f_{2n+1}(x) = 2f_n(x) + 3$ for $n \geq 1 \in \mathbb{N}$. It would be difficult to graph elements of $\mathcal{S}'$ because it is homeomorphic to the Cantor set; instead we graph the ratio $\frac{\text{distinct terms}}{\text{total terms}}$ for the first 500 terms.

Looking at Figure 1, we can see that many of the tested sequences have the same ratio of distinct terms to total terms that is about 0.52. Additionally, 0.52 appears to be an upper bound. We can also see that there are terms with lower ratios and more repetition. We might conjecture that Λ is continuous only at the points with the most distinct terms. That is terms near 0.52. We will appeal to classical real analysis for a proof technique. Recall the Thomae Function $t : (0,1) \to \mathbb{R}$ defined by

$$t(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ p \in \mathbb{N},\ q \in \mathbb{N}, \text{ and } \gcd(p,q) = 1 \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

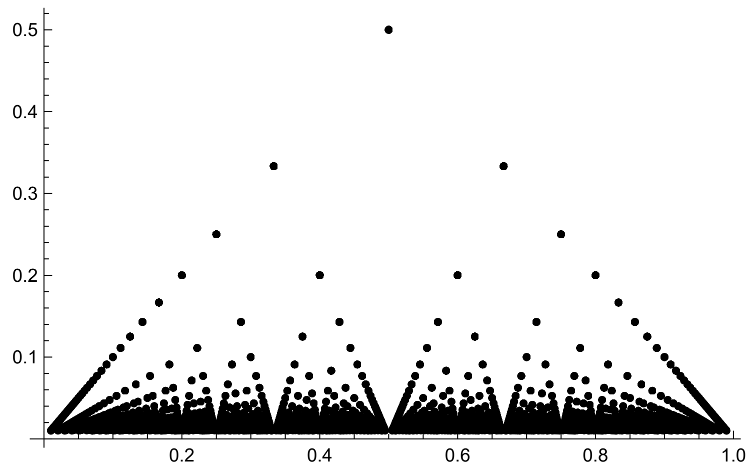


Figure 2: A graph of Thomae's function.

Observe that Figure 2 and Figure 1 bear a striking resemblance. Both of their images accumulate near a horizontal line, acting as a lower or upper bound. It is well known that Thomae's function is continuous at a point x if and only if x is rational. Due to the similarities between the graph of in Figure 1 and the graph of Thomae's function in Figure 2 we hope to find a characterization of points in the domain that correspond to points with little repetition. In order to do this we will differentiate between two types of repetition that can occur in a family of sequences arising from evaluation.

Example 3.15. Let $q \in \mathbb{R}[x]^{\mathbb{N}}$ be defined by $q_1(x) = x$, $q_{2n}(x) = 2q_n(x) + 1$ and $q_{2n+1}(x) = 4q_n(x) + 3$ for $n \geq 1$. The first few values are shown in Table 6.

n	1	2	3	4	5
$q_n(x)$	x	$2x+1$	$4x+3$	$4x+3$	$8x+7$

Table 6: A table showing the first five values of $\{q_n\}_{n=1}^{\infty}$

.

Consider the corresponding family of sequences $\mathrm{ev}(\mathbb{R}) = \{\mathrm{ev}(a) : a \in \mathbb{R}\} \subseteq \mathbb{Q}^{\mathbb{N}}$. These sequences can exhibit two types of repetition. The first is shown by the pair of indices 3 and 4 as $q_3(x) = q_4(x)$ for any x. On the other hand consider the indices 1 and 2. It is clear that $q_1(x) \neq q_2(x)$, but observe that:

$$q_1(-1) = -1 = -2 + 1 = q_2(-1).$$

Then $q_1(-1) = q_2(-1)$ but $q_1(x) \neq q_2(x)$. We can see that some repetitions in the elements of $\mathrm{ev}(\mathbb{R})$ are forced by the polynomial sequence $\{q_n(x)\}_{n=1}^{\infty}$ for every element of $\mathbb{R}$. However, some repetitions only occur in sequences generated by specific evaluations. Only some of the sequences in $\mathrm{ev}(\mathbb{R})$ exhibit these repetitions. We formalize this difference.

Definition 3.16. Let X and Y be sets. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from X to Y. If $i, j \in \mathbb{N}$ so that $i \neq j$ and $f_i = f_j$ then the pair $\{i, j\}$ is an induced repetition relative to $\{f_n\}_{n=1}^{\infty}$.

Definition 3.17. Let X and Y be sets. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from X to Y. If $i, j \in \mathbb{N}$ so that $i \neq j$, $f_i \neq f_j$, and $f_i(x) = f_j(x)$ for some $x \in X$ then the pair $\{i, j\}$ is a shifting repetition relative to $\{f_n\}_{n=1}^{\infty}$. We say that $x \in X$ induces a shifting repetition if there exists $i \neq j \in \mathbb{N}$ so that $f_i \neq f_j$ but $f_i(x) = f_j(x)$.

Example 3.18. Refer back to Example 3.15, we see that $\{3, 4\}$ is an induced repetition relative to $\{q_n\}_{n=1}^{\infty}$. Additionally, $\{1, 2\}$ is a shifting repetition relative to $\{q_n\}_{n=1}^{\infty}$. We also say that -1 induces a shifting repetition in $\{q_n\}_{n=1}^{\infty}$.

Looking back to Figure 1, observe that $\{f_n\}_{n=1}^{\infty}$ has induced repetitions. If $\{f_n\}_{n=1}^{\infty}$ had no induced repetitions we would expect the upper bound on the ratio of new terms to total terms to be 1 instead of 0.52. Because the upper bound is 0.52 every sequence in $\mathrm{ev}(\mathbb{R})$ must have some number of repetitions. This leads us to suspect many of them are induced. For a specific example, note that $f_{22}(x) = -\frac{9}{2} + x = f_{25}(x)$. Then $\{22, 25\}$ is an induced repetition relative to $\{f_n\}_{n=1}^{\infty}$. Additionally, observe that there are many points in Figure 1 accumulating near 0.52. These points are highest in the graph and must have the highest number of distinct terms. We conjecture that the points near 0.52 have little to no shifting repetitions, but they do still have induced repetitions. This way they have as many distinct terms as possible. If this is the case, we can draw inspiration from Thomae's function and conjecture that Λ is continuous at a point x if and only if x does not induce any shifting repetitions. Before proving this characterization, we first characterize the continuity of Thomae's function, as the proof will be similar. This is a standard undergraduate real analysis exercise.

Theorem 3.19. *Thomae's function is continuous at a point $x \in (0, 1)$ if and only if x is irrational.*

Proof. We proceed by showing that Thomae's functon $t(x) : (0, 1) \to \mathbb{R}$ is continuous at a point $c \in (0, 1)$ if and only if c is irrational via the topological formulation of continuity at a point. That is $t(c)$ is continuous at c if for every open set of $\mathbb{R}$ containing $f(c)$ there is an open set U of $(0, 1)$ containing c so that $f(U) \subseteq V$ [4, Theorem 18.1].

We will first show that $t^{-1}([\epsilon, \infty))$ is finite for all $\epsilon > 0$. First let $\epsilon > 0 \in \mathbb{R}$. Suppose that $t(x) > \epsilon$. By the definition of Thomae's function it must be that $x \in (0, 1) \cap \mathbb{Q}$. Observe that $x = p/q$ for some $p, q \in \mathbb{N}$ so that $\gcd(p, q) = 1$. Then $t(p/q) = 1/q > \epsilon$. Due to the archimedian property of the natural numbers there exists $n_0 \in \mathbb{N}$ so that $\frac{1}{n_0} < \epsilon$. It follows that $n \geq n_0 \in \mathbb{N}$ implies that $\frac{1}{n} < \epsilon$. It follows that $\frac{1}{s} \geq \epsilon$ implies $s < n_0$. Note there there are only finitely many such s. Temporally fix $s < n_0$. Using a similar argument, the archimedian property implies that there are finitely many $r \in \mathbb{N}$ so that $\frac{r}{s} < 1$. It follows that $A = \{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{s}{r} < 1, \frac{1}{s} \geq \epsilon\}$ is finite. But notice $t(x) = (p/q) = \frac{1}{q} > \epsilon$. Then $t^{-1}([\epsilon, \infty)) \subseteq A$. Because subsets of finite sets are finite, $t^{-1}([\epsilon,\infty))$ is finite. Now we can prove the original statement.

$\implies$ We show the contrapositive: If $c \in \mathbb{Q}$, then t is not continuous at c. Suppose $c \in \mathbb{Q}$. $c = p/q$ for some $p, q \in \mathbb{N}$ and $\gcd(p,q) = 1$. Then $t(c) = \frac{1}{q}$. Let $\epsilon = 1/2q$. Notice that $B(t(c), \epsilon)$ is an open set of $\mathbb{R}$, containing $t(c)$. Now let U be any open set of $(0,1) \subseteq \mathbb{R}$ containing c. From above we know that $t^{-1}((\epsilon,\infty))$ is finite. Consider $t(U)$. Because U is an open set of $\mathbb{R}$, it has infinitely many elements. For each $u \in U$, either $t(u) \in [0,\epsilon)$ or $[\epsilon, \infty)$. Due to the fact $t^{-1}([\epsilon,\infty))$ is finite and U has infinitely many elements, there exists $u \in t^{-1}([0,\epsilon))$. Then $t(u) \in [0, \epsilon) = [0, \frac{1}{2q})$. It follows that $t(u) \notin B(t(c), \epsilon) = (\frac{1}{2q}, \frac{3}{2q})$. It follows that $t(U) \not\subseteq B(f(c), \epsilon)$ for any open set U, so that $c \in U$. We have shown by definition that t is not continuous at c.

$\impliedby$ It will suffice to use basis open sets. Suppose that c is irrational. By definition of t we have that $t(c) = 0$. Let $\epsilon > 0$ and see that $B(f(c), \epsilon) = B(0, \epsilon)$ is a basis open set containing $f(c)$. From above, we have that $t^{-1}([\epsilon, \infty))$ is finite. Note that finite sets are closed in $(0,1) \subseteq \mathbb{R}$, so $t^{-1}([\epsilon,\infty))$ is closed. It follows that $(0,1) \setminus t^{-1}((0,\epsilon))$ is open and

$$\begin{aligned}
(0,1) \setminus (t^{-1}([\epsilon,\infty)) &= (0,1) \setminus \left(t^{-1}\left(\mathbb{R}^{\geq 0} \setminus [\epsilon, \infty)\right)\right) \\
&= (0,1) \setminus \left((0,1) \setminus t^{-1}([0,\epsilon)\right) \\
&= t^{-1}([0,\epsilon).
\end{aligned}$$

Then $t^{-1}([0,\epsilon))$ is an open set of $(0,1)$ containing c and $t\big(t^{-1}([0,\epsilon)\big) \subseteq [0,\epsilon) \subseteq B(0,\epsilon)$. By the topological definition of continuity at a point, t is continuous at c.

$\square$

We will now characterize Λ using a similar argument. For the sake of clarity we prove Lemmas 3.20, 3.21, and 3.22.

Lemma 3.20. *Let X and Y be sets with $\{f_n\}_{n=1}^\infty$ a sequence of functions from X to Y, let $\mathrm{ev} : X \to Y^{\mathbb{N}}$ be the corresponding evaluation map, let $g : Y^{\mathbb{N}} \to S'$ be the map sending a sequence to its initial index representation, and let $\Lambda = \tau \circ \mathrm{ev}$. If $\hat{f}$ is the initial index representation for $\{f_n\}_{n=1}^\infty$, then $x \in X$ does not induce a shifting repetition relative to $\{f_n\}_{n=1}^\infty$ if and only if $\Lambda(x) = \hat{f}$.*

Proof. $\implies$ We show the contrapositive: if $x \in R$ and $\Lambda(x) \neq \hat{f}$, then x induces a shifting repetition. Suppose that $x \in X$ so that $\Lambda(x) \neq \hat{f}$. Then there exists $j_0 \in \mathbb{N}$ so that $\Lambda(x)_{j_0} \neq \hat{f}_{j_0}$. Because $\mathbb{N}$ is well-ordered let μ be the least such element. Then $i < \mu \in \mathbb{N}$ implies that $\Lambda(x)_i = \hat{f}_i$ and $\Lambda(x)_\mu \neq \hat{f}_\mu$. Observe that if $f_\mu = f_j$ then $f_\mu(x) = f_j(x)$. This implies that $\{j \in \mathbb{N} : f_\mu = f_j\} \subseteq \{j \in \mathbb{N} : f_\mu(x) = f_j(x)\}$ so $\Lambda(x)_\mu \leq \hat{f}_\mu$. Because $\Lambda(x)_\mu \neq \hat{f}_\mu$ we have that $\Lambda(x)_\mu < \hat{f}_\mu$. From the definition of initial index representation we have that $f_{\Lambda(x)_\mu}(x) = f_\mu(x)$. Next, assume for purposes of contradiction that $f_{\Lambda(x)_\mu} = f_\mu$. By definition of initial index representation and Λ, $\hat{f}_\mu \leq \Lambda(x)_\mu$, but $\Lambda(x)_\mu < \hat{f}_\mu$. This is a contradiction. It must be that $f_{\Lambda(x)_\mu} \neq f_\mu$. Finally, $\Lambda(x)_\mu < \hat{f}_\mu \leq \mu$, so $\Lambda(x)_\mu \neq \mu$. Putting it all together, we have $\Lambda(x)_\mu \neq \mu$ so that $f_{\Lambda(x)_\mu} \neq f_\mu$ but $f_{\Lambda(x)_\mu}(x) = f_\mu(x)$. It follows from the definition that $(\Lambda(x)_\mu, \mu)$ is a shifting repetition relative to $\{f_n\}_{n=1}^\infty$. Then x induces a shifting repetition.

$\Longleftarrow$ Suppose that $\Lambda(x) = \hat{f}$. Recall from Proposition 2.19 that

$$f_i(x) = f_j(x) \iff \Lambda(x)_i = \Lambda(x)_j \iff \hat{f}_i = \hat{f}_j \iff f_i = f_j$$

for all $i, j \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ so that $i \neq j$ and $f_i(x) = f_j(x)$. Then it must be that $f_i = f_j$. It follows that x induces no shifting repetitions.

$\square$

In Theorem 3.19, it was crucial to show that the image of Thomae's function had finitely many points outside of $[0, \epsilon)$. We show that the image of Λ has finitely many points outside of an ϵ-ball, centered at a special point.

Lemma 3.21. *Let R be an integral domain, let $\{q_n\}_{n=1}^\infty$ be a sequence of polynomials with coefficients in R, let $\mathrm{ev} : R \to R^{\mathbb{N}}$ be the corresponding evaluation map, let $\tau : R^{\mathbb{N}} \to S'$ be the map sending a sequence in $R^{\mathbb{N}}$ to its initial index representation, and let $\Lambda = \tau \circ \mathrm{ev}$. If $\epsilon > 0$, then $\Lambda^{-1}(S' \setminus B(\hat{q}, \epsilon))$ is finite, where $\hat{q}$ is the initial index representation for $\{q_n\}_{n=1}^\infty$.*

Proof. Let $\epsilon > 0$, $x \in \Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon))$, and $d : \mathcal{S}' \times \mathcal{S}' \to \mathbb{R}$ be the metric on S'. By definition $\Lambda(x) \in \mathcal{S}' \setminus B(\hat{q}, \epsilon)$. It must be the $d(\Lambda(x), \hat{q}) \geq \epsilon$. Let $\mu \in \mathbb{N}$ be the largest element of $\mathbb{N}$ so that $2^{-\mu} < \epsilon$. Note that $\Lambda(x) \neq \hat{q}$ because $d(\Lambda(x), \hat{q}) > 0$. Then there exist $k_0 \in \mathbb{N}$ so that $\Lambda(x)_{k_0} \neq \hat{q}_{k_0}$ and $2^{-k_0} > 2^{-\mu} \implies k_0 < \mu$. By the well-ordering principle on $\mathbb{N}$, let k be the first element so that $\Lambda(x)_k \neq \hat{q}_k$. Then we have $i < k$ implies that $\Lambda(x)_i = \hat{q}$, $\Lambda(x)_k \neq \hat{q}_k$, and $k < \mu$. Observe that for all $j \in \mathbb{N}$ that $q_k = q_j$ implies that $q_k(x) = q_j(x)$. Then $\{j \in \mathbb{N} : q_k = q_j\} \subseteq \{j \in \mathbb{N} : q_k(x) = q_j(x)\}$. The definition of the initial index representation implies that $\Lambda(x)_k \leq \hat{q}_k$. Additionally, $\Lambda(x)_k \neq \hat{q}_k$, which implies that $\Lambda(x)_k < \hat{q}_k$. Then $q_k(x) = q_{\Lambda(x)_k}(x)$ and $q_k \neq q_{\Lambda(x)_k}$. Assume for purposes of contradiction that $q_k = q_{\Lambda(x)_k}$. It follows that $\hat{q}_k \leq \Lambda(x)_k$ but $\Lambda(x)_k < \hat{q}_k$. This is a contradiction, so $q_k \neq q_{\Lambda(x)_k}$. Putting the last two steps together, we get that $q_k - q_{\Lambda(x)_k} \neq 0$ but $q_k(x) - q_{\Lambda(x)_k}(x) = 0$. This shows that x is a root of the non-zero polynomial $q_k - q_{\Lambda(x)_k}$. Note that $x \in \{z \in R : z \text{ is a root of } q_i - q_j \neq 0, i, j \leq \mu \in \mathbb{N}\}$, and $\Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon)) \subseteq \{z \in R : z \text{ is a root of } q_i - q_j \neq 0, i, j \leq \mu \in \mathbb{N}\}$. Because R is an integral domain, any non-zero polynomial with coefficients in R has at most the number of roots in R equal to its degree. There are a finite number of distinct non-zero polynomials of the form $q_i - q_j$ where $i, j \leq \mu$ and each with a finite number of roots. It follows that $\{z \in R : z \text{ is a root } of q_i - q_j \neq 0, i, j \leq \mu\}$ is a finite set. Then $\Lambda^{-1}(S' \setminus B(\hat{q}, \epsilon))$ is finite, being a subset of a finite set. $\square$

Lemma 3.22. *Let X be a T_1 perfect topological space. Then each non-empty open set of X has infinitely many points.*

Proof. Suppose that $V \subseteq X$ is a non-empty open set of X. Assume for purposes of contradiction that V is finite. Then $V = \{x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{N}$. Because X is T_1 there exists an open set $W_i \subseteq X$ so that $x_1 \in W_i$ and $x_i \notin W_i$ for each $2 \leq i \leq n$. Observe that $\bigcap_{i=2}^{n} W_i$ is open and $\bigcap_{i=1}^{n} W_i \cap V = \{x_1\}$. Then $\{x_1\}$ is an open set containing x_1 that does not contain another distinct point. This contradicts that X is perfect. It must be that V is not finite. $\square$

Recall that, R is a topological ring if R is a ring and a topological space such that addition and multiplication are continuous maps from $R \times R$, in the product topology, to R.

Theorem 3.23. *Let R be topological integral domain that is Hausdorff perfect. Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of polynomials with coefficients in R, let $\mathrm{ev} : R \to R^{\mathbb{N}}$ be the corresponding evaluation map, let $\tau : R^{\mathbb{N}} \to \mathcal{S}'$ be the map sending a sequence in $R^{\mathbb{N}}$ to its corresponding initial index representation, and let $\Lambda = \tau \circ \mathrm{ev}$. Given $x \in X$ the following are equivalent:*

1. *Λ is continuous at x.*
2. *x does not induce a shifting repetition relative to $\{q_n\}_{n=1}^{\infty}$.*
3. *$\Lambda(x) = \hat{q}$, where $\hat{q}$ is the initial index representation for $\{q_n\}_{n=1}^{\infty}$.*

Proof. First observe that we have (2) $\iff$ (3) from Lemma 3.20. It remains to show that (1) $\iff$ (2).

Recall for Λ to be continuous at a point $x \in R$ it must be that for any open set V of S' containing $\Lambda(x)$ there must be some open set U containing x so that $\Lambda(U) \subseteq V$.

Let $x \in R$.

$\implies$ We show the contrapositive: if $\Lambda(x) \neq \hat{q}$ then Λ is not continuous at x. Suppose that $\Lambda(x) \neq \hat{q}$. Then $d(\Lambda(x), \hat{q}) = \epsilon$ for some $\epsilon > 0$. From Lemma 3.21 $\Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon)))$ is finite. From Lemma 3.22 we have that each open set in R has infinitely many points. Now let A be an open set of R so that $x \in A$. Because $\Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon))$ is finite and A contains infinitely points, there exists $a \in A$ so that $a \notin \Lambda^{-1}(R \setminus B(\hat{q}, \epsilon))$. It follows that $\Lambda(a) \notin R \setminus B(\hat{q}, \epsilon)$. So $\Lambda(a) \in B(\hat{q}, \epsilon)$. Either $\Lambda(a) \in B(\Lambda(x), \epsilon)$ or not. Suppose $\Lambda(a) \in B(\Lambda(x), \epsilon)$, d is an ultra-metric implies that

$$\epsilon = d(\Lambda(x), \hat{q}) \leq \max\{d(\Lambda(x), \Lambda(a)), d(\Lambda(a), \hat{q})\} < \epsilon.$$

This is a contradiction, it must be that $\Lambda(a) \notin B(\Lambda(x), \epsilon)$ and $\Lambda(A) \nsubseteq B(\Lambda(x), \epsilon)$. Then $B(\Lambda(x), \epsilon)$ is an open set containing $\Lambda(x)$ and for every open set A containing x we have that $\Lambda(A) \nsubseteq B(\Lambda(x), \epsilon)$. By definition, Λ is not continuous at x.

$\Longleftarrow$ Suppose that $\Lambda(x) = \hat{q}$. To show Λ is continuous at x it will suffice to show that for each $\epsilon > 0$ and $B(\hat{q}, \epsilon)$ there is an open subset of R containing x so that Λ maps it into $B(\hat{q}, \epsilon)$. Let $\epsilon > 0$. Lemma 3.21 implies that $\Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon))$ is finite. Because R is T_1 finite sets are closed sets from Lemma 3.22. It follows that $R \setminus \Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon))$ is open and

$$\begin{aligned} R \setminus \left(\Lambda^{-1}(\mathcal{S}' \setminus B(\hat{q}, \epsilon))\right) &= R \setminus \left(R \setminus \Lambda^{-1}(B(\hat{q}, \epsilon))\right) \\ &= \Lambda^{-1}(B(\hat{q}, \epsilon)). \end{aligned}$$

Then $\Lambda^{-1}(B(\hat{q}, \epsilon))$ is an open subset of R containing x and $\Lambda\left(\Lambda^{-1}\left(B(\hat{q}, \epsilon)\right)\right) \subseteq B(\hat{q}, \epsilon)$. It follows from the definition of continuity at a point that Λ is continuous at x.

$\square$

Recall our goal in Section 1. Given topological spaces X and Y and a sequence of continuous functions from X to Y, $\{f_n\}_{n=1}^{\infty}$, we wanted to find a topological space Z and a map $g : Y \to Z$ so that $g \circ \text{ev}$ is continuous. Additionally, $g \circ \text{ev}$ should map each element $x \in X$ to an element in Z that records how terms repeat in $\{f_n(x)\}_{n=1}^{\infty}$.

We found that letting $Z = \mathcal{S}'$, the space of initial index representations, results in a function $\Lambda = g \circ \text{ev} : X \to \mathcal{S}'$ that is continuous at a point in x if and only if x does not induce a shifting repetition. This means that Λ is only continuous at points with the least repetition. These points are often of less interest than points with repetition because they are easy to find. If we let $X = \mathbb{R}$ and $\{f_n\}_{n=1}^{\infty}$ be some polynomial sequence with rational coefficients it is immediate that Λ is continuous at every transcendental number. On the other hand, points that induced shifting repetitions are more difficult to find. Theorem 3.23 reveals that we cannot use continuity to determine more repetition from known shifting repetitions.

4 Diagnosing the Problem with Initial Index Representations

In this section we will explore what properties of $\mathcal{S}'$ made Λ fail to be continuous. Using these properties we hope to construct $\Lambda' = g' \circ \text{ev}'$ with $\text{ev}' : X' \to Y'$, $g' : Y' \to Z'$, and $Z' \subseteq \mathbb{N}^{\mathbb{N}}$ so that Λ' is continuous. In order to build on what we have done so far, we ensure $Z' \subseteq \mathbb{N}^{\mathbb{N}}$ where $\mathbb{N}^{\mathbb{N}}$ is in the order topology from the lexicographical order on $\mathbb{N}^{\mathbb{N}}$.

5 κ-Initial Index Representation

Given a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$. Theorem 3.23 used that polynomials have finitely zeros (over integral rings) to reduce the number of being sent away from $\hat{q}$, ensuring continuity at points x where $\Lambda(x) = \hat{q}$. Generally, there are not enough solutions to $f_n(x) = f_m(x)$ for Λ to be continuous for everywhere. In order to get around this, we consider introducing a small amount of error.

Definition 5.1. Let $a \in \mathbb{R}^{\mathbb{N}}$. Then $a's$ κ-initial index representation is $\phi \in \mathbb{N}^{\mathbb{N}}$ defined by

$$\phi_n = \min\left(\{j \in \mathbb{N} : |a_n - a_j| < \kappa\}\right).$$

Example 5.2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ with the corresponding evaluation map $\mathrm{ev} : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$. Let $\tau : \mathbb{R}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the function sending an element of $\mathbb{R}^{\mathbb{N}}$ to is κ-initial index representation. Finally, let $\Lambda = \tau \circ \mathrm{ev}$.

Λ still fails to be continuous. Suppose that $f_1(x) = 0$, $f_2(x) = 2$, $f_3(x) = x$ and $\epsilon = 1$. Observe that $\Lambda(1) = 1, 2, 3, \ldots$. But for any $x \in (0, 1)$ we have that $\Lambda(x) = 1, 1, 2, \ldots$ and for any $x \in (1, 2)$ we have that $\Lambda(x)_3 = 1, 2, 2, \ldots$. We cannot fix $\Lambda(x)_3$ for any $x \in B(1, \delta)$ no matter how small we make δ. Recall that Λ has codomain $\mathbb{N}^{\mathbb{N}}$ in the lexicographical order. Two elements are close in $\mathbb{N}^{\mathbb{N}}$ if their heads agree for many indices. Because we cannot fix $\Lambda(x)_3$, we cannot ensure that $d(\Lambda(0), \Lambda(x)) < \epsilon$ for some small $\delta > 0$. Then Λ is not continuous at 1, even when mapping into the set of κ-initial index representations.

The upshot of is this: no matter how little we wiggle $x \in B(x_0, \delta)$, if $|f_n(x_0) - f_m(x_0)| \geq \epsilon$ we cannot ensure that $|f_n(x) - f_m(x)| \geq \epsilon$. The problem is that when $x = 1$, $f_3(x) = 2$, which lies exactly $\epsilon = 1$ away from 0. An arbitrarily small change in the value of $f_3(x)$ can result in $|f_3(x) - f_1(x)| < \epsilon$.

From Example 5.2 we cannot fix arbitrary many elements of in the beginning of $\Lambda(x)$ to equal $\Lambda(x_0)$ for x near x_0. If $|f_n(x_0) - f_m(x_0)| \geq \kappa$, we cannot ensure that $|f_n(x) - f_m(x)| \geq \kappa$. The issues arises when $|f_n(x) - f_m(x)| = \kappa$. If $|f_n(x_0) - f_m(x_0)| > \kappa$ we will be able to ensure that $|f_n(x) - f_m(x)| > \kappa$. Similarly if $|f_n(x_0) - f_m(x_0)| < \epsilon$ we will be able to ensure that $|f_n(x) - f_m(x)| < \kappa$. If we can just remove the case where $|f_n(x) - f_m(x)| = \kappa$ we will be able to make Λ continuous.

Theorem 5.3. *Let $\kappa \in \mathbb{R} \backslash \mathbb{Q}$, let $\{f_n\}_{n=1}^{\infty}$ be a continuous function from $\mathbb{Q}$ to $\mathbb{Q}$ with the Euclidean norm, let $\mathrm{ev} : \mathbb{Q} \to \mathbb{Q}^{\mathbb{N}}$ be the corresponding evaluation map, and let $\tau : \mathbb{Q}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the function sending an element of $\mathbb{Q}^{\mathbb{N}}$ to is κ-initial index representation. Then $\Lambda = g \circ \mathrm{ev}$ is continuous.*

Proof. Because $\mathbb{Q}$ and $\mathbb{N}^{\mathbb{N}}$ are metric spaces, it suffices to show metric continuity. Let $\kappa > 0 \in \mathbb{R} \backslash \mathbb{Q}$. Now let $c \in \mathbb{Q}$ and $\epsilon > 0$. There exists $\mu \in \mathbb{N}$ so that $2^{-\mu} < \epsilon$. We will now ensure that the points in $\{f_n(c)\}_{n=1}^{\mu}$ only move slightly c varies. This will ensure that $\{j \leq \mu \in \mathbb{N} : |f_i(c) - f_j(c)| < \kappa\} = \{j \leq \mu \in \mathbb{N} :$

$|f_i(x) - f_j(x)| < \kappa\}$ for $1 \leq i \leq \mu$ and $x \in B(x_0, \delta)$ for some $\delta > 0 \in \mathbb{R}$. If we can do this, we will be able to show that $d(\Lambda(x_0), \Lambda(x)) \leq 2^{\mu+1} < \epsilon$.

Let
$$\gamma = \min\{||f_i(c) - f_j(c)| - \kappa| : i, j \leq \mu\}.$$
Note that $||f_i(c) - f_j(c)| - \kappa| \neq 0$ because $f_i(c) - f_j(c) \in \mathbb{Q}$ for all $i, j \leq \mu$, but $\kappa \in \mathbb{R} \setminus \mathbb{Q}$. It follows that $\gamma > 0$. Each $f_n : \mathbb{Q} \to \mathbb{Q}$ is continuous and there are there are finitely many of them for $1 \leq n \leq \mu$. We may pick $\delta > 0$ so that $|x - c| < \delta$ implies that $|f_n(x) - f_n(c)| < \frac{\gamma}{2}$ for all $1 \leq n \leq \mu$. We now show that if $x \in \mathbb{Q}$, $|x - c| < \delta$ and $n \in \mathbb{N}$ so that $n \leq \mu$, then

$$\{j \leq n \in \mathbb{N} : |f_n(c) - f_j(c)| < \kappa\} = \{j \leq n \in \mathbb{N} : |f_n(x) - f_j(x)| < \kappa\}.$$

We show set inclusion in both directions.

$\Longrightarrow$ Suppose that $j \leq n$ and $|f_n(c) - f_j(c)| < \kappa$. It follows from the triangle inequality that:

$$\begin{aligned}
|f_n(x) - f_j(x)| &\leq |f_n(x) - f_n(c)| + |f_n(c) - f_j(c)| + |f_j(c) - f_j(x)| \\
&\leq \frac{\gamma}{2} + |f_n(c) - f_j(c)| + \frac{\gamma}{2} \\
&= |f_n(c) - f_j(c)| + \gamma \\
&\leq |f_n(c) - f_j(c)| + ||f_n(c) - f_j(c)| - \kappa| \\
&= |f_n(c) - f_j(c)| + \kappa - |f_n(c) - f_j(c)| = \kappa.
\end{aligned}$$

We have that $|f_n(x) - f_j(x)| \leq \kappa$. But observe that $|f_n(x) - f_j(x)| \in \mathbb{Q}$ and $\kappa \in \mathbb{R} \setminus \mathbb{Q}$ so it must be that $|f_n(x) - f_j(x)| < \kappa$.

$\Longleftarrow$ In this direction it is easier to show the contrapositive. Suppose that $j \leq n \in \mathbb{N}$ and $|f_k(c) - f_j(c)| \geq \kappa$. Observe that

$$\begin{aligned}
|f_n(c) - f_j(c)| &\leq |f_n(c) - f_n(x)| + |f_n(x) - f_j(x)| + |f_j(x) - f_j(c)| \\
&\leq |f_n(x) - f_j(x)| + \gamma \\
|f_n(c) - f_j(c)| &= |f_n(x) - f_j(x)| + |f_n(c) - f_j(c)| - \kappa \\
\kappa &\leq |f_n(x) - f_j(x)|.
\end{aligned}$$

We have shown that $|f_n(x) - f_j(x)| \geq \kappa$. Then $|f_n(x) - f_j(x)| < \kappa$ implies that $|f_n(c) - f_j(c)| < \kappa$.

It follows that $\{j \leq n \in \mathbb{N} : |f_n(c) - f_j(c)| < \kappa\} = \{j \leq n \in \mathbb{N} : |f_n(x) - f_j(x)| < \kappa\}$ for all $n \leq \mu \in \mathbb{N}$. This implies that $\Lambda(x)_n = \Lambda(c)_n$ for $1 \leq n \leq \mu$. Then $d(\Lambda(x), \Lambda(c)) \leq 2^{\mu+1} < \epsilon$. We have shown that Λ is continuous c for arbitrary $c \in \mathbb{Q}$. Then $\Lambda : \mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ is continuous. $\square$

It is worrying that the Λ is only continuous when $\kappa > 0 \in \mathbb{R} \setminus \mathbb{Q}$. The failure to use any rational κ indicates that this map is sitting unnaturally. Or at least there may exist a nicer map that is similar this this one. Additionally, we hope to do analysis so we would like our domain to be a complete metric space.

5.1 A Complete Metric Space Where We Can Continuously Approximate Repetition

In Theorem 5.3 we found continuous $\Lambda : \mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ corresponding to evaluation of a polynomial sequence $\{f_n\}_{n=1}^{\infty}$ followed by sending the evaluation to its κ-initial index representation, as long as each $f_n : \mathbb{Q} \to \mathbb{Q}$ is continuous and $\kappa \in \mathbb{R} \setminus \mathbb{Q}$. We would like to construct a similar function to Λ from Theorem 5.3 that does not impose such a restriction on κ. In Theorem 5.3 we were able to show Λ is continuous by only allowing $\mathbb{Q}$-valued functions, but we could not use any value for κ. Because we have already restricted to rational valued functions, we can change the topology on $\mathbb{Q}$ by considering alternate metrics. The only possible metrics on $\mathbb{Q}$ are the standard Euclidean norm, which we have been using, and the p-adic norms for each prime $p \in \mathbb{N}$ [3]. The most obvious difference between the topology induced by the Euclidean norm and the p-adic norms is that $\mathbb{Q}$ is an ultra-metric space under the p-adic norm, but it is not under the Euclidean norm. In an ultra-metric space $d(x,z) \le \max(d(x,y), d(y,z))$. Lastly we note that the p-adic numbers, $\mathbb{Q}_p$ are the metric completion of $\mathbb{Q}$ under the p-adic metric. Then $\mathbb{Q}_p$ could be our complete metric space.

5.1.1 Sending an Evaluation into an Ultra-Metric Space to its κ-Initial Index Representation is Continuous

$\mathbb{Q}_p$ is an example of a non-archimedian field. A non-arch median field is a field $\mathbb{K}$ whose norm $|\cdot|$ satisfies the ultra-metric inequality. That is if $a, b, c \in \mathbb{K}$ then $|a-c| \le \max\{|a-b|, |b-c|\}$. This property makes $\mathbb{K}$ an ultra-metric space. An ultra-metric spaces is just an metric space whose metric, d, also satisfies the ultra-metric inequality. That is $d(a,c) \le \max\{d(a,b), d(b,c)\}$ for all $a, b, c \in \mathbb{K}$.

Lemma 5.4. *Let X be an ultra-metric space with metric d. Then every point in an open ball is a center of the ball. More precisely, let $x \in X$ and $\kappa > 0$. If $y \in B(x,\kappa)$ then $B(x,\kappa) = B(y,\kappa)$.*

Proof. Let $x \in X$, an ultra-metric space. Let $\kappa > 0$ and let $y \in B(x,\kappa)$. Then $d(x,y) < \kappa$. We show set containment in each direction.

$\Longrightarrow$ Suppose that $z \in B(x,\kappa)$. Then $d(x,z) < \kappa$. The ultra-metric inequality implies that $d(y,z) \le \max(d(y,x), d(x,z)) < \kappa$

$\Longleftarrow$ A similar proof shows this direction.

$\square$

Theorem 5.5. *Let X and Y be metric spaces with metrics $d_1 : X \times X \to \mathbb{R}$ and $d_2 : Y \times Y \to \mathbb{R}$. Let $d_3 : S' \times S' \to \mathbb{R}$ be the metric for S', let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from X to Y, let $\mathrm{ev} : X \to Y^{\mathbb{N}}$ be the corresponding evaluation map, let $\tau : Y^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the function sending an element of $Y^{\mathbb{N}}$ to its κ-initial index representation. If $\kappa > 0 \in \mathbb{R}$, then $\Lambda = \tau \circ \mathrm{ev}$ is continuous.*

Proof. Because X and S' are metric spaces, it suffices to show metric continuity. Let $\kappa > 0$, let $c \in X$ and $\epsilon > 0$. Note that there exists $\mu \in \mathbb{N}$ so that $2^{-\mu} < \epsilon$. We will now ensure that the points $\{f_n(c)\}_{n=1}^{\mu}$ only move slightly c varies. Because f_n is continuous for $1 \leq n \leq \mu$ and there are finitely many n to consider, there exists $\delta > 0$ so that $d_1(x, c) < \delta$ implies that $d_2(f_n(c) - f_n(x)) < \kappa$ for $1 \leq n \leq \mu$. Let $x \in \mathbb{Q}_p$ so that $d_1(x, c) < \delta$. This will be enough to show that for $1 \leq n \leq \mu$,

$$\{j \leq n \in \mathbb{N} : d_2(f_n(c) - f_j(c)) < \kappa\} = \{j \leq n \in \mathbb{N} : d_2(f_n(x) - f_j(x)) < \kappa\}.$$

We show set inclusion in both directions.

$\implies$ Suppose that $j \in \{j \leq n \in \mathbb{N} : d_2(f_n(c) - f_j(c)) < \kappa\}$. It follows that $d_2(f_n(c) - f_j(c)) < \kappa$. Then $f_j(c) \in B(f_n(c), \kappa)$ and Lemma 5.4 implies that $B(f_n(c), \kappa) = B(f_j(c), \kappa)$. Because $d(x, c) < \delta$, $d_2(f_j(c) - f_j(x)) < \kappa$ and $d_2(f_n(c) - f_n(x)) < \kappa$. Similarly, lemma 5.4 implies that $B(f_j(x), \kappa) = B(f_j(c), \kappa)$ and $B(f_n(c), \kappa) = B(f_j(c), \kappa)$. Combining these facts, we get $f_j(x) \in B(f_j(x), \kappa) = B(f_j(c), \kappa) = B(f_n(c), \kappa) = B(f_n(x), \kappa)$. We have that $d_2(f_n(x) - f_j(x)) < \kappa$. Then $j \in \{j \leq n \in \mathbb{N} : d_2(f_n(x) - f_j(x)) < \kappa\}$.

$\impliedby$ A similar proof as the forward direction shows the backward direction.

We have shown that $\{j \leq n \in \mathbb{N} : d_2(f_n(c) - f_j(c)) < \kappa\} = \{j \leq n \in \mathbb{N} : d_2(f_n(x) - f_j(x)) < \kappa\}$. Additionally, note that the $n \in \{j \in N : d_2(f_n(c) - f_j(c)) < \kappa\}$. These two facts combined imply that $\min\{j \in N : d_2(f_n(c) - f_j(c)) < \kappa\} = \min\{j \in N : d_2(f_n(x) - f_j(x)) < \kappa\}$. Then $\Lambda(c)_n = \Lambda(x)_n$ for $1 \leq n \leq \mu$. It follows that from the definition of d_3 that $d_3(\Lambda(c), \Lambda(x)) < \epsilon$. We have shown that Λ is continuous at c for arbitrary $c \in X$. Then $\Lambda : X \to S'$ is continuous. $\square$

With Theorem 5.5, we have achieved something close the goal described in Section 1. We have shown that if X is a metric space, Y is an ultra-metric space, $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions from X to Y, and $\tau : Y^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is the function sending a sequence to its κ-initial index representation, then $\Lambda = \tau \circ \mathrm{ev}$ is continuous. We can continuously approximate repetition as we evaluate $\{f_n\}_{n=1}^{\infty}$ with different points in the domain. In order to do this we had to move to totally disconnected topological spaces, and so we lost the ability to use the Intermediate Value Theorem. In retrospect, this is not surprising. However, this result does emphasize the stark difference between Euclidean metric spaces and ultra-metric spaces.

6 Concluding Remarks

In this paper, we set out to continuously measure repetition in a one-parameter family of sequences. In order to do so, we introduced the idea of initial index representations, induced repetition, and shifting repetition. In Theorem 3.23

we found that sending an evaluation to its initial index representation was continuous at a certain evaluation if and only if that evaluation induced as little repetition as possible. In Section 4, we introduced some error through κ-initial index representations. Example 5.2 showed that is is not enough. In the case that two terms of a sequence are separated by exactly κ, there is no way to ensure a small change in the input evaluation does not cause these terms to move closer than κ. Theorem 5.3 shows that restricting κ to lie outside of values attained by the distance between points in our families of sequences ensures the continuity of mapping evaluations of sequences to their κ-initial index representations. We concluded with Theorem 5.5. This result showed that mapping to the κ-initial index representation is continuous when considering a sequence of continuous functions into an ultra-metric space. Overall Theorem 5.5 helps indicate the stark difference in the topology between metric spaces with and without the ultra-metric inequality. This difference can have great consequences on analysis arguments.

In the future, we could consider specific examples of $\Lambda = \tau \circ \mathrm{ev}$, from Theorem 5.5. In particular, let $X = \mathbb{Q}_p$ and $Y = \mathbb{Q}_p$. This gives a continuous endofunction to study, and we could investigate its Mahler expansion.

Appendix A Partial and Total Order Proofs

For the purposes of this section we assume that A and I are non-empty sets so that $|A| \geq |I|$. Additionally, $<$, $\leq$, and $\mathcal{S}$ come from Definitions 2.5, 2.11, and 2.4, respectively.

Lemma A.1. *$<$ is well-defined on $\mathcal{S} \times \mathcal{S}$.*

Proof. Suppose that $[a] < [b]$, we want to show that $[a'] < [b']$. Because $[a] < [b]$ there exists $\mu, j_0 \in I$ so that $a_\mu = a_{j_0}$, $b_\mu = b_k$ implies that $j_0 < k$ for each $k \in I$, and $a_n = a_m \iff b_n = b_m$ for all $m, n < \mu \in I$. From $[a] = [a']$ we have that $a_i = a_j \iff a'_i = a'_j$. Then

$$a_\mu = a_{j_0} \implies a'_\mu = a'_{j_0}.$$

Now suppose that $b'_\mu = b'_k$ for some $k \in I$. From $[b] = [b']$ we have that $b_i = b_j \iff b'_i = b'_j$ for all $i, j \in I$. Then

$$b'_\mu = b'_k \implies b_\mu = b_k \implies j_0 < k.$$

Finally let $m, n < \mu \in I$. Then

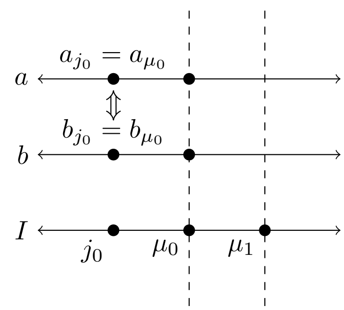
$$b'_n = b'_m \iff b_n = b_m \iff a_n = a_m \iff a'_n = a'_m.$$

By the definition of $<$ we have now shown that $[a'] < [b']$. We have shown that $[a] < [b]$ implies that $[a'] < [b']$. Then $<$ is well-defined on equivalence class representatives. $\square$

Lemma A.2. *$\leq$ is an anti-symmetric relation.*

Proof. We show that $\leq$ is anti-symmetric. Let $[a],[b] \in \mathcal{S}$ so that $[a] \leq [b]$ and $[b] \leq [a]$. We want to show that $[a] = [b]$. Assume for purpose of contradiction that $[a] \neq [b]$, then $[a] < [b]$ and $[b] < [a]$. There exists $\mu_0, \mu_1, j_0, j_1 \in I$ so that $a_{\mu_0} = a_{j_0}$, $a_{\mu_1} = a_{j_1}$, $b_{\mu_0} = b_k$ implies that $j_0 < k$ for all $k \in I$, $b_{\mu_1} = b_{j_1}$ implies that $j_1 < k$, $a_n = a_m \iff b_n = b_m$ for all $n, m < \mu_0 \in I$, and $a_n = a_m \iff b_n = b_m$ for all $n, m < \mu_1 \in I$. Because I is a total order, there are three cases: $\mu_0 < \mu_1$, $\mu_1 < \mu_0$, or $\mu_0 = \mu_1$.

Case 1: Suppose that $\mu_0 < \mu_1$. Note that $b_{\mu_0} = b_{\mu_0}$ so $j_0 < \mu_0 < \mu_1$. From $[a] < [b]$ we have that $a_{\mu_0} = a_{j_0}$ and $j_0 < \mu_0 < \mu_1$implies that $b_{\mu_0} = b_{j_0}$ because $[b] < [a]$. Then $b_{\mu_0} = b_{j_0}$ and $[a] < [b]$ implies that $j_0 < j_0$, but $j_0 = j_0$. This is a contradiction so it must be that $[a] = [b]$.



Case 2: Suppose that $\mu_1 < \mu_0$. Repeat a proof similar to **Case 1** and find a contradiction. Then it must be that $[a] = [b]$.

Case 3: Suppose that $\mu_0 = \mu_1$. Because $b_{\mu_0} = b_{\mu_1} = b_{j_1}$ we have that $j_0 < j_1$. Similarly because $b_{\mu_1} = b_{\mu_0} = b_{j_0}$ we have that $j_1 < j_0$. Then $j_0 < j_1$ and $j_1 < j_0$. This is a contradiction so it must be that $[a] = [b]$.

Each case led to a contradiction so it must be that $[a] = [b]$. Therefore $\leq$ is anti-symmetric. $\square$

Lemma A.3. *$\leq$ is a transitive relation.*

Proof. We show that $\leq$ is transitive. Let $[a],[b],[c] \in \mathcal{S}$ so that $[a] \leq [b]$ and $[b] \leq [c]$. If $[a] = [b]$ or $[b] = [c]$ it is immediate that $[a] \leq [c]$. We may now assume that $[a] < [b]$ and $[b] < [c]$. Then there exist $\mu_0, j_0, \mu_1, j_1 \in I$ so that $a_{\mu_0} = a_{j_0}$, $b_{\mu_1} = b_{j_1}$, $b\mu_0 = k$ implies $j_0 < k$ for all $k \in I$, $c_{\mu_1} = c_k$ implies that $j_1 < k$ for all $k \in I$, $a_n = a_m \iff b_n = b_m$ for all $n, m < \mu_0 \in I$, and $b_n = b_m \iff c_n = c_m$ for all $m, n < \mu_1 \in I$.

Let $\mu = \min\{\mu_0, \mu_1\}$. Then let $n, m < \mu$. It follows that

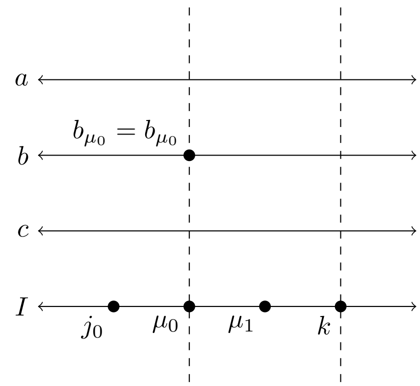
$$a_n = a_m \iff b_n = b_m \iff c_n = c_m.$$

from the fact $n, m < \mu \leq \mu_0$ and $n, m < \mu \leq \mu_1$. It remains to find a $j_2 \in I$ so that $c_\mu = c_k$ implies that $j_2 < k$ for each $k \in I$. Let $k \in I$ so that $c_\mu = c_k$.

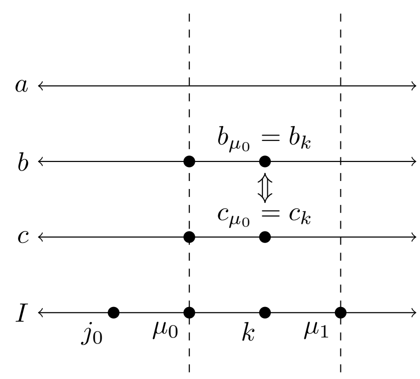
Because I is total order there are three cases: $\mu_0 < \mu_1$, $\mu_1 < \mu_0$, or $\mu_0 = \mu_1$. If $\mu_0 \leq \mu_1$ let $j_2 = j_0$, otherwise let $j_2 = j_1$.

Case 1: Suppose that $\mu_0 < \mu_1$. Then $\mu = \mu_0$ and $j_2 = j_0$. Then $a_\mu = a_{j_2}$. Either $\mu_1 \leq k$ or $k < \mu_1$.

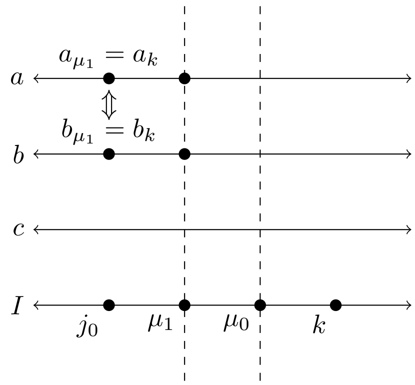
Case 1.1: Suppose that $\mu_1 \leq k$. Observe that $b_{\mu_0} = b_{\mu_0}$ implies that $j_0 < \mu_0$. It follows that $j_2 = j_0 < \mu_0 < \mu_1 \leq k$. Then $j_2 < k$ as desired.



Case 1.2: Suppose that $k < \mu_1$. Because $\mu = \mu_0 < \mu_1$, $k < \mu_1$, and $c_\mu = c_k$ we have that $b_\mu = b_k$. But then $b_{\mu_0} = b_k$ implies that $j_2 = j_0 < k$ as desired.



Case 2: Suppose that $\mu_1 < \mu_0$. Then $\mu = \mu_1$ and $j_2 = j_1$. Observe that $c_{\mu_1} = c_{\mu_1}$ implies that $j_1 < \mu_1$. Then $j_1 < \mu_1 < \mu_0$ and $b_{\mu_1} = b_{j_1}$ implies that $a_{\mu_1} = a_{j_1}$. Then $a_\mu = a_{j_2}$. Recall that $c_\mu = c_{j_2}$. Then $j_2 = j_1 < k$.



Case 3: Suppose that $\mu_0 = \mu_1$. It is immediate that $a_\mu = a_{j_2}$. Then $\mu = \mu_0 = \mu_1$ and $j = j_0$. Observe that $b_{\mu_1} = b_{j_1}$ and $b_{\mu_0} = b_{\mu_1}$ so $j_2 = j_0 < j_1$. Recall that $c_\mu = c_k$. Then $c_{\mu_1} = c_k$ implies that $j_1 < k$. But also $j_2 < j_1 < k$ as desired.

In all cases we found that $a_\mu = a_{j_2}$ and $c_\mu = c_k$ implies that $j_2 < k$. Then $[a] < [c]$ by definition. Then $\leq$ is transitive. $\square$

Proof of Lemma 2.12. First we show that $\leq$ is reflexive. Let $[a] \in \mathcal{S}$. Then $[a] = [a]$ so $[a] \leq [a]$. Then $\leq$ is reflexive. Lemma A.2 and Lemma A.3 imply that $\leq$ is anti-symmetric and transitive. It follows by definition, that $\leq$ is a partial order. $\square$

Proof of Theorem 2.14. From Lemma 2.12 we have that $\leq$ is a partial order. It remains to show that if $[a], [b] \in \mathcal{S}$ then $[a] \leq [b]$ or $[b] \leq [a]$. Either $[a] \leq [b]$ or $[a] \not\leq [b]$.

Case 1: If $[a] \leq [b]$ we are done.

Case 2: Suppose $[a] \not\leq [b]$. Then $[a] \neq [b]$ and $[a] \not< [b]$. From $[a] \neq [b]$ we know that there exists $n_0, m_0 \in I$ so that $a_{n_0} = a_{m_0}$ and $b_{n_0} \neq b_{m_0}$ or $b_{n_0} = b_{m_0}$ and $a_{n_0} \neq b_{m_0}$. Now let

$$\begin{aligned} U = &\{j \in I : a_j = a_k \text{ and } b_j \neq b_k \text{ for some } k \leq j \in I\} \\ &\cup \{j \in I : b_j = b_k \text{ and } a_j \neq a_k \text{ for some } k \leq j \in I\} \end{aligned}$$

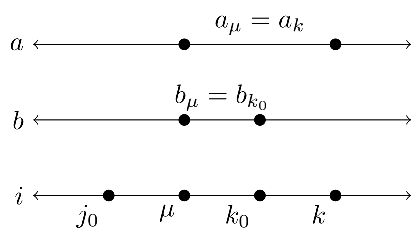
This is non-empty from above and the fact that I is well-ordered let $\mu = \min(U)$. Now let

$$J = \{j \in I : b_\mu = b_j\}.$$

From the fact I is well-ordered let $j_0 = \min(J)$.

We first show that $a_n = a_m \iff b_n = b_m$ for all $n, m < \mu \in I$. Let $m, n \in I$ so that $a_n = a_m$. Without loss of generality, let $m \leq n$. If $b_n \neq b_m$ then $n \in U$ and $n < \mu$violating the minimality of $\mu \in U$. Then it must be that $b_n = b_m$. Similarly we have $b_n = b_m \implies a_n = a_m$. Then $a_n = a_m \iff b_n = b_m$.

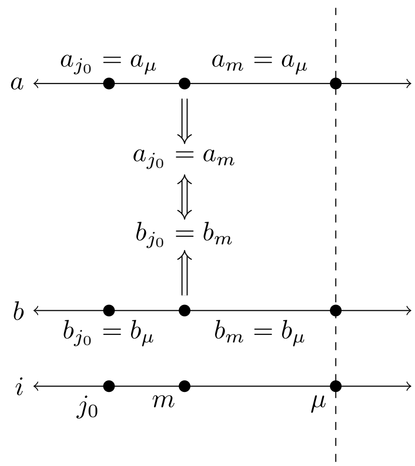
We now show that $a_\mu = a_k$ implies $j_0 < k$. Now because $[a] \not< [b]$ we have that for all $i, j \in I$ so that $a_i = a_j$ there exists $k_0 \in I$ so that $b_i = b_{k_0}$ and $k \leq j$. Now suppose that $a_\mu = a_k$ for some $k \in I$. There there exists $k_0 \in I$ so that $b_\mu = b_{k_0}$ and $k_0 \leq k$. But from minimality of $j_0 \in J$ we have that $j_0 \leq k_0 \leq k$.



We must now rule out that $j_0 = k$. Assume for the purposes of contradiction that there exists $k \in I$ so that $a_\mu = a_k$ and that $k = j_0$. Then $a_\mu = a_{j_0}$. Will will show that $a_n = a_m \iff b_n = b_m$ for $n, m \leq \mu$. We have already shown that $a_n = a_m \iff b_n = b_m$ for $n, m < \mu \in I$ above. It is (almost) a tautology that $a_\mu = a_\mu \iff b_\mu = b_\mu$.

Now suppose that $a_\mu = a_m$ for some $m < \mu$. Then from $[a] \neq [b]$ we know that there exists $k_0 \leq m$ so that $b_\mu = b_{k_0}$. From minimality of $j_0 \in U$ we have that $j_0 \leq k_0 \leq m < \mu$. Then $a_\mu = b_m$ and $b_\mu = j_0$ implies that $a_{j_0} = a_m$ where $j_0, m < \mu$. It follows that $b_{j_0} = b_m$. But recall that $b_\mu = b_{j_0}$. It follows that $b_\mu = b_m$.

Now suppose that $b_\mu = b_m$ for some $m < \mu$. Then we have $b_\mu = b_{j_0}$ and $b_\mu = b_m$. Then $b_{j_0} = b_m$. Additionally, the minimality of $j_0 \in U$ implies that $j_0 \leq m < \mu$. Then $b_{j_0} = b_m$ and $j_0, m < \mu \in I$ implies that $a_{j_0} = a_m$. We have that $a_{j_0} = a_m$ and $a_\mu = a_{j_0}$, from assumption, then $a_\mu = a_m$.



We have shown that $a_\mu = a_m \iff b_\mu = b_m$ for $m < \mu$. Combining this with past results we have that $a_n = a_m \iff b_n = b_m$ for all $n, m \leq \mu \in I$. This contradicts the fact that $\mu \in U$. Then it must be that $a_\mu = a_k$ implies that $j_0 < k$.

We have now shown that $[b] < [a]$ by definition. Then $[b] \leq [a]$

In any case we have $[a] \leq [b]$ or $[b] \leq [a]$. $\square$

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