Math2310 - Fall '22

Syllabus - Lecture 19

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Review

- Change of variables formula

- The Jacobian

- important examples of changes of variables: linear changes of variables

Topics

1 Symmetries (2D)

- Basic intuition

- Proof with change of variables formula

- the symmetry of translation

- rotation symmetry and radial functions

- dilation symmetry

- <u>exmpl</u> Find the area of

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x^2 + 3y^2 < 1 \right\}$$

  - understanding the shape: an ellipse

  - guessing the area by stretching considerations (linear vs area scaling)

  - cooking up a change of variables to prove our claim

2 Basic integrals in 3D

- Fubini's theorem in $3D$

  - <u>exmpl</u> integrate $f(x, y, z) = xyz$ over the

$$\mathcal{D} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \geq 0, y \geq 0, x + y + z \leq 1 \right\}$$

- Fubini as integrating cross sections:

$$\iiint_{\mathcal{D}} f(x, y, z) \mathrm{dVol}(x, y, z) = \int_{z=z_{\min}}^{z_{\max}} \left(\int_{\mathcal{D}_z} f(x, y, z) \mathrm{dArea}(x, y) \right) \mathrm{d}z$$

where $\mathcal{D}_z$ is the "slice" or "cross section". Expressed mathematically as:

$$\mathcal{D}_z := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{D} \right\}$$

Intution: freeze z and slice.

- Intuitive proof of Fubini's theorem: subdividing the volume into cubes and counting one layer at a time.

- `exmpl` find the volume of a pyramid of given the area of the base (arbitrary shape) and height

- Cylindrical coordinates

  - basic idea: Fubini in z then polar

  - full statement: the change of coordinates is

    $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \cos(\theta) \\ \rho \sin(\theta) \\ z \end{pmatrix}$$

  - The Jacobian is ρ so you integrate w.r.t. to

    $$\rho \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}z$$

References

Textbook

- [Ste] Chap 15.3 (complete) - Double integrals in polar coordinates

- [Ste] Chap 15.6 (complete) - Triple Integrals

- [Ste] Chap 15.7 (complete) - Triple Integrals in Cylindrical Coordinates

- [Ste] Chap 15.9 (complete) - Change of variables in Multiple integrals

Videos

- Triple Integrals in Cartesian Coordinates | Volume between Surfaces - YouTube

- Calculus 3: Triple Integrals (3 of 25) Choosing a Coordinate System: Cylindrical - YouTube

Geogebra applets

- Cylindrical coordinates - GeoGebra

- volume of pyramid - fubini - GeoGebra