MA 106 Help Session-2

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These are some of the interesting questions discussed in the Help Session conducted on March 23, 2023.

Problem 1 (Tutorial 3: Q15)

Given n^2 functions $f_{ij}(x)$ each differentiable on the interval (a, b), define $f(x) = det(f_{ij}(x))$ for each$x \in (a, b)$. Let $A(x) = (f_{ij}(x))$. Let $A_i(x)$ be the matrix obtained from $A(x)$ by differentiating the functions in the i^{th} row of A(x). Prove that $f'(x) = \sum_{i=1}^{n} detA_i(x)$

Solution:
We prove it by induction on order of matrix i.e. n.
Base case: $n = 1$
Trivial to see since $f(x) = f_{11}(x)$ and so $f'(x) = f'_{11}(x) = detA_1(x) = \sum_{i=1}^{1} detA_i(x)$
Induction hypothesis: Assume the result $f'(x) = \sum_{i=1}^{n-1} detA_i(x)$ holds true for all $(n-1)\times(n-1)$ matrices.
Now, we show that it is true for all $n \times n$ matrices.
Let A_{jk} denote the sub-matrix of A obtained by removing the j^{th} row and the k^{th} column.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} f_{1i}(x) \det(A_{1i})$$

Differentiate both sides

$$f'(x) = \sum_{i=1}^{n} (-1)^{i+1} (f'_{1i}(x) \det(A_{1i}) + f_{1i}(x)(\det(A_{1i}))')$$

Note that $(\det(A_{1i}))' = \sum_{j=1}^{n-1} \det(A_{1i_j}(x))$ by induction hypothesis as $\forall i$ A_{1i} is $(n-1) \times (n-1)$ matrix .

$$\begin{aligned}
f'(x) &= \sum_{i=1}^{n}(-1)^{i+1}(f'_{1i}(x)\det(A_{1i}) + f_{1i}(x)(\sum_{j=1}^{n-1}\det(A_{1i_j}(x)))) \\
&= \sum_{i=1}^{n}(-1)^{i+1}f'_{1i}(x)\det(A_{1i}) + \sum_{j=1}^{n-1}\sum_{i=1}^{n}(-1)^{i+1}f_{1i}(x)\det(A_{1i_j}(x)) \\
&= \det(A_1(x)) + \sum_{j=1}^{n-1}\det(A_{j+1}(x)) \\
&= \sum_{i=1}^{n}\det(A_i(x))
\end{aligned}$$

Hence proved

Problem 2

Let $\mathbf{A} \in \mathbb{R}^{9\times4}$ and $\mathbf{B} \in \mathbb{R}^{7\times3}$. Is there $\mathbf{X} \in \mathbb{R}^{4\times7}$ such that $\mathbf{X} \neq \mathbf{O}$ but $\mathbf{AXB} = \mathbf{O}$?

Hint: $\mathbf{AXB} = \mathbf{O}$ gives us $9 \times 3 = 27$ *equations* to solve in $4 \times 7 = 28$ *variables.*
Also note that the obtained system is homogeneous. The augmented matrix can have at most 27 pivots and therefore you can find at least one non-null solution to $\mathbf{X}$

Problem 3

Given a square matrix $\mathbf{A}$ of order $n \geq 2$. Represent $\mathbf{A}$ as
(a) Sum of two invertible matrices
(b) Sum of two non-invertible matrices

Solution:
(a) $\mathbf{A} = \mathbf{L} + \mathbf{U}$ where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices, select diagonal entries s.t. in both $\mathbf{L}$ and $\mathbf{U}$, they are all non-zero. Then $\mathbf{L}$ and $\mathbf{U}$ are required invertible matrices. (Determinant of either lower or upper triangular matrix is just the product of diagonal entries)
(b) $\mathbf{A} = \mathbf{L} + \mathbf{U}$ where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices respectively, select diagonal entries s.t. in both $\mathbf{L}$ and $\mathbf{U}$, their is at least one zero entry. Then $\mathbf{L}$ and $\mathbf{U}$ are required non-invertible matrices. (Determinant of either lower or upper triangular matrix is just the product of diagonal entries)