MA 106 Tutorial-1

Guramrit Singh
(210050061)

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All tutorial problems were discussed in class, solutions and approach to some of them which most students found difficult are given here.

Problem 3

If A and B are square matrices, show that $I - AB$ is invertible iff $I - BA$ is invertible.
[Hint: Start from $B(I - AB) = (I - BA)B$.]

Approach:
To show that a square matrix M is invertible, it is sufficient to show that there exists a square matrix N such that $MN = I = NM$. Then N is the inverse of M.

Solution: Only one side of *iff condition* is shown in this solution. Try to workout the other side yourself !!!
To show
If $I - AB$ is invertible then $I - BA$ is also invertible.
Since $I - AB$ is invertible, let C be it's inverse, i.e.

$$(I - AB)C = I = C(I - AB) \tag{1}$$

From equation (1), we can write BA as

$$BA = BIA \tag{2}$$
$$BA = B(I - AB)CA \tag{3}$$

Now using hint and equation (3)

$$I - BA = I - B(I - AB)CA \tag{4}$$
$$I - BA = I - (I - BA)BCA \tag{5}$$
$$(I - BA)(I + BCA) = I \tag{6}$$

From equation (1), we can again write BA as

$$BA = BIA \tag{7}$$
$$BA = BC(I - AB)A \tag{8}$$

Note the following equality

$$A(I - BA) = (I - AB)A \tag{9}$$

Now using equations (8) and (9)

$$I - BA = I - BC(I - AB)A \tag{10}$$
$$I - BA = I - BCA(I - BA) \tag{11}$$
$$(I + BCA)(I - BA) = I \tag{12}$$

Equations (6) and (12) shows us that $I - BA$ is invertible and $I + BCA$ is it's inverse.
Hence proved

Problem 4

Let N = {1, 2,..., n}. By a permutation on n letters we mean a bijective mapping $\sigma : N \to N$. Given a permutation $\sigma : N \to N$ define the permutation matrix P_σ to be the $n \times n$ matrix $((p_{ij}))$ where

$$p_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $P_{\sigma\circ\tau} = P_\tau P_\sigma$. Deduce that all permutation matrices are invertible and

$$P_\sigma^{-1} = P_{\sigma^{-1}} = P_\sigma^T$$

Approach:
Recall that for an $n \times n$ matrix A,

$$e_i^T A = A_i$$

where e_i is the i^{th} standard basis vector of $\mathbb{R}^n$ and A_i is the i^{th} row of A.
Also recall that any **bijective** mapping σ is one-one, onto mapping which has an inverse mapping σ^{-1} such that $\sigma^{-1} \circ \sigma$ is the *Identity* function on the domain of σ i.e.

$$\sigma^{-1} \circ \sigma(x) = x \qquad\qquad ,\forall x \in Domain(\sigma)$$

Solution:
Part 1 Look at entry p_{ij} of $P_{\sigma\circ\tau}$, we have by definition of permutation matrix

$$p_{ij} = \begin{cases} 1 & \text{if } j = \sigma \circ \tau(i) \\ 0 & \text{otherwise.} \end{cases}$$

Now we look at entry p'_{ij} of $P_\tau P_\sigma$
The entry p'_{ij} comes from product of $(P_\tau)_i (P_\sigma)^j$, where for a matrix A, A_i is it's i^{th} row and A^j is it's j^{th} column.
Explanation-1 For $p'_{ij} = 1$, it should be the case that $\tau(i) = \sigma^{-1}(j)$
($\tau(i)$ gives the column of entry corresponding to 1 in i^{th} row of P_τ and $\sigma^{-1}(j)$ gives the row of entry corresponding to 1 in j^{th} column of P_σ)
Hence we get $j = \sigma \circ \tau(i)$
Explanation-2 For $p'_{ij} = 1$, it should be the case that $\sigma \circ \tau(i) = j$
($\tau(i)$ gives the column of entry corresponding to 1 in i^{th} row of P_τ and $\sigma \circ \tau(i)$ gives the column

of entry corresponding to 1 in $\tau(i)^{th}$ row of P_σ. The point to note is that $(P_\tau)_i = e^T_{\tau(i)}$ and thus extracts $\tau(i)^{th}$ row of P_σ on pre-multiplication)

Thus, we get

$$p'_{ij} = \begin{cases} 1 & \text{if } j = \sigma \circ \tau(i) \\ 0 & \text{otherwise.} \end{cases}$$

Since $p_{ij} = p'_{ij}$ $\forall i, j$, it implies

$$P_{\sigma\circ\tau} = P_\tau P_\sigma$$

Part 2 To find P_σ^{-1}, consider the permutation matrix obtained from the bijection σ^{-1}, then as shown in Part-1,

$$P_{\sigma\circ\sigma^{-1}} = P_{\sigma^{-1}} P_\sigma$$

and also

$$P_{\sigma^{-1}\circ\sigma} = P_\sigma P_{\sigma^{-1}}$$

By using the property that $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are Identity function, we get that

$$P_{\sigma^{-1}} P_\sigma = I = P_\sigma P_{\sigma^{-1}}$$

Therefore $P_\sigma^{-1} = P_{\sigma^{-1}}$
The other equality $P_\sigma^{-1} = P_\sigma^T$ can be directly verified by showing that $P_\sigma P_\sigma^T = I = P_\sigma^T P_\sigma$ (Try to verify this yourself !!!)

Problem 5

The matrix $A = \begin{bmatrix} a & i \\ i & b \end{bmatrix}$, where $i^2 = -1$, $a = \frac{1}{2}(1+\sqrt{5})$ and $b = \frac{1}{2}(1-\sqrt{5})$, has the property $A^2 = A$. Describe completely all 2×2 matrices A with complex entries such that $A^2 = A$.

Approach:
Just square the matrix A and you will get 4 equations to solve.

Definition:
A matrix A such that $A^2 = A$ is called an *idempotent* matrix.

Solution:
Take a general 2×2 matrix A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

Equating A^2 and A gives us the following set of equations:

$$a^2 + bc = a \tag{13}$$
$$ab + bd = b \tag{14}$$
$$ac + cd = c \tag{15}$$
$$bc + d^2 = d \tag{16}$$

Subtracting equation (16) from (13) gives us

$$a^2 - d^2 = a - d \tag{17}$$

Solution to (17) is either $a = d$ or $a + d = 1$

Case-1 (a=d)
New set of equations become

$$a^2 + bc = a \tag{18}$$
$$2ab = b \tag{19}$$
$$2ac = c \tag{20}$$

Case-1.1
$b = 0$ and $c = 0$.
Then equation (18) dictates that $a = 0$ or $a = 1$.
We get two solutions here.
Case-1.2
$b = 0$ and $c \neq 0$.
Then equation (20) dictates that $a = \frac{1}{2}$ but it doesn't satisfy equation (18).
No solution here.
Case-1.3
$b \neq 0$ and $c = 0$.
Then equation (19) dictates that $a = \frac{1}{2}$ but it doesn't satisfy equation (18).
No solution here.
Case-1.4
$b \neq 0$ and $c \neq 0$.
Then equation (19) dictates that $a = \frac{1}{2}$ which satisfies equation (20). To satisfy equation (18), $bc = \frac{1}{4}$.
Infinite solutions here, solution are of the form $a = d = \frac{1}{2}$, $bc = \frac{1}{4}$.

Case-2 (a+d=1)
Equations (14) and (15) automatically gets satisfied.
Note that from equations (13) and (16), we get a and d as roots of

$$x^2 - x + bc = 0 \tag{21}$$

So given b, c, we can get a and d such that they are roots of equation (21) (Verify that sum of roots is indeed equal to 1).

Note
Solutions from Case-1.4 are a **subset** of solutions from Case-2 as they have $a + d = 1$ and restricted b, c i.e. $bc = \frac{1}{4}$.

Finally, all solution matrices A are given below

$$A \in \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} \frac{1+\sqrt{1-4bc}}{2} & b \\ c & \frac{1-\sqrt{1-4bc}}{2} \end{bmatrix} \middle| b, c \in \mathbb{C} \right\} \cup \left\{ \begin{bmatrix} \frac{1-\sqrt{1-4bc}}{2} & b \\ c & \frac{1+\sqrt{1-4bc}}{2} \end{bmatrix} \middle| b, c \in \mathbb{C} \right\}$$