tokam2D equations:

$$\partial_t n + [\Phi, n] - g\partial_y (\Phi - n) = C(\Phi - n) + D\nabla^2 n$$

$$\partial_t \Omega + (1+\tau)\, g\partial_y n + \nabla_{\perp i}\left[\Phi, \nabla_{\perp i}(\Phi + \tau n)\right] = C(\Phi - n) + \nu\nabla^2\Omega$$

$$\partial_t n_k + [\Phi, n]_k + ik_y\kappa\Phi_k - igk_y(\Phi_k - n_k) = C(\Phi_k - n_k) - Dk^2 n_k$$

$$\partial_t \Omega_k + i(1+\tau)\, gk_y n_k + \nabla_{\perp i}\left[\Phi, \nabla_{\perp i}(\Phi + \tau n)\right]_k = C(\Phi_k - n_k) - \nu k^2\Omega_k$$

$$\partial_t n_k = -[\Phi, n]_k + [C + i(g-\kappa)k_y]\,\Phi_k - \left(C + igk_y n_k + Dk^2\right) n_k$$

$$\partial_t \Omega_k = -\nabla_{\perp i}\left[\Phi, \nabla_{\perp i}(\Phi + \tau n)\right]_k - (C + i(1+\tau)\, gk_y)\, n_k + \left(C + \nu k^4\right)\Phi_k$$

$$\Omega_k = -k^2(\Phi_k + \tau n_k)$$

$$\Phi_k = -\left(\frac{\Omega_k}{k^2} + \tau n_k\right)$$

so that the actual vorticity equation becomes:

$$\partial_t\nabla_\perp^2\Phi + \left[\Phi + \tau n, \nabla_\perp^2\Phi\right] - \tau\left[\nabla_{\perp i}\Phi, \nabla_{\perp i} n\right] + g\partial_y n - \kappa\tau\partial_y\nabla_\perp^2\Phi + \tau g\partial_y\left[\left(1 - \nabla_\perp^2\right)n + \nabla_\perp^2\Phi\right] = C\left(1 - \tau\nabla_\perp^2\right)(\Phi$$

$$-k^2\partial_t\Phi_k = -\left[\Phi, \nabla_\perp^2\Phi\right]_k + \tau\nabla_{\perp i}\left[\nabla_{\perp i}\Phi, n\right]_k - igk_y n_k - i\tau gk_y\left[\left(1 + k^2\right)n_k - k^2\Phi_k\right] + C\left(1 + \tau k^2\right)(\Phi_k - n_k)$$

$$\partial_t\Phi_k = \frac{\left[igk_y\left[1 + \tau\left(1 + k^2\right)\right] + C\left(1 + \tau k^2\right)\right]}{k^2} n_k - \frac{\left[C\left(1 + \tau k^2\right) + i\tau(g - \kappa)\, k_y k^2\right]}{k^2}\Phi_k$$

$$\begin{aligned}\partial_t\nabla_\perp^2\Phi + \left[\Phi, \nabla_\perp^2\Phi\right] - \tau\left[\nabla_\perp^2\Phi, n\right] - \tau\left[\nabla_{\perp i}\Phi, \nabla_{\perp i} n\right]& \\ + g\partial_y n - \kappa\tau\partial_y\nabla_\perp^2\Phi + \tau g\partial_y\left[\left(1 - \nabla_\perp^2\right)n + \nabla_\perp^2\Phi\right] &= C\left(1 - \tau\nabla_\perp^2\right)(\Phi - n)\end{aligned}$$

$$\begin{aligned}\partial_t\nabla_\perp^2\Phi + \left[\Phi, \nabla_\perp^2\Phi\right] - \tau\nabla_{\perp i}\left[\nabla_{\perp i}\Phi, n\right]& \\ + g\partial_y n - \kappa\tau\partial_y\nabla_\perp^2\Phi + \tau g\partial_y\left[\left(1 - \nabla_\perp^2\right)n + \nabla_\perp^2\Phi\right] &= C\left(1 - \tau\nabla_\perp^2\right)(\Phi - n)\end{aligned}$$

note that here we used:

$$\tau\left[\nabla_\perp^2\Phi, n\right] + \tau\left[\nabla_{\perp i}\Phi, \nabla_{\perp i} n\right] = \tau\nabla_{\perp i}\left[\nabla_{\perp i}\Phi, n\right]$$

or more explicitly:

$$\tau\left[(\partial_{xx} + \partial_{yy})\,\Phi, n\right] + \tau\left[\partial_x\Phi, \partial_x n\right] + \tau\left[\partial_y\Phi, \partial_y n\right] = \tau\partial_x\left(\left[\partial_x\Phi, n\right]\right) + \partial_y\left(\left[\partial_y\Phi, n\right]\right)$$

$$\begin{aligned}\nabla_{\perp i}\left[\nabla_{\perp i}\Phi, n\right] &= \partial_x\left(\partial_{xx}\Phi\partial_y n - \partial_{xy}\Phi\partial_x n\right) + \partial_y\left(\partial_{yx}\Phi\partial_y n - \partial_{yy}\Phi\partial_x n\right) \\ &= \partial_{xx}\left(\partial_x\Phi\partial_y n - \partial_y\Phi\partial_x n\right) - \partial_x\left(\partial_x\Phi\partial_{yx} n - \partial_y\Phi\partial_{xx} n\right) \\ &+ \partial_{yy}\left(\partial_x\Phi\partial_y n - \partial_y\Phi\partial_x n\right) - \partial_y\left(\partial_x\Phi\partial_{yy} n - \partial_y\Phi\partial_{yx} n\right) \\ &= \partial_{xx}\left(\partial_x\Phi\partial_y n - \partial_y\Phi\partial_x n\right) - \partial_x\left(\partial_y\left(\partial_x\Phi\partial_x n\right) - \partial_x\left(\partial_y\Phi\partial_x n\right)\right) \\ &+ \partial_{yy}\left(\partial_x\Phi\partial_y n - \partial_y\Phi\partial_x n\right) - \partial_y\left(\partial_y\left(\partial_x\Phi\partial_y n\right) - \partial_x\left(\partial_y\Phi\partial_y n\right)\right) \\ &= \partial_{xx}\left(\partial_x\Phi\partial_y n\right) - \partial_{yy}\left(\partial_y\Phi\partial_x n\right) - \partial_{xy}\left(\partial_x\Phi\partial_x n - \partial_y\Phi\partial_y n\right)\end{aligned}$$

$$\nabla_{\perp i}\left[\nabla_{\perp i}\Phi,n\right]_k=-k_x^2\left(\partial_x\Phi\partial_y n\right)_k+k_y^2\left(\partial_y\Phi\partial_x n\right)_k+k_xk_y\left(\partial_x\Phi\partial_x n-\partial_y\Phi\partial_y n\right)_k$$

$$\begin{aligned}\left[\Phi,\nabla_\perp^2\Phi\right]_k&=\partial_x\Phi\partial_y\nabla^2\Phi-\partial_y\Phi\partial_x\nabla^2\Phi\\&=\partial_y\left(\partial_x\Phi\nabla^2\Phi\right)-\partial_x\left(\partial_y\Phi\nabla^2\Phi\right)\\&=ik_x\left(\partial_y\Phi\Omega\right)_k-ik_y\left(\partial_x\Phi\Omega\right)_k\end{aligned}$$

$$-k^2\partial_t\Phi_k=-\left[\Phi,\nabla_\perp^2\Phi\right]_k+\tau\nabla_{\perp i}\left[\nabla_{\perp i}\Phi,n\right]_k-igk_yn_k-\tau gik_y\left[\left(1+k^2\right)n_k-k^2\Phi_k\right]+C\left(1+\tau k^2\right)\left(\Phi_k-n_k\right)$$

$$\begin{aligned}\partial_t\Phi_k&=\frac{1}{k^2}\left[\Phi,\nabla_\perp^2\Phi\right]_k-\frac{\tau}{k^2}\nabla_{\perp i}\left[\nabla_{\perp i}\Phi,n\right]_k\\&+\frac{\left[igk_y\left(1+\tau\left(1+k^2\right)\right)+C\left(1+\tau k^2\right)\right]}{k^2}n_k-\frac{\left[\tau\left(g-\kappa\right)ik_yk^2+C\left(1+\tau k^2\right)\right]}{k^2}\Phi_k\end{aligned}$$

so in the end we have the following equations:

$$\begin{aligned}\partial_t\Phi_k&=\frac{\left[igk_y\left[1+\tau\left(1+k^2\right)\right]+C\left(1+\tau k^2\right)\right]}{k^2}n_k-\frac{\left[C\left(1+\tau k^2\right)+i\tau\left(g-\kappa\right)k_yk^2\right]}{k^2}\Phi_k\\&+\frac{\left[\Phi,\nabla_\perp^2\Phi\right]_k}{k^2}-\frac{\tau}{k^2}\nabla_{\perp i}\left[\nabla_{\perp i}\Phi,n\right]_k\\\partial_t n_k&=\left[C+i\left(g-\kappa\right)k_y\right]\Phi_k-\left(C+igk_yn_k+Dk^2\right)n_k-\left[\Phi,n\right]_k\end{aligned}$$