Derivation of Satellite State Estimation Equations

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State vector:

$$\vec{x} = [x, y, z, u, v, w, \mu, J2, C_D, x_{s1}, y_{s1}, z_{s1}, x_{s2}, y_{s2}, z_{s2}, x_{s3}, y_{s3}, z_{s3}],$$

where x, y, z and u, v, w are respectively the position and velocity components in the geocentric equatorial frame, which will be considered an inertial frame for the purposes of our calculations, μ is the gravitational parameter of the earth, J_2 is the second spherical harmonic used in the first approximation to the gravitational potential due to the nonuniform mass distribution of the earth, C_D is the drag coefficient, and x_{si}, y_{si}, z_{si} are the coordinates in the inertial frame of the three ground observation stations. Note, that μ, C_D and J_2 are included as part of the state and so will be, as intended, influenced by the measurements.

We assume that the total force acting on the satellite is

$$F = -\frac{\mu}{r^2}\hat{u}_r - \mu J_2 R^2 \left(\frac{3z}{r^5}\hat{K} + \left(\frac{3}{2r^4} - \frac{15z^2}{2r^6} \right) \hat{u}_r \right) - \frac{1}{2}\rho \frac{C_D A}{m} v_{rel}\vec{v}_{rel},$$

where $\hat{u}_r$ points in the radial direction,

$$\rho = \rho_0 e^{-(r-r_0)/H},$$

and A, m are the surface area used in the drag calculations and the mass of the satellite respectively. The oblateness correction may be found in Prussing and Conway (1993).

Similarly, the relative velocity used for drag is related to the velocity in the inertial system by the angular velocity of earth's rotation as

$$\begin{aligned} \vec{v}_{rel} &= \tfrac{dr}{dt} - \vec{\omega}_E \times \vec{r} = \left(\tfrac{dx}{dt} + \omega_E y \right) \hat{I} + \left(\tfrac{dy}{dt} - \omega_E x \right) \hat{J} + \tfrac{dz}{dt}\hat{K} && (1) \\ &= \left(u + \omega_E y \right) \hat{I} + \left(v - \omega_E x \right) \hat{J} + w\hat{K} && (2) \end{aligned}$$

and

$$v_{rel}^2 = (u + \omega_E y)^2 + (v - \omega_E x)^2 + w^2$$

Here ω_E is the norm of $\vec{\omega}_E$.

Since $\hat{u}_r = \frac{x}{r}\hat{I} + \frac{y}{r}\hat{J} + \frac{z}{r}\hat{K}$, combining these expressions with the Newton's second law and using $\partial_i r = x_i/r$, we obtain expressions for the accelerations in the x, y, z directions:

$$\dot{u} = -\frac{\mu x}{r^3} - \mu J_2 R^2 x \left(\frac{3}{2r^5} - \frac{15z^2}{2r^7} \right) - \frac{1}{2}\rho \frac{C_D A}{m} v_{rel} \left(u + \omega_E y \right) =: F_3,$$

$$\dot{v} = -\frac{\mu y}{r^3} - \mu J_2 R^2 y \left(\frac{3}{2r^5} - \frac{15z^2}{2r^7} \right) - \frac{1}{2}\rho \frac{C_D A}{m} v_{rel} \left(v - \omega_E x \right) =: F_4,$$

$$\dot{w} = -\frac{\mu z}{r^3} - \mu J_2 R^2 z \left(\frac{9}{2r^5} - \frac{15z^2}{2r^7} \right) - \frac{1}{2}\rho \frac{C_D A}{m} v_{rel} w =: F_6.$$

In addition, μ, J_2, and C_D are assumed to be constant for the purposes of the dynamic equation (they will change based on the estimation algorithm), we have

$$\dot{\mu} = \dot{J}_2 = \dot{C_D} = 0.$$

In the following three pages we calculate the derivatives of F with respect to some key state variables, i.e. F_{40} is the derivative of F_4 with respect to the zero state variable, i.e. x.

$$\dot{u} = -\frac{\mu x}{r^3} - \mu J_2 R^2 x \left(\frac{3}{2r^5} - \frac{15z^2}{2r^7} \right) - \frac{1}{2}\rho \frac{C_D A}{m} v_{rel} \left(u + \omega_E y \right) =: F_3$$

with partials,

$$\begin{aligned}
F_{30} &= -\tfrac{\mu}{r^3} + 3\tfrac{\mu x}{r^4} r_x - P_G \left(\left(\tfrac{3}{2r^5} - \tfrac{15z^2}{2r^7} \right) + x r_x \left(-\tfrac{15}{2r^6} + \tfrac{105}{2r^8} z^2 \right) \right) && (3)\\
&\quad + P_D \omega_E (v - \omega_E x)(u + \omega_E y)/v_{rel} + P_D \tfrac{x}{rH} v_{rel}(u + \omega_E y) && (4)\\
&= -\tfrac{\mu}{r^3} + 3\tfrac{\mu x^2}{r^5} - P_G \left(\left(\tfrac{3}{2r^5} - \tfrac{15z^2}{2r^7} \right) + x^2 \left(-\tfrac{15}{2r^7} + \tfrac{105}{2r^9} z^2 \right) \right) && (5)\\
&\quad + P_D \omega_E (v - \omega_E x)(u + \omega_E y)/v_{rel} + P_D \tfrac{x}{rH} v_{rel}(u + \omega_E y), && (6)
\end{aligned}$$

$$\begin{aligned}
F_{31} &= 3\tfrac{\mu x}{r^4} r_y - P_G \left(x r_y \left(-\tfrac{15}{2r^6} + \tfrac{105}{2r^8} z^2 \right) \right) + P_D \left(\omega_E v_{rel} + \omega_E (u + \omega_E y)^2 / v_{rel} \right) && (7)\\
&\quad + P_D \tfrac{y}{rH} v_{rel}(u + \omega_E y) && (8)\\
&= 3\tfrac{\mu x y}{r^5} - P_G x y \left(-\tfrac{15}{2r^7} + \tfrac{105}{2r^9} z^2 \right) - P_D \omega_E \left(v_{rel} + (u + \omega_E y)^2 / v_{rel} \right) && (9)\\
&\quad + P_D \tfrac{y}{rH} v_{rel}(u + \omega_E y), && (10)
\end{aligned}$$

$$\begin{aligned}
F_{32} &= 3\tfrac{\mu x}{r^4} r_z - P_G \left(x r_z \left(-\tfrac{15}{2r^6} + \tfrac{105}{2r^8} z^2 \right) - x \tfrac{15z}{r^7} \right) + P_D \tfrac{z}{rH} v_{rel}(u + \omega_E y) && (11)\\
&= 3\tfrac{\mu x z}{r^5} - P_G x z \left(-\tfrac{45}{2r^7} + \tfrac{105z^2}{2r^9} \right) + P_D \tfrac{z}{rH} v_{rel}(u + \omega_E y), && (12)
\end{aligned}$$

$$F_{33} = -P_D \left(v_{rel} + \frac{(u + \omega_E y)^2}{v_{rel}} \right),$$

$$F_{34} = -P_D \left(\frac{(v - \omega_E x)(u + \omega_E y)}{v_{rel}} \right),$$

$$F_{35} = -P_D \frac{w(u + \omega_E y)}{v_{rel}}.$$

$$F_{36} = -\frac{x}{r^3} - J_2 R^2 x \left(\frac{3}{2r^5} - \frac{15z^2}{2r^7} \right)$$

$$F_{37} = -\mu R^2 x \left(\frac{3}{2r^5} - \frac{15z^2}{2r^7} \right)$$

$$F_{38} = -\frac{1}{2}\rho \frac{A}{m} v_{rel}(u + \omega_E y)$$

$$\dot{v} = -\frac{\mu y}{r^3} - \mu J_2 R^2 y \left(\frac{3}{2r^5} - \frac{15z^2}{2r^7}\right) - \frac{1}{2}\rho\frac{C_D A}{m} v_{rel}\left(v - \omega_E x\right) =: F_4$$

with partials,

$$\begin{aligned} F_{40} &= 3\tfrac{\mu y}{r^4} r_x - P_G\left(y r_x \left(-\tfrac{15}{2r^6} + \tfrac{105z^2}{2r^8}\right)\right) + P_D\omega_E\left(v_{rel} + (v-\omega_E x)^2/v_{rel}\right) \\ &\qquad + P_D\tfrac{x}{rH} v_{rel}(v-\omega_E x) \\ &= 3\tfrac{\mu x y}{r^5} - P_G x y\left(-\tfrac{15}{2r^7} + \tfrac{105}{2r^9}z^2\right) + P_D\omega_E\left(v_{rel} + (v-\omega_E x)^2/v_{rel}\right) \\ &\qquad + P_D\tfrac{x}{rH} v_{rel}(v-\omega_E x), \end{aligned} \tag{13}$$

$$F_{41} = -\tfrac{\mu}{r^3} + 3\tfrac{\mu y}{r^4} r_y - P_G\left(\left(\tfrac{3}{2r^5} - \tfrac{15z^2}{2r^7}\right) + y r_y\left(-\tfrac{15}{2r^6} + \tfrac{105}{2r^8}z^2\right)\right) \tag{14}$$

$$-P_D\omega_E(u+\omega_E y)(v-\omega_E x)/v_{rel} + P_D\tfrac{y}{rH} v_{rel}(v-\omega_E x) \tag{15}$$

$$= -\tfrac{\mu}{r^3} + 3\tfrac{\mu y^2}{r^5} - P_G\left(\left(\tfrac{3}{2r^5} - \tfrac{15z^2}{2r^7}\right) + y^2\left(-\tfrac{15}{2r^7} + \tfrac{105}{2r^9}z^2\right)\right) \tag{16}$$

$$-P_D\omega_E(u+\omega_E y)(v-\omega_E x)/v_{rel} + P_D\tfrac{y}{rH} v_{rel}(v-\omega_E x), \tag{17}$$

$$F_{42} = 3\tfrac{\mu y}{r^4} r_z - P_G\left(y r_z\left(-\tfrac{15}{2r^6} + \tfrac{105}{2r^8}z^2\right) - y\tfrac{15z}{r^7}\right) + P_D\tfrac{z}{rH} v_{rel}(v-\omega_E x) \tag{18}$$

$$= 3\tfrac{\mu y z}{r^5} - P_G y z\left(-\tfrac{45}{2r^7} + \tfrac{105z^2}{2r^9}\right) + P_D\tfrac{z}{rH} v_{rel}(v-\omega_E x), \tag{19}$$

$$F_{43} = -P_D\left(\frac{(u+\omega_E y)(v-\omega_E x)}{v_{rel}}\right),$$

$$F_{44} = -P_D\left(v_{rel} + \frac{(v-\omega_E x)^2}{v_{rel}}\right),$$

$$F_{45} = -P_D\frac{w(v-\omega_E x)}{v_{rel}}.$$

$$F_{46} = -\frac{y}{r^3} - J_2 R^2 y\left(\frac{3}{2r^5} - \frac{15z^2}{2r^7}\right)$$

$$F_{47} = -\mu R^2 y\left(\frac{3}{2r^5} - \frac{15z^2}{2r^7}\right)$$

$$F_{48} = -\frac{1}{2}\rho\frac{A}{m} v_{rel}(v-\omega_E x)$$

$$\dot{w} = -\frac{\mu z}{r^3} - \mu J_2 R^2 z \left(\frac{9}{2r^5} - \frac{15z^2}{2r^7} \right) - \frac{1}{2}\rho \frac{C_D A}{m} v_{rel} w =: F_6.$$

with partials

$$\begin{aligned}
F_{50} \quad &= 3\tfrac{\mu z}{r^4} r_x - P_G \left(z r_x \left(-\tfrac{45}{2r^6} + \tfrac{105z^2}{2r^8} \right) \right) + P_D \left(\omega_E (u - \omega_E x) w / v_{rel} \right) && (20)\\
&\qquad + P_D \tfrac{x}{rH} v_{rel} w && (21)\\
&= 3\tfrac{\mu x z}{r^5} - P_G x z \left(-\tfrac{45}{2r^7} + \tfrac{105}{2r^9} z^2 \right) + P_D \left(\omega_E (u + \omega_E y) w / v_{rel} \right) && (22)\\
&\qquad + P_D \tfrac{x}{rH} v_{rel} w, && (23)
\end{aligned}$$

$$\begin{aligned}
F_{51} \quad &= 3\tfrac{\mu z}{r^4} r_y - P_G \left(z r_y \left(-\tfrac{45}{2r^6} + \tfrac{105z^2}{2r^8} \right) \right) - P_D \left(\omega_E (u + \omega_E y) w / v_{rel} \right) && (24)\\
&\qquad + P_D \tfrac{y}{rH} v_{rel} w && (25)\\
&= 3\tfrac{\mu y z}{r^5} - P_G y z \left(-\tfrac{45}{2r^7} + \tfrac{105z^2}{2r^9} \right) - P_D \left(\omega_E (u + \omega_E y) w / v_{rel} \right) && (26)\\
&\qquad + P_D \tfrac{y}{rH} v_{rel} w && (27)
\end{aligned}$$

$$\begin{aligned}
F_{52} \quad &= -\tfrac{\mu}{r^3} + 3\tfrac{\mu z}{r^4} r_z - P_G \left(\left(\tfrac{9}{2r^5} - \tfrac{15z^2}{2r^7} - z\tfrac{15z}{r^7} \right) + z r_z \left(-\tfrac{45}{2r^6} + \tfrac{105}{2r^8} z^2 \right) \right) && (28)\\
&\qquad + P_D \tfrac{z}{rH} v_{rel} w && (29)\\
&= -\tfrac{\mu}{r^3} + 3\tfrac{\mu z^2}{r^5} - P_G \left(\left(\tfrac{9}{2r^5} - \tfrac{45z^2}{r^7} + \tfrac{105z^2}{2r^9} \right) \right) + P_D \tfrac{z}{rH} v_{rel} w, && (30)
\end{aligned}$$

$$F_{53} = -P_D \left(\frac{(u + \omega_E y) w}{v_{rel}} \right),$$

$$F_{54} = -P_D \left(\frac{(v - \omega_E x) w}{v_{rel}} \right),$$

$$F_{55} = -P_D \left(v_{rel} + \frac{w^2}{v_{rel}} \right).$$

$$F_{56} = -\frac{z}{r^3} - J_2 R^2 z \left(\frac{9}{2r^5} - \frac{15z^2}{2r^7} \right)$$

$$F_{57} = -\mu R^2 z \left(\frac{9}{2r^5} - \frac{15z^2}{2r^7} \right)$$

$$F_{58} = -\frac{1}{2}\rho \frac{A}{m} v_{rel} w$$