Take Home Exam 3

Question 1

Theorem (The Well Ordering Principle): A least element exists in any non empty set of positive integers.

Assume let $A = \{n \mid n \in \mathbb{Z}, \text{ and } 0 < n < 1\}$.

If $A \neq \emptyset$, then by the Well-Ordering Principle A has a smallest element, say $n \in A$.

But then multiplying the inequality $0 < n < 1$ by the positive integer n, we have $0 < n^2 < n < 1$.

However , n^2 is an integer and so $n^2 \in A$, that contradicts n is the smallest element of A. Thus, our original assumption is not correct, so there does not exist an integer n satisfying $0 < n < 1$.

As a result, we have proved that 1 is the smallest positive integer.

Question 2

S(m,n): The number of possible (ordered) solutions to $X_1 + X_2 + . . . + X_m = n$ $\quad X_i \in \{0\} \cup \mathbb{Z}^+$

Basis Step

$S(m,1) \rightarrow X_1 + X_2 + X_3 \ldots + X_m = 1$

If $X_1 = 1$, then $X_2, X_3 \ldots, X_m$ are equal to 0.

$X_2 = 1$, then $X_1, X_3 \ldots, X_m$ are equal to 0.

$X_3 = 1$, then $X_1, X_2 \ldots, X_m$ are equal to 0.

….

$X_m = 1$, then $X_1, X_2 \ldots, X_{m-1}$ are equal to 0.

Therefore, there are m different cases for S(m,1).

According to the formula , $S(m,n) = (n+m-1)! / (n! \,.\, (m-1)!)$

$S(m,1) = (m!) / (m-1)! = m$

This is true for the formula.

$S(1,n) \rightarrow X_1 = n$

Therefore, there is only one case for S(1,n).

According to the formula , $S(m,n) = (n+m-1)! / (n! \,.\, (m-1)!)$

$S(n,1) = (n!) / (n)! = 1$

This is also true for the formula.

Induction Step

Assume $S(m + 1, n)$ and $S(m, n + 1)$ are true.

$S(m + 1, n) = (n+m)! / (n! \,.\, m!)$

$S(m, n+1) = (n+m)! / ((n+1)! \,.\, (m-1)!)$

We can evaluate $S(m+1, n+1)$ in two different situations,

1) When $X_{m+1} = 0$,

$$\underbrace{X_1 + X_2 + X_3 \dots + X_m}_{} + \underbrace{X_{m+1}}_{0} = n+1$$

There are $S(m, n+1)$ solutions for this equation.

2) When $X_{m+1} > 0$, $\quad X_{m+1} \in \mathbb{Z}^+$

$$\underbrace{X_1 + X_2 + X_3 \dots + X_m}_{} + \underbrace{X_{m+1}}_{1,2,3,\dots} = n+1$$

If we reduce both sides with 1,

$$X_1 + X_2 + X_3 \dots + X_m + \underbrace{X_{m+1}}_{0,1,2\dots} = n$$

There are $S(m+1, n)$ solutions for this equation.

As a result, $S(m+1, n+1)$ equation has $S(m, n+1) + S(m+1, n)$ distinct solutions.

According to the formula $(n+m-1)! / (n! \,.\, (m-1)!)$,

$S(m+1, n+1) = (n+m+1)! / ((n+1)! \,.\, m!)$,

According to the induction,

$$\begin{aligned} S(m+1, n+1) &= S(m+1, n) + S(m, n+1) \\ &= [(n+m)! / (n! \,.\, m!)] + [(n+m)! / ((n+1)! \,.\, (m-1)!)] \\ &= (n+m+1)! / ((n+1)! \,.\, m!) \end{aligned}$$

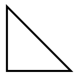
Both results are the same.

So by the mathematical induction, the number of possible solutions of $S(m,n)$ is

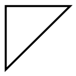
$\rightarrow X_1 + X_2 + X_3 \dots + X_m = n$ is, $\quad (n+m-1)! / (n! \,.\, (m-1)!)$

Question 3

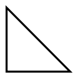
a) We can find 4 different conditions in that we place triangles that are the same size in any rotation.



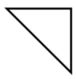
First case, we can place 28 of a triangle like this.



Second case, we can place 21 of a triangle like this .



Third case, we can place 21 of a triangle like this .



Fourth case, we can place 21 of a triangle like this.

When we add them together, 91 triangles can be placed.

b) There are 4^6 functions from a set with six elements to a set with four elements. However, this counts functions with fewer than four elements in the range. We must exclude those functions. To do so, we can use the Inclusion-Exclusion Principle.

$$\binom{4}{0} . 4^6 - \binom{4}{1} . 3^6 + \binom{4}{2} . 2^6 - \binom{4}{3} . 1^6 + \binom{4}{4} . 0^6$$

$$= 4^6 – 4 . 3^6 + 6 . 2^6 - 0$$

$$= 4096 – 2916 + 384 – 4 + 0 = 1560.$$

Question 4

a) a_n is the number of strings which consist of $\Sigma = \{0, 1, 2\}$ with length of n that contain two consecutive symbols that are the same.

We can consider 2 distinct cases while trying to get a_n.

1) When a_{n-1} has two same consecutive symbols, so β can be 3 different value in $\{0, 1, 2\}$.

$$a_n = \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{a_{n-1}} \ | \ \underbrace{\_}_{\beta}$$

Thus, there are $3.a_{n-1}$ situation.

2) When a_{n-1} does not have two same consecutive symbols, we need to find all non-consecutive possibilites and make it consecutive.

$$a_n = \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{a_{n-1}} \mid \underbrace{\_}_{\beta}$$

The number of all possible situations = 3^{n-1}

The number of consecutive situations = a_{n-1}

When we substitute them, we can get the number of non-consecutive situations = $3^{n-1} - a_{n-1}$

In order to reach recurrence relations for a_n, we need to add the probabilities in the two different cases that we examined.

$$a_n = 3.a_{n-1} + 3^{n-1} - a_{n-1} = 3^{n-1} + 2.a_{n-1}$$

b) $a_1 = \{\}$

$a_2 = \{ \{0,0\},\{1,1\},\{2,2\} \}$

$a_3 = \{ \{0,0,0\}, \{1,0,0\}, \{2,0,0\}, \{0,0,1\}, \{0,0,2\}, \{1,1,1\}, \{0,1,1\}, \{2,1,1\}, \{1,1,0\}, \{1,1,2\}, \{2,2,2\}, \{0,2,2\}, \{1,2,2\}, \{2,2,0\}, \{2,2,1\}\}$

$a_1 = 0$

$a_2 = 3^{n-1} + 2.a_{n-1} = 3^1 + 2.a_1 = 3$

$a_3 = 3^{n-1} + 2.a_{n-1} = 3^2 + 2.a_2 = 15$

c) $X_g = X_h + X_p$

$X_h \rightarrow a_n - 2a_{n-1} = 0$ (the characteristic equation)

We need to find characteristic roots that satisfies the characteristic equation.

$r - 2 = 0$ So, the characteristic root r should be 2.
$X_h = A . 2^n$

$X_p \rightarrow a_n - 2a_{n-1} = 3^{n-1}$

We can assume a_n as $= B.3^n$

$X_g = X_h + X_p = A .2^n + B. 3^n$
By solving the inital condition, $a_2 = A .2^2 + B.3^2 = 4.A + 9.B = 3$
$a_3 = A .2^3 + B.3^3 = 8.A + 27.B = 15$
$A = - (3/2) \qquad B= 1$

So, $X_g = - (3/2) . 2^n + 3^n$