Recursive Definitions

Friday, December 3, 2021 10:18 AM

$f(0) = a$

$f(n+1) =$ a function of $f(0), f(1), \dots, f(n-1), f(n)$

e.g.,

$$\begin{cases} f(0) = 0 \\ f(n+1) = f(n) + (n+1) \end{cases}$$

$f(n) = 1 + 2 + 3 + \dots + n$

$$(\underset{\substack{\uparrow \\ f(0)}}{0}, \underset{\substack{\uparrow \\ f(1)}}{0+1}, \underset{\substack{\uparrow \\ f(2)}}{0+1+2}, \dots, 0 + \dots + n, \dots)$$

e.g.,

$$\begin{cases} p(0) = 1 \\ p(n+1) = p(n) \cdot (n+1) \end{cases}$$

$p(n) = 1 \cdot 2 \cdot \dots \cdot n$
$= n!$

e.g.,
$$\begin{cases} a(0) = a_0 \\ a(n+1) = a(n) + d \end{cases}$$

rec. def^{n} of arith. progression

$$(\underset{\substack{\uparrow \\ a_0}}{a_0}, \underset{\substack{\uparrow \\ a_1}}{a_0+d}, \underset{\substack{\uparrow \\ a_2}}{a_0+2d}, \dots, \underset{\substack{\uparrow \\ a(n)}}{a_0+nd}, \dots)$$

Friday, December 3, 2021 10:47 AM

e.g., rec. def^n of multipl. via addition

$$\begin{cases} x \cdot 0 = 0 \\ x \cdot (y+1) = x \cdot y + x \end{cases}$$

e.g., given $a_0, a_1, a_2, \ldots, a_n, \ldots$

a sequence

$$\begin{cases} S(0) = 0 \\ S(n+1) = S(n) + a_{n+1} \end{cases} \Rightarrow \text{rec. def}^n \text{ of } S(n) = \sum_{i=0}^{n} a_i$$

$$\begin{cases} P(0) = 1 \\ P(n+1) = P(n) \cdot a_{n+1} \end{cases} \Rightarrow \text{rec. def}^n \text{ of } P(n) = \prod_{i=0}^{n} a_i$$

e.g., fib. seq.

$0, 1, 1, 2, 3, 5, \ldots$

rec. def^n $\begin{cases} f_0 = 0 \\ f_1 = 1 \end{cases}$ $f_n = f_{n-1} + f_{n-2}$ $n \geq 2$

Friday, December 3, 2021 10:52 AM

$(0, 1, 1, 2, \ldots)$

$f_0 \quad f_1 \quad f_2 \quad f_3$

$f(0) = 0$

$$f(n+1) = \begin{cases} 1 & \text{if } n+1 = 1 \\ f(n) + f(n-1) & \text{otherwise (e.g., } n+1 > 2) \end{cases}$$

Rec. function

function fib(n)
  if n = 0 return 0
  else if n = 1 return 1
    else return f(n-1) + f(n-2)

\# of divisions? → seems exponential. needs a proof!

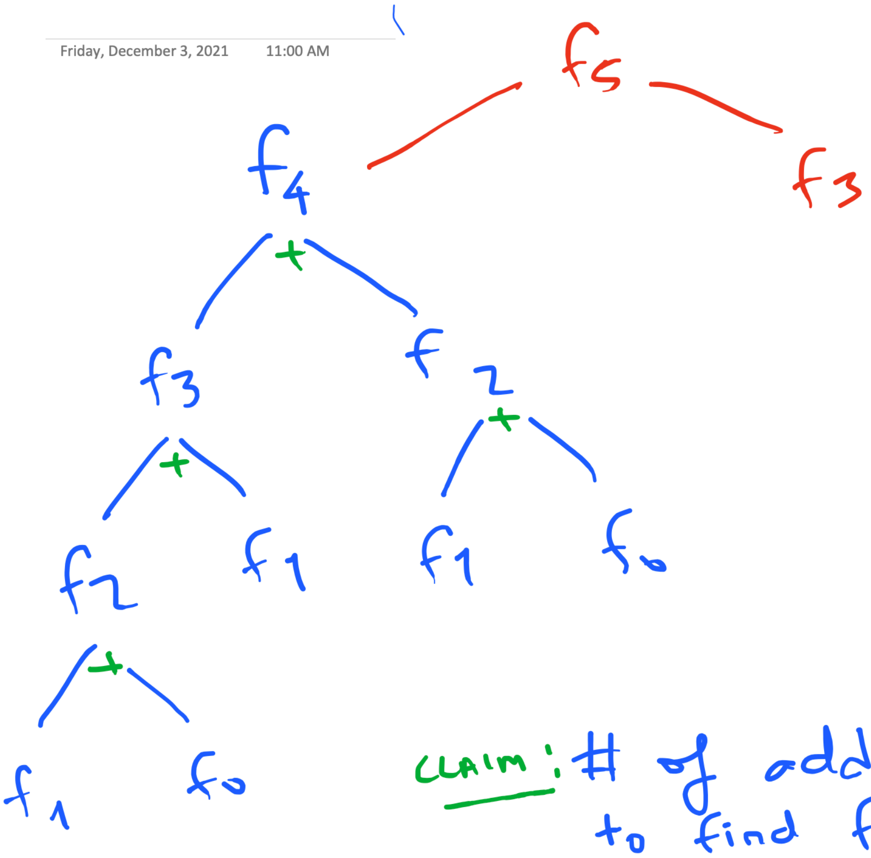
Iterative alg.

procedure fib(n)
 if n = 0 then $y \leftarrow 0$ (f_n)
 else $\{x \leftarrow 0, y \leftarrow 1$
   for i=0 to n-1
     $\{z = x+y, x = y, y \leftarrow z\}\}$

$\longrightarrow O(n)$ additions!

x, y → past two values

Friday, December 3, 2021 11:00 AM



$(n+1)^{st}$ fib. #

CLAIM: # of additions to find f_n $= f_{n+1} - 1$

	# of additions
f_2	1
f_3	2
f_4	4
f_5	6

$\rightarrow \dots$

for	# of additions
f_n $\rightsquigarrow$	$f_{n+1} - 1$

Friday, December 3, 2021 11:04 AM

$(0, 1, 1, 2, 3, \dots)$

Thm $n \geq 3$ $f_n > \alpha^{n-2}$ where f_n is the n^{th} fib. number & $\alpha = \frac{1+\sqrt{5}}{2}$.

Proof (strong induction) $\qquad f_n \sim f_{n-1}, f_{n-2}$

BASE

- $\boxed{n=3}$ $\quad f_3 > \alpha^{3-1} = \alpha = \frac{1+\sqrt{5}}{2}$?

$f_3 = 2$

$$\frac{1+\sqrt{5}}{2} < \frac{1+3}{2} = \frac{4}{2} = 2 = f_3$$

$$f_3 > \frac{1+\sqrt{5}}{2} \checkmark$$

- $\boxed{n=4}$ $\quad f_4 = 3 > \alpha^{4-2} = \alpha^2$?

$$\alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1}{4}(1 + 2\sqrt{5} + 5) = \frac{3+\sqrt{5}}{2}$$

$$\frac{1}{4}(1+2\sqrt{5}+5) = \frac{6+2\sqrt{5}}{4}$$

$$\alpha^2 = \frac{3+\sqrt{5}}{2} < \frac{3+3}{2} = \frac{6}{2} = 3 = f_4 \qquad f_4 > \alpha^2 \checkmark$$

IND. STEP

IND. HYP. Assume that $f_3 > \alpha,\ f_4 > \alpha^2, \ldots,$

$f_n > \alpha^{n-2}$ ✓

f_{n+1} ?

$f_{n+1} = f_n + f_{n-1}$ via rec. def^n of fib. #'s

$$\begin{array}{rl} & f_n > \alpha^{n-2} \\ + & f_{n-1} > \alpha^{n-3} \\ \hline & f_n + f_{n-1} = f_{n+1} > \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha+1) \end{array}$$

$= \alpha^{n-3} \cdot \alpha^2$

$= \alpha^{n-1}$

$\alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} = \alpha + 1 = \frac{1+\sqrt{5}}{2}$

$\square$

Friday, December 3, 2021 11:18 AM

Euclidean alg. to find GCD of two integers

$$\begin{cases} \gcd(a,0) = a \\ \gcd(a,b) = \gcd(b, a \bmod b) \end{cases}$$

Time complexity of Euclid's Alg. ?

Thm

Let $a \geq b > 0$ be integers. # of divisions used by the Euclid's Alg. to find gcd of a, b is $5\times$ the # of decimal digits in b

Friday, December 3, 2021 11:24 AM

$a \geq 0$

gcd (a, b)

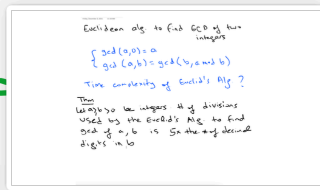
```
while b ≠ 0
{ t = a mod b
  a = b
  b = t
}
return a
```

$b > \frac{a}{2}$ $a \bmod b = a - b < \frac{a}{2}$

$b \leq \frac{a}{2}$ $a \bmod b < b \leq \frac{a}{2}$

for both cases magnitute is at least HALVE

$g(a, b) = gcd(b, \underbrace{a \bmod b}_{< \frac{a}{2}})$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gcd (a, b)

while $b \neq 0$

{ t = a mod b

a = b

b = t

}

return a

$b > \frac{a}{2}$ $\quad a \bmod b = a - b < \frac{a}{2}$

$b \leq \frac{a}{2}$ $\quad a \bmod b < b \leq \frac{a}{2}$

for both cases magnitute is at least HALVE

$g(a,b) = \gcd(b, \underbrace{a \bmod b}_{< \frac{a}{2}})$ → logarithmic drop