

CENG 223

Fall 2021-2022

THE3 - Sample Solutions

Question 1

By the well-ordering property, the set of positive integers has a least element, m . First we note that m cannot be larger than 1: 1 itself is a positive integer. Also m cannot be equal to or smaller than 0: that would make m a non-positive integer. Then we have $0 < m < 1$. Multiplying all sides by positive integer powers of m , we get $0 < m^{n+1} < m^n < m^{n-1} < \dots < m < 1$. Since m is assumed to be a positive integer, by the provided hint, all of its positive integer powers are positive integers as well. Then there should exist an infinite number of positive integers $(\{m^n \mid n \in \mathbb{Z}^+ - \{1\}\})$ that are smaller than m . Hence, a contradiction.

Note that showing the above for only $n = 1$ is also acceptable.

Question 2

$S(1, 1)$ declares number of different solutions to $x_1 = 1$ is 1, which is true as any $x_1 \neq 1$ would violate the equality. So we have the basis step established for both $S(m, 1)$ and $S(1, n)$.

Let us first prove $S(m, 1)$. Assume $S(m - 1, 1)$ is true. Then there are

$$f(m - 1, 1) = \frac{(m - 1 + 1 - 1)!}{1!(m - 1 - 1)!} = m - 1$$

solutions to $x_1 + \dots + x_{m-1} = 1$. For $S(m, 1)$, we have $x_1 + \dots + x_m = 1$. Taking $x_m = 0$ yields $f(m - 1, 1)$ -many solutions, and $x_m = 1$ renders all other x_i vanish, thus, yields only 1 solution. Therefore, $f(m, 1) = f(m - 1, 1) + 1 = (m - 1) + 1 = m = \frac{m!}{1!(m-1)!}$.

Next is $S(1, n)$. Assume $S(1, n - 1)$ is true, *i.e.*, there are $f(1, n - 1) = 1$ solution to $x_1 = n - 1$. Then $S(1, n)$ states that number of solutions to $x_1 = n$ should be $f(1, n) = 1$. For the purpose of induction, convert this problem to $x_1 = (n - 1) + 1$ and define $x'_1 = x_1 - 1$. Then $x'_1 = n - 1$ is the resulting problem which by the induction assumption admits only one solution. Since x_1 is completely determined by x'_1 , indeed we have $f(1, n) = 1$, and as such $S(1, n)$ is proved by induction as well.

For the above part, solutions that did not properly apply mathematical induction will also be accepted.

We now turn to prove $S(m + 1, n + 1)$. Assume $S(m, n + 1)$ and $S(m + 1, n)$ are true. That is,

$$x_1 + \dots + x_m = n + 1 \quad \text{has} \quad f(m, n + 1) = \frac{(n + m)!}{(n + 1)!(m - 1)!} \text{ -many solutions,}$$

$$\text{and} \quad x_1 + \dots + x_{m+1} = n \quad \text{has} \quad f(m + 1, n) = \frac{(n + m)!}{n!m!} \text{ -many solutions.}$$

For $x_1 + \cdots + x_m + x_{m+1} = n + 1$ we can divide the problem into two parts for the sake of induction: $x_{m+1} = 0$ and $x_{m+1} > 0$.

The first case corresponds to $S(m, n + 1)$ and we have $f(m, n + 1)$ -many solutions.

For the second case we can rewrite the problem as $x_1 + \cdots + (x'_{m+1} + 1) = n + 1$ where $x'_{m+1} = x_{m+1} - 1$. That is, $x_1 + \cdots + x'_{m+1} = n$. By doing so we inherently satisfied the condition of $x_{m+1} > 0$ and converted the problem into the one stated by $S(m + 1, n)$. As such we get $f(m + 1, n)$ -many solutions.

Then, the total number of solutions to $x_1 + \cdots + x_{m+1} = n + 1$ is

$$\begin{aligned} f(m, n + 1) + f(m + 1, n) &= \frac{(n + m)!}{(n + 1)!(m - 1)!} + \frac{(n + m)!}{n!m!} = \frac{(n + m)!m}{(n + 1)!m!} + \frac{(n + m)!(n + 1)}{(n + 1)!m!} \\ &= \frac{(n + m)!(n + 1 + m)}{(n + 1)!m!} = \boxed{\frac{(n + m + 1)!}{(n + 1)!m!} = f(m + 1, n + 1)}. \end{aligned}$$

Question 3

a. In each square that do not encapsulate a grid point, we have 4 congruent triangles to the one provided in the figure. As there are $N = 6$ rows of squares with n squares in the n th row, there a total of $N(N + 1)/2 = 21$ squares, meaning 84 triangles are present. There are also congruent triangles appearing at the end of each row of squares, and one at the top. Thus we have $N + 1$ triangles more to count. This gives us a total of $84 + 7 = 91$ triangles.

b. Using the inclusion-exclusion principle, number of onto functions are given by $\sum_{k=0}^n (-1)^k \binom{n}{k} (n - k)^m$ where $m = 6$ and $n = 4$. Then there are

$$\binom{4}{0}4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6 + \binom{4}{4}0^6 = 1560$$

Solutions that use Stirling numbers are also accepted.

Question 4

a. Let a_n denote the number of strings of length n that contain two consecutive symbols that are the same. Then the number of strings of length n that do not satisfy this property is given by $3^n - a_n$.

To acquire a recurrence relation for a_n we can either concatenate any of $\{0, 1, 2\}$ to the strings counted in a_{n-1} , or we can concatenate the last digit of a string counted in $3^{n-1} - a_{n-1}$. With this we establish

$$a_n = 3a_{n-1} + (3^{n-1} - a_{n-1}) = 2a_{n-1} + 3^{n-1}.$$

b. The question of number of strings satisfying the given property makes sense for $n \geq 0$ where $a_0 = 0$, $a_1 = 0$, $a_2 = 3$ and so on. Note that the logic of the established recurrence relation requires a_{n-1} to have at least a digit, so that it can be repeated. Thus, we expect $a_0 = 0$ and $a_1 = 0$ to be sufficient as initial conditions, and the rest of the numbers can be acquired by the recurrence relation. So formal description of the recurrence relation is

$$a_n = 2a_{n-1} + 3^{n-1} \text{ for } n \geq 2, \quad a_0 = a_1 = 0.$$

c. For the homogeneous solution we have $a_n = 2a_{n-1}$, characteristic equation of which yields $r = 2$. Then $a_n^{(h)} = A 2^n$ where A is some constant to be determined later on.

For the particular solution we guess the form of solution as $a_n^{(p)} = B 3^n$. Then

$$B 3^n = 2B 3^{n-1} + 3^{n-1} \leftrightarrow 3B 3^{n-1} = 2B 3^{n-1} + 3^{n-1} \leftrightarrow 3B = 2B + 1 \leftrightarrow B = 1.$$

Thus, $a_n^{(p)} = 3^n$.

Then our solution $a_n = a_n^{(h)} + a_n^{(p)} = A 2^n + 3^n$ can be determined with the initial condition $a_1 = 0$.

$$a_1 = A 2^1 + 3^1 = 2A + 3 = 0 \leftrightarrow A = -3/2 \leftrightarrow \boxed{a_n = -\frac{3}{2} 2^n + 3^n}.$$

Solutions that use a_2, a_3 etc. to determine the undetermined constant A are also accepted.