NATURAL DEDUCTION IN PROPOSITIONAL LOGIC

Logic explores the methods of sound reasoning.

Natural Deduction (ND) formalizes deductive reasoning that takes place in mathematical arguments.

We start with ND in the setting of Propositional Logic. Consider the more comprehensive setting of First Order Logic later.

A rule of ND has the following form:

$$\frac{\alpha_1 \dots \alpha_n}{\beta}$$

Here α_1 through α_n and β are all well-formed formulas of propositional logic (compound propositions). In other words, they are built from propositional variables and constants by applying the connectives to combine simpler formulas into complex ones. We prefer the shorter term *formula*.

The formulas α_1 through α_n are called the *premises*, and β is called the *conclusion*. (Another terminology: α_1 through α_n are called antecedents and β consequent.)

A premise is a proposition that is given. β follows from α_1 through α_n by a single application of the above rule. Thus, we pronounce the line separating the premises and the conclusion as “therefore”, “thus”, “hence” or any alternative wording.

The *provability* relation between formulas $\alpha_1, \dots, \alpha_n$ and β is designated as $\alpha_1, \dots, \alpha_n \vdash \beta$.

It means that starting with $\alpha_1, \dots, \alpha_n$ and applying the deduction rules successively, we arrive at the conclusion β. In a mathematical proof $\alpha_1, \dots, \alpha_n$ play the role of hypotheses, and β the conclusion of the theorem.

Let us first focus on the rules, then consider proofs.

Natural Deduction Rules for Propositional Logic

A deduction system consists of finitely many rules.

Let us start things off with a rather simple rule, the reflexivity rule:

$\dfrac{\alpha}{\alpha}$	Reflexivity

We shall investigate other rules by closely following the structure of formulas.

Given both α and β, it seems natural to conclude their conjunction:

$\dfrac{\alpha\ \beta}{\alpha \wedge \beta}$	$\wedge i$

This rule is known as ∧ introduction as it introduces a conjunction in its conclusion.

Conversely, given a conjunction it seems natural to conclude each one of its conjuncts separately:

$\dfrac{\alpha \wedge \beta}{\alpha}$	$\wedge e$
$\dfrac{\alpha \wedge \beta}{\beta}$	

The above two rules are known as ∧ elimination because they eliminate the conjunction in their premise.

These two rules are known as ∨ introduction:

$\dfrac{\alpha}{\alpha \vee \beta}$	$\vee i$
$\dfrac{\beta}{\alpha \vee \beta}$	

The ∨ elimination is a bit involved.

$$\frac{\alpha \lor \beta \begin{bmatrix}\alpha \\ \dots \\ \gamma\end{bmatrix}\begin{bmatrix}\beta \\ \dots \\ \gamma\end{bmatrix}}{\gamma} \quad \lor e$$

Shown inside "boxes" are sub-proofs. Given the premise $\alpha \lor \beta$, to reach a conclusion γ we need to conclude γ separately from α and from β . These two sub-proofs are shown inside boxes to emphasize that they are performed in separation. (In notation, $\alpha \dots \vdash \gamma$ and $\beta \dots \vdash \gamma$. The ellipses indicate that other assumptions that are in force at that point can be used as well. But this point should wait for our study of proofs.) The idea is actually familiar. Given a disjunction we perform case analysis on each one of the disjuncts: In both cases γ follows; therefore γ follows from the whole disjunction.

Let us now consider the conditional form.

$$\frac{\alpha \ \to \beta \quad \alpha}{\beta} \qquad \to e$$

This rule is also known with its Latin name, *modus ponens*, abbreviated MP.

The introduction rule for conditional is a bit involved.

$$\frac{\begin{bmatrix}\alpha \\ \dots \\ \beta\end{bmatrix}}{\alpha \ \to \beta} \qquad \to i$$

To conclude $\alpha \ \to \beta$ we need a sub-proof starting with the assumption α and concluding β. This sub-proof α ... $\vdash \beta$ is shown in box to emphasize that the assumption α is not available outside this sub-proof. Though we may use available assumptions from outside the box. We say that assumption $\propto$ is discharged as soon as β is reached.

The overall structure of the proofs of most theorems follows this line of reasoning.

What about negation?

$$\frac{\begin{bmatrix}\alpha \\ \dots \\ F\end{bmatrix}}{\neg\alpha} \qquad \neg i$$

In a sub-proof, start with the assumption α and reach a false conclusion, a contradiction, in particular. (In notation, α ... $\vdash F$.) Then you conclude that $\neg\alpha$ must be the case.

$$\frac{\alpha \quad \neg\alpha}{F} \qquad \neg e$$

Having both a proposition and its negation both means contradiction. Recall that a contradiction is a compound proposition that is logically equivalent to F.

And from falsity anything follows:

$\dfrac{F}{\alpha}$	Fe

Finally, double negations cancel out:

$\dfrac{\neg\neg\alpha}{\alpha}$	$\neg\neg e$

This completes our account of essential rules.

One important consideration is that for any rule

$$\frac{\alpha_1 \ldots \alpha_n}{\beta}$$

$\alpha_1 \wedge \ldots \wedge \alpha_n \to \beta$ must be a tautology.

Exercise: Formulate the rules $\leftrightarrow i$ and $\leftrightarrow e$.

There are several derived rules that come handy in many proofs. But our immediate next task is to take a closer look into proofs.