Natural Deduction

Suppose $\phi_1, \ldots, \phi_n$ and ϕ contain N different propositional constants.

To determine whether $\phi_1, \ldots, \phi_n \models \phi$ using the definition of validity, we need to construct a truth table, which has 2^N rows.

As N grows, this quickly becomes impractical.

Question: Can we do better?

Answer: One way we can try to do better is to model human reasoning ...

1

Rules for Conjunction

$$\frac{A \quad B}{A \wedge B} \quad \wedge i$$

$$\frac{A \wedge B}{A} \quad \wedge e_1$$

$$\frac{A \wedge B}{B} \quad \wedge e_2$$

$$(i = \text{Introduction}, \qquad e = \text{Elimination})$$

An Elimination Rule for Implication

$$\frac{A \quad A \longrightarrow B}{B} \quad \longrightarrow e$$

Proofs

A *proof* starts from some premises, and generates new conclusions by repeatedly applying some rules to premises or derived conclusions, until some desired conlusion is reached. We present proofs by listing the sequence of formulae generated down the page, with numbers on the left and proof step justifications on the right.

Example: The door has been disturbed. If the door is disturbed then the alarm will ring. If the alarm rings then the police will come. Therefore, the police will come.

$1:$	D	Premise
$2:$	$D \longrightarrow A$	Premise
$3:$	$A \longrightarrow P$	Premise
$4:$	A	$1, 2, \ \longrightarrow e$
$5:$	P	$3, 4, \ \longrightarrow e$

The rules presented above (and below) are really *schemas*, representing all the instances obtained by substituting a formula for each $A, B, \ldots$. E.g some of the instances of

$$\frac{A \quad A \longrightarrow B}{B} \quad \longrightarrow e$$

are

$$\frac{P \quad P \longrightarrow (Q \vee R)}{Q \vee R} \quad \longrightarrow e$$

$$\frac{(P \longrightarrow Q) \quad (P \longrightarrow Q) \longrightarrow (Q \vee R)}{Q \vee R} \quad \longrightarrow e$$

3

How to figure out whether a formula matches a pattern A *op* B?

In general, this can be done by figuring out the operator at the root of the parse tree of the formula.

Example: $A \longrightarrow B$ matches

- $P \longrightarrow Q$
- $(P \vee Q) \longrightarrow (R \longrightarrow P)$
- $(P \longrightarrow (R \vee S)) \longrightarrow ((P \vee Q) \longrightarrow (R \longrightarrow P))$

but not

- $P \wedge Q$
- $(P \longrightarrow Q) \vee R$
- $\neg(R \longrightarrow S)$

Example: John is coming to the party. If John comes then Yoko will come also. If John and Yoko both come then Paul will not come. Thus, Paul will not come.

Translating to logic, and using the proof rules:

$1:$	J	Premise
$2:$	$J \longrightarrow Y$	Premise
$3:$	$(J \wedge Y) \longrightarrow \neg P$	Premise
$4:$	Y	$1, 2, \ \longrightarrow e$
$5:$	$J \wedge Y$	$1, 4, \ \wedge i$
$6:$	$\neg P$	$3, 5, \ \longrightarrow e$

(Here step 6 uses a complex instance of $\longrightarrow e$.)

4

$\longrightarrow$-Introduction

The rule for introducing $\longrightarrow$ is defined using the notion of proof. If from Assumption A we can construct a proof of B then we may conclude $A \longrightarrow B$.

$$\frac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \longrightarrow B} \quad \longrightarrow i$$

Since the precondition for applying this rule is a *proof* instead of just a formula, we have to extend the notion of proof to allow *subproofs* ....

Example

If John comes to the party then so will Yoko. If Yoko is at the party then Paul will not come. Thus, if John comes to the party then Paul will not be there.

$$\begin{array}{llll} 1: & J \longrightarrow Y & & \text{Premise} \\ 2: & Y \longrightarrow \neg P & & \text{Premise} \\ & \boxed{\begin{array}{lll} 3: & J & \text{Assumption} \\ 4: & Y & 1,3, \ \longrightarrow e \\ 5: & \neg P & 2,4, \ \longrightarrow e \end{array}} & \\ 6: & J \longrightarrow \neg P & & 3-5, \ \longrightarrow i \end{array}$$

Nesting subproofs is allowed:

Example: If Phillipine Airlines flies to Hong Kong then if Qantas does not fly there, its profit will be reduced. Thus, if Qantas does not fly to Hong Kong, then if Phillipine Airlines flies to Hong Kong then the Qantas profit will be reduced.

$$
\begin{array}{lll}
1: & P \longrightarrow (Q \longrightarrow R) & \text{Premise} \\
\quad 2: & Q & \text{Assumption} \\
\quad\quad 3: & P & \text{Assumption} \\
\quad\quad 4: & Q \longrightarrow R \quad 1,3, \;\; \longrightarrow e \\
\quad\quad 5: & R \quad\quad\quad 2,4, \;\; \longrightarrow e \\
\quad 6: & P \longrightarrow R \quad\quad\quad\quad 3-5, \;\; \longrightarrow i \\
7: & Q \longrightarrow (P \longrightarrow R) \quad 2-6, \;\; \longrightarrow i
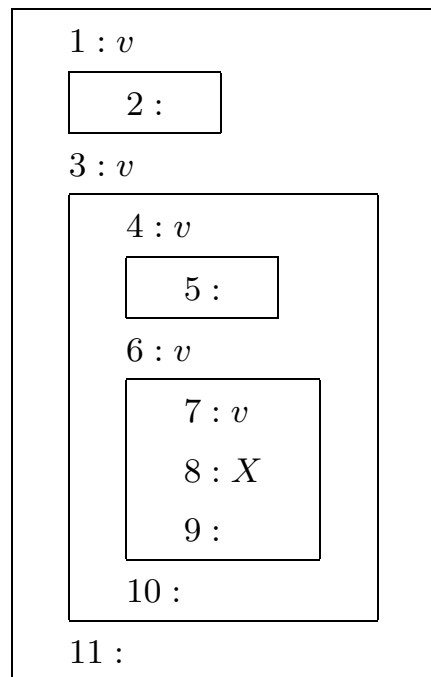\end{array}
$$

Importing formulae into subproofs

Note that when we are in a subproof, we are allowed to make use of, or *import*, premises, assumptions, and results that we have already been established in the outer proof. However, there are some restrictions.

Intuitively, the formulae inside a box are not known to be true, but *depend on the assumptions made in the box*. When we complete a subproof by closing its box, we *discharge* these assumptions. The result of this is that all formulae in the box are no longer available for use.

Thus, we may make use of formulae found by some sequences of moves that are either up inside a box, or up and out of a box, but we may never use a formula reached by a move *into* a box that has already been closed.

$v = $ "visible from X" :

```
┌─────────────────────────────────┐
│ 1 : v                           │
│  ┌──────────────┐               │
│  │   2 :        │               │
│  └──────────────┘               │
│ 3 : v                           │
│  ┌────────────────────────────┐ │
│  │  4 : v                     │ │
│  │   ┌──────────────┐         │ │
│  │   │    5 :       │         │ │
│  │   └──────────────┘         │ │
│  │  6 : v                     │ │
│  │   ┌──────────────┐         │ │
│  │   │    7 : v     │         │ │
│  │   │    8 : X     │         │ │
│  │   │    9 :       │         │ │
│  │   └──────────────┘         │ │
│  │  10 :                      │ │
│  └────────────────────────────┘ │
│ 11 :                            │
└─────────────────────────────────┘
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Rules for Equivalence

$$\frac{A \longrightarrow B \qquad B \longrightarrow A}{A \longleftrightarrow B} \qquad \longleftrightarrow i$$

$$\frac{A \longleftrightarrow B}{A \longrightarrow B} \qquad \longleftrightarrow e_1$$

$$\frac{A \longleftrightarrow B}{B \longrightarrow A} \qquad \longleftrightarrow e_2$$

7

Introduction Rules for Disjunction

$$\frac{A}{A \vee B} \quad \vee i_1$$

$$\frac{B}{A \vee B} \quad \vee i_2$$

Or-Elimination, or Proof by cases

$$
\begin{array}{ccc}
 & A & B \\
 & \vdots & \vdots \\
A \vee B & C & C \\
\hline
 & C &
\end{array}
\quad \vee e
$$

$$
\begin{array}{llll}
1: & P \longrightarrow Q & & \text{Premise} \\
2: & R \longrightarrow S & & \text{Premise} \\
3: & P \vee R & & \text{Premise} \\
\end{array}
$$

$$
\begin{array}{llll}
4: & P & & \text{Assumption} \\
5: & Q & & 1, 4, \ \longrightarrow e \\
6: & Q \vee S & & 5, \ \vee i_1 \\
\end{array}
$$

$$
\begin{array}{llll}
7: & R & & \text{Assumption} \\
8: & S & & 2, 7, \ \longrightarrow e \\
9: & Q \vee S & & 8, \ \vee i_2 \\
\end{array}
$$

$$
\begin{array}{llll}
10: & Q \vee S & & 3, 4 - 6, 7 - 9, \ \vee e \\
\end{array}
$$

Example

Requirements R: A burglar alarm system for a house is to operate as follows. The alarm should not sound unless the system has been armed or there is a fire. If the system has been armed and a door is disturbed, the alarm should ring. Irrespective of whether the system has been armed, the alarm should go off when there is a fire.

Conclusion C: If the door is disturbed then the alarm will ring if and only if the system is armed or there is a fire.

Expressing the requirements as formulas of propositional logic, with

- S = the alarm sounds = the alarm rings

- A = the system is armed

- D = a door is disturbed

- F = there is a fire

we get

Requirements:

1. $S \longrightarrow (A \vee F)$

2. $(A \wedge D) \longrightarrow S$

3. $F \longrightarrow S$

Conclusion: $D \longrightarrow (S \longleftrightarrow (A \vee F))$

$$
\begin{array}{lll}
1: & S \longrightarrow (A \vee F) & \text{Premise} \\
2: & (A \wedge D) \longrightarrow S & \text{Premise} \\
3: & F \longrightarrow S & \text{Premise} \\
\end{array}
$$

4:	D			Assumption

	5:	$A \vee F$		Assumption

		6:	A	Assumption
		7:	$A \wedge D$	$4,6, \ \wedge i$
		8:	S	$2,7 \longrightarrow e$

		9:	F	Assumption
		10:	S	$3,9, \ \longrightarrow e$

	11:	S		$5, 6-8, 9-10, \ \vee e$
12:	$(A \vee F) \longrightarrow S$			$3, 5-8, 9-10, \ \vee e$
13:	$S \longleftrightarrow (A \vee F)$			$1, 12, \ \longleftrightarrow i$

14:	$D \longrightarrow (S \longleftrightarrow (A \vee F))$			$4-13, \ \longrightarrow i$