Mathematical Induction

Tuesday, November 30, 2021 2:03 PM

A method to prove claims requiring universal stmts

$\forall n \geq n_0 \; P(n)$ n is a nonneg. integer

$$\underbrace{P(n_0)}_{\text{BASIS}} \wedge \underbrace{\forall k \geq n_0 \left(P(k) \rightarrow P(k+1) \right)}_{\text{INDUCTIVE STEP}} \vdash \forall n \geq n_0 \; P(n)$$

e.g., Prove by induction that $\forall n \geq 1 \; \sum_{i=1}^{n} (2i-1) = n^2$

n	
1	$1 = 1^2$
2	$1+3 = 2^2$
3	$1+3+5 = 9 = 3^2$
	$\vdots$

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1. Basis $n = 1$

$1 = 1^2$ ✓

LHS of eq^n RHS

$P(k)$ Assumed
⋮
$P(k+1)$ obtained

$P(k) \to P(k+1)$

$P(k)$

2. IND. STEP

IND. HYP. Assume that $\sum_{i=1}^{k} (2i-1) = k^2$

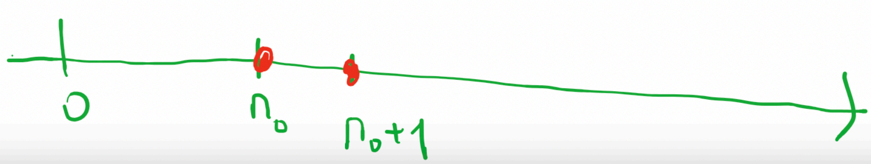
$P(k+1)$:

$$\sum_{i=1}^{k+1} (2i-1) = \underbrace{\sum_{i=1}^{k} (2i-1)}_{\text{ind. hyp}} + (2(k+1)-1)$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

$P(n_0)$ $P(n_0+1)$



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e.g., Let S be a set with n objects. $|P(S)| = 2^n$ ✓ $\Big\} Q(n)$

1. Basis

$n = 1$ $S = \{a\}$ $P(S) = \{\emptyset, \{a\}\}$

2 subsets $= 2^1$ subsets ✓

2. Ind. step

Ind. hyp. Assume that $Q(k)$

"a set with k elements has 2^k subsets"

$Q(k+1)$: consider a set A with $k+1$ elements

$A = \{a_1, a_2, \dots, a_k, a_{k+1}\}$

$A' = A - \{a_{k+1}\} \rightarrow$ IND. HYP $\Rightarrow 2^k$ subsets

Any subset X of A

1) $a_{k+1} \in X \longrightarrow 2^k$

2) $a_{k+1} \notin X \longrightarrow 2^k$

$\Big\}$ by ind. hyp.

$2^k + 2^k = 2 \cdot 2^k = \underline{\underline{2^{k+1}}}$

e.g., Any number composed of 3^n identical digits is divisible by 3^n — $R(n)$

$n=1$ 222, 777, 111 $\rightsquigarrow$ by 3^1

$n=2$ $9 \mid 777777777$

$\vdots$

1. Basis $n=1$ ddd $d \in \{0, 1, \ldots, 9\}$

$3 \mid ddd$ as $\underbrace{d+d+d}_{3 \cdot d}$ is divisible by 3 ✓

2. Ind. step

Assume that $R(K)$ is true.

$R(k+1)$ x is composed of 3^{k+1} identical digits

$x = dddd \ldots d$

Claim $x = y \cdot z$

$\uparrow$ a number with 3^k identical digits

$\overbrace{dd \ldots d}^{x} = \underbrace{ddd \ldots d}_{y} \times z$

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$$\underbrace{d \times 11\cdots1}_{x} = \underbrace{d \times 11\cdots1}_{y} \times z$$

$$\underbrace{111\cdots1}_{3^{k+1}\ 1's} = \underbrace{111\cdots1}_{3^k\ 1's} \times z$$

$$\begin{array}{r|l} \underbrace{10^{3^{k+1}-1}} + 10^{3^{k+1}-2} + \cdots + 1 & 10^{3^k-1} + 10^{3^k-2} + \cdots + 1 \\ \hline -\cancel{10^{3^{k+1}-1}} \mp \cancel{10^{3^{k+1}-2}} \mp \cdots \mp \cancel{10^{2\cdot 3^k}} & 10^{2\cdot 3^k} + 10^{3^k} + 1 \\ \hline 10^{2\cdot 3^k-1} + 10^{2\cdot 3^k-2} + \cdots + 1 & \\ -\cancel{10^{2\cdot 3^k-1}} \mp \cancel{10^{2\cdot 3^k-2}} \mp \cdots \mp \cancel{10^{3^k}} & \\ \hline 10^{3^k-1} + 10^{3^k-2} + \cdots + 1 & \\ 10^{3^k-1} + \cdots + 1 & \\ \hline 0 & \end{array}$$

$$z = 10^{2\cdot 3^k} + 10^{3^k} + 1$$
$$= 100\cdots0100\cdots1$$

$$\underbrace{x} = \underbrace{y} \cdot z$$

aim $3^{k+1} \mid x$

- $3^k \mid y$ due to IND. HYP
- $3 \mid z$

$\Big\}\ 3^{k+1} \mid x$

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e.g., Prove that $n \geq 10$ $\quad 2^n > n^3$

1. Basis.

$n = 10 \qquad 2^{10} = 1024 > 10^3 = 1000$ ✓

2. ind. step.

Assume $2^k > k^3$

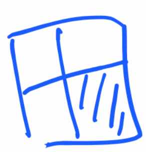
$$2^{k+1} = 2 \cdot 2^k > \left(1+\frac{1}{10}\right)^3 \cdot 2^k \geq \left(1+\frac{1}{k}\right)^3 \cdot 2^k > \left(1+\frac{1}{k}\right)^3 k^3 = (k+1)^3 \quad \blacksquare$$

IND. HYP.

e.g., $2^n \times 2^n$ chessboard with one square
$p(n)$ (removed can be tiled using L-shaped pieces

1. BASIS $p(1)$: 2×2 with one square removed →  is already an L-shaped piece ✓

$n = 1$

 ✓

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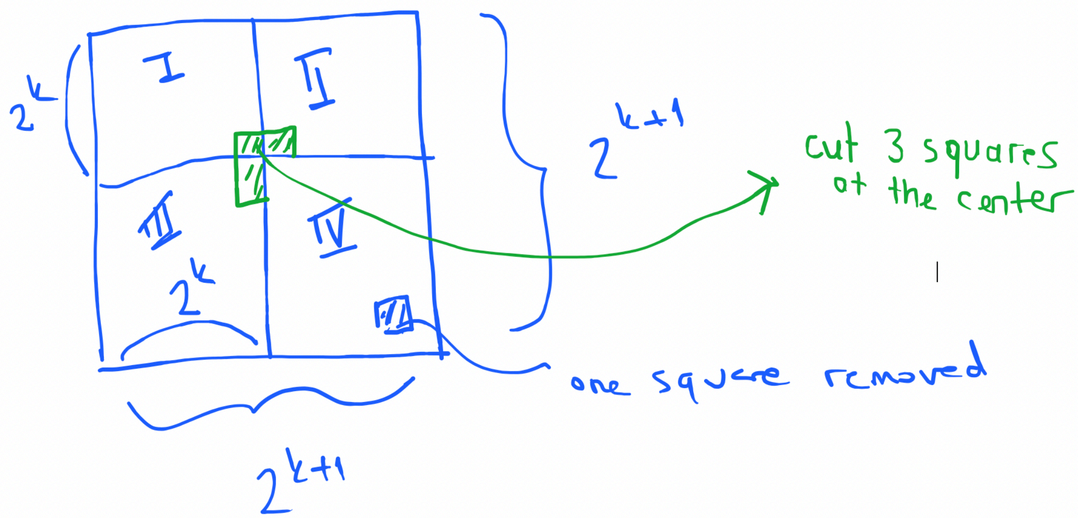
2. IND. STEP. Assume that $p(k)$ is true

$p(k+1)$



Region $\boxed{IV}$ $\rightarrow$ can be tiled (due to IND. HYP)

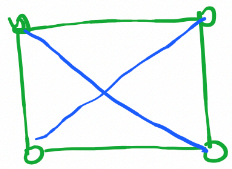
Regions I, II, III without these 3 squares can be tiled. (due to IND. HY)

These 3 squares can be tiled using a single L-shaped piece

$\blacksquare$

e.g.) Any convex polygon with n vertices has $\frac{n(n-3)}{2}$ diagonals. $P(n)$

BASIS. $n=4$



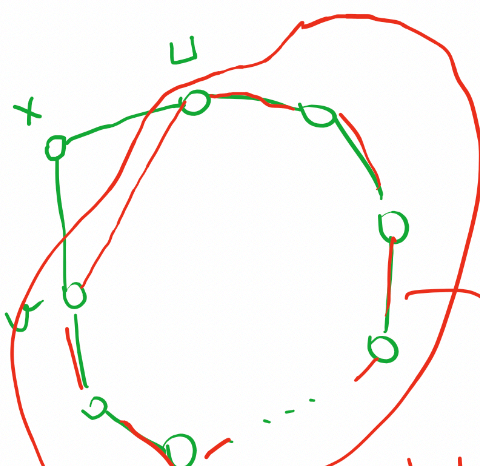
2 diagonals

$$\frac{4\cdot(4-3)}{2} = \frac{4\cdot 1}{2} = 2 \checkmark$$

IND. STEP

IND. HYP. Assume that $P(k)$ is true

$P(k+1)$: k+1 vertex convex polyg. $\rightarrow$



k+1 vertices

a convex polygon with k vertices

$$\frac{(k+1)((k+1)-3)}{2} \; \square$$

$$= \frac{(k+1)(k-2)}{2}$$

$$\# \text{ of diagonal} = \frac{k(k-3)}{2} + \overbrace{1 + k-2}^{k-1} = \frac{k^2-3k+2k-2}{2} = \frac{k^2-k-2}{2}$$

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Strong Induction (2nd principle of math. ind.)

$\forall n \geq n_0 \; p(n)$?

$$\underbrace{p(n_0)}_{\text{Basis}} \wedge \Big(p(n_0+1) \wedge \dots \wedge \underbrace{p(k) \rightarrow p(k+1)}_{\text{Ind. step}}\Big) \vdash \forall n \geq n_0 \; p(n)$$

$\forall k \geq n_0$

e.g., Show that any positive integer ≥ 2 is either prime or a product of primes

1. Basis $n=2$ 2 is a prime number ✓

2. Ind. step $\forall i \; 2 \leq i \leq k \; p(i)$ is true

(Assume that $p(2) \wedge p(3) \wedge p(4) \wedge \dots \wedge p(k)$)

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$k+1$

case I) $k+1$ is a prime number ✓

case II) $k+1$ is a composite number

$k+1 = p \cdot q \qquad 2 \leq p, q < k+1$

$\hookrightarrow p, q \leq k$

from IND. HYP claim holds for p & q.

We conclude that $p \cdot q = k+1$ is written as a product of primes.

$\boxtimes$

$Q(n)$

e.g., for $n \geq 8$, a postage of n cents is possible using unlimited supply of 5 cents & 3 cents

a b

Proof

1. BASIS $n = 8 \to$ a 3¢ stamp & a 5¢ stamp) 8 cents

$n = 9 \rightsquigarrow$ three 3¢ stamps

$n = 10 \rightsquigarrow$ two 5¢ stamps

2. IND. STEP. Assume that $Q(8) \wedge Q(9) \wedge \cdots \wedge Q(k)$ is true.

≥ 11 (postage $k+1$ = postage for k + a 3¢ stamp.

$\boxed{\checkmark}$