CENG223 Lecture Notes, v.3

NATURAL DEDUCTION FOR FIRST ORDER LOGIC

Notation: We use lowercase Greek letters like $\alpha, \beta, \gamma, \varphi, \chi$ to denote the well-formed formulas of FOL, formulas, in short, formed from predicates by using quantifiers and connectives. Formulas of FOL include those of Propositional Logic.

We may explicitly indicate the free variables that may occur in the formula. For example,

$\varphi = \varphi(x) = P(x) \wedge Q(x)$

$\phi = \beta(x, y) = \exists x P(x, y) \longrightarrow Q(x)$

$\chi = \chi(x, y) = \neg R(x)$

The domain of φ is the union of the domains of the predicates P and Q. Similarly, for the other examples.

We keep the ND rules of Propositional Logic. In addition, we have introduction and elimination rules for quantifiers.

Rules for Universal Quantification

Universal elimination rule should look quite familiar. From a universal statement, we can conclude the statement about an arbitrary object in the domain. Hence this rule is also known as universal instantiation.

$$\frac{\forall x \varphi(x)}{\varphi(a)} \quad \forall e$$

A remark on the mechanics of quantifier elimination: A bound variable, like x in $\forall x \varphi(x)$, is not available for substitution. When we remove the universal quantifier we obtain $\varphi(x)$. Now x becomes free, available for substitution by a constant (or a variable or a term involving constants, variables and function symbols). In this rule, it is replaced by a constant, named a, so we have $\varphi(a)$. A constant is a fixed member of the domain; it is denoted by a name (or a term involving names and function symbols only). A constant is not available for substitution.

In order to prove that any object (from the domain) possesses the property φ it suffices to show that $\varphi(a)$ holds for an object a where the name a is fresh. Hence, the $\forall$ introduction rule, requires us to introduce a fresh name (that is, a name that has not been used before within the current scope), and in a sub-proof obtain $\varphi(a)$ to reach the conclusion $\forall x\ \varphi(x)$. Note that the name a is not visible from any point outside the box enclosing the sub-proof.

$$\frac{\boxed{\begin{array}{ll} a & \\ & \vdots \\ & \varphi(a) \end{array}}}{\forall x\ \varphi(x)} \quad \forall i$$

The significance of the name a being fresh is to make sure that there are no assumptions about the object a; it is truly arbitrary. Thus, any conclusion we can draw about a could be drawn about any other object in the domain. This is how we justify the generalization. Hence the other name, universal generalization for the rule.

Examples and Remarks:

Let us see these two rules in action.

Prove that $\vdash\ \forall x\big(\varphi(x) \wedge \gamma(x)\big) \longrightarrow \forall x\, \varphi(x) \wedge \forall x\, \gamma(x)$

1. $\forall x\big(\varphi(x) \wedge \gamma(x)\big)$ assumption

   a fresh name

2. $\varphi(a) \wedge \gamma(a)$ 1, $\forall$e
3. $\varphi(a)$ 2, $\wedge e$
4. $\forall x\, \varphi(x)$ 2 $-$ 3, $\forall i$

   b fresh name

5. $\varphi(b) \wedge \gamma(b)$ 1, $\forall$e
6. $\gamma(b)$ 5, $\wedge e$
7. $\forall x\, \gamma(x)$ 5 $-$ 6, $\forall i$
8. $\forall x\, \varphi(x) \wedge \forall x\, \gamma(x)$ 4,7, $\wedge i$
9. $\forall x\big(\varphi(x) \wedge \gamma(x)\big) \longrightarrow \forall x\, \varphi(x) \wedge \forall x\, \gamma(x)$ 1 $-$ 8, $\longrightarrow i$

Note that we could have used the name a again in place of b, since a is not visible outside the box it is introduced.

Observe that with the proof within the outer box (lines 1 – 8) we have

$\forall x\big(\varphi(x) \wedge \gamma(x)\big) \vdash \forall x\, \varphi(x) \wedge \forall x\, \gamma(x)$

The only difference would be the status in line 1: it is now a premise rather than an assumption. A premise is given to you; it is visible from any line in your proof. An assumption is something you make, and discharge of when you are done with it; it is not effective from any line outside the box in which it is put forth.

Remark: In fact, this follows from the Deduction Theorem, which states

$\vdash \propto \longrightarrow \beta$ iff $\propto \vdash \beta$ for any FOL formulas $\propto$ and β.

Now consider the converse: $\vdash \forall x\, \varphi(x) \land \forall x\, \gamma(x) \longrightarrow \forall x\big(\varphi(x) \land \gamma(x)\big)$

1. $\forall x\varphi(x) \land \forall x\gamma(x)$ assumption

2. $\forall x\, \varphi(x)$ 1,$\land e$

3. $\forall x\, \gamma(x)$ 1,$\land e$

a fresh name

4. $\varphi(a)$ 2, $\forall e$

5. $\gamma(a)$ 3, $\forall e$

6. $\varphi(a) \land \gamma(a)$ 4,5,$\land i$

7. $\forall x\big(\varphi(x) \land \gamma(x)\big)$ $4-6, \forall i$

8. $\forall x\, \varphi(x) \land \forall x\, \gamma(x) \longrightarrow \forall x\big(\varphi(x) \land \gamma(x)\big)$ 1,7,$\longrightarrow i$

Making the same observation as in the preceding example, we have

$\forall x\, \varphi(x) \land \forall x\, \gamma(x) \vdash \forall x\big(\varphi(x) \land \gamma(x)\big)$

Putting them together we arrive at the logical equivalence between the two formulas:

$\forall x\big(\varphi(x) \land \gamma(x)\big) \equiv \forall x\, \varphi(x) \land \forall x\, \gamma(x)$

This exercise shows that universal quantification distributes over conjunction.

Question-1a: Would we have the same logical equivalence in case x does not occur as a free variable in φ or γ (or both)?

Question-2a: What about $\forall x\big(\varphi(x) \lor \gamma(x)\big)$ and $\forall x\, \varphi(x) \lor \forall x\, \gamma(x)$? Consider the two directions, $\vdash$ and $\dashv$, separately.

Rules for Existential Quantification

In order to prove that an object with a property φ exists it suffices to show that $\varphi(a)$ holds for some object a. The existential introduction rule formalizes this idea:

$$\frac{\varphi(a)}{\exists x\ \varphi(x)}\ \exists i$$

Existential elimination requires us to introduce a fresh name a and substitute a for x in $\varphi(x)$. This amounts to assuming the property φ for the object a. Then any conclusion χ that we can obtain based this assumption will be our conclusion. Note that χ must not contain the free variable x.

$$\frac{\exists x\varphi(x) \qquad \boxed{\begin{array}{ll} \mathrm{a} & \varphi(a) \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists e$$

Again, the significance of the name a being fresh is to make sure that there are no assumptions about the object a. After all , $\exists x\varphi(x)$ simply asserts the existence of some object, say a, from the domain, with property φ, not an object with extra properties.

Examples and Remarks:

Let us see these last two rules in action.

Show that $\exists x(\varphi(x) \vee \gamma(x)) \vdash \exists x\, \varphi(x) \vee \exists x\, \gamma(x)$

1. $\exists x(\varphi(x) \vee \gamma(x))$ premise

a fresh name

2. $\varphi(a) \vee \gamma(a)$ assumption, 1

3. $\varphi(a)$ assumption, 2

4. $\exists x\, \varphi(x)$ 3, $\exists i$

5. $\exists x\, \varphi(x) \vee \exists x\, \gamma(x)$ 4, $\vee i$

6. $\gamma(a)$ assumption, 2

7. $\exists x\, \gamma(x)$ 6, $\exists i$

8. $\exists x\, \varphi(x) \vee \exists x\, \gamma(x)$ 7, $\vee i$

9. $\exists x\, \varphi(x) \vee \exists x\, \gamma(x)$ 2, 3 − 5, 6 − 8, $\vee e$

10. $\exists x\, \varphi(x) \vee \exists x\, \gamma(x)$ 1, 2 − 9, $\exists e$

Conversely, show that $\exists x\, \varphi(x) \vee \exists x\, \gamma(x) \vdash \exists x\big(\varphi(x) \vee \gamma(x)\big)$.

$1.\ \exists x\, \varphi(x) \vee \exists x\, \gamma(x)$ premise

$2.\ \exists x\, \varphi(x)$ assumption, 1

a fresh name

$3.\ \varphi(a)$ assumption, 2

$4.\ \varphi(a) \vee \gamma(a)$ 3,$\vee i$

$5.\ \exists x\big(\varphi(x) \vee \gamma(x)\big)$ 4, $\exists i$

$6.\ \exists x\big(\varphi(x) \vee \gamma(x)\big)$ 2,3 – 5, $\exists e$

$7.\ \exists x\, \gamma(x)$ assumption, 1

b fresh name

$8.\ \gamma(b)$ assumption, 7

$9.\ \varphi(b) \vee \gamma(b)$ 8,$\vee i$

$10.\ \exists x\big(\varphi(x) \vee \gamma(x)\big)$ 9, $\exists i$

$11.\ \exists x\big(\varphi(x) \vee \gamma(x)\big)$ 7,8 – 10, $\exists e$

$12.\ \exists x\big(\varphi(x) \vee \gamma(x)\big)$ 1,2 – 6,7 – 11,$\vee e$

Combining these two results, we have $\exists x\big(\varphi(x) \vee \gamma(x)\big) \equiv \exists x\, \varphi(x) \vee \exists x\, \gamma(x)$

In other words, the existential quantifier distributes over disjunction.

Question-1b: Would we have the same logical equivalence in case x does not occur as a free variable in φ or γ (or both)?

Question-2b: What about $\exists x\big(\varphi(x) \wedge \gamma(x)\big)$ and $\exists x\, \varphi(x) \wedge \exists x\, \gamma(x)$? Consider the two directions, $\vdash$ and $\dashv$, separately.

We are ready to consider the interaction of the two quantifiers.

Show $\neg\forall x\beta(x) \equiv \exists x\neg\beta(x)$. Again, we need to establish both directions.

Firstly, show $\neg\forall x\beta(x) \vdash \exists x\neg\beta(x)$

$1. \neg\forall x\beta(x)$ premise

$2. \neg\exists x\neg\beta(x)$ assumption

a fresh name

$3. \neg\beta(a)$ assumption

$4. \exists x\neg\beta(x)$ 3, $\exists$i

$5. F$ 2,4, Fi

$6. \beta(a)$ 3-5,PbC

$7. \forall x\beta(x)$ $3-6, \forall i$

$8. F$ $1,7, \neg e$

$9. \exists x\neg\beta(x)$ $2-8$, PbC

In the above proof, we used a derived deduction rule, designated PbC (Proof by Contradiction, or, its Latin name, *reductio ad absurdum*). The method is to assert the negation of what you want to prove and arrive at contradiction (falsity).

$$\frac{\boxed{\begin{array}{c}\neg\alpha \\ \cdot \\ \cdot \\ \cdot \\ F\end{array}}}{\alpha}\ \text{PbC}$$

Its derivation from basic rules is straightforward:

$$\boxed{\begin{array}{l}1.\,\neg\alpha \\ \ldots \\ k.\,F\end{array}}$$

$k+1.\,\neg\neg\alpha \quad 1-k, \neg i$

$k+2.\,\alpha \quad k+1, \neg\neg e$

Secondly, show $\exists x \neg \beta(x) \vdash \neg \forall x \beta(x)$.

1. $\exists x \neg \beta(x)$ premise

2. $\forall x \beta(x)$ assumption

a fresh name

3. $\neg \beta(a)$ assumption, 1

4. $\beta(a)$ 2, $\forall e$

5. F 3,4, Fi

6. F 1,3 − 5, $\exists e$

7. $\neg \forall x \beta(x)$ 2 − 6, $\neg i$

Observe the handling of name a in the above proof. In line 3, it is introduced as a fresh name (to instantiate the existential). When it is being used in line 4 (to instantiate the universal), it is not fresh. After line 5 (having reached a contradiction), the scope of name a closes.

Exercise: $\neg \exists x \beta(x) \equiv \forall x \neg \beta(x)$