Languages that are and are not context free

CENG 280

Course outline

- Preliminaries: Alphabets and languages
- Regular languages = union, concatenation, Kleene star, intersection and complementation.
- Context-free languages
  - Context-free grammars
  - Parse trees
  - Push-down automaton
  - Push-down automaton - context-free languages
  - Languages that are and that are not context-free, Pumping lemma
- Turing-machines

CFL Closure Properties

Theorem

The context-free languages are closed under union, concatenation, and Kleene-star.

$G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$ with disjoint sets of non-terminals, $(V_1 \setminus \Sigma_1 \cap V_2 \setminus \Sigma_2 = \emptyset)$,

$L_1, L_2 \qquad L_1 \cup L_2$

$L(G_1) \cup L(G_2)$ is CF.

$L(G_u) = L(G_1) \cup L(G_2)$

$L(G_u) = L(G_1) \cup L(G_2)$

$S_u \rightarrow S_1 \mid S_2$

$G_u = (V_u, \Sigma_u, R_u, S_u)$

$V_u = V_1 \cup V_2 \cup \{S_u\}$

$\Sigma_u = \Sigma_1 \cup \Sigma_2$

$R_u = R_1 \cup R_2 \cup \{(S_u, S_1), (S_u, S_2)\}$

$w \in L(G_u)$

$(R_1) \quad S_u \Rightarrow S_1 \Rightarrow \cdots \Rightarrow w$

$(R_2) \quad S_u \Rightarrow S_2 \Rightarrow \cdots \Rightarrow w_2 \in L(G_2)$

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$L = L(G_1)L(G_2) = \{ w \mid w = xy,\ x \in L(G_1), y \in L(G_2) \}$

$G,\quad L(G) = L$

$G = (V, \Sigma, R, S),\quad \Sigma = \Sigma_1 \cup \Sigma_2$

$S \to S_1 S_2 \quad V = \{S\} \cup V_1 \cup V_2$

$R = R_1 \cup R_2 \cup \{(S, S_1 S_2)\}$

$y \in L(G_2)$

$S \Rightarrow S_1 S_2 \Rightarrow \ldots \Rightarrow w \in \Sigma^*$

$x \in L(G_1) \quad\quad xy$

CFL Closure Properties

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L is a CFL, then its Kleene star is also CFL.

L^*

$\underline{\underline{L(G_1)^*}}$

$G = (V, \Sigma, R, S)$

$S \rightarrow e \mid SS_1$

$V = V_1 \cup \{S\}$

$\Sigma = \Sigma_1$,

$R = V_1 \cup \{(S, e), (S, SS_1)\}$

$S \Rightarrow SS_1 \Rightarrow SS_1S_1 \Rightarrow SS_1S_1S_1 \Rightarrow \underbrace{S_1}_{w_1}\underbrace{S_1}_{w_2}\underbrace{S_1}_{w_3}$

CFL Closure Properties

Theorem

The context-free languages are closed under union, concatenation, and Kleene-star.

The proof is constructive. $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$ with disjoint sets of non-terminals, $(V_1 \setminus \Sigma_1 \cap V_2 \setminus \Sigma_2 = \emptyset)$,

- G_U such that $L(G_U) = L(G_1) \cup L(G_2)$
- $G_U = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \to S_1, S \to S_2\}, S)$
- G_C such that $L(G_C) = L(G_1)L(G_2)$
- $G_C = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \to S_1 S_2\}, S)$
- G_K such that $L(G_K) = L(G_1)^{\star})$
- $G_K = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{S \to e, S \to SS_1\}, S)$

CFL Closure Properties

- Context-free languages are not closed under intersection or complementation.
- The complementation proof for regular languages requires a deterministic automaton.
- However, not all context free languages are accepted by deterministic push-down automaton.
- There is proof of the closure under intersection based on construction of product of two automata (Problem 2.3.3).
- This construction can be extended to push-down automata, but the product automaton has two stacks.
- However, the same idea words for FA and PDA (only one stack).

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Theorem

The intersection of a context-free language with a regular language is a context-free language.

CFL Closure Properties

$w \in L(M)$ iff $w \in L(M_1)$ and $w \in L(M_2)$

$L(M) = L(M_1) \cap L(M_2)$

Theorem

The intersection of a context-free language with a regular language is a context-free language.

Proof idea: $M_1 = (K_1, \Sigma, \Gamma_2, \Delta_2, s_1, F_1)$, $M_2 = (K_2, \Sigma, \delta, s_2, F_2)$ with $L = L(M_1)$ and $R = L(M_2)$. Define $M = (K, \Sigma, \Gamma, \Delta, s, F)$

- $K = K_1 \times K_2$,
- $\Gamma = \Gamma_1$,
- $s = (s_1, s_2)$, $F = F_1 \times F_2$
- $\Delta = \{(((q_1, q_2), a, \beta), ((p_1, \delta(q_2, a)), \gamma) \mid ((q_1, a, \beta), (p_1, \gamma)) \in \Delta_1, q_2 \in K_2\} \cup \{(((q_1, q_2), e, \beta), ((p_1, q_2)), \gamma) \mid ((q_1, e, \beta), (p_1, \gamma)) \in \Delta_1, q_2 \in K_2\}$

$(q_1, q_2) \xrightarrow[\beta/\gamma]{a} (p_1, p_2)$

q_1: PDA, q_2: FA

$p_2 = \delta(q_2, a)$

$(q_1, q_2) \xrightarrow[\beta/\gamma]{e} (p_1, q_2)$

Examples

Example

CFL or not: L consists of all strings with equal number of a's and b's, and no two consecutive b's is followed by another b.

L_1

L_2

$L = \underline{L_1} \cap \underline{L_2}$

L_1: $S \rightarrow aSbS \mid bSaS \mid e$ L_1, L_1 CF

L_2: $\mathcal{L}((a \cup b)^* bbb (a \cup b)^*) = \overline{L_2}$

L_2 is regular.

CFL $\cap$ Reg. CFL.

Examples

Example

CFL or not: L is CFL, R is regular, then $L \setminus R$. CFL

$$\underbrace{L}_{\text{CF}} \setminus R = L \cap \underbrace{\overline{R}}_{\text{regular}}$$

Example

CFL or not: L is CFL, R is regular, then $R \setminus L$. CF ?

$\overline{L} = \Sigma^* \setminus L$, → CF closed under complement X

$\{a\} \setminus L$ ~ $\{a\}$ regular / CF

~ $\{\}$