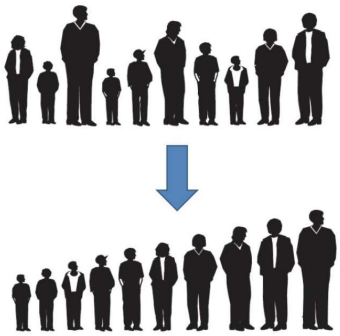
Sorting Algorithms

A Familiar Example



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Sorting

- ***Sorting*** is a process that organizes a collection of data into either ascending or descending order.
- As a human being, we can perform this task quickly
  - Spot the tallest person/shortest person right away
  - Simultaneously compare multiple persons
- A computer cannot see the big picture all at once
  - It's limited to compare two numbers at a time, swap or copy them, and move on to the next pair
  - However, computers can do these really fast.

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Sorting Algorithms

- There are many sorting algorithms, such as:
  - Selection Sort
  - Insertion Sort
  - Bubble Sort
  - Merge Sort
  - Quick Sort
  - Heapsort
  - Radix sort
  - Counting sort
  - …

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Selection Sort

Intuition:

- Round 1: Find the smallest element
- Round 2: Find the second smallest element
- Round 3: Find the third smallest element
- …

A list of *n* elements requires *n-1* passes over the array to completely rearrange the data.

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Selection Sort

- The array is viewed as two sublists, *sorted* and *unsorted*, which are divided by an imaginary wall.
- Find the smallest element from the unsorted sublist and swap it with the element at the beginning of the unsorted data.
- Move the imaginary wall between the two sublists one element ahead.
- Each time we move one element from the unsorted sublist to the sorted sublist, we say that we have completed a sort pass.
- A list of *n* elements requires *n-1* passes to completely rearrange the data.

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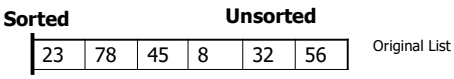
Selection Sort



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Selection Sort

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Selection Sort Algorithm

```
template <class Item>
void selectionSort( Item a[], int n)
{
  for (int i = 0; i < n-1; i++) {
    int min = i;
    // find the index of the next minimum
    for (int j = i+1; j < n; j++)
       if (a[j] < a[min]) min = j;
    // swap a[i] and a[min]
    Item tmp = a[i];
    a[i] = a[min];
    a[min] = tmp;
}
```

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Selection Sort

- Number of swaps?

O(N) in all cases (best case, worst case, average case)

- Number of comparisons?

$N(N-1)/2 = O(N^2)$

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Selection Sort – Analysis

- The inner for loop executes the size of the unsorted part minus 1 (from 1 to n-1), and in each iteration we make one key comparison.
  ➔ # of key comparisons = (n-1)+(n-2)+…+1= n*(n-1)/2
  ➔ **So, Selection sort is $O(n^2)$**

- The best case, the worst case, and the average case of the selection sort algorithm are same.
  ➔ all of them are **$O(n^2)$**

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Selection Sort – Analysis

- The best case, the worst case, and the average case of the selection sort algorithm are all are **$O(n^2)$**
  - This means that the behavior of the selection sort algorithm does not depend on the initial organization of data.
  - Since $O(n^2)$ grows so rapidly, the selection sort algorithm is appropriate only for small n.
  - Although the selection sort algorithm requires $O(n^2)$ key comparisons, it only requires O(n) swaps.
  - A selection sort could be a good choice if data moves are costly but key comparisons are not costly (short keys, long records).

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Comparison of *N*, *logN* and N^2

N	O(LogN)	$O(N^2)$
16	4	256
64	6	4K
256	8	64K
1,024	10	1M
16,384	14	256M
131,072	17	16G
262,144	18	6.87E+10
524,288	19	2.74E+11
1,048,576	20	1.09E+12
1,073,741,824	30	1.15E+18

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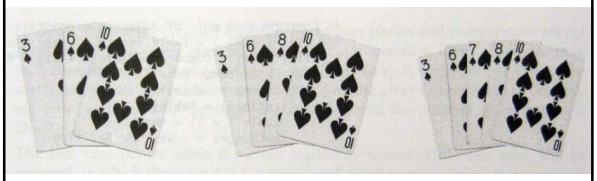
Insertion Sort

How do you sort a set of poker cards?



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Insertion Sort

- Insertion sort is a simple sorting algorithm that is appropriate for small inputs.
  - Most common sorting technique used by card players.
- The array is viewed as two parts: sorted and unsorted.
- In each pass, the first element of the unsorted part is picked up, transferred to the sorted sublist, and inserted at the appropriate place.
- A list of *n* elements will take at most *n-1* passes to sort the data.

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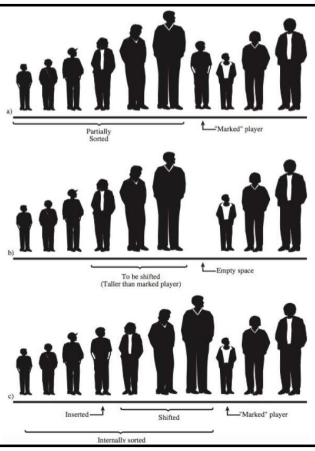

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Insertion Sort

Sorted **Unsorted**

23	78	45	8	32	56	Original List

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Insertion Sort

Sorted **Unsorted**

23	78	45	8	32	56	Original List
23	78	45	8	32	56	After pass 1
23	45	78	8	32	56	After pass 2
8	23	45	78	32	56	After pass 3
8	23	32	45	78	56	After pass 4
8	23	32	45	56	78	After pass 5

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Insertion Sort Algorithm

```
template <class Item>
void insertionSort(Item a[], int n)
{
   for (int i = 1; i < n; i++)
   {
      Item tmp = a[i];

      for (int j=i; j>0 && tmp < a[j-1]; j--)
         a[j] = a[j-1];
      a[j] = tmp;
   }
}
```

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Insertion Sort – Analysis

- Running time depends on not only the size of the array but also the contents of the array.
- ***Best-case:*** ➔ **O(n)**
  - Array is already sorted in ascending order.
  - Inner loop body will not be executed.
  - The number of moves: 2*(n-1) ➔ O(n)
  - The number of comparisons: (n-1) ➔ O(n)
- ***Worst-case:*** ➔ **O(n^2)**
  - Array is in reverse order:
  - Inner loop is executed i-1 times, for i = 1,2, 3,…, n-1
  - The number of moves: 2*(n-1)+(1+2+...+n-1)= 2*(n-1)+ n*(n-1)/2 ➔ O(n^2)
  - The number of key comparisons: (1+2+...+n-1)= n*(n-1)/2 ➔ O(n^2)
- Average-case: ➔ **O(n^2)**
  - We have to look at all possible initial data organizations.
- **So, Insertion Sort is O(n^2)**

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Bubble Sort

Intuition

- Round 1: Find the largest element
- Round 2: Find the largest smallest element
- Round 3: Find the largest smallest element
- …

Bubble sort finds the largest element in each round by repeatedly comparing every two adjacent numbers, and swapping them if the one on the left is larger.

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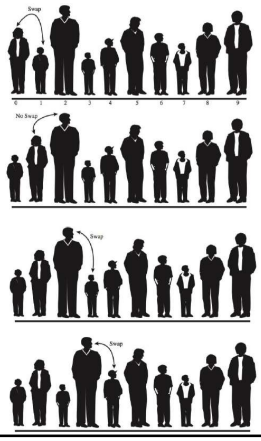

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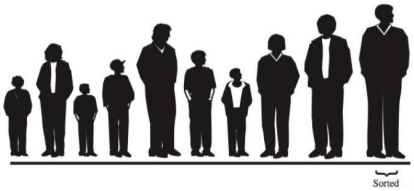
Bubble Sort

- After one pass, we find the largest number in that round



- It is like the 'biggest' bubble floats to the top of the surface, hence the name 'bubble sort'

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Bubble Sort

- The array is viewed as two sublists: sorted and unsorted.
- The largest element is bubbled from the unsorted list and moved to the sorted sublist.
- After that, the wall moves one element back, increasing the number of sorted elements and decreasing the number of unsorted ones.
- Each time an element moves from the unsorted part to the sorted part one sort pass is completed.
- Given a list of n elements, bubble sort requires up to n-1 passes over the array to sort the data.

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Bubble Sort

23	78	45	8	32	56

Original List

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Bubble Sort

23	78	45	8	32	56	Original List
23	45	8	32	56	78	After pass 1
23	8	32	45	56	78	After pass 2
8	23	32	45	78	56	After pass 3
8	23	32	45	56	78	After pass 4

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Bubble Sort Algorithm

```
template <class Item>
void bubleSort(Item a[], int n)
{
   bool swapped = true;

   while(swapped){
      swapped = false;
      for (int i=1; i<n; i++)
         if (a[i-1] > a[i]){
            Item temp = a[i];
            a[i] = a[i-1];
            a[i-1] = temp;
            swapped = true;     // signal exchange
         }
   }
}
```

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Bubble Sort – Analysis

- ***Best-case:*** ➔ **O(n)**
  - Array is already sorted in ascending order.
  - The number of moves: 0 ➔ O(1)
  - The number of comparisons: (n-1) ➔ O(n)
- ***Worst-case:*** ➔ **$O(n^2)$**
  - Array is in reverse order:
  - Outer loop is executed n-1 times,
  - The number of moves: 3*(1+2+...+n-1) = 3 * n*(n-1)/2 ➔ $O(n^2)$
  - The number of comparisons: n(n-1) ➔ $O(n^2)$
- Average-case: ➔ **$O(n^2)$**
  - We have to look at all possible initial data organizations.
- **So, Bubble Sort is $O(n^2)$**

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Recap: $O(N^2)$ Sorting Algorithms

- **Bubble sort** uses repeated comparisons and swaps to find the biggest element in each pass, and positions it toward the end of the array. (https://www.youtube.com/watch?v=lyZQPjUT5B4)
- **Selection sort** reduces the number of swaps by only peforming one swap at the end of each pass. (https://www.youtube.com/watch?v=Ns4TPTC8whw)
- **Insertion Sort** eliminates swaps and replaces them with copies, which are 3 times faster (https://www.youtube.com/watch?v=ROalU379l3U)
- They are all quadratic cost: **$O(N^2)$** in the worst and average cases.

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Merge Sort

- Merge sort algorithm is a divide-and-conquer sorting algorithm.
- It is a recursive algorithm.
- Cost is O(NlogN), much faster than simple sorting algorithms.
- Requires additional memory space
  - A temporary array as large as the input array
  - So it is not an in-place sorting algorithm

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Merging Two Sorted Arrays

- This is the key step in Merge Sort.
- Assume two subsets of the array (Left and Right) are already sorted
- Merge them into array `temp` such that `temp` contains all elements from Left and Right, and remains sorted.
- Note the two subsets may have different sizes. In fact one of them may be empty! Must correctly handle all cases!
- Example:

A: 23 47 81 95

B: 7 14 39 55 62 74

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A: 23 47 81 95

B: 7 14 39 55 62 74

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Merging Two Sorted Arrays

```
const int MAX_SIZE = maximum-number-of-items-in-array;
void merge_TwoArrays(T A[], int sizeA,
                     T B[], int sizeB, T C[]) {
   int ai=0, bi=0, ci=0;   // separate indices for arrays
   while (ai < sizeA && bi < sizeB){
      if (A[ai] < B[bi])
         C[ci++] = A[ai++];
      else
         C[ci++] = B[bi++];
   }
   while (ai < sizeA) C[ci++] = A[ai++];
   while (bi < sizeB) C[ci++] = B[bi++];
}
```

In the actual MergeSort, arrays A and B will be subarrays of the original array. So they will be from the same array, just at different starting locations.

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Merging Two Sorted Arrays

- Is it possible that both Left and Right have remaining elements after the first while loop?
- What happens if Left (A) is empty to begin with?
- How many copy (assignment) instructions?

  Size ofLeft (A) + size of Right (B)

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Merge Sort

- Once we know how to do the merge, Merge Sort is quite simple:
  - **Divide** the array into two halves
  - Sort each half (**Conquer**). How? Recursion!
  - Call merge() to merge two halves.
- What's the base case of the recursion?
  - When there is only 1 element left to sort, it's trivially sorted, so return immediately.

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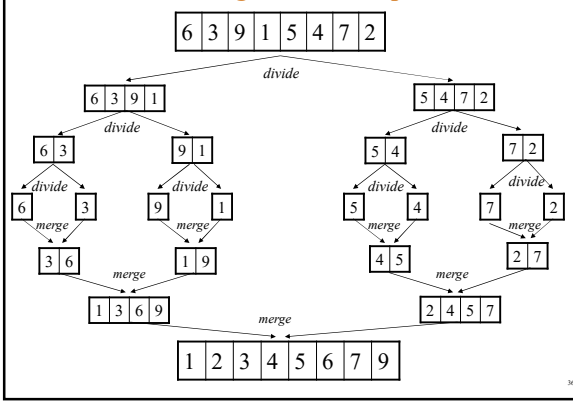
Mergesort - Example



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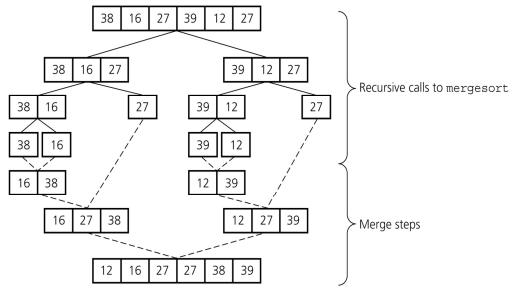
Example



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Mergesort

```
void mergesort(T theArray[], int first, int last) {
   if (first < last) {
      int mid = (first + last)/2;        // index of midpoint
      mergesort(theArray, first, mid);
      mergesort(theArray, mid+1, last);

      // merge the two halves
      merge(theArray, first, mid, last);
   }
}  // end mergesort
```

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Merge

```
void merge(T theArray[], int first, int mid, int last){
   T tempArray[MAX_SIZE];  // temporary array
   int first1 = first;     // beginning of first subarray
   int last1 = mid;        // end of first subarray
   int first2 = mid + 1;   // beginning of second subarray
   int last2 = last;       // end of second subarray
   int index = first1; // next available location in tempArray
   while (first1 <= last1) && (first2 <= last2) {
      if (theArray[first1] < theArray[first2]) {
         tempArray[index++] = theArray[first1++];
      else
         tempArray[index++] = theArray[first2++];
   }
   while (first1 <= last1) tempArray[index++]= theArray[first1++];
   while (first2 <= last2) tempArray[index++] = theArray[first2++];
   // copy the result back into the original array
   for (index = first; index <= last; ++index)
      theArray[index] = tempArray[index];
}  // end merge
```

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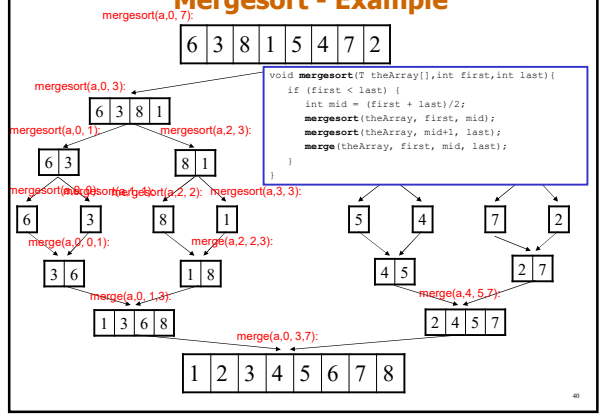
Mergesort - Example



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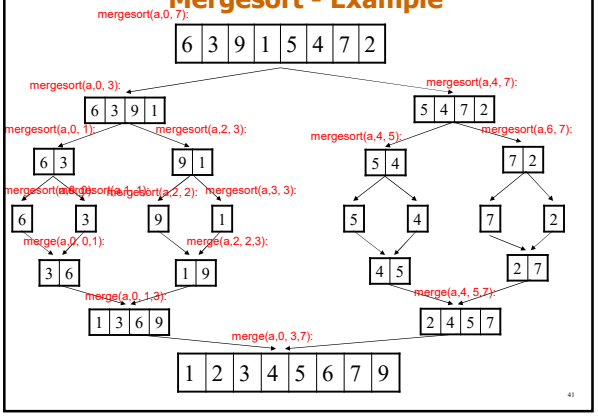
Mergesort - Example



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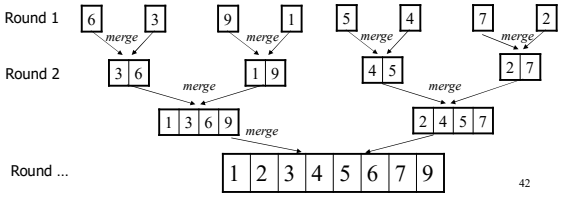
Merge Sort Cost Analysis

- The input array has N elements (just assume N is a power of 2). What is the cost of Merge Sort?
  - How much work (i.e. comparisons + copies) is involved in each round, in terms of N?
  - How many rounds are there?



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Merge Sort Cost Analysis

- The input array has N elements (just assume N is a power of 2). What is the cost of Merge Sort?
  - How much work (i.e. comparisons + copies) is involved in each round, in terms of N?
    - **O(N)**
  - How many rounds are there?
    - **$\log_2$ N**
- Therefore the total cost of Merge Sort is **O(N log N)**

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Mergesort - Analysis

The complexity of mergesort can be defined using the following recurrence equation:

$T(n) = 2T(n/2) + n$

Solving this equation:

$T(1) = 1$

$T(n) = 2\,T(n/2) + n$

1st subst: $= 2\,(\,2\,T(n/4) + n/2) + n$

$= 4\,T(n/4) + 2n$

2nd subst: $= 4\,(2\,T(n/8) + n/4) + 2n$

$= 8\,T(n/8) + 3n$

In general: $2^k\,T(n/2^k) + kn$

When $k = \log_2 n$:

$= 2^{\log n}\,T(n/2^{\log n}) + n \log n$

=> O(nlogn)

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Quicksort

- Like mergesort, quicksort is also based on the *divide-and-conquer* paradigm.
- But it uses this technique in a somewhat opposite manner, as all the hard work is done *before* the recursive calls.
- It works as follows:
  1. First, it partitions an array into two parts using a selected pivot element,
  2. Then, it sorts the parts independently,
  3. Finally, it combines the sorted subsequences by a simple concatenation.

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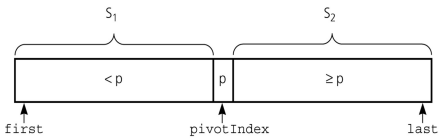
Partition

- Partitioning places the pivot in its correct position within the array.



- Arranging the array elements around the pivot p generates two smaller sorting problems.
  - sort the left section of the array, and sort the right section of the array.
  - when these two smaller sorting problems are solved recursively, our bigger sorting problem is solved.

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Partition – Choosing the pivot

- First, we have to select a pivot element among the elements of the given array before partitioning.
- Which array item should be selected as pivot?
- Somehow we have to select a pivot, and we hope that we will get a good partitioning.
- Possible options:
  - choose a pivot randomly.
  - choose the first or last element as a pivot (it may not give a good partitioning).
  - use different techniques to select the pivot.

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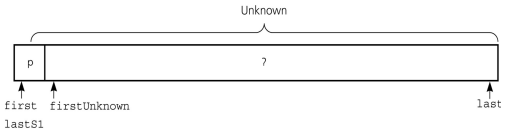
Partitioning

Initial state of the array



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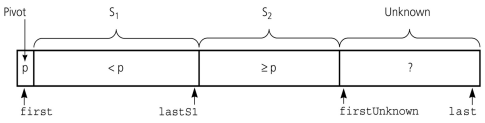
Partitioning (cont.)

Invariant for the partition algorithm



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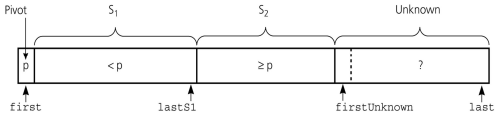
Partitioning (cont.)

Moving `theArray[firstUnknown]` ***into S_2 by incrementing*** `firstUnknown`.



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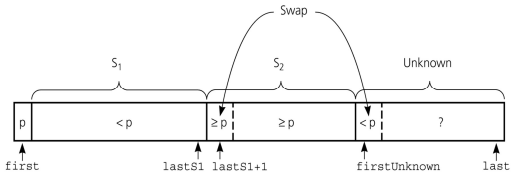
Partitioning (cont.)

Moving `theArray[firstUnknown]` ***into S_1 by swapping it with*** `theArray[lastS1+1]` ***and by incrementing both*** `lastS1` ***and*** `firstUnknown`.



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Partition

```
int partition(int[] array, int first, int last) {
   int pivot = array[first]; // chooses the first element
   int lastS1 = first;
   int firstUnknown = first +1;
   while (firstUnknown <= last) {
      if (array[firstUnknown] < pivot) {
         lastS1++;  //the end of 1st part is incremented
         swap(array, firstUnknown, lastS1); //elements swapped
      }
      firstUnknown++;
   }
   swap(array, first, lastS1);  //places pivot in proper loc.
   return lastS1;
}
```

Return the position of the pivot element after partition

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Partition Example

- Input : 36, 5, 48, 51, 10, 7, 22
- Pivot : 36

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Quicksort

```
void quicksort( int theArray[], int first, int last) {
   int pivotIndex;
   if (first < last) {
      pivotIndex = partition(theArray, first, last);
      quicksort(theArray, first, pivotIndex-1);
      quicksort(theArray, pivotIndex+1, last);
   }
}
```

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Quicksort – Analysis

Worst case: If we always select the smallest or largest element as the pivot, we'll not be able to divide the array into similar size partitions

- In that case, the complexity of the quicksort can be defined by:

  $T(n) = n + T(1) + T(n-1)$

- This gives $O(n^2)$ complexity

Best case: Partitions are equal sized:

$T(n) = n + 2T(n/2)$ (same as mergesort)

- This gives O(nlogn) complexity

Average case: quicksort has been proven to have O(nlogn) complexity

- It also does not need an extra array like mergesort
- Therefore, it is the most popular sorting algorithm

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Heapsort

- The **max heap** can be used to sort *N* items by
  - inserting every item into a max heap and
  - extracting every item by calling `deleteMax` *N* times, thus sorting the result.
- An algorithm based on this idea is *heapsort*.
- First, think about a trivial implementation of Heap Sort:

```
for(i=0; i<n; i++) maxheap.insert(array[i]);
for(i=n-1; i>=0; i--) array[i]=maxheap.remove();
```

- It is an **O(N logN)** worst-case sorting algorithm.
- Is there additional storage involved here? **Yes the heap!**

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Heap Sort

- To eliminate the additional storage, recall that a heap is stored in an array. Since we already have an input array, the same array can be used as a heap. How?

1. Convert the input array to a heap in place (heapify) (using buildheap algorithm that we discussed before)
2. Repeatedly remove the root elemet from the heap and move it towards the end of the array.
   - This is concentually similar to Selection sort, but each round costs only O(log N) due to heap operations.

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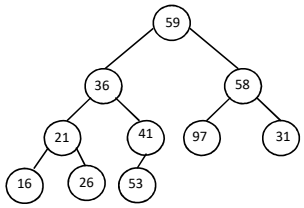
Heapify Example

- Input array and how it looks like as a heap – this is not a valid heap yet

[0]	59
[1]	36
[2]	58
[3]	21
[4]	41
[5]	97
[6]	31
[7]	16
[8]	26
[9]	53



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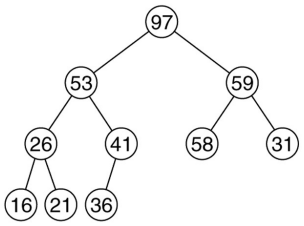
Heapsort

Max heap for the input sequence 59,36,58,21,41,97,31,16,26,53



97	53	59	26	41	58	31	16	21	36
0	1	2	3	4	5	6	7	8	9

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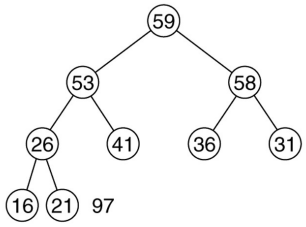
Heapsort

Heap after the first deleteMax operation



59	53	58	26	41	36	31	16	21	97
0	1	2	3	4	5	6	7	8	9

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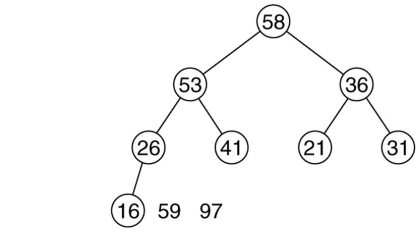
Heapsort

Heap after the second deleteMax operation



58	53	36	26	41	21	31	16	59	97
0	1	2	3	4	5	6	7	8	9

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Implementation

- In the implementation of heapsort, the root is stored in position 0.
- Thus there are some minor changes in the code that we have seen earlier:
  - Since we use max heap, the logic of comparisons is changed from > to <.
  - For a node in position i, the parent is in (i-1)/2, the left child is in 2i+1 and right child is next to left child.
  - Bubbling down needs the current heap size which is lowered by 1 at every deletion.

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The `heapsort` routine

```
template <class T>
void heapsort( vector<T> & a )
{
  for( int i = a.size()/2-1; i >= 0; i-- )  // heapify
    bubbleDown( a, i, a.size() );

  for( int j = a.size()-1; j > 0; j-- )  {
     swap( a[0], a[j] );              // deleteMax
     bubbleDown(a, 0, j-1);
  }
}
```

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Modified bubbleDown

```
template <class T>
void bubbleDown(vector<T> & arr, int hole, int theSize )
{
  int child;
  T tmp = arr[ hole ];

  for( ; hole * 2 + 1 <= theSize; hole = child ) {
    child = hole * 2 + 1;
    if( child != theSize && arr[child + 1] > arr[child])
       child++;
    if( arr[ child ] > tmp )
       arr[ hole ] = arr[ child ];
    else
       break;
  }
  arr[ hole ] = tmp;
}
```

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Analysis of Heapsort

- It is an *O(N log N)* algorithm.
  - First phase: Build heap O(N)
  - Second phase: N deleteMax operations: O(NlogN).
- Detailed analysis shows that, the average case for heapsort is poorer than quick sort.
  - Quicksort's worst case however is far worse.
- An average case analysis of heapsort is very complicated, but empirical studies show that there is little difference between the average and worst cases.
  - Heapsort usually takes about twice as long as quicksort.
  - Heapsort therefore should be regarded as something of an insurance policy:
  - On average, it is more costly, but it avoids the possibility of $O(N^2)$.

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O(N log N) Sorting Summary

- **Merge Sort**
  - A key step is to merge two already sorted sub-arrays.
  - Requires an additional storage
- **Quick Sort**
  - A key step is to partition around a pivot element
  - Sorts in place
  - Does not guarantee log-linear time in worst-case scenarios.
- **Heap Sort**
  - Leverages heap's efficient insert/remove operations
  - The same input array is split into a heap region and sorted region
  - Sorts in place

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Radix Sort

- Radix sort algorithm is different from other sorting algorithms that we have discussed.
  - It does not use key comparisons to sort an array.
- The radix sort treats each data item as a character string.
- Algorithm:
  - First group data items according to their rightmost character, and put these groups into order w.r.t. this rightmost character.
  - Then, combine these groups.
  - Repeat these grouping and combining operations for all other character positions in the data items from the rightmost to the leftmost character position.
  - At the end, the sort operation will be completed.

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Radix Sort – Example

[mom, dad, god, fat, bad, cat, mad, pat, bar, him] ← original list

[dad,god,bad,mad] [mom,him] [bar] - [fat,cat,pat] ← group strings by rightmost letter

[dad,god,bad,mad,mom,him,bar,fat,cat,pat] ← combine groups

[dad,bad,mad,bar,fat,cat,pat] [him] [god,mom] ← group strings by middle letter

[dad,bad,mad,bar,fat,cat,pat,him,god,mom] ← combine groups

[bad,bar] [cat] [dad] [fat] [god] [him] [mad,mom] [pat] ← group strings by first letter

[bad,bar,cat,dad,fat,god,him,mad,mom,par] ← combine groups

(SORTED)

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Radix Sort – Example

0123, 2154, 0222, 0004, 0283, 1560, 1061, 2150	Original integers
(1560, 2150) (1061) (0222) (0123, 0283) (2154, 0004)	Grouped by fourth digit
1560, 2150, 1061, 0222, 0123, 0283, 2154, 0004	Combined
(0004) (0222, 0123) (2150, 2154) (1560, 1061) (0283)	Grouped by third digit
0004, 0222, 0123, 2150, 2154, 1560, 1061, 0283	Combined
(0004, 1061) (0123, 2150, 2154) (0222, 0283) (1560)	Grouped by second digit
0004, 1061, 0123, 2150, 2154, 0222, 0283, 1560	Combined
(0004, 0123, 0222, 0283) (1061, 1560) (2150, 2154)	Grouped by first digit
0004, 0123, 0222, 0283, 1061, 1560, 2150, 2154	Combined (sorted)

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Radix Sort - Algorithm

```
// pseudocode
radixSort(theArray:ItemArray, n:integer,  d:integer)
// sort n d-digit integers in the array theArray
   for (j=d down to 1) {
       Initialize 10 groups to empty
       Initialize a counter for each group to 0
       for (i=0 through n-1) {
           k = jth digit of theArray[i]
           Place theArrar[i] at the end of group k
           Increase kth counter by 1
       }
       Replace the items  in theArray with all the items in group 0,
       followed by all the items in group 1, and so on.
   }
```

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Radix Sort -- Analysis

- The radix sort algorithm requires 2*n*d moves to sort n strings of d characters each.
  ➔ So, Radix Sort is **O(n)**
- Although the radix sort is O(n), it is not appropriate as a general-purpose sorting algorithm.
  - Its memory requirement is ***d * original size of data*** (because each group should be big enough to hold the original data collection.)
  - For example, we need 26 groups to sort strings of uppercase letters in English.
  - The radix sort is more appropriate for a linked list than an array. (we will not need the huge memory in this case)

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Counting sort

```
void countingSort(int arr[],int n, int RANGE){
   int count[RANGE]={0}, i, output[n];
   // store in count array the count of individual numbers
   for(i=0;i<n;i++)
      count[arr[i]]++;
   //change count[i] so that it contains the actual position
   //of numbers
   for(i=1; i < RANGE; i++)
      count[i]+=count[i-1];
   //build the output array
   for(i=0; i<n; i++){
      output[count[arr[i]]-1] = arr[i];
      count[arr[i]]--;
    }
    //copy the output array to array
    for(i=0;i<n;i++)
        arr[i]=output[i];
}
```

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Example

- 1, 4, 1, 2, 7, 5, 2

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Comparison of Sorting Algorithms

	Worst case	Average case
Selection sort	n^2	n^2
Bubble sort	n^2	n^2
Insertion sort	n^2	n^2
Mergesort	n * log n	n * log n
Quicksort	n^2	n * log n
Counting sort	n	n
Treesort	n^2	n * log n
Heapsort	n * log n	n * log n

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