Take Home Exam 2

Question 1

$(A \cup B) \setminus (A \cap B) = \{ x \mid x \in (A \cup B) \land x \notin (A \cap B) \}$ by definition of set difference

$= \{ ((x \in A) \lor (x \in B)) \land (x \notin (A \cap B)) \}$ by definition of union

$= \{ ((x \in A) \lor (x \in B)) \land \neg(x \in (A \cap B)) \}$ by definition of $\notin$

$= \{ ((x \in A) \lor (x \in B)) \land \neg((x \in A) \land (x \in B)) \}$ by definition of instersection

$= \{ ((x \in A) \lor (x \in B)) \land (\neg(x \in A) \lor \neg(x \in B)) \}$ by the first De Morgan's law

$= \{ ((x \in A) \lor (x \in B)) \land ((x \notin A) \lor (x \notin B)) \}$ by definition of $\notin$

$= \{ \underbrace{(((x \in A) \lor (x \in B)) \land (x \notin A))}_{1} \lor \underbrace{(((x \in A) \lor (x \in B)) \land (x \notin B))}_{2} \}$ by the Distributive law

1

$= (((x \in A) \land (x \notin A)) \lor ((x \in B) \land (x \notin A)))$ by the Distributive law

$= (F \lor ((x \in B) \land (x \notin A)))$ by the Negation law

$= (x \in B) \land (x \notin A)$ by the Identity law

2

$= (((x \in A) \land (x \notin B)) \lor ((x \in B) \land (x \notin B)))$ by the Distributive law

$= (((x \in A) \land (x \notin B)) \lor F)$ by the Negation law

$= (x \in A) \land (x \notin B)$ by the Identity Law

$= \{ ((x \in B) \land (x \notin A)) \lor ((x \in A) \land (x \notin B)) \}$ by union of 1 and 2

$= \{ (B \setminus A) \lor (A \setminus B) \}$ by definition of set difference

$= (B \setminus A) \cup (A \setminus B)$ by definition of union

$= (A \setminus B) \cup (B \setminus A)$ by the Commmutative law for union

Question 2

$\{f \mid f : f \subseteq \mathbb{N} \times \{0,1\}, f$ is a function$\} \setminus \{ f \mid f : \{0, 1\} \rightarrow \mathbb{N}, f$ is a function$\}$ is uncountable.

$\{f \mid f : f \subseteq \mathbb{N} \times \{0,1\}, f$ is a function$\}$

$f \subseteq \mathbb{N} \times \{0,1\}$

$1 \rightarrow \{0, a_1\}, \{1, a_2\}, \{2, a_3\}, \{3, a_4\}, \{4, a_5\}$ ...

$2 \rightarrow \{0, b_1\}, \{1, b_2\}, \{2, b_3\}, \{3, b_4\}, \{4, b_5\}$ ...

$3 \rightarrow \{0, c_1\}, \{1, c_2\}, \{2, c_3\}, \{3, c_4\}, \{4, c_5\}$ ...

... continues

x = $\{0, x_1\}, \{1, x_2\}, \{2, x_3\}, \{3, x_4\}, \{4, x_5\}$ ... (x $\in \mathbb{N} \times \{0,1\}$)

$x_1 \neq a_1$

$x_2 \neq b_2$

$x_3 \neq c_3$

… continues

By design x $\in \mathbb{N} \times \{0,1\}$ and it is missed by the enumaration.

So, there does not exist on enumeration counting each element in $\mathbb{N} \times \{0,1\}$.

Therefore, $\{f \mid f : f \subseteq \mathbb{N} \times \{0,1\}, f$ is a function$\}$ is an uncountably infinite.

$\{ f \mid f : \{0, 1\} \rightarrow \mathbb{N}, f$ is a function$\}$

$f : \{0, 1\} \rightarrow \mathbb{N}$

	0	1	2	3	…
0	(0,0)	(0,1)	(0,2)	(0,3)	…
1	(1,0)	(1,1)	(1,2)	(1,3)	…

We have a tuple (a,b) which is a subset of $\{0, 1\} \rightarrow \mathbb{N}$

$\underbrace{(0,0)}_{a+b=0}$ $\underbrace{(0,1), (1,0)}_{a+b=1}$ $\underbrace{(1,1), (0,2)}_{a+b=2}$ ….

We have an enumaration algorithm for visiting each element in the set of $\{0, 1\} \rightarrow \mathbb{N}$.

We cannot find a set which cannot be visited by using this method.

So, if we count each set exactly once, f will be a bijection.

Therefore, $\{ f \mid f : \{0, 1\} \rightarrow \mathbb{N}, f$ is a function$\}$ is countably infinite.

Assume A is uncountable and B is countable sets and A\B is countable set.

Since A\B is countable and B is countable , $(A\backslash B) \cup B$ is also countable.

$(A\backslash B) \cup B = \{x \mid x \in (A\backslash B) \vee x \in B\}$ by definition of union

$\{x \mid (x \in A \wedge x \notin B) \vee x \in B\}$ by definition of set difference

$\{x \mid (x \in A \vee x \in B) \wedge (x \notin B \vee x \in B)\}$ by the Distributive law

$\{x \mid (x \in A \vee x \in B) \wedge T\}$ by the Negation law

$\{x \mid (x \in A \vee x \in B)\}$ by the Identity law

$A \cup B$ by definition of union

Thus, $(A\backslash B) \cup B = A \cup B$

According to the $A \subseteq A \cup B$, if $A \cup B$ is countable , A should also be countable set.
There is a contradiction.

Therefore, A\B is uncountably infinite.

Question 3

$f(n) = 4^n + 5n^2. \log(n) \notin O(2^n)$ for $x>2$ and $n>2$

Assume $4^n + 5n^2. \log(n) \in O(2^n)$,

By definition of BIG-O notation $f(n)$ is $O(g(n))$, if there are constants C and k such that :

$|f(n)| \leq C . |g(n)|$, $n \geq k$

$|4^n + 5n^2. \log(n)| \leq C . |2^n|$

Since $5n^2. \log(n)$ is positive for all $n>2$, $|4^n + 5n^2. \log(n)| \geq |4n|$

$|4n| \leq |4^n + 5n^2. \log(n)| \leq C . |2^n|$

$|4^n| \leq C . |2^n|$, $n \geq k$

$4^n / 2^n \leq C$

$(4/2)^n \leq C$

$2^n \leq C$

$n. \log(2) \leq \log(C)$

$n \leq \log(C) / \log(2)$

'C' is a positive constant, but we need to find:
k such that, for any $n \geq k$ should satisfy $n \leq \log(C) / \log(2)$

It is not the case that $n \leq \log(C) / \log(2)$ for all $n \geq k$, because n can be arbitrarily large.

So, $4^n + 5n^2 \log(n) \notin O(2^n)$.

Question 4

$(2x - 1)^n - x^2 \equiv -x-1 \bmod (x-1)$

$(2x - 1)^n - x^2 +x +1 \equiv 0 \bmod (x-1)$

$(2x - 1)^n - x^2 +x +1 \equiv 0 \bmod (x-1)$

If $a \equiv b \bmod m$, then $a^k \equiv b^k \bmod m$ for any positive integer k

If $(2x-1) \equiv 1 \bmod (x-1)$

Then, $(2x-1)^n \equiv 1^n \bmod (x-1)$

So, $(2x-1)^n \equiv 1 \bmod (x-1)$

$(2x - 1)^n - (x^2 – x -1) \equiv 0 \bmod (x-1)$

$x^2 -x -1 \bmod (x-1) \equiv -1$

$(2x - 1)^n - (x^2 – x -1) \equiv 0 \bmod (x-1)$

$1 - (-1) \equiv 0 \bmod (x-1)$

$2 \equiv 0 \bmod (x-1)$

For given two positive integers x and n such that x > 2 and n > 2,

If $(x-1) \mid 2$

(x-1) can be ‘2’ or ‘-2’

As a result, x can be ‘3’ or ‘-1’.

If x>2, then x = 3.