# Grammars

CENG 280, 2019

# Grammar, unrestricted grammar, rewriting system

## Definition

A grammar is a quadruple $G = (V, \Sigma, R, S)$ where

- $V$ is an alphabet,
- $\Sigma \subset V$ is the set of **terminal** symbols, $V \setminus \Sigma$ is called the set of **non-terminal** symbols
- $S \in V \setminus \Sigma$ is the start symbol, and
- $R$ is a finite subset of $V^\star(V \setminus \Sigma)V^\star \times V^\star$ and it is the set of **rules**.

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- We write $u \to v$ if $(u, v) \in R$.
- We write $u \Rightarrow_G v$ if and only if for some $w_1, w_2 \in V^\star$ and for some rule $u' \to v' \in R$, $w_1 u' w_2 = w_1 v' w_2$.

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- A **derivation** is a sequence of the form $w_0 \Rightarrow_G w_1 \Rightarrow_G \ldots \Rightarrow_G w_n$.

### Example

$L = \{a^n b^n c^n \mid n \geq 1\}$, write $G = (V, \Sigma, R, S)$ that generates $L$, i.e., $L = L(G)$.

## Grammars

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(**if**) Given TM $M$, define $G$ that generates the language semi-decided by $M$. $G$ will simulate backward computations of $M$ (read from the book).

# Compute

**Definition**

Let $G = (V, \Sigma, R, S)$ be a grammar, and $f : \Sigma^\star \to \Sigma^\star$ be a function. $G$ **computes** $f$ if for all $w, v \in \Sigma^\star$,

$$SwS \Rightarrow_G^\star v \text{ if and only if } v = f(w)$$

A function is **grammatically computable** if and only if there is a grammar $G$ that computes it.

**Theorem**

*A function $f : \Sigma^\star \to \Sigma^\star$ is recursive if and only if it is grammatically computable.*

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$L = \{ww \mid w \in \{a,b\}^\star\}$, write $G = (V, \Sigma, R, S)$ that generates $L$, i.e., $L = L(G)$.

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$L = \{a^{3^n} \mid n \geq\}$, write $G = (V, \Sigma, R, S)$ that generates $L$, i.e., $L = L(G)$.