CENG 384 - Signals and Systems for Computer Engineers
20222
Written Assignment 3 Solutions

May 19, 2023

1.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \tag{1}$$

$$\int_{s=-\infty}^{t} x(s)ds = \int_{s=-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}s} ds \tag{2}$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{s=-\infty}^{t} e^{jk\frac{2\pi}{T}s} ds \tag{3}$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{jk\frac{2\pi}{T}}\right) e^{jk\frac{2\pi}{T}s}\Big|_{s=-\infty}^{t} \tag{4}$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{jk\frac{2\pi}{T}}\right) e^{jk\frac{2\pi}{T}t} \tag{5}$$

2. The properties of the Fourier Series can be easily used for this question:

   (a) $x(t)x(t) \xleftrightarrow{\text{FS}} \sum_{l=-\infty}^{\infty} a_l a_{k-l}$ (Multiplication property)

   (b) $x_e(t) = \frac{1}{2}[x(t) - x(-t)] \xleftrightarrow{\text{FS}} \frac{a_k}{2} + \frac{a_{-k}}{2}$ (Linearity and time reversal)

   (c) $x(t+t_0) + x(t-t_0) \xleftrightarrow{\text{FS}} a_k e^{jk\frac{2\pi}{T}t_0} + a_k e^{-jk\frac{2\pi}{T}t_0} = 2a_k cos\left(\frac{k2\pi t_0}{T}\right)$ (Time shift and linearity)

3.

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jkw_0t}dt \tag{6}$$

$$= \frac{1}{4}\left[\int_{-2}^{-1} x(t)e^{-jk\frac{\pi}{2}t}dt + \int_0^1 x(t)e^{-jk\frac{\pi}{2}t}dt\right] \tag{7}$$

$$= \frac{1}{4}\left[\int_{-2}^{-1} (-2)e^{-jk\frac{\pi}{2}t}dt + \int_0^1 2e^{-jk\frac{\pi}{2}t}dt\right] \tag{8}$$

$$= \frac{1}{4}\left[\frac{-2e^{-jk\frac{\pi}{2}t}}{jk\frac{\pi}{2}}\Big|_{-2}^{-1} + \frac{2e^{-jk\frac{\pi}{2}t}}{-jk\frac{\pi}{2}}\Big|_0^1\right] \tag{9}$$

$$= \frac{e^{jk\frac{\pi}{2}}}{jk\pi} - \frac{e^{jk\pi}}{jk\pi} - \frac{e^{-jk\frac{\pi}{2}}}{jk\pi} + \frac{1}{jk\pi} \tag{10}$$

Therefore,

$$x(t) = \sum_{k=-\infty}^{\infty}\left[\left(\frac{e^{jk\frac{\pi}{2}}}{jk\pi} - \frac{e^{jk\pi}}{jk\pi} - \frac{e^{-jk\frac{\pi}{2}}}{jk\pi} + \frac{1}{jk\pi}\right)e^{jk\frac{\pi}{2}t}\right] \tag{11}$$

4. (a)

$$x(t) = 1 + \frac{1}{j2}(e^{jw_0t} - e^{-jw_0t}) + (e^{jw_0t} - e^{-jw_0t}) + \frac{1}{2}(e^{j(2w_0t+\frac{\pi}{4})} + e^{-j(2w_0t+\frac{\pi}{4})}) \tag{12}$$

$$= 1 + (1 + \frac{1}{j2})e^{jw_0t} + (1 - \frac{1}{j2})e^{-jw_0t} + \frac{1}{2}e^{j\frac{\pi}{4}}e^{j2w_0t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2w_0t} \tag{13}$$

$$a_0 = 1 \qquad a_k = 0,\ |k| > 2 \tag{14}$$

$$a_1 = 1 + \frac{1}{j2} \qquad a_{-1} = 1 - \frac{1}{j2} \tag{15}$$

$$a_2 = \frac{1}{2}e^{j\frac{\pi}{4}} \qquad a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}} \tag{16}$$

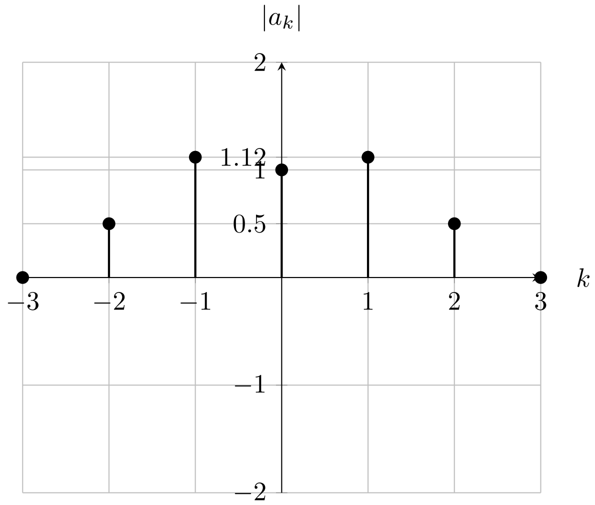


Figure 1: $|a_k|$ vs. k for $x(t)$

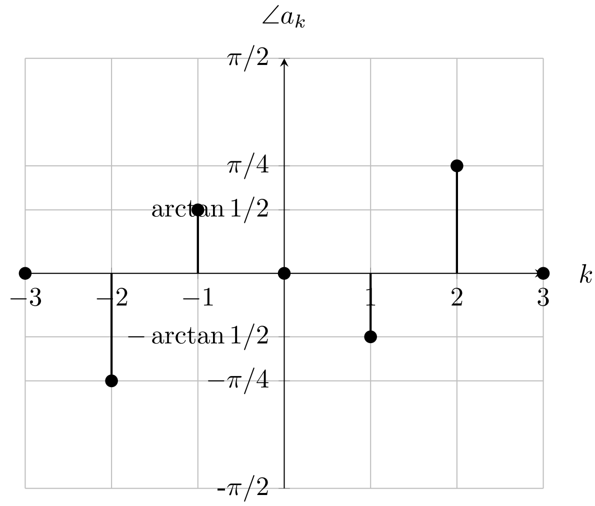


Figure 2: $\angle a_k$ vs. k for $x(t)$

(b) We know that the output of the system will be $H(jw)e^{jwt}$ when the output is e^{jwt} because e^{jwt} is the eigenfunction of the system. Substitute input and output into the differential equation. We obtain,

$$jwH(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt} \tag{17}$$

$$\implies H(jw) = \frac{1}{1+jw} \tag{18}$$

which is the eigenvalue of the system.

(c)

$$b_k = a_k H(jk2\pi) \tag{19}$$

$$b_0 = 1H(0) = 1 \tag{20}$$

$$b_1 = (1 - \frac{1}{2}j)H(j2\pi) = (1 - \frac{1}{2}j)(\frac{1}{1+j2\pi}) \tag{21}$$

$$b_{-1} = (1 + \frac{1}{2}j)H(j2\pi)H(-j2\pi) = (1 + \frac{1}{2}j)H(j2\pi)(\frac{1}{1-j2\pi}) \tag{22}$$

$$b_2 = \frac{1}{2}e^{j\frac{\pi}{4}}H(j4\pi) = \frac{\sqrt{2}}{4}(1+j)(\frac{1}{1+j4\pi}) \tag{23}$$

$$b_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}}H(j4\pi) = \frac{\sqrt{2}}{4}(1-j)(\frac{1}{1-j4\pi}) \tag{24}$$

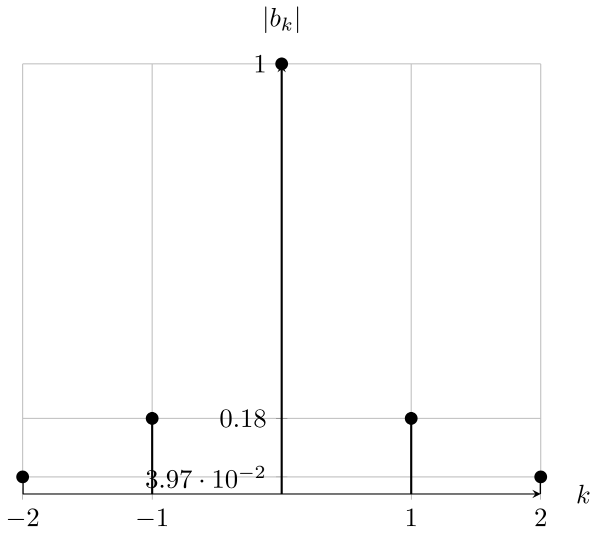


Figure 3: $|b_k|$ vs. k for $x(t)$

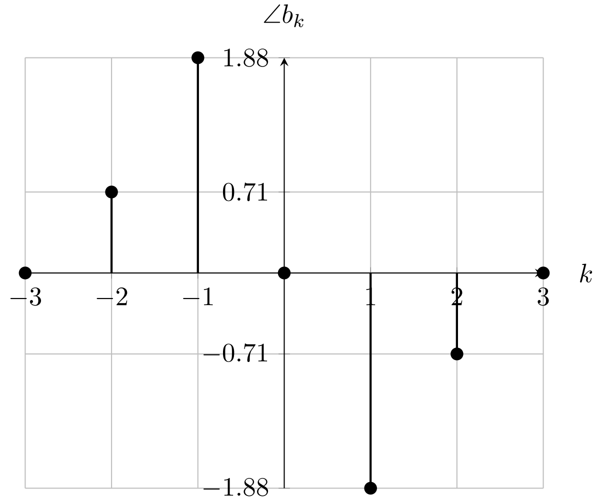


Figure 4: $\angle b_k$ vs. k for $x(t)$

(d)

$$y(t) = \sum_{-2}^{2} b_k e^{jk2\pi t} \tag{25}$$

$$= 1 + (1 + \frac{1}{2}j)(\frac{1}{1 - j2\pi})e^{-j2\pi t} + (1 - \frac{1}{2}j)(\frac{1}{1 + j2\pi})e^{j2\pi t} \tag{26}$$

$$+ \frac{\sqrt{2}}{4}(1 - j)(\frac{1}{1 - j4\pi})e^{-j4\pi t} + \frac{\sqrt{2}}{4}(1 + j)(\frac{1}{1 + j4\pi})e^{j4\pi t} \tag{27}$$

5. Let's first name the coefficients for each signal.

$$x[n] \overset{\text{FS}}{\longleftrightarrow} a_k$$

$$y[n] \overset{\text{FS}}{\longleftrightarrow} b_k$$

$$x[n]y[n] \overset{\text{FS}}{\longleftrightarrow} d_k$$

(a)

$$x[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}), \quad \omega_0 = \frac{\pi}{2}$$

$$a_1 = \frac{1}{2j} = -\frac{j}{2}, \quad a_{-1} = -\frac{1}{2j} = \frac{j}{2}.$$

(b)

$$y[n] = 1 + \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}), \quad \omega_0 = \frac{\pi}{2}$$

$$b_0 = 1, \quad b_1 = b_{-1} = \frac{1}{2}.$$

(c) Multiplication property:

$$x[n]y[n] \overset{\text{FS}}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l}$$

$$d_k = \sum_{l=0}^{3} a_l b_{k-l}, \quad \text{since} \quad N = 4$$

$$= \cancel{\underbrace{a_0 b_k}_{a_0=0}} + a_1 b_{k-1} + \cancel{\underbrace{a_2 b_{k-2}}_{a_2=0}} + a_3 b_{k-3}$$

$$= a_1 b_{k-1} + \underbrace{a_3 b_{k-3}}_{a_3=a_{-1}}$$

$$= a_1 b_{k-1} + a_{-1} b_{k-3}$$

$$\begin{aligned} d_0 &= a_1 b_{-1} + a_{-1} b_{-3} \\ &= a_1 b_{-1} + a_{-1} b_1 \\ &= \frac{1}{4j} - \frac{1}{4j} \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_2 = d_{-2} &= a_1 b_1 + a_{-1} b_{-1} \\ &= \frac{1}{4j} - \frac{1}{4j} \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_1 &= a_1 b_0 + \cancel{\underbrace{a_{-1} b_{-2}}_{b_{-2}=0}} \\ &= \frac{1}{2j} = -\frac{j}{2} \end{aligned}$$

$$\begin{aligned} d_{-1} &= \cancel{\underbrace{a_1 b_{-2}}_{b_{-2}=0}} + \underbrace{a_{-1} b_{-4}}_{b_{-4}=b_0} \\ &= a_{-1} b_0 \\ &= -\frac{1}{2j} = \frac{j}{2} \end{aligned}$$

(d)

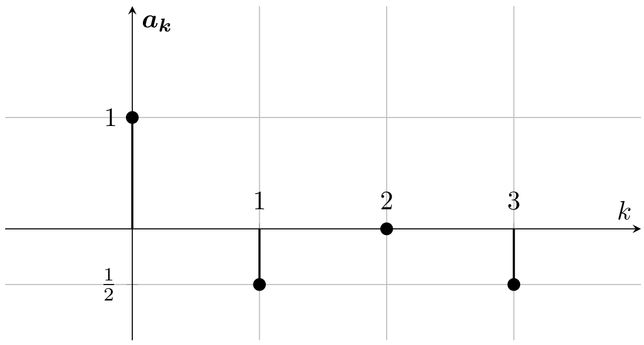
$$\begin{aligned} x[n]y[n] &= (\sin\frac{\pi}{2}n)(1 + \cos\frac{\pi}{2}n) \\ &= \sin\frac{\pi}{2}n + \frac{1}{2}\cancel{\underbrace{\sin \pi n}_{\text{always } 0}} \\ &= \sin\frac{\pi}{2}n \\ &= \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}) \end{aligned}$$

$$d_1 = \frac{1}{2j} = -\frac{j}{2}, \quad d_{-1} = -\frac{1}{2j} = \frac{j}{2}.$$

As you can see the results are the same as the ones found in part c.

6. (a) $N = 4 \quad w_0 = \frac{2\pi}{4} = \frac{\pi}{2} \quad$ so $\quad a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jkw_0 n}$

$$\begin{aligned} a_0 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{0} = \frac{1}{4}[0+1+2+1] = 1 \\ a_1 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}n} = -\frac{1}{2} \\ a_2 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n} = 0 \\ a_3 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{3\pi}{2}n} = -\frac{1}{2} \end{aligned}$$



$a_k = a_{k+N}$ so for $k > 3$, a_k will repeat with $N = 4$

The magnitude of spectral coefficients:

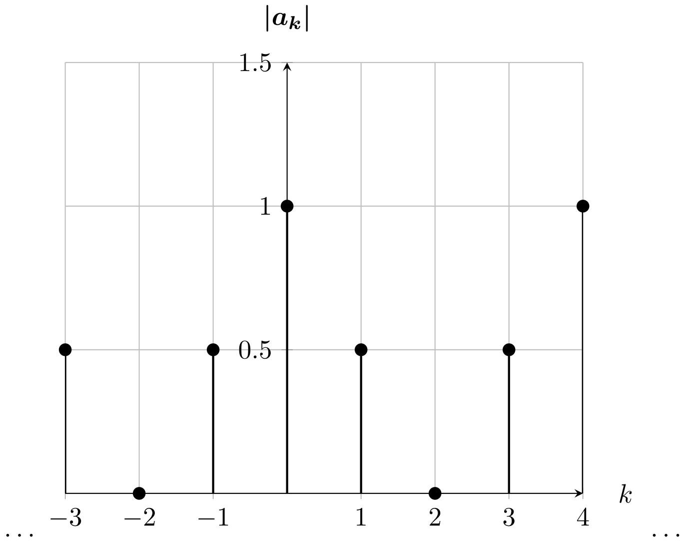


Figure 5: k vs. $|a_k|$.

Phase of the spectral coefficients:

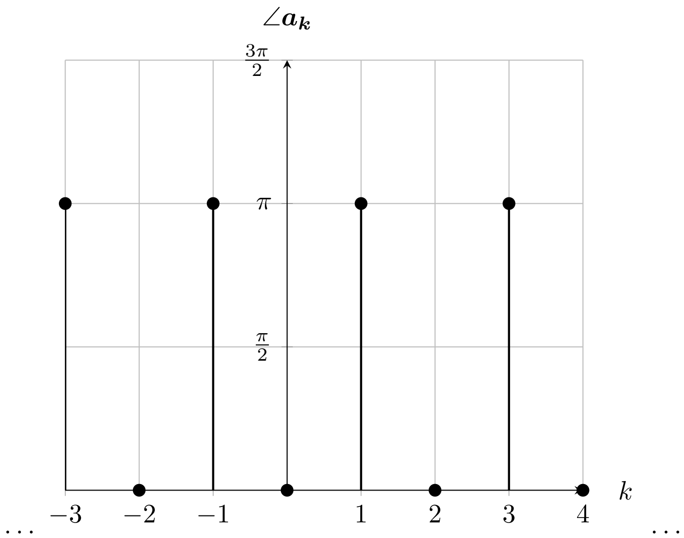


Figure 6: k vs. $\angle a_k$.

(b) i.

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta[n - 3 + N \cdot k] \quad k \in Z, \ N = 4$$

ii.

$$a_0 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^0 = \frac{1}{4}[0+1+2+0] = \frac{3}{4}$$
$$a_1 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\frac{\pi}{2}n} = \frac{-j}{4} - \frac{1}{2}$$
$$a_2 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\pi n} = \frac{1}{4}$$
$$a_3 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\frac{3\pi}{2}n} = \frac{j}{4} - \frac{1}{2}$$

The magnitude of spectral coefficients:

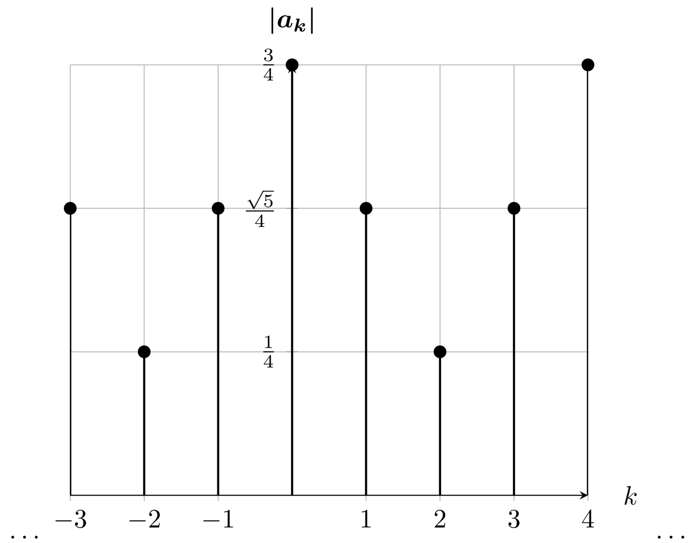


Figure 7: k vs. $|a_k|$.

Phase of spectral coefficients:

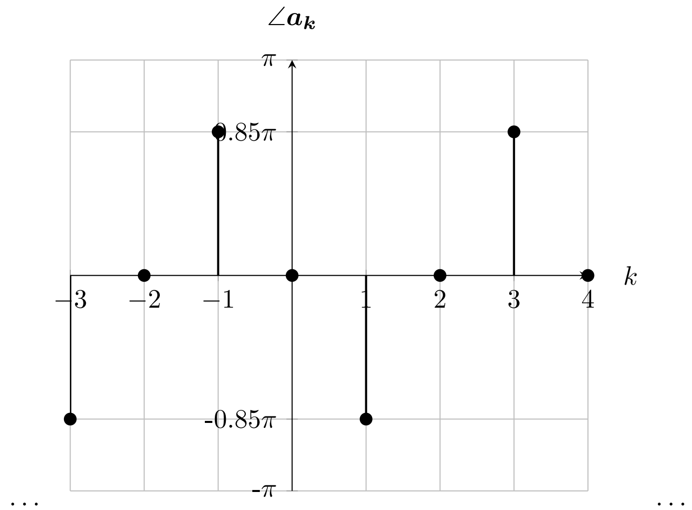


Figure 8: k vs. $\angle a_k$.

7. (a) Let $x(t) \longleftrightarrow a_k$. It's fundamental frequency $\omega_0 = \frac{2\pi}{T} = 2K$. For $x(t) = y(t)$ to be true, $X(j\omega)$ should not have any frequency component for $|\omega| > 80$ since the given system is a LPF and $Y(j\omega) = X(j\omega)H(j\omega)$. That is, all a_k should be zero for $|\omega| = |k\omega_0| > 80 \Rightarrow |k| > \frac{80}{2K} = \frac{40}{K}$.

   Answer: $a_k = 0$ for $\forall |k| > \frac{40}{K}$ and k is integer.

   (b) For $x(t) \neq y(t)$ to be true, there must be some non-zero a_k for $|\omega| > 80$.

Answer: $a_k \neq 0$ for $\exists |k| > \frac{40}{K}$ and k is integer.

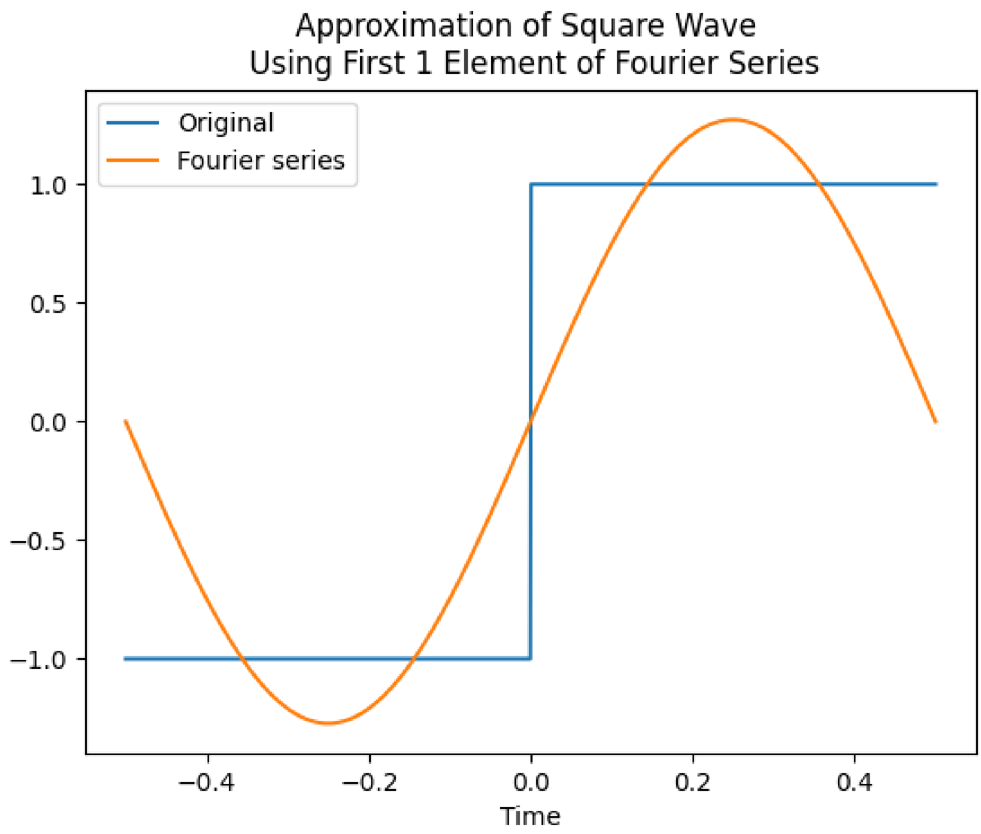


Figure 9: Approximation of square wave using first 1 element of Fourier Series

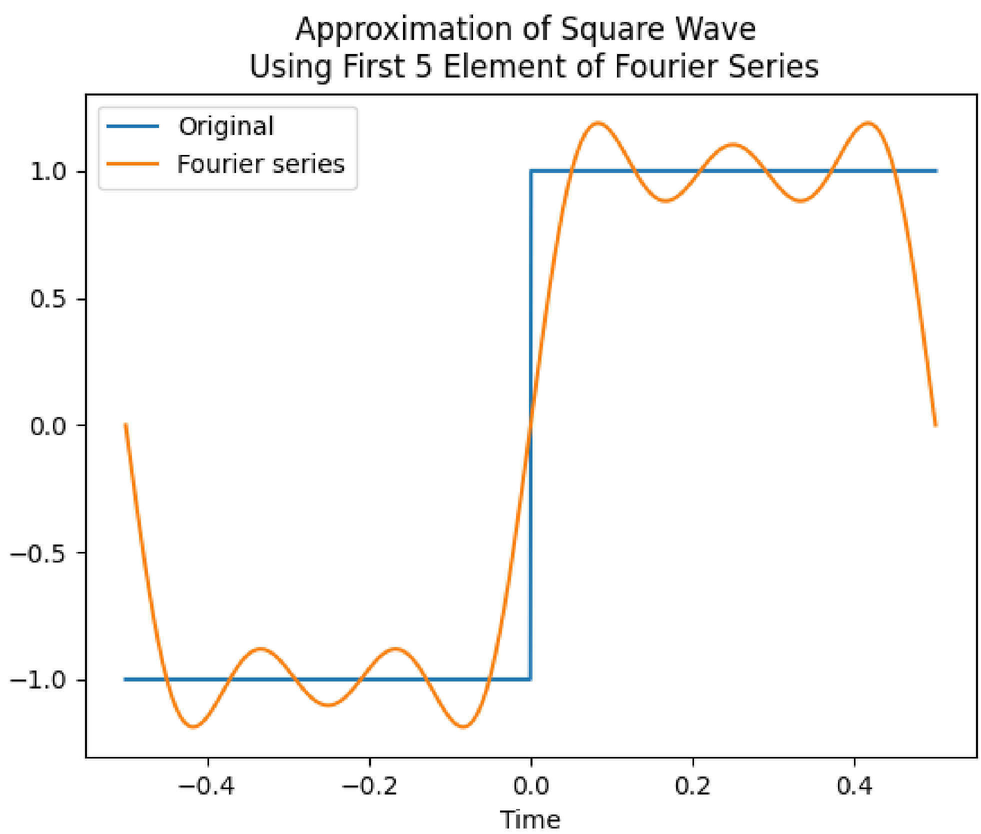


Figure 10: Approximation of square wave using first 5 element of Fourier Series

8. As n gets higher, the approximated function are closer to the original one.

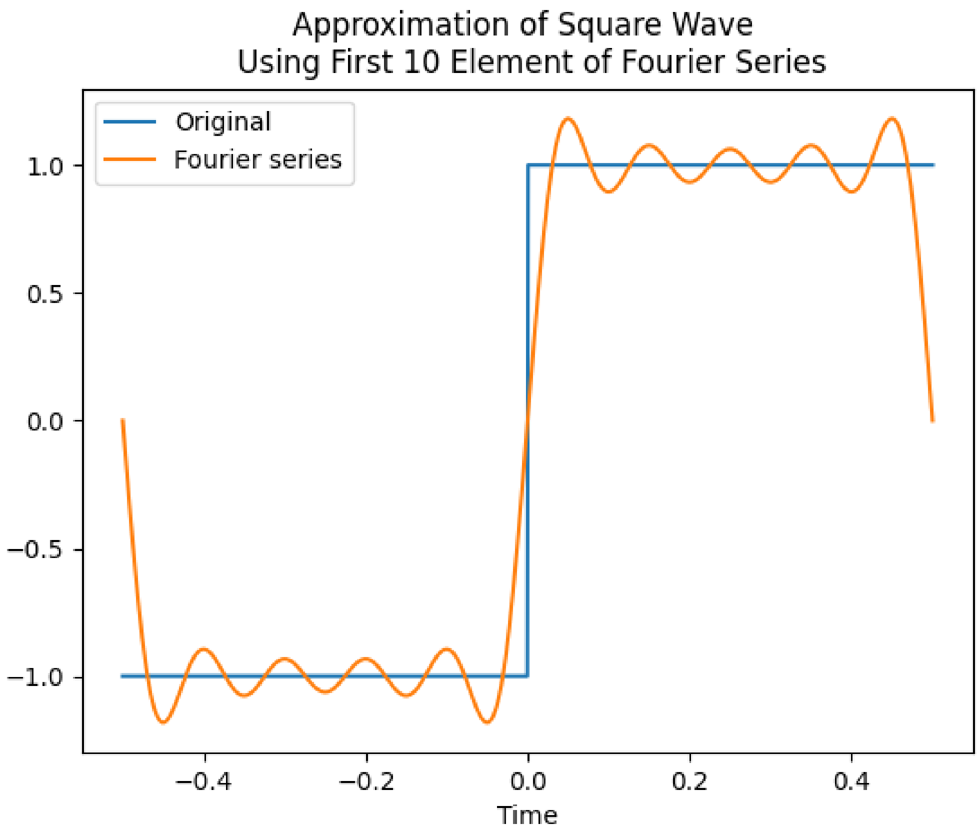


Figure 11: Approximation of square wave using first 10 element of Fourier Series

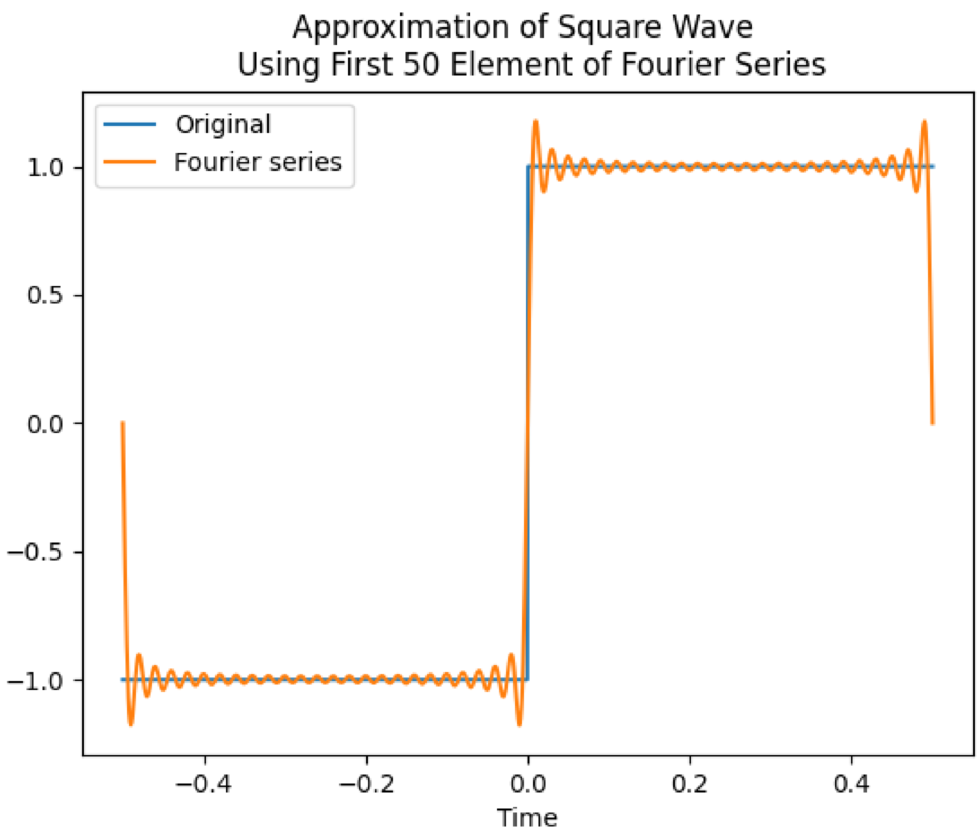


Figure 12: Approximation of square wave using first 50 element of Fourier Series

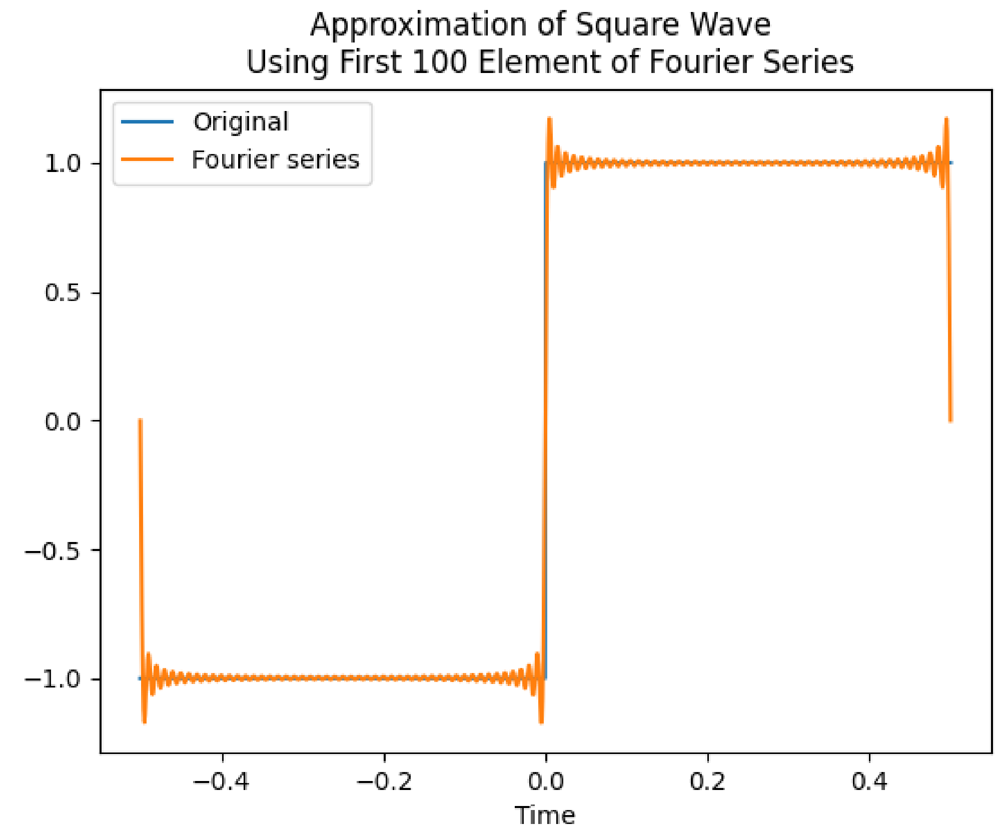


Figure 13: Approximation of square wave using first 100 element of Fourier Series

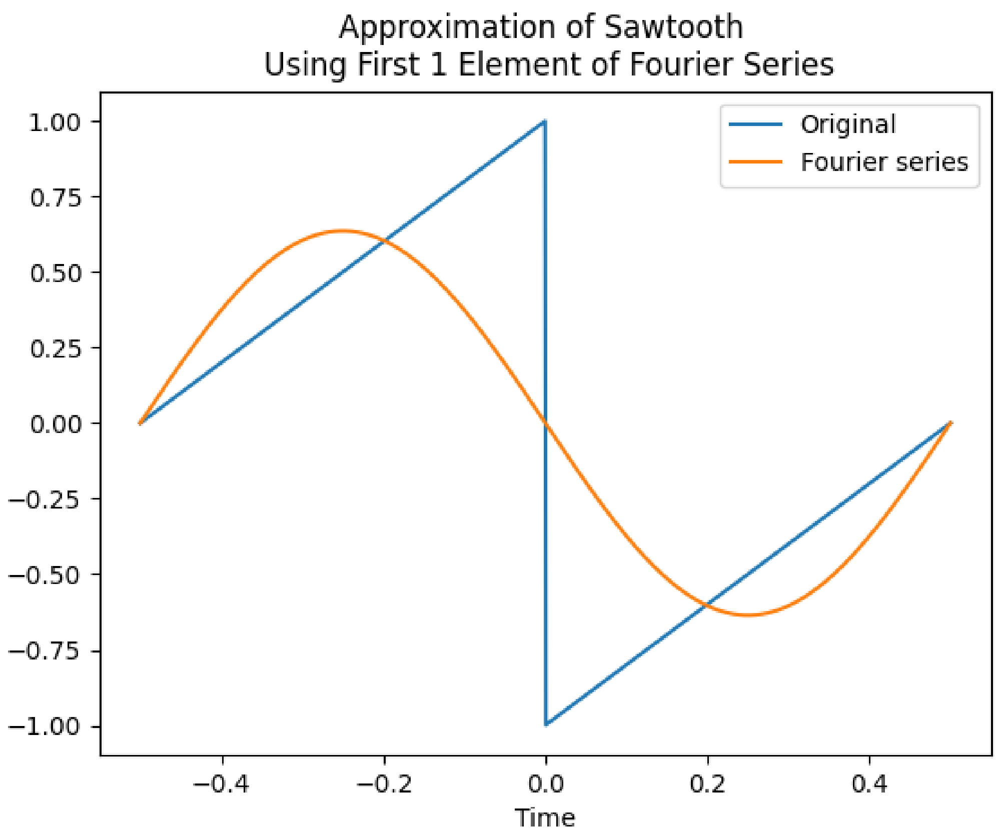


Figure 14: Approximation of sawtooth using first 1 element of Fourier Series

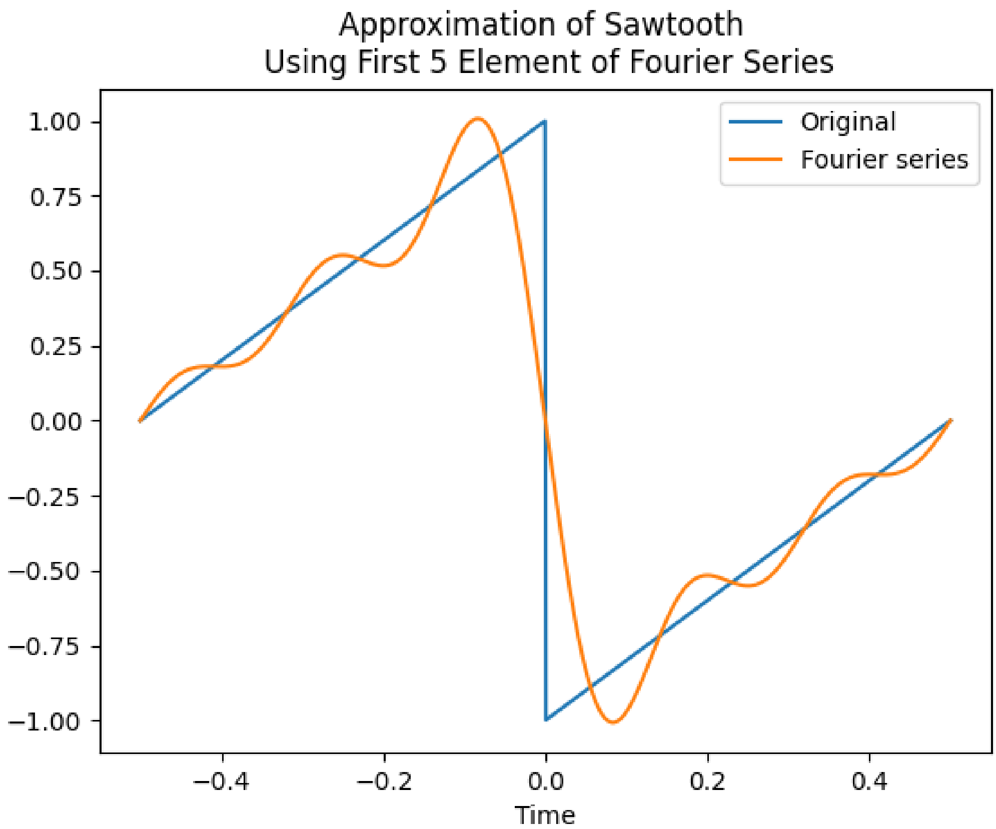


Figure 15: Approximation of sawtooth using first 5 element of Fourier Series

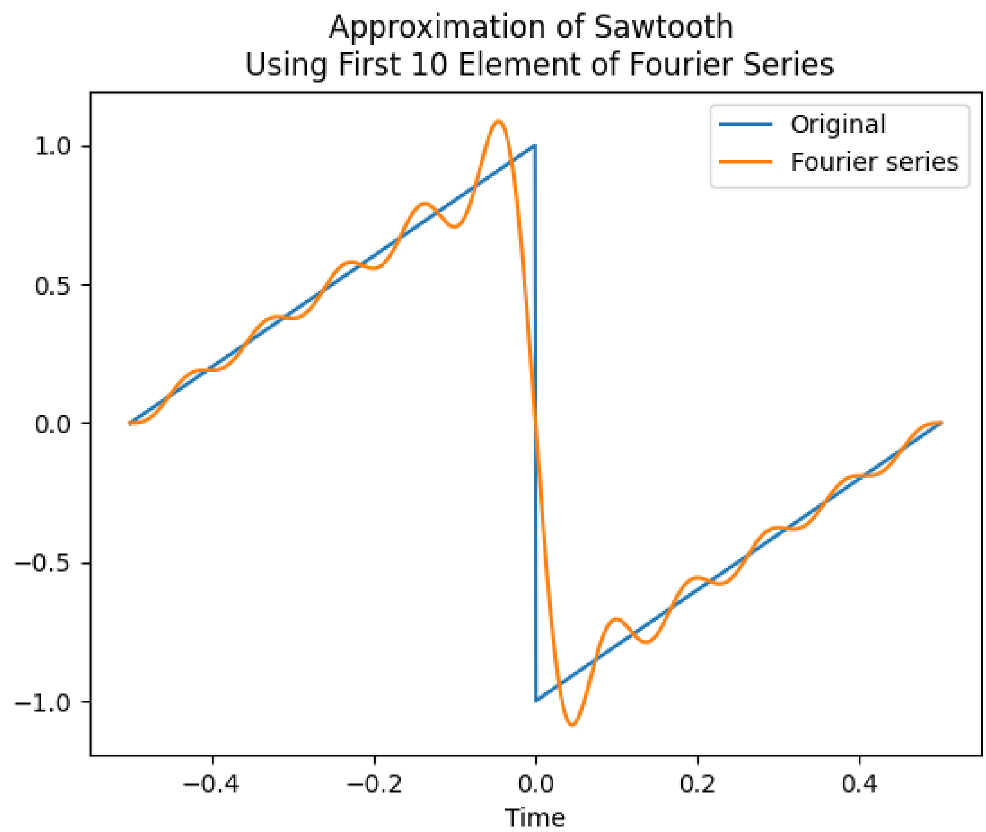


Figure 16: Approximation of sawtooth using first 10 element of Fourier Series

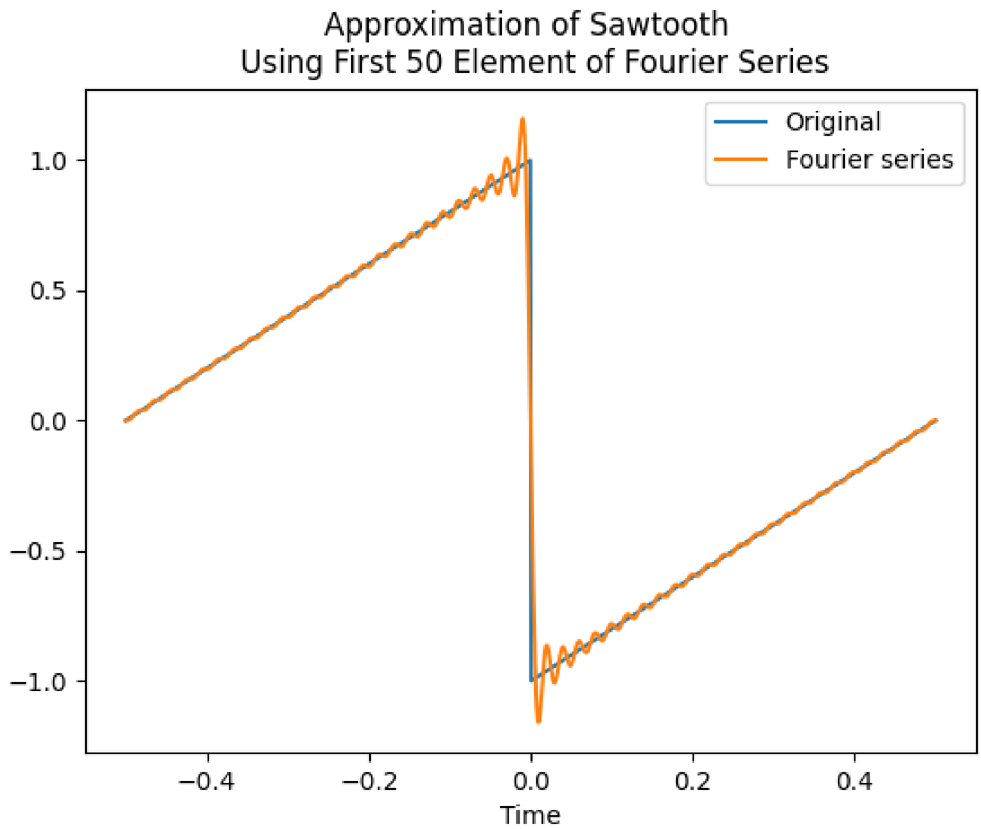


Figure 17: Approximation of sawtooth using first 50 element of Fourier Series

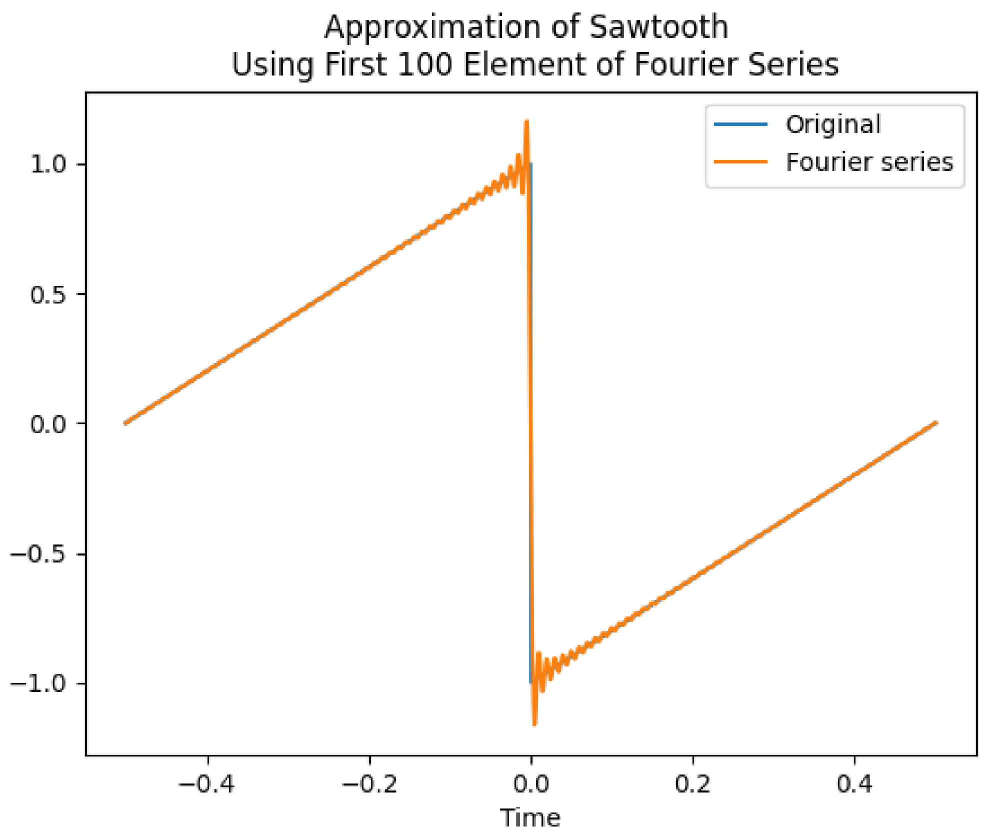


Figure 18: Approximation of sawtooth using first 100 element of Fourier Series