Cardinality of Sets

ex

1	2	3	4	...
2	4	6	8	....

"countably infinite"

$f(x) = 2x$

(1) 1-to-1

$f(x) = 2x$
$f(y) = 2y$

$2x = 2y \longrightarrow x = y$

(2) onto $\quad k$ even integer

$2t = k$

$f(t) = 2t = k$

ex The set of all integers $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$

$\underset{\uparrow 1}{0}, \underset{\uparrow 2}{1}, \underset{\uparrow 3}{-1}, \underset{\uparrow 4}{2}, -2, \ldots$

(countably infinite)

$$f(x) = \begin{cases} -\frac{(x-1)}{2} & x \text{ is odd} \\ +x/2 & x \text{ is even} \end{cases}$$

show that $f(x)$ is 1-to-1 and onto

ex The set of positive rational numbers a/b

denominator \ nominator	1	2	3	4	...
1	1/1	2/1	3/1	...	
2	1/2	2/2	3/2	...	
3	1/3	2/3	3/3	...	
4	1/4	2/4	3/4	...	
⋮	⋮	⋮	⋮		

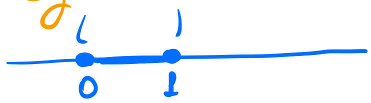
$\underbrace{\frac{1}{1}}_{a+b=2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{a+b=3}, \underbrace{\frac{3}{1}, \frac{2}{2}, \frac{1}{3}}_{a+b=4}, \ldots$

countably infinite

"uncountable sets"

Thm There are more $\overbrace{\text{real numbers in } (0,1)}^{\text{uncountable}}$ then there are natural numbers in $\underbrace{\text{all of } \mathbb{N}}_{\text{countable}}$.

Proof: By contradiction



$0.a_1a_2a_3\ldots$ such that $a_i \in \{0, 1, 2, \ldots, 8, 9\}$

claim: $(0,1)$ is countable (we will contradict)

There exists a 1-to-1 correspondence between $\mathbb{N}$ and $(0,1)$

Suppose we have it

$1 \rightarrow .\underline{\underline{a_1}}a_2a_3\ldots$ a

$2 \rightarrow .b_1\underline{\underline{b_2}}b_3\ldots$ b

$3 \rightarrow .c_1c_2\underline{\underline{c_3}}\ldots$ c

$\vdots$

Now construct a real number in $(0,1)$ that is missed by the enumeration.

$x = \underline{0.x_1x_2x_3\ldots} \in (0,1)$

$x_1 \neq \underline{\underline{a_1}}, 0, 9 \Rightarrow x \neq a$ (the first element of enumeration)

$x_2 \neq b_2, 0, 9 \Rightarrow x \neq b$ second

$x_3 \neq c_3, 0, 9 \Rightarrow x \neq c$ third

$\vdots$

By design $x \in (0,1)$ and it is missed by the enumeration.

So there does not exist an enumeration counting each element in $(0,1)$.

Schröder-Bernstein Thm

$|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

$\underbrace{|A| \leq |B|}_{\text{1-to-1}}$ and $\underbrace{|B| \leq |A|}_{\text{onto}}$

Ex $\Sigma = \{0,1\}$

Σ^* : the set of all finite strings over Σ

$\underbrace{e}_{0}, \underbrace{0, 1}_{1}, \underbrace{00, 01, 10}_{2}, \ldots \underbrace{\quad}_{3}$

$\Rightarrow \underbrace{\Sigma^*}$ is countable

the set of all binary strings

$\Rightarrow$ the number of computer programs is countable

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ex The set of all infinite binary strings

$\Rightarrow$ uncountable

" diagonalization argument

Claim

$1 \rightarrow a_1 = a_{11} \; a_{12} \; a_{13} \; \ldots$

$2 \rightarrow a_2 = a_{21} \; a_{22} \; a_{23} \; \ldots$

$3 \rightarrow a_3 = \vdots \; \ldots$

any enumeration

$a_{ij} \in \{0,1\}$

$b_i \neq a_{ii}$

$b = b_1 b_2 b_3 \ldots$

$\begin{cases} b_i = 0 & \text{if } a_{ii} = 1 \\ b_i = 1 & \text{if } a_{ii} = 0 \end{cases}$

$b \neq a_i \quad i \in \mathbb{N}$

ex $P(\mathbb{N})$ is uncountable

$\Rightarrow$ use diagonalization. (read from book)

Growth Functions + complexity of algorithms

input = A
if $x > 0$ (1)
$x = -x$ (2) } 2
else
$x = 2x$

$3n+2$

for $i = 0$ to len (A)
op1
op2 } $2n + n$
end

$O(n)$

$3n^2 + 5n + \log n$ } $O(n^2)$

Def $f(x)$ is $O(g(x))$ if there are constants C and k (constants) such that

$$|f(x)| \leq C\,|g(x)| \qquad x \geq k$$

f is $O(g)$ if $\exists\ \underbrace{C, k}_{\text{witnesses}}$ s.t. $\forall x\ (x > k \rightarrow |f(x)| \leq C|g(x)|$

ex $\quad f(x) = x^2 + 10x + 2 \quad O(x^2)$

find C, k s.t.

$\rightsquigarrow \quad |x^2 + 10x + 2| \leq C|x^2| \quad x > k$

$x > 0$

$$x^2 + \underbrace{10x}_{\leq 10x^2} + \underbrace{2}_{\leq 2x^2} \leq C x^2$$

$$\leq x^2 + 10x^2 + 2x^2 = 13x^2 \quad x \geq 1$$

$C = 13, \quad k = 1$

$x^2 + 10x + 2 \leq 13x^2 \quad x \geq 1$

ex $\quad f(x) = x$ is $O(x^2)$?

$x \leq x^2$ for $x > 1 \qquad C = 1, \quad k = 1$

ex $\quad f(x) = x^2$ is not $O(x)$ (prove that)

(contradiction) Assume x^2 is $O(x)$

Then there exists C, k s.t.

$x^2 \leq C x$ for all $x > k$

$x < C$ for all $x > k \quad \perp$

it can not hold for all x

Thm $\quad f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$f(x)$ is $O(x^n)$

proof $\quad x > 1 = k$

$$|f(x)| = |a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0|$$

$$\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \dots + |a_1| x + |a_0|$$

$$\leq x^n \left(|a_n| + \underbrace{\frac{|a_{n-1}|}{x}}_{< |a_{n-1}|} + \dots + \underbrace{\frac{|a_1|}{x^{n-1}}}_{< a_1} + \frac{|a_0|}{x^n} \right) \leq$$

$$\leq x^n \underbrace{\left(\sum_{i=0}^{n} |a_i| \right)}_{C} \qquad k = 1$$

ex Factorial function $\quad n! = n \cdot n-1 \cdot \dots \cdot 1$

$$n! \leq \underbrace{n \cdot n \cdot \dots \cdot n}_{n} = n^n$$

$$\log n! \leq \log n^n \leq n \log n$$

$$\Rightarrow \log n! \text{ is } O(n \log n)$$

$1, \quad \log n, \quad n, \quad n \log n, \quad n^2, \quad n^c, \quad 2^n, \quad n!$

constant

ex $\quad n < 2^n$ for each $n \in N_+$

$$\log(n) < n$$

ex $\quad f(x) = \underbrace{2 \log(n)}_{O(n)} + \underbrace{n^2 + 2n^3}_{O(n^3)} \qquad O(n^3)$

$C_1 = 2$, $k_1 = 1$ $\qquad C_2 = 3$, $k_2 = 1$ $\qquad C = 5$, $k = 1$

$$
\begin{aligned}
|f_1(x) + f_2(x)| &\leq |f_1(x)| + |f_2(x)| \\
&\leq C_1 |g_1(x)| + C_2 |g_2(x)| \\
&\leq (C_1 + C_2) \max(|g_1(x)|, |g_2(x)|) \\
&\leq (C_1 + C_2) |g(x)| \\
&\leq C |g(x)|
\end{aligned}
$$

Thm: Suppose that $f_1(x)$ is $O(g_1(x))$, and $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$

$\rightsquigarrow$ generalization on what we saw for the polynomials.

= "look at the leading term"

ex $f(x) = 2n^2 \log(n!) + n^3$

$\underbrace{2n^2}_{n^2} \underbrace{\log(n!)}_{n \log n}$ $\quad$ n^3

$\underbrace{n^2 \quad n\log n}_{n^3 \log n}$

$O(n^3 \log n)$

Big-O $\rightsquigarrow$ upper bound $O(\ )$

Big-omega $\rightsquigarrow$ lower bound $\Omega(\ )$

Big-Theta $\rightsquigarrow$ upper + lower $\Theta(\ )$

Def f, g $\quad f(x)$ is $\Omega(g(x))$ if $\exists\, C, k$

s.t. $\quad |f(x)| \geq C\,|g(x)| \quad x > k$

" f is big-omega of g "

$\Rightarrow$ $f(x)$ is $\Omega(g(x))$ then $g(x)$ is $O(f(x))$

$$|f(x)| \geq C\,|g(x)|$$

$$|g(x)| \leq \tfrac{1}{C}\,|f(x)|$$

ex $\quad f(x) = x^4 + 3x^2 + 1$

$$|x^4 + 3x^2 + 1| > C\,|g(x)|$$

$g(x) =$ x, x^2, x^3, x^4

Def $\quad f(x), g(x)$

$f(x)$ is $\Theta(g(x))$ if $\quad f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$

f is big-theta of $g(x)$

f is of order of $g(x)$

f and g are of the same order.

$$c_1 |g(x)| \le |f(x)| \le c_2 |g(x)|$$

ex Show $f(x) = a x^2 + b x \log(x)$ is $\Theta(x^2)$

$a, b \in \mathbb{R}_+$

$$c_1 x^2 \le a x^2 + b x \log(x) \le c_2 x^2$$

$c_2 = a + b$

$c_1 = a$