Natural Deduction for Predicate Calculus

Reading: Huth and Ryan Section 2.2.3, 2.2.4, 2.3

Definition: A term t is *free for X in ϕ* if no variable in t becomes bound when t is substituted for X in ϕ.

Examples:

Z is free for X in $p(X, Y)$

Z is free for X in $\exists Y(p(X, Y))$

Z is *not* free for X in $\exists Z(p(X, Z))$

More examples:

$f(U, V)$ is free for X in $p(X, Y)$

$f(U, V)$ is free for X in $\forall U(q(U)) \longrightarrow \forall Y(p(X, Y))$

$f(U, V)$ is not free for X in $p(X, Y) \longrightarrow \forall U(q(X, U))$

Eliminating Universal Quantification:

$$\frac{\forall X(\phi(X))}{\phi(t)} \forall X - e$$

where t is a term that is *free for X in ϕ*.

An example of what goes wrong if we don't check that t is free for X in ϕ.

Note Y is not free for X in $\exists Y(\texttt{father}(X, Y)$

$1:$	$\forall X \exists Y(\texttt{father}(X, Y))$	Premise
$2:$	$\exists Y(\texttt{father}(Y, Y))$	$1, \forall X - e(incorrect)$

Remark: It is always possible to make t free for X in ϕ by renaming variables.

Example:

$f(U, V)$ is not free for X in $\phi = \exists U(q(X, U))$

If we rename U to W in ϕ, we get the formula $\exists W(q(X, W))$ .... this is logically equivalent to ϕ.

$f(U, V)$ is free for X in $\exists W(q(X, W))$

The rule $\forall X - e$ "explained" ....

Recall that if the universe is a finite set of objects $a_1, \ldots, a_n$ then

$$\forall X(\phi(X)) \equiv (\phi(a_1) \wedge \ldots \wedge \phi(a_n))$$

Compare:

$$\frac{\forall X(\phi(X))}{\phi(a_i)} \qquad \frac{(\phi(a_1) \wedge \ldots \wedge \phi(a_n))}{\phi(a_i)}$$

A variant of a problem by Bertrand Russell:

1. The barber shaves all and only those men in this town who do not shave themselves.
2. Thus, the barber is not a man.

Introducing universal quantification

$$\dfrac{\boxed{\begin{array}{l} X_0 \\ \qquad\qquad \vdots \\ \qquad\qquad \phi[X \mapsto X_0] \end{array}}}{\forall X \phi} \quad \forall X - i$$

Here X_0 is a variable that does not occur anywhere outside the box.

Intuitively, the first line of the box says "let X_0 be an arbitrary object". The fact that it appears nowhere else means that we make no assumptions about X_0 whatsoever.

We then do some reasoning (the "$\vdots$") and conclude $\phi[X \mapsto X_0]$.

Since we assumed nothing about X_0, we can run the same argument for *any* particular value for X instead of X_0.

1. All Microsoft product has bugs.
2. Thus, if all our software is Microsoft product then all our software has bugs.

$$\neg \forall X(\phi(X)) \quad \vdash \exists X(\neg \phi(X))$$

Introducing existential quantification

$$\frac{\phi[X \mapsto t]}{\exists X(\phi)} \quad \exists X - i$$

where t is a term that is free for X in ϕ

Eliminating existential quantification

$$\frac{\exists X(\phi(X)) \qquad \boxed{\begin{array}{lcl} X_0 & \phi(X_0) & \text{Asmptn} \\ & \vdots & \\ & \chi & \end{array}}}{\chi} \qquad \exists X - e$$

Here X_0 is a variable that does not occur anywhere outside the box.

Intuitively, $\exists X(\phi(X))$ tells us that some value of X satisfies $\phi(X)$, but we don't know which value.

The box says: "let's temporarily use X_0 a name for the value of X satisfying $\phi(X)$, and see what we can conclude from $\phi(X_0)$"

We need to make sure that the name X_0 is not already being used as a name for something else, for there is no guarantee that other object satisfies $\phi(X)$.

$$\neg\forall X(\phi(X)) \ \vdash \exists X(\neg\phi(X))$$

Some problems, by Lewis Carroll:

1. Babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.
4. Thus, babies cannot manage crocodiles.

1. Everyone who is sane can do logic.
2. No lunatics are fit to serve on a jury.
3. None of your sons can do logic.
4. Thus, none of your sons is fit to serve on a jury.