Take Home Exam 4

Question 1

$a_n = a_{n-1} + \mathbf{2^n}$ **,** $\mathbf{n \geq 1}$

a_0 **= 1**

$\sum_{n=1}^{\infty} a_n \cdot x^n = \sum_{n=1}^{\infty} (a_{n-1} + 2^n) x^n$

$A(x) - a_0 = x \cdot \sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1} + \sum_{n=1}^{\infty} 2^n \cdot x^n$

$A(x) - a_0 = x \,.\, A(x) + \sum_{n=1}^{\infty} (2x)^{n}$

$B(x) = \sum_{n=0}^{\infty} (2x)^{n}$

$A(x) - a_0 = x \,.\, A(x) + \underbrace{\sum_{n=1}^{\infty} (2x)^{n}}_{B(x) - b_0}$

$b_0 = (2x)^0 = 1$

$A(x) - 1 = x \,.\, A(x) + \frac{1}{1-2x} - 1$

$A(x) \,.\, (1-x) = \frac{1}{1-2x}$

$A(x) = \frac{1}{(1-x)\cdot(1-2x)}$

$A(x) = \frac{A}{(1-x)} + \frac{B}{(1-2x)} = \frac{1}{(1-x)\cdot(1-2x)}$, A = -1 B = 2

$A(x) = \frac{-1}{(1-x)} + \frac{2}{(1-2x)}$

(-1, -1, -1, -1 …. , -1, ….) $\Leftrightarrow$ $\frac{-1}{(1-x)}$

(2, 4, 8, 16, …., 2^{n+1}, ….) $\Leftrightarrow$ $\frac{2}{(1-2x)}$

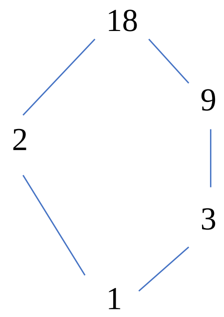
(1, 3, 7, 15, …., 2^{n+1}-1, …) $\Leftrightarrow$ $\frac{-1}{(1-x)} + \frac{2}{(1-2x)} = A(x)$

$A(n) = 2^{n+1} - 1$

Question 2

$R = \{ (a,b) \mid a \text{ divides } b \}$ on $A = \{ 1, 2, 3, 9, 18\}$

a) Hasse diagram of R



b) Matrix representation of R is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

c) For the poset to be called a lattice, each pair of elements must have both a unique least upper bound and a unique greatest lower bound.
Then, in poset (A,R), we can see every pair of elements has both a least upper bound and a greatest lower bound. Hence, this poset can be called a lattice.

d) Symetric Closure of $R = R \cup R^{-1} \{ (b,a) \mid (a,b) \in R \}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{S(R)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

e) The integers 2 and 9 are incomparable, because $2 \nmid 9$ and $9 \nmid 2$. The integers 3 and 18 are comparable, because $3 \mid 18$.

Question 3

a) A relation has ordered pairs (a,b). For anti-symmetric relation,

$a \neq b\ ,\qquad a\,R\,b \rightarrow b\,\cancel{R}\,a$

A = $\{a_1, a_2, a_3, \ldots a_n\}$, R $\subseteq$ A×A

a_{11} , a_{22}, a_{33}, …. , a_{nn} can be both 0 or 1.

So, for (a_n,a_n), total number of ordered pairs = n,

The total number of relation of diagonal part = 2^n.

From the remaining part,

For $1 \leq i \leq n$, $1 \leq j \leq n$ and $i \neq j$, a_{ij} and a_{ji} cannot be both 1,

To find number of different a_{ij}, (total number of ordered pairs– diagonal pairs) /2

$$\left(\underbrace{\text{total number of ordered pairs}}_{n^2} - \underbrace{\text{diagonal pairs}}_{n}\right) / 2 = n\,.\,(n-1)\,/2$$

There are 3 possible interpretation that makes a_{ij} symetric.

$(a_{ij} = 0\ \ a_{ji} = 0\ ,\ a_{ij} = 1\ \ a_{ji} = 0\ ,\ a_{ij} = 0\ \ a_{ji} = 1\ ,\ \underbrace{a_{ij} = 1\ \ a_{ji} = 1}_{\text{This is not anti-symmetric.}})$.

So, there are $3^{\frac{n\cdot(n-1)}{2}}$ option from here.

As a result, total number of anti-symmetric relation is $2^n\,.\,3^{\frac{n\cdot(n-1)}{2}}$.

b)

A = $\{a_1, a_2, a_3, \ldots a_n\}$, R $\subseteq$ A×A

Since all diagonal elements are part of the reflexive relation, (a_{11} , a_{22}, a_{33}, …. , a_{nn}) are all 1.

From the remaining part,
For $1 \leq i \leq n$, $1 \leq j \leq n$ and $i \neq j$, there are $(n^2 - n)/2$ many different a_{ij}.

For anti-symmetric relation, there are 3 possibilities for each of the remaining $(n^2 - n)/2$ elements.

$(a_{ij} = 0\ \ a_{ji} = 0\ ,\ a_{ij} = 1\ \ a_{ji} = 0\ ,\ a_{ij} = 0\ \ a_{ji} = 1\ ,\ \underbrace{a_{ij} = 1\ \ a_{ji} = 1}_{\text{This is not anti-symmetric.}})$.

Thus, we get $3^{\frac{n\cdot(n-1)}{2}}$ binary relations which are reflexive and antisymmetric.