Recurrence Relations (Construction)

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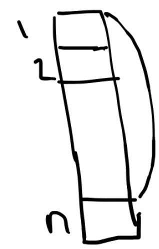
$f(0) = a$

$f(n+1) \Rightarrow$ a function of $f(n), f(n-1), \dots f(0)$

e.g., $\gcd(a, 0) = a$

$\gcd(\underset{a \geq b}{a, b}) = \gcd(b, a \bmod b)$

e.g., Bubble Sort Alg. (in memory)



rec. formula $\rightarrow$ $a_n = a_{n-1} + (n-1)$ $n > 0$ # of comparison & swap operations to sort n #'s

initial (boundary cond's) $\rightarrow$ $a_0 = 0$

$\rightarrow$ Solving a rec. rel^{n} $\quad a_n = \dfrac{n(n-1)}{2}$

$$\Big(\underset{\uparrow a_0}{0}, \underset{\uparrow a_1}{0}, \underset{\uparrow a_2}{1}, \dots\dots, \underset{\uparrow a_n}{\frac{n(n-1)}{2}}, \dots\Big)$$

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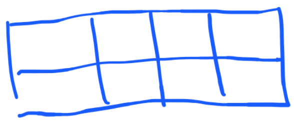
e.g., Determine # of ways to tile a $2 \times n$ chessboard with 2×1 & 1×2 dominoes.

a_n

for instance



2×4 chessboard

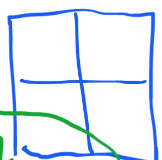
2×1 domino

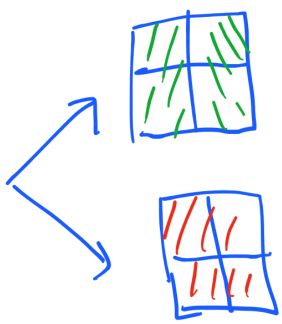
1×2 domino

- $n=1$

can be tiled using a 2×1 domino $\quad a_1 = 1$

- $n=2$





using 2 2×1 dominoes

using 2 1×2 dominos

$a_2 = 2$

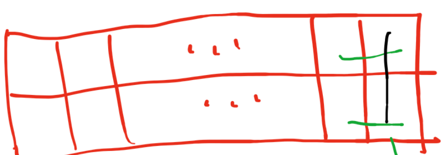
$a_n = a_{n-1} + a_{n-2}$

$a_1 = 1$

$a_2 = 2$

$n \geq 3$

consider n ?



(1) last column can be a 2×1 domino $\rightarrow a_{n-1}$

(2) last two columns with two 1×2 dominos $\rightarrow a_{n-2}$

} mutually excl.

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e.g., Construct a rec. rel^n for the max. # of regions (b_n) defined by n lines in the plane (J. Steiner)

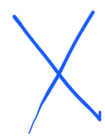
$n=0$

$b_0 = 1$

$n=1$

$b_1 = 2$

$n=2$

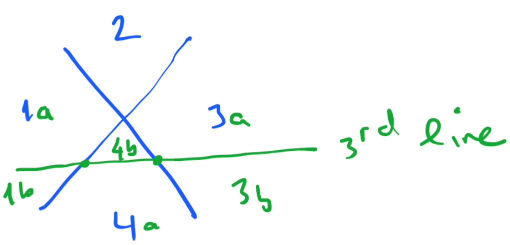


$b_2 = 4$

... seems 2^n (exp.) but NOT!

in order to have max. # of regions we need to intersect n^{th} line with $n-1$ existing lines yielding new intersection points

$n=3$



3 new regions

rec. formula → $b_n = b_{n-1} + n \quad n > 0$

↑ # of new regions

init. cond. → $b_0 = 1$

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e.g., Construct a rec. rel a_n for # of strings of decimal digits with even # of 0's

0..9

$n = 1$ _ $a_1 = 9$ 1, 2, 3, …, 9

$n = 2$ _ _ $10 \cdot 10 = 100$

0 _ → 9 + 9 ← _ 0   invalid ones

two cases

Disjoint cases:

i) a valid string of length $n-1$ + a non-zero digit ⇒ a valid string of length n (length n)

ii) invalid string → $n-1$ + 0 ⇒ a valid string of length n

$$a_n = \underbrace{a_{n-1} \cdot 9}_{(i)} + \underbrace{(10^{n-1} - a_{n-1})}_{(ii)} \cdot 1$$

$$\Rightarrow \begin{cases} a_n = 8a_{n-1} + 10^{n-1} & n \geq 2 \\ a_1 = 9 \end{cases}$$

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e.g., Let $S = \{1, 2, 3, \dots, n\}$ (when $n=0$ $S = \emptyset$)

Construct rec. rel.ⁿ a_n for number of subsets of S that do not contain consecutive numbers.

n	a_n	S	subsets of S
0	1	$\emptyset$	$\emptyset$
1	2	$\{1\}$	$\emptyset, \{1\}$
2	3	$\{1,2\}$	$\emptyset, \{1\}, \{2\}$
3	5	$\{1,2,3\}$	$\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}$

$$\begin{cases} a_n = a_{n-1} + a_{n-2} & n \geq 3 \\ a_0 = 1 \\ a_1 = 2 \end{cases}$$

$n \geq 2$ $\quad S = \{1, 2, 3, \dots, n-2, n-1, n\}$

Any subset A of S without 2 consecutive numbers

DISJOINT CASES

i) $n \in A$. $\quad A = \{\dots, n\} \to n-1 \notin A$ $\quad A - \{n\}$ would be counted in a_{n-2}

ii) $n \notin A$. $\quad A = \{\dots\} \to A$ would be counted in a_{n-1}

Solving Recurrence Relations

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Assume linear rec. rel^n's with constant coeff's

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$$

- C_i's are constant coeff's
- linear

degree is k

a linear rec rel^n with c.c's require k boundary/initial conditions to determine any a_n.

Sol^n
- find homog. sol^n $a_n^{(h)}$ (sol^n to homog. eq^n) replace $f(n)$ with 0
- find part sol^n $a_n^{(p)}$ → solve the eq^n with $f(n)$ RHS of eq^n.

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$C_0 a_n^{(h)} + C_1 a_{n-1}^{(h)} + \cdots + C_k a_{n-k}^{(h)} = 0$$

$$+ \quad C_0 a_n^{(p)} + C_1 a_{n-1}^{(p)} + \cdots + C_k a_{n-k}^{(p)} = f(n)$$

$$C_0\left(a_n^{(h)} + a_n^{(p)}\right) + C_1\left(a_{n-1}^{(h)} + a_{n-1}^{(p)}\right) + \cdots = f(n)$$

Finding Homogeneous Solution

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Homog. sol^n (a_n^h) $\rightsquigarrow$ $\underbrace{A\alpha_1^n}$ where α_1 is a char. root & A is a constant

$$C_0 a_n^h + C_1 a_{n-1}^h + \cdots + C_k a_{n-k}^h = 0$$

$$C_0 \cancel{A} \alpha_1^n + C_1 \cancel{A} \alpha_1^{n-1} + \cdots + C_k \cancel{A} \alpha_1^{n-k} = 0$$

$$\alpha_1^{n-k}\left(C_0 \alpha_1^k + C_1 \alpha_1^{k-1} + \cdots + C_k\right) = 0$$

$$C_0 \alpha_1^k + C_1 \alpha_1^{k-1} + \cdots + C_k = 0 \Leftarrow \boxed{\text{CHAR. EQ}^{\text{N}}}$$

- determine α_1

if roots are distinct then

$$a_n^{(h)} = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_k \alpha_k^n$$

where $\alpha_1, \alpha_2 \ldots \alpha_k$ are distinct roots for char. eq^n

* A_i's are constants to be determined using initial conditions

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e.g., Consider $a_n + a_{n-1} - 6a_{n-2} = 0 \quad n \geq 0$

$a_0 = -1 \quad a_1 = 8 \quad k = 2$

CHAR. EQ.^{n} $\alpha^2 + \alpha^1 - 6 = 0 \Rightarrow (\alpha - 2)(\alpha + 3) = 0$

$\alpha \quad -2$

$\alpha \quad 3$

$\alpha_1 = 2$

$\alpha_2 = -3$

$a_n^h = A \cdot 2^n + B(-3)^n$

determine A & B using initial cond's

$n = 0 \quad a_0 = -1 = A \cdot 2^0 + B(-3)^0 \Rightarrow A + B = -1$

$n = 1 \quad a_1 = 8 = A \cdot 2^1 + B(-3)^1 \quad 2A - 3B = 8$

$A = 1 \quad B = -2$

$a_n^{(h)} = 2^n - 2(-3)^n$ ✓

$a_n = a_n^{(h)} + a_n^{(p)} = a_n^{(h)}$

$a_n^{(p)} = 0$ as $f(n) = 0$

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e.g.; rec. rel. for fib. seq.

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2$$

$$f_0 = 0 \quad f_1 = 1$$

$$f_n - f_{n-1} - f_{n-2} = 0$$

char. eq$^{\underline{n}}$ $\alpha^2 - \alpha - 1 = 0$

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \qquad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

$$a_n^h = A \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + B \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

determine A & B using f_0 & f_1

$$A = \frac{1}{\sqrt{5}} \qquad B = -\frac{1}{\sqrt{5}}$$

$\Rightarrow$ Roots with multiplicity m

Let α_1 be a root of multiplicity m

a poly$^{\underline{n}}$ of degree $m-1$

Sol$^{\underline{n}}$ $\underbrace{(A_1 n^{m-1} + A_2 n^{m-2} + \cdots + A_m)}_{\text{a poly}^{\underline{n}} \text{ of degree } m-1} \cdot \alpha_1^n$

Finding Particular Solution

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e.g., Consider the diff. eq^n

$$4a_n - 20a_{n-1} + 17a_{n-2} - 4a_{n-3} = 0$$

CHAR EQ. $4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$

char. roots $\Rightarrow \frac{1}{2}, \frac{1}{2}, 4$

mult. 2

$\downarrow$

a poly^n of degree 1

$$a_n^{(h)} = (An+B)\left(\frac{1}{2}\right)^n + C4^n$$

PARTICULAR SOL^N

no generic method for obtaining part. sol^n

$$C_0 a_n + \cdots + C_k a_{n-k} = f(n)$$

consider the function type

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① if $f(n)$ is a polyⁿ then $a_n^{(p)}$ will be a polyⁿ

e.g., $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

$a_n^{(p)} = (An^2 + Bn + C)$

$(An^2 + Bn + C) + 5\left(A(n-1)^2 + B(n-1) + C\right) + 6\left(A(n-2)^2 + B(n-2) + C\right) = 3n^2$

determine constants

coeff. of n^2 → $12A = 3$ $\qquad A = \frac{1}{4}$

coeff. of n → $34A - 12B = 0$ $\qquad B = \frac{17}{24}$

const. → $29A - 17B + 12C = 0$ $\qquad C = \frac{115}{288}$