NFA to DFA Construction

Theorem 2.2.1: *For each nondeterministic finite automaton, there is an equivalent deterministic finite automaton.*

It remains to show that M' is deterministic and equivalent to M. The demonstration that M' is deterministic is straightforward: we just notice that δ' is single-valued and well defined on all $Q \in K'$ and $a \in \Sigma$, by the way it was constructed. (That $\delta'(Q, a) = \emptyset$ for some $Q \in K'$ and $a \in \Sigma$ does not mean δ' is not well defined; $\emptyset$ is a member of K'.)

We are now ready to define formally the deterministic automaton $M' = (K', \Sigma, \delta', s', F')$ that is equivalent to M. In particular,

$$\begin{aligned} K' &= 2^K, \\ s' &= E(s), \\ F' &= \{Q \subseteq K : Q \cap F \neq \emptyset\}, \end{aligned}$$

and for each $Q \subseteq K$ and each symbol $a \in \Sigma$, define

$$\delta'(Q, a) = \bigcup \{E(p) : p \in K \text{ and } (q, a, p) \in \Delta \text{ for some } q \in Q\}.$$

We now *claim* that for any string $w \in \Sigma^*$ and any states $p, q \in K$,

$$(q, w) \vdash_M^* (p, e) \text{ if and only if } (E(q), w) \vdash_{M'}^* (P, e)$$

for some set P containing p. From this the theorem will follow easily: To show that M and M' are equivalent, consider any string $w \in \Sigma^*$. Then $w \in L(M)$ if and only if $(s, w) \vdash_M^* (f, e)$ for some $f \in F$ (by definition) if and only if $(E(s), w) \vdash_{M'}^* (Q, e)$ for some Q containing f (by the claim above); in other words, if and only if $(s', w) \vdash_{M'}^* (Q, e)$ for some $Q \in F'$. The last condition is the definition of $w \in L(M')$.

We prove the claim by induction on $|w|$.

Basis Step. For $|w| = 0$ —that is, for $w = e$— we must show that $(q, e) \vdash_M^* (p, e)$ if and only if $(E(q), e) \vdash_{M'}^* (P, e)$ for some set P containing p. The first statement is equivalent to saying that $p \in E(q)$. Since M' is deterministic, the second statement is equivalent to saying that $P = E(q)$ and P contains p; that is, $p \in E(q)$. This completes the proof of the basis step.

Induction Hypothesis. Suppose that the claim is true for all strings w of length k or less for some $k \geq 0$.

Induction Step. We prove the claim for any string w of length $k+1$. Let $w = va$, where $a \in \Sigma$, and $v \in \Sigma^*$.

For the *only if* direction, suppose that $(q,w) \vdash_M^* (p,e)$. Then there are states r_1 and r_2 such that

$$(q,w) \vdash_M^* (r_1,a) \vdash_M (r_2,e) \vdash_M^* (p,e).$$

That is, M reaches state p from state q by some number of moves during which input v is read, followed by one move during which input a is read, followed by some number of moves during which no input is read. Now $(q,va) \vdash_M^* (r_1,a)$ is tantamount to $(q,v) \vdash_M^* (r_1,e)$, and since $|v| = k$, by the induction hypothesis $(E(q),v) \vdash_{M'}^* (R_1,e)$ for some set R_1 containing r_1. Since $(r_1,a) \vdash_M (r_2,e)$, there is a triple $(r_1,a,r_2) \in \Delta$, and hence by the construction of M', $E(r_2) \subseteq \delta'(R_1,a)$. But since $(r_2,e) \vdash_M^* (p,e)$, it follows that $p \in E(r_2)$, and therefore $p \in \delta'(R_1,a)$. Therefore $(R_1,a) \vdash_{M'} (P,e)$ for some P containing p, and thus $(E(q),va) \vdash_{M'}^* (R_1,a) \vdash_{M'} (P,e)$.

To prove the other direction, suppose that $(E(q), va) \vdash_{M'}^* (R_1, a) \vdash_{M'} (P, e)$ for some P containing p and some R_1 such that $\delta'(R_1, a) = P$. Now by the definition of δ', $\delta'(R_1, a)$ is the union of all sets $E(r_2)$, where, for some state $r_1 \in R_1$, (r_1, a, r_2) is a transition of M. Since $p \in P = \delta'(R_1, a)$, there is some particular r_2 such that $p \in E(r_2)$, and, for some $r_1 \in R_1$, (r_1, a, r_2) is a transition of M. Then $(r_2, e) \vdash_M^* (p, e)$ by the definition of $E(r_2)$. Also, by the induction hypothesis, $(q, v) \vdash_M^* (r_1, e)$ and therefore $(q, va) \vdash_M^* (r_1, a) \vdash_M (r_2, e) \vdash_M^* (p, e)$.

This completes the proof of the claim and the theorem. ■