Pushdown Automata

CENG 280

Course outline

- Preliminaries: Alphabets and languages
- Regular languages
- Context-free languages
  - Context-free grammars
  - Parse trees
  - Push-down automaton
  - Push-down automaton - context-free languages
  - Languages that are and that are not context-free, Pumping lemma
- Turing-machines

Definition

Pushdown automaton is a sextuple $M = (K, \Sigma, \Gamma, \Delta, s, F)$ where

- K is a finite set of states,
- Σ is an alphabet (input symbols)
- Γ is an alphabet (stack symbols)
- $s \in K$ is the initial state
- $F \in K$ is the set of final states, and,
- $\Delta \subset (K \times (\Sigma \cup \{e\}) \times \Gamma^\star) \times (K \times \Gamma^\star)$ is a finite transition relation.

- If $((p, a, \delta), (q, \gamma)) \in \Delta$, then when M is in state p, if it reads $a \in \Sigma$ (or if a is e without reading a symbol) and if the top of the stack is δ, it enters state q and replaces δ with γ.
- $((p, a, \delta), (q, \gamma))$ is called a transition of M.
- Since Δ is a relation, several transitions can be applicable at a point. The machine chooses non-deterministically from the applicable transitions

Example

Write a grammar G such that $L(G) = \{w \in \{0,1\}^\star \mid$ the number of $0's$ in w is different than the number of $1's\}$. Write a PDA M such that $L(M) = L(G)$. Write a derivation generating 001 and a computation accepting 001. $(s,001,e) \vdash_M (q,001,\bot) \vdash (q,01,0\bot) \vdash_M (q,1,00\bot) \vdash_M (q,e,0\bot) \vdash_M (f,e,\bot) \vdash_M (f,e,e)$

$001 \in L(M)$

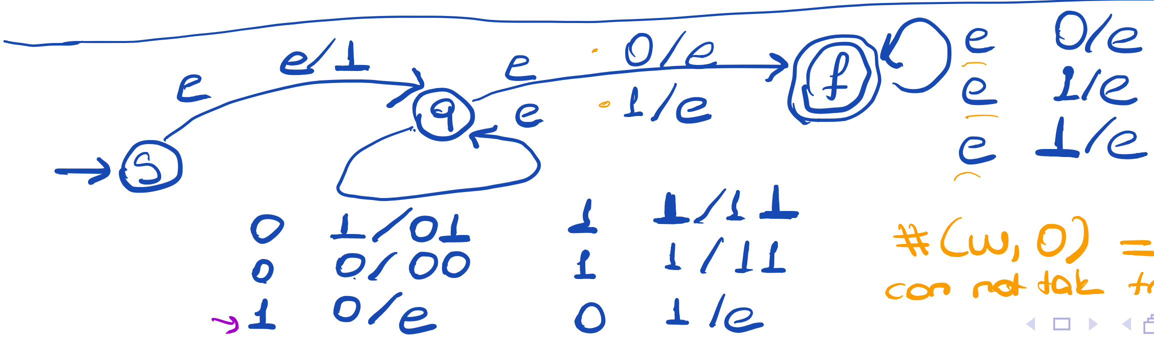
$S \to T0T \mid P1P$

$T \to 0T1T \mid 1T0T \mid TT \mid e, \mid 0T$

generates strings with equal # of 0's and 1's

$P \to 0P1P \mid 1P0P \mid PP \mid e \mid 1P$

$S \Rightarrow T0T \Rightarrow 0T \Rightarrow 00T1T \Rightarrow 001T \Rightarrow 001$



$\#(w,0) = \#(w,1)$ can not take transition to f

Pushdown automata and context-free grammars

Theorem

The class of languages accepted by pushdown automata is exactly the class of context free languages.

Pushdown automata and context-free grammars

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Lemma

Each context free language is accepted by some pushdown automaton.

L, $L = L(G)$, M, $L(M) = L(G)$

Lemma

If a language is accepted by a pushdown automaton, then it is a context-free language.

$L = L(M)$, G $L(G) = L(M)$

Pushdown automata and context-free grammars

given G write M

Lemma

Each context free language is accepted by some pushdown automaton.

Constructive proof: Given a CFG $G = (V, \Sigma, R, S)$, construct a PDA $M = (K, \Sigma, \Gamma, \Delta, s, F)$ such that $L(G) = L(M)$.

Pushdown automata and context-free grammars

Lemma

Each context free language is accepted by some pushdown automaton.

Constructive proof: Given a CFG $G = (V, \Sigma, R, S)$, construct a PDA $M = (K, \Sigma, \Gamma, \Delta, s, F)$ such that $L(G) = L(M)$.

- $K = \{s, q\}$, $F = \{q\}$
- $\Gamma = V$
- Δ :
  1. $((s, e, e), (q, S))$ (push the start symbol)
  2. $((q, e, A), (q, x))$ for each rule $A \to x \in R$ (replace the top nonterminal with a corresponding rule)
  3. $((q, a, a), (q, e))$ for each symbol $a \in \Sigma$ (pop the topmost symbol if it matches the next input symbol)

Type

pop push to stack

Type

Σ^*

Mimics the leftmost derivation of the input string.

Pushdown automata and context-free grammars

Example

Construct a PDA that accepts $L(G)$, where $G = (V, \Sigma, R, S)$,
$V = \{a, b, c, S\}$, R : $\Sigma = \{a, b, c\}$ $\rightarrow ((s, e, e), (q, S))$
$K = \{s, q\}$ $F = \{q\}$
$\Gamma = V$

$((q, a, a), (q, e))$
$((q, b, b), (q, e))$
$((q, c, c), (q, e))$

$S \rightarrow aSa$ $((q, e, S), (q, aSa))$
$S \rightarrow bSb$ $((q, e, S), (q, bSb))$
$S \rightarrow c$ $\rightarrow ((q, e, S), (q, c))$

type 2

Show the computation along "aca"

$(s, aca, e) \vdash_M (q, aca, S) \vdash_M (q, aca, aSa) \vdash_M$
$(q, ca, Sa) \vdash_M (q, ca, ca) \vdash_M (q, a, a) \vdash_M (q, e, e)$

Lemma

Each context free language is accepted by some pushdown automaton.

To complete the proof, we need to show that $L(M) = L(G)$.
Claim: let $w \in \Sigma^\star$ and $\alpha \in (V \setminus \Sigma)V^\star \cup \{e\}$, then:

$$S \overset{L}{\Rightarrow}^\star w\alpha \quad \text{iff} \quad (q, w, S) \vdash_M^\star (q, e, \alpha)$$

$w \in L(G)$ iff $w \in L(M)$

Why this claim is sufficient for language equivalence?

$\alpha = e$ $\quad S \overset{L}{\Rightarrow}^* w \quad$ iff $(q, w, S) \vdash_M^* (q, e, e)$

$w \in L(G) \quad (s, w, e) \vdash_M (q, w, S) \quad w \in L(M)$

Proof in two parts:

(1) $S \overset{L}{\Rightarrow}^\star w\alpha$ implies $(q, w, S) \vdash_M^\star (q, e, \alpha)$

(2) $(q, w, S) \vdash_M^\star (q, e, \alpha)$ implies $S \overset{L}{\Rightarrow}^\star w\alpha$

$*$ (1) $S \overset{L}{\Rightarrow}^{\star} w\alpha$ implies $(q, w, S) \vdash_M^{\star} (q, e, \alpha)$

$\Rightarrow$

By induction on the length of the derivation.

- Basis step, derivation length is 0.

$S \overset{*}{\Rightarrow} S$ $\quad w\alpha = S$ $\quad w = e$ $\quad , \; (q, e, S) \vdash_M^{*} (q, e, S)$

$\in \Sigma^*$ $\quad , \quad \alpha = S$

- **IH**: If $S \overset{L}{\Rightarrow}^{\star} w\alpha$ by a derivation of length n or less, then $(q, w, S) \vdash_M^{\star} (q, e, \alpha)$

$S \overset{L}{\Rightarrow}^{*} w\alpha$ by a derivation of length $n+1$

- **IS**: Show the implication holds for a derivation of length $n+1$.

$S = u_0 \overset{L}{\Rightarrow} u_1 \overset{L}{\Rightarrow} u_2 \overset{L}{\Rightarrow} \cdots \Rightarrow u_n \overset{L}{\Rightarrow} u_{n+1} \overset{L}{=} w\alpha$

show $(q, w, S) \vdash_M^{*} (q, e, \alpha)$

$x \in \Sigma^*$

$A \in V \setminus \Sigma$, $A \to \gamma \in R$ $\; \underbrace{x A \beta}_{u_n} \overset{L}{\Rightarrow} x \underbrace{\gamma}_{u_{n+1}} \beta$

$\beta \in V^*$

$w \in L(G) \Rightarrow$
$w \in L(M)$

(1) $S \overset{L}{\Rightarrow}^{\star} w\alpha$ implies $(q, w, S) \vdash_M^{\star} (q, e, \alpha)$

By induction on the length of the derivation. $(q, xy, S) \vdash_M^{*} (q, y, y\alpha) \vdash_M^{*} (q, e, \cancel{\alpha})$

IH: If $S \overset{L}{\Rightarrow}^{\star} w\alpha$ by a derivation of length n or less, then $(q, w, S) \vdash_M^{\star} (q, e, \alpha)$

IS: Show the implication holds for a derivation of length $n + 1$.

$\underbrace{xA\beta}_{u_n} \overset{L}{\Rightarrow} \underbrace{x\gamma\beta}_{u_{n+1}} = w\alpha \quad w \in \Sigma^{*}$

$w = xy$

$x\gamma\beta = xy\alpha$

$\gamma\beta = y\alpha$

$S \overset{L}{\Rightarrow}^{*} xA\beta$ derivation of length n

read x

$^{*}(q, x, S) \vdash_M^{*} (q, e, A\beta)$ By IH

$^{*}(q, e, A\beta) \vdash_M (q, e, \underbrace{\gamma\beta}_{y\alpha})$ By constr. of M, Type 2

$^{*}(q, y, y\alpha) \vdash_M^{*} (q, e, \alpha)$ via Type 3 transitions

$w \in L(M)$ then $w \in L(G)$

(2) $(q, w, S) \vdash_M^\star (q, e, \alpha)$ implies $S \overset{L}{\Rightarrow}^\star w\alpha$

By induction on the number of type-2 transitions. $A \to \gamma$ $((q, e, A), (q, \gamma))$

Basis step, 0 type-2s transition.

$(q, \underset{e}{w}, S) \vdash_M^\star (q, e, \underset{S}{\alpha})$ $w = e$, $\alpha = S$ $S \overset{L\star}{\Rightarrow} eS = S$

IH: If $(q, w, S) \vdash_M^\star (q, e, \alpha)$ by a computation of with n push (type 2) transitions, then $S \overset{L}{\Rightarrow}^\star w\alpha$

IS: Show the implication holds for a computation $n+1$ type-2 transitions.

$(q, w, S) \vdash_M^\star (q, e, \alpha)$ $n+1$ type 2 transitions $\gamma\beta = y\alpha$

$(q, w, S) \vdash_M^\star (q, y, A\beta) \vdash_M (q, y, \gamma\beta) \vdash_M^\star (q, e, \alpha)$

$w = xy$ read x n $(A \to \gamma \in R)$ $n+1$th T2 $y\alpha$ only T3 transition

Pushdown automata and context-free grammars

(2) $(q, w, S) \vdash_M^\star (q, e, \alpha)$ implies $S \overset{L}{\Rightarrow}^\star w\alpha$

By induction on the number of type-2 transitions.
IH: If $(q, w, S) \vdash_M^\star (q, e, \alpha)$ by a computation of with n push (type 2) transitions, then $S \overset{L}{\Rightarrow}^\star w\alpha$ **IS**: Show the implication holds for a computation $n+1$ type-2 transitions.

$\boxed{S \overset{L}{\Rightarrow}^\star x A\beta \quad \text{By IH}}$

$\rightarrow (q, xy, S) \vdash_M^\star (q, y, A\beta)$

$\rightarrow (q, x, S) \vdash_M^\star (q, e, A\beta)$ $n+2$ transitions

$S \overset{L}{\Rightarrow}^\star x A\beta \overset{L}{\Rightarrow} x\gamma\beta = w\alpha$

$A \to \gamma$

$= xy\alpha$

w

Pushdown automata and context-free grammars

$G \qquad \underline{M} \qquad L(G) = L(M)$

$w \in L(G) \Rightarrow w \in L(M),$
$w \in L(M) \Rightarrow w \in L(G)$

Theorem

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$M, \quad G \quad$ s.t. $\quad L(M) = L(G)$

Pushdown automata and context-free grammars

Lemma

If a language is accepted by a pushdown automaton, then it is a context-free language.

Proof idea: For any $M = (K, \Sigma, \Gamma, \Delta, s, F)$, there exists $G = (V, \Sigma, R, S)$ with $L(M) = L(G)$.

Pushdown automata and context-free grammars

Lemma

If a language is accepted by a pushdown automaton, then it is a context-free language.

Proof idea: For any $M = (K, \Sigma, \Gamma, \Delta, s, F)$, there exists $G = (V, \Sigma, R, S)$ with $L(M) = L(G)$.

- Convert M to M' such that M' is a simple automaton: transitions of M' satisfies the following property (if $q \neq s$)

$$((q, a, \beta), (p, \gamma)) : \qquad \beta \in \Gamma \cup \{e\}, |\gamma| \leq 2$$

- Prove that for each M, there exists a simple M' with $L(M) = L(M')$
- Construct a grammar $G = (V, \Sigma, R, S)$ from M'.
- Prove that $L(G) = L(M')$ (thus $L(G) = L(M)$)

Pushdown automata and context-free grammars

Convert M to M' such that M' is a simple automaton: transitions of M' satisfies the following property (if $q \neq s$)

$$((q, a, \beta), (p, \gamma)) : \qquad \beta \in \Gamma \cup \{e\}, |\gamma| \leq 2$$

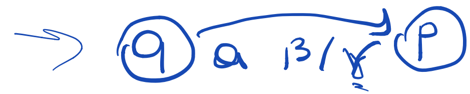
and $L(M) = L(M')$.

- $M = (K, \Sigma, \Gamma, \Delta, s, F)$, define $M' = (K', \Sigma, \Gamma \cup \{Z\}, \Delta', s', \{f'\})$
- $K' = K \cup \{s, f'\}$ $\bigcup$ ......
- $\Delta' = \Delta \cup \{((s', e, e), (s, Z))\} \cup \{((f, e, Z), (f', e) \mid f \in F\}$
- Replace each transition violating the requirement with a series of transitions, for each intermediate transition also add the intermediate states to K'.

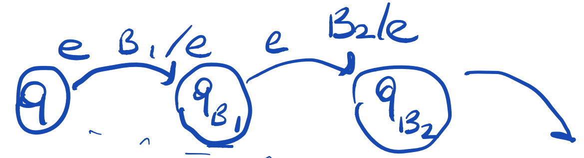
Replace the transitions violating the requirement that $|\beta| \leq 1$.

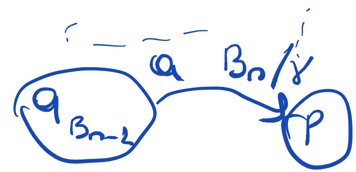
$$((q, a, \beta), (p, \gamma)) \in \Delta', \qquad \beta = B_1 \dots B_n, n > 1$$

Replace $((q, a, \beta), (p, \gamma))$ with



$$((q, e, B_1), (q_{B_1}, e))$$
$$((q_{B_1}, e, B_2), (q_{B_1 B_2}, e))$$
$$\dots$$
$$((q_{B_1 \dots B_{n-2}}, e, B_{n-1}), (q_{B_1 \dots B_{n-1}}, e))$$
$$((q_{B_1 \dots B_{n-1}}, a, B_n), (p, \gamma)),$$

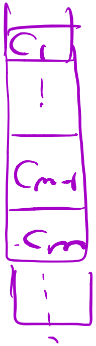




Add $q_{B_1}, \dots, q_{B_1 \dots B_{n-1}}$ to K' $((q, a, bb), (p, a))$

$((q, e, b), (q', e)), ((q', a, b), (p, a))$

Replace the transitions violating the requirement that $|\gamma| \leq 2$.



$$((q, a, \beta), (p, \gamma)) \in \Delta', \qquad \gamma = C_1 \ldots C_m, m > 2$$

Replace $((q, a, \beta), (p, \gamma))$ with

$$\begin{array}{l}
((q, a, \beta), (r_1, C_m)) \\
((r_1, e, e), (r_2, C_{m-1})) \\
\ldots \\
((r_{m-2}, e, e), (r_{m-1}, C_2)) \\
((r_{m-1}, e, e), (p, C_1))
\end{array}$$

Add $r_1, \ldots, r_{m-1}$ to K'

M is simple and $L(M) = L(M')$?

Construct $G = (V, \Sigma, R, S)$ from $M' = (K', \Sigma, \Gamma \cup \{Z\}, \Delta', s', \{f'\})$

non-terminal

- $V = \Sigma \cup \{S\} \cup \{\boxed{< q, A, p >} \mid q, p \in K', A \in \Sigma \cup \{e, Z\}\}$
- Rules R
  1. The rule $S \rightarrow < s, Z, f' >$
  2. For each $((q, a, B), (r, C)) \in \Delta'$ with $B, C \in \Gamma \cup \{e\}$ and for each $p \in K'$, add rule $< q, B, p > \rightarrow a < r, C, p >$
  3. For each $((q, a, B), (r, C_1 C_2))\Delta'$ with $B, C \in \Gamma \cup \{e\}$ and for each $p, p' \in K'$, add rule $< q, B, p > \rightarrow a < r, C_1, p' >< p', C_2, p >$
  4. For each $q' \in K'$, add $< q, e, q > \rightarrow e$

$L(G) = L(M')$

Proof: Home study.