$f : A \to B \qquad A \subseteq \mathbb{R}, \ B \subseteq \mathbb{R} \qquad$ (real, integers ...)

increasing / strictly increasing / decreasing / strictly decreasing.

$\forall x \, \forall y \quad x < y \to f(x) \leq f(y)$ increasing

$\forall x \, \forall y \quad x < y \to f(x) < f(y)$ strictly increasing

$f(x) = x + 1$

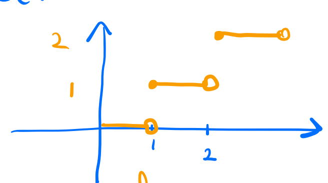
$\forall x \, \forall y \quad x < y \to f(x) \geq f(y)$ decreasing

$\forall x \, \forall y \quad x < y \to f(x) > f(y)$ strictly decreasing

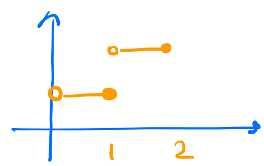
Floor / ceil

floor : $\mathbb{R} \to \mathbb{Z}$ $\qquad \lfloor x \rfloor$ the largest integer $\leq x$

ceil : $\mathbb{R} \to \mathbb{Z}$ $\qquad \lceil x \rceil$ the smallest integer $\geq x$



floor



ceil

$\lfloor 2.4 \rfloor = 2 \qquad \lceil 2.4 \rceil = 3$

1-to-1 / injective $\quad \forall x \, \forall y \quad f(x) = f(y) \to x = y$

onto / surjective

$f : A \to B$ is surjective / onto if

$\forall y \, \exists x \ f(x) = y$

codomain of f is equals to the image of A.

1-to-1 and onto / bijective

$f : A \to B$ is bijective if it is both surjective and injective.

$|A| = |B|$

To prove that

- a function is 1-to-1
  for arbitrary x, y $f(x) = f(y) \rightarrow x = y$
- a function is not 1-to-1
  find $x, y \in A$ such that
  $f(x) = f(y)$ and $x \neq y$
- a function is onto
  consider an arbitrary element $y \in B$ show that
  there exists $x \in A$ s. t. $f(x) = y$
- a function is not onto
  find a particular $y \in B$ such that $f(x) \neq y$ for
  each $x \in A$

$f(x) = x + 1$ $y \in \mathbb{Z}$ $f(y-1) = y$

$f: \mathbb{Z} \rightarrow \mathbb{Z}$ onto $\underline{\underline{y - 1 \in \mathbb{Z}}}$

$f(x) = x^2 + 1$ * not onto ,

$f: \mathbb{N} \rightarrow \mathbb{N}$

$x^2 + 1 = 0$ $x^2 = -1$ $\}$ no $x \in \mathbb{N}$ satisfies this equality

* 1-to-1
  $f(x) = f(y) \rightarrow x^2 + 1 = y^2 + 1$
  $\Rightarrow x = y$

Inverse functions & compositions

$f: A \to B$ be a bijection

$f^{-1}: B \to A$ is the inverse of f and it is defined as

$f^{-1}(b) = a$ if $f(a) = b$

ex) $f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$ is f invertable?

f^{-1} $f(2) = 4$ $f^{-1}(4) = ?$ not well defined

$f(-2) = 4$

Composition Let $g: A \to B$ and $f: B \to C$

The composition of f and g is denoted by $f \circ g$

$(f \circ g)(a) = f(g(a))$

$f \circ g : A \to C$ $(f \circ g)(A) \subseteq C$

ex $f(x) = x - 1$ $g(x) = 2x$ $f: \mathbb{Z} \to \mathbb{Z}$
$g: \mathbb{Z} \to \mathbb{Z}$

$(f \circ g)(x) = f(g(x)) = f(2x) = 2x - 1$

$(g \circ f)(x) = g(\underline{f(x)}) = g(x-1) = 2x - 2$

$f \circ g \neq g \circ f$

$(f^{-1} \circ g)(x) \stackrel{?}{=} f^{-1}(g(x)) = f^{-1}(2x) = 2x+1$

$f^{-1}(x) = x+1$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = x$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(x+1) = x$

$\underline{f : A \to B}$ $\quad$ • $\underline{f^{-1} \circ f} : A \to A$

$\qquad\qquad\qquad\qquad I_A$

$f^{-1} : B \to A$ $\quad$ • $f \circ f^{-1} : B \to B$

$\qquad\qquad\qquad\qquad I_B$

ex $h : A \to B$, $g : B \to C$, $f : C \to D$

$(f \circ g) \circ h = f \circ (g \circ h)$

$(f \circ g) \circ h(x) = (f \circ g)(h(x))$ defn. of composition

$= \underline{f(g(h(x)))}$

$f \circ (\underline{g \circ h})(x) = f \circ (g(h(x)))$

$= \underline{f(g(h(x)))}$

→ the derivation is for an arbitrary x

ex Let $g : A \to B$, $f : B \to C$ be functions.

if $(f \circ g)$ is one-to-one, then g is one-to-one

Show that g is 1-to-1, $g(x) = g(y)$ then $x = y$

$(f \circ g)(x) = (f \circ g)(y)$ then $x = y$

Assume that g is not 1-to-1

$\exists\, a_1, a_2 \in A$ s.t. $g(a_1) = g(a_2)$ and $a_1 \neq a_2$

$\underline{f(g(a_1))} = \underline{f(g(a_2))}$ but $\underline{a_1 \neq a_2}$

Thus g is 1-to-1.

CARDINALITY of SETS

of elements in a finite set

$|\{1, 2\}| = 2$

contably infinite ⇝ the same cardinality as the set of positive integers

uncountably infinite ⇝ (1) no computer program can be written to list each element
(2) no enumeration method can count each element.

<u>Def:</u> A set S is finite with cardinality $n \in \mathbb{N}$ if there exists a bijection from S to $\{0, ..., n-1\}$

<u>Def</u> Two sets A and B have the same cardinality iff there is a bijection from A to B,

$f: A \to B$ one-to-one $|A| \leq |B|$

$f: A \to B$ onto $|B| \leq |A|$

$\Rightarrow |A| = |B|$)

(Shröder-Bernstein theorem)