Forming Particular Solution to Rec.Rel.s

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$c_0 a_n + \dots + c_k a_{n-k} = f(n)$ ← linear rec.rel. with const. coeffi.

$f(n)$ Form	Particular Sol^{n} form
① a poly^{n} $A_k n^k + A_{k-1} n^{k-1} + \dots + A_0$	a poly^{n} $B_k n^k + B_{k-1} n^{k-1} + \dots + B_0$
② β^n β: a constant	$A \cdot \beta^n$ (A: a constant)
③ $(A_k n^k + \dots + A_0) \cdot \beta^n$	$(B_k n^k + \dots + B_0)\beta^n$ if β is not a char. root
④ $(A_k n^k + \dots + A_0)\beta^n$	$n^{m}(B_k n^k + \dots + B_0)\beta^n$ if β is a char. root with multiplicity m

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e.g., $a_n + a_{n-1} = 3n2^n$

determine part. sol$^{\underline{n}}$ $a_n^{(p)}$

char eq$^{\underline{n}}$ $\alpha + 1 = 0 \Rightarrow \alpha = -1$

$a_n^{(p)} = (An+B)\,2^n$

$(An+B)2^n + (A(n-1)+B).2^{n-1} = 3n2^n$

$\left.\begin{array}{l} \frac{3}{2}A = 3 \\ -\frac{1}{2}A + \frac{3}{2}B = 0 \end{array}\right\}$ $A = 2$, $B = \frac{2}{3}$

multip. is 1

e.g., $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$

determine part. sol$^{\underline{n}}$ $a_n^{(p)}$

$\alpha^2 - 5\alpha + 6 = 0$ $(\alpha-2)(\alpha-3) = 0$

$\alpha_1 = 2$, $\alpha_2 = 3$

$a_n^{(p)} = n^1 . A . 2^n + (Bn + C)$

$A = -2$ $C = 7/4$

$B = 1/2$

Forming total solution to rec. rel.s

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given a linear rec. rel. with c.c's

1) find homog. sol^n $a_n^{(h)}$

2) find part. sol^n $a_n^{(p)}$ & determine constants

3) form $a_n = a_n^{(h)} + a_n^{(p)}$

- Then determine constants due to homog. sol^n using boundary conditions

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e.g., $a_n = a_{n-1} + (n-1)$ $\Rightarrow$ $a_n - a_{n-1} = \underbrace{(n-1)}.1^n$

$a_0 = 0$

1) $a_n^{(h)} = ?$ $\alpha - 1 = 0$ $\alpha = 1$ $a_n^{(h)} = A \cdot 1^n$

2) $a_n^{(p)} = ?$ $a_n^{(p)} = n^1(Bn+C) = n(Bn+C) = \underbrace{Bn^2 + Cn}$

$n(Bn+C) - (B\underbrace{(n-1)^2} + C(n-1)) = n - 1$

$(B(n^2-2n+1) + Cn - C)$

$B\cancel{n^2} + C\cancel{n} - B\cancel{n^2} + 2Bn - B - C\cancel{n} + C = n - 1$

$\underbrace{2Bn} - \underbrace{B + C} = \underbrace{n} - \underbrace{1}$

$2B = 1 \Rightarrow B = 1/2$

$-B + C = -1 \Rightarrow -1/2 + C = -1$

$C = -1 + 1/2 = -1/2$

3) $a_n = \underbrace{a_n^{(h)}} + \underbrace{a_n^{(p)}} = \underbrace{A} + \underbrace{\frac{1}{2}(n^2 - n)}$

B.C's $a_0 = 0 = A + \frac{1}{2}(0^2 - 0) \Rightarrow \boxed{A = 0}$

$a_n = \frac{n(n-1)}{2}$

Generating Functions

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- Ordinary generating function (OGF)
  - given an infinite seq. $(f_0\ f_1\ f_2 \cdots)$ we define gen. of disc. numeric function f as $f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots + f_n x^n + \cdots$

$$= \sum_{i=0}^{\infty} f_i x^i = F(x)$$

x is treated as a placeholder

e.g.,

$$\underbrace{(\overset{0^{th}}{0}, \overset{1^{st}}{0}, \overset{2^{nd}}{0}, \ldots, \overset{n^{th}}{0}, \ldots)}_{\text{INF. SEQ.}} \Longleftrightarrow \underbrace{0 + 0 \cdot x + 0 \cdot x^2 + \cdots}_{\text{Generating Function}} = \overset{\text{closed form exp.}}{0}$$

$$(1, 1, 0, \cdots, 0, \cdots) \Longleftrightarrow 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots = 1 + x$$

$$(1, 2, 3, 0, \cdots, 0, \cdots) \Longleftrightarrow 1 + 2x + 3x^2 + 0 \cdot x^3 + 0 \cdot x^4 + \cdots = 3x^2 + 2x + 1$$

$$(1, 1, 1, \ldots, 1, \ldots) \Longleftrightarrow 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x} \quad (\text{where } x < 1)$$

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$$1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots = y$$

$$- \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \cdots = xy$$

$$1 = y - xy \Rightarrow y = \frac{1}{1-x}$$

$$(1,-1,1,-1,\dots) \Longleftrightarrow 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x}$$

$$= (-1)^0 x^0 + (-1) x + (-1)^2 x^2 + (-1)^3 x^3 + \cdots$$

$$1 + \underbrace{(-x)}_{=y} + \underbrace{(-x)^2}_{=y^2} + \underbrace{(-x)^3}_{=y^3} + \cdots$$

$$1 + y + y^2 + y^3 + \cdots = \frac{1}{1-y} = \frac{1}{1+x}$$

$$(1,a,a^2,a^3,\dots) \Longleftrightarrow 1 + \underbrace{ax}_{=y} + \underbrace{a^2x^2}_{=y^2} + \underbrace{a^3x^3}_{=y^3} + \cdots = \frac{1}{1-ax}$$

$$1 + y + y^2 + \cdots = \frac{1}{1-y} = \frac{1}{1-ax}$$

$$(1,0,1,0,\dots) \Longleftrightarrow 1 + 0\cdot\cancel{x} + 1\cdot x^2 + 0\cdot\cancel{x^3} + 1\cdot x^4 + \cdots$$

$$1 + \underbrace{x^2}_{=y} + x^4 + x^6 + \cdots = \frac{1}{1-x^2} \quad \checkmark$$