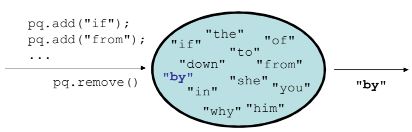
Priority Queues and Heaps

Prioritization problems

- **print jobs:** e.g.Lab printers constantly accept and complete jobs from all over the building. We want to print faculty jobs before staff before student jobs, and grad students before undergrad, etc.
- **ER scheduling:** Scheduling patients for treatment in the ER. A gunshot victim should be treated sooner than a guy with a cold, regardless of arrival time. How do we always choose the most urgent case when new patients continue to arrive?
- *key operations we want:*
  - ***add** an element (print job, patient, etc.)*
  - ***get/remove** the **most "important"** or "urgent" element*

Priority Queue ADT

- **priority queue**: A collection of ordered elements that provides fast access to the minimum (or maximum) element.
  - `add` adds in order
  - `peek` returns **minimum** or "highest priority" value
  - `remove` removes/returns **minimum** value
  - `isEmpty`, `clear`, `size` O(1)



priority queue

Question: PQ interface

What's the output? Assume lower value means higher priority.

```
PriorityQueue q;
q.add(35);
q.add(25);
cout << q.remove() << " " ;
q.add(15);
cout << q.peek() << " ";
q.add(5);
cout <<q.remove() << " ";
cout << q.remove() << " ";
```

Simple Implementations

- Array
- Sorted array
- A simple linked list
- A sorted linked list
- A binary search tree
- Binary Heap

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Array?

- Consider using an unfilled array to implement a priority queue.
  - `add`: Store it in the next available index.
  - `peek`: Loop over elements to find minimum element.
  - `remove`: Loop over elements to find min. Shift to remove.

```
queue.add(9);
queue.add(23);
queue.add(8);
queue.add(-3);
queue.add(49);
queue.add(12);.
queue.remove();
```

index	0	1	2	3	4	5	6	7	8	9
value	9	23	8	-3	49	12	0	0	0	0
size	6									

- How efficient is `add`? `peek`? `remove`?
  - O(1), O(*N*), O(*N*)
  - (`peek` must loop over the array; `remove` must shift elements)

Sorted array?

- Consider using a *sorted* array to implement a priority queue.
  - `add`: Store it in the proper index to maintain sorted order.
  - `peek`: Minimum element is in index [0].
  - `remove`: Shift elements to remove min from index [0].

```
queue.add(9);
queue.add(23);
queue.add(8);
queue.add(-3);
queue.add(49);
queue.add(12);
queue.remove();
```

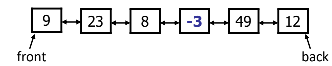
index	0	1	2	3	4	5	6	7	8	9
value	-3	8	9	12	23	49	0	0	0	0
size	6									

  - How efficient is `add`? `peek`? `remove`?
    - O(*N*), O(1), O(*N*)
    - (`add` and `remove` must shift elements)

Linked list?

- Consider using a doubly linked list to implement a priority queue.
  - `add`: Store it at the end of the linked list.
  - `peek`: Loop over elements to find minimum element.
  - `remove`: Loop over elements to find min. Unlink to remove.

```
queue.add(9);
queue.add(23);
queue.add(8);
queue.add(-3);
queue.add(49);
queue.add(12);
queue.remove();
```

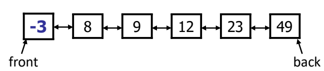


  - How efficient is `add`? `peek`? `remove`?
    - O(1), O(*N*), O(*N*)
    - (`peek` and `remove` must loop over the linked list)

Sorted linked list?

- Consider using a *sorted* linked list to implement a priority queue.
  - `add`: Store it in the proper place to maintain sorted order.
  - `peek`: Minimum element is at the front.
  - `remove`: Unlink front element to remove.

```
queue.add(9);
queue.add(23);
queue.add(8);
queue.add(-3);
queue.add(49);
queue.add(12);
queue.remove();
```

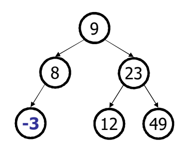


  - How efficient is `add`? `peek`? `remove`?
    - O(*N*), O(1), O(1)
    - (`add` must loop over the linked list to find the proper insertion point)

Binary search tree?

- Consider using a binary search tree to implement a PQ.
  - `add`: Store it in the proper BST L/R - ordered spot.
  - `peek`: Minimum element is at the far left edge of the tree.
  - `remove`: Unlink far left element to remove.

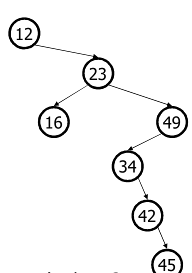
```
queue.add(9);
queue.add(23);
queue.add(8);
queue.add(-3);
queue.add(49);
queue.add(12);
queue.remove();
```



  - How efficient is `add`? `peek`? `remove`?
    - O(log *N*), O(log *N*), O(log *N*)...?
    - (good in theory, but the tree tends to become unbalanced to the right)

Unbalanced binary tree

```
queue.add(9);
queue.add(23);
queue.add(8);
queue.add(-3);
queue.add(49);
queue.add(12);
queue.remove();
queue.add(16);
queue.add(34);
queue.remove();
queue.remove();
queue.add(42);
queue.add(45);
queue.remove();
```



  - Simulate these operations. What is the tree's shape?
  - A tree that is *unbalanced* has a height close to *N* rather than log *N*, which breaks the expected runtime of many operations.

Binary Heap

- The binary heap is the classic method used to implement priority queues.
- We usually use the term **heap** to refer to binary heap.
- Heap is different from the term *heap* used in dynamic memory allocation.
- Heap has two properties:
  1. Structure property
  2. Ordering property

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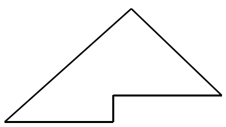
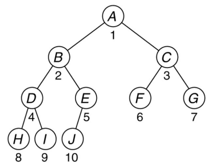
Structure Property

- A **binary heap** is a *complete binary tree*
- A **complete binary tree** is a tree where every level is full except possibly the lowest level, which must be filled from left to right





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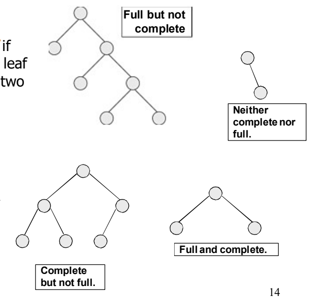
Full and Complete Binary Trees

Here are two important types of binary trees.

Definition: a binary tree T is *full* if each node is either a leaf or possesses exactly two child nodes.

Definition: a binary tree T with n levels is *complete* if all levels except possibly the last are completely full, and the last level has all its nodes to the left side.



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Heap-Order Property

- In a heap (a.k.a **min heap**), for every node X with parent P, the key in P is <u>smaller</u> than or equal to the key in X.
- Thus the minimum element is always at the root.
  - Thus we get the operation peek in constant time.

Q. Is a heap a BST? How are they related?

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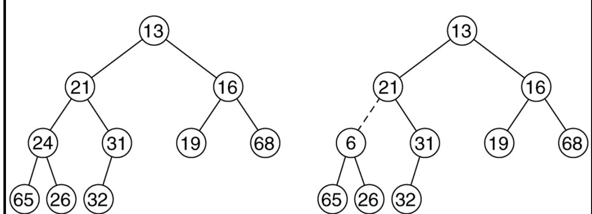
Two complete trees: (a) a heap; (b) not a heap



(a) (b)

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Max Heap

- Heap order property is slightly different :
- In a **max heap** for every node X with parent P, the key in P is <u>greater</u> than or equal to the key in X.
- <u>The default is **min heap**.</u>

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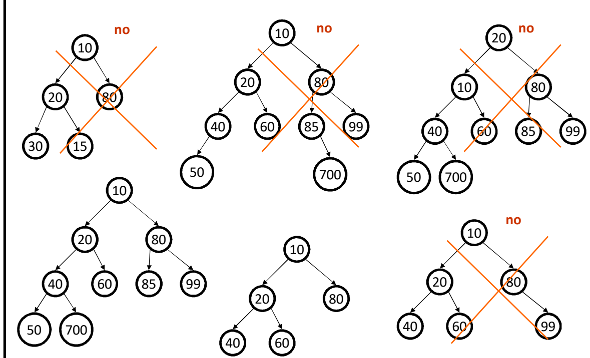
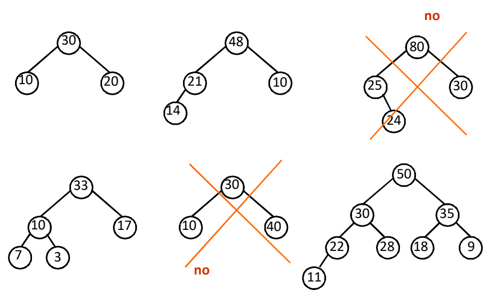
Which are min-heaps?



Which are max-heaps?



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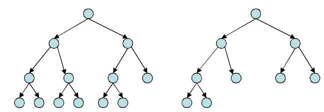
Heap height and runtime

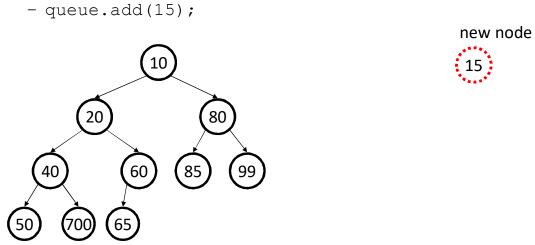
- The height of a complete tree is always $\log_2 N$.
- Because of this, if we implement a priority queue using a heap, we can provide the following runtime guarantees:
  - `add`: O(log *N*)
  - `peek`: O(1)
  - `remove`: O(log *N*)



n-node complete tree of height h:
$2^h \leq n \leq 2^{h+1} - 1$
$h = \lfloor \log n \rfloor$

The add operation

- When an element is added to a heap, where should it go?
  - Must insert a new node while maintaining heap properties.
  - `queue.add(15);`



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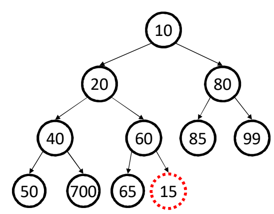
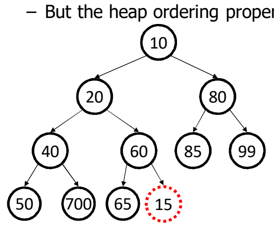


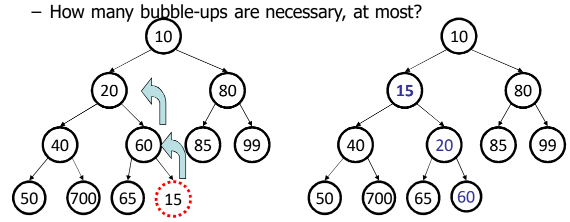
The add operation

- When an element is added to a heap, it should be initially placed as the *rightmost leaf* (to maintain the completeness property).
  - But the heap ordering property becomes broken!



"Bubbling up" a node

- **bubble up**: To restore heap ordering, the newly added element is shifted ("bubbled") up the tree until it reaches its proper place.
  - Weiss (textbook): *"percolate up"* by swapping with its parent
  - How many bubble-ups are necessary, at most?



Bubble-up exercise

- Draw the tree state of a min-heap after adding these elements:
  - 6, 50, 11, 25, 42, 20, 104, 76, 19, 55, 88, 2

Bubble-up exercise

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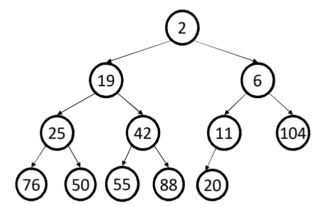
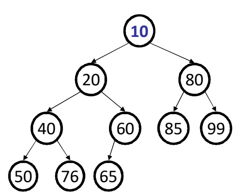
  - 6, 50, 11, 25, 42, 20, 104, 76, 19, 55, 88, 2



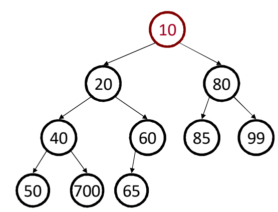
The peek operation

- A peek on a min-heap is trivial to perform.
  - because of heap properties, minimum element is always the root
  - O(1) runtime
- Peek on a max-heap would be O(1) as well (return max, not min)



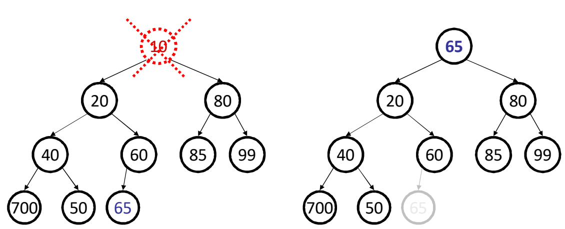
The remove operation

- When an element is removed from a heap, what should we do?
  - The root is the node to remove. How do we alter the tree?
  - `queue.remove();`



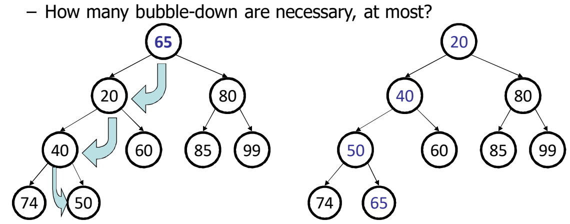
The remove operation

- When the root is removed from a heap, it should be initially replaced by the *rightmost leaf* (to maintain completeness).
  - But the heap ordering property becomes broken!



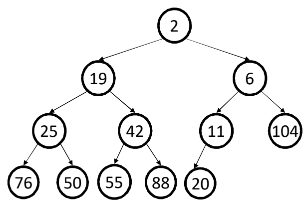
"Bubbling down" a node

- **bubble down**: To restore heap ordering, the new improper root is shifted ("bubbled") down the tree until it reaches its proper place.
  - Weiss: *"percolate down"* by swapping with its *__smaller__* child (why?)
  - How many bubble-down are necessary, at most?

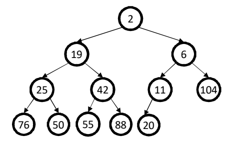


Bubble-down exercise

- Suppose we have the min-heap shown below.
- Show the state of the heap tree after remove has been called 3 times, and which elements are returned by the removal.

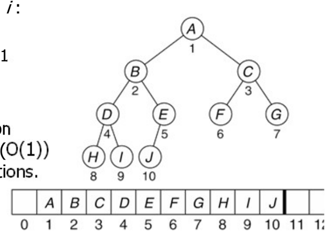


Bubble-down exercise



Array heap implementation

- Though a heap is conceptually a binary tree, since it is a *complete* tree, when implementing it we actually can "cheat" and just *use an array*!
  - index of root = 1 (leave 0 empty to simplify the math)
  - for any node *n* at index *i* :
    - index of *n*.left = $2i$
    - index of *n*.right = $2i + 1$
    - parent index of *n*?
  - This array representation is elegant and efficient (O(1)) for common tree operations.

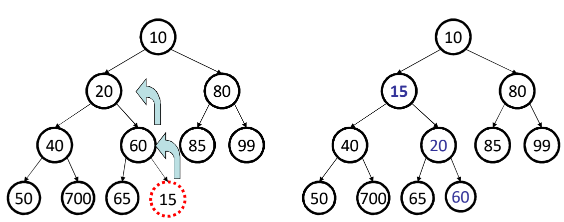


Implementing HeapPQ

```
template <class T>
class HeapPriorityQueue
{
  public:
    HeapPriorityQueue( int capacity = 100 );
    bool isEmpty( ) const;
    const T & peek( ) const;
    void add(  T & x );
    T remove( );
    void makeEmpty( );
    void print();

  private:
    int theSize;       // Number of elements in heap
    vector<T> arr;     // The heap array
    void buildHeap( );
    void bubbleDown( int hole );
};
```

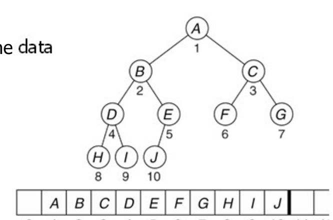
Implementing add

- Let's write the code to add an element to the heap:



Resizing a heap

- What if our array heap runs out of space?
  - What must we do here?
  - We must enlarge it.
  - We can simply copy the data into a larger array.

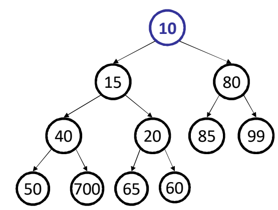


Implementing add

```
// Adds the given value to this priority queue in order.
template <class T>
void HeapPriorityQueue<T>::add(const  T & x )
{
    arr[ 0 ] = x;           // initialize sentinel
    if( theSize + 1 == arr.size( ) )
        arr.resize( arr.size( ) * 2 + 1 );

    // Bubble up
    int hole = ++theSize;
    for( ; x < arr[ hole / 2 ]; hole /= 2 )
        arr[ hole ] = arr[ hole / 2 ];
    arr[ hole ] = x;
}
```

Implementing peek

- Let's write code to retrieve the minimum element in the heap:

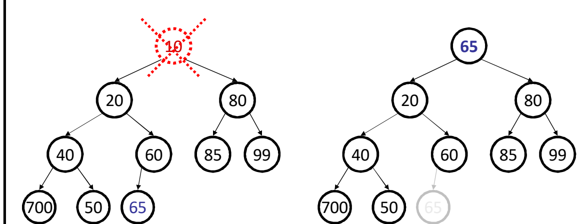


Implementing peek

```
// Returns the minimum element in this priority queue.
// precondition: queue is not empty
template <class T>
const T& HeapPriorityQueue<T>::peek() const{
    return arr[1];
}
```

Implementing remove

- Let's write code to remove the minimum element in the heap:



Implementing remove

```
// Remove the smallest item from the priority queue.
// Throw UnderflowException if empty.

template <class T>
T HeapPriorityQueue<T>::remove( )
{
    if( isEmpty( ) )
        throw UnderflowException( );
    T tmp = arr[1];
    arr[ 1 ] = arr[ theSize-- ];
    bubbleDown( 1 );
    return tmp;
}
```

Bubble down

```
// Internal method to bubble down in the heap.
// hole is the index at which the percolate begins.
template <class T>
void HeapPriorityQueue<T>::bubbleDown( int hole )
{
  int child;
  T tmp = arr[ hole ];

  for( ; hole * 2 <= theSize; hole = child )
  {
    child = hole * 2;
    if( child != theSize && arr[child + 1] < arr[child])
       child++;
    if( arr[ child ] < tmp )
       arr[ hole ] = arr[ child ];
    else
       break;
  }
  arr[ hole ] = tmp;
}
```

Summary: PQ Implementation Alternatives

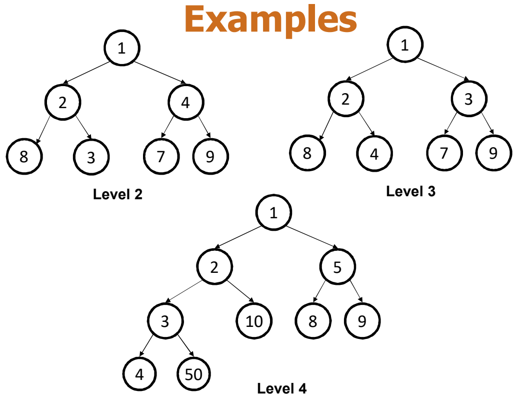
	Add()	Remove()	Peek()
Array	O(1)	O(n)	O(n)
Sorted array	O(n)	O(n)	O(1)
A simple linked list	O(1)	O(n)	O(n)
A sorted linked list	O(n)	O(1)	O(1)
A binary search tree	O(n)	O(n)	O(n)
AVL	O(log n)	O(log n)	O(log n)
Binary Heap	O(log n)	O(log n)	O(1)

Question

Which nodes in a min-heap could possibly contain the **fourth-smallest** element in the heap, assuming there are no equal elements? (root is at level 1)

a) anywhere except level 1 (root)
b) anywhere in level 2 and 3, but not 4
c) any where in level 3, or in the left half of level 4
d) anywhere in level 2, 3, or 4
e) anywhere in level 3, 4, 5

Examples



Example

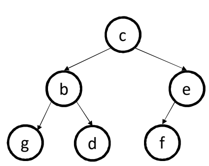
- The array representation of a min-heap is as follows: [13,28,71,29,40,84,87,45] What are the min-heap contents after adding the value 21?

[13, 21, 71, 28, 40, 84, 87, 45, 29]

Question

If we add the value "a" to this heap, what does the heap look like (array representation) afterwards?

a) c b e g d f a
b) a b c g d f e
c) c b a g d f e
d) a b c e g f c



Example

- Draw the tree for the min-heap that results from inserting 5, 8, 4, 9, 12, 6, 7, 3 in that order into an initially empty heap. Write your answer as the array representation of the resulting min-heap.

[3, 4, 5, 8, 12, 6, 7, 9]

Building a Heap

- Inserting N items into an empty heap one by one: *O(NlogN)* worst case.
- However we can build a heap out of N items in linear time *O(N)* by applying a bubble down routine to nodes in reverse level order.

`buildHeap` method

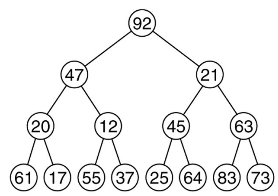
```
// Establish heap-order property from an arbitrary
// arrangement of items. Runs in linear time.

template <class Comparable>
void BinaryHeap<Comparable>::buildHeap( )
{
    for( int i = theSize / 2; i > 0; i-- )
        bubbleDown( i );
}
```

Example

- Given the following integers, build a heap in O(n) time using buildHeap method.

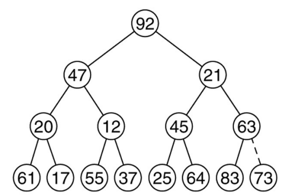
  92, 47, 21, 20, 12, 45, 63, 61, 17, 55, 37, 25, 64, 83, 73

Implementation of the linear-time buildHeap method



(a)

Initial complete tree



(b)

After bubbleDown(7)

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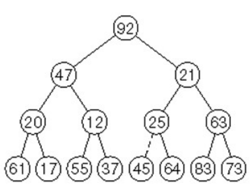
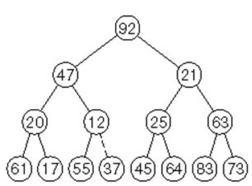
(a) After bubbleDown(6); (b) after bubbleDown(5)



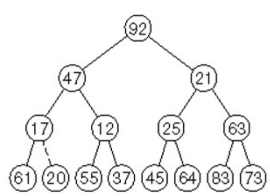
(a)



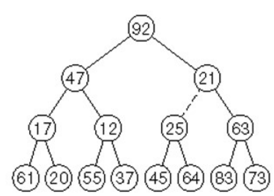
(b)

53

(a) After bubbleDown(4); (b) after bubbleDown(3)



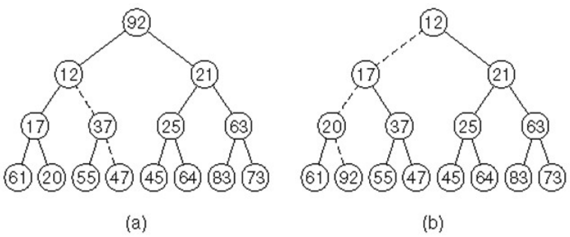
(a)



(b)

54

(a) After bubbleDown(2); (b) after bubbleDown(1) and buildHeap terminates



55

Analysis of `buildHeap`

- The linear time bound of `buildHeap`, can be shown by computing the sum of the heights of all the nodes in the heap, which is the maximum number of dashed lines.
- For the perfect binary tree of height h containing $N = 2^{h+1} - 1$ nodes, the sum of the heights of the nodes is $N - H - 1$.
- Thus it is $O(N)$.

Question

Given the following numbers, construct a min-heap using the build-heap method: 9 8 7 5 3 1

What is the array representation of the heap?

a) 1 5 3 9 7 8
b) 1 3 5 7 8 9
c) 3 5 1 9 8 7
d) 1 3 7 5 8 9

Exercise

- Find the **k**th largest element in an unsorted array. Note that it is the kth largest element in the sorted order, not the kth distinct element.
- Example:

Input: [3,2,1,5,6,4] and k = 2

Output: 5

Algorithm

```
//Using min-heap
int kthSmallest(int arr[], int n, int k) {
   // Assuming there is a constructor to build a min heap of n
   // elements from a given array: O(n) time
   HeapPriorityQueue<int> mh (arr, n);

   // Do extract min (n-k) times
   for (int i = 0; i < n-k ; i++)
      mh.remove();

   // Return root
   return mh.peek();
}   // alternatively use max-heap and delete k-1 elements
```