Algorithm Analysis

Outline

- Run time analysis
- Growth rate of functions
- Big-O notation
- Examples
- Worst-case, best-case and average case analysis of algorithms

Algorithm

- An ***algorithm*** is a set of instructions to be followed to solve a problem.
  - There can be more than one solution (more than one algorithm) to solve a given problem.
  - An algorithm can be implemented using different programming languages on different platforms.
- An algorithm must be correct. It should correctly solve the problem.
  - e.g. For sorting, this means even if (1) the input is already sorted, or (2) it contains repeated elements.
- Once we have a correct algorithm for a problem, we have to determine the efficiency of that algorithm.

Algorithmic Performance

There are *two aspects* of algorithmic performance:

- Time
  - Instructions take time.
  - How fast does the algorithm perform?
  - What affects its runtime?
- Space
  - Data structures take space
  - What kind of data structures can be used?
  - How does choice of data structure affect the runtime?

➢ We will focus on time:

- How to estimate the time required for an algorithm
- How to reduce the time required

Analysis of Algorithms

- When we analyze algorithms, we employ mathematical techniques that analyze algorithms independently of *specific implementations, computers, or data.*
- To analyze algorithms:
  - First, we start to count the number of significant operations in a particular solution to assess its efficiency.
  - Then, we will express the efficiency of algorithms using growth functions.

The Running Time of Algorithms

- Each operation in an algorithm (or a program) has a cost.
  ➔ Each operation takes a certain of time.

`count = count + 1;` ➔ take a certain amount of time, but it is constant

A sequence of operations:

```
count = count + 1;      Cost: c1
sum = sum + count;      Cost: c2
```

➔ Total Cost = $c_1 + c_2$

Run Time Analysis

Example: Simple If-Statement

	Cost	Times
`if (n < 0)`	c1	1
`  absval = -n`	c2	1
`else`		
`  absval = n;`	c3	1

Total Cost <= c1 + max(c2,c3)

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Run Time Analysis (cont.)

Example: Simple Loop

	Cost	Times
`i = 1;`	c1	1
`sum = 0;`	c2	1
`while (i <= n) {`	c3	n+1
`  i = i + 1;`	c4	n
`  sum = sum + i;`	c5	n
`}`		

Total Cost = c1 + c2 + (n+1)*c3 + n*c4 + n*c5

➔ The time required for this algorithm is proportional to n

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Exercise

Which of these loops takes constant time regardless of the value of n?

(a) for (i=n/10; i<n; i++) sum+=i;

(b) for (i=0; i<n; i+=n/10) sum++;

(c) for (i=n; i<2*n; i++) sum--;

(d) for (i=0; i<n; i+=10) sum++;

(e) for (i=0; i<n/10; i+=10) sum*=2;

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Run Time Analysis (cont.)

Example: Nested Loop

	Cost	Times
`i=1;`	c1	1
`sum = 0;`	c2	1
`while (i <= n) {`	c3	n+1
`  j=1;`	c4	n
`  while (j <= n) {`	c5	n*(n+1)
`  sum = sum + i;`	c6	n*n
`  j = j + 1;`	c7	n*n
`  }`		
`  i = i +1;`	c8	n
`}`		

Total Cost = c1 + c2 + (n+1)*c3 + n*c4 + n*(n+1)*c5+n*n*c6+n*n*c7+n*c8

➔ The time required for this algorithm is proportional to n^2

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Example: Nested Loop

Problem: given n numbers in an array A, calculate the sum of all **distinct** pairwise multiplications.

```
// A is an array of integers of size n
double sum = 0.0;
for (int i=0; i < n; i++) {
   for (int j=i; j < n; j++) {
      sum += A[i]*A[j];
   }
}
return sum;
```

$$n + (n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n+1)}{2}$$

➔ The time required for this algorithm is proportional to n^2

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Algorithm Growth Rates

- We measure an algorithm's time requirement as a function of the *problem size*.
  - Problem size depends on the application: e.g. number of elements in a list for a sorting algorithm, the number disks for towers of hanoi.
- The most important thing to learn is how quickly the algorithm's time requirement grows as a function of the problem size.
  - Algorithm A requires time proportional to n^2.
  - Algorithm B requires time proportional to n.
- An algorithm's proportional time requirement is known as ***growth rate***.
- We can compare the efficiency of two algorithms by comparing their growth rates.

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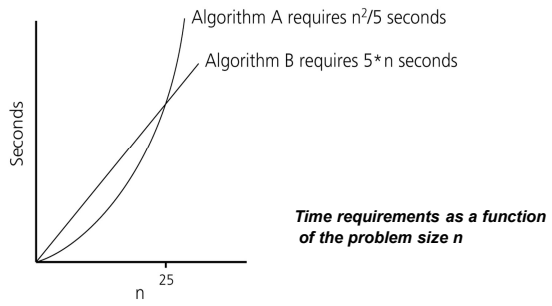
Algorithm Growth Rates (cont.)



Time requirements as a function of the problem size n

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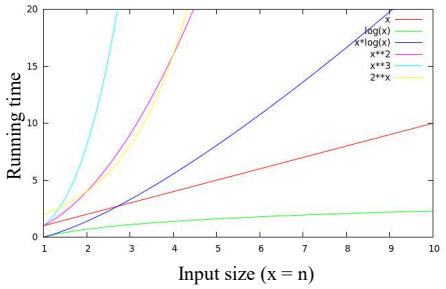
Running Times for Small Inputs



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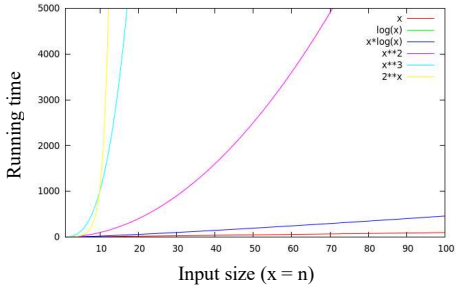
Running Times for Large Inputs



Big-O Notation

- **Big O notation** is a mathematical notation that describes the limiting behavior of a function when the argument tends towards a particular value or infinity.
- We use Big O notation to describe the computation time (**complexity)** of algorithms using algebraic terms.
- O stands for '**order**', as in 'order of magnitude'.

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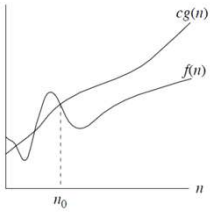
O – Notation (Formally)

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.
If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

Big-O Example

- If an algorithm requires $2n^2–3n+10$ seconds to solve a problem size n and constants c and n_0 exist such that
  $2n^2–3n+10 \le cn^2$ for all $n \ge n_0$.
- In fact, for c = 3 and $n_0 = 3$:
  $2n^2–3n+10 \le 3n^2$ for all $n \ge 3$.
- Thus, we say that the algorithm requires no more than $3n^2$ steps for $n \ge 3$, so it is $O(n^2)$
  - The fastest growing term is $2n^2$.
  - The constant 2 can be ignored

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Order of Terms

- If we graph $0.0001n^2$ against $10000n$, the linear term would be larger for a long time, but the quadratic one would eventually catch up (here at $n = 10^8$).
- In calculus we know that
- $\lim_{n\to\infty} \frac{10000n}{0.0001n^2} = \lim_{n\to\infty} \frac{10^8}{n} = 0$
- As you can see, any quadratic (with a positive leading coefficient) will eventually beat any linear. So the linear term in a quadratic function eventually does not matter.

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Order of Terms

- Consider the function $\mathbf{n^4 + 100\,n^2 + 500 = O(n^4)}$

n	n^4	$100n^2$	500	f(n)
1	1	100	500	601
10	10,000	10,000	500	20,500
100	100,000,000	1,000,000	500	101,000,500
1000	1,000,000,000,000	100,000,000	500	1,000,100,000,500

- The growth of a polynomial in n, as n increases, depends primarily on the degree (the highest order term), and not on the leading constant or the low-order terms

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Big-O Summary

- Write down the cost function (i.e. number of instructions in terms of the problem size n)
  - Specifically, focus on the loops and find out how many iterations the loops run
- Find the highest order term
- Ignore the constant scaling factor.
- Now you have a Big-O notation.

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Exercise 1

What is the complexity in Big-O notation?

Code Fragment

```
i=1;
while (i<n) {
   j = 1;
   while (j<100) {
      j=j+1;
   }
   i=i+1;
}
```

O(n)

(a) O(n)
(b) $O(n^2)$
(c) $O(\log_2 n)$
(d) $O(2^n)$
(e) O(n/2)

Exercise 2

What is the complexity in Big-O notation?

Code Fragment

```
int count = 0;
for (int i =1; i<n; i*=2)
    count++;
```

O(logn)

(a) O(n)
(b) $O(n^2)$
(c) $O(\log_2 n)$
(d) $O(2^n)$
(e) O(n/2)

Logarithmic Cost O(log n)

```
for(int i=1; i < n; i*=2) {…}
for(int i=1; i < n; i<<=1) {…}
for(int i=n; i>0; i/=3) {…}
for(int i=n; i>0; i>>=2) {…}
```

base 2

base 3

base 4

The base does not matter, because:

$O(\log_2 n) = O(\ln n)/O(\ln 2) = O(\ln n)$

Change of base -> Base e (natural log)

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Common Growth Rates

Function	Growth Rate Name
c	Constant
$log\ N$	Logarithmic
log^2N	Log-squared
N	Linear
$N\ log\ N$	Log-linear (Linearithmic)
N^2	Quadratic
N^3	Cubic
2^N	Exponential

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Growth-Rate Functions

- If an algorithm takes 1 second to run with the problem size 8, what is the time requirement (approximately) for that algorithm with the problem size 16?
- If its order is:

O(1) ➔ T(n) = 1 second
O($\log_2$n) ➔ T(n) = (1*$\log_2$16) / $\log_2$8 = 4/3 seconds
O(n) ➔ T(n) = (1*16) / 8 = 2 seconds
O(n*$\log_2$n) ➔ T(n) = (1*16*$\log_2$16) / 8*$\log_2$8 = 8/3 seconds
O(n^2) ➔ T(n) = (1*16^2) / 8^2 = 4 seconds
O(n^3) ➔ T(n) = (1*16^3) / 8^3 = 8 seconds
O(2^n) ➔ T(n) = (1*2^{16}) / 2^8 = 2^8 seconds = 256 seconds

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Exercise 3

What is the complexity in Big-O notation?

Code Fragment

```
i=1;
while (i<N*N)
    i = i+(N/2);
```

O(n)

Exercise 4

What is the complexity in Big-O notation?

Code Fragment

```
i=1;
while (i<N*N)
  i = 2*i;
```

O(logn)

Exercise 5

What is the complexity in Big-O notation?

Code Fragment

```
i=1;
while (i<N) {
   j = 1;
   while (j<N)
     j=j+1;
   i=2*i;
}
```

O(n.logn)

Exercise 6 (hard)

What's the complexity in Big-O notation?

Code Fragment

```
int i, j, count=0;
for (i=1; i < N; i*=2) {
  for (j=0; j < i; j++) {
    count ++;
  }
}
```

O(n)

Exercise 7

What is the complexity in Big-O notation?

Code Fragment

```
x = 1;
for (i = 0; i <= N-1; i++) {
   for (j = 1; j <= x; j++)
     cout << j << endl;
   x = x * 2;
}
```

O(2^n)

Outline

- Run time analysis
- Growth rate of functions
- Big-O notation
- Examples
- Worst-case, best-case and average case analysis of algorithms

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What to Analyze

- An algorithm can require different times to solve different problems of the same size.
- Eg. Searching an item in an array of n elements using sequential search.
- ➔ Cost : 1, 2, 3, … n

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What to Analyze

- ***Worst-Case Analysis*** –The maximum amount of time that an algorithm require to solve a problem of size n.
  - This gives an upper bound for the time complexity of an algorithm.
  - Normally, we try to find worst-case behavior of an algorithm.
- ***Best-Case Analysis*** –The minimum amount of time that an algorithm require to solve a problem of size n.
  - The best case behavior of an algorithm is NOT so useful.
- ***Average-Case Analysis*** –The average amount of time that an algorithm require to solve a problem of size n.
  - Sometimes, it is difficult to find the average-case behavior of an algorithm.
  - We have to look at all possible data organizations of a given size n, and their distribution probabilities of these organizations.
  - ***Worst-case analysis is more common than average-case analysis.***

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Sequential Search

```
int sequentialSearch(const int a[], int item, int n)
{
   for (int i = 0; i < n && a[i]!= item; i++);

   if (i == n)
       return -1;
   return i;
}
```

Unsuccessful Search: ➔ O(n)

Successful Search:

Best-Case: *item* is in the first location of the array ➔O(1)

Worst-Case: *item* is in the last location of the array ➔O(n)

Average-Case: The number of key comparisons 1, 2, ..., n

$$\frac{\sum_{i=1}^{n} i}{n} = \frac{(n^2+n)/2}{n}$$ ➔ O(n)

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Binary Search

```
int binarySearch(int a[], int size, int x)
{
   int low =0;
   int high = size -1;
   int mid;          // mid will be the index of
                     // target when it's found.
   while (low <= high) {
     mid = (low + high)/2;
     if (a[mid] < x)
        low = mid + 1;
     else if (a[mid] > x)
          high  = mid - 1;
     else
          return mid;
   }
   return -1;
}
```

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Binary Search – Analysis

- For an unsuccessful search:
  - The number of iterations in the loop is $\lfloor \log_2 n \rfloor + 1$
    → $O(\log_2 n)$
- For a successful search:
  - ***Best-Case:*** The number of iterations is 1. → $O(1)$
  - ***Worst-Case:*** The number of iterations is $\lfloor \log_2 n \rfloor + 1$ → $O(\log_2 n)$
  - ***Average-Case:*** The avg. # of iterations $< \log_2 n$ → $O(\log_2 n)$

```
0  1  2  3  4  5  6  7   ← an array with size 8
3  2  3  1  3  2  3  4   ← # of iterations
```

The average # of iterations = $21/8 < \log_2 8$

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How much better is $O(\log_2 n)$?

n	***$O(\log_2 n)$***
16	4
64	6
256	8
1024 (1KB)	10
16,384	14
131,072	17
262,144	18
524,288	19
1,048,576 (1MB)	20
1,073,741,824 (1GB)	30

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Analysis of Recursive Functions

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Recursive functions

```
int naivePower(int x, int n){
   if (n == 0)
      return 1;
   else
      return (x * naivePower(x, n - 1));
```

How can we write the running time?

=> Recurrence relations

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Recurrence Relation

- A recurrence relation is an equation that recursively defines a function's values in terms of earlier values
- Very useful for analyzing an algorithm's running time!

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Recurrence relations for naïvepower code

$T(0) = c_1$
$T(n) = c_2 + T(n - 1)$

If only we had an expression for $T(n - 1)$…

Recurrence relations for code

$T(0) = c_1$
$T(n) = c_2 + T(n - 1)$

Expanded:
$T(n) = c_2 + (c_2 + T(n - 2))$
$= c_2 + (c_2 + (c_2 + T(n - 3)))$
$= c_2 + (c_2 + (c_2 + (c_2 + T(n - 4))))$
…
$= k\,c_2 + T(n - k)$
After n expansions: $n\,c_2 + T(0) = n\,c_2 + c_1$
=> O(n)

Another Example

```
int betterPower(int x, int n):
   if (n == 0)
      return 1;
   else if (n == 1)
      return x;
   else
      return betterPower(x * x, n/2);
```

(assume for simplicity that n is a power of 2)
How can we write the running time?

Recurrence relation for code

$T(0) = c_1$
$T(1) = c_2$
$T(n) = c_3 + T(n/2)$
(for simplification we'll assume n is a power of 2)

If only we had an expression for T(n/2)…

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Recurrence relation for code

$T(0) = c_1$, $T(1) = c_2$, …, $T(n) = c_3 + T(n/2)$
Expanded:

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Recurrence relation for code

$T(0) = c_1$, $T(1) = c_2$, …, $T(n) = c_3 + T(n/2)$
Expanded:
$T(n) = c_3 + T(n/2)$
$= c_3 + c_3 + T(n/4)$
$= c_3 + c_3 + c_3 + T(n/8)$
…..
$= kc_3 + T(n/(2^k))$

What should k be in order for us to get down to T(1)?

$2^k = n$, so $k = \log n$

$= c_3 * \log n$

=> O(log n)

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Hanoi Towers Problem

```
void hanoi(int n, char source, char dest, char spare) {      Cost
  if (n > 0) {                                               c1
    hanoi(n-1, source, spare, dest);                         c2
    cout << "Move top disk from pole " << source             c3
         << " to pole " << dest << endl;
    hanoi(n-1, spare, dest, source);                         c4
} }
```

CENG 213 Data Structures

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Hanoi Towers

- What is the cost of `hanoi(n,'A','B','C')`?

when n=0
T(0) = c1

when n>0
T(n) = c1 + c2 + T(n-1) + c3 + c4 + T(n-1)
= 2*T(n-1) + (c1+c2+c3+c4)
= **2*T(n-1) + c** ← recurrence equation for the growth-rate function of hanoi-towers algorithm

- Now, we have to solve this recurrence equation to find the growth-rate function of hanoi-towers algorithm

Hanoi Towers (cont.)

T(n) = 2*T(n-1) + c
= 2 * (2*T(n-2)+c) + c
= 2 * (2* (2*T(n-3)+c) + c) + c
= 2^3 * T(n-3) + (2^2+2^1+2^0)*c (assuming n>2)
when substitution repeated i-1th times
= 2^i * T(n-i) + (2^{i-1}+ ... +2^1+2^0)*c
when i=n
= 2^n * T(0) + (2^{n-1}+ ... +2^1+2^0)*c
= 2^n * c1 + ($\sum_{i=0}^{n-1} 2^i$)*c

= 2^n * c1 + (2^n-1)*c = 2^n*(c1+c) – c ➔ So, the growth rate function is **O(2^n)**

Exercise

- What is the running time of recursive Fibbonacci function?

```
int Fib(int n){
   if (n == 1) return 1;
   if (n == 2) return 1;
   return Fib(n-2) + Fib(n-1);
}
```

O(2^n)

Linked Lists

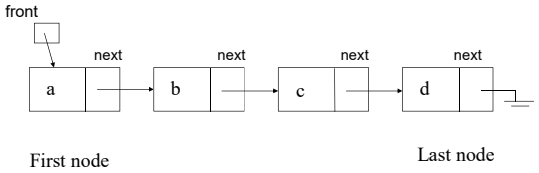
Linked List Basics

- Linked lists and arrays are similar since they both store collections of data.
- The *array's* features all follow from its strategy of allocating the memory for all its elements in one block of memory.
- *Linked lists* use an entirely different strategy: linked lists allocate memory for each element separately and only when necessary.

Linked List Basics

- Linked lists are used to store a collection of information (like arrays)
- A linked list is made of nodes that are pointing to each other
- We only know the address of the first node (head)
- Other nodes are reached by following the “next” pointers
- The last node points to NULL

Linked Lists



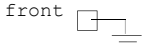
First node

Last node

Empty List

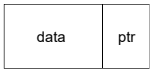
Empty Linked list is a single pointer having the value `nullptr`.

```
front = nullptr;
```



Linked List Basics

- Each node has (at least) two fields:
  - Data
  - Pointer to the next node



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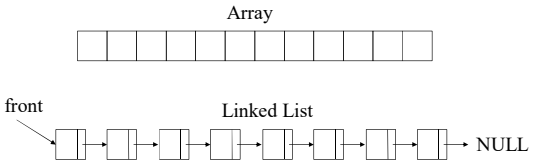
Linked List vs. Array

- In a linked list, nodes are not necessarily contiguous in memory (each node is allocated with a separate "new" call)
- Compare this to arrays which are contiguous



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Linked List vs. Array

- **Advantages of Arrays**
  - Can directly select any element
  - No memory wasted for storing pointers
- **Disadvantages of Arrays:**
  - Fixed size (cannot grow or shrink dynamically)
  - Need to shift elements to insert an element to the middle
  - Memory wasted due to unused elements
- **Advantages of Linked Lists:**
  - Dynamic size (can grow and shrink as needed)
  - No need to shift elements to insert into the middle
  - Size can exactly match the number of elements (no wasted memory)
- **Disadvantages of Linked Lists**
  - Cannot directly select any element (need to follow ptrs)
  - Extra memory usage for storing pointers

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Linked List vs. Array

- In general, we use linked lists if:
  - The number of elements that will be stored cannot be predicted at compile time
  - Elements may be inserted in the middle or deleted from the middle
  - We are less likely to make random access into the data structure (because random access is expensive for linked lists)

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Linked List Applications in CS

- Implementation of stacks, queues, graphs, hash table etc.
- Dynamic memory allocation : linked list of free blocks.
- Maintaining directory of names
- Performing arithmetic operations on long integers
- Manipulation of polynomials by storing constants in the node of linked list
- Representing sparse matrices

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Applications of linked list in real world

- *Image viewer* – Previous and next images are linked, hence can be accessed by next and previous button.
- *Previous and next page in web browser* – We can access previous and next url searched in web browser by pressing back and next button since, they are linked as linked list.
- *Music Player* – Songs in music player are linked to previous and next song. you can play songs either from starting or ending of the list.
- *Multiplayer games* - use a circular list to swap between players in a loop.

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