The Sum and Product Rule

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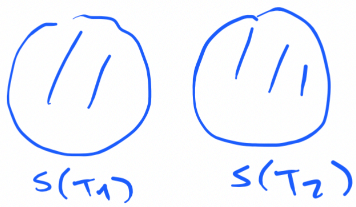
$T \begin{cases} T_1 \\ T_2 \end{cases}$

Thm (sum rule)

if each sol^n of a task T is either a sol^n of T_1 or T_2 & $S(T_1) \cap S(T_2) = \emptyset$ then

$$|S(T)| = |S(T_1)| + |S(T_2)|$$

Proof



$$|S(T)| = |S(T_1)| + |S(T_2)|$$

Thm (Principle of Inc/Exc)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof

elements in $A \cap B$ are counted twice in $|A| + |B|$

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$$(A\cap B)\cup(A\cap C)$$

$$\|$$

$$|A\cup B\cup C| = |A\cup(B\cup C)| = |A| + |B\cup C| - |A\cap(B\cup C)|$$

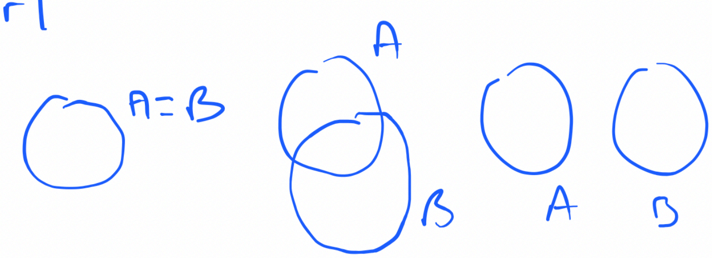
$$= |A|+|B|+|C| - |B\cap C| - \left(|A\cap B| + |A\cap C| - |A\cap B\cap C|\right)$$

$$= |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

✓

$$\bigcup_{i=1}^{r} A_i = A_1\cup A_2\cup \cdots \cup A_r = \sum_{i=1}^{r}|A_i| - \sum_{1\leq i<j\leq r}|A_i\cap A_j| + \sum_{1\leq i<j<k\leq r}|A_i\cap A_j\cap A_k|$$

$$+ \cdots + (-1)^{r-1}|A_1\cap A_2\cap\cdots\cap A_r|$$



- $|A\cup B| \leq |A|+|B|$
- $|A\cap B| \leq \min(|A|,|B|)$
- $|A\oplus B| = |A| + |B| - 2|A\cap B|$

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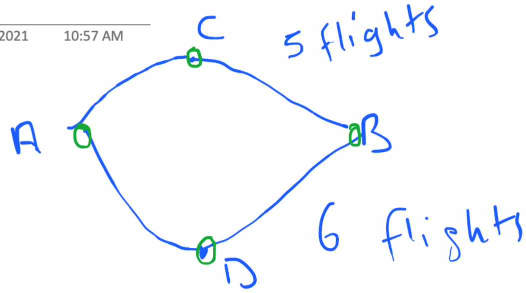
e.g.,



$5+6$

e.g., # of ways to place a pawn & a rook on a 8 x 8 chessboard.

$64 \cdot 63$

Thm (Product rule)

if each sol^n is a pair & we have n_1 & n_2 choices to fill in the coordinates then there are $n_1 \cdot n_2$ sol^n

k-tuple $\langle \overset{n_1}{\underset{1^{st}}{\_}}, \overset{n_2}{\_}, \_, \dots, \overset{n_k}{\underset{k^{th}}{\_}} \rangle$ $\quad n_1 \cdot n_2 \cdot \dots \cdot n_k$

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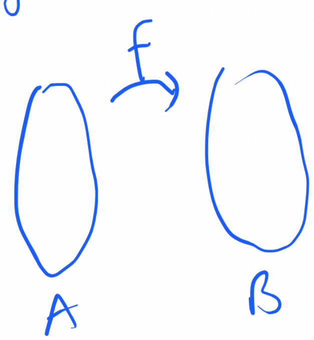
e.g., constraint: rook does not threaten the pawn

rook $\leadsto 64 \cdot (64 - \underbrace{7 - 7 - 1}_{}) $

$\underbrace{}_{48}$

e.g., # of functions from A to B where $|A| = m$, $|B| = n$



m-positions

$\langle \underset{-}{n}, \underset{-}{n}, \ldots, \underset{-}{n} \rangle$

1st obj in A

the last obj in A

$\underbrace{n \cdot n \cdot n \cdots n}_{m\ n\text{'s}} = n^m$

# of 1-to-1 functions?

1) $m > n \leadsto 0$

2) $m \leq n \leadsto$ m-tuple $\langle \overset{\overset{1st}{\downarrow}}{-}, -, -, \ldots, \overset{\downarrow m\text{th}}{-} \rangle$

$n \cdot (n-1) \cdot (n-2) \cdots (n-m+1)$

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e.g., # of passwords of length 4, 5, 6 composed of decimal digits?

P_4 P_5 P_6 sets of passwords of leng. 4, 5, 9

$$|P_4 \cup P_5 \cup P_6| = \underbrace{|P_4|+|P_5|+|P_6|}_{} = 10^4+10^5+10^6$$

↑ sum rule ↑ product rule

e.g., A password is <u>6 to 8</u> char. long. Each char is a digit, a lower case letter, or an upper case letter. There will be at least one digit. # of passwords?

P_6, P_7, P_8

$$|P_6|+|P_7| \cup |P_8| = |P_6 \cup P_7 \cup P_8|$$

$|P_6| = 62^6 - 52^6$ ← # of invalid passwords

↑ # of strings of length 6

$|P_7| = 62^7 - 52^7$

$|P_8| = 62^8 - 52^8$

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Permutations

A perm. is an ordered k-tuple of distinct objects (k-permutation)

e.g., $A = \{1, 2, 3\}$

3-perm's: $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$

↳ 6 tuples

2-perm's: $(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)$

↳ 6 tuples

Thm # of r-perm's of a set with n objects

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof

product rule ⇝ $(\underset{n}{-}, \underset{n-1}{-}, \underset{n-2}{-}, \ldots, \underset{n-r+1}{-})$ r positions $\rightarrow \frac{n!}{(n-r)!}$ $\square$

e.g., # of ways to award 3 medals to 20 players

$P(20,3) = 20 \cdot 19 \cdot 18$

Combinations

r-comb. of a set is a subset with r elements

e.g., $A = \{1,2,3\}$

2-comb's : $\{1,2\}, \{1,3\}, \{2,3\}$

↑ 2-subsets

3-comb : $\{\underline{1}, \underline{2}, \underline{3}\} = A$

Thm # of r-comb's of a set with n objects

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

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Proof

$$P(n,r) = \underbrace{C(n,r)}_{\text{choosing } r\text{-comb's } (r\text{-subset})} \cdot P(r,r)$$

counting r-perm's

THEN

ordering the subsets of r elements chosen!

Thm $C(n,k) = C(n,n-k) \quad 0 \leq k \leq n$

Proof

- each selection of k objects has a unique complement
- $C(n,k) = \dfrac{n!}{(n-k)! \cdot k!} = \dfrac{n!}{(n-(n-k))! \cdot (n-k)!} = C(n,n-k)$

$\square$

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Thm (PASCAL'S IDENTITY)

$$C(n+1, k) = C(n, k-1) + C(n, k)$$

Proof

Let A be a set with $n+1$ objects $|A| = n+1$

$$A = \{a_1, a_2, a_3, \ldots, a_{n+1}\}$$

Pick up an element x in A $\quad B = A - \{x\}$

Any k-comb / k-subset S of A:

\# of subsets:

i) $x \in S \rightarrow S = \{x, \underbrace{-, -, -, \ldots, -}_{k-1 \text{ obj's}}\} \leftarrow C(n, k-1)$

$+$

ii) $x \notin S \rightarrow S = \{\underbrace{-, -, \ldots, -}_{k \text{ objects}}\} \leftarrow C(n, k)$

Mutually excl.

$+$

$C(n+1, k) \rightsquigarrow \boxed{C(n, k-1) + C(n, k)}$

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Thm $\sum_{k=0}^{n} C(n,k) = 2^n$

Proof

$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

\# of 0-subsets

\# of 1-subsets

\# of 2-subset

\# of n-subset

\# of subsets of a set with n objects

Proof revisited (using only product rule)

Thm $2^n = |P(S)|$ where $|S| = n$

$\Rightarrow$ Proof

$S = \{a_1, a_2, \ldots, a_n\}$

a binary string $x_1 x_2 \cdots x_n$ $x_i = 0$ or 1

$$\begin{array}{ccccccccc} & & & & 1 & & & & \leftarrow n=0 \\ & & & 1 & & 1 & & & \leftarrow n=1 \\ & & 1 & & 2 & & 1 & & \leftarrow n=2 \\ & 1 & & 3 & & 3 & & 1 & \leftarrow n=3 \\ 1 & & 4 & & 6 & & 4 & & 1 \leftarrow n=4 \end{array}$$

C_3^1 C_3^2 $\oplus$ C_4^2

$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \cdots\cdots\ \binom{n}{n-1}\ \binom{n}{n} \leftarrow n^{th}$ row

$\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$, $\binom{n}{n-1} = n$, $\binom{n}{n} = 1$

$\underbrace{\binom{n}{k},\ C(n,k),\ C_n^k}$

alternative notations!

e.g., $\{2, 3, 4\}$

$000 \to \emptyset$

$101 \to \{2, 4\}$

$111 \to \{2, 3, 4\}$

2^n $\overset{?}{=}$

<u>Thm</u> (BINOMIAL THM)

$$(x+y)^n = C_n^0 x^n + C_n^1 x^{n-1} \cdot y + C_n^2 x^{n-2} \cdot y^2 + \cdots + C_n^n y^n$$

$$= \sum_{i=0}^{n} C_n^i \cdot x^{n-i} \cdot y^i$$

<u>Proof</u> $(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot(x+y)\cdots(x+y)}_{n \text{ terms}}$

Coeff of term $x^{n-k} \cdot y^k$ is C_n^k ?

- $(n-k)$ x's must be picked up $C(n, n-k)$
- unselected terms $(n-(n-k) = k)$ many will contribute to y's.

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e.g., $(x+y)^3 = (x+y)(x+y)(x+y)$

$$= xxx + \underbrace{xxy + xyx + yxx}_{\binom{3}{1}x^2y} + \underbrace{xyy + yxy + yyx}_{\binom{3}{2}xy^2} + yyy$$

$\binom{3}{0}x^3$ $\binom{3}{3}y^3$

Thm

$$\sum_{k=1}^{n} (-1)^{k+1} C_n^k = 1$$

Proof (using binomial thm)

$$0 = 0^n = (1-1)^n = \sum_{k=0}^{n} C_n^k \cdot 1^{n-k} \cdot (-1)^k = C_n^0 + \sum_{k=1}^{n} C_n^k \cdot (-1)^k$$

$$C_n^0 = -\sum_{k=1}^{n} C_n^k (-1)^k = \sum_{k=1}^{n} (-1)^{k+1} C_n^k \quad \square$$

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e.g., $\sum_{k=0}^{n} k\,C(n,k) = n\,2^{n-1}$ Prove this claim using math. induction (assume $n \geq 1$)

(exercise)

- BASIS $n=1$

⋮

- IND. STEP

⋮