Equivalence Relations

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Given a binary rel.^{n} R on A

R is an equ. rel. if it is

refl, symmetric & transitive

e.g., . $a = b$ on $\mathbb{Z}$ (integers)

. $a \equiv b \pmod{m}$ on $\mathbb{Z}$

. a is relative of b on the set of indv's.

e.g., Consider R on $\mathbb{Z}$ aRb where $ab \geq 0$. Is R an equ. rel.? No!

- refl. $a.a = a^2 \geq 0$ ✓ $a \in \mathbb{Z}$
- sym. $a.b \geq 0 \rightarrow b.a \geq 0$ ✓
- trans. $a > 0,\ b = 0,\ c < 0$ ← counter example

  $a.b \geq 0$ $b.c \geq 0$ but

  $a.c < 0$ ✗

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R is an equ. rel. on A

$[a]_R = \{ b \mid aRb \}$ equivalence class of a

e.g.,

- $[2]_{\text{mod } 5} = \{ \dots -13, -8, -3, 2, 7, 12, 17, \dots \}$
- $[a]_{=} = \{ a \}$

countably inf.

5 equ. classes

equ. rel. = on $\mathbb{Z}$

$A = B$

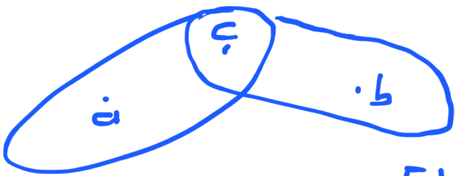
$\parallel$

$A \subseteq B \wedge B \subseteq A$

$\forall y (y \in A \to y \in B) \wedge \forall y (y \in B \to y \in A)$

Thm $[a]_R \cap [b]_R \neq \emptyset \to [a]_R = [b]_R$

Proof



[a] [b]

$\forall x [x \in [a] \leftrightarrow xRa \leftrightarrow xRc \leftrightarrow xRb \leftrightarrow x \in [b])$ □

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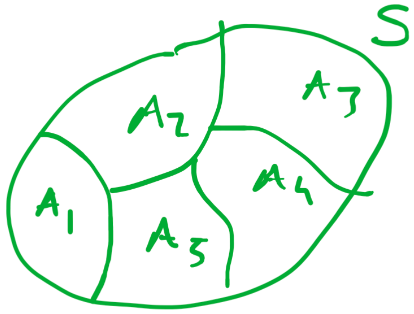
Defn (Partition of a set)

Given a set S, a partition of S is a collection $A_i \subseteq S$, $A_i \neq \emptyset$ s.t

1) $A_i \cap A_j = \emptyset \quad \forall i \neq j$
2) $\bigcup_i A_i = S$



e.g., $S = \{1,2,3,4,5,6\}$

$$\{\underbrace{\{1,2\}}_{A_1}, \underbrace{\{3\}}_{A_2}, \underbrace{\{4,5,6\}}_{A_3}\} \qquad \{\overline{12}, \overline{3}, \overline{456}\}$$

↑ a block

Relation (equ) → two objects in the same block are related

(no two objects in two diff. blocks are related)

equ. rel.

$$R = \{1,2\}\times\{1,2\} \cup \{3\}\times\{3\} \cup \{4,5,6\}\times\{4,5,6\}$$

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Thm Every partition specifies an equ. rel.

$aRb \iff a,b \in A_i$ for some i

Proof

Show that R is an equ. rel.

- refl? aRa, since $a \in A_i$ for some i ✓

- symm? $aRb \to a \in A_i \wedge b \in A_i$ for some i

  $\to b,a \in A_i \to bRa$ ✓

- transitive? $\left.\begin{array}{l} aRb \to a,b \in A_i \\ bRc \to b,c \in A_j \end{array}\right\} \Rightarrow A_i = A_j$ ✓

e.g., cong. rel. (mod 3)

$$\left.\begin{array}{l} [0] = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} \\ [1] = \{\ldots, -5, -2, 1, 4, 7, \ldots\} \\ [2] = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\} \end{array}\right\} \text{3 equ classes}$$

forms a partition of $\mathbb{Z}$

Partial Orderings

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A binary relation R on A is called partial ordering rel. if it is refl., antisym. & transitive.

aRb
(a,b) ⇝ $a \leq b$

e.g., ① $x \leq y$ on $\mathbb{Z}$ is a partial order

- ref $x \leq x$
- antisymm $x \neq y \to x \leq y$ ⇝ $\cancel{y \leq x}$
- transitive $x \leq y \wedge y \leq z \to x \leq z$ ✓

② $x \mid y$ on positive integers

$(2,8) \in R$ $2 \mid 8$
$(3,5) \notin R$ $3 \nmid 5$

③ $x \subseteq y$ on $x, y \subseteq A$

partial order? two obj's are either related or not!

$(S, \leq)$ is a total order (a linear order) if every two elements in S are comparable

($a, b \in S$ are comparable if either $a \leq b$ or $b \leq a$.)

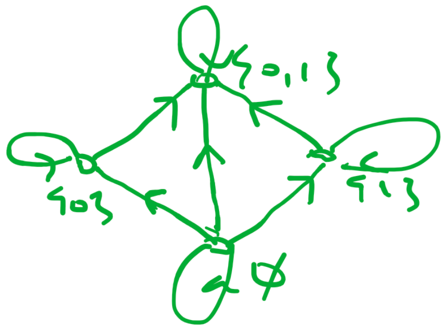
$(A, R) \longrightarrow$ partially ordered set (POSET)

the base set ↑ ↑ a partial ordering rel.

$(S, \leq)$ is <u>well-ordered</u> if it is a partially ordered & every nonempty subset of S has a least element

e.g., $(P(\{0,1\}), \subseteq)$

$\phi, \{0\}, \{1\}, \{0,1\}$



not - total order

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- $a \leq b$ on $\mathbb{Z}$ → a linear (total) order
  not well-ordered

- $a \leq b$ on $\mathbb{Z}^+$ → a linear order
  well-ordered

- $\mathbb{Z}^+ \times \mathbb{Z}^+$ with lexicographical ordering

  $\left((a_1, a_2) \leq (b_1, b_2) \iff a_1 < b_1 \lor a_1 = b_1 \land a_2 \leq b_2 \right)$

$(\mathbb{Z}^+ \times \mathbb{Z}^+, \leq)$ ⤳ Prove that it's well-ordered

a good exercise!

Hasse Diagrams

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- Compact graphical representation of POSET's

eliminate

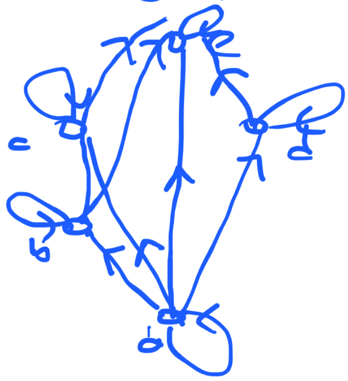
1) loops

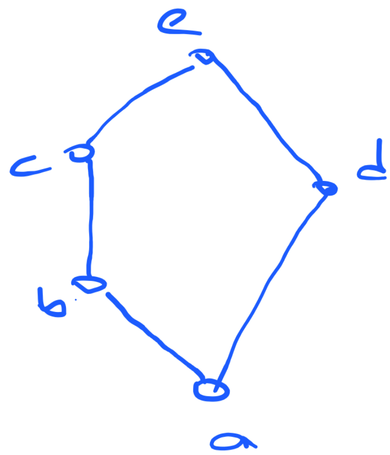
2) directed arc's that can be inferred

3) directions on arcs by introducing bottom to top ordering

e.g., Partial order rel. R on A (A, R)

$$M_R = \begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$





Hasse D.

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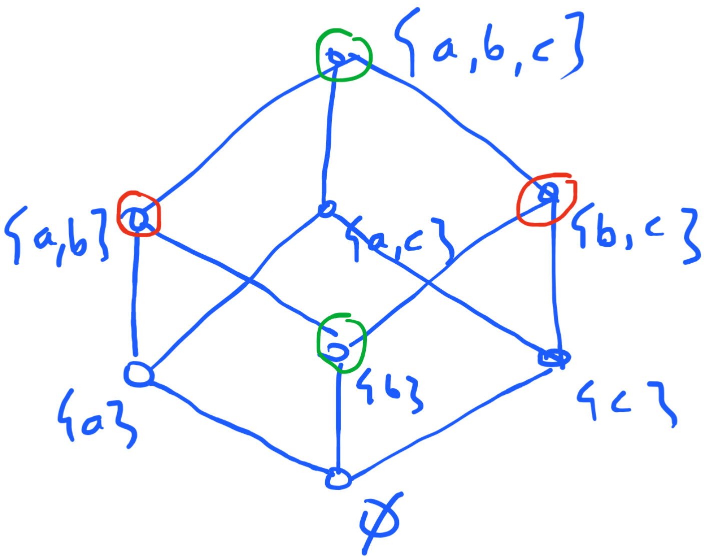
e.g., $(P(\{a,b,c\}), \subseteq)$ ← a POSET

Hasse D.



{a,b} & {b,c} are not related!

Hasse D.

Maximal element — a is maximal if no $b \neq a$ $a \leq b$

minimal element — a is minimal if no $b \neq a$ $b \leq a$

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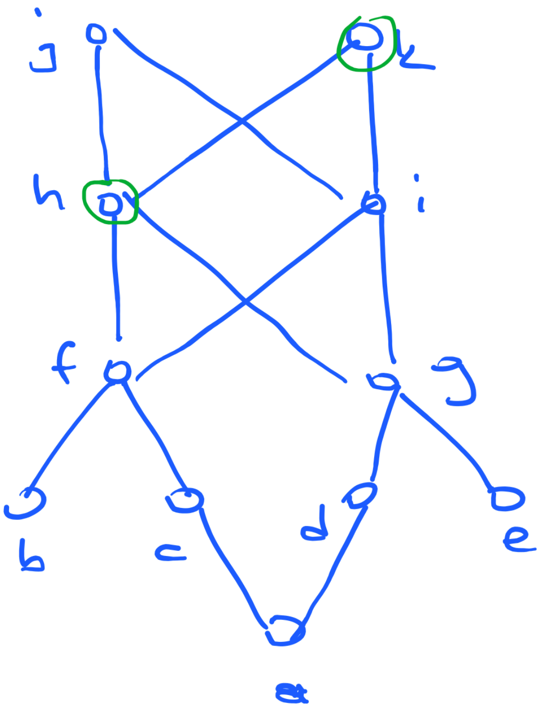

maximal elements → j, k

minimal elements → b, a, e

upper bound of x & y is z
if $x \le z$ & $y \le z$

e.g., f, g → h, i, j, k UB's

k, i → k

Least UB of x, y is z s.t
z is an UB & no z' $z' \le z$

e.g., LUB of f & g → h, i

c & d → h, i

lower bounds, greater
lower bound defined similarly
(GLB)

e.g., GLB of h & i → f, g   GLB of h & k → h

c & d → a

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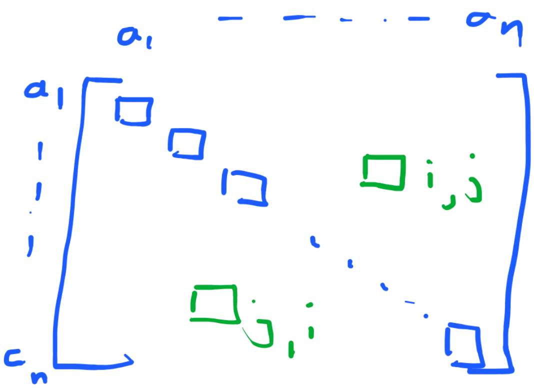
What's the # of antisym. rel^n's on a set with n elements?

$A = \{a_1, a_2, \cdots, a_n\}$

$R \subseteq A \times A$

$a \neq b$

$aRb \to b \not R a$



$$2^n \cdot \underbrace{3^{n(n-1)/2}}$$

i,j	j,i
0	0
0	1
1	1
1	0

not antisymm. (row 1 1)

Three possibilities for each pair

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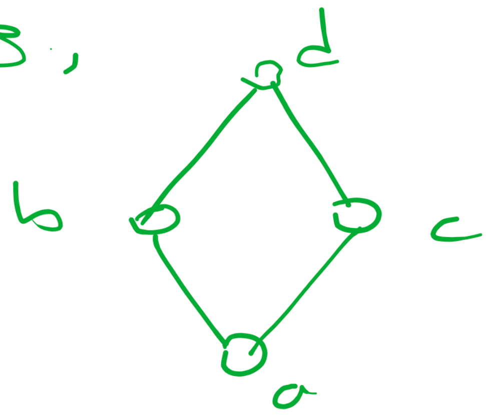
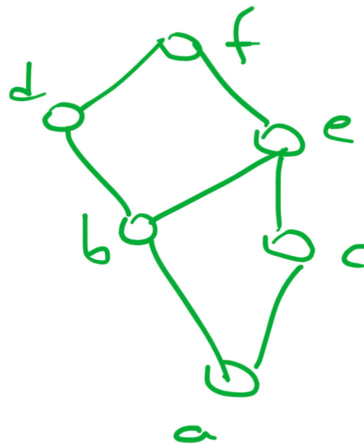
A partial order rel. is a <u>lattice</u> if for every pair of elements there is a unique LUB & a unique GLB.

e.g.,



→ a lattice



→

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$p \to q \vdash \neg p \lor q$

1.	$p \to q$	premise
2	$\neg(\neg p \lor q)$	Assumed
3	p	Assumed
4	q	$\to e$, 1,3
5	$\neg p \lor q$	$\lor i$ 4
6	$\bot$	$\neg e$, 2,5
7	$\neg p$	$\neg i$ 3-6
8	$\neg p \lor q$	$\lor i$ 7
9	$\bot$	$\neg e$, 2,8
10.	$\neg\neg(\neg p \lor q)$	$\neg i$ 2-9
11	$\neg p \lor q$	$\neg\neg e$, 10