Sets → operators
Functions → terminology
Countability

set: unordered collection of objects

- list the members $\{a, b, c\}$ $\{1, 2\}$
- set builder notation $\{ x \mid \underline{P(x)} \}$
  wff well-formed-formula

$\{ x \in \mathbb{R} \mid x \geq 0 \land x < 2 \}$ $[0, 2)$

cardinality $|\{0, 1\}| = 2$

$|\{\}| = 0$

element of $a \in \{a, b\}$

$c \notin \{a, b\}$

<u>subset</u> $A \subseteq B$ iff $\forall x \, (x \in A \rightarrow x \in B)$

$\emptyset$ $\{\}$ empty set

$\emptyset \subseteq A$ $\forall x \, (x \in \emptyset \rightarrow x \in A)$

$\{a\} \nsubseteq \{b\}$

To show that two sets are equal (A, B)

- show that $A \subseteq B$ and $B \subseteq A$
- $\forall x \, (x \in A \rightarrow x \in B) \land \forall x \, (x \in B \rightarrow x \in A)$ is a tautology.

Power sets : The set of all subsets of a set A is called the power set of A, denote it by $\underline{P}(A)$

$P(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

$P(\emptyset) = \{\emptyset\}$ $\quad |P(\emptyset)| = 1$

$|A| = n$ then $|P(A)| = 2^n$

$\{\ \}$ $\quad 2^0 \quad \{\ \}$

$\{0\}$ $\quad \{\ \}\ \{0\}$ by 2

$\{0,1\}$ $\quad \{\ \}\ \{0\}\ \{1\}\ \{0,1\}$ by 2

⋮

ex/ $\{a,b\} = \{b,a\}$ the order is not important.

The ordered n-tuple $(a_1, a_2, \dots, a_n)$

the first element is a_1

the second --- a_2

⋮

"order is important"

Cartesian product

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

$A = \{a,b\}$, $B = \{1,2,4\}$

$A \times B = \{(a,1), (a,2), (a,4), (b,1), (b,2), (b,4)\}$

set of ordered 2-tuples.

$|A \times B| = |A| \times |B|$

$A \times B \overset{?}{\cancel{=}} B \times A$ $\qquad |A \times B| = |B \times A|$

$B \times A = \{ (1, a), \ldots \}$

$A_1 \times A_2 \times \ldots \times A_n =$

$\{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i$ for each $i = 1, \ldots, n \}$

ex/ $A = \{a, b, c\}$ $\quad B = \{ \}$ $\quad C = \{2\}$

$|A \times B \times C| = |A| \times |B| \times |C|$

$= 3 \times 0 \times 2 = 0$

Basic Set operations: $\cup, \cap, \setminus, \oplus$

union $\quad A \cup B = \{ x \mid x \in A \vee x \in B \}$

ex $A = \{1, 2, a\}$ $\quad B = \{a, b\}$

$A \cup B = \{1, 2, a, b\}$

intersection $\quad A \cap B = \{ x \mid x \in A \wedge x \in B \}$

ex: $A \cap B = \{a\}$

* A and B are disjoint if $A \cap B = \emptyset$

* $|A \cup B| = |A| + |B| - |A \cap B|$

↳ the principle of "inclusion and exclusion"

difference $\quad A \setminus B = \{ x \mid x \in A \wedge x \notin B \}$

(book notation) * $A - B = \{ x \mid x \in A \wedge x \notin B \}$

ex: $A - B = \{1, 2\}$

$B - A = \{b\}$

complement $\quad U$: universal set

complement $\overline{A} = U - A$

$\overline{A} = \{x \in U \mid x \notin A\}$

ex $A = \{x \in N \mid x^2 > 7\}$

$\overline{A} = \{x \in \mathbb{N} \mid x^2 \leq 7\}$

$\overline{A} = \{0, 1, 2\}$

symmetric difference

$A \oplus B = \{x \mid (x \in A \vee x \in B) \wedge x \notin A \cap B\}$

$= (A - B) \cup (B - A)$

Set identities

universe

$A \cup U = U$ $\quad$ domination laws $\quad \left(\begin{array}{l} p \vee T = T \\ p \wedge F = F \end{array}\right)$

$A \cap \emptyset = \emptyset$

$A \cup \emptyset = A$ $\quad$ identity laws $\quad \left(\begin{array}{l} p \vee F = p \\ p \wedge T = p \end{array}\right)$

$A \cap U = A$

$A \cup A = A$
$A \cap A = A$

idempotent laws

$\left(\begin{array}{l} p \vee p = p \\ p \wedge p = p \end{array} \right)$

$\overline{(\overline{A})} = A$ complementation law $(\neg\neg p = p)$

$A \cup B = B \cup A$
$A \cap B = B \cap A$

commutative laws ?

$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

associative laws ? ?

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

distributive law

$A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

absorbtion law

$\overline{A \cap B} = \bar{A} \cup \bar{B}$
$\overline{A \cup B} = \bar{A} \cap \bar{B}$

De Morgan's

$A \cup \bar{A} = U$ complement laws

$A \cap \bar{A} = \emptyset$

Set equivalence proofs

LHS = RHS

1. definitions and logic equivalences
2. LHS $\subseteq$ RHS and RHS $\subseteq$ LHS

   $x \in LHS \rightarrow x \in RHS$ - - - - -

3. Membership table (resembles the truth table)

Example De Morgan's law

(1) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

$\overline{A \cup B} = \{ x \mid x \notin (A \cup B) \}$ Defn. of complement.

$= \{ x \mid \neg (x \in (A \cup B)) \}$ Defn. of $\notin$

$= \{ x \mid \neg (x \in A \vee x \in B) \}$ Defn. of $\cup$

$= \{ x \mid \neg(x \in A) \wedge \neg (x \in B) \}$ De Morgan's rule for logical operators

$= \{ x \mid x \notin A \wedge x \notin B \}$ Defn. of $\notin$

$= \{ x \mid x \in \bar{A} \wedge x \in \bar{B} \}$ Defn. of complement

$= \bar{A} \cap \bar{B}$ defn. of intersection.

(2) LHS $\subseteq$ RHS and RHS $\subseteq$ LHS

$\Rightarrow$ show that $x \in \overline{A \cup B}$ then $x \in \bar{A} \cap \bar{B}$

Suppose that $x \in \overline{A \cup B}$.

By defn. of complement $x \notin A \cup B$

By defn of union $\neg (x \in A \vee x \in B)$

By De. Morgans law for propositions

$$\neg(x \in A) \wedge \neg(x \in B)$$

By defn. of $\notin$ symbol $x \notin A \wedge x \notin B$

By defn. of complement $x \in \bar{A} \wedge x \in \bar{B}$

By defn. of intersection $x \in \bar{A} \cap \bar{B}$

$$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$$

$\Leftarrow$ Prove that $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Show that $x \in \bar{A} \cap \bar{B}$ then $x \; \overline{A \cup B}$

$\Rightarrow$ complete yourself.

(3) membership method

$\cup$

	A	B	$\bar{A}$	$\bar{B}$	$\bar{A} \cap \bar{B}$	$A \cup B$	$\overline{A \cup B}$
(1)	1	1	0	0	0	1	0
(2)	1	0	0	1	0	1	0
(3)	0	1	1	0	0	1	0
(4)	0	0	1	1	1	0	1

ex $A \cap (B - A) = \emptyset$

$= \{x \mid x \in A \wedge x \in (B - A)\}$ Defn. of intersection

$= \{x \mid x \in A \wedge (x \in B \wedge x \notin A)\}$ Defn. of set diff.

$= \{x \mid \underbrace{(x \in A \wedge x \notin A)}_{\perp} \wedge x \in B\}$

$= \{x \mid F\} = \emptyset$

ex * $A - B = A \cap \overline{B}$

$= \{x \mid x \in A \wedge x \notin B\}$ defn. of difference

$= \{x \mid x \in A \wedge x \in \overline{B}\}$ complement

$= A \cap \overline{B}$ intersection

ex $A - (B \cup C) = (A - B) \cap (A - C)$ *

$= A \cap \overline{(B \cup C)}$

$= A \cap (\overline{B} \cap \overline{C})$ De Morgans

$= A \cap A \cap (\overline{B} \cap \overline{C})$ idempotent laws

$= (A \cap \overline{B}) \cap (A \cap \overline{C})$ commutative

$= (A - B) \cap (A - C)$ By *

Functions

Let A, B be non-empty sets.

$f: A \to B$ (f maps A to B)

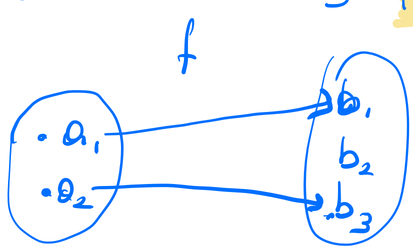
domain codomain

$$\forall x \, (x \in A \to \exists y \, (y \in B \wedge f(x) = y)) \wedge$$

$$\forall x, \forall y \, \forall z \, (x \in A \wedge y \in B \wedge z \in B \wedge f(x) = y \wedge f(x) = z \to y = z) \quad (\wedge \; ?)$$

$f(a) = b$ — b is the image of a; a is the pre-image of b

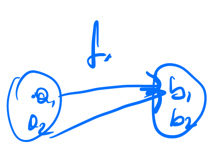
$\bigcup_{a \in A} f(a)$ } range / image of A } not necessarily the codomain B



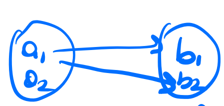
range $\{b_1, b_3\}$

$\Rightarrow$ how many different functions from A to B

$$\underset{a_1}{|B|} \quad \underset{a_2}{|B|} \quad \cdots \quad \underset{a_n}{|B|} = |B|^{|A|}$$



function



not a function

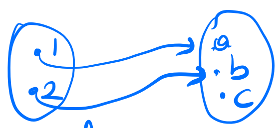
$f: A \to B, \quad C \subseteq A$

$$f(C) = \{ f(a) \mid a \in C \}$$

one - to - one / 1 - to - 1 / injective

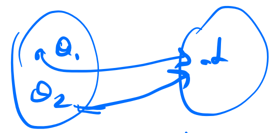
$f : A \to B$ is injective if

$\forall x \forall y \ (f(x) = f(y) \to x = y)$



f_1

injective



not injective