Natural	Deduction	for	Predicate	Calculus
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Reading: Huth and Ryan Section 2.2.3, 2.2.4, 2.3

Definition: A term t is *free for* X *in* ϕ if no variable in t becomes bound when t is substituted for X in ϕ .

Examples:

Z is free for X in p(X,Y)

Z is free for X in $\exists Y(p(X,Y))$

Z is not free for X in $\exists Z(p(X,Z))$

More examples:

f(U,V) is free for X in p(X,Y)

f(U, V) is free for X in $\forall U(q(U)) \longrightarrow \forall Y(p(X, Y))$

f(U,V) is not free for X in $p(X,Y) \longrightarrow \forall U(q(X,U))$

Eliminating Universal Quantification:

$$\frac{\forall X(\phi(X))}{\phi(t)} \forall X - e$$

where t is a term that is free for X in ϕ .

An example of what goes wrong if we don't check that t is free for X in ϕ .

Note Y is not free for X in $\exists Y (father(X, Y))$

1: $\forall X \exists Y (\mathtt{father}(X, Y))$

Premise

2: $\exists Y (father(Y,Y))$ $1, \forall X - e(incorrect)$

Remark: It is always possible to make t free for X in ϕ by renaming variables.

Example:

$$f(U,V)$$
 is not free for X in $\phi = \exists U(q(X,U))$

If we rename U to W in ϕ , we get the formula $\exists W(q(X,W))$ this is logically equivalent to ϕ .

$$f(U,V)$$
 is free for X in $\exists W(q(X,W))$

The rule $\forall X-e$ "explained"

Recall that if the universe is a finite set of objects a_1, \ldots, a_n then

$$\forall X(\phi(X)) \equiv (\phi(a_1) \land \dots \land \phi(a_n))$$

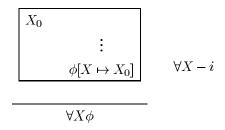
Compare:

$$\frac{\forall X(\phi(X))}{\phi(a_i)} \qquad \frac{(\phi(a_1) \land \dots \land \phi(a_n))}{\phi(a_i)}$$

A variant of a problem by Bertrand Russell:

- 1. The barber shaves all and only those men in this town who do not shave themselves.
- 2. Thus, the barber is not a man.

Introducing universal quantification



Here X_0 is a variable that does not occur anywhere outside the box.

Intuitively, the first line of the box says "let X_0 be an arbitrary object". The fact that it appears nowhere else means that we make no assumptions about X_0 whatsoever.

We then do some reasoning (the ":") and conclude $\phi[X \mapsto X_0]$. Since we assumed nothing about X_0 , we can run the same argument for any particular value for X instead of X_0 . 1. All Microsoft product has bugs.

2. Thus, if all our software is Microsoft product then all our software has bugs.

 $\neg \forall X(\phi(X)) \vdash \exists X(\neg \phi(X))$

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Introducing existential quantification

$$\frac{\phi[X \mapsto t]}{\exists X(\phi)} \quad \exists X - i$$

where t is a term that is free for X in ϕ

Eliminating existential quantification

$$\begin{array}{c|cccc} X_0 & \phi(X_0) & \operatorname{Asmptn} \\ & \vdots & & \\ & X & & \\ \hline & \chi & & \\ \hline & & \chi & & \\ \hline \end{array}$$

Here X_0 is a variable that does not occur anywhere outside the box.

Intuitively, $\exists X(\phi(X))$ tells us that some value of X satisfies $\phi(X)$, but we don't know which value.

The box says: "let's temporarily use X_0 a name for the value of X satisfying $\phi(X)$, and see what we can conclude from $\phi(X_0)$ "

We need to make sure that the name X_0 is not already being used as a name for something else, for there is no guarantee that other object satisfies $\phi(X)$.

$$\neg \forall X (\phi(X)) \vdash \exists X (\neg \phi(X))$$

Some problems, by Lewis Carroll:

- 1. Babies are illogical.
- 2. Nobody is despised who can manage a crocodile.
- 3. Illogical persons are despised.
- 4. Thus, babies cannot manage crocodiles.

- 1. Everyone who is sane can do logic.
- 2. No lunatics are fit to serve on a jury.
- 3. None of your sons can do logic.
- 4. Thus, none of your sons is fit to serve on a jury.