Reasoning about Negation

Rules for Double Negation

$$\frac{A}{\neg\neg A}\quad \neg\neg i$$

$$\frac{\neg\neg A}{A}\quad \neg\neg e$$

Contradictions: $\perp$

$\perp$ is a shorthand for $\phi \wedge \neg\phi$, where ϕ is *any* formula.

Observation: For every formula ψ, the argument $\phi \wedge \neg\phi \models \psi$ is valid.

A	$A \wedge \neg A$
T	F
F	F

(There can be no line of the truth table where $\phi \wedge \neg\phi$ is true, hence there can be no counterexample to validity.)

Rules for $\perp$

$$\frac{A \qquad \neg A}{\perp}\ \neg e$$

$$\frac{\perp}{A}\ \perp e$$

(Here A can be *any formula.*)

$\neg$-Introduction

The rule for introducing negation also makes use of subproofs:

$$\frac{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}{\neg A}\ \neg i$$

Modus Tollens

We can use $\neg i$ to prove a rule that is very useful for using negative facts:

$$\frac{A \longrightarrow B \quad \neg B}{\neg A}\ MT$$

Proof:

1 :	$A \longrightarrow B$	Premise
2 :	$\neg B$	Premise
3 :	A	Assumption
4 :	B	$1, 3, \longrightarrow e$
5 :	$\bot$	$2, 4, \neg e$
6 :	$\neg A$	$3 - 5\ \neg i$

Example

1 :	$(S \vee G) \longrightarrow P$	Premise
2 :	$P \longrightarrow A$	Premise
3 :	$\neg A$	Assumption
4 :	G	Assumption
5 :	$S \vee G$	$4, \ \vee i_2$
6 :	P	$1, 5, \ \longrightarrow e$
7 :	A	$2, 6, \ \longrightarrow e$
8 :	$\bot$	$3, 7, \ \neg e$
9 :	$\neg G$	$4 - 8, \ \neg i$
10 :	$\neg A \longrightarrow \neg G$	$3 - 9, \ \longrightarrow i$

A useful variant of $\neg i$:

$$\frac{\begin{array}{c}\neg A \\ \vdots \\ \bot\end{array}}{A}\ \text{RAA}$$

This rule is also known as *proof by contradiction* or *reductio ad absurdum* (RAA).

Law of the excluded Middle

$$\frac{}{A \vee \neg A}\ \text{LEM}$$