CENG 384 - Signals and Systems for Computer Engineers
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Homework 2

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1. (a) $\int x(t) - 5y(t)\,dt = y(t)$

   (b) By getting the differential of the equation above, we get:

   $$y'(t) + 5y(t) = x(t)$$

   Solving the characteristic equation gives us:

   $$r + 5 = 0$$

   $$r = -5$$

   The homogeneous solution is found as $y_h = Ke^{-5t}$

   We know that the system is linear. Therefore, we can find the particular solutions from:

   $$x(t) = (e^{-t} + e^{-3t})u(t) = e^{-t}u(t) + e^{-3t}u(t)$$

   $$x_1(t) = e^{-t}u(t) \text{ and } x_2(t) = e^{-3t}u(t)$$

   We need to find x_1 and x_2 After that, we need to add them.

   For the first equation:

   $$x_1(t) = e^{-t}u(t), \text{ so } x_1(t) = 0 \text{ for } t < 0 \text{ and } e^{-t} \text{ for } t > 0.$$

   Transfer function for $x_1(t)$ is $H(\lambda) = \dfrac{1}{\lambda + 5}$. $\lambda = -1$ and $H(-1) = \dfrac{1}{4}$.

   Hence, the particular solution for $x_1(t) = \dfrac{1}{4}e^{-t}u(t)$.

   We can get the particular solution for $x_2(t)$ following the same steps.

   $$x_2(t) = e^{-3t}u(t) \text{ and } H(-3) = \frac{1}{2}$$

   Therefore, particular solution for $x_2(t)$ is $\dfrac{1}{2}e^{-3t}u(t)$

   $$y(t) = y_H(t) + y_P(t) = Ke^{-5t} + \frac{1}{4}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

   Finally, we need to find K.

   $$y(0) = K + \frac{1}{4} + \frac{1}{2} = 0$$

   $$\text{K} = \frac{-3}{4}$$

   Therefore, $y(t) = \dfrac{-3}{4}e^{-5t} + \dfrac{1}{4}e^{-t}u(t) + \dfrac{1}{2}e^{-3t}u(t)$

2. (a)

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k].h[n-k] \\
&= \sum_{k=-\infty}^{\infty} (2\delta[k].\delta[n-k-1] + 2\delta[k+1].\delta[n-k-1] + \delta[k+1].2\delta[n-k+1]) \\
&= 2\sum_{k=-\infty}^{\infty} \delta[k].\delta[n-k-1] + 4\sum_{k=-\infty}^{\infty} \delta[k].\delta[n-k+1] + \sum_{k=-\infty}^{\infty} \delta[k+1].\delta[n-k-1] + 2\sum_{k=-\infty}^{\infty} \delta[k+1].\delta[n-k+1] \\
&= 2.\delta[n-1] + 4.\delta[n+1] + \delta[n] + 2.\delta[n+2]
\end{aligned}
$$

$$y[n] = 2.\delta[n-1] + \delta[n] + 4.\delta[n+1] + 2.\delta[n+2]$$

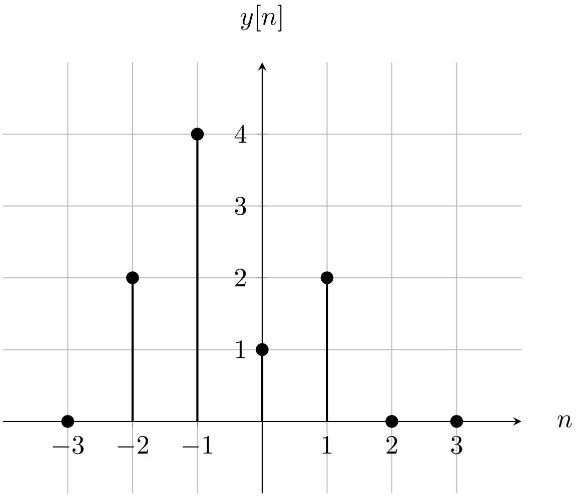

Figure 1: $y[n] = x[n] * h[n]$.

(b) $\frac{dx(t)}{dt} = \delta(t-1) + \delta(t+1)$

Since the Convolution is distributive,

$$
\begin{aligned}
y(t) &= \frac{dx(t)}{dt} * h(t) \\
&= h(t) * (\delta(t-1) + \delta(t+1)) \\
&= (h(t) * \delta(t-1)) + (h(t) * \delta(t+1)) \\
&= h(t-1) + h(t+1) \\
y(t) &= e^{-(t-1)} sin(t-1).u(t-1) + e^{-(t+1)} sin(t+1).u(t+1)
\end{aligned}
$$

3. (a) $y(t) = x(t)h(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) e^{-2(t-\tau)} u(t-\tau) d\tau$

Since $u(\tau)$ and $u(t-\tau)$ does not overlap when $t < 0$, we do not need to consider that interval.

But, we need to consider when $t > 0$, because they overlap in that interval.

So, we need to change the limits of the integral in the interval 0 to t.

$y(t) = \int_0^t e^{-t} e^{-2(t-\tau)}\, d\tau$

$= e^{-2t} \int_0^t e^{\tau}\, d\tau$

$= e^{-2t}(e^t - 1)$

$= (e^{-t} - e^{-2t})u(t)$

(b) $x(t) = u(t) - u(t-1)$

$= \delta(t-1)$

$y(t) = x(t) * h(t)$

$= \delta(t-1) * h(t)$

$= h(t-1)$

$y(t) = e^{3(t-1)}.u(t-1)$

4. (a) $y[n] - y[n-1] - y[n-2] = 0, y[0] = 1$ and $y[1] = 1$

$r^2 - r - 1 = 0$

$r_1 = \dfrac{1+\sqrt{5}}{2}$

$r_2 = \dfrac{1-\sqrt{5}}{2}$

$y[n] = A(\dfrac{1+\sqrt{5}}{2})^n + B(\dfrac{1-\sqrt{5}}{2})^n$

$y[0] = A + B = 1$

$y[1] = A\dfrac{1+\sqrt{5}}{2} + B\dfrac{1-\sqrt{5}}{2} = 1$

$y[1] - \dfrac{y[0]}{2} = \dfrac{\sqrt{5}}{2}(A - B) = \dfrac{1}{2}$

$A - B = \dfrac{1}{\sqrt{5}}$

$A + B = 1$

$A = \dfrac{1 + \dfrac{1}{\sqrt{5}}}{2} = \dfrac{\sqrt{5}+1}{2\sqrt{5}}$

$B = \dfrac{1 - \dfrac{1}{\sqrt{5}}}{2} = \dfrac{\sqrt{5}-1}{2\sqrt{5}}$

Therefore:

$y[n] = \dfrac{\sqrt{5}+1}{2\sqrt{5}}(\dfrac{1+\sqrt(5)}{2})^n + \dfrac{\sqrt{5}-1}{2\sqrt{5}}(\dfrac{1-\sqrt(5)}{2})^n$

(b) $y^(3)(t) - 6y''(t) + 13y'(t) - 10y(t) = 0, y''(0) = 3, y'(0) = \dfrac{3}{2} and, y(0) = 1.$

$r^3 - 6r^2 + 13r + 10 = 0$

$= (r-2)(r-(2+j))(r-(2-j))$

$y(t) = Ae^{2t} + Be^{(2+j)t} + Ce^{(2-j)t}$

$y(0) = A + B + C = 1$

$y'(0) = 2A + (2+j)B + (2-j)C = \dfrac{3}{2}$

$= 2(A+B+C) + j(B-C)$

$j(B-C) = \dfrac{-1}{2}$

$y''(0) = 4A + (2+j)^2B + (2-j)^2C = 3$

$= 4A + 3B + 3C + 4j(B - C)$

$A + 4j(B - C) = 0$

$A = 2$

$B + C = -1$

$B - C = \frac{j}{2}$

$B = \frac{j-2}{4} C = \frac{-j-2}{4}$

$y(t) = 2e^{2t} + \frac{j-2}{4} e^{(2+j)t} + \frac{-j-2}{4} e^{(2-j)t}$

$= e^{2t}(2 + \frac{j-2}{4}(cos(t) + jsin(t)) + \frac{-j-2}{4}(cos(t) - jsin(t)))$

$y(t) = e^{2t}[2 - \frac{1}{2}(sin(t) + 2cos(t))]$

5. (a) $y_p(t) = Kx(t) = Kcos(5t)$

K $= H(\lambda) = \frac{\sum_{k=0}^{M} b_k \lambda^k}{\sum_{k=0}^{N} a_k \lambda^k}$

According to the Euler's formula :

$cos(5t) = \frac{(e^{j5t} + e^{-j5t})}{2}$

$y_p(t) = Kcos(5t) = \frac{K.e^{j5t} + K.e^{-j5t}}{2}$

For $x_1(t) = e^{j5t}, \lambda$ is $(j5t)$, so using the formula of the Transfer function:

We have $H(j5) = \frac{j5}{(j5)^2 + 5(j5) + 6}$

For $x_1(t)$, the particular solution is $H(j5)e^{j5t}$

For $x_2(t)$, the particular solution is $H(-j5)e^{-j5t}$

For $x(t) = \frac{x_1(t) + x_2(t)}{2}$, the particular solution is:

$y_p(t) = \frac{H(j5)e^{j5t} + H(-j5)e^{-j5t}}{2}$

(b) Assume $y_h(t) = Ce^{st} => Cs^2e^{st} + 5Cse^{st} + 6Ce^{st} = \emptyset$

$y_h(t) = C(s^2 + 5s + 6)e^{st}$

So, (S = -3) or (S = -2)

$y_h(t) = C_1e^{-3t} + C_2e^{-2t}$

(c) $y(t) = y_h(t) + y_p(t)$

Use initially at rest condition is $y(0) = 0$ and $y'(0) = 0$

$y(0) = y_h(0) + y_p(0) = 0$

$y(0) = C_1e^{-3.0} + C_2e^{-2.0} + \frac{H(j.5)e^{j.5.0} + H(-j.5)e^{-j.5.0}}{2} = 0$

$C_1 + C_2 + \frac{H(5j)}{2} + \frac{H(-5j)}{2} = 0$

$H(5j) = \frac{5j}{25j - 19}$ and $H(-5j) = \frac{-5j}{-25j - 19}$

$$C_1 + C_2 = \frac{5j.(25j+19) + 5j(25j-19)}{-(25^2+19^2).2}$$

$$C_1 + C_2 = \frac{-250}{-986.2} = \frac{125}{986}$$

$$y'(0) = -3C_1e^{-3.0} + -2C_2e^{-2.0} + \frac{(5j)H(j.5)e^{j.5.0} + (-5j)H(-j.5)e^{-j.5.0}}{2} = 0$$

$$-3C_1 + -2C_2 + \frac{(5j)H(5j)}{2} + \frac{(-5j)H(-5j)}{2} = 0$$

$$(5j).H(5j) = \frac{-25}{25j-19} \quad \text{and} \quad (-5j).H(-5j) = \frac{25}{-25j-19}$$

$$3C_1 + 2C_2 = \frac{-950}{-986.2} = \frac{475}{986}$$

$$C_1 = \frac{225}{986} \quad \text{and} \quad C_2 = \frac{-100}{986}$$

$$y(t) = (\tfrac{225}{986})e^{-3t} + (\tfrac{-100}{986})e^{-2t} + \frac{H(j.5)e^{j.5.t} + H(-j.5)e^{-j.5.t}}{2}$$

6. (a) $h_0[n] - \frac{1}{2}h_0[n-1] = \delta[n] \rightarrow h_0[n] = \frac{1}{2}h_0[n-1] + \delta[n]$

If the system is initially at rest:

$h_0[n] = 0$ for all $n < 0$.

$h_0[n] = \frac{1}{2}h_0[n-1] + \delta[0] = 0 + 1 = 1$

$\delta[n] = 0$ for all $n > 0$. Thus for $n > 0$,

$h_0[n] = \frac{1}{2^n}.u[n]$

(b) $h[n] = h_0[n].h_0[n] = \sum_{k=-\infty}^{\infty} \frac{1}{2^k}.u[k].\frac{1}{2^{n-k}.u[n-k]}$

If $n < 0$ and $u[n-k]$ do not overlap, then $h[n] = 0$.

If $n \geq 0$ and $0 \geq u[n-k] \geq n$, then Limits of the Sum can be changed.

$h[n] = \sum_{k=0}^{n} \frac{1}{2^k}.\frac{1}{2^{n-k}} = \sum_{k=0}^{n} \frac{1}{2^n} = \frac{n}{2^n}$

As a result, $h[n] = \frac{n}{2^n}.u[n]$

(c) $w[n] = y[n] - \frac{y[n-1}{2}]$

Since the System is the LTI system:

$$\frac{y[n-1]}{2} - \frac{y[n-2]}{4} = \frac{w[n-1]}{2}$$

We have two equations now. When we subtract them from each other,

$$x[n] = y[n] - y[n-1] + \frac{y[n-2]}{4} = w[n] - \frac{w[n-1]}{2}$$

$$x[n] = y[n] - y[n-1] + \frac{y[n-2]}{4}$$

7.

```
import matplotlib.pyplot as plt
import numpy as np

def convolution(signal1, start1, signal2, start2):
    len_x = len(signal1)
    len_h = len(signal2)
    len_y = len_x+len_h+1
    y = np.zeros(len_y)
    for n in range(len_y):
        for k in range(len_x):
            if n - k >= 0 and n - k < len_h:
                y[n] += signal1[k] * signal2[n - k]

    return y


filename = "hw2_signal.csv"
data = np.loadtxt(filename, delimiter=",")
startIndex = int(data[0])
signalList = data[1:]
N = 20  # will change
h = []
len_x = len(signalList)
for i in range(0, len_x):
    if (0 <= i+startIndex and i+startIndex <= N-1):
        h.append(1/N)
    else:
        h.append(0)
lst = convolution(signalList, startIndex, h, startIndex)
n = np.arange(startIndex, startIndex+len(signalList)*2+1)

plt.stem(n, lst, linefmt='b-', markerfmt='bo', label="y[n]")
plt.legend()
plt.show()
```

a) The effect of convolution with $\delta[n-5]$ is shifting the signal to the right by 5.

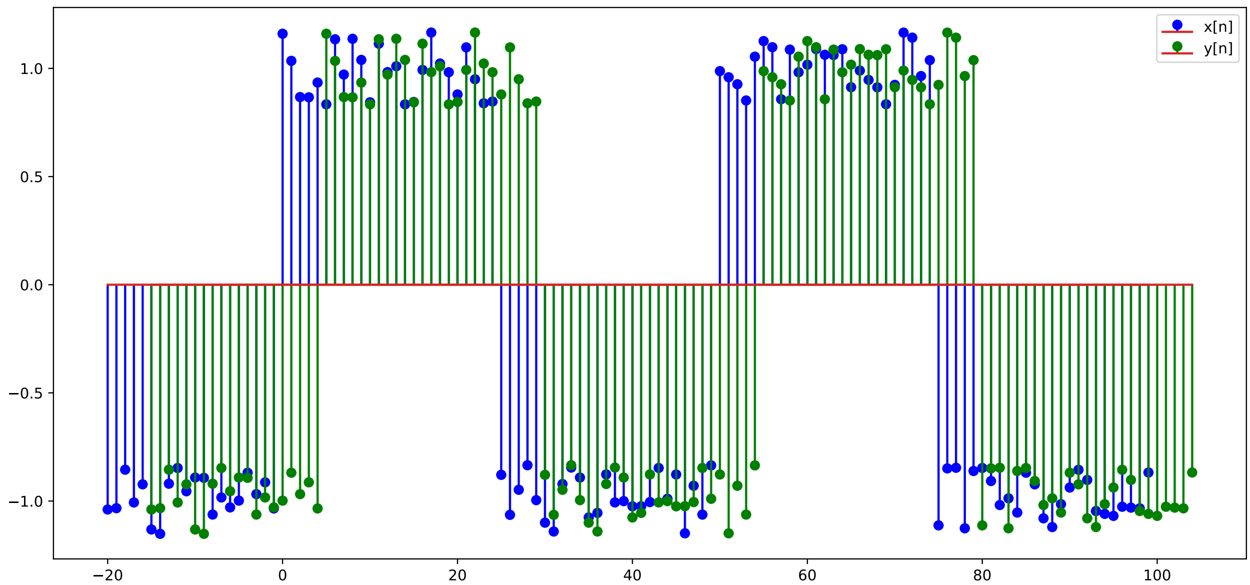

Figure 2: y[n]

b) The effect of m[n] is that it produces a smoothed version of the input signal by averaging adjacent samples. The smoothing effect is more prominent as the length of the filter N increases.

The differences between different N values are that a larger N will result in a smoother output signal with more attenuation of high-frequency components.

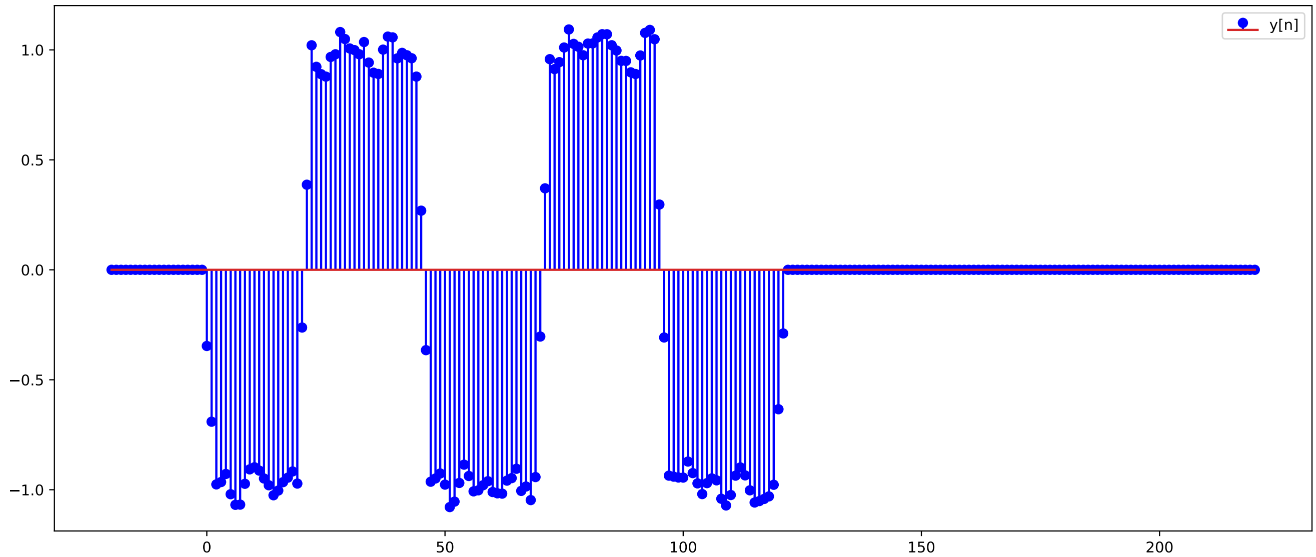

Figure 3: N=3

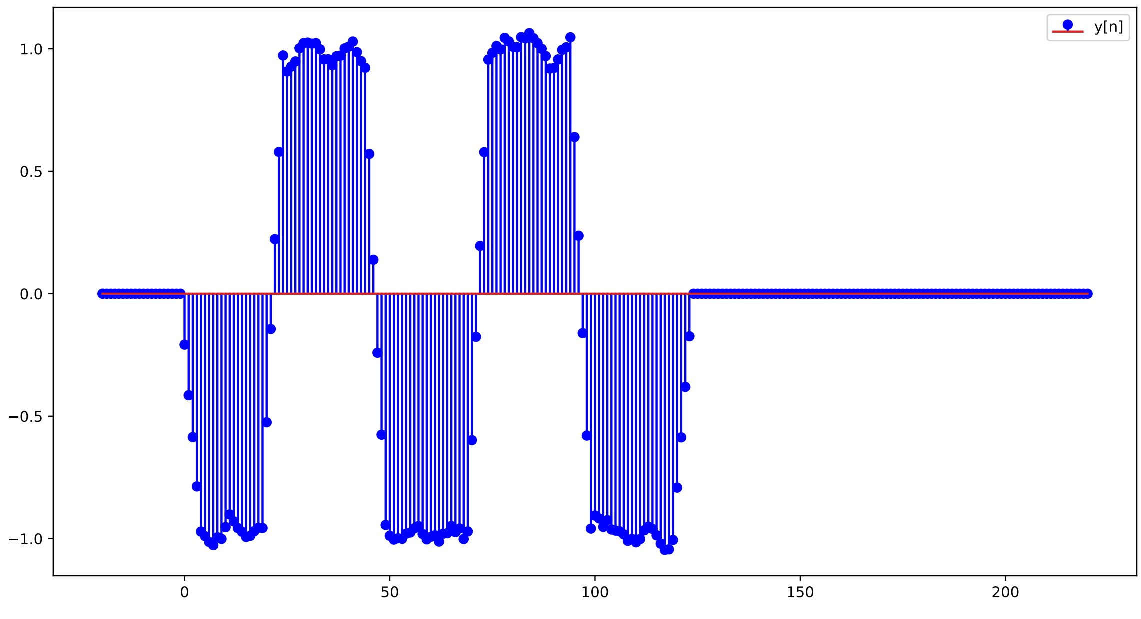

Figure 4: N=5

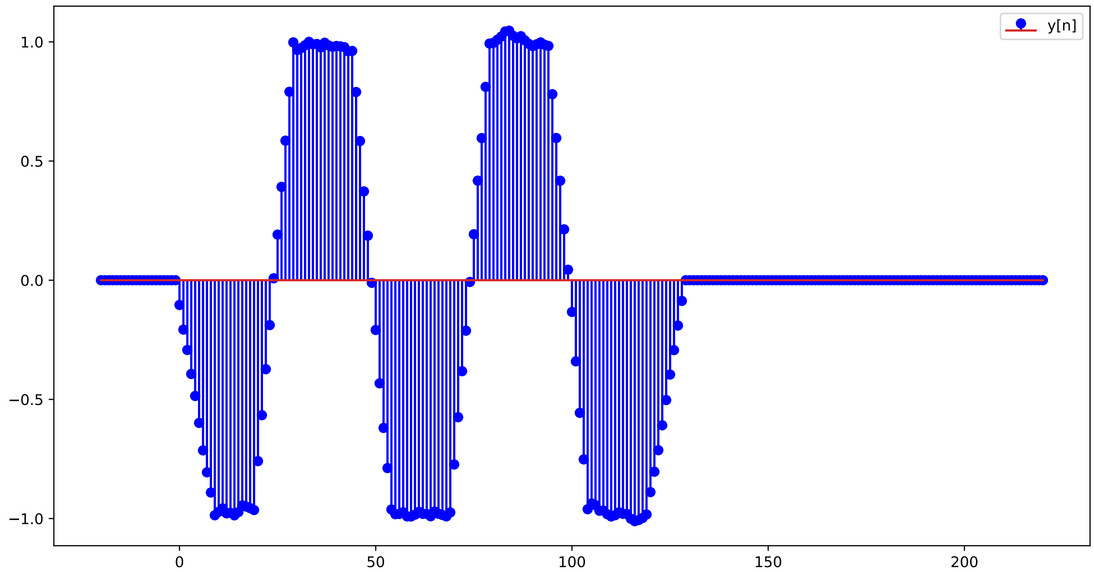

Figure 5: N=10

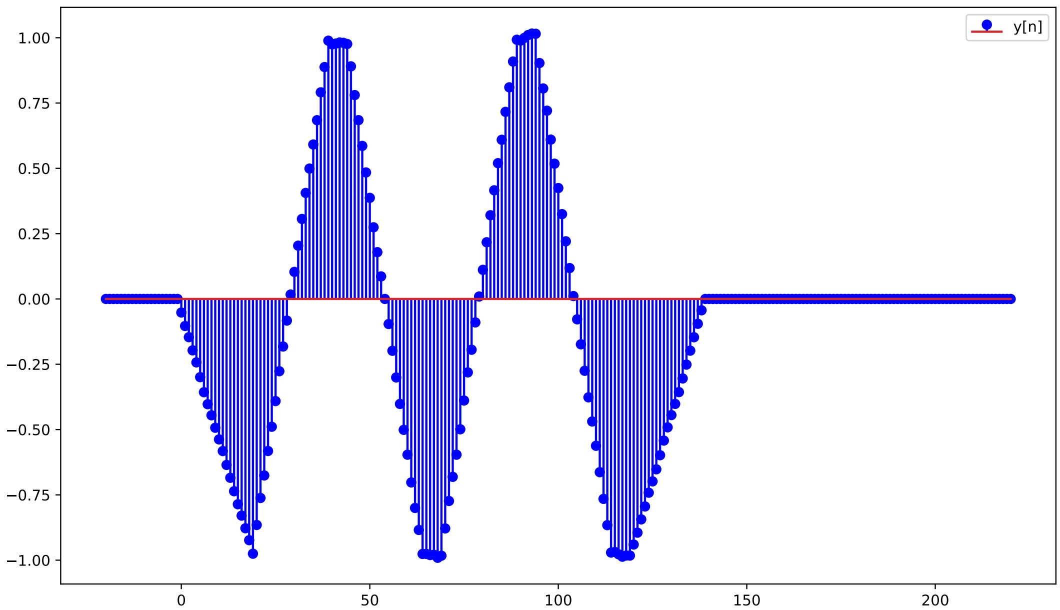

Figure 6: N=20