Ceng 111 – Fall 2020
Week 12

Iteration, Complexity

Credit: Some slides are from the “Invitation to Computer Science” book by G. M. Schneider, J. L. Gersting and some from the “Digital Design” book by M. M. Mano and M. D. Ciletti.

Today

- Finalize iteration with a final example
- Recursion vs. iteration
- Complexity

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Administrative Notes

- Live sessions schedule change
  - Tue 13:40 Session (a.k.a. common session)
  - Wed 10:40 Session
- Social session
- The labs
- Office hours: Tue 10:30
- Lab Exam 1: 23 Dec
- Final: 30 January 13:30

More examples for iteration: Counting Sort

“To count or not to count: That’s what counts”

-- a CENG111 proverb ☺

```python
def csort(A):
        # Assume that the numbers are in the range 1,...,k
        k = max(A)
        C = [0] * k

        # Count the numbers in A
        for x in A:
                C[x-1] += 1

        # Accumulate the counts in C
        for i in range(1, k):
                C[i] += C[i-1]

        # Place the numbers into correct locations
        B = [0] * len(A)
        for x in A:
                B[C[x-1]-1] = x
                C[x-1] -= 1

        return B
```

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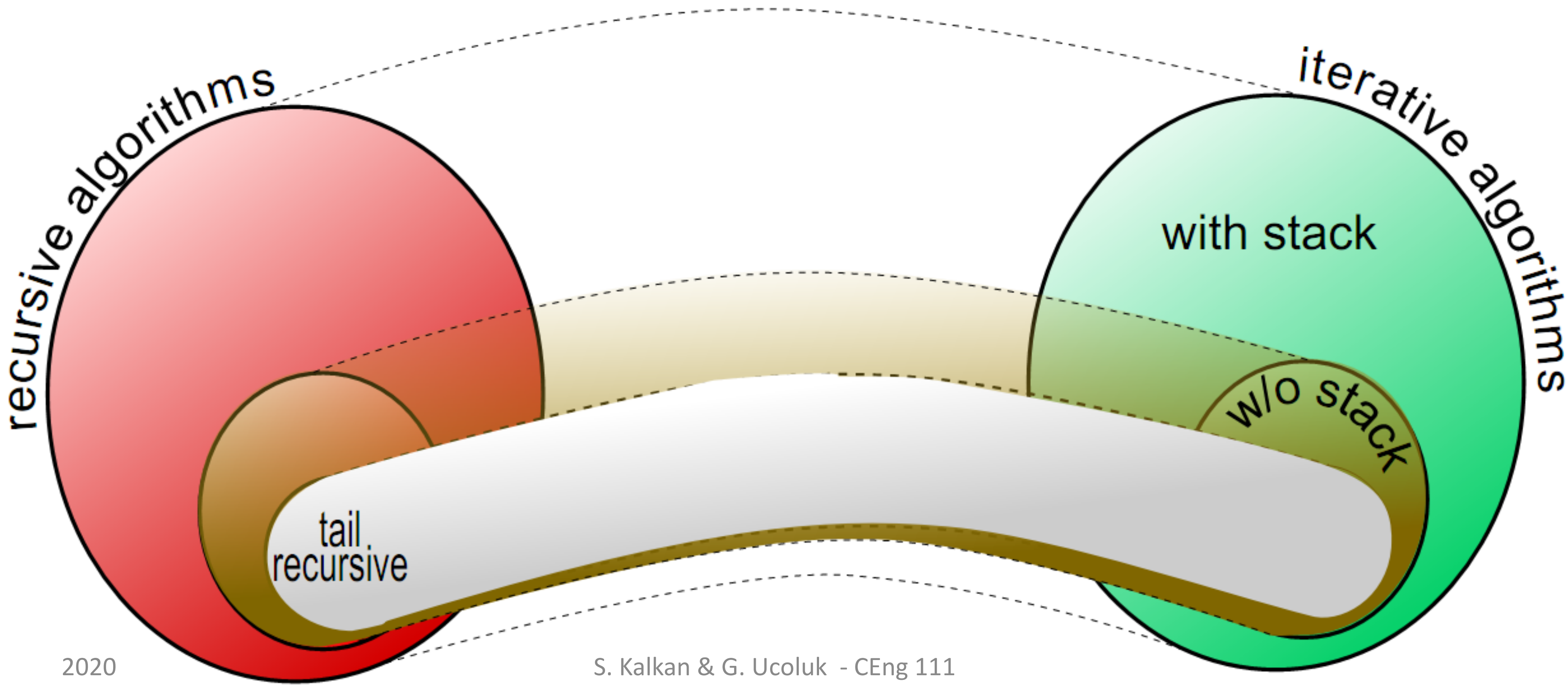
TAIL RECURSION VS. RECURSION VS. ITERATION

Recursion vs. Iteration

- Any recursive algorithm can be transformed into an iterative algorithm.
- The reverse is also true.



Recursion vs. Iteration

- Which one is better?
- Better in what way?
  - Resource-wise:
    1. Iteration without stacks
    2. Iteration with stacks
    3. Recursion
  - Implementation-wise:
    1. Recursion
    2. Iteration without stack
    3. Iteration with stack

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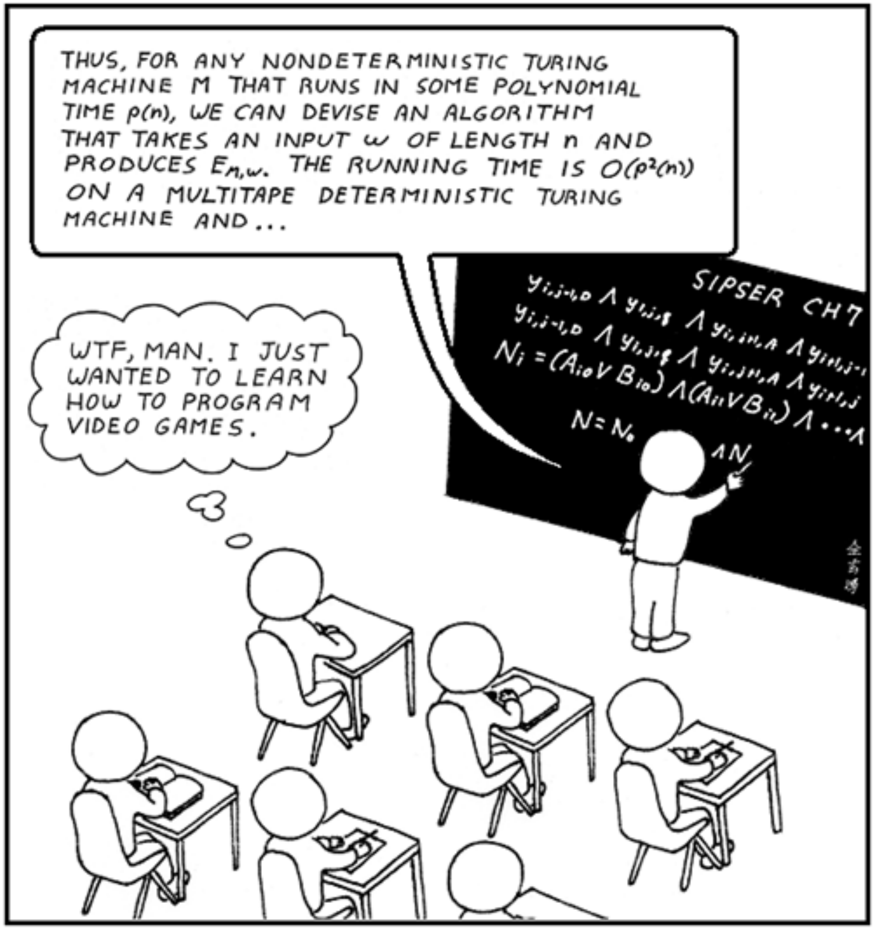
TIME ANALYSIS OF ALGORITHMS



How do you compare these two algorithms?

Bubble sort

```
def f(List):
    length = len(List)
    changed = True
    while changed:
        changed = False
        i = 0
        while i < length-1:
            if List[i] > List[i+1]:
                (List[i], List[i+1]) = (List[i+1], List[i])
                changed = True
            i += 1
    return List
```

Count sort

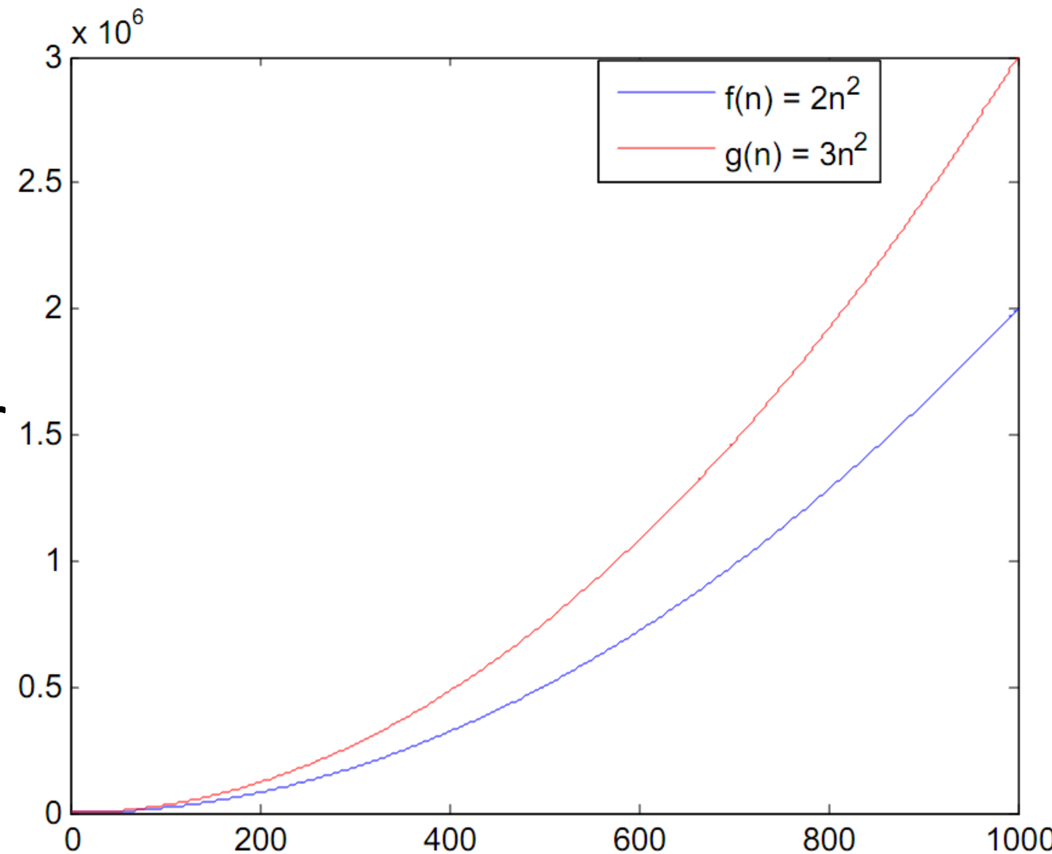
```
def csort(A):
        # Assume that the numbers are in the range 1,...,k
        k = max(A)
        C = [0] * k

        # Count the numbers in A
        for x in A:
                C[x-1] += 1

        # Accumulate the counts in C
        i = 1
        while i < k:
                C[i] += C[i-1] i += 1

        # Place the numbers into correct locations
        B = [0] * len(A)
        for x in A:
                B[C[x-1]-1] = x C[x-1] -= 1

        return B
```

Analyzing Performance of Algorithms

- How do you compare the performance of algorithms?
  1. Implement them and count the time they take?
  2. Count the number of main steps that affect the performance and depend on the size of the data.

Measuring Complexity

- There are several measures for complexity.
- A measure for complexity is basically a bound for the running time of an algorithm.
- Look at $f(n) = 2n^2$
- $f(n)$ is bounded by $3n^2$



Measuring complexity

- Consider again $f(n) = 2n^2$
- There are several functions that can bound f(n):
  1. $3n^2, 4n^2, 6n^2, \ldots$
  2. $n^3, 2n^3, 3n^3, \ldots$
  3. $n^4, 2n^4, 3n^4, \ldots$
  4. …
  5. …
- In computational complexity, we are interested in the most "suitable" bounding function.

Big-O Notation; O()

f(n) is O(g(n)) if and only if there exists a real constant c>0, and a positive integer n_0, such that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

- Example: for $f(n) = 2n^2$, $g(n) = n^2$.
  - f(n) is $O(g(n)) = O(n^2)$
  - But, it is also $O(n^3)$ and $O(n^4)$
  - We *prefer* the *smallest*.
  - For example:

$$f(n) = 9\log n + 5(\log n)^3 + 3n^2 + 2n^3 \in O(n^3).$$

Other notations for computational complexity: $\Omega()$ Notation

f(n) is $\Omega(g(n))$ if and only if there exists a real constant $c > 0$, and positive integer n_0, such that

$$c|g(n)| \leq |f(n)| \text{ for all } n \geq n_0$$

- Lower boundary for f(n).
- $2n = \Omega(n)$
- $n^2 = \Omega(n^2)$

Other notations for computational complexity: $\Theta()$ notation

- $f(n) \in \Theta(g(n))$ if and only if there exists positive real constants c_1 and c_2, and a positive integer n_0, such that

$$c_1|g(n)| \leq |f(n)| \leq c_2|g(n)|$$

- Lower and upper boundary for f(n).
- $2n = \Theta(n)$
- $n^2 = \Theta(n^2)$

Example: We want to show that $1/2n^2 + 3n = \Theta(n^2)$.

Solution: $f(n) = 1/2n^2 + 3n$, $g(n) = n^2$

To show desired result, we need determine positive constants c_1, c_2, and n_0 such that

$0 \leq c_1 . n^2 \leq 1/2n^2 + 3n \leq c_2 . n^2$ for all $n \geq n_0$.

Dividing by n^2, we get $0 \leq c_1 \leq 1/2 + 3/n \leq c_2$

$c_1 \leq 1/2 + 3/n$ holds for any value of $n_0 \geq 1$ by choosing

$c_1 \leq 7/2$

$1/2 + 3/n \leq c_2$ holds for any value of $n_0 \geq 1$ by choosing

$c_2 \geq 7/2$

Thus, by choosing $c_1 = 3$ and $c_2 = 8$ and $n_0 = 1$, we can verify $1/2n^2 + 3n = \Theta(n^2)$.

Certainly other choices for the constants exist.

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Relationships among O, Ω, and Θ - Notations

- f(n) is $O(g(n))$ iff g(n) is $\Omega(f(n))$
- f(n) is $\Theta(g(n))$ iff f(n) is $O(g(n))$ and f(n) is $\Omega(g(n))$
- $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

Properties

1. We can ignore low-order terms in an algorithm's growth-rate function.
   - If an algorithm is $O(n^3+4n^2+3n)$, it is also $O(n^3)$.
   - We only use the higher-order term as algorithm's growth-rate function.

2. We can ignore a multiplicative constant in the higher-order term of an algorithm's growth-rate function.
   - If an algorithm is $O(5n^3)$, it is also $O(n^3)$.

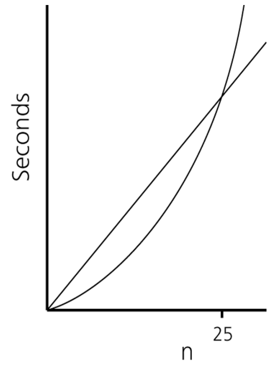
3. $O(f(n)) + O(g(n)) = O(f(n)+g(n))$
   - We can combine growth-rate functions.
   - If an algorithm is $O(n^3) + O(4n^2)$, it is also $O(n^3 +4n^2)$ ➔ So, it is $O(n^3)$.
   - Similar rules hold for multiplication.

Properties

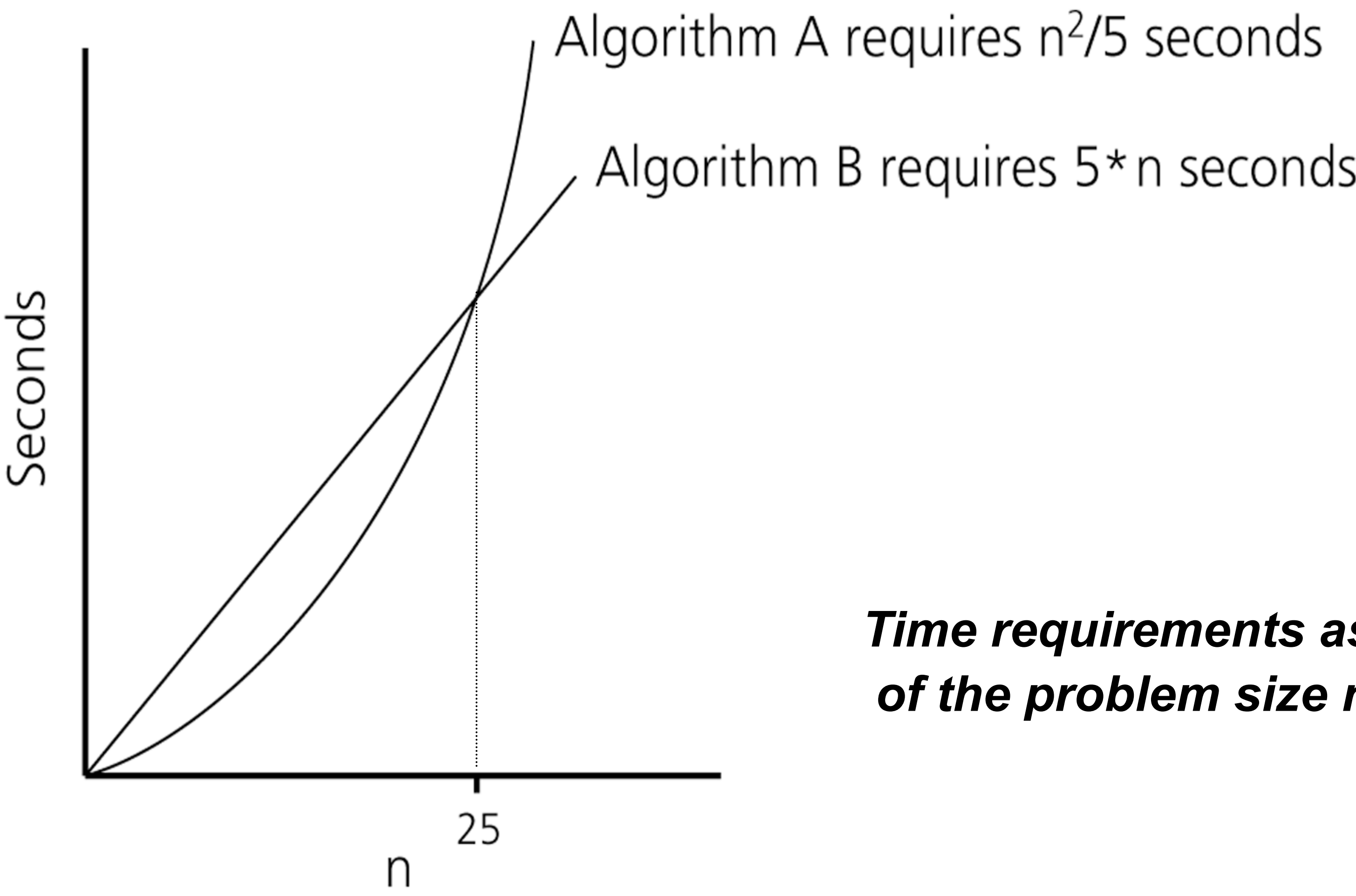
- A = B implies that B = A, right?
- But, f(n) = O(g(n)) does not imply O(g(n)) = f(n).
- We prefer to see the '=' operator here as a membership operation:
  - f(n) = O(g(n)) implies that f(n) ϵ O(g(n)).
  - That means O(g(n)) is a set.

Problems with growth rate analysis

- An algorithm with a smaller growth rate will not run faster than one with a higher growth rate for ~~all n~~, but only for all 'large enough' n!
- Algorithms with identical growth rates may have strikingly different running times because of the constants in the running time functions.
  - The value of n where two growth rates are the same is called the *break-even point*.



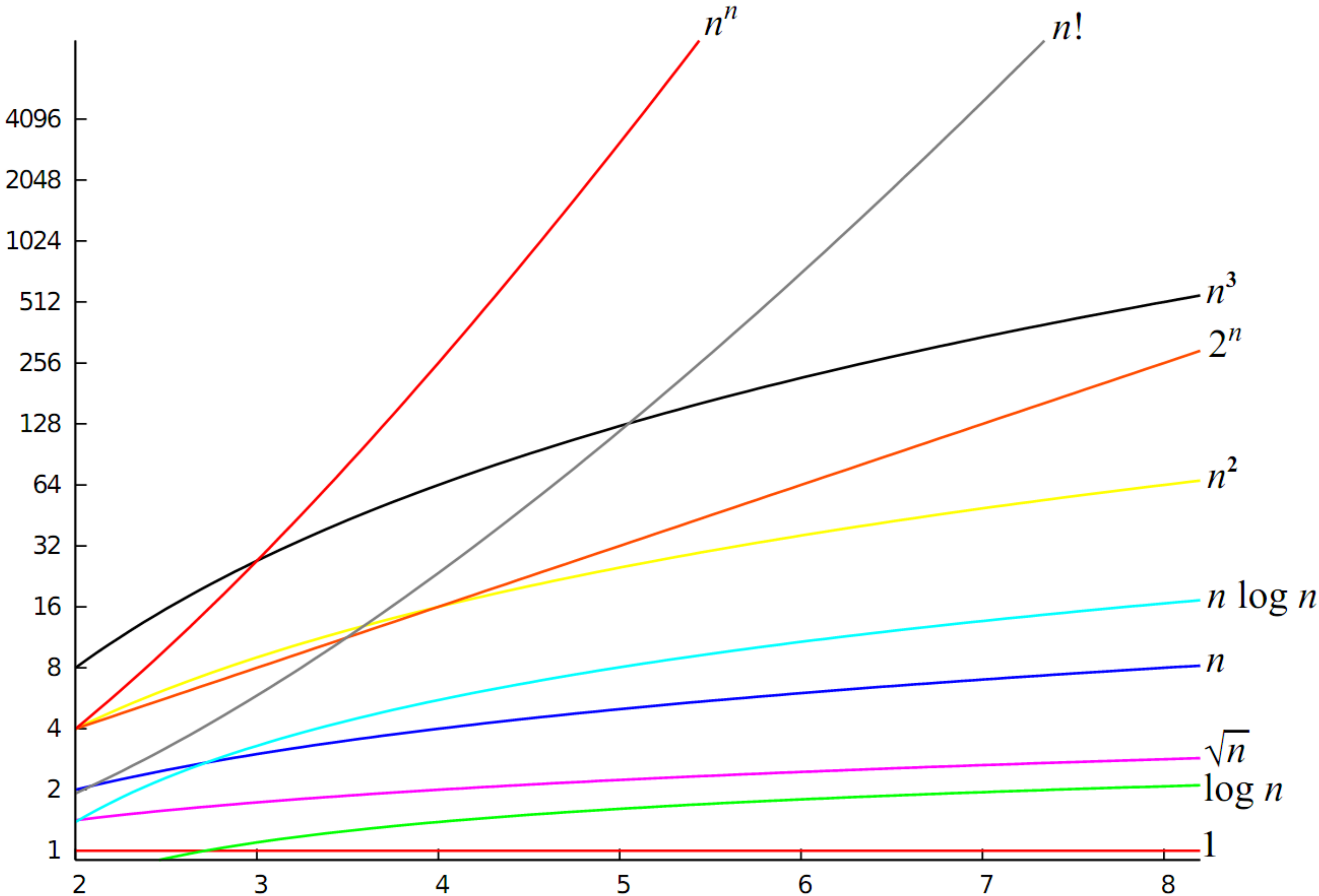
Problems with growth rate analysis



Time requirements as a function of the problem size n

Notation	Name	Example
$O(1)$	constant	Determining if a number is even or odd; using a constant-size lookup table or hash table
$O(\log n)$	logarithmic	Finding an item in a sorted array with a binary search or a balanced search tree as well as all operations in a Binomial heap.
$O(n^c),\ 0 < c < 1$	fractional power	Searching in a kd-tree
$O(n)$	linear	Finding an item in an unsorted list or a malformed tree (worst case) or in an unsorted array; Adding two n-bit integers by ripple carry.
$O(n \log n) = O(\log n!)$	linearithmic, loglinear, or quasilinear	Performing a Fast Fourier transform; heapsort, quicksort (best and average case), or merge sort
$O(n^2)$	quadratic	Multiplying two *n*-digit numbers by a simple algorithm; bubble sort (worst case or naive implementation), shell sort, quicksort (worst case), selection sort or insertion sort
$O(n^c),\ c > 1$	polynomial or algebraic	Tree-adjoining grammar parsing; maximum matching for bipartite graphs
$L_n[\alpha, c],\ 0 < \alpha < 1 = e^{(c+o(1))(\ln n)^\alpha(\ln \ln n)^{1-\alpha}}$	L-notation or sub-exponential	Factoring a number using the quadratic sieve or number field sieve
$O(c^n),\ c > 1$	exponential	Finding the (exact) solution to the traveling salesman problem using dynamic programming; determining if two logical statements are equivalent using brute-force search
$O(n!)$	factorial	Solving the traveling salesman problem via brute-force search; finding the determinant with expansion by minors.

From Wikipedia


n^n
$n!$
n^3
2^n
n^2
$n \log n$
n
$\sqrt{n}$
$\log n$
1
4096
2048
1024
512
256
128
64
32
16
8
4
2
1
2
3
4
5
6
7
8

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Function	n: 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Importance of Developing Efficient Algorithms

Sequential search vs Binary search

Array size	No. of comparisons by seq. search	No. of comparisons by bin. search
128	128	8
1,048,576	1,048,576	21
~4.10^9	~4.10^9	33

Execution times for algorithms with the given time complexities:

n	f(n)=n	nlgn	n^2	2^n
20	0.02 μs	0.086 μs	0.4 μs	1 ms
10^6	1μs	19.93 ms	16.7 min	31.7 years
10^9	1s	29.9s	31.7 years	!!! centuries

Anaysis of Algorihms, A.Yazici

Other Notations

o-Notation: f(n) is o(g(n)), "little-oh of g of n" is the following set:

$o(g(n)) = \{f(n)$: for all positive real constant $c > 0$, there exists a constant $n_0 \geq 0$ such that $0 \leq |f(n)| < c|g(n)|$ for all $n \geq n_0\}$

- We use o-notation to denote an upper bound that is not asymptotically tight, whereas O-notation may be asymptotically tight. Intuitively, in the o-notation, the function f(n) becomes insignificant relative to g(n) as n approaches infinity, that is,

  $$\lim_{n \to \infty} f(n)/g(n) = 0$$

  Ex: $n^2/2 \in o(n^3)$, since $\lim_{n \to \infty} (n^2/2)/n^3 = \lim_{n \to \infty} 1/2n = 0$

  Ex: $2n = o(n^2)$, but $2n^2 \neq o(n^2)$

Proposition: $f(n) \in o(g(n)) \Rightarrow O(f(n)) \subset O(g(n))$

Other Notations

ω - Notation: f(n) is $\omega(g(n))$, "little-omega of g of n", is the following set:

$\omega(g(n))$ = {f(n) : for all positive real constant c > 0, there exists a constant $n_0 \geq 0$ such that $0 \leq c|g(n)| < |f(n)|$ for all $n \geq n_0$}

ω-notation denotes a lower bound that is not asymptotically tight. The relation $f(n) = \omega(g(n))$ implies that,

$$\lim_{n \to \infty} f(n)/g(n) = \infty, \text{ if the limit exists.}$$

That is, f(n) becomes arbitrarily large relative to g(n) as n approaches infinity.

Ex: $n^2/2 = \omega(n)$, since $\lim_{n \to \infty} (n^2/2)/n = \infty$,

but $n^2/2 \neq \omega(n^2)$

Other notations

~ - Notation: Given the function g(n), we define ~g(n) to be the set of all functions f(n) having the property that

$$\lim_{n \to \infty} f(n)/g(n) = 1,$$

If $f(n) \in {\sim}g(n)$, then we say that f(n) is *strongly* asymptotic to g(n) and denote this by writing $f(n) \sim g(n)$.

Ex: $n^2 = \sim(n^2)$, since $\lim_{n \to \infty} n^2/n^2 = 1$,

Property: $f(n) \sim g(n) \Rightarrow f(n) \in \Theta(g(n))$

Family of Bachmann–Landau notations

Notation	Naming	Meaning
$f(n)$ is $O(g(n))$	• Big Omicron • Big O • Big Oh	f is bounded above by g (up to constant factor, as it was with Θ) asymptotically
$f(n)$ is $\Omega(g(n))$	• Big Omega	f is bounded below by g (up to constant factor, as it was with Θ) asymptotically
$f(n)$ is $\Theta(g(n))$	• Big Theta	f is bounded both above and below by g (up to constant factors) asymptotically
$f(n)$ is $o(g(n))$	• Small Omicron • Small O • Small Oh	f is dominated by g asymptotically
$f(n)$ is $\omega(g(n))$	• Small Omega	f dominates g asymptotically
$f(n) \sim \quad (g(n))$	• on the order of • twiddles	f is equal to g asymptotically