METU | Department of Computer Engineering | Fall 2020

CENG 223—Discrete Computational Structures

FINAL | Jan 18, 2021

A. Birturk, F. Polat, H. Oguztuzun | 25 minutes

Q-1) {*15 points*} For $n \geq 1$, let $S = \{x_1, x_2, x_3, \cdots, x_n\}$ and

$$T = \{X | X \subseteq S \land \exists i\, (1 \leq i \leq n-1 \land x_i \in X \land x_{i+1} \in X)\}$$

Construct a recurrence relation a_n where $a_n = |T|$.

Any element of T, a subset of S, can be characterized by a binary string of length n with two consecutive 1's

e.g., $S = \{a, b, c, d, e\}$

1 0 1 1 0 $\longleftrightarrow$ $X = \{a, c, d\}$ so $X \in T$

Any such binary string can be one of the foll. forms:

mutually exclusive

1) $\underbrace{[\ \ \cdots\ \ |1|0]}_{n}$ a_{n-1} many such binary strings

2) $\underbrace{[\ \ \cdots\ \ |0|1]}_{n}$ a_{n-2} " " "

3) $\underbrace{[\ \ \cdots\ \ |1|1]}_{n}$ 2^{n-2} " " "

$$a_n = a_{n-1} + a_{n-2} + 2^{n-2} \quad n \geq 3$$

grading

$a_1 = 0$

$a_2 = 1$

grading: boundary conditions must be well explained

a_1	a_2	a_3	a_4	a_5	a_6	. . .
"	"	"	"	"	"	
0	1	3	8	19	43	. . .

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Q-2) {*20 points*} Solve the recurrence relation $a_n = a_{n-2} + 2n - 3$ $n \geq 2$ with initial conditions $a_0 = 0, a_1 = 0$ using *generating functions*. No credit will be given if you attempt to solve it with some other method.

$$\underbrace{\sum_{n=2}^{\infty} a_n x^n}_{A(x)-a_0-a_1x} = \sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 2nx^n - \sum_{n=2}^{\infty} 3x^n = \underbrace{x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}}_{x^2(A(x))} + \underbrace{2x\sum_{n=2}^{\infty} nx^{n-1}}_{2x\left(\frac{1}{(1-x)^2}-1\right)} - \underbrace{3x^2\sum_{n=2}^{\infty} x^{n-2}}_{3x^2\frac{1}{1-x}}$$

note $1+x+x^2+\dots = \frac{1}{1-x}$

$\underline{1}+\underline{2}x+\underline{3}x^2+\dots = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

$$A(x) - \overset{0}{\cancel{a_0}} - \overset{0}{\cancel{a_1}}x = x^2A(x) + 2x\left(\frac{1}{(1-x)^2}-1\right) - \frac{3x^2}{(1-x)}$$

$$A(x)(1-x^2) = \frac{2x}{(1-x)^2} - \underset{(1-x)^2}{\frac{2x}{1}} - \underset{(1-x)}{\frac{3x^2}{(1-x)}}$$

$$= \frac{2x - 2x(1-x^2) - 3x^2(1-x)}{(1-x)^2} = \frac{2x - 2x(1-2x+x^2) - 3x^2 + 3x^3}{(1-x)^2}$$

$$= \frac{\cancel{2x} - \cancel{2x} + 4x^2 - 2x^3 - 3x^2 + 3x^3}{(1-x)^2} = \frac{x^2(1+x)}{(1-x)^2}$$

$$\underline{\underline{A(x)}} = \frac{x^2(1+x)}{(1-x)^2(1-x^2)} = \frac{x^2\cancel{(1+x)}}{(1-x)^3\cancel{(1+x)}} = \boxed{\frac{x^2}{(1-x)^3}}$$

grading: 10pts

$\langle 1, 1, 1, \dots \rangle \longleftrightarrow \frac{1}{1-x}$

$\langle 1, 2, 3, \dots \rangle \longleftrightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

$\langle 2.1, 3.2, 4.3, \dots \rangle \longleftrightarrow \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{2}{(1-x)^3}$

grading: 10pts

$\langle 2\frac{.1}{2}, \frac{3.2}{2}, \frac{4.3}{2}, \dots \rangle \longleftrightarrow \frac{1}{(1-x)^3}$ (divide by 2)

$\langle 0, 0, \frac{2.1}{2}, \frac{3.2}{2}, \frac{4.3}{2}, \dots, \frac{n(n-1)}{2}, \dots \rangle \longleftrightarrow \frac{x^2}{(1-x)^3}$ shift twice

0th, 1st, 2nd, nth

$$\boxed{a_n = \frac{n(n-1)}{2}}$$