Natural deduction for predicate logic

Readings: Section 2.3.

In this module, we will extend our previous system of natural deduction for propositional logic, to be able to deal with predicate logic. The main things we have to deal with are equality, and the two quantifiers (existential and universal).

All of the rules from propositional logic carry over to predicate logic, and there are six new rules (introduction and elimination for each of the new features). We will also introduce one derived rule not in the textbook, for convenience.

Equality

The rule for introducing equality is fairly simple: it simply says that any term t is equal to itself, and that no premises are needed to conclude this.

$$\frac{}{t = t} =\text{i}$$

The "elimination" rule for equality is more interesting; it describes how substitution may be used in formulas.

If t_1 and t_2 are free for z in ϕ, then we may substitute the term t_2 for t_1 in ϕ. But our definition of substitution substitutes a term for a variable. Hence the following phrasing:

$$\frac{t_1 = t_2 \quad \phi[t_1/z]}{\phi[t_2/z]} =\text{e}$$

Of course, any variable can be used in this rule, not just z.

In using this rule, there are four important components: the equality $t_1 = t_2$, the formula $\phi[t_1/z]$, the variable z involved in the substitution, and the formula ϕ. We extend the textbook's notation to put all four of these as justification for any use of the rule.

Any notion of equality should be reflexive, symmetric, and transitive. Let's prove these properties using our new rules. Reflexivity follows immediately from the $=$i rule. The sequent schema expressing symmetry is $t_1 = t_2 \vdash t_2 = t_1$, where t_1 and t_2 represent specific terms.

$$
\begin{array}{lll}
1 & t_1 = t_2 & \text{premise} \\
2 & t_1 = t_1 & =\text{i} \\
3 & t_2 = t_1 & =\text{e } 1,\ 2,\ z,\ z = t_1
\end{array}
$$

Note that the justification for line 3 cites $t_1 = t_2$ as the equality to be eliminated, z as the variable to be substituted for in the formula $z = t_1$, $t_1 = t_1$ as the result of substituting t_1 for z, and $t_2 = t_1$ as the result of substituting t_2 for z.

The sequent expressing transitivity is $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$.

$$
\begin{array}{lll}
1 & t_1 = t_2 & \text{premise} \\
2 & t_2 = t_3 & \text{premise} \\
3 & t_1 = t_3 & =\text{e } 2,\ 1,\ z,\ t_1 = z
\end{array}
$$

The justification for line 3 cites $t_2 = t_3$ as the equality to be eliminated, z as the variable to be substituted for in the formula $t_1 = z$, $t_1 = t_2$ as the result of substituting t_2 for z, and $t_1 = t_3$ as the result of substituting t_3 for z.

The $=$e rule allows us to substitute the right-hand side of an equality for the left-hand side. It does not say anything about substituting the left-hand side of an equality for the right-hand side. In order to do this, we must switch around the equality using the three lines on the previous slide and then apply $=$e.

This is annoying enough that we will introduce a derived rule not in the textbook, $=$e$_R$, which looks like this.

$$\frac{t_1 = t_2 \quad \phi[t_2/z]}{\phi[t_1/z]} =\text{e}_R$$

The sequent schema corresponding to this derived rule is
$t_1 = t_2, \phi[t_2/z] \vdash \phi[t_1/z]$. The following proof schema justifies
the rule, basically by writing out what we'd have to do if we didn't
have the rule.

1    $t_1 = t_2$ premise

2    $t_1 = t_1$ =i

3    $t_2 = t_1$ =e 1, 2, z, $z = t_1$

4    $\phi[t_2/z]$ premise

5    $\phi[t_1/z]$ =e 3, 4, z, ϕ

Universal quantification

We can think of the universal quantifier $\forall$ as a generalization of $\wedge$. The natural deduction rules for $\forall$ can also be viewed as generalizations of the rules for $\wedge$. To eliminate a conjunction, we just choose one of the two conjuncts. To eliminate a quantifier, we choose one of the many "values" quantified over, namely any term t (which is free for x in ϕ):

$$\frac{\forall x \phi}{\phi[t/x]} \ \forall x \ \text{e}$$

$\forall$-introduction is not quite so simple.

Recall that our rule for $\wedge$-introduction was:

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \, \wedge\mathsf{i}$$

This suggests that to introduce a quantifier $\forall x$, we must prove the

formula being quantified for all possible "values". This seems

impossible. Let's look at a proof we did informally on the first day to

see how to proceed. We will phrase it slightly differently (but in a

fashion which will be familiar to you).

Thm: Every even natural number is the sum of two odd natural numbers whose difference is at most 2.

Proof: Let n be an even natural number. Then n is of the form $2k$, for some $k \geq 1$. If k is odd, then we can write $n = k + k$, and the two k's satisfy the theorem. If k is even, we can write $n = (k - 1) + (k + 1)$, and the numbers $k - 1$ and $k + 1$ satisfy the theorem. $\square$

If we look closely at this proof, we are taking an arbitrary even natural number, which we are calling n, and expressing it as the sum of two odd natural numbers whose difference is at most 2. That allows us to conclude that the property holds for all natural numbers n, since there is nothing special about n (that is, it doesn't appear in the statement of the theorem or anywhere else outside the proof).

This suggests that to prove a formula of the form $\forall x \phi$, we can prove ϕ with some arbitrary but **fresh** variable x_0 substituted for x. That is, we want to prove the formula $\phi[x_0/x]$.

On the previous slide, we used n as a fresh variable, but in our formal proofs, we adopt the convention of using subscripts (for example, x_0 as the fresh variable used to prove a $\forall x$ statement).

The word "fresh" means that the variable has never been used before in the proof. Furthermore, it will not be used once $\phi[x_0/x]$ has been proved. It is "local" to this part of the proof. As we did with assumptions, we enforce this locality by surrounding this part of the proof with a proof box, putting the fresh variable as a label in the top left corner.

We label the top of the proof box with the fresh variable, but so far we only know what the last line inside the proof box should look like: $\phi[x_0/x]$. The first line can be anything we consider useful. In certain cases, we might even leave it blank, for clarity (for example, if we are immediately going to open another proof box inside it).

This results in the following rule:

$$
\frac{\begin{array}{|l|}
\hline
x_0 \\
\quad \vdots \\
\quad \phi[x_0/x] \\
\hline
\end{array}}{\forall x \phi} \;\forall x\mathsf{i}
$$

We'll use this to prove

$$\forall x (P(x) \vee Q(x)), \forall x (\neg P(x)) \vdash \forall x Q(x).$$

	1	$\forall x (P(x) \lor Q(x))$	premise
	2	$\forall x (\neg P(x))$	premise
x_0	3	$P(x_0) \lor Q(x_0)$	$\forall x$e 1
	4	$P(x_0)$	assumption
	5	$\neg P(x_0)$	$\forall x$e 2
	6	$\bot$	$\neg$e 4, 5
	7	$Q(x_0)$	$\bot$e 6
	8	$Q(x_0)$	assumption
	9	$Q(x_0)$	$\lor$e 3, 4 − 7, 8
	10	$\forall x Q(x)$	$\forall x$i 3, 9

In the proof on the previous slide, we used $\forall$-elimination to create a statement using the fresh variable, which appeared on the first line of the proof box labelled with the fresh variable x_0. We could also have chosen to list a premise that did not mention the fresh variable, or chosen to use some other rule, or left the first line blank.

Fresh variables are involved in one other proof rule to be discussed shortly.

Existential quantification

The rules for $\exists$ parallel those for $\vee$. Here are the rules for $\vee$:
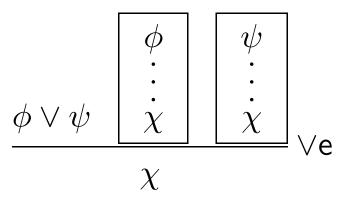
$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \vee i_2$$

If we think of $\exists x \phi$ as a sort of large disjunction over all "values" of x, this suggests that we can extend a formula with one such "value" to an existential quantification.
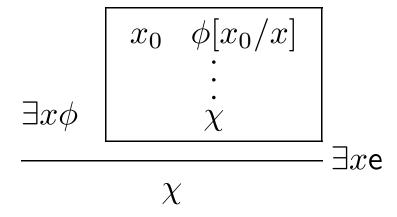
This yields the following rule for $\exists$-introduction (where t is free for x in ϕ):

$$\frac{\phi[t/x]}{\exists x \phi} \exists x i$$

The rule for eliminating disjunction was complicated.

$$\frac{\phi \vee \psi \quad \boxed{\begin{array}{c} \phi \\ \vdots \\ \chi \end{array}} \quad \boxed{\begin{array}{c} \psi \\ \vdots \\ \chi \end{array}}}{\chi} \vee e$$

This suggests that to eliminate an existential quantification $\exists x$ and conclude χ, we should be able to conclude it starting from a formula with any "value" substituted for x. We do this by substituting a fresh variable x_0 which does not occur in χ (or anywhere outside its box).

$$\frac{\exists x \phi \quad \boxed{\begin{array}{c} x_0 \quad \phi[x_0/x] \\ \vdots \\ \chi \end{array}}}{\chi} \exists x e$$

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To eliminate an existential quantification, we open a box and introduce a fresh variable. Again, we use the convention that in eliminating a formula of the form $\exists x \phi$, our fresh variable should be a subscripted version of x.

This rule dictates the first line of the box. The first line needs to be ϕ with the fresh variable substituted for the quantified variable to be eliminated; this line is labelled as an assumption. The last line should be whatever statement we hope to take outside the box (as is the case for the proof boxes used in or-elimination). That statement cannot involve the fresh variable, as it is not permitted to appear outside the proof box.

You should keep the analogy between the propositional connectives $\wedge, \vee$ and the quantifiers $\forall, \exists$ in mind, as the similarity in their introduction and elimination rules can help in developing proofs.

Sometimes you can prove a sequent of predicate logic by first figuring out how to prove the corresponding sequent of propositional logic.

The next slide provides a proof of the sequent
$$\forall x (P(x) \vee Q(x)), \exists x (\neg P(x)) \vdash \exists x Q(x).$$

	1	$\forall x(P(x) \vee Q(x))$	premise
	2	$\exists x(\neg P(x))$	premise
x_0	3	$\neg P(x_0)$	assumption
	4	$P(x_0) \vee Q(x_0)$	$\forall x$e 1
	5	$P(x_0)$	assumption
	6	$\bot$	$\neg$e 3, 5
	7	$Q(x_0)$	$\bot$e 6
	8	$Q(x_0)$	assumption
	9	$Q(x_0)$	$\vee$e 4, 5 − 7, 8
	10	$\exists x Q(x)$	$\exists x$i 9
	11	$\exists x Q(x)$	$\exists x$e 2, 3 − 10

Quantifier equivalences

Earlier we suggested that the statement "No students love CS 245" had at least two possible translations, $\neg \exists x (S(x) \wedge L(x, c))$ and $\forall x (S(x) \rightarrow \neg L(x, c))$, which we intuitively believed to be semantically equivalent.

We should be able to prove the logical equivalence of these formulas using our system of natural deduction. Theorem 2.13 (page 117) in the text covers a number of such equivalences. They are grouped into four broad categories, and a representative proof is presented for each one.

In lecture we will cover a couple of these proofs, leaving others to reading, tutorial examples, or as exercises.

DeMorgan-style equivalences

These quantifier equivalences deal with moving negation from within a quantifier to outside, or vice-versa. They are analogous to DeMorgan's laws for propositional logic.

Note that not all of these equivalences are intuitionistically valid.

The example we will prove is $\exists x \neg \phi \vdash \neg \forall x \phi$. This is a sequent schema parameterized by an arbitrary formula ϕ, and as we did before, we must present a proof schema which is not a formal proof according to our rules, but becomes one for any specific ϕ.

The analogous propositional law is $\neg p_1 \vee \neg p_2 \vdash \neg(p_1 \wedge p_2)$, and we prove this first to demonstrate the parallels.

$$1 \quad \neg p_1 \lor \neg p_2 \qquad\qquad\qquad\qquad \text{premise}$$

2	$\neg p_1$	assumption	$\neg p_2$	assumption
3	$p_1 \land p_2$	assumption	$p_1 \land p_2$	assumption
4	p_1	$\land e_1\ 3$	p_2	$\land e_1\ 3$
5	$\bot$	$\neg e\ 2, 4$	$\bot$	$\neg e\ 2, 4$
6	$\neg(p_1 \land p_2)$	$\neg i\ 3 - 5$	$\neg(p_1 \land p_2)$	$\neg i\ 3 - 5$

$$7 \quad \neg(p_1 \land p_2) \qquad\qquad\qquad\qquad \lor e\, 1, 2 - 6$$

The premise in line 1 is analogous to an existential quantification, and the assumption in line 2 is analogous to the first line in existential elimination, where a substitution occurs.

If we start with the premise $\exists x (\neg \phi)$, then to use existential elimination, we must open a proof box, introduce a fresh variable x_0, and start the proof box with the assumption $(\neg \phi)[x_0/x]$.

But since we are doing this for an arbitrary ϕ, we cannot actually make the substitution. Instead, we have to reason about the effect of this substitution. In particular, we need to argue that $(\neg \phi)[x_0/x]$ is the same as $\neg(\phi[x_0/x])$. For any specific ϕ, the two resulting formulas are identical.

This leads to two lines in the proof schema which for any specific ϕ would be identical in the resulting formal proof (and therefore the second of the two lines can be removed).

	1	$\exists x(\neg\phi)$	premise
x_0	2	$(\neg\phi)[x_0/x]$	assumption
	3	$\neg(\phi[x_0/x])$	identical
	4	$\forall x\phi$	assumption
	5	$\phi[x_0/x]$	$\forall x$e
	6	$\bot$	$\neg$e $3,5$
	7	$\neg\forall x\phi$	$\neg$i $4-6$
	8	$\neg\forall x\phi$	$\exists x$e $1, 2-7$

Notation such as square brackets and the justification "identical" can occur only in proof schemas of this sort, not in formal proofs.

Distribution without freedom

These quantifier equivalences allow one to move a logical connective involving a formula ϕ (in which some variable x may be free) and a formula ψ (in which x is not free) into or out of a quantification over x. Here is an example:

$$(\exists x \phi) \wedge \psi \dashv\vdash \exists x (\phi \wedge \psi)$$

Intuitively, the fact that x is not free in ψ means that there is no interference from the quantifier. On the following slide, we give a proof schema showing the forward direction of this equivalence, leaving the other direction as an exercise.

	1	$(\exists x \phi) \wedge \psi$	premise
	2	$\exists x \phi$	$\wedge e_1 \ 1$
	3	ψ	$\wedge e_2 \ 1$
x_0	4	$\phi[x_0/x]$	assumption
	5	$\phi[x_0/x] \wedge \psi$	$\wedge i \ 3, 4$
	6	$(\phi \wedge \psi)[x_0/x]$	identical
	7	$\exists x (\phi \wedge \psi)$	$\exists x i \ 6$
	8	$\exists x (\phi \wedge \psi)$	$\exists x e \ 2, 4 - 7$

Where did we use the fact that x was not free in ψ?

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Other equivalences

The book also lists distributive laws when both formulas involve quantification over a free variable, such as

$$\forall x \phi \wedge \forall x \psi \dashv\vdash \forall x (\phi \wedge \psi).$$

There are commutative laws for quantifiers, such as

$$\forall x \forall y \phi \dashv\vdash \forall y \forall x \phi.$$

In the exercises, there are also "rewriting" rules such as

$$\forall x \forall y P(x, y) \dashv\vdash \forall u \forall v P(u, v).$$

Some of these you will see in tutorial, others in homework, and others you should work on for practice.

	1	$\neg \forall x \phi$	premise
	2	$\neg \exists x \neg \phi$	assumption
x_0	3		
	4	$\neg(\phi[x_0/x])$	assumption
	5	$(\neg\phi)[x_0/x]$	identical
	6	$\exists x \neg \phi$	$\exists$i 5
	7	$\bot$	$\neg$e 2, 6
	8	$\phi[x_0/x]$	PBC 4–7
	9	$\forall x \phi$	$\forall$i 4–8
	10	$\bot$	$\neg$e 9, 1
	11	$\exists x \neg \phi$	PBC 2–10

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