

Solutions to Practice Problems

Pumping Lemma

1. $L = \{a^k b^k \mid k \geq 0\}$

see notes

2. $L = \{a^k \mid k \text{ is a prime number}\}$

Proof by contradiction:

Let us assume L is regular. Clearly L is infinite (there are infinitely many prime numbers). From the pumping lemma, there exists a number n such that any string w of length greater than n has a “repeatable” substring generating more strings in the language L . Let us consider the first prime number $p \geq n$.

From the pumping lemma the string of length p has a “repeatable” substring. We will assume that this substring is of length $k \geq 1$. Hence:

$$\begin{array}{ll} a^p & \in L \\ a^{p+k} & \in L \\ a^{p+2k} & \in L, \text{ etc.} \end{array} \quad \begin{array}{l} \text{and} \\ \text{as well as} \end{array}$$

It should be relatively clear that $p + k$, $p + 2k$, etc., cannot all be prime but let us add k p times, then we must have:

$$a^{p+pk} \in L, \text{ of course} \quad a^{p+pk} = a^{p(k+1)}$$

so this would imply that $(k+1)p$ is prime, which it is not since it is divisible by both p and $k+1$.

Hence L is not regular.

3. $L = \{a^n b^{n+1}\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^{p+1}$. Its length is $2p+1 \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p-k} b^{p+1}$ must also be in L but it is not of the right form. Hence the language is not regular.

Note that the repeatable string needs to appear in the first n symbols to avoid the following situation:

assume, for the sake of argument that $n = 20$ and you choose the string $a^{10} b^{11}$ which is of length larger than 20, but $|xy| \leq 20$ allows xy to extend past b , which means that y could contain some b 's. In such case, removing y (or adding more y 's) could lead to strings which still belong to L .

4. $L = \{a^n b^{2^n}\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^{2^p}$. Its length is $3p \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p-k} b^{2^p}$ must also be in L

but it is not of the right form. Hence the language is not regular.

5. TRAILING-COUNT as any string s followed by a number of a 's equal to the length of s .

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $b^p a^p$. Its length is $2p \geq p$. Since the length of xy cannot exceed p , y must be of the form b^k for some $k > 0$. From the pumping lemma $b^{p-k} a^p$ must also be in L but it is not of the right form. Hence the language is not regular.

6. EVENPALINDROME = { all words in PALINDROME that have even length}
Same as #2 above, choose $a^n b b a^n$.

7. ODDPALINDROME = { all words in PALINDROME that have odd length}
Same as #2 above, choose $a^n b a^n$.

8. DOUBLESQUARE = { $a^n b^n$ where n is a square }

Assume DOUBLESQUARE is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^{p^2} b^{p^2}$. Its length is $2p^2 \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. Let us add y p times. From the pumping lemma $a^{p^2+p^2k} b^{p^2} = a^{p(p+k)} b^{p^2}$ must also be in L but it is not of the right form. Hence the language is not regular.

9. $L = \{ w \mid w \in \{a, b\}^*, w = w^R \}$

Proof by contradiction:

Assume L is regular. Then the pumping lemma applies.

From the pumping lemma there exists an n such that every $w \in L$ longer than n can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq n$.

Let us choose the palindrome $a^n b a^n$.

Again notice that we were clever enough to choose a string which:

- has a center mark which is not a (otherwise when we remove or add y we would be left with an acceptable string)*
- has a first portion on length n which is all a 's (so that when we remove or add y it will create an imbalance).*

Its length is $2n + 1 \geq n$. Since the length of xy cannot exceed n , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{n-k} b a^n$ must also be in L but it is not a palindrome.

Hence L is not regular.

10. $L = \{ w \in \{a, b\}^* \mid w \text{ has an equal number of } a\text{'s and } b\text{'s} \}$

Let us show this by contradiction: assume L is regular. We know that the language generated by $a^* b^*$ is regular. We also know that the intersection of two regular languages is regular. Let $M = \{ a^n b^n \mid n \geq 0 \} = L(a^* b^*) \cap L$. Therefore if L

is regular M would also be regular. but we know that M is not regular. Hence, L is not regular.

11. $L = \{ w w^R \mid w \in \{a, b\}^* \}$
see # 7

12. $L = \{ 0^n \mid n \text{ is a power of } 2 \}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose

$n = 2^p$. Since the length of xy cannot exceed p , y must be of the form 0^k for some $0 < k \leq p$. From the pumping lemma 0^m where $m = 2^p + k$ must also be in L . We have

$$2^p < 2^p + k \leq 2^p + p < 2^{p+1}$$

Hence this string is not of the right form. Hence the language is not regular.

13. $L = \{ a^{2^k} w \mid w \in \{a, b\}^*, |w| = k \}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^{2^p} b^p$. Its length is $3p \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{2^p+k} b^p$ must also be in L but it is not of the right form since the number of a 's cannot be twice the number of b 's (Note that you must subtract not add, otherwise some a 's could be shifted into w). Hence the language is not regular.

14. $L = \{ a^k w \mid w \in \{a, b\}^*, |w| = k \}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^p$. Its length is $2p \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p+k} b^p$ must also be in L but it is not of the right form since the number of a 's cannot be equal to the number of b 's (Note that you must subtract not add, otherwise some a 's could be shifted into w). Hence the language is not regular.

15. $L = \{ a^n b^l \mid n \leq l \}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^p$. Its length is $2p \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p+k} b^p$ must also be in L but it is not of the right form since the number of a 's exceeds the number of b 's (Note that you must add not subtract, otherwise the string would be OK). Hence the language is not regular.

16. $L = \{ a^n b^l a^k \mid k = n + l \}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let

us choose $a^p b a^{p+1}$. Its length is $2p+2 \geq p$. Since the length of xy cannot exceed p , y must be of the form a^m for some $m > 0$. From the pumping lemma $a^{p-m} b a^{p+1}$ must also be in L but it is not of the right form. Hence the language is not regular.

17. $L = \{v a^{k+1} \mid v \in \{a, b\}^*, |v| = k\}$

Assume L is regular. From the pumping lemma there exists an n such that every $w \in L$ such that $|w| \geq n$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $b^n a^{n+1}$. Its length is $2n+1 \geq n$. Since the length of xy cannot exceed n , y must be of the form b^k for some $k > 0$. From the pumping lemma if we add two y to the original string $b^{n+2k} a^{n+1}$ must also be in L but that string is of length $2n+2k+1$ and v would have to be b^{n+k} to fit the pattern the rest of the string would then be $b^k a^{k+1}$ which is not of the right form. Hence the language is not regular.

18. $L = \{v a^{2k} \mid v \in \{a, b\}^*, |v| = k\}$

Assume L is regular. From the pumping lemma there exists a n such that every $w \in L$ such that $|w| \geq n$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq n$. Let us choose $b^n a^{2n}$. Its length is $3n \geq n$. Since the length of xy cannot exceed n , y must be of the form b^k for some $k > 0$. From the pumping lemma $b^{n+k} a^{2n}$ must also be in L but it is not of the right form since the number of a 's exceeds the number of b 's and we cannot move any b 's on the a side (Note that you must add not subtract, otherwise the string would be OK by shifting a 's to the b side). Hence the language is not regular.

19. $L = \{ww \mid w \in \{a, b\}^*\}$

Assume L is regular. From the pumping lemma there exists an n such that every $w \in L$ such that $|w| \geq n$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq n$. Let us choose $a^n b^n a^n b^n$. Its length is $4n \geq n$. Since the length of xy cannot exceed n , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{n+k} b^n a^n b^n$ must also be in L but it is not of the right form since the middle of the string would be in the middle of the b which prevents a match with the beginning of the string. Hence the language is not regular.

20. $L = \{a^n \mid n \geq 0\}$

Proof by contradiction:

Let us assume L is regular. From the pumping lemma, there exists a number p such that any string w of length greater than p has a "repeatable" substring generating more strings in the language L . Let us consider $a^p!$ (unless $p < 3$ in which case we chose $a^{3!}$). From the pumping lemma the string w has a "repeatable" substring. We will assume that this substring is of length $k \geq 1$. From the pumping lemma $a^{p!-k}$ must also be in L . For this to be true there must be j such that $j! = m! - k$. But this is not possible since when $p > 2$ and $k \leq m$ we have

$$m! - k > (m - 1)!$$

Hence L is not regular.

21. $L = \{a^n b^l \mid n \neq l\}$

Proof by contradiction:

Let us assume L is regular. From the pumping lemma, there exists a number p

such that any string w of length greater than p has a “repeatable” substring generating more strings in the language L . Let us consider $n = p!$ and $l = (p+1)!$. From the pumping lemma the resulting string is of length larger than p and has a “repeatable” substring. We will assume that this substring is of length $k \geq 1$. From the pumping lemma we can add y $i-1$ times for a total of i y s. If we can find an i such that the resulting number of a ’s is the same as the number of b ’s we have won. This means we must find i such that:

$$\begin{aligned} m! + (i-1)k &= (m+1)! & \text{or} \\ (i-1)k &= (m+1)m! - m! &= m * m! & \text{or} \\ i &= (m * m!) / k + 1 \end{aligned}$$

but since $k < m$ we know that k must divide $m!$ and that $(m * m!) / k$ must be an integer. This proves that we can choose i to obtain the above equality. Hence L is not regular.

22. $L = \{a^n b^l a^k \mid k > n + l\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b a^{p+2}$. Its length is $2p+3 \geq p$. Since the length of xy cannot exceed p , y must be of the form a^m for some $m > 0$. From the pumping lemma $a^{p+2m} b a^{p+2}$ must also be in L but it is not of the right form since $p+2m+1 > p+2$. Hence the language is not regular.

23. $L = \{a^n b^l c^k \mid k \neq n + l\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^{p!} b^{p!} a^{(p+1)!}$. Its length is $2p! + (p+1)! \geq p$. Since the length of xy cannot exceed p , y must be of the form a^m for some $m > 0$. From the pumping lemma any string of the form $xy^i z$ must always be in L . If we can show that it is always possible to choose i in such a way that we will have $k = n + l$ for one such string we will have shown a contradiction. Indeed we can have

$$p! + (i-1)m + p! = (p+1)!$$

if we have $i = 1 + ((p+1)! - 2p!) / m$ Is that possible? only if m divides $((p+1)! - 2p!)$

$((p+1)! - 2 * (p)! = (p+1-2)p!$ and since $m \leq p$ m is guaranteed to divide $p!$.

Hence i exists and the language is not regular.

24. $L = \{a^n b^l a^k \mid n = l \text{ or } l \neq k\}$

Proof by contradiction:

Let us assume L is regular. From the pumping lemma, there exists a number p such that any string w of length greater than p has a “repeatable” substring generating more strings in the language L . Let us consider $w = a^p b^p a^p$. From the pumping lemma the string w , of length larger than p has a “repeatable” substring. We will assume that this substring is of length $m \geq 1$. From the pumping lemma we can remove y and the resulting string should be in L . However, if we remove y we get $a^{p-m} b^p a^p$. But this string is not in L since $p-m \neq p$ and $p = p$.

Hence L is not regular.

25. $L = \{a^n b a^{3n} \mid n \geq 0\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b a^{3p}$. Its length is $4p+1 \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p-k} b a^{3p}$ must also be in L but it is not of the right form. Hence the language is not regular.

26. $L = \{a^n b^n c^n \mid n \geq 0\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^p c^p$. Its length is $3p \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p-k} b^p c^p$ must also be in L but it is not of the right form. Hence the language is not regular.

27. $L = \{a^i b^n \mid i, n \geq 0, i = n \text{ or } i = 2n\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^p$. Its length is $2p \geq p$. Since the length of xy cannot exceed p , y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p-k} b^p$ must also be in L but it is not of the right form. Hence the language is not regular.

28. $L = \{0^k 10^k \mid k \geq 0\}$

Assume L is regular. From the pumping lemma there exists an n such that every $w \in L$ such that $|w| \geq n$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq n$. Let us choose $0^n 10^n$. Its length is $2n+1 \geq n$. Since the length of xy cannot exceed n , y must be of the form 0^p for some $p > 0$. From the pumping lemma $0^{n-p} 10^n$ must also be in L but it is not of the right form. Hence the language is not regular.

29. $L = \{0^n 1^m 2^n \mid n, m \geq 0\}$

Assume L is regular. From the pumping lemma there exists a p such that every $w \in L$ such that $|w| \geq p$ can be represented as $x y z$ with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $0^p 12^p$. Its length is $2p+1 \geq p$. Since the length of xy cannot exceed p , y must be of the form 0^p for some $p > 0$. From the pumping lemma $0^{n-p} 12^n$ must also be in L but it is not of the right form. Hence the language is not regular.