Generating Functions (Continued)

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$$\underbrace{\langle a_0, a_1, a_2, \dots, a_n, \dots \rangle}_{\text{inf. seq.}} \leftrightarrow \underbrace{A(x) = \sum_{i=0}^{\infty} a_i x^i}_{\text{gen. func.}}$$

inf. seq. = disc. numeric function

Operations:

1) scaling

Thm if $(a_0, a_1, \dots) \leftrightarrow A(x)$ then

$(ca_0, ca_1, \dots, ca_n, \dots) \leftrightarrow cA(x)$

Proof

$$(ca_0, ca_1, \dots) \leftrightarrow \sum_{i=0}^{\infty} ca_i x^i = c \underbrace{\sum_{i=0}^{\infty} a_i x^i}_{A(x)}$$

2) Addition

Thm if $(a_0, a_1, \dots, a_n, \dots) \leftrightarrow A(x)$ &

$(b_0, b_1, \dots, b_n, \dots) \leftrightarrow B(x)$

then $(a_0+b_0, a_1+b_1, \dots, a_n+b_n, \dots) \leftrightarrow A(x)+B(x)$

Proof

(exercise)

e.g., $(2,0,2,0,\dots) \leftrightarrow ?$

$$(1,1,1,\dots) \leftrightarrow \frac{1}{1-x}$$

$$+\ (1,-1,1,\dots) \leftrightarrow \frac{1}{1+x}$$

$$(2,0,2,\dots) \leftrightarrow \frac{1}{1-x}+\frac{1}{1+x} = \frac{1+x+1-x}{1-x^2} = \frac{2}{1-x^2}$$

$$(1,0,1,0,\dots) \leftrightarrow \frac{1}{2}\cdot\frac{2}{1-x^2} = 1/1-x^2$$

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3) Shifting Right

Thm if $(a_0, a_1, \ldots, a_n, \ldots) \Leftrightarrow A(x)$

then $(\underbrace{0, 0, \ldots, 0}_{k\ 0\text{'s}}, a_0, a_1, \ldots) \Leftrightarrow x^k \cdot A(x)$

Proof

$$(0, 0, \ldots, 0, a_0, a_1, \ldots) \Leftrightarrow 0 + 0 \cdot x + \cdots 0 \cdot x^{k-1} + a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \ldots$$

$$= x^k \underbrace{(a_0 + a_1 x + a_2 x^2 + \ldots)}_{A(x)} \qquad \square$$

4) Differentiation

Thm

if $(a_0, a_1, \dots, a_n, \dots) \leftrightarrow A(x)$

then $(a_1, 2a_2, 3a_3, \dots, na_n, \dots) \leftrightarrow \frac{d}{dx} A(x)$

Proof

$$(a_1, 2a_2, 3a_3, \dots) \leftrightarrow a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$= \frac{d}{dx}\underbrace{(a_0 + a_1 x + a_2 x^2 + \dots)}_{}$$

$$\frac{d}{dx} A(x) \quad \square$$

Two effects

1) multiply each term with its index

2) shifting the seq. left (once)

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$(1,1,1,\ldots) \Longleftrightarrow \frac{1}{1-x}$

$(1,2,3,\ldots) \Longleftrightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

e.g., Determine closed form expr. for the generating function of $(0,1,4,9,\ldots)$

shift right $\rightarrow \overset{0}{(0}, \overset{1}{1}, \overset{2}{2}, \overset{3}{3}, \ldots) \Longleftrightarrow x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

take derivative $\rightarrow (1.1, 2.2, 3.3, \ldots) \Longleftrightarrow \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) =$

$$\frac{1\cdot(1-x)^{\cancel{2}} - x\cdot 2\cancel{(1-x)}(-1)}{\cancel{(1-x)}^4}$$

$$= \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

shift right $\rightarrow$ $\boxed{(0,1,4,9,\ldots) \Longleftrightarrow x \cdot \frac{(1+x)}{(1-x)^3}}$

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5) Product

Thm: if $(a_0, a_1, a_2, \dots) \longleftrightarrow A(x)$ &

$(b_0, b_1, b_2, \dots) \longleftrightarrow B(x)$

then

$$(c_0, c_1, c_2, \dots) \longleftrightarrow C(x) = A(x) \cdot B(x)$$

where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$

Proof

$$
\begin{array}{c|c|cccc}
 & & & B(x) & & \\
 & & b_0x^0 & b_1x^1 & b_2x^2 & \dots \\
\hline
 & a_0x^0 & a_0b_0x^0 & a_0b_1x^1 & a_0b_2x^2 & \dots \\
A(x) & a_1x^1 & a_1b_0x^1 & a_1b_1x^2 & a_1b_2x^3 & \dots \\
 & a_2x^2 & a_2b_0x^2 & a_1\;x^3 & a_1b_2x^4 & \dots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$c_0 = a_0 b_0$

$c_1 = a_1 b_0 + a_0 b_1$

$c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$

$\vdots$

convolution operation $\rightarrow$ signal processing

(conv. of two signal seq.s)

e.g., Solve the rec. relⁿ for Bubble sort using generating functions.

$a_n = a_{n-1} + n - 1 \quad n \geq 1$

$a_0 = 0$

$(a_0, a_1, a_2, \ldots)$

$(0, 0, 1, \ldots, a_n, \ldots)$

?

$$\sum_{i=1}^{\infty} a_n x^n = \sum_{i=1}^{\infty} (a_{n-1} + n - 1) x^n$$

$$A(x) - a_0 x^0 = x \sum_{i=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{i=1}^{\infty} n x^{n-1} - \sum_{i=1}^{\infty} x^n$$

$(1, 1, \ldots)$

$\leftrightarrow 1 + x + x^2$

$= \frac{1}{1-x}$

$$A(x) = x \cdot A(x) + x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right) - \left(\frac{1}{1-x} - 1\right)$$

$$A(x) - x A(x) = \frac{x}{(1-x)^2} - \frac{1}{(1-x)} + 1$$

$$A(x) = \frac{x}{(1-x)^3} - \frac{1}{(1-x)^2} + \frac{1}{(1-x)}$$

$$(1, 1, 1, \ldots, 1, \ldots) \longleftrightarrow \boxed{\frac{1}{1-x}}$$

$$\times(-1) \qquad -1 \times \; (1, 2, 3, \ldots, n+1, \ldots) \longleftrightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \boxed{\frac{1}{(1-x)^2}}$$

$$(2.1, 3.2, \ldots, (n+2)(n+1), \ldots) \longleftrightarrow \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{2}{(1-x)^3}$$

$$\left(\frac{2.1}{2}, \frac{3.2}{2}, \ldots, \frac{(n+2)(n+1)}{2}, \ldots\right) \longleftrightarrow \frac{1}{(1-x)^3}$$

$$\left(0, \frac{2.1}{2}, \frac{3.2}{2}, \ldots, \frac{n(n+1)}{2}, \ldots\right) \longleftrightarrow \boxed{\frac{x}{(1-x)^2}}$$

$$1 - (n+1) + \frac{n(n+1)}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$$

$$\left(0, 0, 1, \ldots, \frac{n(n-1)}{2}, \ldots\right)$$

$$a_n = \frac{n(n-1)}{2} \qquad n^{th} \text{ term}$$

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e.g., $f_n = f_{n-1} + f_{n-2} \quad n \geq 2$

$f_0 = 0 \quad f_1 = 1$

Solve this rec. rel. using gen. functions.

$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2} (f_{n-1} + f_{n-2}) x^n = x \sum_{n=2} f_{n-1} x^{n-1} + x^2 \sum_{n=2} f_{n-2} x^{n-2}$$

$$F(x) - \underbrace{\cancel{f_0 x^0}}_{0} - \underbrace{f_1 x^1}_{1 \cdot x} = x \cdot (F(x) - \underbrace{f_0 x^0}_{0}) + x^2 \cdot F(x)$$

$$F(x) - x = x F(x) + x^2 F(x)$$

$$\boxed{F(x) = \frac{x}{1 - x - x^2}} \Leftarrow \text{closed form expr.}$$

↳ use partial fractions

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \qquad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

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$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha_1 x} + \frac{B}{1-\alpha_2 x} = \frac{A - A\alpha_2 x + B - B\alpha_1 x}{(1-\alpha_1 x)(1-\alpha_2 x)}$$

$$\left.\begin{aligned} A + B &= 0 \\ -A\alpha_2 - B\alpha_1 &= 1 \end{aligned}\right\} \text{ solve for } A \,\&\, B \qquad A = \frac{1}{\sqrt{5}} \qquad B = -\frac{1}{\sqrt{5}}$$

$$F(x) = \frac{x}{1-x-x^2} = \boxed{\frac{1}{\sqrt{5}}\left(\underbrace{\frac{1}{1-\alpha_1 x}} - \underbrace{\frac{1}{1-\alpha_2 x}}\right)}$$

$$(\alpha_1^0, \alpha_1^1, \alpha_1^2, \ldots, \alpha_1^n, \ldots)$$

$$(\alpha_2^0, \alpha_2^1, \alpha_2^2, \ldots, \alpha_2^n, \ldots)$$

$$\left(\frac{1}{\sqrt{5}}(\alpha_1^0 - \alpha_2^0), \frac{1}{\sqrt{5}}(\alpha_1^1 - \alpha_2^1), \ldots, \frac{1}{\sqrt{5}}(\alpha_1^n - \alpha_2^n), \right.$$

nth

$$f_n = \frac{1}{\sqrt{5}}(\alpha_1^n - \alpha_2^n) = \boxed{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

Relations

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A binary relation R from A to B $(A, B$: sets$)$

is $R \subseteq A \times B$

notation: $(x,y) \in R$, xRy

$$|A \times B| = |A| \cdot |B|$$

e.g.) - a is a parent of b

- $a \mid b$
- $a \leq b$

$\vdots$

representing a rel? $\rightarrow$ - list the elements

- bit matrix

$R \subseteq A \times B$

$$\begin{array}{c|ccccc} & b_1 & b_2 & \cdots & b_j & \\ \hline a_1 & & & & & \\ a_2 & & & & & \\ \vdots & & & & & \\ a_i & & & & \delta_{ij} & \\ \vdots & & & & & \end{array}$$

$$\delta_{i,j} = \begin{cases} 1 & \text{if } a_i R b_j \\ 0 & \text{othw.} \end{cases}$$

# of binary relations? $\rightarrow$ $2^{|A| \cdot |B|}$ ✓

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Consider a binary rel. R from A to A

(a binary rel. R on A)

① Reflexive

R is refl. if $\forall x \in A \ xRx$

e.g., $x \le y$, $n | m$, $n \equiv m \pmod 5$ ✓

a is mother of b ✗

a has a joint paper with b ✗

② Symmetric

R is symmetric if $\forall x, y \in A \ (xRy \rightarrow yRx)$

$x \le y$, $n | m$, $n \equiv m \pmod 5$, a is mother of b

✗ ✗ ✓ ✗

③ antisymm.

R is antisymm. if $\forall x y \in A \ (xRy \wedge yRx \rightarrow x = y)$

$x \le y$ $x | y$ a is mother of b

✓ ✓ ✓

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- any rel^{n} that is both sym & antisym ?
  - $n \equiv m \pmod 3$? ✗
  - $id(A) = \{(x,x) \mid x \in A\}$ → ✓

    identity rel^{n}

- any rel. that is not sym & antisymm.

  e.g., $A = \{1,2,3\}$

  $R = \{(1,1),(1,2),(2,3),(3,2)\}$

  - not sym ✓
  - not antisymm. ✓

- transitive

  R is transitive if $\forall a,b,c \in A\ (aRb \wedge bRc \rightarrow aRc)$

  e.g., $x \le y$, $a \mid b$, $n \equiv m \pmod 3$, $id(A)$ ... ✓

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e.g., what's the # of different reflexive binary relations on a set with n elements?

A: a set $\quad |A| = n$ $\qquad 2^{n^2}$ diff. binary rel$^{\underline{n}}$'s

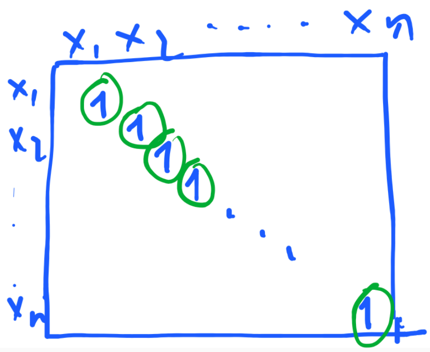
$R \subseteq A \times A$

$$\left\{ \begin{array}{l} \text{"}\{(x,x) \mid x \in A\} \cup R' \\ \\ 2^{(n^2 - n)} \end{array} \right.$$

$R' \subseteq A \times A$

# of diff R's?

$A = \{x_1, x_2, \ldots x_n\}$



all bit matrices having 1 on their diagonals

$2^{n^2 - n}$ ← $n^2 - n$ entries to fill in with a 1 or a 0.