Student Information

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Answer 1

a)

Two continuous random variables are independent if the joint pdf is a product of marginal pdfs.

Joint PDF : $\frac{1}{\pi}$

Hence the product of marginal PDFs is :

$$f_X(x)f_y(y) = \frac{4}{\pi^2}\sqrt{(1-x^2)(1-y^2)}, \qquad -1 \leq x, y \leq 1$$

Clearly, this is not equal to the joint PDF, and therefore, the two random variables are dependent.

b)

The Marginal PDF of X be found :

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2}, \qquad \mathbf{-1} \leq x \leq 1$$

The Marginal PDF of Y be found :

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx = \frac{2}{\pi}\sqrt{1-y^2}, \qquad -1 \leq y \leq 1$$

c)

$$\mu = \mathbf{E(X)} = \int_{-1}^{1} x f_X(x)\,dx = \int_{-1}^{1} x\frac{2}{\pi}\sqrt{1-x^2}dx = \mathbf{0}$$

d)

$$\sigma^2 = \mathbf{Var(X)} = \int_{-1}^{1} x^2 f(x)dx \text{ - } \mu^2 = \int_{-1}^{1} x^2\frac{2}{\pi}\sqrt{1-x^2}dx = \frac{1}{4} - 0 = \frac{1}{4}$$

Answer 2

a)

Joint Density Function:

$$f_{T_a} = 1/100 \qquad 0 \le t_a \le 100$$

$$f_{T_b} = 1/100 \qquad 0 \le t_b \le 100$$

$$f_{T_a,T_b}(t_a,t_b) = \frac{d^2}{dt_a dt_b} F_{t_a,t_b}(t_a,t_b) = f_{T_a}(t_a).f_{T_b}(t_b)$$

$$f_{T_a,T_b}(t_a,t_b) = \frac{1}{100}.\frac{1}{100} = \frac{1}{10000}$$

$$f(t_a,t_b) = \begin{cases} \frac{1}{100}.\frac{1}{100} & 0 \le t_a \le 100 \quad 0 \le t_b \le 100 \\ 0 & otherwise \end{cases}$$

Joint CDF :

Since T_a and T_b independent :

$$F_{T_a,T_b}(t_a,t_b) = P\{T_a \le t_a \cap T_b \le t_b\}$$

$$F_{T_a,T_b}(t_a,t_b) = F_{T_a}(t_a).F_{T_b}(t_b).$$

$$F_{T_a}(t_a) = \int_0^{100} f(t_a)\, dt_a$$

$$F_{T_b}(t_b) = \int_0^{100} f(t_b)\, dt_b$$

$$F_{T_a,T_b}(t_a,t_b) = \int_0^{t_a}\int_0^{t_b} \frac{1}{100}.\frac{1}{100}\, dt_a dt_b \qquad 0 \le t_a \le 100 \qquad 0 \le t_b \le 100$$

$$F_{T_a,T_b}(t_a,t_b) = \frac{t_a.t_b}{10000}$$

We can verify this formula with :

$$\mathbf{F}_{T_a,T_b}(t_a,t_b) = \int_0^{100}\int_0^{100} \frac{1}{100}.\frac{1}{100}\, dt_a dt_b \mathbf{= 1}$$

b)

We need to find A pushes the button in the first 10 seconds and subject B in the last 10 seconds with :

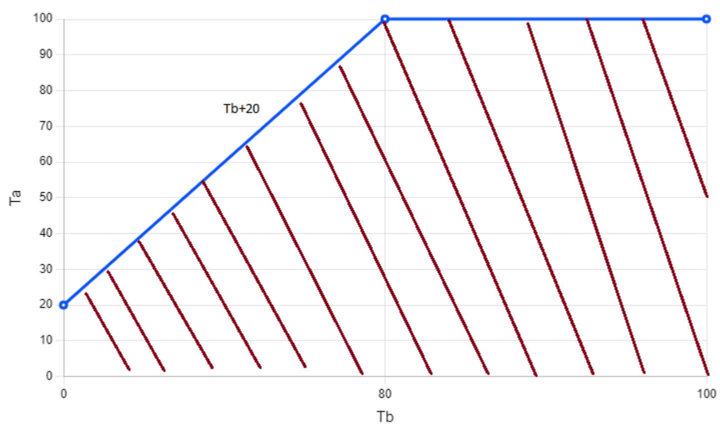
$$P\{\mathbf{T}_a \leq 10\} = F_{T_a}(10) = \int_0^{10} \frac{1}{100}\, dt_a = \frac{10}{100}$$

$$P\{\mathbf{T}_b \geq 90\} = 1 - F_{T_b}(90) = \int_0^{90} \frac{1}{100}\, dt_b = 1 - \frac{90}{100} = \frac{10}{100}$$

Since $P(T_a \leq 10)$ **and** $P(T_b \geq 90)$ **are independent :**

$$\mathbf{P\{}T_a \leq 10\}.P\{\mathbf{T}_b \geq 90\} = F_{T_a}(10).F_{T_b}(90) = \frac{10}{100}.\frac{10}{100} = \frac{1}{100}$$

c)



Graph 1.1

According to the Graph 1.1 , we can divide this question into 2 parts :

1) In the first part we should consider that A pushes at most 20 seconds after B for T_b **between 0 and 80 seconds.** T_a **can take all the values between** $[0, t_b + 20]$ **So,**

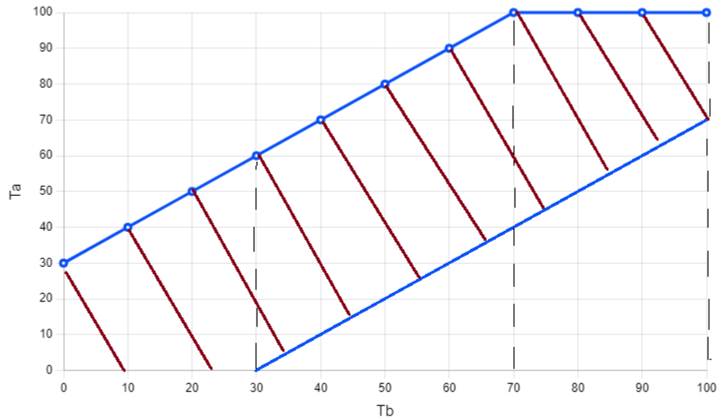
$$\int_0^{80} \frac{t_b + 20}{10000} dt_b = \mathbf{0.48} \qquad 0 \leq T_b \leq 80$$

2) In the second part we should consider that when T_b **greater than 80 seconds,** T_a **can take values between**$[0, 100]$ **. So,**

$$\int_{80}^{100} \frac{100}{10000} dt_b = \mathbf{0.20} \qquad 80 \leq T_b \leq 100$$

As a result, when we sum this independent probabilities we can get :
$$\mathbf{P(1) + P(2) = 0.48 + 0.20 = 0.68}$$

d)



Graph 1.2

According to the Graph 1.2, we can divide this question into 3 parts :

1) In the first part we should consider that A can pushes a maximum of 30 seconds before or after from B. So, T_a can take all the values between $[0, T_b + 30]$

$$\int_0^{30} \frac{t_b + 30}{10000} dt_b = \mathbf{0.135} \qquad \textbf{when} \qquad 0 \leq T_b \leq 30$$

2) In the second part we should consider that when T_b is between $(30, 70)$, T_a can take all the values between $[T_b - 30, T_b + 30]$. So,

$$\int_{30}^{70} \frac{(t_b + 30) - (t_b - 30)}{10000} dt_b = \int_{30}^{70} \frac{60}{10000} dt_b = \mathbf{0.240} \qquad \textbf{when} \qquad 30 \leq T_b \leq 70$$

3) In the third part we should consider that when T_b is greater than 70, T_a can take all tha values between $[T_b - 30, 100]$. So,

$$\int_{70}^{100} \frac{100 - (t_b - 30)}{10000} dt_b = \int_{70}^{100} \frac{130 - t_b}{10000} dt_b = \mathbf{0.135} \qquad \textbf{when} \qquad 70 \leq T_b \leq 100$$

As a result, when we sum this independent probabilities we can get :

$$\mathbf{P(1) + P(2) = 0.135 + 0.240 + 0.135 = 0.51}$$

Answer 3

a)

$F_{X_i}(x) = P(X_i \leq x) = 1 - e^{-\lambda_i x_i}$

Let T = min $\{X_1, X_2, ..., X_n\}$**. Then the cumulative distribution function of T is:**

$F_T(y) = P(T \leq t)$

$= \mathbf{1 - P(T} \geq t)$

$= 1 - P(min\{X_1, X_2, ..., X_n\} \geq t)$

$= 1 - P(X_1 \geq t, X_2 \geq t, ..., X_n \geq t)$

$= 1 - P(X_1 \geq t)P(X_2 \geq t)...P(X_n \geq t)$

$= 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} ... e^{-\lambda_n t}$

$= 1 - e^{-\lambda_1 t - \lambda_2 t ... - \lambda_n t}$

$= 1 - e^{-\sum_{i=1}^{n} \lambda_i t}$ t > 0

b)

In the Exponential Distribution E(x) $= \dfrac{1}{\lambda}$

Let T = min $\{C_1, C_2, ..., C_{10}\}$

$F_T(t) = P(T \leq t)$

$= \mathbf{1 - P(T} \geq t)$

$= 1 - e^{-\sum_{n=1}^{10} \lambda_n y}$

$\mathrm{e}^{-\sum_{n=1}^{10} \lambda_n t} = \mathrm{e}^{-0.1.t}.\ \mathrm{e}^{-0.2.t} ...\ \mathrm{e}^{-1.t} = \mathrm{e}^{-5.5t}$

CDF of T: $F_T(t) = 1 - \mathrm{e}^{-5.5t}$

PDF of T: $F_T'(t) = f_T(t) = 5.5\ \mathrm{e}^{-5.5t}$

E(x) $= \int_0^{\infty} x.(5.5)\mathrm{e}^{-5.5t} dx = 0.18$

Answer 4

a)

The number X of participants are undergraduate students has Binomial distribution with

n = 100, p = 0.74, $\mu = np$ **= 74, and** $\sigma = \sqrt{np.(1-p)}$**= 4.386**

Applying the Central Limit Theorem with the continuity correction:

$$P\{X \geq 70\} = P\{X > 69.5\} = \mathbf{1} \textbf{ - } P\{X < 69.5\}$$

$$P\{X < 69.5\} = P\{\frac{X - 74}{\sqrt{74.0.26}}\} < P\{\frac{69.5 - 74}{\sqrt{74.0.26}}\} = \Phi(-1.02591)$$

$$P\{X > 69.5\} = 1 - \Phi(-1.02591)$$

$$\Phi(-1.02591) = 0.1515$$

$$P\{X < 69.5\} == 1 - 0.1525 = 0.8485$$

b)

The number X of participants are pursuing a doctoral degree has Binomial distribution with

n = 100, p = 0.10, $\mu = np$ **= 10, and** $\sigma = \sqrt{np.(1-p)}$**= 3**

Applying the Central Limit Theorem with the continuity correction:

$$P\{X \leq 5\} = P\{X < 5.5\} = P\{\frac{X - 10}{\sqrt{10.0.9}}\} < P\{\frac{5.5 - 10}{\sqrt{10.0.9}}\} = \Phi(-1.5)$$

$$\Phi(-1.5) = 0.066807$$