Permutations and Combinations with Repetetions

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Thm

\# of r-perm's of n objects with repetitions is n^r

Proof

r positions & n alternatives for each

$\underline{n}\ \underline{n} \dots \underline{n}$ → by product rule $n \cdot n \cdot n \cdots n = n^r$

↑ 1st ↑ rth

e.g., # of passwords of length 10 with only Eng. letters → 26^{10}

Thm

\# of r-comb's of n objects with repetition is $C(n+r-1, r)$

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Proof r-comb's = r-subsets

$* \; * \; * \; * \; \cdots \; *$ r $*$'s

n objects

(each obj type is unlimited)

e.g., a committee with 4 members from a pool of engineers (E), lawyers (L), teachers (T)

$C(6,4) = C(6,2)$

$\underbrace{*}_{\#\text{ of E's}} | \underbrace{* \; *}_{\#\text{ of L}} | \underbrace{*}_{\#\text{ of T}}$ $\Rightarrow$ 1E, 2L, 1T

$| \, | * * * *$ $\Rightarrow$ 4T's

- introduce n-1 separators (|) to this string $\Rightarrow$ an encoding of the committee

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$** | *** | * . - - - | * \rightarrow$ a possible encoding

# of such strings = # of r-comb's of n obj's with repet.

$$C(\underbrace{r+n-1}_{\text{length of binary string}}, \underbrace{\overset{n-1}{\overset{\shortparallel}{r}}}_{\text{\# of *'s or \# of 1's}})$$

e.g., # of sol'n's to $x+y+z=100$ where $x, y, z \geq 0$ (integers)

$r=100$

$n=3$

$$\underbrace{\circ\ \circ\ \circ}_{x} | \underbrace{\circ\ \circ\ - -}_{y} | \underbrace{\cdots\ \circ}_{z} \leftarrow 100 \text{ beans}$$

$C(102, 100) = C(102, 2)$

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Thm # of diff. perm's of n obj's with $n_1, n_2, \ldots, n_k$ indistinguishable objects of types $1, 2, \ldots, k$ is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Proof

if all obj's were distinct, we would have

- $n!$ diff. perm's.

- for obj of type k there are n_k instances

$n_k!$ of orderings are not counted once in $n!$

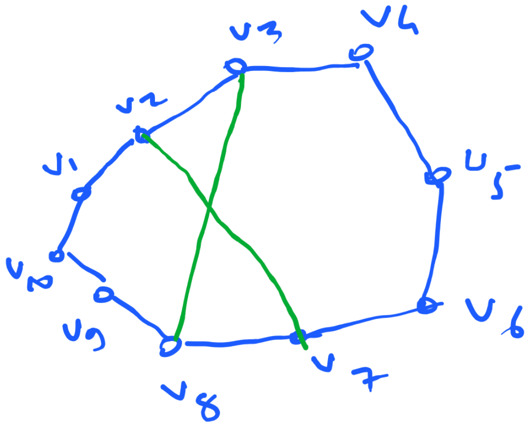
$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

e.g., the number of different permutations of letters of BİLGİSAYAR

i_2 i_1

$$\frac{10!}{2!\cdot 2!}$$ ✓

e.g., consider a convex decagon where no 3 diagonals meet at the same point inside the polygon. # of line segments on the diagonals



First count # of diagonal

- $\frac{10\cdot(10-3)}{2} = 35$
- $C(10,2) - 10 = \frac{10\cdot 9}{2} = 45 - 10 = 35$

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a digonal with k intersection points has $k+1$ line segments

$$\#\text{ of line segments} = 35 + \#\text{ of intersection points} \times 2$$

$$= 35 + C(10,4) \times 2$$

$$C(10,4) = 210$$

$$= 35 + 420 = 455$$

line segments!

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Thm $n \geq 1$ $\displaystyle\sum_{k=0}^{n} k C_n^k = n \cdot 2^{n-1}$

BASIS $n=1$ $\displaystyle\sum_{k=0}^{1} k C_1^k = 0\cdot \underbrace{C_1^0}_{1} + 1\cdot \underbrace{C_1^1}_{1} = 1 = 1\cdot 2^{1-1} = 1$ ✓

IND. STEP

Assume that $\displaystyle\sum_{k=0}^{n} k C_n^k = 0\cdot \cancel{C_n^0} + \sum_{k=1}^{n} C_n^k = n\cdot 2^{n-1}$ ← IND. HYP.

Consider n+1 ?

$$\sum_{k=0}^{n+1} k C_{n+1}^k = \sum_{k=1}^{n+1} k C_{n+1}^k = \sum_{k=1}^{n} k \boxed{C_{n+1}^k} + (n+1)\cdot \underbrace{C_{n+1}^{n+1}}_{1}$$

Pascal's identity

$$= \sum_{k=1}^{n} k\left(C_n^{k-1} + C_n^k\right) + (n+1)$$

$$= \sum_{k=1}^{n} k C_n^{k-1} + \sum_{k=1}^{n} k C_n^k + (n+1) = \sum_{k=1}^{n} (k-1) C_n^{k-1} + \sum_{k=1}^{n} C_n^{k-1} + \sum_{k=1}^{n} k C_n^k + (n+1)$$

Sum of nth row of Pascal's triangle exc. C_n^n

IND. HYP

IND HYP

$$= n\cdot 2^{n-1} - n\underbrace{C_n^n}_{1} + 2^n - \underbrace{C_n^n}_{1} + n\,2^{n-1} + n + 1$$

$$= 2\cdot n\cdot 2^{n-1} - \cancel{n} + 2^n - \cancel{1} + \cancel{n} + \cancel{1} = n\cdot 2^n + 2^n = (n+1)\,2^n$$