Take Home Exam 3

**Question 1**

Theorem (The Well Ordering Principle): A least element exists in any non empty set of positive integers.

Assume let A = {n | n ∈ , and 0 < n < 1}.

If A ∅, then by the Well-Ordering Principle A has a smallest element, say n∈ A.

But then multiplying the inequality 0 < n < 1 by the positive integer n, we have 0 < n2< n < 1.

However , n2 is an integer and so n2 ∈A, that contradicts n is the smallest element of A. Thus, our original assumption is not correct, so there does not exist an integer n satisfying 0 < n < 1.

As a result, we have proved that 1 is the smallest positive integer.

**Question 2**

**S(m,n): T**he number of possible (ordered) solutions to X1 +X2 +. . . + Xm = n Xi {0}

**Basis Step**

S(m,1)  X1 +X2+X3 … +Xm = 1

If X1 = 1, then X2,X3…,Xm are equal to 0.

X2 = 1, then X1,X3…,Xm are equal to 0.

X3 = 1, then X1,X2…,Xm are equal to 0.

….

Xm = 1, then X1,X2…,Xm-1 are equal to 0.

Therefore, there are m different cases for S(m,1).

According to the formula , S(m,n) = (n+m-1)! / (n! . (m-1)!)

S(m,1) = (m!) / (m-1)! = m

**This is true for the formula.**

S(1,n)  X1 = n

Therefore, there is only one case for S(1,n).

According to the formula , S(m,n) = (n+m-1)! / (n! . (m-1)!)

S(n,1) = (n!) / (n)! = 1

**This is also true for the formula.**

**Induction Step**

Assume S(m + 1, n) and S(m, n + 1) are true.

S(m + 1, n) = (n+m)! / (n! . m!)

S(m, n+1) = (n+m)! / ((n+1)! . (m-1)!)

We can evaluate S(m+1, n+1) in two different situations,

1. When Xm+1 = 0 ,

X1 +X2+X3 … +Xm +Xm+1 = n+1

0

There are S(m, n+1) solutions for this equation.

1. When Xm+1 > 0, Xm+1

X1 +X2+X3 … +Xm +Xm+1 = n+1

1,2,3,…

If we reduce both sides with 1,

X1 +X2+X3 … +Xm +Xm+1 = n

0,1,2…

There are S(m+1, n) solutions for this equation.

As a result, S(m+1, n+1) equation has S(m, n+1) + S(m+1, n) distinct solutions.

According to the formula (n+m-1)! / (n! . (m-1)!),

S(m+1, n+1) = (n+m+1)! / ((n+1)! . m!),

According to the induction,

S(m+1, n+1) = S(m+1, n) + S(m, n+1)

= [(n+m)! / (n! . m!)] + [(n+m)! / ((n+1)! . (m-1)!)]

= (n+m+1)! / ((n+1)! . m!)

Both results are the same.

So by the mathematical induction, the number of possible solutions of S(m,n) is

X1 +X2+X3 … +Xm = n is, (n+m-1)! / (n! . (m-1)!)

**Question 3**

a) We can find 4 different conditions in that we place triangles that are the same size in any rotation.

First case, we can place 28 of a triangle like this.

Second case, we can place 21 of a triangle like this .

Third case, we can place 21 of a triangle like this .

Fourth case, we can place 21 of a triangle like this.

When we add them together, 91 triangles can be placed.

b) There are 46 functions from a set with six elements to a set with four elements. However, this counts functions with fewer than four elements in the range. We must exclude those functions. To do so, we can use the Inclusion-Exclusion Principle.

46 - 36 +26 - 16 + 06

= 46 – 4 . 36 + 6 . 26 - 0

= 4096 – 2916 + 384 – 4 + 0 = 1560.

**Question 4**

1. an is the number of strings which consist of Σ = {0, 1, 2} with length of n that contain two consecutive symbols that are the same.

We can consider 2 distinct cases while trying to get an.

1. When an-1 has two same consecutive symbols, so β can be 3 different value in {0, 1, 2}.

an = \_ \_ \_ \_ \_ \_ \_ | \_ Thus, there are 3.an-1 situation.

an-1 β

1. When an-1 does not have two same consecutive symbols, we need to find all non-consecutive possibilites and make it consecutive.

an = \_ \_ \_ \_ \_ \_ \_ | \_ The number of all possible situations = 3n-1

an-1 β The number of consecutive situations = an-1

When we substitute them, we can get the number of non-consecutive situations = 3n-1 - an-1

In order to reach recurrence relations for an, we need to add the probabilities in the two different cases that we examined.

an = 3.an-1  + 3n-1 - an-1 = 3n-1 + 2.an-1

b) a1 = {}

a2  = { {0,0},{1,1},{2,2} }

a3  = { {0,0,0}, {1,0,0}, {2,0,0}, {0,0,1}, {0,0,2}, {1,1,1}, {0,1,1}, {2,1,1}, {1,1,0}, {1,1,2}, {2,2,2}, {0,2,2}, {1,2,2}, {2,2,0}, {2,2,1}}

a1 = 0

a2 = 3n-1 + 2.an-1  = 31 + 2.a1  = 3

a3 = 3n-1 + 2.an-1  = 32 + 2.a2  = 15

c) Xg = Xh  + Xp

Xh  an - 2an-1 = 0 (the characteristic equation)

We need to find characteristic roots that satisfies the characteristic equation.

r - 2 = 0 So, the characteristic root r should be 2.

Xh  = A . 2n

Xp  an - 2an-1 = 3n-1

We can assume an as = B.3 n

Xg = Xh  + Xp = A .2n + B. 3n

By solving the inital condition, a2 = A .22 + B.3 2  = 4.A + 9.B = 3 a3 = A .23 + B.3 3 = 8.A + 27.B = 15

A = - (3/2) B= 1

So, Xg = - (3/2) . 2n + 3 n