A STUDY OF BILEVEL INTEGER QUADRATIC PROGRAMMING PROBLEMS

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CERTIFICATE

This is to certify that the thesis entitled "**A study of Bilevel Integer Quadratic Programming Problems**", being presented in partial fulfillment of the requirements for the award of the degree of Masters of Science in Mathematics and Computing and submitted to **Department of Mathematics**, Thapar Institute of Engineering and Technology, Patiala, is a review work carried out under the supervision of **Dr. Vikas Sharma** .
The matter presented in this thesis has not been submitted for the award of any other degree from this or any other institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

Bilevel Programming Problem (BPP) is an intricate optimization problem which adhere to hierarchical structure and used for solving real life problems. In this dissertation, an algorithm for solving Quadratic Bilevel Integer Programming Problem is proposed. We have also reviewed an algorithm used for ranking the integer feasible solutions of Quadratic Integer Programming Problem. The thesis contains three major chapters and Bibliography in end.

First chapter consists of the introduction to Quadratic Programming Problem, Quadratic Integer Programming Problem, Bilevel Programming Problem and Quadratic Bilevel Integer Programming Problem. Some basic results and applications of these problems are discussed.

Second chapter elaborates the algorithm discussed by Renu and Puri [1]. The chapter delineates the algorithm used for ranking the integer feasible solutions of Quadratic Integer Programming Problem. We have outlined the problem formulation and the theoretical framework. Two numerical examples are discussed to explain the algorithm. One example is taken from Renu and Puri [1] and other is randomly generated in Matlab. We have also provided computational results for a large number of variables and constraints that were not provided in the paper.

In the third chapter an algorithm is proposed for finding an optimal solution of Quadratic Bilevel Integer Programming Problem (QBIPP). Also the algorithm works well for its special cases like integer linear bilevel programming problems. This algorithm can also be used to solve Linear Bilevel Integer Programming Problem (BLIPP). We have also created a code in Matlab to find optimal solution of this problem. We have shown the effectiveness of the proposed algorithm by solving QBIPP discussed in Nacera Maachou and Mustapha Moulaı [2], Ritu Narang and SR Arora [3] and BLIPP discussed in James T Moore and Jonathan F Bard [4]. One randomly generated QBIPP in Matlab is also solved. In addition, we have provided computational results for a large number of variables and constraints for both QBIPP and BLIPP.

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Chapter 1

Introduction

1.1 Quadratic Programming Problem

Quadratic programming problems (QPP) are a special class of non-linear programming problems that contain a quadratic objective function subject to linear constraints. QPP had garnered significant attention in the domain of the optimization field due to its structure, which enables it to proficiently model intricate real-life situations. For example, QPP is applied in finance to optimize portfolios and regulate the trade-off between return and risk. Markowitz [5] suggested that the main goal of an investor or banker in a portfolio optimization is to have the maximum expected return and the minimum risk (variance). The expected return (average return) is the earning that a banker anticipates from a portfolio or investment. Risk is measured by variance. The classical portfolios model [6] can be formulated as:

$$\begin{aligned} & min \sum_{j=1}^{m} \sum_{k=1}^{m} \sigma_{jk} X_j X_k \\ & \text{s.t.} \sum_{j=1}^{m} \mu_j X_j = \rho \\ & \sum_{j=1}^{m} X_j = 1 \\ & X_j \geq 0 \qquad j = 1, 2, ..., m \end{aligned} \tag{1.1}$$

where X_j is the fraction of a given capital that should be invested in each asset j, m is the total number of assets, σ_{jk} is the covariance of returns of both assets i and j, μ_j is the expected return of asset j and ρ is the required return.

A general QPP can be formulated as:

$$\begin{aligned} \min_{X} \; & X^T AX + C^T X \\ \text{s.t. } & BX \le b \qquad (1.2) \\ & X \ge 0 \end{aligned}$$

where X is the m-dimensional of variables, A is an $m * m$ real symmetric matrix, C is an m-dimensional vector, b is an n-dimensional vector and B is an $n * m$ matrix.

QPP can be convex or non-convex based on the matrix A. If the matrix A is positive semi-definite (i.e. $X^T AX \ge 0$ for all $X \ge 0$), then the QPP is convex, whereas if the matrix A is indefinite (i.e. there exist X_1 and X_2 such that $X_1{}^T AX_1 < 0$ and $X_2{}^T AX_2 > 0$), then the QPP is non convex.

Quadratic Integer Programming Problems (QIPP) are a special class of QPP in which decision variables are constrained or restricted to integer values. The structure of QIPP made them NP-hard, i.e.,in polynomial time, no known algorithm has been able to solve all instances effectively. QIPP is very useful in real-life applications that require integer solutions such as scheduling and resource allocation. QIPP can be formulated as:

$$\begin{aligned} \min_{x} \; & X^T AX + C^T X \\ \text{s.t. } & BX \le b \qquad (1.3) \\ & X \ge 0 \text{ and an integer vector} \end{aligned}$$

where X is the m-dimensional integer vector, A is an $m * m$ real symmetric matrix, C is an m-dimensional vector, b is an n-dimensional vector and B is an $n * m$ matrix.

Many diverse fields such as finance [5], economics [7], marketing [8] and production and operatons [9] have utilized QPP to model and solve problems.

1.1.1 Some methods to solve Quadratic Programming Problem

The majority of methods for solving QPP can be classified as either active-set methods or interior methods. The methods to solve the QPP depend on its convexity. Active-set methods is used for obtaining solutions of both convex and nonconvex QPP. Interior point methods are suitable for solving large convex QPP. Due to complex nature of QIPP there are only few algorithms to solve them such as Branch and Bound Algorithm [10], Enumeration Algorithm [11] and Ranking Algorithm [1].

1. **Active Set Methods [12, 13, 14] :** The active-set methods are iterative methods in which feasible solutions are searched along the faces and edges of a feasible region, and some of the inequality constraints are made to undergo temporary binding (i.e., the left side of the constraints is made equal to the right side) in each iteration. These constraints that undergo binding are referred to as the active set. The first step in the algorithm is finding an initial feasible solution for the QPP and the active set that satisfies this solution, and then it is checked whether the feasible solution is optimal or not. If the solution is optimal, the desired result is found; otherwise, an optimal solution can be found by changing, one by one, the constraints in the active set. The solution is obtained in a few steps because, with a change in a constraint in the active set, the objective function value decreases and there is no repetition of the active set. Some of the algorithms that uses active set method to solve QPP are Wolfe's algorithm [15], Dantzig's Algorithm [16, 17] and Beale's Algorithm [18].

2. **Interior-Point Methods [12, 19]:** Interior point methods are also known as barrier, trajectory-following, or path-following methods. Unlike boundary-based methods, these methods solve QPP by computing solutions that lie inside the feasible region by following a continuous path towards the optimal solution. A positive scalar parameterizing the path can be understood as a perturbation of the problem's optimal conditions. At each iteration, this parameter acts as a regularization parameter for the linear equations, which is important for the stability and convergence of the method.

3. **Branch and Bound (B&B) Algorithm [10] :** This algorithm functions as a tree search procedure in which every node on the tree is a different relaxed QPP sub-instance. When the integer constraint is relaxed in a QIPP, meaning the decision variables are allowed to assume any continuous values, then the QIPP becomes a relaxed QPP. In this structure, the tree root corresponds to the original relaxed QPP. By the branching method, each parent or primitive node produces two child nodes. The nodes that have no child nodes are called the leaf nodes. The algorithm starts with finding the solution of relaxed QPP which acts as initial bound for the objective function then by branching, the main QPP is broken down into two smaller sub-problems (nodes) with updated constraints. For each sub-problem (node), bounds are calculated to find the best optimal solution and if there is no improvement in the solution then that node is discarded. This process continues till the feasible solution is found.

4. **Enumeration Algorithm [11] :** This algorithm also known as Brute Force or Exhaustive Search, is used for solving QIPP that contains smaller feasible region or where other intricate algorithms don't work. This algorithm creates and evaluates every possible combination or permutation of variables in a systematic manner. The algorithm exhaustively compares all potential solutions to find the optimal solution that either maximizes or minimizes the QIPP.

5. **Ranking Algorithm [1]:** In this algorithm, corresponding to a QIPP, we construct a linear integer programming problem (LIPP), which yields limits on values of the quadratic objective function. To rank the integer feasible solutions of QIPP according their objective function values, we systematically scan the feasible integer solutions of the associated integer programming problem.

1.2 Bilevel Programming Problem

Bilevel Programming Problem (BPP) is an intricate optimization problem which adhere to hierarchical structure. It involves two interconnected optimization problems namely leader (upper level) and follower (lower level). The leader's main aim is to optimize its objective function while taking into account the follower's optimal solution, which itself is a main part of the leader's decision.

Bilevel Programming Problem can be formulated as:

$$\begin{aligned}
&\min_{Y \in S} F(Y) \\
&\text{where } z \text{ solves} \\
&\min_{z} f(Y) \\
&s.t.\ P_1 y + P_2 z \leq b\ y \\
&\quad ,\ z \geq 0
\end{aligned} \tag{1.4}$$

where $Y = (y, z) \in R^M$; $M = m_1 + m_2$, $y \in R^{m1}$, $z \in R^{m2}$ are leader and follower variables respectively. $F(Y)$ and $f(Y)$ are upper and lower level objective function respectively. $P_1 \in R^{n*m1}$, $P_2 \in R^{n*m2}$ and $S \in R^M$.

The set $S \subset R^M$ defines the bounded common constraint region :

$$S = \{X \in R^M : P_1 y + P_2 z \leq b;\ x,\ z \geq 0\}$$

In lower level programming problem, the follower chooses z in order to minimize its objective function for a given y value.

$$\begin{aligned}
&\min_{z} f(Y) \\
&s.t.\ P_2 z \leq b - P_1 y\ z \\
&\quad \geq 0
\end{aligned} \tag{1.5}$$

Assuming that $M(y)$ is a point-to-point mapping and non-empty, let $M(Y)$ represent the set of optimal feasible solutions to (1.5).

The set of feasible values where the leader minimizes its objective function is called as inducible region (IR).

$$IR = \{(y, z) : (y, z) \in S, z \in M(y)\}$$

In a Bilevel Programming Problem, an objective function at both levels can be linear, quadratic, integer linear, linear fractional, or integer quadratic. If the objective function at one level or both levels is quadratic, then BPP is called a Quadratic Bilevel Programming Problem (QBPP). QBPP gets transformed into the Quadratic Bilevel Integer Programming Problem (QBIPP) if integer restrictions are imposed on variables at both levels.

Many diverse fields such as revenue management [20], energy management [21], congestion management [22], Stackelberg-Nash game [23], and hazardous waste management [24] have utilized BPP to model and solve problems. In revenue management [20], BPP plays an essential role in optimizing inventory and pricing decisions. For example, a firm or service provider (leader) sets prices, while the customers (follower) react on the basis of those prices either they are purchasing it or not.

In energy management [21], BPP plays a crucial role in optimizing decision making between different stakeholders. For example, in power generation and pricing, a utility company (leader) adjusts wholesale electricity prices while taking into the account the optimal consumption behavior of consumers (follower).

BPP also finds applications in the area of transportation and telecommunications [22]. For example, a network operator or planner (leader) adjusts policies or infrastructure investments in a way, which help in reducing congestion, while the users or customers (follower) decide their behaviors depending on these decisions.

Stackelberg Nash game [23] is a strategic structure that combines the notion of Stackelberg competition and Nash equilibrium. It focus on hierarchical interconnections between leaders and followers. In this model, the initial decision is made by the leader which involves setting prices or production levels, while taking into the account leader decision, follower reacts with its optimal strategies.

BPP plays a crucial role in the management of hazardous materials [24]. It aids in optimizing decision-making that balances cost and safety. It involves categorizing, analyzing, and ensuring adherence to regulations in order to mitigate health risks.

1.2.1 Some methods to solve Bilevel Programming Problem

Algorithms such descent methods, penalty method, Karush-Kuhn-Tucker (KKT) approach and extreme point approach are commonly employed to solve bilevel programming problems. There are only a few algorithms to solve QBIPP, which are branch and bound algorithm [2], an algorithm for solving indefinite QBIPP with bounded variables [3], algorithm to solve QBIPP using the dual simplex approach and Gomory cut [29].

1. **Descent Methods [25, 26] :** Descent methods are used iteratively for solving Bilevel Programming Problem by moving in the direction of steepest descent (or gradient descent). These methods can be compared to climbing down a hill for searching the effective and best route. It means solving one part of the problem, then adjusting another part based on the obtained solution, repeating until we find the optimal solution.

2. **Penalty Method [27]:** Penalty method is used for solving Bilevel Programming Problem by adding penalties to the objective function of leader based on violations of follower constraints, efficacious converting the problem into single level programming problem.

3. **Karush-Kuhn-Tucker (KKT) approach [28]:** KKT approach involves implementing optimality conditions to change BPP into single-level optimization problem. Both leader and follower should satisfy set of necessary conditions which are generated by KKT approach by incorporating the follower constraints. This transformation allows the problem to be solved using standard optimization techniques, making it manageable.

4. **Extreme Point Approach [28]:** Extreme Point Approach is a systematic approach used for solving BPP by locating the extreme (corner) points of the feasible region which is defined by follower and leader constraints. Extreme points are obtained through linear programming. For each and every corner point, firstly the optimal solution for leader is found. After that optimal response for follower is evaluated while taking into account the follower's optimal solution. Then both optimal solutions of leader and follower are compared. If they are same then the process stops, otherwise other corner point is considered.

5. **Branch and Bound Method [2]:** This method is used for solving indefinite QBIPP. In this method, the upper level indefinite BPP is solved, the optimal solution of which lie in the set of efficient solutions of corresponding BPP. Branch and bound approach can used for finding set of efficient solutions. The obtained integer optimal solution is tested for optimality of the main problem by solving the lower level problem. The obtained integer optimal solution is used for solving lower level problem for testing the optimality of the main problem. If this solution is not optimal for the indefinite QBIPP, a cut is again added to the upper level problem and a new set of efficient solutions is found and after that new integer solution of the upper level is obtained and the process continues till new solution is found.

1.3 Summary of Thesis

In first chapter, we have discussed Quadratic Programming Problem, Quadratic Integer Programming Problem, Quadratic Bilevel Integer Programming Problem and Bilevel Programming Problem. We have listed applications and important algorithms for solving these problems. We have also stated problem formulation of Quadratic Programming Problem and Bilevel Programming Problem. Going further in this thesis, we will delineates, in length the ranking algorithm used for ranking the feasible integer solution of QIPP. We will also discuss about the proposed algorithm for solving Quadratic Bilevel Integer Programming Problem.

In the second chapter, we have discussed the problem formulation and the theoretical framework of a QIPP. We have explained an algorithm which is used for ranking the integer solutions of a QIPP. A Linear Integer Programming Problem (LIPP) is generated corresponding to QIPP to provide bounds on value of quadratic objective function. The integer feasible solutions of LIPP are ranked in the increasing order of value of the objective function. We present two numerical examples to explain the algorithm from which One is randomly generated in Matlab. We have also provided computational results for a large number of variables and constraints.

In the third chapter, we have proposed a new algorithm for finding an optimal solution of Quadratic Bilevel Integer Programming Problem. We have given problem formulation of QBIPP. This algorithm can also be utilized to solve Linear Bilevel Integer Programming Problem (BLIPP). We have also created a code in Matlab to find optimal solution of this problem. We have shown the effectiveness of the proposed algorithm by solving two QBIPP and one BLIPP which are taken from [2, 3, 4] . One randomly generated QBIPP in Matlab is also solved. In addition, we have provided computational results for a large number of variables and constraints for both QBIPP and BLIPP.

Chapter 2

Ranking In Quadratic Integer Programming Problem

In mathematical programming, Quadratic Integer Programming has been considered as a significant field. We can express many non-linear programming problems as Quadratic Integer Programming Problem (QIPP) using a quadratic objective function and linear constraints.

QIPP with complicated nonlinear restrictions can be formulated as:

$$\begin{aligned} &\min_{X} F(X) = X^T AX + C^T X \\ &\text{s.t. } BX \leq b \; h(X) \\ &\quad \leq 0 \\ &\quad X \geq 0 \text{ and integers.} \end{aligned} \tag{2.1}$$

where h(X) represents complicated nonlinear restrictions. $X \in R^N$, $b \in R^m$, $B \in R^{m*N}$, $C^T \in R^N$, h is a nonlinear r-dimensional vector function and $A \in R^{N*N}$ is real symmetric matrix

An integer Quadratic Programming Problem can be defined as:

$$
\begin{aligned}
&\min_{X} \; F(X) = X^T A X + C^T X \\
&\text{s.t. } BX \le b \\
&\quad X \ge 0 \text{ and integers.}
\end{aligned}
\tag{2.2}
$$

Let the set S represents feasible solutions of QIPP, which is assumed to be nonempty and bounded.

$$S = \{X \in R^N : BX = b, \; X \ge 0 \text{ and integers}\}$$

Linear Integer Programming Problem (LIPP) that provides lower bounds on objective function of QIPP can be constructed as:

$$\min_{X \in S} M(X) = (C^T + P)X \tag{2.3}$$

where $P^i = i^{th}$ component of $(P)^T \in R^N$ defined as $P^i = \min_{X \in S} X^T A^i \; (i = 1, 2, 3, ..., N)$ and A^i is the i^{th} column of A. Also $M(X) \le F(X) \quad \forall X \in S$

Some Basis Notations

L^j = Set of j^{th} best integer solutions of (2.3).

Q^k = Set of k^{th} best integer solutions of (2.2).

M^n = Maximum value of (2.3).

M^j = The j^{th} best objective function value of (2.3).

F^k = The k^{th} best objective function value of (2.2).

2.1 Theoretical Developments

Proposition 1. *[1] If* $M^j \ge \min\{F(X) : X \in \bigcup_{i=1}^{j} L^i\} = F(X^*)$, *then* X^* *is an optimal solution of QIPP.*

Proof. Since $F(X^*) = \min\{F(X) : X \in \bigcup_{i=1}^{j} L^i\}$, we have

$$F(X) \geq F(X^*) \ \forall \ X \in \bigcup_{i=1}^{j} L^i \tag{2.4}$$

Now,

$$M^j \geq \min\{F(X) : X \in \bigcup_{i=1}^{j} L^i\} \ \ (given) \tag{2.5}$$

We know that M^j is j^{th} best objective function value of (2.3). So,

$$M^p > M^j \ \ \forall \ \ p \geq j+1 \tag{2.6}$$

Comparing (2.5) and (3.2), we have

$$M^p > M^j \geq \min\{F(X) : X \in \bigcup_{i=1}^{j} L^i\}$$

Also $F(X) \geq G(X)$. Hence,

$$F^p \geq M^p > M^j \geq \min\{F(X) : X \in \bigcup_{i=1}^{j} L^i\}$$

$$\Longrightarrow \ F^p \geq \min\{F(X) : X \in \bigcup_{i=1}^{j} L^i\} \ = \ F(X^*)$$

Now,

$$F^p \geq F(X^*), \quad p \geq k+1 \tag{2.7}$$

From (2.4) and (2.7), we conclude that $f(X^*)$ has the least value. Thus, X^* becomes an optimal solution of QIPP. □

Corollary 1. [1] If $M^1 = \min\{F(X) : X \in L^1\} = F(X^*)$, then X^* is an optimal solution of QIPP and $Q^k = \{X \in L^1 : F(X) = M^1\}$.

Proposition 2. *[1] If* $M^j \geq \min\{F(X) : F(X) > F^{k-1}, X \in \bigcup_{i=1}^{j} L^i\} = F(X')$*, then* X'

is considered to be the k^{th} best integer solution of QIPP.

Proof. Since $F(X') = \min\{F(X) : F(X) > F^{k-1}, X \in \bigcup_{i=1}^{S} L^i\}$. So for $X \in \bigcup_{i=1}^{j} L^i$ which satisfy $F(X) > F^{k-1}$ we get,

$$F(X) \geq F(X') \tag{2.8}$$

Also, $F^p \geq M^p \quad (\because \ M(X) \leq F(X))$

Therefore, Let $p \geq q+1$ then

$F^p \geq M^p \geq M^q \geq F(X')$ (by the hypothesis)

That is,

$$F^p \geq F(X'), \qquad p \geq q+1 \tag{2.9}$$

From (2.8) and (2.9), it is clear that X' is a K^{th} best integer solution of QIPP. □

2.2 Numerical Illustrations

Example 1. Consider the following QIPP taken from [1]

$$\begin{aligned} \min_{X} F(X) &= 5x + 12y - 2x^2 - y^2 \\ \text{s.t. } 2x + y &\leq 10 \\ 4x + 5y &\geq 20 \\ x, y &\geq 0 \text{ and integers} \end{aligned} \tag{2.10}$$

where $X = (x, y)^T$ and satisfies the nonlinear additional constraint

$$2x + 8y - 2y^2 \geq 15 \tag{2.11}$$

solution. Here QIPP is

$$\min_{X \in S} F(X) = 5x + 12y - 2x^2 - y^2 \tag{2.12}$$

Algorithm 1: Ranking Algorithm

Output: Q^k

// Set of k^{th} Optimal solution of BQIPP

Initialization $j = 1, a = 1, k = 1$

$F(X) = X^T AX + CX$ // QIPP $\min\limits_{X \in S} F(X)$

$F^N = \max\limits_{X \in S} F(X)$ $S = \{X \in R^N : BX = b, X \geq 0$ *and an integer vector*$\}$

for $i = 1$ *to* N **do**

  $P(1, i) = \min\limits_{X \in S} X^T A(:, i)$ // LIPP $\min\limits_{X \in S} M(X)$

$M(X) = (C + P)X$

$M^n = \max\limits_{X \in S} M(X)$

$L^1 = \{Z \in S : M(Z) = \min\limits_{X \in S} M(X)\}$

$M^1 = M(Z)$ for $Z \in L^1$

while $M^j < \min\{F(X) : X \in \cup_{i=1}^{j} L^i\}$ **do**

  $j = j + 1$

  $S_j = \{X \in S : M(X) \geq M^{j-1} + 1\}$

  $L^j = \{Z \in S_j : M(Z) = \min\limits_{X \in S_j} M(X)\}$

  $M^j = M(Z)$ for $Z \in L^j$

  // j^{th} best objective function value of LIPP

$F(X') = \min\{F(X) : X \in \cup_{i=1}^{l} L^i\}$

$Q^1 = \{X \in \cup_{i=1}^{l} L^i : F(X) = F(X')\}$ // Optimal solution of QIPP

$k = k + 1$

while $M^l < M^n$ **do**

  while $M^j < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^i\}$ **do**

    $j = j + 1$

    $S_j = \{X \in S : M(X) \geq M^{j-1} + 1\}$

    $L^j = \{Z \in S_j : M(Z) = \min\limits_{X \in S_j} M(X)\}$

    $M^j = M(Z)$ for $Z \in L^j$

  $F(X^*) = \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^i\}$

  $Q^k = \{X \in \cup_{i=1}^{j} V^i : F(X) = F(X^*)\}$ // k^{th} best solution of QIPP

  $k = k + 1$

$r = k - 1$

$k = r + a$

repeat

  $F^k = \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^i\}$;

  $Q^k = \{X \in \cup_{i=1}^{j} L^i : F(X) = F^k\}$;

  $a = a + 1$

  $k = r + a$

until $k \leq j$

where $S = \{(x, y) \in R^2 : 2x + y \le 10,\ 4x + 5y \ge 20;\ x, y \ge 0 \text{ and integers}\}$

Here $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$, $C = \begin{pmatrix} 5 & 12 \end{pmatrix}$

First, we will construct a LIPP corresponding to QIPP.

$P^1 = \min\limits_{X \in S} \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \min\limits_{X \in S}(-2x) = -10.$

$P^2 = \min\limits_{X \in S} \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \min\limits_{X \in S}(-y) = -10.$

Thus, the related LIPP is

$$\min_{X \in S} M(X) = (C + P)X = -5x + 2y \tag{2.13}$$

Loop 1: Set $j = 1, k = 1$. On solving (2.13), we get $L^1 = (5, 0)$, $M^1 = -25$ and $F^1 = -25$.

Thus, $M^1 = \min\{F(X) : X \in L^1\} = F^1 = -25$

Hence, the optimal integer solution of QIPP is $Q^1 = \{(5, 0)\}$. Since Q^1 does not satisfy additional constraint (2.11), we will find second best integer solution of QIPP.

Loop 2: Set $k = 2, j = 2$. Let $S_2 = \{X \in S : M(X) \ge M^{j-1} + 1 = -24\}$. Now, solving $\min\limits_{X \in S_2} M(X)$, we get $L^2 = \{(4, 1)\}$, $M^2 = -18$ and $F^2 = -1$. As $M^2 < \min\{F(X) : F(X) > F^{k-1},\ X \in \cup_{i=1}^{j} L^i\} = -1$, we find next integer solution of LIPP.

Loop 2.1 Set $j = 3$. Let $S_3 = \{X \in S : M(X) \ge M^{j-1} + 1 = -17\}$. Now, solving $\min\limits_{X \in S_3} M(X)$, we get $L^3 = \{(4, 2)\}$, $M^3 = -16$ and $F^3 = 8$. As $M^3 < \min\{F(X) : F(X) > F^{k-1},\ X \in \cup_{i=1}^{j} L^i\} = -1$, we find next integer solution of LIPP.

Loop 2.2 Set $j = 4$. Let $S_4 = \{X \in S : M(X) \ge M^{j-1} + 1 = -15\}$. Now, solving $\min\limits_{X \in S_4} M(X)$, we get $L^4 = \{(3, 2)\}$, $M^4 = -11$ and $F^4 = 17$. As $M^4 < \min\{F(X) : F(X) > F^{k-1},\ X \in \cup_{i=1}^{j} L^i\} = -1$, we find next integer solution of LIPP.

Loop 2.3 Set $j = 5$. Let $S_5 = \{X \in S : M(X) \geq M^{j-1} + 1 = -10\}$. Now, solving $\min_{X \in S_5} M(X)$, we get $L^5 = \{(3, 3)\}$, $M^5 = -9$ and $F^5 = 24$. As $M^5 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = -1$, we find next integer solution of LIPP.

Loop 2.4 Set $j = 6$. Let $S_6 = \{X \in S : M(X) \geq M^{j-1} + 1 = -8\}$. Now, solving min $M(X)$ $_{X \in S_6}$, we get $L^6 = \{(3, 4)\}$, $M^6 = -7$ and $F^6 = 29$. As $M^6 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = -1$, we find next integer solution of LIPP.

Loop 2.5 Set $j = 7$. Let $S_7 = \{X \in S : M(X) \geq M^{j-1} + 1 = -6\}$. Now, solving min $M(X)$ $_{X \in S_7}$, we get $L^7 = \{(2, 3)\}$, $M^7 = -4$ and $F^7 = 29$. As $M^7 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = -1$, we find next integer solution of LIPP.

Loop 2.6 Set $j = 8$. Let $S_8 = \{X \in S : M(X) \geq M^{j-1} + 1 = -3\}$. Now, solving min $M(X)$ $_{X \in S_8}$, we get $L^8 = \{(2, 4)\}$, $M^8 = -2$ and $F^8 = 34$. As $M^8 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = -1$, we find next integer solution of LIPP.

Loop 2.7 Set $j = 9$. Let $S_9 = \{X \in S : M(X) \geq M^{j-1} + 1 = -1\}$. Now, solving min $M(X)$ $_{X \in S_9}$, we get $L^9 = \{(2, 5)\}$, $M^9 = 0$ and $F^9 = 37$. As $M^9 > \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = -1 = F^2$, the second best integer solution of QIPP is $Q^2 = (4, 1)$. Since Q^2 does not satisfy additional constraint (2.11), we will find third best integer solution of QIPP.

Loop 3: Set $k = 3$, $j = 10$. Let $S_10 = \{X \in S : M(X) \geq M^{j-1} + 1 = 1\}$. Now, solving $\min_{X \in S_{10}} M(X)$, we get $L^{10} = \{(2, 6)\}$, $M^{10} = 2$ and $F^{10} = 38$. As $M^{10} < min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 8$, we find next integer solution of LIPP.

Loop 3.1 Set $j = 11$. Let $S_{11} = \{X \in S : M(X) \geq M^{j-1} + 1 = 3\}$. Now, solving $\min_{X \in S_{11}} M(X)$, we get $L^{11} = \{(1, 4)\}$, $M^{11} = 3$ and $F^{11} = 35$. As

$M^{11} < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 8$, we find next integer solution of LIPP.

Loop 3.2 Set $j = 12$. Let $S_{12} = \{X \in S : M(X) \geq M^{j-1} + 1 = 4\}$. Now, solving $\min_{X \in S_{12}} M(X)$, we get $L^{12} = \{(1, 5)\}$, $M^{12} = 5$ and $F^{12} = 38$. As $M^{12} < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 8$, we find next integer solution of LIPP.

Loop 3.3 Set $j = 13$. Let $S_{13} = \{X \in S : M(X) \geq M^{j-1} + 1 = 6\}$. Now, solving $\min_{X \in S_{13}} M(X)$, we get $L^{13} = \{(1, 6)\}$, $M^{13} = 7$ and $F^{13} = 39$. As $M^{13} < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 8$, we find next integer solution of LIPP.

Loop 3.4 Set $j = 14$. Let $S_{14} = \{X \in S : M(X) \geq M^{j-1} + 1 = 8\}$. Now, solving $\min_{X \in S_{14}} M(X)$, we get $L^{14} = \{(0, 4)\}$, $M^{14} = 8$ and $F^{14} = 32$. As $M^{14} \geq \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 8 =: F^3$, the third best integer solution of QIPP is $Q^3 = (4, 2)$. Since $Q^3 = (4, 2)$ satisfy the additional constraint (2.11), it becomes an optimal solution of problem (2.10).

Example 2. Consider the problem

$$\begin{aligned} &\min F(X) = 5x + 57y + 82x^2 + 80y^2 + 354xy \\ &\text{s.t. } 13x + 2y \leq 14 \; 12x \\ &\quad + 2y \leq 37 \\ &\quad x, y \geq 0 \text{ and integers} \end{aligned} \tag{2.14}$$

where $X = (x, y)^T$ and satisfies the nonlinear additional constraint

$$2x + 9y + y^2 \geq 15 \tag{2.15}$$

solution. Here QIPP is

$$\min_{X\in S} F(X) = 5x + 57y + 82x^2 + 80y^2 + 354xy \tag{2.16}$$

where $S = \{(x, y) \in R^2 : 13x + 2y \leq 14,\ 12x + 2y \leq 37;\ x, y \geq 0 \text{ and integers}\}$

Here $A = \begin{bmatrix} 82 & 177 \\ 177 & 80 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 57 \end{bmatrix}$

First, we will construct a LIPP corresponding to QIPP.

$P^1 = \min_{X\in S} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 82 \\ 177 \end{bmatrix} = \min_{X\in S}(82x + 177y) = 0.$

$P^2 = \min_{X\in S} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 177 \\ 80 \end{bmatrix} = \min_{X\in S}(177x + 80y) = 0.$

Thus, the related LIPP is

$$\min_{X\in S} M(X) = (C + P)X = 5x + 57y \tag{2.17}$$

$$M^n = \max_{X\in S} M(X) = 399$$

Loop 1: Set $k = 1, j = 1$. On solving (2.17), we get $L^1 = (0, 0)$, $M^1 = 0$ and $F^1 = 0$.

Thus, $M^1 = min\{F(X) : X \in L^1\} = F^1 = 0$

Hence, the optimal integer solution of QIPP is $Q^1 = \{(0, 0)\}$. Since Q^1 does not satisfy additional constraint (2.15), we will find second best integer solution of QIPP.

Loop 2: Set $k = 2, j = 2$. Let $S_2 = \{X \in S : M(X) \geq M^{j-1} + 1 = 1\}$. Now, solving $\min_{X\in S_2} M(X)$, we get $L^2 = \{(1, 0)\}$, $M^2 = 5$ and $F^2 = 87$. As $M^2 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$, we find next integer solution of LIPP.

Loop 2.1 Set $j = 3$. Let $S_3 = \{X \in S : M(X) \geq M^{j-1} + 1 = 6\}$. Now, solving $\min_{X\in S_3} M(X)$, we get $L^3 = \{(0, 1)\}$, $M^3 = 57$ and $F^3 = 137$. As $M^3 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$, we find next integer solution of LIPP.

Loop 2.2 Set $j = 4$. Let $S_4 = \{X \in S : M(X) \geq M^{j-1} + 1 = 58\}$. Now, solving $\min_{X \in S_4} M(X)$, we get $L^4 = \{(0,2)\}$, $M^4 = 114$ and $F^4 = 434$. As $M^4 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$, we find next integer solution of LIPP.

Loop 2.3 Set $j = 5$. Let $S_5 = \{X \in S : M(X) \geq M^{j-1} + 1 = 115\}$. Now, solving $\min_{X \in S_5} M(X)$, we get $L^5 = \{(0,3)\}$, $M^5 = 171$ and $F^5 = 891$. As $M^5 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$, we find next integer solution of LIPP.

Loop 2.4 Set $j = 6$. Let $S_6 = \{X \in S : M(X) \geq M^{j-1} + 1 = 172\}$. Now, solving $\min_{X \in S_6} M(X)$, we get $L^6 = \{(0,4)\}$, $M^6 = 228$ and $F^6 = 1508$. As $M^6 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = -1$, we find next integer solution of LIPP.

Loop 2.5 Set $j = 7$. Let $S_7 = \{X \in S : M(X) \geq M^{j-1} + 1 = 229\}$. Now, solving $\min_{X \in S_7} M(X)$, we get $L^7 = \{(0,5)\}$, $M^7 = 285$ and $F^7 = 2285$. As $M^7 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$, we find next integer solution of LIPP.

Loop 2.6 Set $j = 8$. Let $S_8 = \{X \in S : M(X) \geq M^{j-1} + 1 = 286\}$. Now, solving $\min_{X \in S_8} M(X)$, we get $L^8 = \{(0,6)\}$, $M^8 = 342$ and $F^8 = 3222$. As $M^8 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$, we find next integer solution of LIPP.

Loop 2.7 Set $j = 9$. Let $S_9 = \{X \in S : M(X) \geq M^{j-1} + 1 = 343\}$. Now, solving $\min_{X \in S_9} M(X)$, we get $L^9 = \{(0,7)\}$, $M^9 = 399$ and $F^9 = 4319$. As $M^9 < \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$

Since, $M^9 = M^n = 399$. Set $a = 1$ and $r = k-1 = 1$. SO, $k = r+a == 2$.

$F^2 = \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 87$

$Q^2 = \{X \in \cup_{i=1}^{j} L^j : F(X) = F^k\} = \{(1, 0)\}$

Since Q^2 does not satisfy additional constraint (2.15), we will find third best integer solution of QIPP.

Loop 3 Set $a = a + 1$, $k = r + a = 3$. Now,

$F^3 = \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 137$

$Q^3 = \{X \in \cup_{i=1}^{j} L^j : F(X) = F^k\} = \{(0, 1)\}$

Since Q^3 does not satisfy additional constraint (2.15), we will find fourth best integer solution of QIPP.

Loop 4 Set $a = a + 1$, $k = r + a = 4$. Now,

$F^4 = \min\{F(X) : F(X) > F^{k-1}, X \in \cup_{i=1}^{j} L^j\} = 434$

$Q^3 = \{X \in \cup_{i=1}^{j} L^j : F(X) = F^k\} = \{(0, 2)\}$

Since Q^4 satisfy additional constraint (2.15), it becomes an optimal solution of problem (2.14).

2.3 Computational Results

We successfully implemented the ranking algorithm in Matlab. An Intel(R) Core(TM) i3-8130U CPU running at 2.20GHz with 8 GB RAM is used to run the program on an HP laptop. Let n and c denote the number of variables and number of constraints respectively. In QIPP, the right-hand side of the constraints has been randomly generated by taking an integer value from set [0,50], and all the coefficients of the variables of the constraints and objective function have randomly taken. The ranking algorithm is used to solve a set of ten randomly generated QIPP for every case.

2.3.1 Effect of number of variables on the processing time

Tables 2.1, 2.2 and 2.3 show the mean processing time with change in the number of variables for $c = 5$, 15 and 15 respectively.

For a given number of constraints, Figure 2.1, 2.2 and 2.3 illustrates how the number of variables impacts the processing time.

Table 2.1: Processing time to find ranked solutions when $c = 5$

No of Variables (n)	Mean Time (in sec)
10	6.8629454
15	7.5999631
20	22.1908174
25	15.2540041
30	17.1451515
35	41.2893977
40	61.2720661
45	66.1806797
50	79.43398425

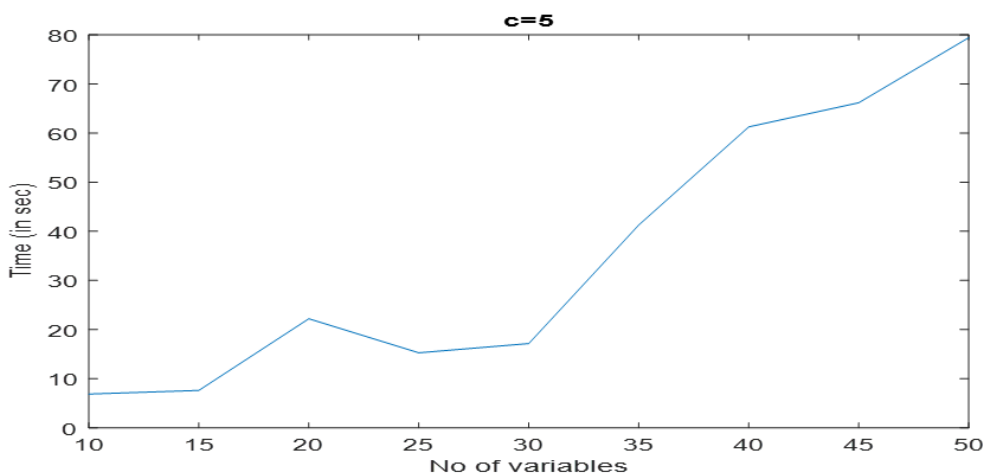

Figure 2.1: Impact of number of variables on mean time when $c = 5$

Table 2.2: Processing time to find ranked solutions when $c = 15$

No of Variables (n)	Mean Time (in sec)
20	0.809704
25	0.839423333
30	0.925664
35	0.9542085
40	1.0882818
45	1.13748
50	1.2710703

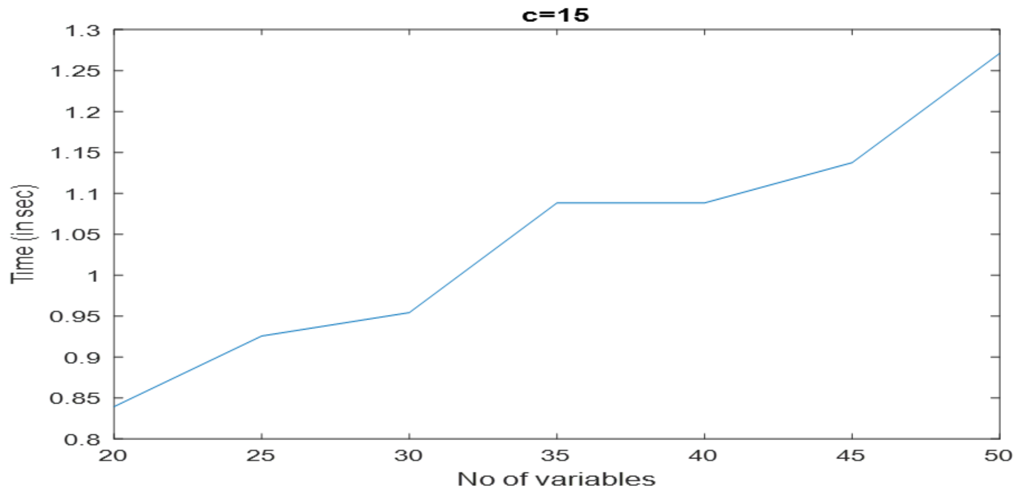

Figure 2.2: Impact of number of variables on mean time when $c = 15$

Table 2.3: Processing time to find ranked solutions when $c = 25$

No of Variables (n)	Mean Time (in sec)
30	1.1467793
35	1.2258636
40	1.2867009
45	1.3452437
50	1.4609751

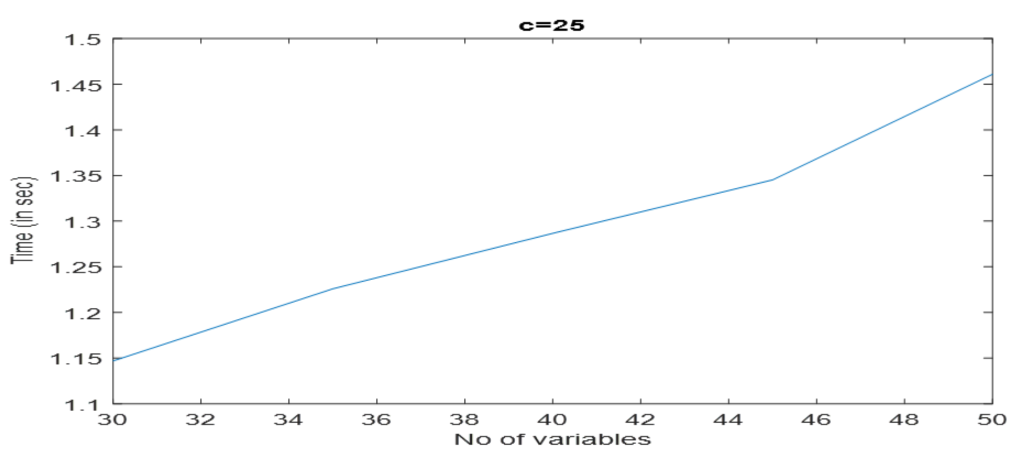

Figure 2.3: Impact of number of variables on mean time when $c = 25$

2.3.2 Effect of number of constraints on the processing time

Tables 2.4, 2.5 and 2.6 show the mean processing time with change in the number of constraints for n = 25, 35 and 45 respectively.

For a given number of constraints, Figure 2.4, 2.5 and 2.6 illustrates how the number of constraints impacts the processing time.

Table 2.4: Processing time to find ranked solutions when n = 25

No of constraints (c)	Mean Time (in sec)
4	47.9371476
8	25.9703232
12	1.230537
16	0.8575968
20	0.8617589
24	0.8547446

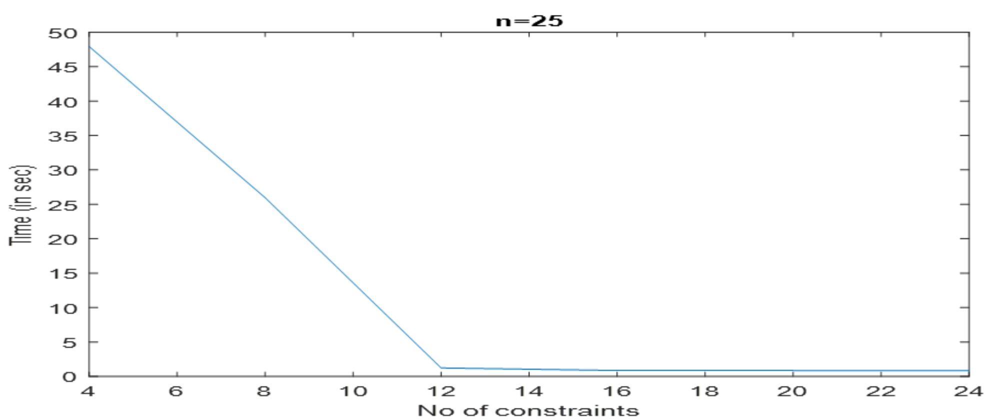

Figure 2.4: Impact of number of constraints on mean time when n = 25

Table 2.5: Processing time to find ranked solutions when n = 35

No of constraints (c)	Mean Time (in sec)
4	53.7525508
8	2.1407036
12	1.1156097
16	1.0691838
20	0.929727
24	0.9284047
28	0.9309534
32	0.9324296

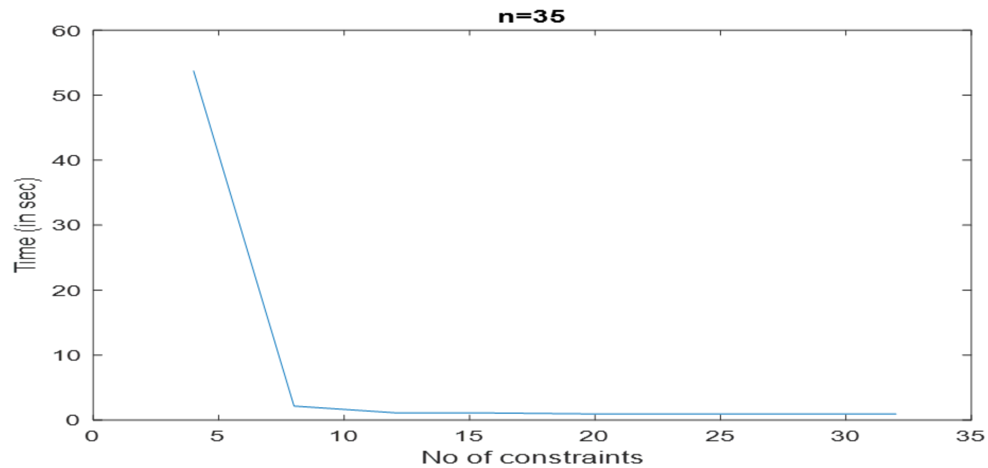

Figure 2.5: Impact of number of constraints on mean time when $n = 35$

Table 2.6: Processing time to find ranked solutions when $n = 45$

No of constraints (c)	Mean Time (in sec)
4	9.584463
8	4.3444667
12	1.1526537
16	1.0759491
20	1.0453172
24	1.0330215
28	1.0336911
32	1.042654
36	1.0360309
40	1.0340075
44	1.022898

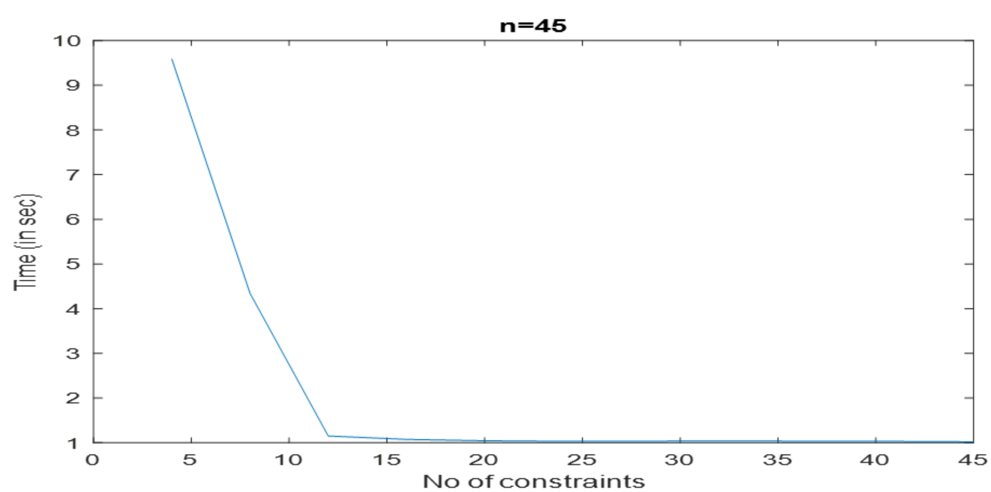

Figure 2.6: Impact of number of constraints on mean time when $n = 45$

Chapter 3

Quadratic Bilevel Integer Programming Problem

3.1 Quadratic Bilevel Integer Programming Problem

Quadratic Bilevel Integer Programming Problem (QBIPP) is stated as:

$$\min_{Y \in S} F(Y) = Y^T B_1 Y + D_1 Y + c_1$$

where z **solves**

$$\min_{z} f(Y) = Y^T B_2 Y + D_2 Y + c_2 \tag{3.1}$$

$$\text{s.t. } P_1 y + P_2 z \leq b$$

$$y, z \geq 0 \text{ and integers}$$

where $Y = (y, z) \in R^M$; $M = m_1 + m_2$, $y \in R^{m1}$, $z \in R^{m2}$ are leader and follower variables respectively. B_1, $B_2 \in R^{M*M}$ are real symmetric matrices. $D_1^T, D_2^T \in R^M$, $P_1 \in R^{n*m1}$, $P_2 \in R^{n*m2}$, $S \in R^M$ and $b \in R^n$.

The $S \subset R^M$ defines the bounded common constraint region:

$$S = \{X \in R^M : P_1 y + P_2 z \leq b;\ x, z \geq 0 \text{ and integers}\}$$

In quadratic lower level integer programming problem (QLIPP), the follower chooses z in order to minimize its objective function for a given y value.

$$\begin{aligned} &\min_{z} z^T B_2 z + D_2 z + c_2 \\ &\text{s.t. } P_2 z \le b - P_1 y \qquad (3.2) \\ &\qquad z \ge 0 \text{ and integers} \end{aligned}$$

$M(y)$ represents the set of optimal feasible solutions of (3.2). The constraint region of QLIPP is defined as

$$W = \{z \in R^{m2} : P_2 z \le b - P_1 z,\ z \ge 0 \ \textit{and integers}\}$$

The set of feasible values where the leader minimizes its objective function is called as inducible region (IR) is given as:

$$IR = \{(y, z) : (y, z) \in S, z \in M(y)\}$$

3.2 Notations

As stated in previous chapter, we can construct a linear integer programming problem (LIPP) that corresponds to a QIPP.

First, we consider quadratic upper-level quadratic integer programming problem (QUIPP)

$$\min_{Y \in S} F(Y) = Y^T B_1 Y + D_1 Y + c_1 \qquad (3.3)$$

At the upper level, one can generate a linear integer programming problem (LIPP) for a QIPP, which can be expressed as:

$$\min_{Y \in S} R_1(Y) = (D_1 + \mathbf{Q}_1)X + c_1 \qquad (3.4)$$

where $Q_1{}^i$ = i^{th} component of $(Q_1)^T \in R^M$ defined as $Q_1{}^i = \min_{Y \in S} Y^T B_1{}^i$ ($i = 1, 2, 3, ..., M$) and B_1^i is the i^{th} column of B_1.

Similarly, for a given y, we can construct a lower level integer programming problem (LLIPP) that corresponds to a QLIPP as stated below

$$\min_z f(Y) = Y^T B_2 Y + D_2 Y + c_2 \tag{3.5}$$

Lower level linear integer programming problem is

$$\min_z R_2(Y) = (D_2 + Q_2)Y + c_2 \tag{3.6}$$

where $Q_2{}^i$ = i^{th} component of $(Q_2)^T \in R^{m_2}$ defined as $Q_2{}^i = \min_{Y \in S} Y^T B_2{}^i$($i = 1, 2, 3, ..., m_2$) and B_2^i is the i^{th} column of B_2.

3.2.1 Some basis Notations

$L_1{}^l$ = Set of l^{th} best integer feasible solutions of (3.4).

L_2 = Set of optimal integer feasible solutions of (3.6).

$Q_1{}^k$ = Set of k^{th} best integer feasible solutions of (3.3).

Q_2 = Set of optimal integer feasible solutions of (3.5).

F^k = The k^{th} best objective function value of (3.3).

F^N = Maximum value of (3.3) .

Y^k = An element of k^{th} best solution of (3.3).

3.3 Special Case: Linear Bilevel Integer Programming Problem

QBIPP reduces into a linear bilevel integer programming problem (LBIPP) if the upper and lower level objective functions are linear (i.e. $B_1 = \varnothing$ *and* $B_2 = \varnothing$). Since B_1 =

$\varnothing$ *and* $B_2 = \varnothing$, problem (2.5) reduces to

$$\min_{Y \in S} F(Y) = D_1 Y + c_1$$

where z solves

$$\min_{z} f(Y) = D_2 Y + c_2 \tag{3.7}$$

s.t. $P_1 y + P_2 z \le b$

$y,\ z \ge 0$ and integers

For all $i = 1, 2, 3, \ldots, M$, we get $Q_1^i = \min_{Y \in S} Y^T B_1^l = \varnothing$ since upper level objective function is linear ($B_1 = \varnothing$). As a result, problem (3.4) simplify to the following problem.

$$\min_{Y \in S} R_1(Y) = D_1 Y + c_1 \tag{3.8}$$

Similarly, $Q_2^i = \min_{Y \in S} Y^T B_2^l = \varnothing$ for all $i = 1, 2, 3, \ldots, m_2$, since lower level objective function is linear ($B_2 = \varnothing$). As a result, problem (3.6) simplify to the following problem.

$$\min_{z} R_2(X) = D_2 Y + c_2 \tag{3.9}$$

It is clear from above equations that $F(Y) = R_1(Y)$ and $f(Y) = R_2(Y)$ for LBIPP.

3.4 Algorithm

Algorithm to obtain optimal solution of QBIPP involves following steps:

Step 1: Set $k = 1$. Solve the problem (3.3) and find Q_1^k by applying the ranking algorithm (1). Go to Step 2

Step 2: For each $Y' = (y', z') \in Q_1^k$ do the following:

Step 2(a): Put $y = y'$ in the problem (3.5) and then solve the obtained problem by applying the ranking algorithm (1) to find Q_2.

Step 2(b): For each $z^* \in Q_2$ do the following:

Let $Y^* = (y', z^*)$. If $Y' = Y^*$ then Y' is the optimal solution of QBIPP and then stop the algorithm. Otherwise, go to Step 3.

Step 3: Set $k = k + 1$ and find k^{th} best integer solutions (Q_1^k) of the problem (3.3) by ranking algorithm. Go to Step 2.

Remark: As stated above, in the case of LBIPP, $R_1(Y) = F(Y)$ and $R_2(Y) = f(Y)$, implies that we can find ranked solutions by solving the main problem; there is no need of forming any corresponding linear integer programming problem. So, for finding the solution of LBIPP, we have to rank solutions of $F(Y)$ and $f(y)$. We can modify above algorithm to find solution of LBIPP by replacing k by i, Q_1^k by L_1^i and Q_2 by L_2.

3.5 Numerical Illustrations

In this section, we have solved three BQIPP and one BLIPP utilizing the suggested algorithm. In example 1, we have solved a general three-variable BQIPP that did not factorise into two linear functions. The effectiveness of the algorithm is shown by solving example 2 and 3 taken from [2] and [3]. Additionally, we have also solved BLIPP in example 4, taken from [4].

In the examples, for each updated value of i/k we are considering a loop.

3.5.1 General BQIPP is solved using proposed algorithm

Example 1. Consider the problem

$$\min_{y_1, z_1, z_2} F(y_1, z_1, z_2) = -44y_1^2 - 70z_1^2 - 32z_2^2 - 196y_1z_1 - 236y_1z_2 - 276z_1z_2 - 12y_1 - 23z_1 - 2z_2$$

where $z = (z_1, z_2)$ solves

$$\min_{z_1, z_2} f(y_1, z_1, z_2) = -122y_1^2 - 110z_1^2 - 192z_2^2 - 200y_1z_1 - 124y_1y_2 - 162z_1z_2 - 12z_1 - 42z_2 + 10$$

s.t. $(y_1, z_1, z_2) \in S$

$$(3.10)$$

where $S = \{(y_1, z_1, z_2) : 15y_1 + 18z_1 + 13z_2 \leq 40,\ 12y_1 + 8z_1 + 13z_2 \leq 50,\ 9y_1 + 3z_1 + 6z_2 \leq 50;\ y_1,\ z_1,\ z_2 \geq 0 \text{ and integers}\}$

Solution. Consider the following UQIPP and LQIPP

$$\min_{y_1, z_1, z_2} F(y_1, z_1, z_2) = -44y_1^2 - 70z_1^2 - 32z_2^2 - 196y_1z_1 - 236y_1z_2 - 276z_1z_2 - 12y_1 - 23z_1 - 2z_2$$

s.t. $(y_1, z_1, z_2) \in S$

(3.11)

$$\min_{z_1, z_2} f(y_1, z_1, z_2) = -122y_1^2 - 110z_1^2 - 192z_2^2 - 200y_1z_1 - 124y_1y_2 - 162z_1z_2 - 12z_1 - 42z_2 + 10$$

s.t. $(z_1, z_2) \in W$

(3.12)

where $W = \{(z_1, z_2))\ 18z_1 + 13z_2 \leq 40 - 14y_1,\ 8z_1 + 13z_2 \leq 50 - 12y_1,\ 3z_1 + 6z_2 \leq 50 - 9y_1;\ z_1,\ z_2 \geq 0 \text{ and integers}\}$

Loop 1: Set $k = 1$. After solving the problem (3.15) we get $Q_1^1 = (o, 1, 1)$. For $Y' = (0, 1, 1) \in Q_1^1$. Put $y_1 = 0$ in the problem (3.16) and after solving the obtained problem we get $Q_2 = (0, 3)$. Now for $z^* = (0, 3) \in Q_2$. Let $Y^* = (0, 0, 3)$. Since $Y^* \neq Y'$.

Loop 2: Set $k = 2$ and solve the problem (3.15) to get second best solution i.e $Q_1^2 = (1, 1, 0)$. For $Y' = (1, 1, 0) \in Q_1^2$. Put $y_1 = 1$ in the problem (3.16) and then after solving the obtained problem we get $Q_2 = (0, 1)$. Now for $z^* = (0, 1) \in Q_2$. Let $Y^* = (1, 0, 1)$. Since $Y^* \neq Y'$.

Loop 3: Set $k = 3$ and solve the problem (3.15) to get second best solution i.e $Q_1^3 = (0, 2, 0)$. For $Y' = (0, 2, 0) \in Q_1^3$. Put $y_1 = 0$ in the problem (3.16) and then after solving the obtained problem we get $Q_3 = (0, 3)$. Now for $z^* = (0, 3) \in Q_3$. Let $Y^* = (0, 0, 3)$. Since $Y^* \neq Y'$.

Loop 4: Set $k = 4$ and solve the problem (3.15) to get third best solution i.e $Q_1^4 = (1, 0, 1)$. For $Y' = (1, 0, 1) \in Q_1^4$. Put $y_1 = 1$ in the problem (3.16) and then after solving the obtained problem we get $Q_4 = (0, 1)$. Now for $z^* = (1, 1) \in Q_2$. Let $Y^* = (1, 0, 1)$. Since $Y^* = Y'$. Therefore $Y = (1, 1, 1)$ is the optimal solution of problem (3.10).

3.5.2 QBIPP covered in [2] is solved by applying the proposed approach

Example 2. Consider the problem [2]

$$\begin{aligned}
&\max_{y_1, z_1, z_2} F(y_1, z_1, z_2) = (3y_1 - z_1 + 2)(-y_1 + z_1 - z_2 + 4) \\
&\text{where } z = (z_1, z_2) \text{ solves} \\
&\max_{z} f(y_1, z_1, z_2) = (y_1 + z_1 + 5z_2 - 1)(y_1 + z_1 - 3z_2 + 4) \\
&\text{s.t. } (y_1, z_1, z_2) \in S
\end{aligned} \tag{3.13}$$

where $S = \{(y_1, z_1, z_2) : 4y_1 + 3z_1 + 4z_2 \leq 20,\ y_1 + 2z_1 - z_2 \leq 3,\ 3y_1 + z_1 - z_2 \leq 4;\ y_1 ;\ z_1 ,\ z_2 \geq 0 \text{ and integers}\}$

Solution. First, solve the product in problem (3.13)

$$\begin{aligned}
&\max_{y_1, z_1, z_2} F(y_1, z_1, z_2) = -3y_1^2 - z_1^2 + 4y_1z_1 - 3y_1z_2 + z_1z_2 + 10y_1 - 2z_1 - 2z_2 + 8 \\
&\text{where } z = (z_1, z_2) \text{ solves} \\
&\max_{z} f(y_1, z_1, z_2) = y_1^2 + z_1^2 - 15z_2^2 + 2y_1z_1 + 2y_1z_2 + 2z_1z_2 + 3y_1 + 3z_1 + 23z_2 - 4 \\
&\text{s.t. } (y_1, z_1, z_2) \in S
\end{aligned} \tag{3.14}$$

Now, convert the problem from maximum to minimum.

$$\begin{aligned}
&\min_{y_1, z_1, z_2} F^1(y_1, z_1, z_2) = 3y_1^2 + z_1^2 - 4y_1z_1 + 3y_1z_2 - z_1z_2 - 10y_1 + 2z_1 + 2z_2 - 8 \\
&\text{where } z = (z_1, z_2) \text{ solves} \\
&\min_{z} f^1(y_1, z_1, z_2) = -y_1^2 - z_1^2 + 15z_2^2 - 2y_1z_1 - 2y_1z_2 - 2z_1z_2 - 3y_1 - 3z_1 - 23z_2 + 4 \\
&\text{s.t. } (y_1, z_1, z_2) \in S
\end{aligned}$$

Consider the following QUIPP and QLIPP

$$\min_{y_1, z_1, z_2} F^1(y_1, z_1, z_2) = 3y_1^2 + z_1^2 - 4y_1z_1 + 3y_1z_2 - z_1z_2 - 10y_1 + 2z_1 + 2z_2 - 8$$

$$\text{where } z = (z_1, z_2) \text{ solves} \tag{3.15}$$

$$\text{s.t. } (y_1, z_1, z_2) \in S$$

$$\min_{z} f^1(y_1, z_1, z_2) = -y_1^2 - z_1^2 + 15z_2^2 - 2y_1z_1 - 2y_1z_2 - 2z_1z_2 - 3y_1 - 3z_1 - 23z_2 + 4$$

$$\text{s.t. } z = (z_1, z_2) \in W \tag{3.16}$$

where $W = \{(z_1, z_2) : 3z_1 + 4z_2 \leq 20 - 4y_1,\ 2z_1 - z_2 \leq 3 - y_1,\ z_1 - z_2 \leq 4 - 3y_1;\ z_1,\ z_2 \geq 0 \text{ and integers}\}$

Loop 1: Set $k = 1$. After solving the problem (3.15) we get $Q_1^1 = (1, 1, 0)$. For $Y' = (1, 1, 0) \in Q_1^1$. Put $y_1 = 1$ in the problem (3.16) and after solving the obtained problem we get $Q_2 = (1, 1)$. Now for $z^* = (1, 1) \in Q_2$. Let $Y^* = (1, 1, 1)$. Since $Y^* \neq Y'$.

Loop 2: Set $k = 2$ and solve the problem (3.15) to get second best solution i.e $Q_1^2 = (1, 0, 0)$. For $Y' = (1, 0, 0) \in Q_1^2$. Put $y_1 = 1$ in the problem (3.16) and then after solving the obtained problem we get $Q_2 = (1, 1)$. Now for $z^* = (1, 1) \in Q_2$. Let $Y^* = (1, 1, 1)$. Since $Y^* \neq Y'$.

Loop 3: Set $k = 3$ and solve the problem (3.15) to get third best solution i.e $Q_1^3 = (1, 1, 1)$. For $Y' = (1, 1, 1) \in Q_1^3$. Put $y_1 = 1$ in the problem (3.16) and then after solving the obtained problem we get $Q_2 = (1, 1)$. Now for $z^* = (1, 1) \in Q_2$. Let $Y^* = (1, 1, 1)$. Since $Y^* = Y'$. Therefore $Y = (1, 1, 1)$ is the optimal solution of problem (3.13).

3.5.3 QBIPP covered in [3] is solved by applying the proposed approach

Example 3. Consider the problem [3]

$$\begin{aligned}
&\max_{y_1,z_1,z_2} F(y_1, z_1, z_2) = (2y_1 + z_1 + 1)(y_1 + 2z_1 + 2)\\
&\text{where } z = (z_1, z_2) \text{ solves}\\
&\max_{z} f(y_1, z_1, z_2) = (y_1 + z_1 + 1)(z_1 + 2z_2 + 1)\\
&\text{s.t. } (y_1, z_1, z_2) \in S
\end{aligned} \tag{3.17}$$

where $S = \{(y_1, z_1, z_2) : 2y_1 + z_1 + z_2 \le 20,\ 4y_1 + 7z_1 \le 75,\ 4y_1 - z_1 \le 22;\ 6 \le y_1 \le 8,\ 5 \le z_1 \le 7,\ 0 \le z_2 \le 1 \text{ and integers}\}$

Solution. First, Solve product in the problem (3.17)

$$\begin{aligned}
&\max_{y_1,z_1,z_2} F(y_1, z_1, z_2) = 2y_1^2 + 2z_1^2 + 5y_1z_1 + 5y_1 + 4z_1 + 2\\
&\text{where } z = (z_1, z_2) \text{ solves}\\
&\max_{z} f(y_1, z_1, z_2) = z_1^2 + y_1z_1 + 2y_1z_2 + 2z_1z_2 + y_1 + 5z_1 + 8z_2 + 4\\
&\text{s.t. } (x_1, y_1, y_2) \in S
\end{aligned} \tag{3.18}$$

Now, convert the problem from maximum to minimum.

$$\begin{aligned}
&\min_{y_1,z_1,z_2} F^1(y_1, z_1, z_2) = -2y_1^2 - 2z_1^2 - 5y_1z_1 - 5y_1 - 4z_1 - 2\\
&\text{where } z = (z_1, z_2) \text{ solves}\\
&\min_{z} f^1(y_1, z_1, z_2) = -z_1^2 - y_1z_1 - 2y_1z_2 - 2z_1z_2 - y_1 - 5z_1 - 8z_2 - 4\\
&\text{s.t. } (y_1, z_1, z_2) \in S
\end{aligned}$$

Consider the following QUIPP and QLIPP

$$\begin{aligned}
&\min_{y_1,z_1,z_2} F^1(y_1, z_1, z_2) = -2y_1^2 - 2z_1^2 - 5y_1z_1 - 5y_1 - 4z_1 - 2\\
&\text{s.t. } (y_1, z_1, z_2) \in S
\end{aligned} \tag{3.19}$$

$$\min_{z} f^1(y_1, z_1, z_2) = -z_1^2 - y_1 z_1 - 2y_1 z_2 - 2z_1 z_2 - y_1 - 5z_1 - 8z_2 - 4$$
$$\text{s.t. } z = (z_1, z_2) \in W \tag{3.20}$$

where $W = \{(z_1, z_2) : z_1 + z_2 \leq 20 - 2y_1,\ 7z_1 \leq 75 - 4y_1,\ -z_1 \leq 22 - 4y_1;\ 5 \leq z_1 \leq 7,\ 0 \leq z_2 \leq 1 \text{ and integers}\}$

Loop 1: Set $k = 1$. After solving the problem (3.19) we get $Q_2^1 = (7, 6, 0)$. For $Y' = (7, 6, 0) \in Q^1$. Put $y_1 = 7$ in the problem (3.20) and after solving the obtained problem we get $Q_2 = (6, 0)$. Now for $z^* = (6, 0) \in Q_2$. Let $Y^* = (7, 6, 0)$. Since $Y^* = Y'$.Therefore $Y = (7, 6, 0)$ is the optimal solution of problem (3.17).

3.5.4 LBIPP covered in [4] is solved by applying the proposed approach

Example 4. Consider the problem [4]

$$\max_{y_1, z_1} F(y_1, z_1) = y_1 + 10z_1$$
$$\text{where } z_1 \text{ solves}$$
$$\max_{z_1} f(y_1, z_1) = -z_1 \tag{3.21}$$
$$\text{s.t. } (y_1, z_1) \in S$$

where $S = \{(y_1, z_1) : -25y_1 + 20z_1 \leq 30,\ y_1 + 2z_1 \leq 10,\ 2y_1 + 10z_1 \geq 15,\ 2y_1 - z_1 \leq 15,\ y_1, z_1 \geq 0 \text{ and integers}\}$

Solution. Firstly, convert the problem from maximum to minimum.

$$\min_{y_1, z_1} F^1(y_1, z_1) = -y_1 - 10z_1$$
$$\text{where } z_1 \text{ solves}$$
$$\min_{z_1} f^1(y_1, z_1) = z_1$$
$$\text{s.t. } (y_1, z_1) \in S$$

Consider the QUIPP and QLIPP

$$\min_{y_1, z_1} F^1(y_1, z_1) = -y_1 - 10z_1$$
$$\text{s.t. } (y_1, z_1) \in S \tag{3.22}$$

$$\min_{z_1} f^1(y_1, z_1) = z_1$$
$$\text{s.t. } z_1 \in W \tag{3.23}$$

where $W = \{z_1 : 20z_1 \leq 30 + 25y_1,\ 2z_1 \leq 10 - y_1,\ 10z_1 \geq 15 - 2y_1,\ z_1 \leq 15 - 2y_1,\ z_1 \geq 0 \text{ and integers}\}$

Loop 1: Set $i = 1$. After solving the problem (3.22) we get $L_1^1 = (2, 4)$. For $Y' = (2, 4) \in L_1^1$. Put $y_1 = 2$ in the problem (3.23) and after solving the obtained problem we get $L_2 = 2$. Now for $z_1^* = 2 \in L_2$. Let $Y^* = (2, 2)$. Since $Y^* \boxminus Y'$.

Loop 2: Set $i = 2$ and solve the problem (3.22) to get second best solution i.e $L_1^2 = (4, 3)$. For $Y' = (4, 3) \in L_1^2$. Put $y_1 = 4$ in the problem (3.23) and then after solving the obtained problem we get $L_2 = 1$. Now for $z_1^* = 1 \in L_2$. Let $Y^* = (4, 1)$. Since $Y^* \boxminus Y'$.

Loop 3: Set $i = 3$ and solve the problem (3.22) to get third best solution i.e $L_1^3 = (3, 3)$. For $Y' = (3, 3) \in V^3$. Put $y_1 = 3$ in the problem (3.23) and then after solving the obtained problem we get $L_2 = 1$. Now for $z_1^* = 1 \in L_2$. Let $Y^* = (3, 1)$. Since $Y^* \boxminus Y'$.

Loop 4: Set $i = 4$and solve the problem (3.22) to get third best solution i.e $L_1^4 = (2, 3)$. For $Y' = (2, 3) \in L_1^4$. Put $y_1 = 2$ in the problem (3.23) and then after solving the obtained problem we get $L_2 = 2$. Now for $z_1^* = 2 \in L_2$. Let $Y^* = (2, 2)$. Since $Y^* \boxminus Y'$.

Loop 5: Set $i = 5$ and solve the problem (3.22) to get third best solution i.e $L_1^5 = (6, 2)$. For $Y' = (6, 2) \in L_1^5$. Put $y_1 = 6$ in the problem (3.23) and then after solving the obtained problem we get $L_2 := 1$. Now for $z_1^* = 1 \in L_2$. Let $Y^* = (6, 1)$. Since $Y^* \neq Y'$.

Loop 6: Set $i = 6$ and solve the problem (3.22) to get third best solution i.e $L_1^6 = (5, 2)$. For $Y' = (5, 2) \in L_1^6$. Put $y_1 = 5$ in the problem (3.23) and then after solving the obtained problem we get $L_2 = 1$.Now for $z_1^* = 1 \in L_2$. Let $Y^* = (5, 1)$. Since $Y^* \neq Y'$.

Loop 7: Set $i = 7$ and solve the problem (3.22) to get third best solution i.e $L_1^7 = (4, 2)$. For $Y' = (4, 2) \in L_1^7$. Put $y_1 = 4$ in the problem (3.23) and then after solving the obtained problem by ranking algorithm we get $L_2 = 1$. Now for $z_1^* = 1 \in L_2$. Let $Y^* = (4, 1)$. Since $Y^* \neq Y'$.

Loop 8: Set $i = 8$ and solve the problem (3.22) to get third best solution i.e $L_1^8 = (3, 2)$. For $Y' = (3, 2) \in L_1^8$. Put $y_1 = 3$ in the problem (3.23) and then after solving the obtained problem we get $L_2 = 1$. Now for $z_1^* = 1 \in L_2$. Let $Y^* = (3, 1)$. Since $Y^* \neq Y'$.

Loop 9: Set $i = 9$ and solve the problem (3.22) to get third best solution i.e $L_1^9 = (2, 2)$. For $Y' = (2, 2) \in L_1^9$. Put $y_1 = 2$ in the problem (3.23) and then after solving the obtained problem we get $L_2 = 1$. Now for $z_1^* = 1 \in L_2$. Let $Y^* = (2, 2)$. Since $Y^* = Y'$. Therefore $Y = (2, 2)$ is the optimal solution of problem (3.21).

3.6 Computational Results

We successfully implemented the proposed algorithm in Matlab. An Intel(R) Core(TM) i3-8130U CPU running at 2.20GHz with 8 GB RAM is used to run the program on an HP laptop. Let n, c and l denote the number of variables, number of constraints and number of leader variables in QBIPP/LBIPP respectively. In QBIPP/LBIPP, the constraints on the right side has been randomly generated by taking an integer value from set [0,50], and all the variables coefficients of the constraints and objective functions at upper and lower levels have randomly taken integer values from the set [0,20] and [0,100] respectively. Also, l has taken a random integer value from the set [1,n-1].

3.6.1 Computational Results for QBIPP

The proposed algorithm is used to solve a set of ten randomly generated QBIPP for every case.

Tables 1 and 2 show the mean and median processing time with changes in the number of variables for c = 5 and 15. For a given number of constraints, Figure 1 and 2 illustrates how the number of variables impacts the processing time.

Table 3 and 4 shows the mean and median processing time with change in the number of constraints for n = 25 and 35. Figure 3 and 4 depicts the impact of number of constraints on processing time.

Table 5 shows the mean and median processing time with the change in a number of leader variables for n = 50, c = 10. Figure 3 depicts the impact of the number of leader variables on processing time in the case of fixed n = 50, c = 10.

Table 3.1: Processing time to find solution of BQIPP when c = 5

No of Variables (n)	Mean Time (in sec)	Median Time (in sec)
10	0.8704381	0.8547135
15	2.6846465	0.947384
20	1.7405972	1.281208
25	2.910769	1.315445
30	3.0679229	1.6124125
35	1.9335984	1.7155085
40	12.0407656	5.305055
45	8.5911494	5.5913935
50	3.9888406	3.703602

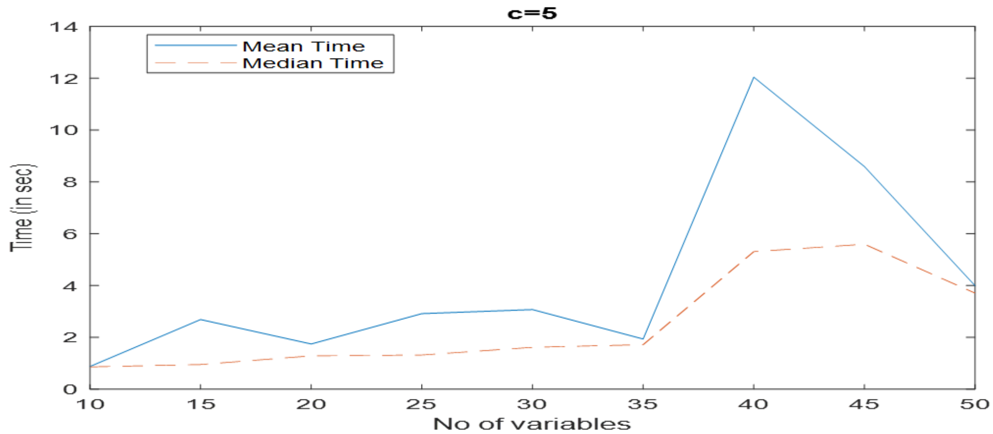

Figure 3.1: Effect of the number of variables (n) on processing when c=5

Table 3.2: Processing time to find solution of BQIPP when c = 15

No of Variables (n)	Mean Time (in sec)	Median Time (in sec)
20	1.1322523	1.08772
25	1.1382342	1.0842865
30	1.2671716	1.169107
35	1.9194351	1.490351
40	2.6088072	2.583777
45	4.1357105	3.859834
50	3.0090485	2.253644

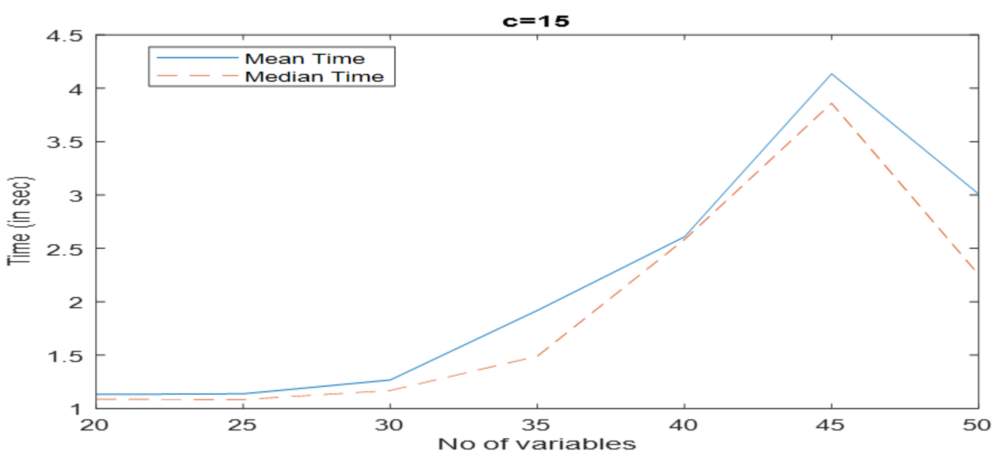

Figure 3.2: Effect of the number of variables (n) on processing when c=15

Table 3.3: Processing time to find solution of BQIPP when c = 25

No of Variables (n)	Mean Time (in sec)	Median Time (in sec)
30	1.2961176	1.2925785
35	1.6737705	1.4279015
40	1.684752	1.576623
45	2.491809	1.93338
50	3.046288	2.5205795

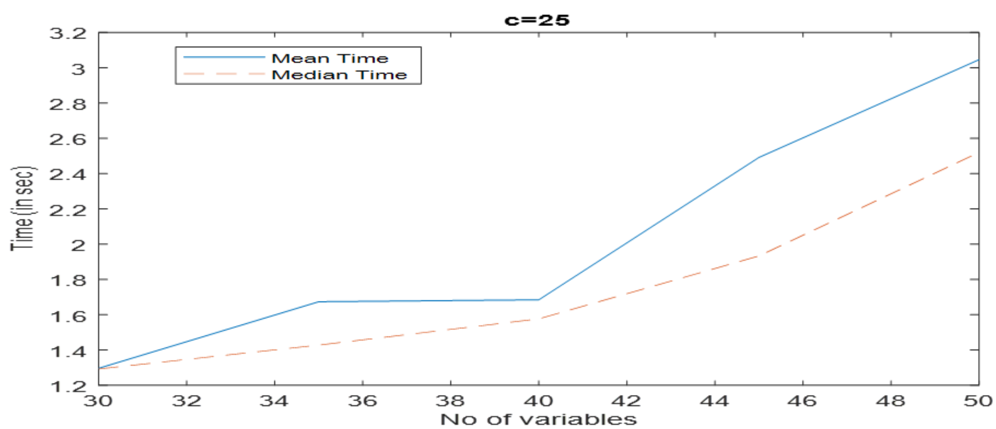

Figure 3.3: Effect of the number of variables (n) on processing when c=25

Table 3.4: Processing time to find solution of BQIPP when n = 25

No of constraints (c)	Mean Time (in sec)	Median Time (in sec)
4	2.3879834	1.899927
8	2.5798866	1.0416495
12	1.2929596	1.2047155
16	1.5711736	1.314488
20	1.0886039	1.087654
24	1.1839815	1.1611695

Table 3.5: Processing time to find solution of BQIPP when n = 35

No of constraints (c)	Mean Time (in sec)	Median Time (in sec)
4	5.720804	2.75003
8	3.2951405	2.5549685
12	1.9620735	1.7223915
16	2.1808143	2.0825195
20	1.7103677	1.5668875
24	1.5294743	1.4248375
28	1.866528	1.678558
32	1.7726681	1.692961

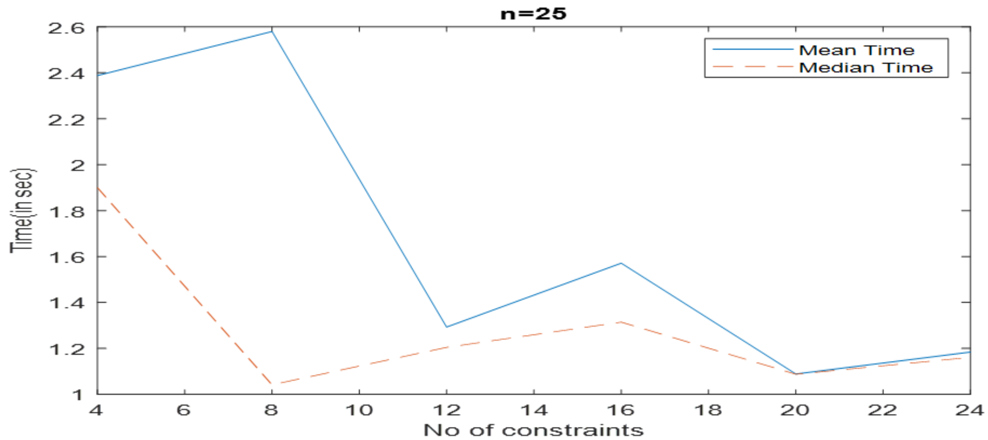

Figure 3.4: Effect of the number of constraints (c) on processing time when n = 25

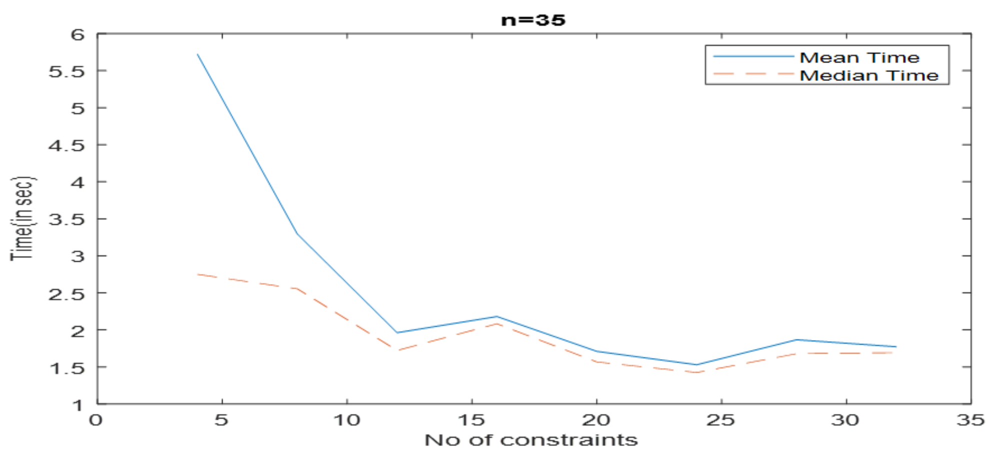

Figure 3.5: Effect of the number of constraints (c) on processing time when n = 35

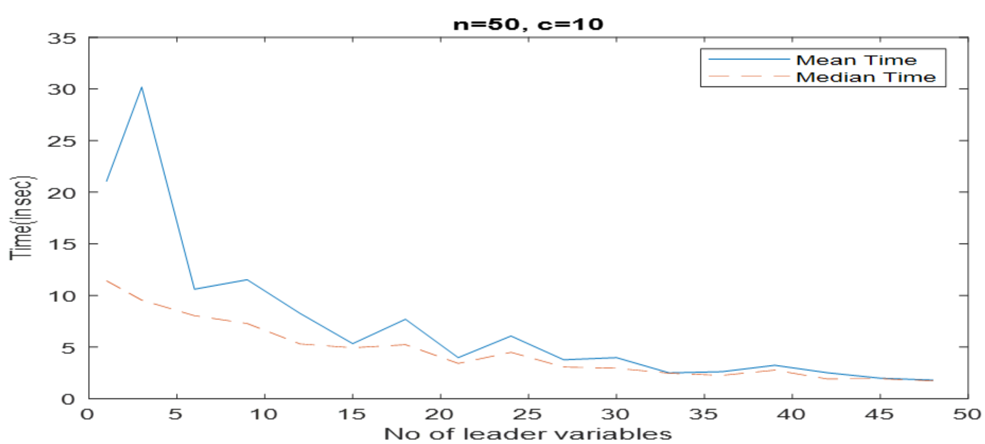

Figure 3.6: Effect of the number of leader variables (l) on processing time when n=35 and c=10

Table 3.6: Processing time to find solution of BQIPP when n=50 and c=10

No of leader variables (l)	Mean Time (in sec)	Median Time (in sec)
1	21.0541885	11.4174445
3	30.1789284	9.5430925
6	10.6001056	8.039086
9	11.5231405	7.280019
12	8.2656435	5.3029695
15	5.3277753	4.931709
18	7.6919403	5.230147
21	3.9725071	3.4164325
24	6.0711845	4.490029
27	3.7675484	3.066983
30	3.9757523	2.946286
33	2.4969462	2.464297
36	2.6129939	2.258268
39	3.2353366	2.7638865
42	2.5083326	1.906404
45	1.9807574	1.9322095
48	1.7971335	1.736444

3.6.2 Computational Results for LBIPP

The proposed algorithm is used to solve a set of ten randomly generated LBIPP for every case.

Tables 7 show the mean and median processing time with changes in the number of variables for $c = 25$. For a given number of constraints, Figure 7 illustrates how the number of variables impacts the processing time.

Table 8 shows the mean and median processing time with the change in a number of constraints for $n = 35$. Figure 8 depicts the impact of a number of constraints on processing time for $n = 35$.

Table 9 shows the mean and median processing time with the change in a number of leader variables for $n = 50$, $c = 10$. Figure 9 depicts the impact of the number of leader variables on processing time in the case of fixed $n = 50$, $c = 10$.

Table 3.7: Processing time to find solution of BLIPP when c = 25

No of Variables (n)	Mean Time (in sec)	Median Time (in sec)
30	1.4295768	0.7518285
35	0.7183792	0.702954
40	0.7584304	0.7254635
45	0.7057757	0.6923285
50	0.699291	0.6985315

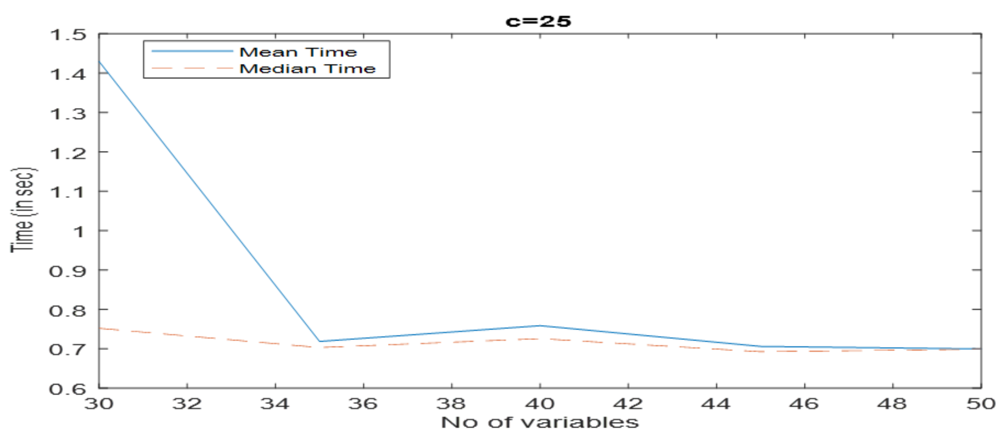

Figure 3.7: Effect of the number of variables (n) on processing when c=25

Table 3.8: Processing time to find solution of BLIPP when n = 35

No of constraints (c)	Mean Time (in sec)	Median Time (in sec)
4	1.1358184	0.784575
8	0.806978	0.7065415
12	0.6752651	0.672761
16	0.6771155	0.6631785
20	0.6891633	0.6817595
24	0.6995882	0.689837
28	0.7164973	0.6824915
32	0.712553	0.692694

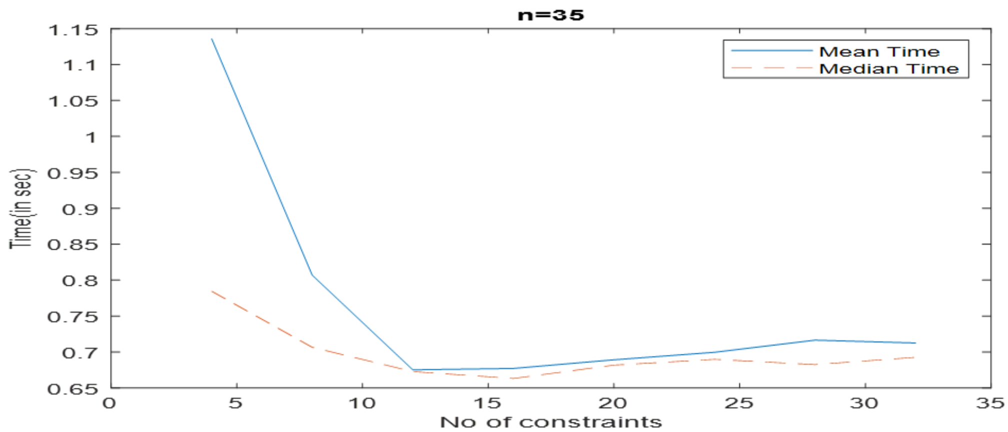

Figure 3.8: Effect of the number of constraints (c) on processing time when n = 3

Table 3.9: Processing time to find solution of BLIPP when n=50 and c=10

No of leader variables (l)	Mean Time (in sec)	Median Time (in sec)
1	0.578277	0.532831
3	0.5322037	0.471186
6	0.694458	0.4549335
9	0.527755	0.4677285
12	0.5212072	0.4550575
15	0.6256304	0.4631885
18	0.5523444	0.467252
21	0.5112213	0.4636455
24	0.7689209	0.466386
27	0.6887789	0.4765685
30	0.5028115	0.452618
33	0.5049527	0.460003
36	0.4540868	0.4540335
39	0.5417979	0.4590915
42	0.523402	0.490318
45	0.4821146	0.481175
48	0.5640286	0.5386995

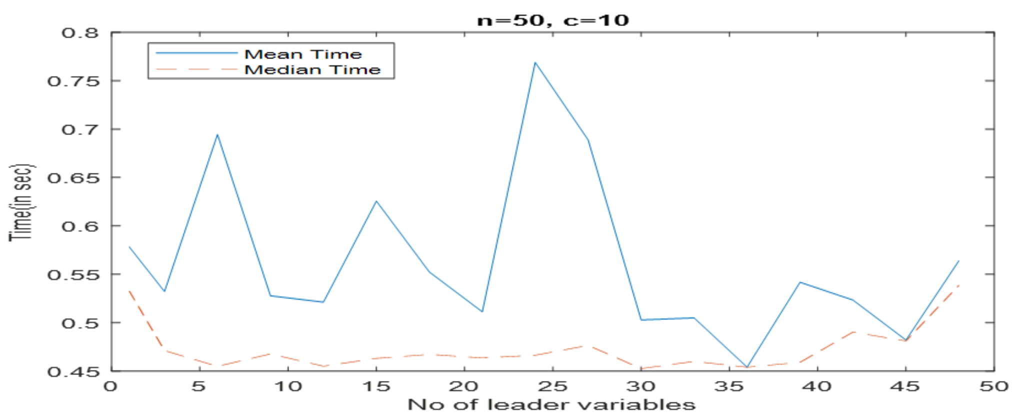

Figure 3.9: Effect of the number of leader variables (l) on processing time when n=35 and c=10

Bibliography

[1] Gupta, R., Bandopadhyaya, L., Puri, M. C. (1996). Ranking in quadratic integer programming problems. European journal of operational research, 95(1), 231-236.

[2] Maachou, N., Moula¨ı, M. (2022). Branch and cut method for solving integer indefinite quadratic bilevel programs. Annals of Operations Research, 1-31.

[3] Narang, R., Arora, S. R. (2009). Indefinite quadratic integer bilevel programming problem with bounded variables. Opsearch, 46, 428-448.

[4] Moore, J. T., Bard, J. F. (1990). The mixed integer linear bilevel programming problem. Operations research, 38(5), 911-921.

[5] Ravipati, A. (2012). Markowitz's portfolio selection model and related problems (Doctoral dissertation, Rutgers University-Graduate School-New Brunswick).

[6] Cesarone, F., Scozzari, A., Tardella, F. (2011). Portfolio selection problems in practice: a comparison between linear and quadratic optimization models. arXiv preprint arXiv:1105.3594.

[7] Chen, J. T. (1980). Cost-Volume-Profit Analysis in Stochastic Programming Models. Decision Sciences, 11(4), 632-647.

[8] McCarl, B. A., Moskowitz, H., Furtan, H. (1977). Quadratic programming applications. Omega, 5(1), 43-55.

[9] Koopmans, T. C., Beckmann, M. (1957). Assignment problems and the location of economic activities. Econometrica: journal of the Econometric Society, 53-76.

[10] Buchheim, C., Caprara, A., Lodi, A. (2012). An effective branch-and-bound algorithm for convex quadratic integer programming. Mathematical programming, 135, 369-395.

[11] McBride, R. D., Yormark, J. S. (1980). An implicit enumeration algorithm for quadratic integer programming. Management Science, 26(3), 282-296.

[12] Wright, S. J. (2006). Numerical optimization.

[13] Biegler, L. T., Grossmann, I. E., Westerberg, A. W. (1997). Systematic methods for chemical process design.

[14] Wong, E. (2011). Active-set methods for quadratic programming. University of California, San Diego.

[15] Wolfe, P. (1959). The simplex method for quadratic programming. Econometrica: Journal of the Econometric Society, 382-398.

[16] Van de Panne, C., Whinston, A. (1964). The simplex and the dual method for quadratic programming. Journal of the Operational Research Society, 15(4), 355-388.

[17] Boot, J. C. (1964). Quadratic programming: algorithms, anomalies, applications.

[18] Beale, E. M. L. (1959). On quadratic proramming. Naval Research Logistics Quarterly, 6(3), 227-243.

[19] Boyd, S., Vandenberghe, L. (2004). Convex optimization. Cambridge university press.

[20] Côté, J. P., Marcotte, P., Savard, G. (2003). A bilevel modelling approach to pricing and fare optimisation in the airline industry. Journal of Revenue and Pricing Management, 2, 23-36.

[21] Haurie, A., Loulou, R., Savard, G. (1992). A two-player game model of power cogeneration in New England. IEEE Transactions on Automatic Control, 37(9), 1451-1456.

[22] Larsson, T., Patriksson, M. (1998). Side constrained traffic equilibrium models—traffic management through link tolls. In Equilibrium and advanced transportation modelling (pp. 125-151). Boston, MA: Springer US.

[23] Sherali, H. D., Soyster, A. L., Murphy, F. H. (1983). Stackelberg-Nash-Cournot equilibria: characterizations and computations. Operations Research, 31(2), 253-276.

[24] Kara, B. Y., Verter, V. (2004). Designing a road network for hazardous materials transportation. Transportation Science, 38(2), 188-196.

[25] Kolstad, C. D., Lasdon, L. S. (1990). Derivative evaluation and computational experience with large bilevel mathematical programs. Journal of optimization theory and applications, 65, 485-499.

[26] Savard, G., Gauvin, J. (1994). The steepest descent direction for the nonlinear bilevel programming problem. Operations Research Letters, 15(5), 265-272.

[27] Shimizu, K., Aiyoshi, E. (1981). A new computational method for Stackelberg and min-max problems by use of a penalty method. IEEE Transactions on Automatic Control, 26(2), 460-466.

[28] Dempe, S. (2002). Foundations of bilevel programming. Springer Science Business Media.