ME431 SYSTEM DYNAMICS

HOMEWORK – 2DOF Spring Mass System

08.12.2021

GitHub link is available for the MATLAB and all files related to the assignment.

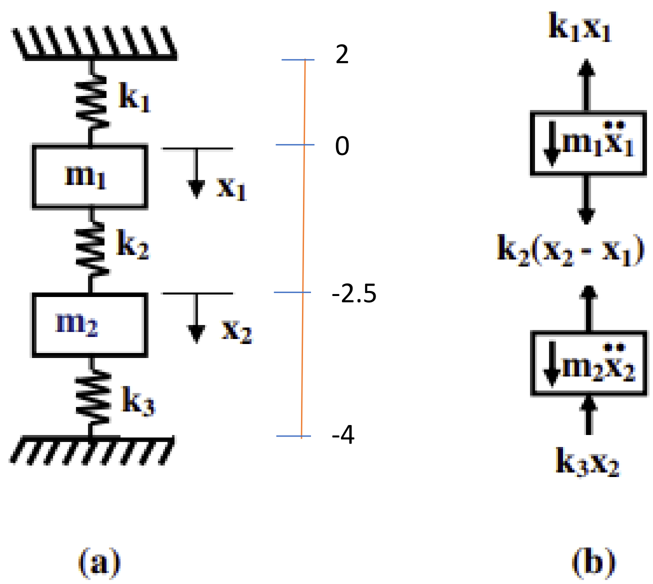

Figure 1

Assumed $x_2 > x_1$

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 + (-k_2)x_2 = 0 \qquad (1)$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1)$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 + (-k_2)x_1 = 0 \qquad (2)$$

Matrix format of (1) and (2) is written as (3)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3)$$

Harmonic Motion Equations

The two harmonic motions have the same frequency

$\emptyset = 0$

$x_2 = 1$

$$x_1(t) = x_1 \sin(wt + \emptyset) \tag{4}$$

$$x_2(t) = x_2 \sin(wt + \emptyset) \tag{5}$$

$$\ddot{x}_1(t) = -x_1 w^2 \sin(wt + \emptyset) \tag{6}$$

$$\ddot{x}_2(t) = -x_2\, w^2 \sin(wt + \emptyset) \tag{7}$$

Substituting (3) with the (6) and (7), the following equation can be written

$$-w^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 - m_1 w^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2 w^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

Solution of (8) is written as (9)

$$det \begin{bmatrix} k_1 + k_2 - m_1 w^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2 w^2 \end{bmatrix} = 0 \tag{9}$$

$$(w^2)^2 m_1 m_2 + w^2(-m_1 k_2 - m_1 k_3 - m_2 k_2 - m_2 k_1) + k_1 k_2 + k_1 k_3 + k_2 k_3 = 0$$

There are four solution of w, the real two are taken as w_1 and w_2

$$w = -\frac{\sqrt{\frac{k_1}{m_1} - \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}}{\sqrt{2}}$$

$$w = \frac{\sqrt{\frac{k_1}{m_1} - \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}}{\sqrt{2}}$$

$$w = -\frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}}{\sqrt{2}}$$

$$w = \frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}}{\sqrt{2}}$$

$$w_1 = \frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{m_1m_2} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}}{\sqrt{2}}$$

$$w_2 = -\frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{m_1m_2} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}}{\sqrt{2}}$$

Substituting (8) with the w_1 and w_2,
There are two solutions for the system but the natural frequency for both mass is identical.

$$\begin{bmatrix} k_1 + k_2 - m_1w^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2w^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1\left(\frac{m_2k_1 + m_2k_2 - m_1k_2 - m_1k_3 + \sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{2m_2}\right) - k_2x_2 = 0$$

For w_1, Normal Modes 1 (Solution 1)

$$r_1 = \frac{x_1}{x_2} = \frac{m_2k_1 + m_2k_2 - m_1k_2 - m_1k_3 + \sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{2m_2k_2}$$

For w_2, Normal Modes 2 (Solution 2)

$$r_2 = \frac{x_1}{x_2} = \frac{3m_2k_1 + 3m_2k_2 - m_1k_2 - m_1k_3 + \sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{2m_2k_2}$$

Table 1

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 1	1	1	1	1	1	1	3	1	1	1.732	-1.732
Measurement 2	2	1	1.2	1.2	2	-0.586	1.413	1	1	1.869	-1.869
Measurement 3	1.5	0.5	1.4	1.6	1.6	0.767	2.642	1	1	2.542	-2.542
Measurement 4	0.5	1.5	1.6	2.5	2.5	1.237	2.877	1	1	2.929	-2.929
Measurement 5	0.8	1	2	1.5	1	1.736	4.07	1	1	2.300	-2.300
Measurement 6	2	2	2	1.4	1.5	0.671	3.10	1	1	1.719	-1.719
Measurement 7	1.4	0.8	2.3	2.3	2	0.981	2.981	1	1	2.703	-2.703
Measurement 8	1.6	2.5	2.1	2.5	1.5	0.772	2.612	1	1	2.124	-2.124
Measurement 9	1.8	0.9	2	1.8	2	0.334	2.445	1	1	2.327	-2.327
Measurement 10	2.5	2	1	1.5	2	-0.244	1.422	1	1	1.586	-1.586

Natural frequencies (w1 and w2) of the system and amplitudes (x1 and x2) are shown in Table 1 for ten different measurements. Amplitude of the x2 is taken as 1 therefore, x1 is equal to the amplitude ratio (r).

MATLAB CODE

```
%%% 2 DOF Spring Mass System

global m1 m2 k1 k2 k3 w1 w2 r1 r2 x11 x21 x12 x22 t x1_loc x2_loc

% [m1, m2, k1, k2, k3] = deal(1, 2, 1.5, 2, 2.5) ;  % test

% random values for m mass, k spring constant
m1 = [1; 2; 1.5; 0.5; 0.8; 2; 1.4; 1.6; 1.8; 2.5] ;
m2 = [1; 1; 0.5; 1.5; 1; 2; 0.8; 2.5; 0.9; 2]     ;
k1 = [1; 1.2; 1.4; 1.6; 2; 2; 2.3; 2.1; 2; 1]     ;
k2 = [1; 1.2; 1.6; 2.5; 1.5; 1.4; 2.3; 2.5; 1.8; 1.5] ;
k3 = [1; 2; 1.6; 2.5; 1; 1.5; 2; 1.5; 2; 2]         ;

% natural frequency w1 and w2 for two modes
w1 =    (sqrt( k1./m1 + ...
        sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 + k2.*k3))./(m1.*m2) ...
        + k2./m1 + k2./m2 + k3./m2 ) )/sqrt(2) ;


w2 =    -(sqrt( k1./m1 + ...
        sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 + k2.*k3))./(m1.*m2) ...
        + k2./m1 + k2./m2 + k3./m2 ) )/sqrt(2) ;


% amplitude ratio r1 and r2 for two modes
r1 =    ( sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 + k2.*k3))./(m1.*m2)...
        + m2.*k1 + m2.*k2 - m1.*k2 - m1.*k3)./(2.*m2.*k2) ;


r2 =    ( sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 + k2.*k3))./(m1.*m2)...
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        + 3.*m2.*k1 + 3.*m2.*k2 - m1.*k2 - m1.*k3)./(2.*m2.*k2) ;


t   =   (0:0.1:10) ;            % t is time in seconds
x1_loc = 0  ;                   % x1_loc is the location of x1
x2_loc = -2.5  ;                   % x2_loc is the location of x2

x11 = x1_loc + r1 .* sin(w1.*t)  ;      % x11 and x21 are displacement for Mode 1

x21 = x2_loc + 1 .* sin(w1.*t)   ;      % Amplitude 2 is taken as 1
                                        % so Amplitude 1 equals the r1 amplitude ratio A1/A2


x12 = x1_loc + r2 .* sin(w2.*t)  ;      % x12 and x22 are displacement for Mode 2

x22 = x2_loc + 1 .* sin(w2.*t)   ;


%% GRAPH for 10 DIFFERENT VALUES
i = 1   ;

while i < 11

    x11 =   x1_loc + r1(i,1) * sin(w1(i,1).*t) ;
    x21 =   x2_loc + 1 * sin(w1(i,1).*t) ;

    x12 =   x1_loc + r2(i,1) * sin(w2(i,1).*t)  ;
    x22 =   x2_loc + 1 .* sin(w2(i,1).*t)    ;

    figure
    graph = tiledlayout(2,1);

    nexttile

    plot(t, x11, 'r-') ;
    hold on
    plot(t, x21, 'b-') ;

    hold off
    legend('Mass 1 (X1)','Mass 2 (X2)')

    title({sprintf('Measurement %i', i);'Mode 1'})
    xlabel('Time (sec)')
    ylabel('Displacement (m)')


    nexttile

    plot(t, x12, 'r-') ;
    hold on
    plot(t, x22, 'b-') ;

    title('Mode 2')
    xlabel('Time (sec)')
    ylabel('Displacement (m)')

    hold off
    legend('Mass 1 (X1)','Mass 2 (X2)')

    i = i + 1 ;
end

```
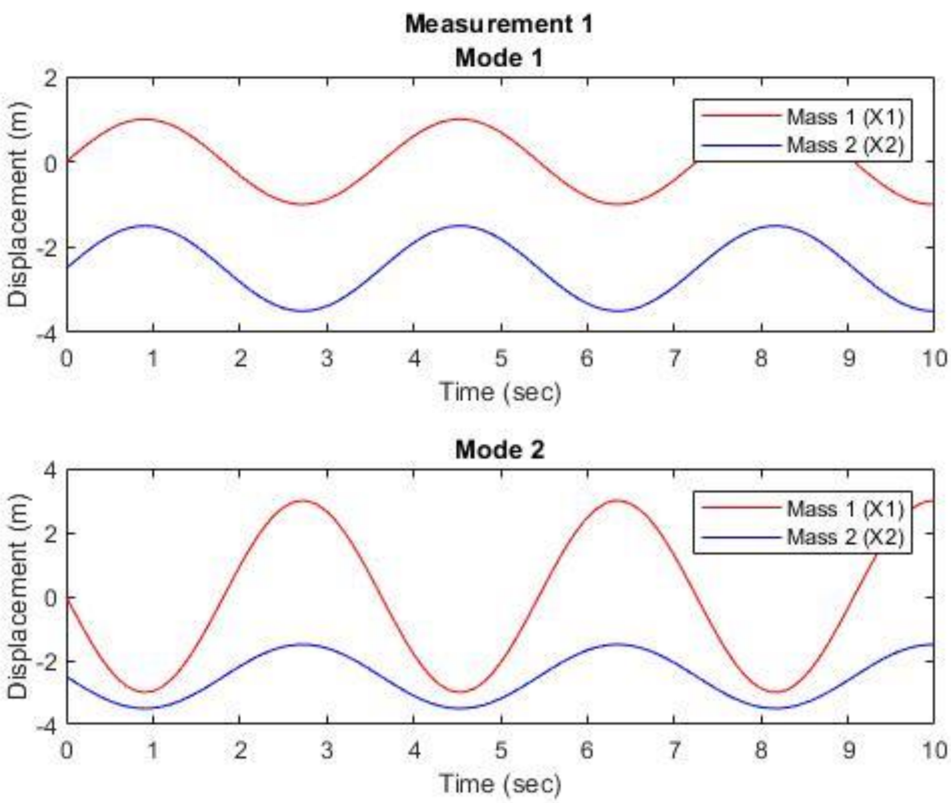

The following graphs represent the motion of the spring and mass system. There are two figures for each measurement that one is for first solution (Mode 1) while the other for the second solution (Mode 2). The initial conditions for the location of two masses are shown in Figure 1 **(x1 = 0, x2 = -2.5)**

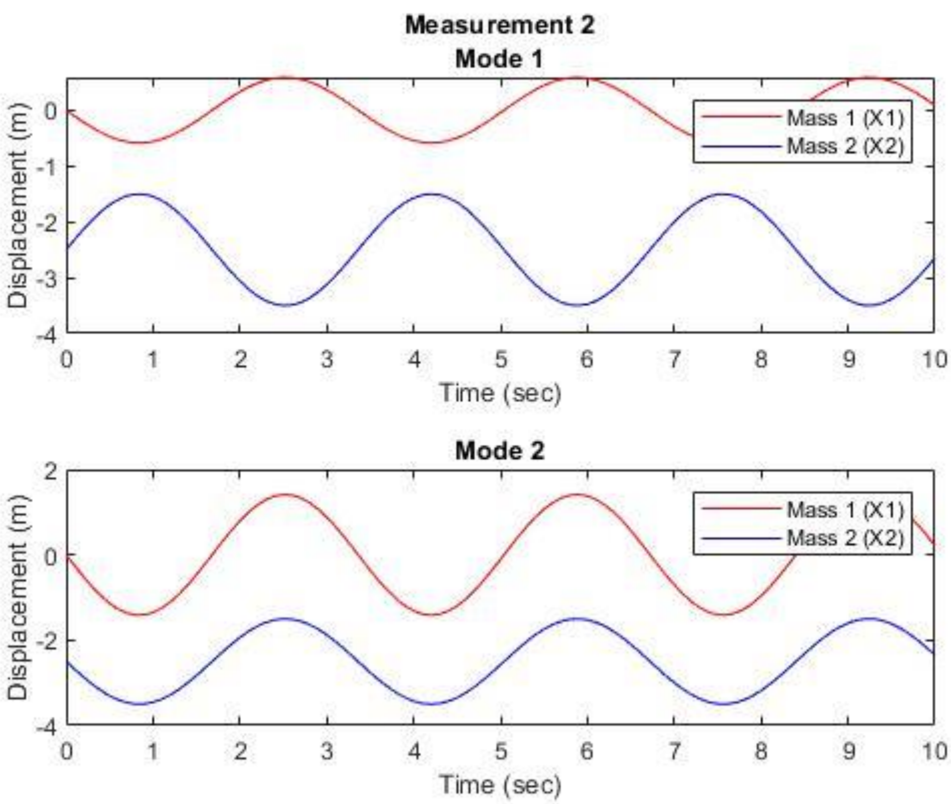
Measurement 1

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 1	1	1	1	1	1	1	3	1	1	1.732	-1.732



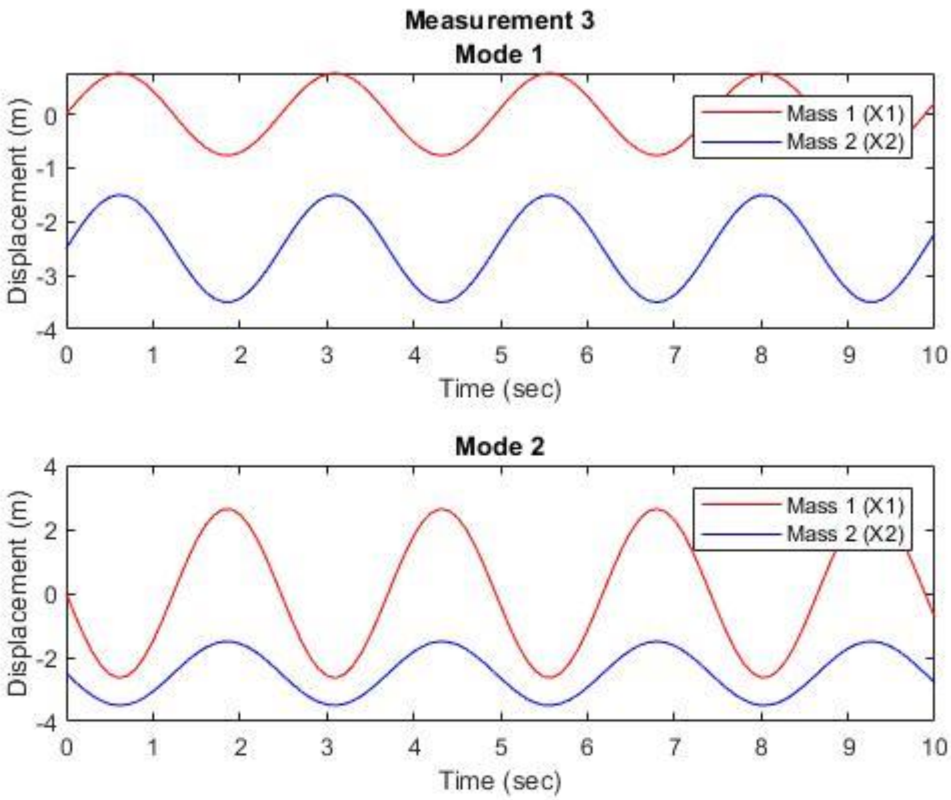
Measurement 2

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 2	2	1	1.2	1.2	2	-0.586	1.413	1	1	1.869	-1.869



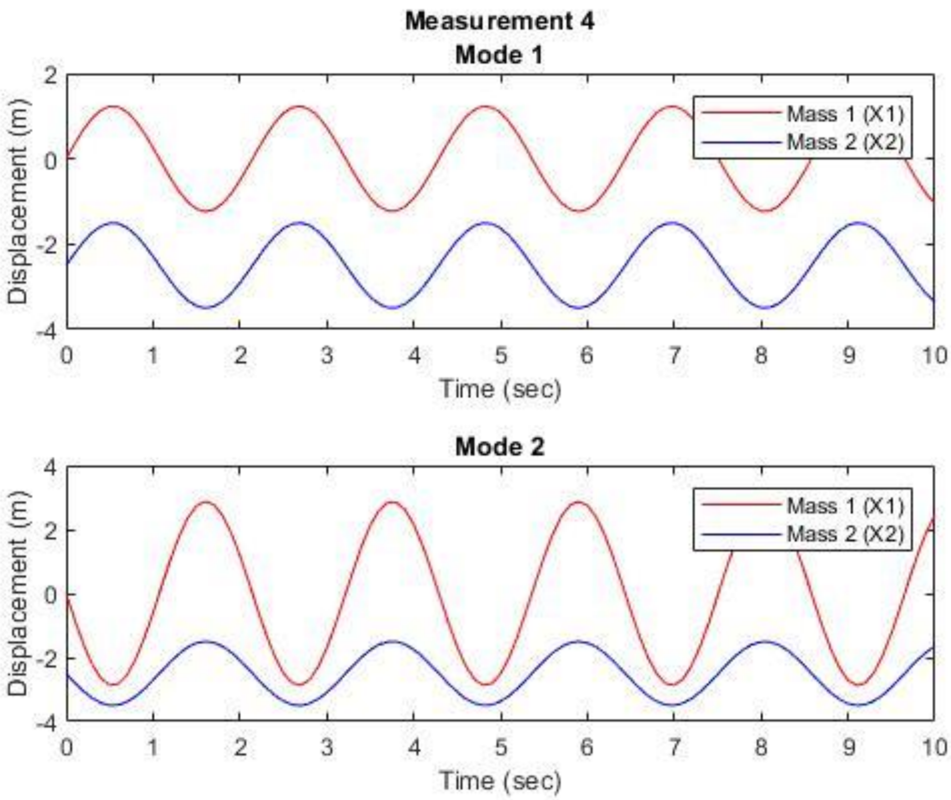
Measurement 3

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 3	1.5	0.5	1.4	1.6	1.6	0.767	2.642	1	1	2.542	-2.542



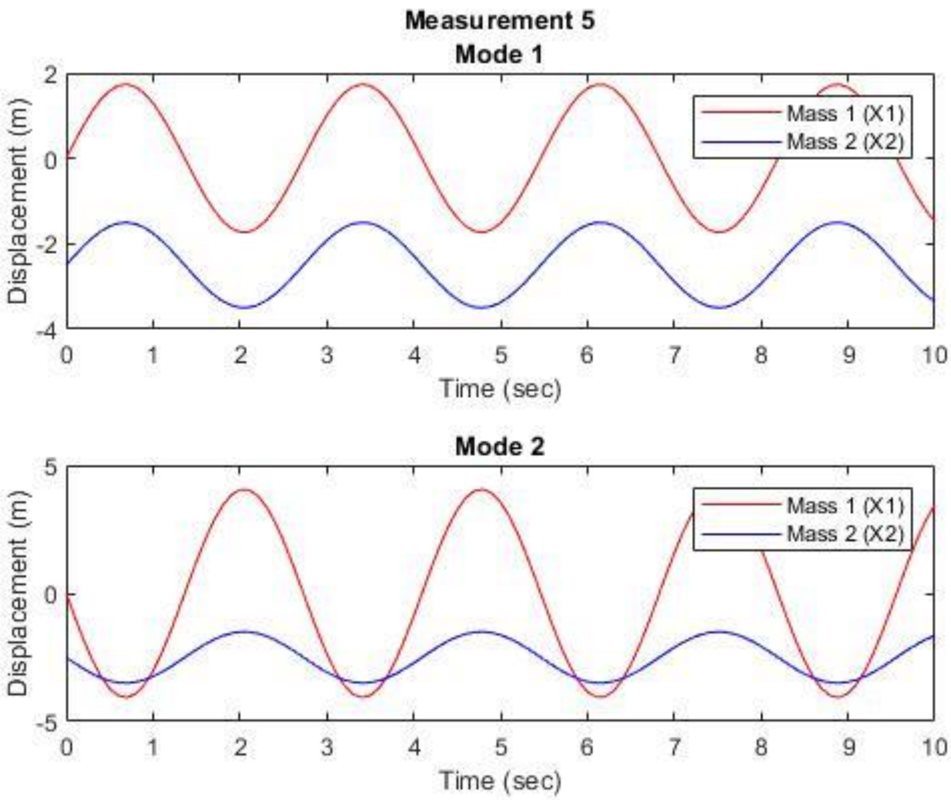
Measurement 4

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 4	0.5	1.5	1.6	2.5	2.5	1.237	2.877	1	1	2.929	-2.929



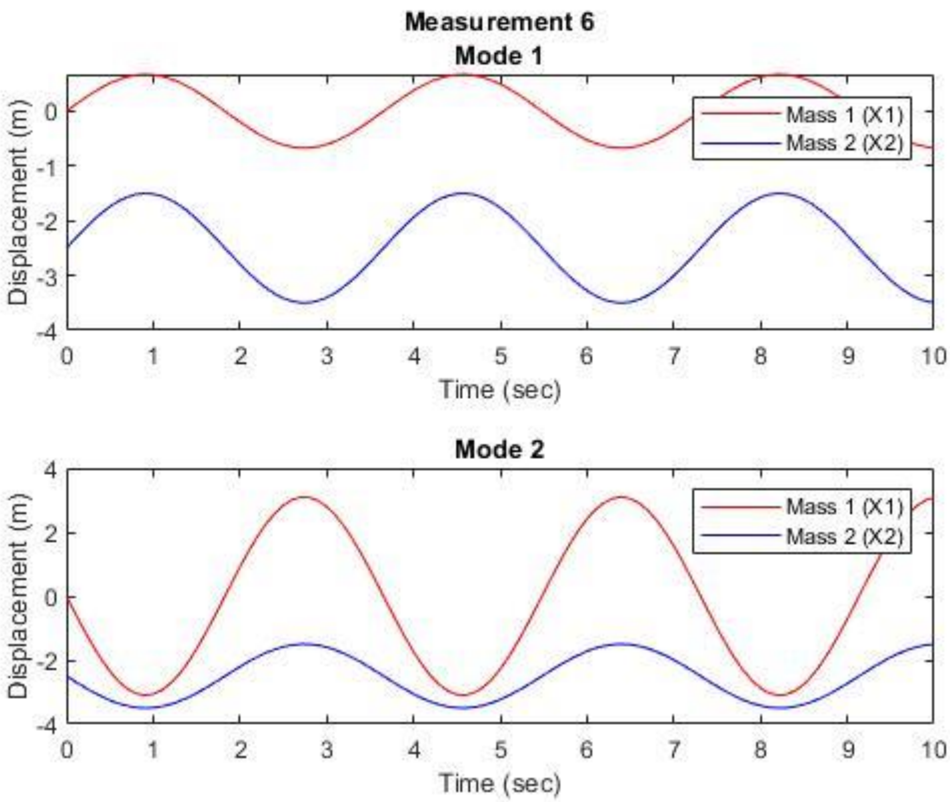
Measurement 5

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 5	0.8	1	2	1.5	1	1.736	4.07	1	1	2.300	-2.300



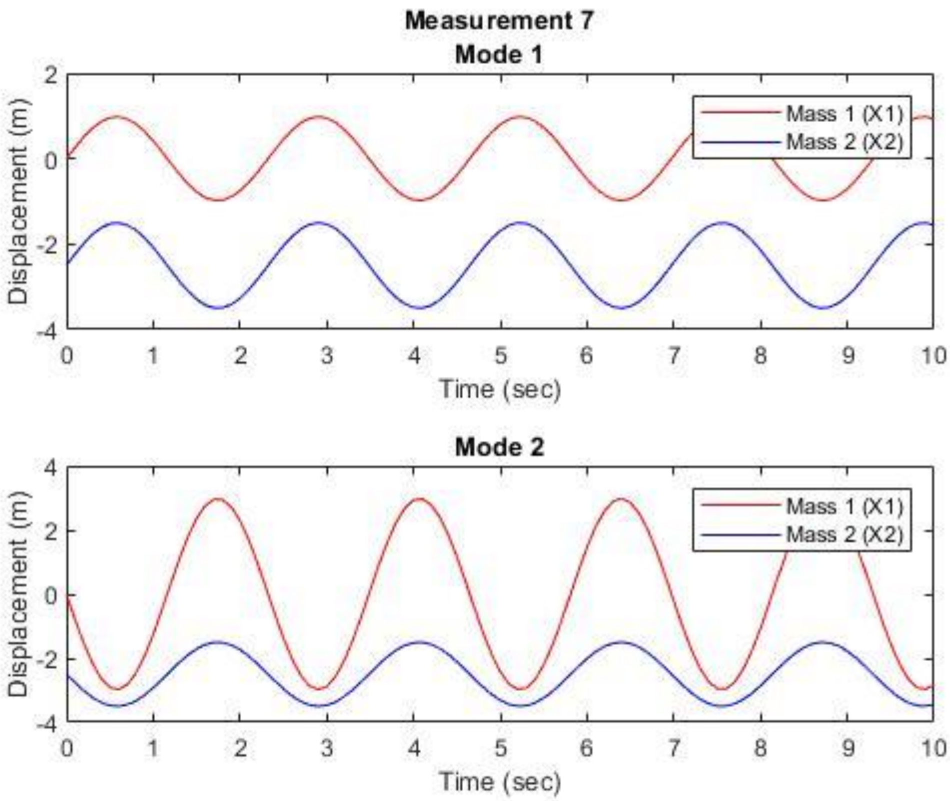
Measurement 6

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 6	2	2	2	1.4	1.5	0.671	3.10	1	1	1.719	-1.719



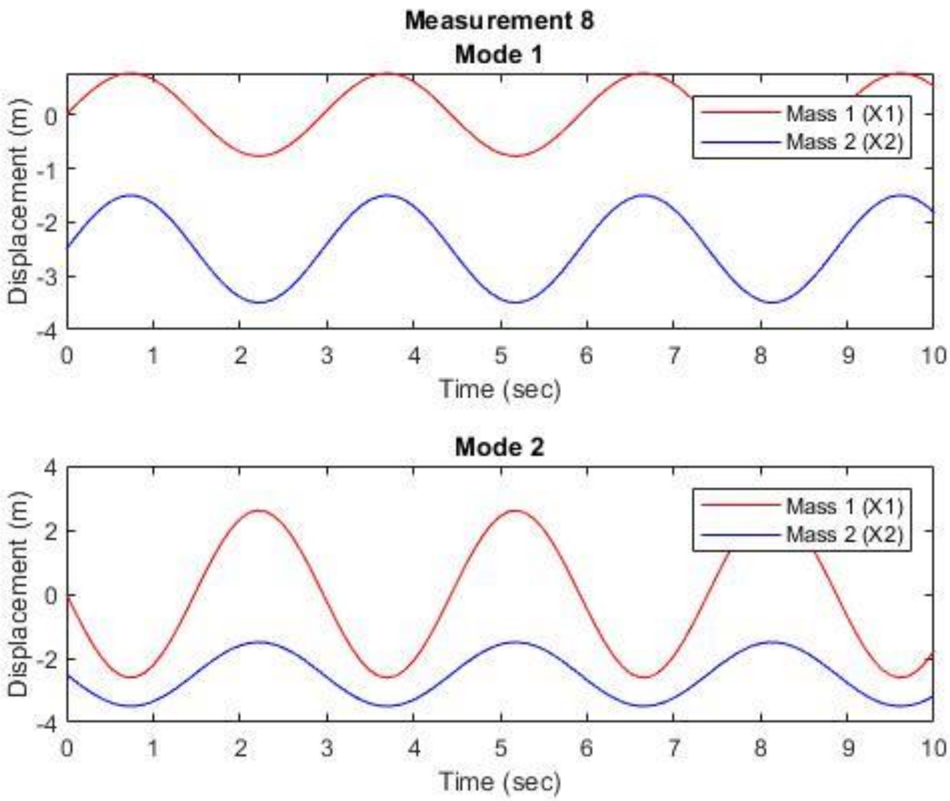
Measurement 7

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 7	1.4	0.8	2.3	2.3	2	0.981	2.981	1	1	2.703	-2.703



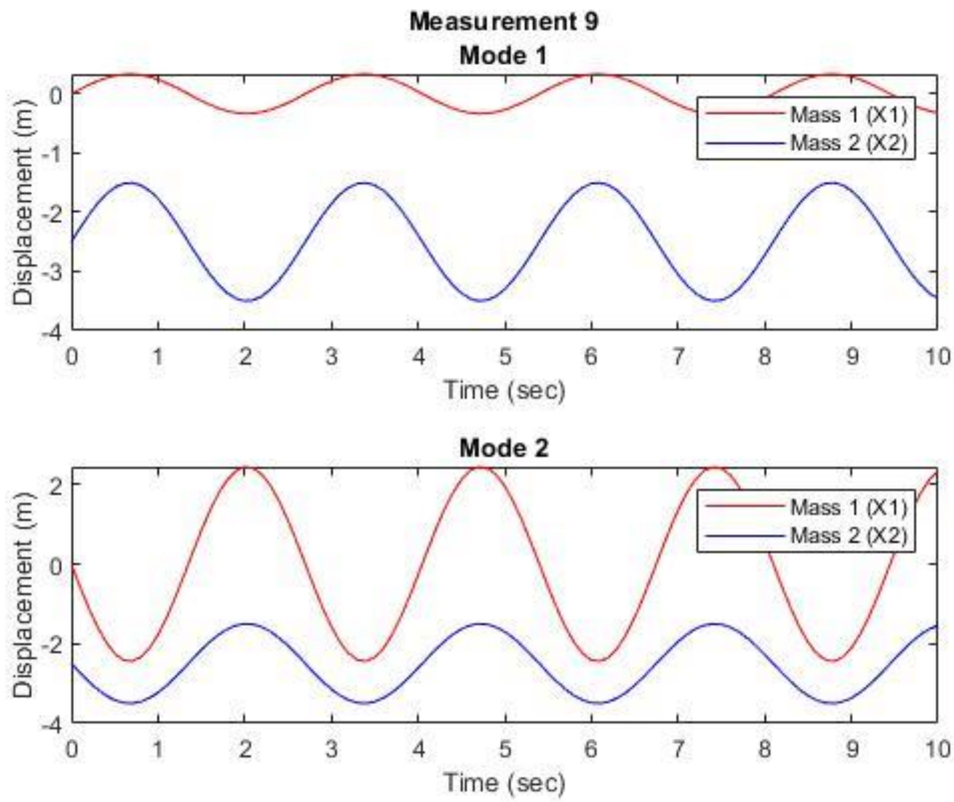
Measurement 8

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 8	1.6	2.5	2.1	2.5	1.5	0.772	2.612	1	1	2.124	-2.124



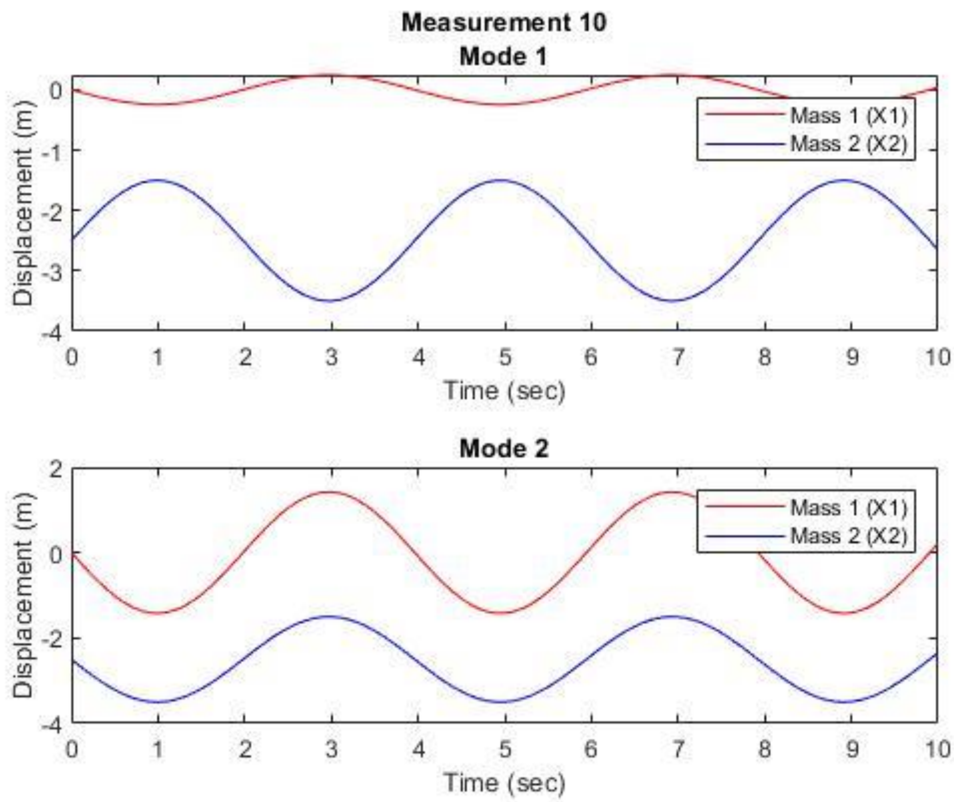
Measurement 9

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 9	1.8	0.9	2	1.8	2	0.334	2.445	1	1	2.327	-2.327



Measurement 10

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		$\ddot{x}_1\ (w_1)$	$\ddot{x}_2\ (w_2)$
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 10	2.5	2	1	1.5	2	-0.244	1.422	1	1	1.586	-1.586



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[1] J. Vandiver, and David Gossard. *2.003SC Engineering Dynamics.* Fall 2011. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

[2] "4-2 natural frequencies of two degree of Freedom System - vibrations of linear double-degree-of-freedom systems 1: The relation between mode shape and frequency," *Coursera*. [Online]. Available: https://www.coursera.org/lecture/introduction-basic-vibrations/4-2-natural-frequencies-of-two-degree-of-freedom-system-UgBN7. [Accessed: 05-Dec-2021].

[3] "Vibrations of two degree of Freedom Systems." [Online]. Available: https://d13mk4zmvuctmz.cloudfront.net/assets/main/study-material/notes/mechanical_engineering_mechanical-vibrations_vibrations-of-two-degree-of-freedom-systems_notes.pdf. [Accessed: 05-Dec-2021].

[4] "Student corner," *Sri Indu Institute of Engineering & Technology*. [Online]. Available: https://siiet.ac.in/mech/student-corner/. [Accessed: 05-Dec-2021].

[5] "(w^2)^2+m_1+m_2+++w^2+(-m_1+k_2+-+m_1+k_3+-+m_2+k_2+-+m_2+k_1)+++k_1+k_2+++k_1+k_3+++k_2+k_3+=+0 - wolfram: Alpha," *WolframAlpha computational knowledge AI*. [Online]. Available: https://www.wolframalpha.com/input/?i=%28w%5E2%29%5E2%2Bm_1%2Bm_2%2B%2B%2Bw%5E2%2B%28-m_1%2Bk_2%2B-%2Bm_1%2Bk_3%2B-%2Bm_2%2Bk_2%2B-%2Bm_2%2Bk_1%29%2B%2B%2Bk_1%2Bk_2%2B%2B%2Bk_1%2Bk_3%2B%2B%2Bk_2%2Bk_3%2B%3D%2B0. [Accessed: 06-Dec-2021].