

ME431 SYSTEM DYNAMICS

HOMEWORK – 2DOF Spring Mass System

08.12.2021

[GitHub link](#) is available for the MATLAB and all files related to the assignment.

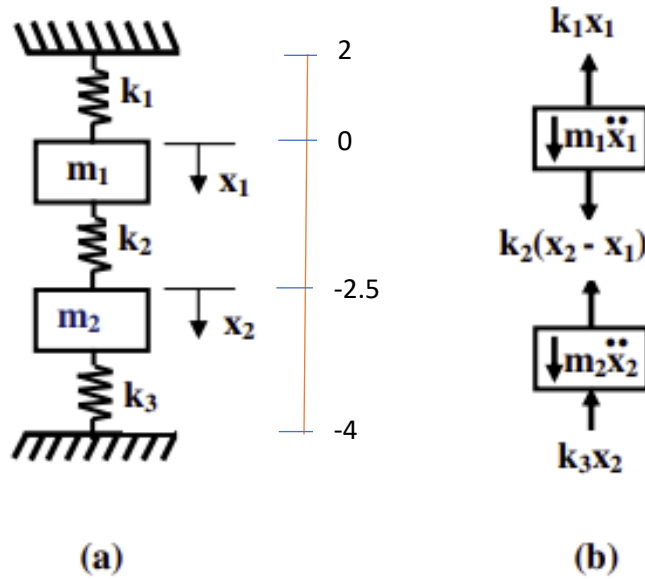


Figure 1

Assumed $x_2 > x_1$

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 + (-k_2)x_2 = 0 \quad (1)$$

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 + (-k_2)x_1 = 0 \quad (2)$$

Matrix format of (1) and (2) is written as (3)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

Harmonic Motion Equations

The two harmonic motions have the same frequency

$$\phi = 0$$

$$x_2 = 1$$

$$x_1(t) = x_1 \sin(\omega t + \phi) \quad (4)$$

$$x_2(t) = x_2 \sin(\omega t + \phi) \quad (5)$$

$$\ddot{x}_1(t) = -x_1 \omega^2 \sin(\omega t + \phi) \quad (6)$$

$$\ddot{x}_2(t) = -x_2 \omega^2 \sin(\omega t + \phi) \quad (7)$$

Substituting (3) with the (6) and (7), the following equation can be written

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2 \omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (8)$$

Solution of (8) is written as (9)

$$\det \begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2 \omega^2 \end{bmatrix} = 0 \quad (9)$$

$$(\omega^2)^2 m_1 m_2 + \omega^2 (-m_1 k_2 - m_1 k_3 - m_2 k_2 - m_2 k_1) + k_1 k_2 + k_1 k_3 + k_2 k_3 = 0$$

There are four solution of ω , the real two are taken as ω_1 and ω_2

$$\omega = \frac{-\sqrt{\frac{k_1}{m_1} - \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2}} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}{\sqrt{2}}$$

$$\omega = \frac{\sqrt{\frac{k_1}{m_1} - \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2}} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}{\sqrt{2}}$$

$$\omega = \frac{-\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2}} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}{\sqrt{2}}$$

$$\omega = \frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2)^2 - 4(k_1 k_2 + k_3 k_2 + k_1 k_3) m_1 m_2}}{m_1 m_2}} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}{\sqrt{2}}$$

$$w_1 = \frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{m_1m_2}} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}{\sqrt{2}}$$

$$w_2 = -\frac{\sqrt{\frac{k_1}{m_1} + \frac{\sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{m_1m_2}} + \frac{k_2}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_2}}{\sqrt{2}}$$

Substituting (8) with the w_1 and w_2 ,

There are two solutions for the system but the natural frequency for both mass is identical.

$$\begin{bmatrix} k_1 + k_2 - m_1w^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2w^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$x_1 \left(\frac{m_2k_1 + m_2k_2 - m_1k_2 - m_1k_3 + \sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{2m_2} \right) - k_2x_2 = 0$$

For w_1 , Normal Modes 1 (Solution 1)

$$r_1 = \frac{x_1}{x_2} = \frac{m_2k_1 + m_2k_2 - m_1k_2 - m_1k_3 + \sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{2m_2k_2}$$

For w_2 , Normal Modes 2 (Solution 2)

$$r_2 = \frac{x_1}{x_2} = \frac{3m_2k_1 + 3m_2k_2 - m_1k_2 - m_1k_3 + \sqrt{(-m_1k_2 - m_1k_3 - m_2k_2 - m_2k_1)^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2}}{2m_2k_2}$$

Table 1

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 1	1	1	1	1	1	1	3	1	1	1.732	-1.732
Measurement 2	2	1	1.2	1.2	2	-0.586	1.413	1	1	1.869	-1.869
Measurement 3	1.5	0.5	1.4	1.6	1.6	0.767	2.642	1	1	2.542	-2.542
Measurement 4	0.5	1.5	1.6	2.5	2.5	1.237	2.877	1	1	2.929	-2.929
Measurement 5	0.8	1	2	1.5	1	1.736	4.07	1	1	2.300	-2.300
Measurement 6	2	2	2	1.4	1.5	0.671	3.10	1	1	1.719	-1.719
Measurement 7	1.4	0.8	2.3	2.3	2	0.981	2.981	1	1	2.703	-2.703
Measurement 8	1.6	2.5	2.1	2.5	1.5	0.772	2.612	1	1	2.124	-2.124
Measurement 9	1.8	0.9	2	1.8	2	0.334	2.445	1	1	2.327	-2.327
Measurement 10	2.5	2	1	1.5	2	-0.244	1.422	1	1	1.586	-1.586

m: kg, k: N/m, x: m, w: rad/sec

Natural frequencies (w_1 and w_2) of the system and amplitudes (x_1 and x_2) are shown in Table 1 for ten different measurements. Amplitude of x_2 is taken as 1 therefore, x_1 is equal to the amplitude ratio (r).

MATLAB CODE

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1. %%% 2 DOF Spring Mass System
2.
3. global m1 m2 k1 k2 k3 w1 w2 r1 r2 x11 x21 x12 x22 t x1_loc x2_loc
4.
5. % [m1, m2, k1, k2, k3] = deal(1, 2, 1.5, 2, 2.5) ; % test
6.
7. % random values for m mass, k spring constant
8. m1 = [1; 2; 1.5; 0.5; 0.8; 2; 1.4; 1.6; 1.8; 2.5] ;
9. m2 = [1; 1; 0.5; 1.5; 1; 2; 0.8; 2.5; 0.9; 2] ;
10. k1 = [1; 1.2; 1.4; 1.6; 2; 2; 2.3; 2.1; 2; 1] ;
11. k2 = [1; 1.2; 1.6; 2.5; 1.5; 1.4; 2.3; 2.5; 1.8; 1.5] ;
12. k3 = [1; 2; 1.6; 2.5; 1; 1.5; 2; 1.5; 2; 2] ;
13.
14. % natural frequency w1 and w2 for two modes
15. w1 = (sqrt( k1./m1 + ...
16.         sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 + k2.*k3))./(m1.*m2)
17.         + k2./m1 + k2./m2 + k3./m2 ) )/sqrt(2) ;
18.
19.
20. w2 = -(sqrt( k1./m1 + ...
21.         sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 + k2.*k3))./(m1.*m2)
22.         + k2./m1 + k2./m2 + k3./m2 ) )/sqrt(2) ;
23.
24.
25. % amplitude ratio r1 and r2 for two modes
26. r1 = ( sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 +
27.         k2.*k3))./(m1.*m2)...
28.         + m2.*k1 + m2.*k2 - m1.*k2 - m1.*k3)./(2.*m2.*k2) ;
29.

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30. r2 = ( sqrt( (-m1.*k2-m1.*k2-m2.*k2-m2.*k1).^2 -4.*(k1.*k2 + k1.*k3 +
    k2.*k3))./(m1.*m2)...
31.      + 3.*m2.*k1 + 3.*m2.*k2 - m1.*k2 - m1.*k3)./(2.*m2.*k2) ;
32.
33.
34. t = (0:0.1:10) ;           % t is time in seconds
35. x1_loc = 0 ;               % x1_loc is the location of x1
36. x2_loc = -2.5 ;           % x2_loc is the location of x2
37.
38. x11 = x1_loc + r1 .* sin(w1.*t) ; % x11 and x21 are displacement for Mode 1
39.
40. x21 = x2_loc + 1 .* sin(w1.*t) ; % Amplitude 2 is taken as 1
41.                                     % so Amplitude 1 equals the r1 amplitude ratio A1/A2
42.
43.
44. x12 = x1_loc + r2 .* sin(w2.*t) ; % x12 and x22 are displacement for Mode 2
45.
46. x22 = x2_loc + 1 .* sin(w2.*t) ;
47.
48.
49. %% GRAPH for 10 DIFFERENT VALUES
50. i = 1 ;
51.
52. while i < 11
53.
54.     x11 = x1_loc + r1(i,1) * sin(w1(i,1).*t) ;
55.     x21 = x2_loc + 1 * sin(w1(i,1).*t) ;
56.
57.     x12 = x1_loc + r2(i,1) * sin(w2(i,1).*t) ;
58.     x22 = x2_loc + 1 .* sin(w2(i,1).*t) ;
59.
60.     figure
61.     graph = tiledlayout(2,1);
62.
63.     nexttile
64.
65.     plot(t, x11, 'r-') ;
66.     hold on
67.     plot(t, x21, 'b-') ;
68.
69.     hold off
70.     legend('Mass 1 (X1)', 'Mass 2 (X2)')
71.
72.     title(sprintf('Measurement %i', i); 'Mode 1'})
73.     xlabel('Time (sec)')
74.     ylabel('Displacement (m)')
75.
76.
77.     nexttile
78.
79.     plot(t, x12, 'r-') ;
80.     hold on
81.     plot(t, x22, 'b-') ;
82.
83.     title('Mode 2')
84.     xlabel('Time (sec)')
85.     ylabel('Displacement (m)')
86.
87.     hold off
88.     legend('Mass 1 (X1)', 'Mass 2 (X2)')
89.
90.     i = i + 1 ;
91. end
92.

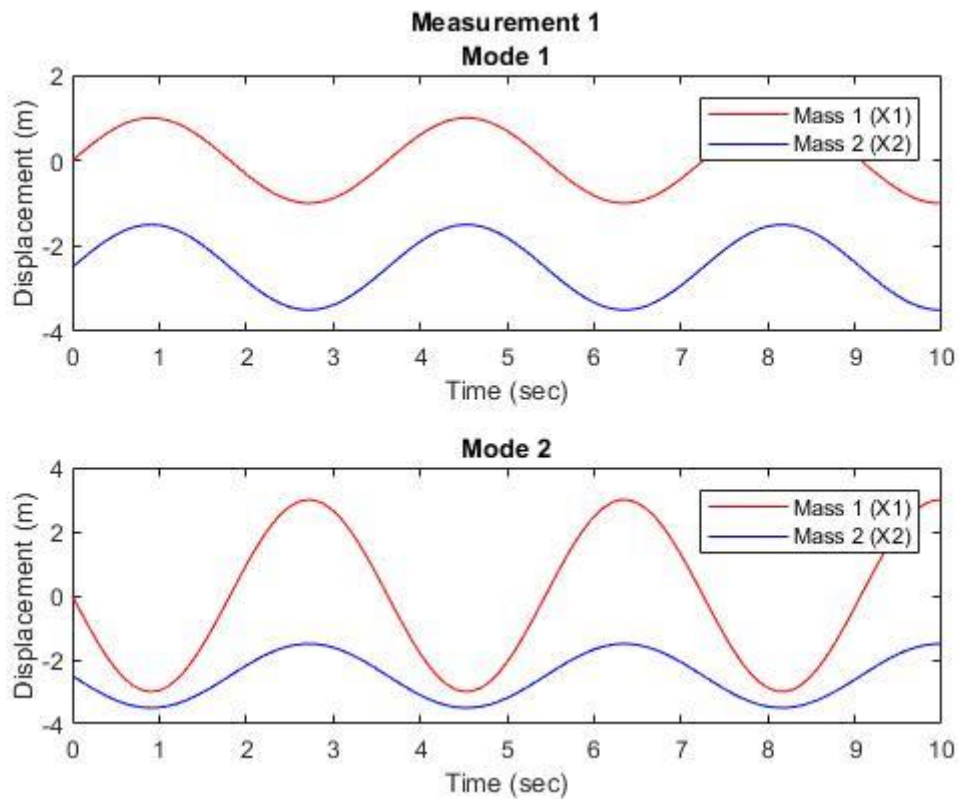
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The following graphs represent the motion of the spring and mass system. There are two figures for each measurement that one is for first solution (Mode 1) while the other is for the second solution (Mode 2). The initial conditions for the location of two masses are shown in Figure 1. ($x_1 = 0$ m, $x_2 = -2.5$ m)

Measurement 1

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 1	1	1	1	1	1	1	3	1	1	1.732	-1.732

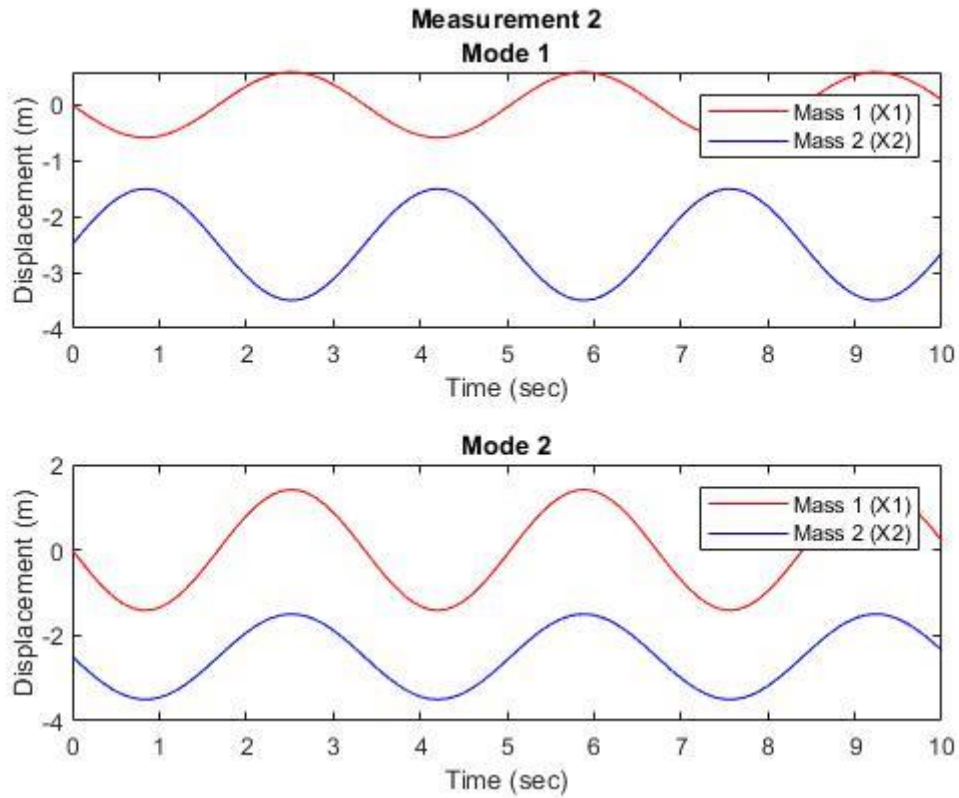
m: kg, k: N/m, x: m, w: rad/sec



Measurement 2

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 2	2	1	1.2	1.2	2	-0.586	1.413	1	1	1.869	-1.869

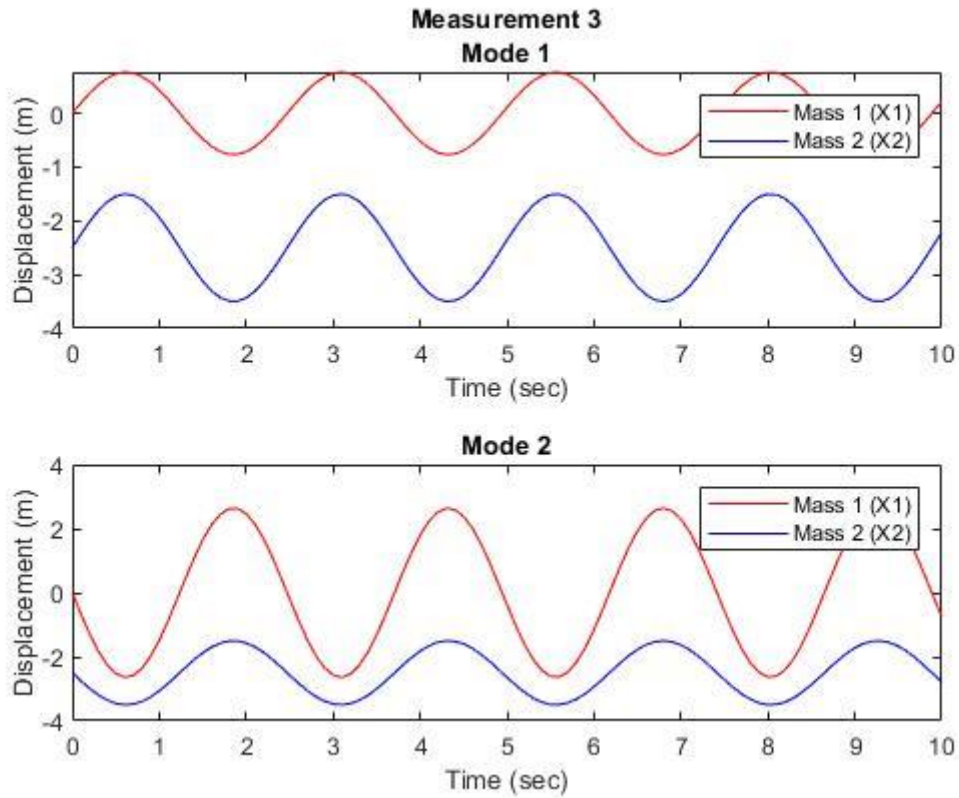
m: kg, k: N/m, x: m, w: rad/sec



Measurement 3

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 3	1.5	0.5	1.4	1.6	1.6	0.767	2.642	1	1	2.542	-2.542

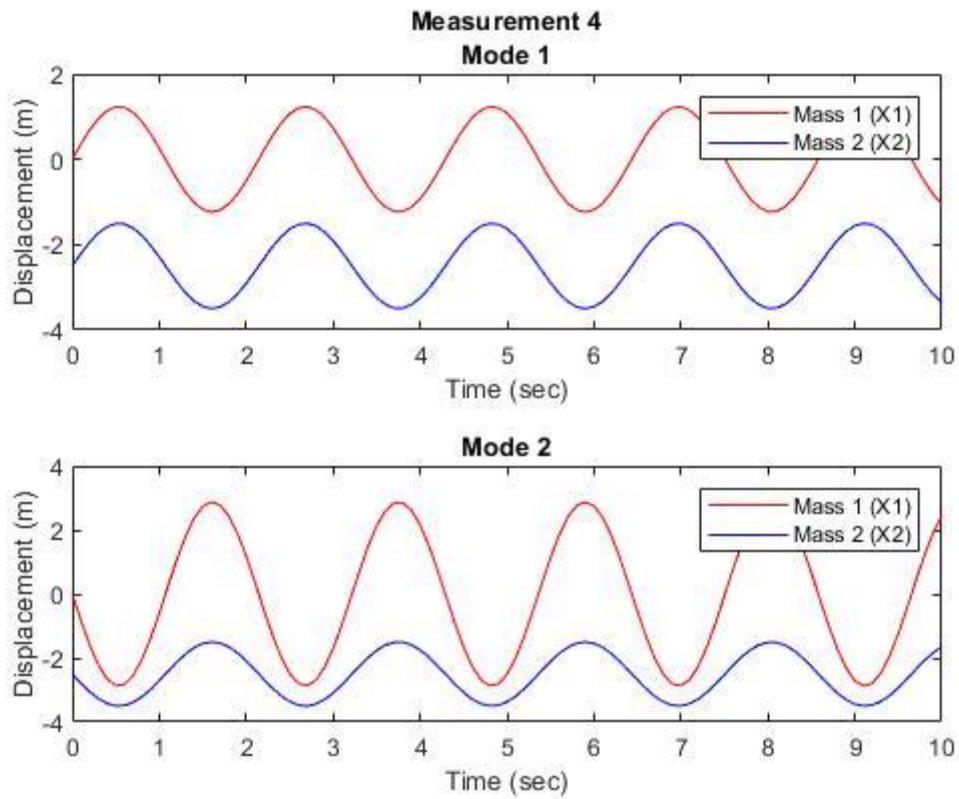
m: kg, k: N/m, x: m, w: rad/sec



Measurement 4

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 4	0.5	1.5	1.6	2.5	2.5	1.237	2.877	1	1	2.929	-2.929

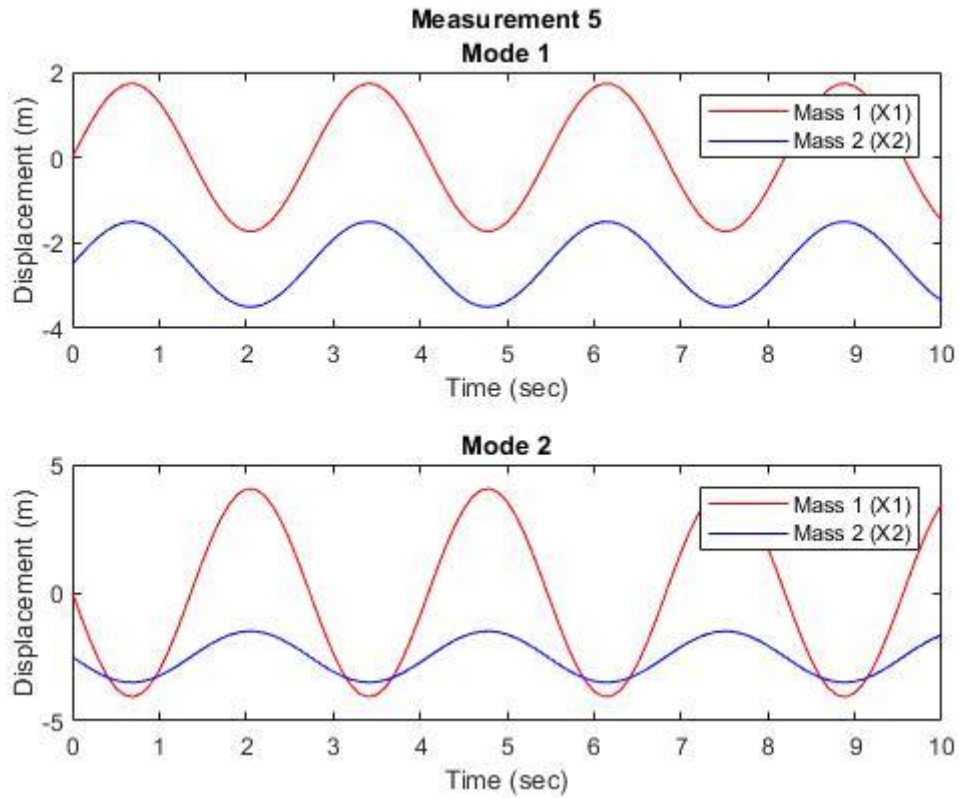
m: kg, k: N/m, x: m, w: rad/sec



Measurement 5

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 5	0.8	1	2	1.5	1	1.736	4.07	1	1	2.300	-2.300

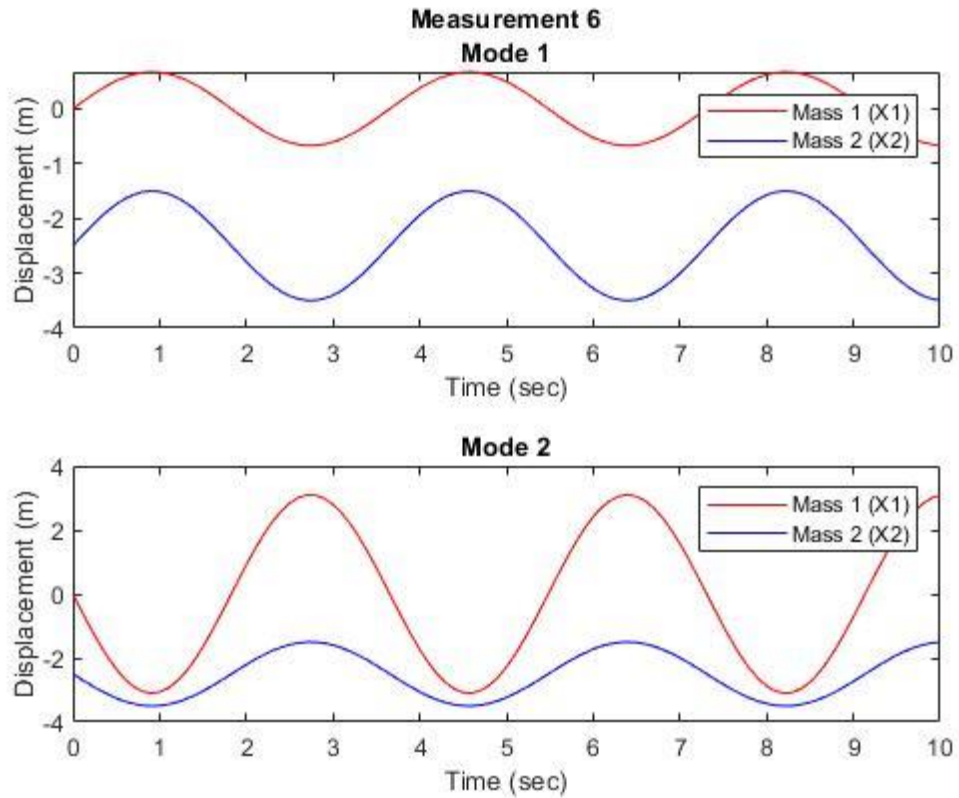
m: kg, k: N/m, x: m, w: rad/sec



Measurement 6

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 6	2	2	2	1.4	1.5	0.671	3.10	1	1	1.719	-1.719

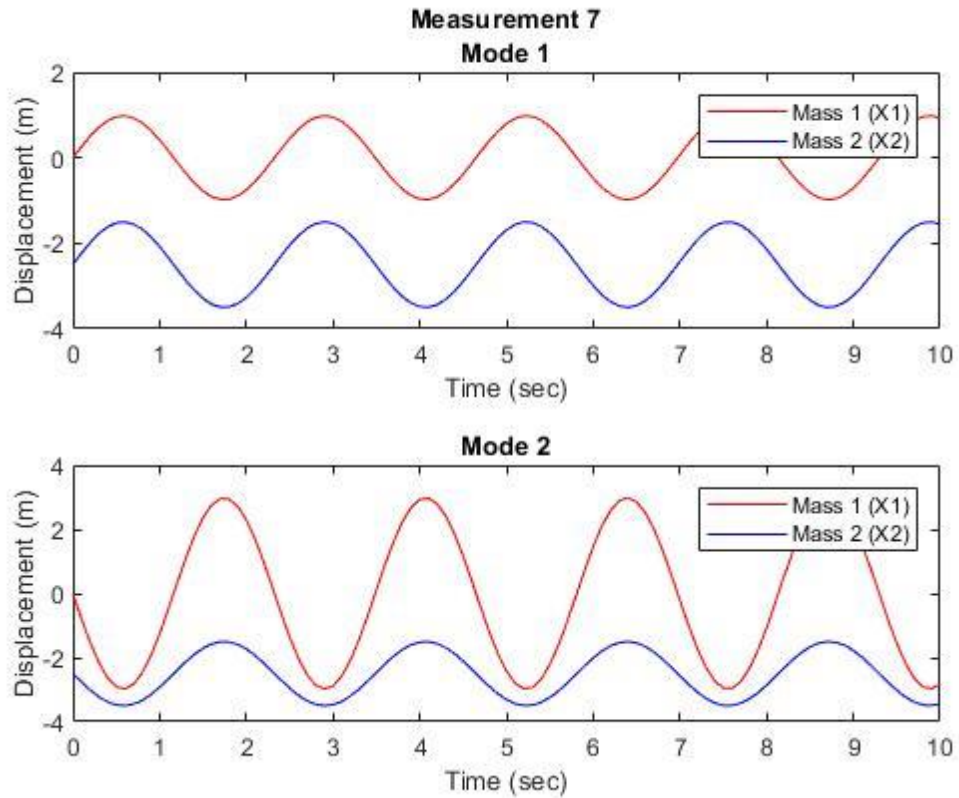
m: kg, k: N/m, x: m, w: rad/sec



Measurement 7

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 7	1.4	0.8	2.3	2.3	2	0.981	2.981	1	1	2.703	-2.703

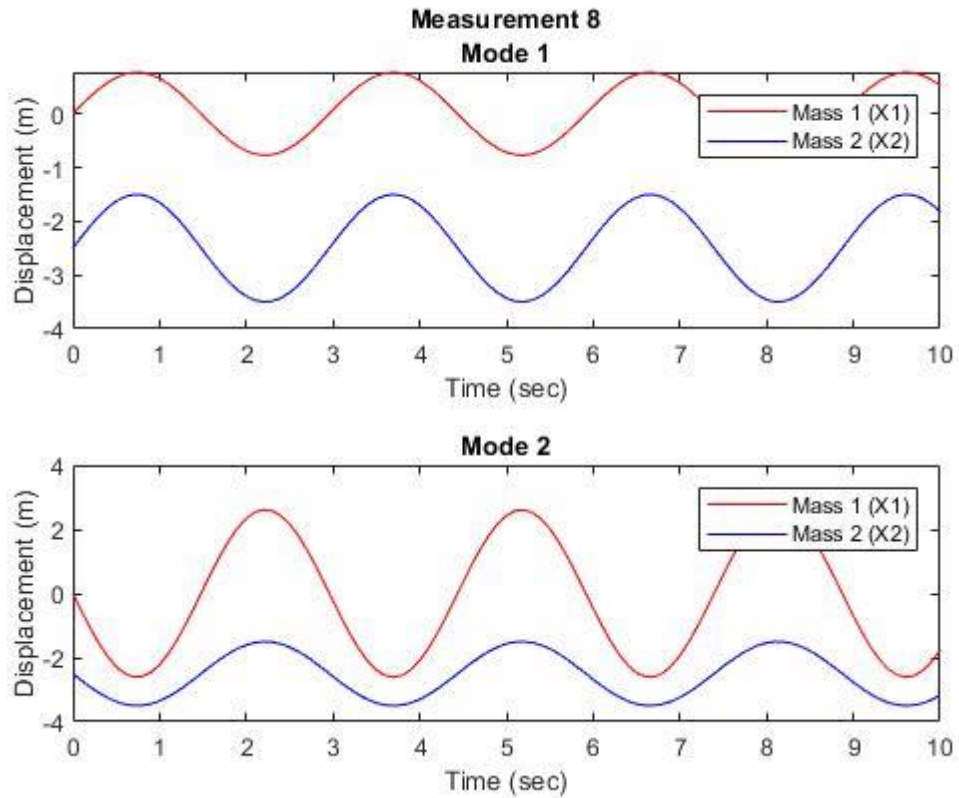
m: kg, k: N/m, x: m, w: rad/sec



Measurement 8

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 8	1.6	2.5	2.1	2.5	1.5	0.772	2.612	1	1	2.124	-2.124

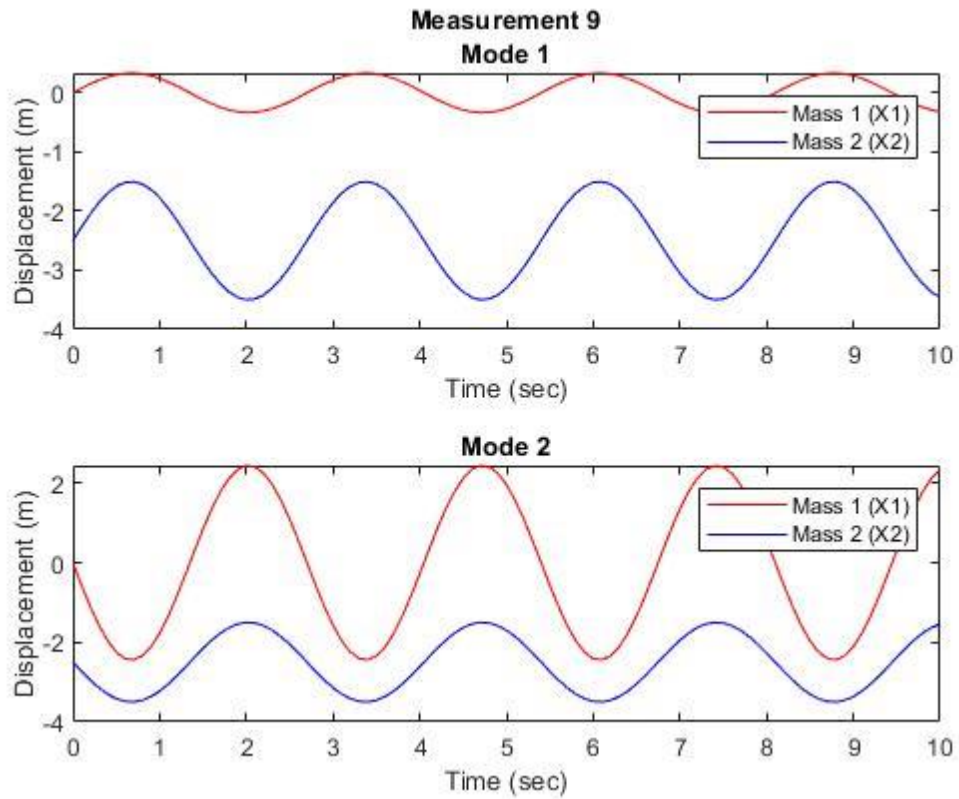
m: kg, k: N/m, x: m, w: rad/sec



Measurement 9

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 9	1.8	0.9	2	1.8	2	0.334	2.445	1	1	2.327	-2.327

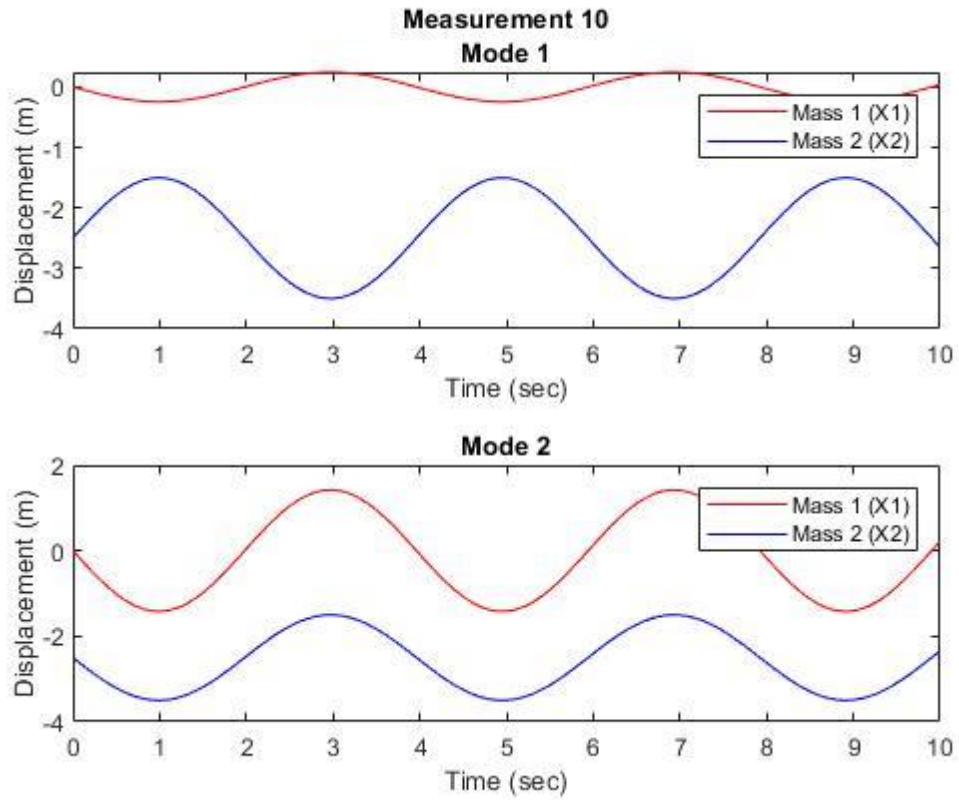
m: kg, k: N/m, x: m, w: rad/sec



Measurement 10

	m_1	m_2	k_1	k_2	k_3	x_1		x_2		w_1	w_2
						Mode 1	Mode 2	Mode 1	Mode 2		
Measurement 10	2.5	2	1	1.5	2	-0.244	1.422	1	1	1.586	-1.586

m: kg, k: N/m, x: m, w: rad/sec



References

- [1] J. Vandiver, and David Gossard. *2.003SC Engineering Dynamics*. Fall 2011. Massachusetts Institute of Technology: MIT OpenCourseWare, <https://ocw.mit.edu>. License: [Creative Commons BY-NC-SA](#).

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- [4] “Student corner,” *Sri Indu Institute of Engineering & Technology*. [Online]. Available: <https://siiet.ac.in/mech/student-corner/>. [Accessed: 05-Dec-2021].

- [5] “ $(w^2)^2 + m_1 + m_2 + w^2 + (-m_1 + k_2 + m_1 + k_3 + m_2 + k_2 + m_2 + k_1) + k_1 + k_2 + k_1 + k_3 + k_2 + k_3 = 0$ - wolfram: Alpha,” *WolframAlpha computational knowledge AI*. [Online]. Available: https://www.wolframalpha.com/input/?i=%28w%5E2%29%5E2%2Bm_1%2Bm_2%2B%2B%2Bw%5E2%2B%28-m_1%2Bk_2%2B-%2Bm_1%2Bk_3%2B-%2Bm_2%2Bk_2%2B-%2Bm_2%2Bk_1%29%2B%2B%2Bk_1%2Bk_2%2B%2Bk_1%2Bk_3%2B%2B%2Bk_2%2Bk_3%2B%3D%2B0. [Accessed: 06-Dec-2021].