Numerical Solutions of Convection-Diffusion Equations by Hybrid Discontinuous Galerkin Methods

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ABSTRACT

A hybrid discontinuous Galerkin method (HDG) for numerical solutions of the steady and time-dependent convection-diffusion equations is presented in this paper. The HDG is proposed as a local discontinuous Galerkin method (LDG) hybridizable method. Numerical results show that the HDG methods can achieve accurate approximate results, and keep local conservation well for strong discontinuous problems.

INTRODUCTION

Over the past several years, significant progresses have been made in developing the discontinuous Galerkin finite element methods for numerical solutions of convection-diffusion problems. Among these methods, local discontinuous Galerkin method (LDG) and hybrid discontinuous Galerkin method (HDG) are most popular and important discontinuous Galerkin finite element methods. The LDG method was introduced by Cockburn and Shu (1998) for the general convection-diffusion problems. This scheme was an extension of the Runge-Kutta discontinuous Galerkin (RKDG) method developed by Cockburn and Shu (1989) for nonlinear hyperbolic systems. The LDG method is one of the most popular discontinuous Galerkin methods, because of their properties of local conservativity, high degree of locality, high parallelizability, and high-order formal accuracy. However, there are some drawbacks for this method. The stabilization parameters are empirical.-

The HDG method was first introduced by Cockburn *et al.* (2009) for symmetric

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second-order elliptic problems, and then extended to steady-state convection-diffusion equations (Cockburn *et al.*, 2009). Nguyen *et al.* (2009) proposed a different approximation for the total flux, and extended this method to time-dependent convection-diffusion problems. The HDG method is proposed as a LDG-hybridizable method. First, by using the LDG method on each element, the approximate scalar variable and the flux are expressed in terms of an approximate trace along the element boundary. Then, a unique value for the trace at element boundaries is obtained by enforcing the continuity of the normal component of the flux. This leads to a global equation system like other hybrid method. Finally, the approximate scalar variable and the flux are solved by the element-wise sweep according to the element boundary traces. The distinctive feature of the HDG method is that the globally coupled degrees of freedom are those of an approximation of the solution defined only on the boundaries of the elements. The amount of calculation work is significantly reduced in comparison with other standard discontinuous Galerkin methods. Another advantage of the HDG method is that the approximate scalar variable converges at a rate of $k+1$ when polynomials of order k are used. This method exhibits optimal convergence properties.

In this paper, both the LDG method and the HDG methods are applied to solve the steady and time-dependent convection-diffusion equations. The similarities and the differences between the LDG method and the HDG methods for numerical solutions of convection-diffusion problems are described in the following sections. Numerical results are provided to assess the convergence and accuracy of these two methods. The HDG method has the better convergence properties than the LDG method. Concluding remarks are drawn in the end of the paper.

PROBLEM STATEMENT AND NOTATION

The convection-diffusion equation

Consider the steady-state convection-diffusion equation for simplicity here, and then extend to time-dependent problems:

$$\nabla\cdot\left(\mathbf{c}u\right)-\nabla\cdot\left(\kappa\nabla u\right)=f,\ \ \text{in }\Omega;\quad u=g_D,\ \ \text{on }\partial\Omega \tag{1}$$

where $\Omega\in R^d$ is a d dimension domain with boundary $\partial\Omega$, f is a source term, κ is a positive diffusivity coefficient, and $\mathbf{c}$ is a velocity vector field.

For applying the LDG method and the HDG methods, we introduce the auxiliary variable $\mathbf{q}$, then rewrite the above equation as a set of first-order equations.

$$\kappa^{-1}\mathbf{q}+\nabla u=0,\ \nabla\cdot\left(\mathbf{c}u+\mathbf{q}\right)=f,\ \ \text{in }\Omega,\quad u=g_D,\ \ \text{on }\partial\Omega \tag{2}$$

Definition and operators of mesh and trace

Let E_h denote the collection of all the disjoint elements that partition Ω. For any element $K\in E_h$, let ∂K be the boundary of K, and $e=\partial K\bigcap\partial\Omega$ is the boundary face if e is non-zero. For any two element $K^+\in E_h$ and $K^-\in E_h$, $e=\partial K^+\bigcap\partial K^-$ is the interior face between K^+ and K^- if e is non-zero. Let ε_h^0 and ε_h^∂ denote the collection of interior and boundary faces respectively, and $\varepsilon_h = \varepsilon_h^0 \bigcup \varepsilon_h^\partial$ denote the union of all the faces.

Let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the outward unit normal vectors of ∂K^+ and ∂K^- respectively. If $e = \partial K^+ \bigcap \partial K^-$ is an interior face, let $\left(\mathbf{q}^\pm, u^\pm\right)$ be the trace of $\left(\mathbf{q}, u\right)$ on e from the interior of K^+ and K^-. Then, we introduce the definition of mean values $\left\{\{\cdot\}\right\}$ and jumps $[\![\cdot]\!]$ on an interior boundary as follows:

$$\left\{\{\mathbf{q}\}\right\} = \left(\mathbf{q}^+ + \mathbf{q}^-\right)/2,\quad \left\{\{u\}\right\} = \left(u^+ + u^-\right)/2,\quad [\![\mathbf{q}\cdot\mathbf{n}]\!] = \mathbf{q}^+\cdot\mathbf{n}^+ + \mathbf{q}^-\cdot\mathbf{n}^-,\quad [\![u\mathbf{n}]\!] = u^+\mathbf{n}^+ + u^-\mathbf{n}^-$$

For a boundary face $e = \partial K^+ \bigcap \partial\Omega$, we set:

$$\left\{\{\mathbf{q}\}\right\} = \mathbf{q}^+,\quad \left\{\{u\}\right\} = u^+,\quad [\![\mathbf{q}\cdot\mathbf{n}]\!] = \mathbf{q}^+\cdot\mathbf{n}^+,\quad [\![u\mathbf{n}]\!] = u^+\mathbf{n}^+$$

The approximation spaces

Let $p\left(D\right)$ denote the space of polynomials which spanned by shape functions of the domain D. D may be an element or a face. The shape functions depend on the element type and the degree of freedom. Four different 2D elements will be test, as shown in figure 1.

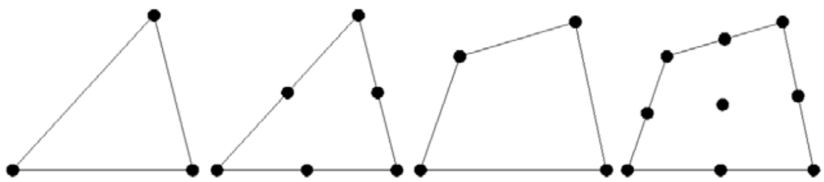


Figure 1. Diffident elements (from left to right: triangle linear element, triangle quadratic element, quadrilateral bi-linear element, quadrilateral bi-quadratic element).

The discontinuous finite element space is defined as follows:

$$W_h = \left\{w \in L^2\left(\Omega\right) : w\big|_K \in p\left(K\right),\ \forall K \in E_h\right\}; M_h = \left\{\mu \in L^2\left(\varepsilon_h\right) : \mu\big|_e \in p\left(e\right),\ \forall e \in \varepsilon_h\right\}$$

The LDG method and the HDG method

In this section, the LDG method and the HDG methods are described in a unified way. Multiplying equation (2) by weight function $\mathbf{v}$ and w, and integrating over the element K_j , one has:

$$\int_{K_j} \mathbf{v}\cdot\left(\kappa^{-1}\mathbf{q} + \nabla u\right) dV = 0;\ \int_{K_j} w\nabla\cdot\left(\mathbf{c}u + \mathbf{q}\right) dV = \int_{K_j} wf dV$$

Integration-by-parts once yields the following weak formulation:

$$\int_{K_j} \kappa^{-1}\mathbf{v}\cdot\mathbf{q} dV - \int_{K_j} \left(\nabla\cdot\mathbf{v}\right) u dV + \int_{\partial K_j} \mathbf{n}\cdot\mathbf{v} u dS = 0;\quad -\int_{K_j} \nabla w\cdot\left(\mathbf{c}u + \mathbf{q}\right) dV + \int_{\partial K_j} w\mathbf{n}\cdot\left(\mathbf{c}u + \mathbf{q}\right) dS = \int_{K_j} fw dV$$

Let the numerical trace $\widehat{\mathbf{c}u} + \hat{\mathbf{q}}$ and $\hat{u}$ be approximations to $\mathbf{c}u + \mathbf{q}$ and u over ∂K_j respectively. Now, we seek an approximation

$$\left(\mathbf{q}_\mathbf{h}, u_h, \widehat{\mathbf{c}u_h} + \hat{\mathbf{q}}_h, \hat{u}_h\right) \in \left(W_h\right)^d \times W_h \times \left(M_h\right)^d \times M_h$$

Such that for all $K_j \in E_h$,

$$\begin{aligned}
&\int_{K_j} \kappa^{-1}\mathbf{v}\cdot\mathbf{q}_h dV - \int_{K_j} (\nabla\cdot\mathbf{v}) u_h dV + \int_{\partial K_j} \mathbf{n}\cdot\mathbf{v}\hat{u}_h dS = 0 \qquad \forall \mathbf{v}\in\left(p\left(K_j\right)\right)^d \\
&-\int_{K_j} \nabla w\cdot(\mathbf{c}u_h + \mathbf{q}_h) dV + \int_{\partial K_j} w\mathbf{n}\cdot\left(\widehat{\mathbf{c}u_h} + \hat{\mathbf{q}}_h\right) dS = \int_{K_j} fwdV \qquad \forall w\in p\left(K_j\right)
\end{aligned} \tag{3}$$

The LDG method

In the LDG method, $(\mathbf{q_h}, u_h)$ is unknown, $\left(\widehat{\mathbf{c}u_h} + \hat{\mathbf{q}}_h, \hat{u}_h\right)$ is explicitly defined as follows:

$$\widehat{\mathbf{c}u_h} = \begin{cases} \mathbf{c}u_h^+ & if \quad \mathbf{c}\cdot\mathbf{n}\geq 0 \\ \mathbf{c}u_h^- & if \quad \mathbf{c}\cdot\mathbf{n} < 0 \end{cases};\ \hat{\mathbf{q}}_h = \{\{q_h\}\} - C_{11}[\![u_h\mathbf{n}]\!] - \mathbf{C}_{12}[\![\mathbf{q}_h\cdot\mathbf{n}]\!];\ \hat{u}_h = \{\{u_h\}\} - C_{22}[\![\mathbf{q}_h\cdot\mathbf{n}]\!] + \mathbf{C}_{12}\cdot[\![u_h\mathbf{n}]\!]$$

Then an element-by-element iteration, with successive substitution, is used to solve the system. The iteration sweep is from element K_1 to element K_n. When solving the equation (3) over element K_j, the data of neighbor elements is consider as known and the boundary can be specified. After a round of iteration, the L^2-norm relative error will be calculated to determine if the convergence of the solution is achieved. If not, a new round of iteration needs to be applied.

The HDG method

In the HDG method, $(\mathbf{q}_h, u_h, \hat{u}_h)$ is unknown, and the numerical trace $\widehat{\mathbf{c}u_h} + \hat{\mathbf{q}}_h$ is considered as: $\widehat{\mathbf{c}u_h} + \hat{\mathbf{q}}_h = \mathbf{c}\hat{u}_h + \mathbf{q}_h + \tau(u_h - \hat{u}_h)\mathbf{n}$, where τ is the local stabilization parameter. Since there is a more unknown quantity $\hat{u}_h$, the conservativity condition by enforcing the continuity of the normal component of the numerical flux is introduced,

$$\int_e \left[\!\left[\left(\widehat{\mathbf{c}u_h} + \hat{\mathbf{q}}_h\right)\cdot\mathbf{n}\right]\!\right]\mu dS = 0 \qquad \forall\mu\in p(e), \forall e\in\varepsilon_h^0 \tag{4}$$

In order to get a global matrix only for $\hat{u}_h$, we define $\left(\mathbf{q}_h^{m,f}, u_h^{m,f}\right)$ as follows. Once the numerical trace $\hat{u}_h = m$ and the source term f are determined, the pair $\left(\mathbf{q}_h^{m,f}, u_h^{m,f}\right)\in\left(W_h(K)\right)^d\times W_h(K)$ satisfied the following equation can be found,

$$\begin{aligned}
&\int_{K_j} \kappa^{-1}\mathbf{v}\cdot\mathbf{q}_h^{m,f} dV - \int_{K_j} (\nabla\cdot\mathbf{v}) u_h^{m,f} dV + \int_{\partial K_j} \mathbf{n}\cdot\mathbf{v}mdS = 0 \\
&-\int_{K_j} \nabla w\cdot\left(\mathbf{c}u_h^{m,f} + \mathbf{q}_h^{m,f}\right) dV + \int_{\partial K_j} w\mathbf{n}\cdot\left(\widehat{\mathbf{c}u_h}^{m,f} + \hat{\mathbf{q}}_h^{m,f}\right) dS = \int_{K_j} fwdV
\end{aligned} \tag{5}$$

where $\widehat{\mathbf{c}u_h}^{m,f} + \hat{\mathbf{q}}_h^{m,f} = \mathbf{c}m + \mathbf{q}_h^{m,f} + \tau(u_h^{m,f} - m)\mathbf{n}$. It is easy to see that $\mathbf{q}_h = \mathbf{q}_h^{\hat{u}_h,f}$, $u_h = u_h^{\hat{u}_h,f}$. Consider equation (4) together, such that

$$\int_e \left[\!\left[\left(\widehat{\mathbf{c}u_h}^{\hat{u}_h,f} + \hat{\mathbf{q}}_h^{\hat{u}_h,f}\right)\cdot\mathbf{n}\right]\!\right]\mu dS = 0 \qquad \forall\mu\in p(e), \forall e\in\varepsilon_h^0$$

Let

$$a_h(\eta,\mu) = -\int_{\varepsilon_h^0} \left[\!\left[\left(\widehat{\mathbf{c}u_h}^{\eta,0} + \hat{\mathbf{q}}_h^{\eta,0}\right)\cdot\mathbf{n}\right]\!\right]\mu dS; \quad b_h(\mu) = \int_{\varepsilon_h^0} \left[\!\left[\left(\widehat{\mathbf{c}u_h}^{0,f} + \hat{\mathbf{q}}_h^{0,f}\right)\cdot\mathbf{n}\right]\!\right]\mu dS$$

So the solution of $\hat{u}_h$ satisfies

$$a_h\left(\hat{u}_h, \mu\right) = b_h\left(\mu\right) \qquad \forall \mu \in M_{\mathrm{h}}$$

If $\mathbf{A}$ is the sparse matrix which associated with the bilinear form $a_h\left(\cdot,\cdot\right)$, $\mathbf{F}$ is the matrix associated with the linear form $b_h\left(\cdot\right)$, we can solve the system $\mathbf{AX} = \mathbf{F}$ and get the solution $\hat{u}_h$. Then according to (5), an element-by-element procedure can be applied to get the $\left(\mathbf{q_h}, u_h\right)$.

The local stabilization parameter

There are many choices of the stabilization parameter C_{11}, C_{12}, C_{22} and τ, one can see the reference paper (Castillo *et al.,* 2000) for details. In this paper, we set $C_{12} = C_{22} = 0$, $C_{11} = C\kappa / h$ for the LDG method, where h is the minimum length of all the face, C is a const depend on the element type: $C = 2$ for triangle linear element, $C = 4$ for triangle quadratic element, $C = 1$ for quadrilateral bi-linear element, and $C = 2$ for quadrilateral bi-quadratic element. For the HDG method $\tau = \left|\mathbf{c}\cdot\mathbf{n}\right| + \kappa / l$ is adopted, where l is the length of current integration face, and $\mathbf{n}$ is the unit normal vector.

NUMERICAL EXAMPLES

In this section, both triangle mesh and quadrilateral mesh are adopted (see figure 2). The variable h is defined as the minimum length of all the faces.

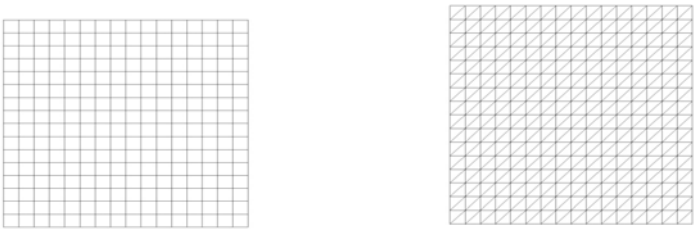


Figure 2. Different mesh types ($h = 1/16$): quadrilateral (left), triangle (right).

Steady convection-diffusion example

In this example, a classic steady convection-diffusion problem is considered in $\Omega = \left(0,1\right)^2$. We select $\kappa = 1$, $\mathbf{c} = \left(-5, -10\right)$, and $f = 0$. The boundary condition is $g_D = \left(\frac{1-e^{-5x}}{1-e^{-5}}\right)\left(\frac{1-e^{-10y}}{1-e^{-10}}\right)$. It can be seen that $\left(\frac{1-e^{-5x}}{1-e^{-5}}\right)\left(\frac{1-e^{-10y}}{1-e^{-10}}\right)$ is also the exact solution.

The results comparison of the LDG method and the HDG methods is shown in table 1. In the table, the error and the order of convergence of u_h in L^2-norm are presented. It can be seen that both the LDG and the HDG can get high accuracy for all types of mesh. Quadratic and bi-quadratic element mesh can get better result than linear and bi-linear, even when the comparison is conducted between the linear/bi-linear mesh of $h = 1/(2n)$ and the quadratic/bi-quadratic mesh of $h = 1/n$. The table also shows that the HDG method gets better convergence properties than the LDG method. When polynomials of order k are used, the HDG converges at a rate of order $k+1$, which is better than the LDG method.

Time-dependent convection-diffusion example

On the basis of the steady convection-diffusion problem, time-dependent problems can be solved by the backward Euler scheme. In this example, we consider a discontinuous convection dominant time dependent problem in $\Omega=(0,1)^2$, we select $\kappa=1e-20$, $\mathbf{c}=(0.25,0.25)$, $f=0$, and the boundary condition is $g_D=0$. The initial condition is:

$$u(x,y,0)=\begin{cases}\sin(4\pi x)\sin(4\pi y) & \text{if } (0<x<0.25, 0<y<0.25)\\ 0 & \text{otherwise}\end{cases}$$

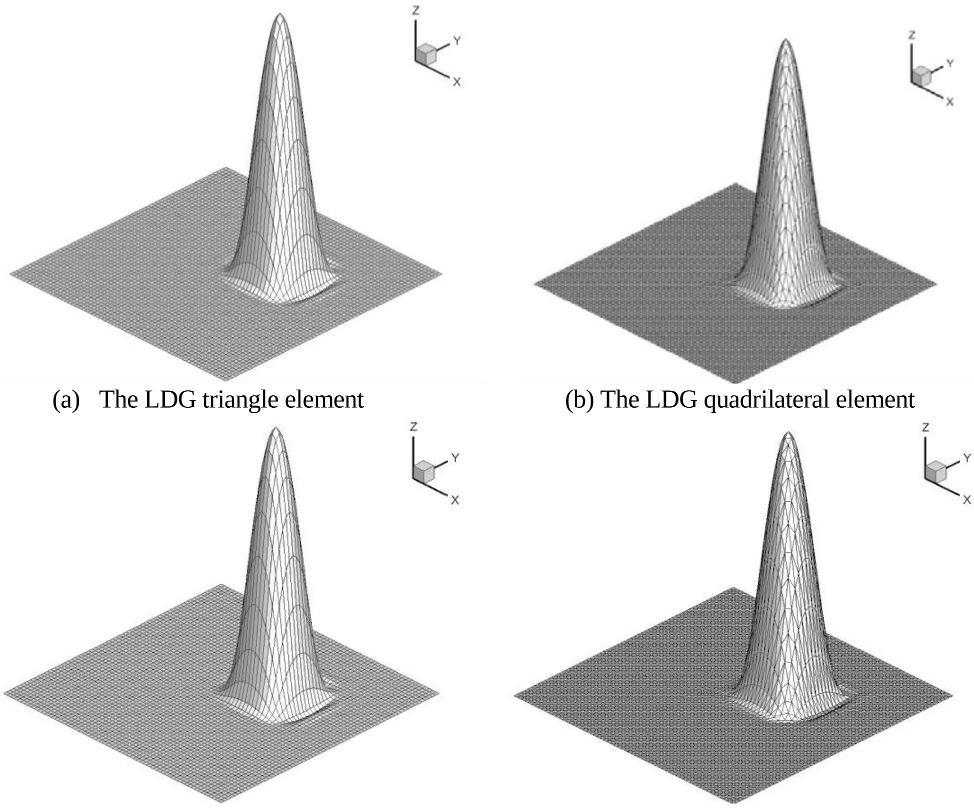
We use both linear triangle and bi-linear quadrilateral mesh, and set $h=1/64$, $\Delta t=0.00078125$. The results comparison is shown in figure 3. The consuming time comparison of the LDG and HDG methods are presented in table 2. From the figures, it can be seen that the wave can keep steep when it propagates along the mesh and there are no nonphysical oscillations. It shows that both the LDG method and the HDG method can keep local conservation excellently, and the conservation is independent of the element type. From the table 2, we can see that the HDG method costs the less computational time than the LDG method, even 60% reduction for the quadrilateral mesh.

Table 1. Results comparison of the LDG and the HDG methods.

LDG				HDG			
Triangle Mesh		$\Vert u-u_h\Vert_{L^2(\Omega)}$		Triangle Mesh		$\Vert u-u_h\Vert_{L^2(\Omega)}$	
	1/h	Error	Order		1/h	Error	Order
Linear Element	16	4.31e-3	--	Linear Element	16	5.29e-3	--
	32	1.30e-3	1.72		32	1.33e-3	1.99
	64	3.56e-4	1.86		64	3.33e-4	2.00
Quadratic Element	8	6.23e-4	--	Quadratic Element	8	1.30e-3	--
	16	9.82e-5	2.66		16	1.42e-4	3.19
	32	1.39e-5	2.82		32	1.38e-5	3.36
Quadrilateral Mesh		$\Vert u-u_h\Vert_{L^2(\Omega)}$		Quadrilateral Mesh		$\Vert u-u_h\Vert_{L^2(\Omega)}$	
	1/h	Error	Order		1/h	Error	Order
Bi-Linear Element	16	6.01e-3	--	Bi-Linear Element	16	5.65e-3	--
	32	2.07e-3	1.53		32	1.42e-3	1.99
	64	6.21e-4	1.73		64	3.53e-4	2.01
Bi-Quadratic Element	8	9.71e-4	--	Bi-Quadratic Element	8	1.42e-3	--
	16	1.79e-4	2.43		16	1.56e-4	3.19
	32	2.69e-5	2.73		32	1.51e-5	3.37

CONCLUSIONS

The local discontinuous Galerkin method (LDG) and the hybrid discontinuous Galerkin method (HDG) are applied to solve the steady and time-dependent convection-diffusion problems in this paper. Steady and time-dependent convection-diffusion problems are solved respectively. It is showed that (a) both the LDG and the HDG methods can get accurate approximate results; (b) both methods can keep local conservation well for strong discontinuous problems; (c) the HDG method has the better convergence properties due to the introduction of auxiliary variable. When polynomials of order k are used, and the HDG method converges at a rate of order $k+1$, which is better than the LDG method. Moreover, the HDG method costs the less computational time. We will extend the HDG method to the nonlinear convection-diffusion and incompressible Navier-Stokes equations in the future works, thanks to its excellent convergence properties.



(a) The LDG triangle element (b) The LDG quadrilateral element

(c) The HDG triangle element (d) The HDG quadrilateral element

Figure 3. Numerical result at T=2s.

Table 2. Time consuming comparison of the LDG and HDG methods.

Time (s)	LDG	HDG
Quadrilateral Mesh	1803	723
Triangle Mesh	1778	942

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REFERENCES

1. Castillo, P., Cockburn, B., Perugia, I., and Schotzau, D. (2000), An a priori error analysis of the local discontinuous galerkin method for elliptic problems. *SIAM J. Numer. Anal.* 38, pp. 1676–1706.
2. Cockburn, B., Dong, B., Guzmán, J., Restelli, M., and Sacco, R. (2009), A hybridizable discontinuous Galerkin method for steady-state convection-diffusion-reaction problems. *SIAM J. Sci. Comput.* 31(5), pp. 3827–3846.
3. Cockburn B., Gopalakrishnan J., and Lazarov R. (2009), Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second-order elliptic problems. *SIAM J. Numer. Anal.* 47(2), pp. 1319–1365.
4. Cockburn, B., and Shu, C.W. (1989), TVB Runge-Kutta local projection, discontinuous Galerkin finite element method for scalar conservation laws II: General framework. *Math. Comp.* 52, pp. 411–435.
5. Cockburn, B., and Shu, C.W. (1991), The Runge-Kutta local projection P1-discontinuous Galerkin method for scalar conservation laws. *Math. Anal. Numer.* 25, pp. 337–361.
6. Cockburn, B., and Shu, C.W. (1998), The local discontinuous Galerkin time-dependent method for convection-diffusion systems. *SIAM J. Numer. Anal.* 35, pp. 2440–2463.
7. Nguyen, N.C., Peraire, J., and Cockburn, B. (2009), An implicit high-order hybridisable discontinuous Galerkin method for linear convection-diffusion equations. *J. Comput. Phys.* 228, pp. 3232–3254.
8. Nguyen, N.C., Peraire, J., and Cockburn, B. (2009), An implicit high-order hybridisable discontinuous Galerkin method for nonlinear convection-diffusion equations. *J. Comput. Phys.* 228, pp. 8841–8855.
9. Nguyen, N.C., Peraire, J., and Cockburn, B. (2010), A hybridisable discontinuous Galerkin method for Stokes flow. *Comput. Methods Appl. Mech. Engrg*. 199, pp. 582–597.
10. Nguyen, N.C., Peraire, J., and Cockburn, B. (2011), An implicit high-order hybridisable discontinuous Galerkin method for the incompressible Navier-Stokes equations. *J. Comput. Phys.* 230, pp. 1147–1170.