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CMPE462 - Spring '20
Linear Algebra Assessment Solutions

1. (10 Points) **Solution 1**:

- Matrix A
  Column vectors of A:

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix}, v_2 = \begin{bmatrix}2\\4\end{bmatrix} \tag{1}$$

These vectors are linearly dependent, since 2*v_1 = v_2. Therefore, v_1 is basis for the column space. So column space for A is any vector of the form:

$$\begin{bmatrix}c_1\\2c_1\end{bmatrix} \tag{2}$$

where c_1 is scalar.

- Matrix B
  Column vectors of B:

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix}, v_2 = \begin{bmatrix}2\\0\end{bmatrix}, v_3 = \begin{bmatrix}3\\4\end{bmatrix} \tag{3}$$

For matrix B, v_1 and v_3 are linearly independent, while v_2 is a linear combination of v_1, since 2*v_1 = v_2. Therefore, v_1 and v_3 are basis for the column space. So column space for B is any vector of the form:

$$\begin{bmatrix}c_1 + 3c_3\\4c_3\end{bmatrix} \tag{4}$$

where c_1, c_3 are scalars.

2. (10 points)

In order to classify a subset of R^n as subspace, these conditions must be satisfied:

(a) Subspace must contain origin.

(b) Given x, y are elements of subspace S, x+y is also element of S. In other words, S is closed under addition.

(c) Given x is element of subspace S and c is any real number, cx is also element of S. In other words, S is closed under scalar multiplication.

- **Solution 2(a):** Yes.
  Any line through (0,0,0) is a subspace of R^3. Because of the fact that it contains origin, it holds first condition. Assuming x,y are vectors on that line, if we add them resulting vector is still on that line. Again, if we multiply a vector on that line with any real number, resulting vector is still on that line. So, it is a subspace.
- **Solution 2(b):** No.
  Quarter plane is not a subspace. Since, it does not hold 3rd condition. For example, if our plane is vectors(x,y) whose components are positive or zero. When we multiply vector with negative real number, it does not element of subspace. Although other conditions are satisfied, it is not a subspace.

3. (10 points)

- **Solution 3(a):**

$$v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \tag{5}$$

If we use standart Euclidean inner product for R^3, we can see that results are 0, which means they form an orthogonal set.

$$v_1^T v_2 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{6}$$

$$v_1^T v_3 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{7}$$

$$v_2^T v_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{8}$$

However, since v_1 and v_3 has length other than 1 they do not form an orthonormal set.

$$||v_1|| = \sqrt[2]{(-2)^2 + 0^2 + 1^2} = \sqrt[2]{5} \tag{9}$$

$$||v_2|| = \sqrt[2]{0^2 + 1^2 + 0^2} = \sqrt[2]{1} \tag{10}$$

$$||v_3|| = \sqrt[2]{2^2 + 0^2 + 4^2} = \sqrt[2]{20} \tag{11}$$

- **Solution 3(b):** If we make each vectors' length 1, they form an orthonormal set. In order to make their length 1, alter the vector elements by dividing them to current vector length, which is found above.

$$v_1 = \begin{bmatrix} -2/\sqrt[2]{5} \\ 0 \\ 1/\sqrt[2]{5} \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 2/\sqrt[2]{20} \\ 0 \\ 4/\sqrt[2]{20} \end{bmatrix} \tag{12}$$

4. (10 points) **Solution 4:**

$$x = \begin{bmatrix} x_1 \\ x_2 \\ * \\ * \\ * \\ x_m \end{bmatrix} \tag{13}$$

$$y^T = \begin{bmatrix} y_1 & y_2 & * & * & * & y_n \end{bmatrix} \tag{14}$$

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & * & * & * & x_1y_n \\ x_2y_1 & x_2y_2 & * & * & * & x_2y_n \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ x_my_1 & x_my_2 & * & * & * & x_my_n \end{bmatrix} \tag{15}$$

The rank of a matrix is the maximum number of linearly independent column vectors. We can see that columns are linearly dependent. Each column vector can be written as linear combination of first column vector. For example, $v_2 = (y_2/y_1)^*v_1$. Therefore, basis for the column space is constructed only using one column vector. So their rank is 1.

5. (10 points) **Solution 5:**

X is mxn matrix composed from n vectors, where x is vector with m elements.

$$X = \begin{bmatrix} | & | & * & * & * & | \\ | & | & * & * & * & | \\ x_1 & x_2 & * & * & * & x_n \\ | & | & * & * & * & | \\ | & | & * & * & * & | \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} & * & * & * & x_{n1} \\ x_{12} & x_{22} & * & * & * & x_{n2} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ x_{1m} & x_{2m} & * & * & * & x_{nm} \end{bmatrix} \tag{16}$$

Y^T is pxn matrix composed from n vectors, where y is vector with p elements.

$$Y^T = \begin{bmatrix} | & | & * & * & * & | \\ | & | & * & * & * & | \\ y_1 & y_2 & * & * & * & y_n \\ | & | & * & * & * & | \\ | & | & * & * & * & | \end{bmatrix} = \begin{bmatrix} y_{11} & y_{21} & * & * & * & y_{n1} \\ y_{12} & y_{22} & * & * & * & y_{n2} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ y_{1p} & y_{2p} & * & * & * & y_{np} \end{bmatrix} \tag{17}$$

$$Y = \begin{bmatrix} - & - & y_1 & - & - \\ - & - & y_2 & - & - \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ - & - & y_n & - & - \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & * & * & * & y_{1p} \\ y_{21} & y_{22} & * & * & * & y_{2p} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ y_{n1} & y_{n2} & * & * & * & y_{np} \end{bmatrix} \tag{18}$$

$$XY = \begin{bmatrix} x_{11}y_{11} + ... + x_{n1}y_{n1} & x_{11}y_{12} + ... + x_{n1}y_{n2} & * & * & * & x_{11}y_{1p} + ... + x_{n1}y_{np} \\ x_{12}y_{11} + ... + x_{n2}y_{n1} & x_{12}y_{12} + ... + x_{n2}y_{n2} & * & * & * & x_{12}y_{1p} + ... + x_{n2}y_{np} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ x_{1m}y_{11} + ... + x_{nm}y_{n1} & x_{1m}y_{12} + ... + x_{nm}y_{n2} & * & * & * & x_{1m}y_{1p} + ... + x_{nm}y_{np} \end{bmatrix} \tag{19}$$

$$XY = \sum_{i=1}^{n} x_i y_i^T = \begin{bmatrix} x_{11}y_{11} & x_{11}y_{12} & * & * & * & x_{11}y_{1p} \\ x_{12}y_{11} & x_{12}y_{12} & * & * & * & x_{12}y_{1p} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ x_{1m}y_{11} & x_{1m}y_{12} & * & * & * & x_{1m}y_{1p} \end{bmatrix} + ... + \begin{bmatrix} x_{n1}y_{n1} & x_{n1}y_{n2} & * & * & * & x_{n1}y_{np} \\ x_{n2}y_{n1} & x_{n2}y_{n2} & * & * & * & x_{n2}y_{np} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ x_{nm}y_{n1} & x_{nm}y_{n2} & * & * & * & x_{nm}y_{np} \end{bmatrix} \tag{20}$$

Each matrix dimension is mxp. So summation of those matrices will also result in mxp matrix. Resulting matrix is equal to XY.

6. (10 points) **Solution 6:**
For symmetric matrices, matrix is equal to its transposed matrix. In order words, below equation must be satisfied.

$$A = A^T \tag{21}$$

In our problem, X^TX is symmetric since its transpose is equal to itself,

$$(X^TX)^T = X^TX \tag{22}$$

A symmetric matrix is positive semi-definite, if and only if,

$$x^TAx >= 0 \tag{23}$$

In our case it is symmetric matrix, and it holds above equation. Here is the proof.

$$x^TX^TXx = (Xx)^TXx \tag{24}$$

It is an inner product of transpose of Xx with Xx. We know that result of inner product of matrix with its transpose must be greater or equal to 0. Result can be 0 if matrix is zero-matrix, otherwise it is greater than 0. As a result we can conclude that X^TX is both symmetric and positive semi-definite matrix.

A symmetric matrix is positive definite, if below equation is hold for x is not zero vector.

$$x^TAx > 0 \tag{25}$$

In our case following condition must be satisfied,

$$(Xx)^T(Xx) > 0 \tag{26}$$

Result can be greater than 0, if Xx is not zero matrix. In other words, if Xx is not zero matrix for x is not zero vector, X^TX can be positive definite. Xx can be zero matrix for x is not zero vector, when X has linearly dependent columns. Thus, X^TX is positive definite when X's columns are linearly independent, which means when it is fully ranked.

7. (15 points)

- (5 points) **Solution 7(a)**
When it is a low rank matrix, c can not be calculated. Matrix can be invertible if it is fully ranked. Since it is low rank matrix, we can not invert it. Thus, c can not be calculated.
- (10 points) **Solution 7(b)**
From Question6, we know that A^TA is symmetric and positive semi-definite matrix. One of the properties of positive semi-definite matrices is their eigenvalues are greater than or equal to 0.

$$(A^TA)v_1 = \lambda_1 v_1 \tag{27}$$

$$\lambda_1 >= 0 \tag{28}$$

If we add positive term to $A^T A$, result matrix has eigenvalue greater than 0, which means it become positive definite.

$$(A^T A + \lambda)v_1 = \lambda_1 v_1 + \lambda v_1 = (\lambda_1 + \lambda)v_1 \tag{29}$$

$$\lambda_1 + \lambda > 0 \tag{30}$$

If a matrix is positive definite, again from Question6 we can say that its columns are linearly independent, it is fully ranked, so it can be invertible. Thus, when that matrix is invertible, c can be calculated.

8. (25 points) **Solution 8:**
We are expected to minimize square of l2 norm of a vector. l2 norm of a vector is calculated as below,

$$||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{31}$$

The function in our problem is (x-y) and y = 1.1 and it is 1 dimensional, so we are expected to minimize following function,

$$f(x) = (x - 1.1)^2 \tag{32}$$

$$f(x) = (x^2 - 2.2x + 1.21) \tag{33}$$

If we take the derivative of function and make it equal to 0, we can get local points.

$$df/dx = (2x - 2.2) = 0 \tag{34}$$

$$2x = 2.2 \tag{35}$$

$$x = 1.1 \tag{36}$$

If we take second derivative of function, we can conclude whether given point is local maxima or minima.

$$d^2 f/dx^2 = 2 \tag{37}$$

Since it is greater than zero, it is local minima. As a result since x is natural number, we need to check closest natural numbers to 1.1.

Closest natural numbers to 1.1 is 1 and 2. We need to get results of f(1) and f(2) in order to determine which one is more smaller.

$$f(1) = (1 - 1.1)^2 \tag{38}$$

$$f(1) = (-0.1)^2 = 0.01 \tag{39}$$

$$f(2) = (2 - 1.1)^2 \tag{40}$$

$$f(2) = (0.9)^2 = 0.81 \tag{41}$$

Since f(1) is less than f(2). Result of problem is x = 1.