CMPE 462: Machine Learning (Spring 2020)
Linear Algebra Assessment (counts for 2 quizzes = 4%)
Due February 18, 2020 by 14:00pm on Moodle

Machine learning is an interdisciplinary field that involves the following topics: linear algebra, probability, and numerical optimization. Before proceeding in this class, make sure you have sufficient linear algebra background.

- Study the following questions and show your solution steps clearly.
- Submit 4 deliverables. Every file you will submit online must be named with your student ID. Your name/surname should not appear in any part of the assignment.
    1. Submit a .zip file through Moodle via **Linear Algebra Assessment** assignment. The .zip file must consist of .pdf and .tex files of your solutions. Since other file formats will not be accepted and receive zero, use the already provided .tex file as the solution template.
    2. Submit only .pdf of your solutions via **Linear Algebra Assesment - Turnitin** assignment in Moodle.
    3. Submit signed Academic Integrity Document as a .pdf through the assignment **Academic Integrity Document** in Moodle. The assignments of the students who do not submit the signed Academic Integrity Document will not be evaluated.
    4. Last, leave a printed/stapled copy of your solutions to Rıza Özçelik's mailbox at the BM secretary office.
- For any submission related questions mail to `riza.ozcelik@boun.edu.tr` whereas for content related questions mail to `inci.baytas@boun.edu.tr`

Important Note: All the assignments will be checked by plagiarism softwares. Students who attempt cheating, which includes copying answers from other students or internet, will automatically FAIL the class.

1. (10 points) Describe the column spaces for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

2. (10 points)
    (a) Is any line through (0,0,0) a subspace of $\mathbb{R}^3$. Explain.
    (b) Is quarter-plane (e.g., vectors (x,y) whose components are positive or zero) a subspace?

3. (10 points) Given the three vectors $v_1 = (-2, 0, 1), v_2 = (0, 1, 0)$ and $v_3 = (2, 0, 4)$ in $\mathbb{R}^3$.
    - Show that they form an orthogonal set under the standard Euclidean inner product for $\mathbb{R}^3$ but not an orthonormal set.
    - Turn them into a set of vectors that will form an orthonormal set of vectors under the standard Euclidean inner product for $\mathbb{R}^3$.

4. (10 points) Given $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, show that the rank of matrix xy^T is one.

5. (10 points) Given $X = [x_1, x_2, \cdots, x_n] \in \mathbb{R}^{m\times n}$ where $x_i \in \mathbb{R}^m$ for all i, and $Y^T = [y^1, y^2, \cdots, y^n] \in \mathbb{R}^{p\times n}$ where $y^i \in \mathbb{R}^p$ for all i. Show that

$$XY = \sum_{i=1}^{n} x_i (y^i)^T.$$

6. (10 points) Given $X \in \mathbb{R}^{m\times n}$, show that the matrix $X^T X$ is symmetric and positive semi-definite. When is it positive definite?

7. (15 points) Consider $c = \left(A^T A\right)^{-1} A^T b$, where $b \in \mathbb{R}^N$, $A \in \mathbb{R}^{N\times d}$ and $N > d$.

   (a) (5 points) When $A^T A$ is a low rank matrix, c cannot be calculated. Why?

   (b) (10 points) Show why can adding a positive term $c = \left(A^T A + \lambda \mathbf{I}\right)^{-1} A^T b$ overcome the problem in (a)?

8. (25 points) An *Euclidean projection* of a d-dimensional point $y \in \mathbb{R}^d$ to a set Ω is given by the following optimization problem:

$$x^* = \arg\min_x \|x - y\|_2^2, \quad \text{subject to: } x \in \Omega \tag{1}$$

   where Ω is the *feasible set*, $\|\cdot\|_2$ is the ℓ_2 norm of a vector, and $x^* \in \mathbb{R}^d$ is the projected vector. What is x^* if $y = 1.1$ and $\Omega = \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.